



Essential Spectrum for Maxwell's Equations

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Abstract. We study the essential spectrum of operator pencils associated with anisotropic Maxwell equations, with permittivity ε , permeability μ and conductivity σ , on finitely connected unbounded domains. The main result is that the essential spectrum of the Maxwell pencil is the union of two sets: namely, the spectrum of the pencil $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$, and the essential spectrum of the Maxwell pencil with constant coefficients. We expect the analysis to be of more general interest and to open avenues to investigation of other questions concerning Maxwell's and related systems.

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1. Introduction

In this paper we consider the essential spectrum of linear operator pencils arising from the Maxwell system

$$\begin{cases} \operatorname{curl} H = -i(\omega\varepsilon + i\sigma)E & \text{in } \Omega, \\ \operatorname{curl} E = i\omega\mu H & \text{in } \Omega, \end{cases} \quad (1)$$

where $\Omega \subseteq \mathbb{R}^3$ is a finitely connected domain, with boundary condition

$$\nu \times E = 0 \text{ on } \partial\Omega$$

if Ω has a boundary. In these equations ω is the pencil spectral parameter, ε the electric permittivity, μ the magnetic permeability and σ is the conductivity; ν is the unit normal to the boundary.

Lassas [15] already studied this problem on a bounded domain with $C^{1,1}$ boundary, so in this article our primary concern is to treat unbounded domains which provides additional sources for essential spectrum. However, even for bounded domains, we are able to relax the required boundary regularity to Lipschitz continuity. Like Lassas we allow the permittivity, permeability and conductivity to be tensor valued (i.e., we allow anisotropy); however, we make

the physically realistic assumption that, at infinity, these coefficients approach isotropic constant values.

Maxwell systems in infinite domains are usually studied in the context of scattering, with a Silver–Müller radiation condition imposed at infinity, see, e.g., [17, p. 10] and [5, 6]. Scattering theory is sometimes regarded as the study of solutions when the spectral parameter lies in the essential spectrum, though the fact that the Maxwell system already has non-trivial essential spectrum in bounded domains indicates that such an interpretation involves local conditions as well as the study of radiation to infinity. The case of zero conductivity $\sigma \equiv 0$ is substantially simpler, both for bounded and unbounded domains. However, it is also physically unrealistic in numerous applications, including imaging [7, 11, 13, 14].

The main technical difficulty in dealing with the essential spectrum of Maxwell systems in infinite domains is the fact that compactly supported perturbations to the coefficients do change the essential spectrum, as is clear even for bounded domains from [15]. This means that techniques such as Glazman decomposition, useful for Schrödinger operators, are no longer helpful. We use instead a Helmholtz decomposition inspired by [1, 3] together with further decompositions of the resulting 2×2 block-operator matrices. As in [2], this approach allows us to substantially reduce Maxwell’s system to an elliptic problem. The main result is stated in Theorem 6: the essential spectrum of the Maxwell pencil is the union of two sets: namely the spectrum of the pencil $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ acting between suitable spaces, together with the essential spectrum of the Maxwell pencil with constant coefficients. The spectral geometric question of how the topology of Ω at infinity is reflected in the essential spectrum of a constant coefficient Maxwell operator is also interesting, and an avenue for future work.

Our original motivation for the investigations in this paper came from our study of inverse problems in a slab for the Maxwell system with conductivity. However, knowledge of the essential spectrum has much more fundamental importance. It is a first step toward determination of the absolutely continuous subspace of an operator and hence the behavior of its semi-group, as required, e.g., for the study of Vlasov–Maxwell systems. It can also be a key component in the analysis of certain types of homogenization problem.

2. Main Result

We shall study the Maxwell system on a finitely connected domain $\Omega \subseteq \mathbb{R}^3$. Prototype examples include exterior domains $\Omega := \mathbb{R}^3 \setminus \overline{\Omega'}$ in which Ω' has finitely many simply connected components; the case of an infinite slab, $\Omega = \mathbb{R}^2 \times (0, 1)$, or a half space $\Omega = \mathbb{R}^2 \times (0, \infty)$; domains with cylindrical ends, such as waveguides; and indeed the case $\Omega = \mathbb{R}^3$ (see Assumption 14 and Proposition 15 below for more details). The boundary $\partial\Omega$, if non-empty, will be of Lipschitz type, and the coefficients ε , σ and μ will be assumed to be symmetric matrix-valued functions in $L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ such that for some $\Lambda > 0$ and every $\eta \in \mathbb{R}^3$

$$\Lambda^{-1}|\eta|^2 \leq \eta \cdot \varepsilon \eta \leq \Lambda|\eta|^2, \quad \Lambda^{-1}|\eta|^2 \leq \eta \cdot \mu \eta \leq \Lambda|\eta|^2, \quad 0 \leq \eta \cdot \sigma \eta \leq \Lambda|\eta|^2 \quad (2)$$

almost everywhere in Ω .

As already mentioned, the case of bounded domains was treated by Lassas [15] under slightly stronger regularity assumptions; for infinite domains we assume that all the coefficients have a ‘value at infinity’ in the precise sense that

$$\lim_{x \rightarrow \infty} \mu(x) = \mu_0 I, \quad \lim_{x \rightarrow \infty} \varepsilon(x) = \varepsilon_0 I, \quad \lim_{x \rightarrow \infty} \sigma(x) = \sigma_0 I, \quad (3)$$

for some scalar values $\mu_0 > 0$, $\varepsilon_0 > 0$ and $\sigma_0 \geq 0$. To allow a unified treatment of unbounded and bounded domains, it is convenient to assign values to ε_0 , μ_0 and σ_0 when Ω is bounded, and we choose

$$\varepsilon_0 := 1, \quad \mu_0 := 1; \quad \sigma_0 := 0, \quad (\Omega \text{ bounded}). \quad (4)$$

Several function spaces arise commonly in the study of Maxwell systems; to fix notation, we denote

$$\begin{aligned} \mathcal{H}(\text{curl}, \Omega) &:= \{u \in L^2(\Omega; \mathbb{C}^3) : \text{curl } u \in L^2(\Omega; \mathbb{C}^3)\}, \\ \mathcal{H}(\text{div}, \Omega) &:= \{u \in L^2(\Omega; \mathbb{C}^3) : \text{div } u \in L^2(\Omega)\}, \end{aligned}$$

equipped with the canonical norms

$$\begin{aligned} \|u\|_{\mathcal{H}(\text{curl}, \Omega)}^2 &= \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\text{curl } u\|_{L^2(\Omega; \mathbb{C}^3)}^2, \\ \|u\|_{\mathcal{H}(\text{div}, \Omega)}^2 &= \|u\|_{L^2(\Omega; \mathbb{C}^3)}^2 + \|\text{div } u\|_{L^2(\Omega; \mathbb{C})}^2. \end{aligned}$$

If $\partial\Omega$ is non-empty, then we let ν denote the outward unit normal vector, and define

$$\mathcal{H}_0(\text{curl}, \Omega) = \{u \in \mathcal{H}(\text{curl}, \Omega) : \nu \times u|_{\partial\Omega} = 0\},$$

with the understanding that when $\Omega = \mathbb{R}^3$ then $\mathcal{H}_0(\text{curl}, \Omega) = \mathcal{H}(\text{curl}, \Omega)$.

We start by considering, in the Hilbert space

$$\mathcal{H}_1 := \mathcal{H}_0(\text{curl}, \Omega) \oplus \mathcal{H}(\text{curl}, \Omega), \quad (5)$$

the operator pencil $\omega \mapsto V_\omega$ defined from (1) in the space \mathcal{H}_1 by

$$\begin{aligned} V_\omega : \mathcal{H}_1 &\longrightarrow L^2(\Omega; \mathbb{C}^3)^2, \\ (E, H) &\longmapsto (\text{curl } H + i(\omega\varepsilon + i\sigma)E, \text{curl } E - i\omega\mu H). \end{aligned} \quad (6)$$

Our aim is to study the essential spectrum of the pencil V_ω .

Definition 1. Let H_1 and H_2 be two Hilbert spaces. For each $\omega \in \mathbb{C}$, let $L_\omega : H_1 \rightarrow H_2$ be a bounded linear operator. Adapting the definitions in [9, Ch. I, §4], we say that $\omega \in \mathbb{C}$ lies in the

1. essential spectrum $\sigma_{e,1}$ of the pencil $\omega \mapsto L_\omega$ if L_ω is not semi-Fredholm (an operator is semi-Fredholm if its range is closed and its kernel or its cokernel is finite-dimensional);
2. essential spectrum $\sigma_{e,2}$ of the pencil $\omega \mapsto L_\omega$ if L_ω is not in the class \mathcal{F}_+ of semi-Fredholm operators with finite-dimensional kernel;

3. essential spectrum $\sigma_{e,3}$ of the pencil $\omega \mapsto L_\omega$ if L_ω is not in the class \mathcal{F} of Fredholm operators with finite-dimensional kernel and cokernel;
4. essential spectrum $\sigma_{e,4}$ of the pencil $\omega \mapsto L_\omega$ if L_ω is not Fredholm with index zero, where $\text{ind } L_\omega = \dim \ker L_\omega - \dim \text{coker } L_\omega$.

When these essential spectra coincide, we shall use the notation σ_{ess} . With an abuse of terminology, we shall refer to the essential spectrum of L_ω and write $\omega \in \sigma_{e,k}(L_\omega)$.

For the Maxwell pencil V_ω , all essential spectra $\sigma_{e,k}(L_\omega)$, $k = 1, \dots, 4$, coincide.

Lemma 2. *We have*

$$\sigma_{e,1}(V_\omega) = \sigma_{e,2}(V_\omega) = \sigma_{e,3}(V_\omega) = \sigma_{e,4}(V_\omega).$$

Proof. By definition, we always have

$$\sigma_{e,1}(V_\omega) \subseteq \sigma_{e,2}(V_\omega) \subseteq \sigma_{e,3}(V_\omega) \subseteq \sigma_{e,4}(V_\omega).$$

It remains to prove that $\sigma_{e,4}(V_\omega) \subseteq \sigma_{e,1}(V_\omega)$.

Take $\omega \notin \sigma_{e,1}(V_\omega)$. Thus V_ω is semi-Fredholm, namely the range of V_ω is closed and its kernel or its cokernel is finite-dimensional. A direct calculation using integration by parts yields, thanks to the symmetry of the coefficients ε , μ and σ ,

$$\langle V_\omega(u), u' \rangle_{L^2(\Omega; \mathbb{C}^3)^2} = \langle u, \overline{V_\omega(u')} \rangle_{L^2(\Omega; \mathbb{C}^3)^2}, \quad u, u' \in \mathcal{H}_1.$$

Thus, since \mathcal{H}_1 is dense in $L^2(\Omega; \mathbb{C}^3)^2$, for $u \in \mathcal{H}_1$ we have

$$\begin{aligned} u \in \ker V_\omega &\iff \langle V_\omega(u), u' \rangle_{L^2(\Omega; \mathbb{C}^3)^2} = 0 \text{ for all } u' \in L^2(\Omega; \mathbb{C}^3)^2 \\ &\iff \langle V_\omega(u), u' \rangle_{L^2(\Omega; \mathbb{C}^3)^2} = 0 \text{ for all } u' \in \mathcal{H}_1 \\ &\iff \langle u, \overline{V_\omega(u')} \rangle_{L^2(\Omega; \mathbb{C}^3)^2} = 0 \text{ for all } u' \in \mathcal{H}_1 \\ &\iff \langle u, \overline{V_\omega(u')} \rangle_{L^2(\Omega; \mathbb{C}^3)^2} = 0 \text{ for all } u' \in \mathcal{H}_1 \\ &\iff u \in \text{coker } \overline{V_\omega}. \end{aligned}$$

Hence, $\ker V_\omega = \text{coker } \overline{V_\omega}$, and so

$$\dim \ker V_\omega = \dim \text{coker } \overline{V_\omega} = \dim \text{coker } V_\omega,$$

implying that V_ω is a Fredholm operator with index zero, namely $\omega \notin \sigma_{e,4}(V_\omega)$. \square

Remark 3. When H_1 and H_2 are separable, infinite-dimensional Hilbert spaces, the statement $\omega \in \sigma_{e,2}(L_\omega)$ is equivalent to the statement that there exists a *Weyl singular sequence* (u_n) in H_1 with $\|u_n\|_{H_1} = 1$ and $u_n \rightarrow 0$ in H_1 such that $\|L_\omega u_n\|_{H_2} \rightarrow 0$. This follows from [9, Ch. IX, Thm. 1.3], which covers the case $H_1 = H_2$, by using a straightforward argument involving the isomorphism between H_1 and H_2 .

Finally, we introduce some homogeneous Sobolev spaces which are required for the Helmholtz decomposition for unbounded domains. For bounded domains, these coincide with the usual Sobolev spaces.

- Definition 4.** (1) (Ω unbounded) The homogeneous Sobolev spaces $\dot{H}_0^1(\Omega)$ and $\dot{H}^1(\Omega)$ are the completions of the Schwartz spaces $\mathcal{D}(\Omega)$ and $\mathcal{D}(\bar{\Omega})$, respectively, with respect to the norm $\|u\| := \|\nabla u\|_{L^2(\Omega)}$.
- (2) (Ω bounded) In this case we define the homogeneous Sobolev spaces to coincide with the usual Sobolev spaces: $\dot{H}_0^1(\Omega) = H_0^1(\Omega)$ and $\dot{H}^1(\Omega) = H^1(\Omega)$.

Remark 5. (a) Note that this definition does not coincide with Definition 1.31 in [4], which uses Fourier transforms to define $\dot{H}^s(\mathbb{R}^d)$ and results in spaces which are not complete if $s \geq d/2$. Our definition follows Dautray and Lions [8]. For clarity, we use our definition directly in ‘‘Appendix A’’.

(b) If K is any compact subset of Ω with non-empty interior and Ω is bounded, then the usual H^1 -norm is equivalent to

$$\|u\|^2 := \|u\|_{L^2(K)}^2 + \|\nabla u\|_{L^2(\Omega)}^2, \quad (7)$$

see Maz'ya [16]. In the case when Ω is unbounded, the norms on \dot{H}^1 and \dot{H}_0^1 may be shown to be equivalent to the norm defined in (7), for any compact $K \subset \Omega$ with non-empty interior. Thus an equivalent definition of $\dot{H}^1(\Omega)$, valid for bounded and unbounded Ω , is the closure of $\mathcal{D}(\Omega)$ in the norm (7). However, for unbounded Ω , this is no longer equivalent to the H^1 -norm; e.g., the function given in polar coordinates by $u(r) = 1/(r+1)^{3/2}$ does not lie in $H^1(\mathbb{R}^3)$ but lies in $\dot{H}^1(\mathbb{R}^3)$.

We are now ready to state our main result.

Theorem 6. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain satisfying Assumption 14 (given below) and $\varepsilon, \sigma, \mu \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$ satisfy (2). Assume (3) if Ω is unbounded, and (4) if Ω is bounded. We have*

$$\sigma_{ess}(V_\omega) = \sigma_{ess}(\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)) \cup \sigma_{ess}(V_\omega^0),$$

where $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ acts from $\dot{H}_0^1(\Omega; \mathbb{C})$ to its dual $\dot{H}^{-1}(\Omega; \mathbb{C})$ and V_ω^0 is the Maxwell pencil with constant coefficients ε_0, μ_0 and σ_0 .

Thanks to this result, the essential spectrum of the Maxwell pencil is decomposed into two parts.

- The essential spectrum of the operator $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$: this component depends on the coefficients ε and σ directly. In particular, in the case when the coefficients ε and σ are continuous, it consists of the closure of the set of $\omega = i\nu$, $\nu \in \mathbb{R}$, for which $\nu\varepsilon + \sigma$ is indefinite at some point in Ω : see Proposition 27. As in Lemma 2, a direct calculation shows that for this operator all essential spectra coincide, and so the notation σ_{ess} is unambiguous.
- The essential spectrum of the constant coefficient Maxwell pencil,

$$V_\omega^0(E, H) = (\operatorname{curl} H + i(\omega\varepsilon_0 + i\sigma_0)E, \operatorname{curl} E - i\omega\mu_0 H),$$

which is determined by the geometry of Ω . This can be computed explicitly in many cases of interest: we provide several examples below. It is

worth observing that 0 always belongs to $\sigma_{ess}(V_\omega^0)$, since $\{0\} \oplus \nabla \dot{H}^1(\Omega) \subseteq \ker V_0^0$.

In the next examples, we will calculate the essential spectrum of V_ω^0 , for different choices of domains Ω .

Example 7. The simplest case to consider in the calculation of $\sigma_{ess}(V_\omega^0)$ is when Ω is bounded. By (4) we have $\varepsilon_0 = \mu_0 = 1$ and $\sigma_0 = 0$. Thus, the pencil is self-adjoint, and we have

$$\sigma_{ess}(V_\omega^0) = \{0\},$$

see [12, 15, 17].

Example 8. We consider here the case of the full space $\Omega = \mathbb{R}^3$. We can make use of the Fourier transform to obtain a simple expression of this operator. Writing $E(x) = \int_{\mathbb{R}^3} \hat{E}(\xi) e^{ix \cdot \xi} d\xi$, the expression of the operator $\text{curl } E$ in the Fourier domain is given by the multiplication operator $iC(\xi) \hat{E}(\xi)$, where

$$C(\xi) = \begin{pmatrix} 0 & -\xi_3 & \xi_2 \\ \xi_3 & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix}. \tag{8}$$

Writing $\text{curl } H$ in a similar way, we immediately see that V_ω^0 is represented, in the Fourier domain, by the multiplication by the matrix

$$A_\omega(\xi) = \begin{pmatrix} i(\omega\varepsilon_0 + i\sigma_0)I & iC(\xi) \\ iC(\xi) & -i\omega\mu_0 I \end{pmatrix}.$$

A direct calculation gives

$$\det(A_\omega(\xi)) = k_\omega(|\xi|^2 - k_\omega)^2, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0).$$

By a standard argument, $\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : \det(A_\omega(\xi)) = 0 \text{ for some } \xi \in \mathbb{R}^3\}$, so that

$$\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : k_\omega \geq 0\} = \left\{ a - \frac{\sigma_0}{2\varepsilon_0} i : a \in \mathbb{R} \right\} \cup \left\{ ib : b \in \left[-\frac{\sigma_0}{\varepsilon_0}, 0 \right] \right\}.$$

In the particular case, when the conductivity at infinity is zero, i.e., $\sigma_0 = 0$, we simply have $\sigma_{ess}(V_\omega^0) = \mathbb{R}$.

Example 9. Let us look at the case of the slab $\Omega = \{x = (x', x_3) \in \mathbb{R}^3 : 0 < x_3 < L\}$, for some $L > 0$. The derivation is very similar to the one presented above for the full space, the only difference being that the continuous Fourier transform in the third variable becomes a Fourier series. As a consequence, the continuous variable ξ_3 is replaced by a discrete variable $n = 0, 1, \dots$. More precisely,

$$E_j(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \hat{E}_j(\xi', n) e^{ix' \cdot \xi'} \sin\left(\frac{n\pi}{L} x_3\right) d\xi', \quad j = 1, 2,$$

$$E_3(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \hat{E}_3(\xi', n) e^{ix' \cdot \xi'} \cos\left(\frac{n\pi}{L} x_3\right) d\xi',$$

and, analogously,

$$H_j(x) = \sum_{n=0}^{\infty} \int_{\mathbb{R}^2} \hat{H}_j(\xi', n) e^{ix' \cdot \xi'} \cos\left(\frac{n\pi}{L} x_3\right) d\xi', \quad j = 1, 2,$$

$$H_3(x) = \sum_{n=1}^{\infty} \int_{\mathbb{R}^2} \hat{H}_3(\xi', n) e^{ix' \cdot \xi'} \sin\left(\frac{n\pi}{L} x_3\right) d\xi';$$

the range of n in each summation has been determined by the boundary conditions on $x_3 = 0$ and $x_3 = L$. Compared to the full space in Example 8, the continuous frequency variable $\xi \in \mathbb{R}^3$ has become $\xi := (\xi', \frac{n\pi}{L}) \in \mathbb{R}^2 \times (\frac{\pi}{L}\mathbb{N})$. By calculations similar to those for the full space, we see that the essential spectrum is the set of $\omega \in \mathbb{C}$ such that for some $\xi \in \mathbb{R}^2 \times (\frac{\pi}{L}\mathbb{N})$

$$k_\omega(|\xi|^2 - k_\omega)^2 = 0, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0),$$

and it is easy to see that this coincides with the essential spectrum for the full space problem.

Example 10. We now compute the essential spectrum of V_ω^0 in a cylinder $\Omega = \{x \in \mathbb{R}^3 : 0 < x_2 < L_1, 0 < x_3 < L_2\}$. As above, let us expand E and H in Fourier coordinates as

$$E_1(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_1(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi,$$

$$E_2(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_2(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi,$$

$$E_3(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{E}_3(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi$$

and

$$H_1(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_1(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi,$$

$$H_2(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_2(n, \xi) \sin\left(\frac{\pi n_1}{L_1} x_2\right) \cos\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi,$$

$$H_3(x_1, x_2, x_3) = \sum_{n \in \mathbb{N}^2} \int_{\mathbb{R}} \hat{H}_3(n, \xi) \cos\left(\frac{\pi n_1}{L_1} x_2\right) \sin\left(\frac{\pi n_2}{L_2} x_3\right) e^{i\xi x_1} d\xi.$$

In order to guarantee uniqueness of the expansions, set

$$\begin{aligned} \hat{H}_2(0, n_2, \xi) &= 0, & \hat{H}_3(n_1, 0, \xi) &= 0, & \hat{E}_1(0, n_2, \xi) &= 0, \\ \hat{E}_1(n_1, 0, \xi) &= 0, & \hat{E}_2(n_1, 0, \xi) &= 0, & \hat{E}_3(0, n_2, \xi) &= 0, \end{aligned} \quad (9)$$

for every $n \in \mathbb{N}^2$ and $\xi \in \mathbb{R}$.

A direct calculation gives that the operators $E \mapsto \text{curl } E$ and $H \mapsto \text{curl } H$ may be written in Fourier coordinates as the multiplication operators by the matrices

$$C\left(i\xi, \frac{\pi}{L_1} n_1, \frac{\pi}{L_2} n_2\right) \quad \text{and} \quad C\left(i\xi, -\frac{\pi}{L_1} n_1, -\frac{\pi}{L_2} n_2\right),$$

respectively, where the matrix C is defined in (8). As a consequence, in the Fourier domain, V_ω^0 is a multiplication operator represented by the matrix

$$A_\omega(n, \xi) = \begin{pmatrix} i(\omega\varepsilon_0 + i\sigma_0)I & C(i\xi, -\frac{\pi}{L_1}n_1, -\frac{\pi}{L_2}n_2) \\ C(i\xi, \frac{\pi}{L_1}n_1, \frac{\pi}{L_2}n_2) & -i\omega\mu_0 I \end{pmatrix}.$$

A further calculation yields

$$\det(A_\omega(n, \xi)) = k_\omega(\xi^2 + \frac{\pi^2}{L_1^2}n_1^2 + \frac{\pi^2}{L_2^2}n_2^2 - k_\omega)^2, \quad k_\omega = \omega\mu_0(\omega\varepsilon_0 + i\sigma_0).$$

If ω is such that $\det(A_\omega(n, \xi)) \neq 0$ for every $n \in \mathbb{N}^2$ and $\xi \in \mathbb{R}$, then ω does not belong to the essential spectrum of V_ω^0 . On the other hand, suppose that ω is such that $\det(A_\omega(n, \xi)) = 0$ for some $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$. If $n_1 = n_2 = 0$, it is easy to see that there are no nonzero elements of $\ker A_\omega(n, \xi)$ satisfying (9). On the other hand, the vector $(0, \omega\mu_0 L_2, 0, \pi i, 0, \xi L_2)$ belongs to $\ker A_\omega(0, 1, \xi)$ and satisfies (9) (and similarly if $n_1 = 1$ and $n_2 = 0$). As a consequence, we have that

$$\sigma_{ess}(V_\omega^0) = \{\omega \in \mathbb{C} : k_\omega = 0 \text{ or } k_\omega \geq \frac{\pi^2}{L^2}\}, \quad L = \max(L_1, L_2).$$

In the particular case when $\sigma_0 = 0$, this set takes the simpler form

$$\sigma_{ess}(V_\omega^0) = \left(-\infty, -\frac{\pi}{L\sqrt{\varepsilon_0\mu_0}}\right] \cup \{0\} \cup \left[\frac{\pi}{L\sqrt{\varepsilon_0\mu_0}}, +\infty\right).$$

Note that this set approaches the essential spectrum for the slab as $L \rightarrow +\infty$. This is expected: as L increases the cylinder becomes larger and larger in one direction.

3. Helmholtz Decomposition and Related Operators

We shall treat both bounded and unbounded Lipschitz domains $\Omega \subseteq \mathbb{R}^3$. The latter are our primary interest, as the bounded case has already been studied by Lassas [15], albeit under slightly stronger assumptions on the boundary regularity. However, in the definitions which follow, we deal with both cases.

The first decomposition result which we require is true without restrictions on the topology of Ω . Although it is standard, we present a proof since it shows how the homogeneous Sobolev spaces arise in a natural way.

Lemma 11. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

1. *The space $L^2(\Omega; \mathbb{C}^3)$ admits the following orthogonal decompositions:*

$$L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}_0^1(\Omega) \oplus \mathcal{H}(\operatorname{div} 0, \Omega), \quad (10a)$$

$$L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}^1(\Omega) \oplus \mathcal{H}_0(\operatorname{div} 0, \Omega), \quad (10b)$$

in which

$$\mathcal{H}(\operatorname{div} 0, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{div} u = 0\},$$

$$\mathcal{H}_0(\operatorname{div} 0, \Omega) = \{u \in L^2(\Omega; \mathbb{C}^3) : \operatorname{div} u = 0, \nu \cdot u|_{\partial\Omega} = 0\}.$$

2. The spaces $\mathcal{H}_0(\text{curl}, \Omega)$ and $\mathcal{H}(\text{curl}, \Omega)$ admit the orthogonal decompositions

$$\mathcal{H}_0(\text{curl}, \Omega) = \nabla \dot{H}_0^1(\Omega; \mathbb{C}) \oplus (\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div } 0, \Omega)), \quad (11a)$$

$$\mathcal{H}(\text{curl}, \Omega) = \nabla \dot{H}^1(\Omega; \mathbb{C}) \oplus (\mathcal{H}(\text{curl}, \Omega) \cap \mathcal{H}_0(\text{div } 0, \Omega)). \quad (11b)$$

Proof. (1) The operator $\nabla: \dot{H}_0^1(\Omega) \rightarrow L^2(\Omega; \mathbb{C}^3)$ is an isometry, and so $\nabla \dot{H}_0^1(\Omega)$ is closed in $L^2(\Omega; \mathbb{C}^3)$. It remains to prove that $(\nabla \dot{H}_0^1(\Omega))^\perp = \mathcal{H}(\text{div } 0, \Omega)$. Suppose that $\phi \in (\nabla \dot{H}_0^1(\Omega))^\perp$; then $\langle \phi, \nabla v \rangle = 0$ for all $v \in \mathcal{D}(\Omega)$, which means that $\langle \text{div } \phi, v \rangle = 0$ for all $v \in \mathcal{D}(\Omega)$. This proves that $\phi \in \mathcal{H}(\text{div } 0, \Omega)$. Conversely, if $\phi \in \mathcal{H}(\text{div } 0, \Omega)$ then for any $v \in \mathcal{D}(\Omega)$ we have $0 = \langle \text{div } \phi, v \rangle = \langle \phi, \nabla v \rangle$. Taking the closure in the $\dot{H}_0^1(\Omega)$ -topology shows that $\langle \phi, \nabla v \rangle = 0$ for all $v \in \dot{H}_0^1(\Omega)$, which proves (10a).

Analogously, $\nabla \dot{H}^1(\Omega)$ is closed in $L^2(\Omega; \mathbb{C}^3)$. To prove (10b) suppose that $\phi \in (\nabla \dot{H}^1(\Omega))^\perp$; then certainly $\text{div } \phi = 0$ since $(\nabla \dot{H}^1(\Omega))^\perp \subseteq (\nabla \dot{H}_0^1(\Omega))^\perp$. Thus for all $v \in \mathcal{D}(\bar{\Omega})$, we have $0 = \langle \phi, \nabla v \rangle = \int_{\partial\Omega} (\nu \cdot \phi) \bar{v} \, ds$. This means that $\nu \cdot \phi = 0$ on $\partial\Omega$ and so $\phi \in \mathcal{H}_0(\text{div } 0, \Omega)$. The proof that any $\phi \in \mathcal{H}_0(\text{div } 0, \Omega)$ lies in $(\nabla \dot{H}^1(\Omega))^\perp$ is straightforward.

(2) The decompositions (11) follow immediately from (10) by taking the appropriate subspaces. \square

To decompose the Maxwell pencil we need to decompose the spaces $\mathcal{H}(\text{div } 0, \Omega)$ and $\mathcal{H}_0(\text{div } 0, \Omega)$ further, by using vector potentials in some suitable spaces, which we now introduce.

Definition 12. Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.

- The space $\dot{X}_T(\Omega)$ is the completion of $\mathcal{H}(\text{curl}, \Omega) \cap \mathcal{H}_0(\text{div } 0, \Omega)$ with respect to the seminorm $\|u\| := \|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)} + \|u \cdot \nu\|_{H^{-1/2}(\partial\Omega)}$.
- The space $\dot{X}_N(\Omega)$ is the completion of $\mathcal{H}_0(\text{curl}, \Omega) \cap \mathcal{H}(\text{div } 0, \Omega)$ with respect to the seminorm $\|u\| := \|\text{curl } u\|_{L^2(\Omega)} + \|\text{div } u\|_{L^2(\Omega)} + \|u \times \nu\|_{H^{-1/2}(\partial\Omega)}$.
- The space $K_T(\Omega)$ is the kernel of the curl operator restricted to $\dot{X}_T(\Omega)$, namely

$$K_T(\Omega) = \{u \in \dot{X}_T(\Omega) : \text{curl } u = 0\}.$$

- The space $K_N(\Omega)$ is the kernel of the curl operator restricted to $\dot{X}_N(\Omega)$, namely

$$K_N(\Omega) = \{u \in \dot{X}_N(\Omega) : \text{curl } u = 0\}.$$

The spaces $K_T(\Omega)$ and $K_N(\Omega)$ are closed in $\dot{X}_T(\Omega)$ and in $\dot{X}_N(\Omega)$, respectively, and so we can consider the quotient spaces

$$\dot{X}_T(\Omega)/K_T(\Omega), \quad \dot{X}_N(\Omega)/K_N(\Omega).$$

The curl operator is well-defined and injective on these spaces. To avoid cumbersome notation, we will in the following identify $\text{curl } \psi$ for $\psi \in \dot{X}_T(\Omega)/K_T(\Omega)$ or $\psi \in \dot{X}_N(\Omega)/K_N(\Omega)$ with the vector in $L^2(\Omega; \mathbb{C}^3)$ given by curl acting on any

representative of the equivalence class ψ . The curl operator maps these quotient spaces into the space of divergence-free fields, with appropriate boundary conditions.

Lemma 13. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain.*

1. *The space $\text{curl}(\dot{X}_T(\Omega)/K_T(\Omega))$ is contained in $\mathcal{H}(\text{div } 0, \Omega)$.*
2. *The space $\text{curl}(\dot{X}_N(\Omega)/K_N(\Omega))$ is contained in $\mathcal{H}_0(\text{div } 0, \Omega)$.*

Proof. Part (1) follows immediately from $\text{div} \circ \text{curl} = 0$. Part (2) follows from the identities $\text{div} \circ \text{curl} = 0$ and $(\text{curl } u) \cdot \nu = \text{div}_{\partial\Omega}(u \times \nu)$ on $\partial\Omega$ [17, (3.52)]. \square

We make the following assumption.

Assumption 14. The spaces $K_T(\Omega)$ and $K_N(\Omega)$ are finite-dimensional and

$$\mathcal{H}(\text{div } 0, \Omega) = \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \oplus K_N(\Omega), \quad (12a)$$

$$\mathcal{H}_0(\text{div } 0, \Omega) = \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \oplus K_T(\Omega). \quad (12b)$$

This assumption is verified in many cases of theoretical and practical interest.

Proposition 15. *Assumption 14 is verified in any of the following cases:*

1. $\Omega = \mathbb{R}^3$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
2. Ω is a bounded Lipschitz domain, satisfying Hypothesis 3.3 of [4];
3. Ω is a C^2 exterior domain, satisfying assumptions (1.45) of [8, Chapter IXA];
4. Ω is the half space $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0\}$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
5. Ω is the slab $\{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < L\}$ for some $L > 0$ (with $K_T(\Omega) = K_N(\Omega) = \{0\}$);
6. Ω is a cylinder $\mathbb{R} \times \Omega'$, where $\Omega' \subseteq \mathbb{R}^2$ is a simply connected bounded domain of class $C^{1,1}$ or piecewise smooth with no re-entrant corners (with $K_T(\Omega) = K_N(\Omega) = \{0\}$).

Remark 16. We have decided not to provide the details of the assumptions of parts (2) and (3), since they are rather lengthy and are not needed for the rest of the paper. In simple words, these assumptions require $\partial\Omega$ to be a finite union of connected surfaces and that there exist a finite number of *cuts* within Ω which divide it into multiple simply connected domains. The number of cuts is given by $\dim K_T(\Omega)$, and the number of connected components of $\partial\Omega$ by $\dim K_N(\Omega) + 1$. Thus, for simply connected domains with connected boundaries, the decomposition is even simpler: $K_T(\Omega)$ and $K_N(\Omega)$ are trivial and can be omitted.

Proof. (1) The decompositions (12a) and (12b) coincide, and simply follow from the identity $\hat{u}(\xi) = -\xi \times \left(\frac{\xi \times \hat{u}}{|\xi|^2}\right)$, valid for every divergence-free field u (which implies $\xi \cdot \hat{u} = 0$), where \hat{u} denotes the Fourier transform of u . Alternatively, this is also a consequence of Proposition 29 and Lemma 30.

- (2) This part is proved in [4] (see also [8, Chapter IXA] and [10, Chapter I, §3] for the smooth case). The construction of the spaces $K_T(\Omega)$ and $K_N(\Omega)$ is described explicitly.
- (3) The decompositions in this case are proved in [8, Chapter IXA].
- (4)–(5)–(6) The arguments are standard and explicit, but it is not easy to find precise statements in the literature. We detail the derivation in “Appendix A”, which contains a general construction for a larger class of cylinders. \square

Combining (10) and (12), we obtain that the space $L^2(\Omega; \mathbb{C}^3)$ admits the following orthogonal decompositions:

$$L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}_0^1(\Omega; \mathbb{C}) \oplus \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \oplus K_N(\Omega), \quad (13a)$$

$$L^2(\Omega; \mathbb{C}^3) = \nabla \dot{H}^1(\Omega; \mathbb{C}) \oplus \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \oplus K_T(\Omega). \quad (13b)$$

In view of these decompositions, to every vector field in $L^2(\Omega; \mathbb{C}^3)$, we can associate the unique vector potentials in $\dot{X}_T(\Omega)/K_T(\Omega)$ and in $\dot{X}_N(\Omega)/K_N(\Omega)$.

Lemma 17. *Let $\Omega \subseteq \mathbb{R}^3$ be a Lipschitz domain satisfying Assumption 14. There exist bounded operators $T_N : L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{X}_N(\Omega)/K_N(\Omega)$ and $T_T : L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{X}_T(\Omega)/K_T(\Omega)$ such that*

$$\begin{aligned} T_N \text{curl} \Phi &= \Phi, \quad \Phi \in \dot{X}_N(\Omega)/K_N(\Omega), \\ T_N \nabla q &= 0, \quad q \in \dot{H}^1(\Omega; \mathbb{C}); \quad T_N f = 0, \quad f \in K_T(\Omega); \\ T_T \text{curl} \Phi &= \Phi, \quad \Phi \in \dot{X}_T(\Omega)/K_T(\Omega), \\ T_T \nabla q &= 0, \quad q \in \dot{H}_0^1(\Omega; \mathbb{C}); \quad T_T f = 0, \quad f \in K_N(\Omega). \end{aligned} \quad (14)$$

Proof. In view of (13b), every $F \in L^2(\Omega; \mathbb{C}^3)$ admits a unique decomposition into three orthogonal vectors,

$$F = \nabla q + \text{curl} \Phi + f,$$

with $q \in \dot{H}^1(\Omega; \mathbb{C})$, $\Phi \in \dot{X}_N(\Omega)/K_N(\Omega)$ and $f \in K_T(\Omega)$. We define T_N by $T_N F = \Phi$, so that $T_N \text{curl} \Phi = \Phi$ for all $\Phi \in \dot{X}_N(\Omega)/K_N(\Omega)$. By the closed graph theorem, T_N is bounded. The definition of T_T follows similarly by using the other Helmholtz decomposition (13a). \square

4. Proof of the Main Result

In a first part, we introduce a series of equivalent reformulations of our problem to obtain a form where the two contributions to the essential spectrum in our main result can easily be separated.

Decomposing \mathcal{H}_1 using (11) and (12) allows us to transform the Maxwell operator V_ω . More precisely, for $E \in \mathcal{H}_0(\text{curl}, \Omega)$ and $H \in \mathcal{H}(\text{curl}, \Omega)$ consider the decompositions

$$E = \nabla q_E + \Psi_E + h_N, \quad H = \nabla q_H + \Psi_H + h_T, \quad (15)$$

where $q_E \in \dot{H}_0^1(\Omega; \mathbb{C})$, $q_H \in \dot{H}^1(\Omega; \mathbb{C})$, $\Psi_E \in \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega))$, $\Psi_H \in \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega))$, $h_T \in K_T(\Omega)$ and $h_N \in K_N(\Omega)$. We

now wish to discard the contribution coming from $K_T(\Omega)$ and $K_N(\Omega)$. To this end, we introduce the space

$$\begin{aligned} \mathcal{H}_2 &= \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \times \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \\ &\quad \times \mathcal{H}(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \end{aligned}$$

equipped with the canonical product norm

$$\begin{aligned} \|(u_1, u_2, \Psi_1, \Psi_2)\|_{\mathcal{H}_2}^2 &= \|u_1\|_{L^2(\Omega)}^2 + \|u_2\|_{L^2(\Omega)}^2 + \|\Psi_1\|_{\mathcal{H}(\operatorname{curl}, \Omega)}^2 \\ &\quad + \|\Psi_2\|_{\mathcal{H}(\operatorname{curl}, \Omega)}^2. \end{aligned} \quad (16)$$

Define the projection map

$$W: \mathcal{H}_1 \rightarrow \mathcal{H}_2, \quad W(E, H) = (\nabla q_E, \nabla q_H, \Psi_E, \Psi_H),$$

where E, H are given by (15), and its right inverse $W^{-1}: \mathcal{H}_2 \rightarrow \mathcal{H}_1$ by

$$W^{-1}(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) = (\nabla q_E + \Psi_E, \nabla q_H + \Psi_H).$$

Since the decompositions in (11) and (12) are orthogonal, for any $(E, H) \in \mathcal{H}_1$ we have

$$\|(E, H)\|_{\mathcal{H}_1}^2 = \|W(E, H)\|_{\mathcal{H}_2}^2 + \|(h_N, h_T)\|_{L^2(\Omega)^2}^2. \quad (17)$$

Instead of the operator V_ω , we consider

$$\tilde{V}_\omega = V_\omega \circ W^{-1}: \mathcal{H}_2 \rightarrow L^2(\Omega; \mathbb{C}^3)^2.$$

This does not change the essential spectrum, as the following lemma shows.

Lemma 18. *We have $\sigma_{e,2}(V_\omega) = \sigma_{e,2}(\tilde{V}_\omega)$.*

Proof. Using that W^{-1} is an isometry we immediately obtain that $\sigma_{e,2}(\tilde{V}_\omega) \subseteq \sigma_{e,2}(V_\omega)$. It remains to show the reverse inclusion.

Let $\omega \in \sigma_{e,2}(V_\omega)$. By Remark 3, there exists a sequence of functions $u_n = (\nabla q_{E,n} + \Psi_{E,n} + h_{N,n}, \nabla q_{H,n} + \Psi_{H,n} + h_{T,n})$ in \mathcal{H}_1 , $\|u_n\|_{\mathcal{H}_1} = 1$, $u_n \rightharpoonup 0$ in \mathcal{H}_1 such that $\|V_\omega u_n\|_{L^2} \rightarrow 0$. Then there exists $c > 0$ such that $\|W u_n\|_{\mathcal{H}_2} \geq c$ for all sufficiently large n . This follows from the fact that otherwise by (17) we would have that, at least on a subsequence, $P_{NT} u_n := (h_{N,n}, h_{T,n})$ satisfies $\|P_{NT} u_n\|_{\mathcal{H}_1} \rightarrow 1$. However, the range of P_{NT} is the finite-dimensional space $K_N(\Omega) \times K_T(\Omega)$. This contradicts that $u_n \rightharpoonup 0$ in \mathcal{H}_1 , which implies that $(h_{N,n}, h_{T,n}) \rightarrow 0$ in \mathcal{H}_1 .

Set $\tilde{u}_n = W u_n / \|W u_n\|_{\mathcal{H}_2}$. Then, $\|\tilde{u}_n\|_{\mathcal{H}_2} = 1$ and

$$\tilde{V}_\omega \tilde{u}_n = \frac{V_\omega(\nabla q_{E,n} + \Psi_{E,n}, \nabla q_{H,n} + \Psi_{H,n})}{\|W u_n\|_{\mathcal{H}_2}} = \frac{V_\omega u_n - V_\omega(h_{N,n}, h_{T,n})}{\|W u_n\|_{\mathcal{H}_2}} \rightarrow 0$$

in $L^2(\Omega; \mathbb{C}^3)^2$. Finally, for any $\varphi \in (\mathcal{H}_2)'$ we have $\varphi \circ W \in (\mathcal{H}_1)'$, so

$$\varphi(\tilde{u}_n) = \frac{(\varphi \circ W) u_n}{\|W u_n\|_{\mathcal{H}_2}} \rightarrow 0,$$

and hence ω is in the $\sigma_{e,2}$ essential spectrum of \tilde{V}_ω . \square

By definition of \tilde{V}_ω and (6), we obtain

$$\tilde{V}_\omega(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) = \begin{pmatrix} \operatorname{curl} \Psi_H + iM_\omega \nabla q_E + iM_\omega \Psi_E \\ \operatorname{curl} \Psi_E - i\omega M_\mu \nabla q_H - i\omega M_\mu \Psi_H \end{pmatrix}, \quad (18)$$

where $M_\omega F = (\omega\varepsilon + i\sigma)F$ and $M_\mu F = \mu F$.

In order to simplify this operator even further, we need the following elementary result.

Lemma 19. *Let P_H denote the orthogonal projection onto the space H .*

1. *The map $\zeta_1: L^2(\Omega; \mathbb{C}^3) \rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\dot{X}_T(\Omega)/K_T(\Omega)) \times K_N(\Omega)$ defined by*

$$F \longmapsto (\operatorname{div} F, T_T F, P_{K_N(\Omega)} F)$$

is an isomorphism, where $\dot{H}^{-1}(\Omega; \mathbb{C})$ denotes the dual of $\dot{H}_0^1(\Omega; \mathbb{C})$.

2. *The map $\zeta_2: L^2(\Omega; \mathbb{C}^3) \rightarrow \left(\nabla \dot{H}^1(\Omega; \mathbb{C})\right)' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times K_T(\Omega)$ given by*

$$F \longmapsto (h(F), T_N F, P_{K_T(\Omega)} F),$$

where $h: L^2(\Omega; \mathbb{C}^3) \rightarrow \left(\nabla \dot{H}^1(\Omega; \mathbb{C})\right)'$ is defined by

$$\langle h(F), \nabla q \rangle := \int_{\Omega} F \cdot \nabla q \, dx, \quad (19)$$

is an isomorphism.

Proof. (1) Take $(\phi, \Phi, f) \in \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\dot{X}_T(\Omega)/K_T(\Omega)) \times K_N(\Omega)$. We need to show that there exists a unique $F \in L^2(\Omega; \mathbb{C}^3)$ such that $\zeta_1(F) = (\phi, \Phi, f)$. We use the Helmholtz decomposition (13a) and look for F of the form $F = \nabla q + \operatorname{curl} \tilde{\Phi} + f_N$, with $q \in \dot{H}_0^1(\Omega; \mathbb{C})$, $\tilde{\Phi} \in \dot{X}_T(\Omega)/K_T(\Omega)$ and $f_N \in K_N(\Omega)$. First, since $P_{K_N(\Omega)} F = f_N$, choose $f_N = f$. Now note that

$$\operatorname{div} F = \phi \iff \Delta q = \phi,$$

which is uniquely solvable for $q \in \dot{H}_0^1(\Omega; \mathbb{C})$ by the Lax–Milgram theorem.

Further,

$$T_T F = \Phi \iff \tilde{\Phi} = \Phi,$$

which is clearly uniquely solvable for $\tilde{\Phi} \in \dot{X}_T(\Omega)/K_T(\Omega)$. This shows that $\zeta_1(\nabla q + \operatorname{curl} \tilde{\Phi} + f_N) = (\phi, \Phi, f)$, as desired.

(2) The map ζ_2 is well-defined since $\nabla \dot{H}^1(\Omega) \subseteq L^2(\Omega; \mathbb{C}^3)$. We now show that ζ_2 is an isomorphism. Take

$$(\varphi, \Phi, f) \in \left(\nabla \dot{H}^1(\Omega; \mathbb{C})\right)' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times K_T(\Omega).$$

We use the Helmholtz decomposition (13b) and look for F of the form $F = \nabla p + \operatorname{curl} \tilde{\Phi} + f_T$, with $p \in \dot{H}^1(\Omega; \mathbb{C})$, $\tilde{\Phi} \in \dot{X}_N(\Omega)/K_N(\Omega)$ and $f_T \in K_T(\Omega)$. Then $T_N F = \tilde{\Phi}$ and $P_{K_T(\Omega)} F = f_T$, and so $\tilde{\Phi}$ and f_T are uniquely determined by $\tilde{\Phi} = \Phi$ and $f_T = f$.

It remains to show that p can be chosen so that $\nabla p + \operatorname{curl} \Phi + f = \varphi$ or $\nabla p = \varphi - \operatorname{curl} \Phi - f$ in $(\nabla \dot{H}^1(\Omega; \mathbb{C}))'$. Thus we need to find p such that

$$\int_{\Omega} \nabla p \cdot \nabla q \, dx = \int_{\Omega} (\varphi - \operatorname{curl} \Phi - f) \cdot \nabla q \, dx, \quad q \in \dot{H}^1(\Omega; \mathbb{C}).$$

Using that $L^2(\Omega; \mathbb{C}^3) \subseteq (\nabla \dot{H}^1(\Omega))'$, this is uniquely solvable for p using the Lax–Milgram theorem.

This shows that $\zeta_2(\nabla p + \operatorname{curl} \Phi + f) = (\varphi, \Phi, f)$, as desired. \square

Now, define $\zeta = \begin{pmatrix} \zeta_1 & 0 \\ 0 & \zeta_2 \end{pmatrix}$ and $\tilde{\zeta}(F_1, F_2) = (\operatorname{div} F_1, h(F_2), T_N F_2, T_T F_1)$, i.e., $\tilde{\zeta}$ contains the parts of ζ not in $K_N(\Omega) \oplus K_T(\Omega)$. Let

$$\mathcal{H}_3 = \dot{H}^{-1}(\Omega; \mathbb{C}) \times (\nabla \dot{H}^1(\Omega; \mathbb{C}))' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega)).$$

Set

$$\tilde{V}_\omega = \tilde{\zeta} \circ \tilde{V}_\omega : \mathcal{H}_2 \rightarrow \mathcal{H}_3.$$

Lemma 20. *We have $\sigma_{e,2}(\tilde{V}_\omega) = \sigma_{e,2}(\tilde{V}_\omega)$.*

Proof. We use the characterization of the essential spectrum by Weyl singular sequences given in Remark 3.

Since $\tilde{\zeta}$ is continuous, the inclusion $\sigma_{e,2}(\tilde{V}_\omega) \subseteq \sigma_{e,2}(\tilde{V}_\omega)$ is immediate. Let us now show the reverse inclusion.

Take $\omega \in \sigma_{e,2}(\tilde{V}_\omega)$. Let $(u_n)_n \subseteq \mathcal{H}_2$ be a singular sequence, namely $\|u_n\|_{\mathcal{H}_2} = 1$, $u_n \rightharpoonup 0$ in \mathcal{H}_2 and $\tilde{\zeta}(\tilde{V}_\omega(u_n)) \rightarrow 0$ in \mathcal{H}_3 . Since $K_N(\Omega) \oplus K_T(\Omega)$ is finite-dimensional, we have $\pi(\tilde{V}_\omega(u_n)) \rightarrow 0$, where π is the projection onto $K_N(\Omega) \oplus K_T(\Omega)$, since $\pi \circ \tilde{V}_\omega$ is compact. This implies that $\zeta(\tilde{V}_\omega(u_n)) \rightarrow 0$, whence $\tilde{V}_\omega(u_n) \rightarrow 0$ since ζ is an isomorphism. Thus $(u_n)_n$ is a singular sequence for \tilde{V}_ω , and so $\omega \in \sigma_{e,2}(\tilde{V}_\omega)$. \square

Combining Lemmata 18 and 20, we obtain $\sigma_{e,2}(V_\omega) = \sigma_{e,2}(\tilde{V}_\omega)$. The corresponding identity for the essential spectrum $\sigma_{e,4}$ follows from the following abstract result.

Lemma 21. *Let X, Y and Z be Hilbert spaces with $\dim Y < +\infty$ and $T : X \oplus Y \rightarrow Z \oplus Y$ be a bounded linear operator. Define $T' : X \rightarrow Z$ by*

$$T' = P_Z \circ T \circ i_X,$$

where $P_Z : Z \oplus Y \rightarrow Z$ is the orthogonal projection $(z, y) \mapsto z$ and $i_X : X \rightarrow X \oplus Y$ is the canonical immersion $x \mapsto (x, 0)$. Then T is a Fredholm operator with index 0 if and only if T' is a Fredholm operator with index 0.

Proof. Observe that

$$\begin{aligned} T(x, y) &= (P_Z T(x, y), P_Y T(x, y)) \\ &= (P_Z T(x, y), 0) + (0, P_Y T(x, y)) \\ &= (P_Z T(x, 0), 0) + (P_Z T(0, y), 0) + (0, P_Y T(x, y)). \end{aligned}$$

Since the index is invariant under compact perturbations and the operator $(x, y) \mapsto (P_Z T(0, y), 0) + (0, P_Y T(x, y))$ is finite rank ($\dim Y < +\infty$), we have that T is Fredholm with index 0 if and only if

$$T'' : X \oplus Y \rightarrow Z \oplus Y, \quad (x, y) \mapsto (P_Z T(x, 0), 0),$$

is Fredholm with index 0.

Let $T''' = T'' \circ i_X : X \rightarrow Z \oplus Y$. Since $\dim \ker T'' = \dim \ker T''' + \dim Y$ and $\dim \operatorname{coker} T'' = \dim \operatorname{coker} T'''$ we have that T'' is Fredholm with index 0 if and only if T''' is Fredholm with

$$\operatorname{ind}(T''') = \dim \ker T''' - \dim \operatorname{coker} T''' = -\dim Y.$$

Finally, observe that $T' = P_Z \circ T'''$. Since the range of T''' is contained in $Z \oplus \{0\}$ we have $\dim \ker T''' = \dim \ker T'$ and $\dim \operatorname{coker} T''' = \dim \operatorname{coker} T' + \dim Y$. Hence, T''' is Fredholm with $\operatorname{ind}(T''') = -\dim Y$ if and only if T' is Fredholm with index 0. This concludes the proof. \square

Lemma 22. *We have $\sigma_{e,4}(V_\omega) = \sigma_{e,4}(\tilde{V}_\omega)$.*

Proof. Recall that $V_\omega : \mathcal{H}_1 \rightarrow L^2(\Omega; \mathbb{C}^3)^2$. By (11) and (12), we have $\mathcal{H}_1 = \mathcal{H}_2 \oplus (K_N(\Omega) \oplus K_T(\Omega))$. By Lemma 19, we can identify

$$L^2(\Omega; \mathbb{C}^3)^2 = \mathcal{H}_3 \oplus (K_N(\Omega) \oplus K_T(\Omega)),$$

(disregarding isomorphisms). Since $K_N(\Omega) \oplus K_T(\Omega)$ is finite-dimensional, the result is an immediate consequence of Lemma 21 applied with $X = \mathcal{H}_2$, $Z = \mathcal{H}_3$, $Y = K_N(\Omega) \oplus K_T(\Omega)$, $T = V_\omega$ and $T' = \tilde{V}_\omega$. \square

Now, recalling that $\Psi_H \in \dot{X}_T(\Omega)$ and $\Psi_E \in \dot{X}_N(\Omega)$, by (14) and (18) we have that

$$\begin{aligned} \tilde{V}_\omega(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H) &= \tilde{\zeta} \begin{pmatrix} \operatorname{curl} \Psi_H + iM_\omega \nabla q_E + iM_\omega \Psi_E \\ \operatorname{curl} \Psi_E - i\omega M_\mu \nabla q_H - i\omega M_\mu \Psi_H \end{pmatrix} \\ &= \begin{pmatrix} i \operatorname{div}(M_\omega \nabla q_E) + i \operatorname{div}(M_\omega \Psi_E) \\ -i\omega h(M_\mu \nabla q_H) - i\omega h(M_\mu \Psi_H) \\ [\Psi_E] - i\omega T_N M_\mu \nabla q_H - i\omega T_N M_\mu \Psi_H \\ [\Psi_H] + iT_T M_\omega \nabla q_E + iT_T M_\omega \Psi_E \end{pmatrix}, \end{aligned} \quad (20)$$

in which $[\cdot]$ denotes the equivalence class in the appropriate quotient space.

In order to compute the essential spectrum of \tilde{V}_ω we now decompose the coefficients in the Maxwell system. As a consequence of our Hypotheses (3, 4), whether Ω be bounded or unbounded, for each $\delta > 0$ the Maxwell coefficients admit a decomposition

$$\mu = \mu_0 + \mu_c + \mu_\delta, \quad \varepsilon = \varepsilon_0 + \varepsilon_c + \varepsilon_\delta, \quad \sigma = \sigma_0 + \sigma_c + \sigma_\delta, \quad (21)$$

in which the terms μ_0 , ε_0 and σ_0 are constant and do not depend on δ , the terms μ_c , ε_c and σ_c are compactly supported, and the terms μ_δ , ε_δ , σ_δ are essentially bounded, with

$$m_\delta := \max(\|\mu_\delta\|_{L^\infty(\Omega)}, \|\varepsilon_\delta\|_{L^\infty(\Omega)}, \|\sigma_\delta\|_{L^\infty(\Omega)}) < \delta; \quad (22)$$

here the norms are defined by $\|a\|_{L^\infty(\Omega)} := \text{ess sup}_{x \in \Omega} \|a(x)\|_2$ for $a \in L^\infty(\Omega; \mathbb{R}^{3 \times 3})$, where $\|A\|_2$ denotes the induced norm $\sup_{v \in \mathbb{R}^3 \setminus \{0\}} \frac{|Av|}{|v|}$ for $A \in \mathbb{R}^{3 \times 3}$.

In the expression for \tilde{V}_ω appearing in (20) the Maxwell coefficients appear linearly in the multiplication operators M_μ (multiplication by μ) and M_ω (multiplication by $\omega\varepsilon + i\sigma$). The decomposition (21) of the coefficients is partially reflected in the following decomposition of \tilde{V}_ω :

$$\tilde{V}_\omega = \tilde{V}_{\omega,0} + \tilde{V}_{\omega,c} + \tilde{V}_{\omega,\delta},$$

in which

$$\tilde{V}_{\omega,0} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} = \begin{pmatrix} i \operatorname{div}((\omega\varepsilon + i\sigma)\nabla q_E) \\ -i\omega h(\mu\nabla q_H) \\ -i\omega T_N(\mu\nabla q_H) + [\Psi_E] - i\omega T_N(\mu_0\Psi_H) \\ iT_T((\omega\varepsilon + i\sigma)\nabla q_E) + iT_T((\omega\varepsilon_0 + i\sigma_0)\Psi_E) + [\Psi_H] \end{pmatrix}, \quad (23)$$

$$\tilde{V}_{\omega,c} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} = \begin{pmatrix} i \operatorname{div}((\omega(\varepsilon_0 + \varepsilon_c) + i(\sigma_0 + \sigma_c))\Psi_E) \\ -i\omega h((\mu_0 + \mu_c)\Psi_H) \\ -i\omega T_N(\mu_c\Psi_H) \\ iT_T((\omega\varepsilon_c + i\sigma_c)\Psi_E) \end{pmatrix}$$

and

$$\tilde{V}_{\omega,\delta} \begin{pmatrix} \nabla q_E \\ \nabla q_H \\ \Psi_E \\ \Psi_H \end{pmatrix} = \begin{pmatrix} i \operatorname{div}((\omega\varepsilon_\delta + i\sigma_\delta)\Psi_E) \\ -i\omega h(\mu_\delta\Psi_H) \\ -i\omega T_N(\mu_\delta\Psi_H) \\ iT_T((\omega\varepsilon_\delta + i\sigma_\delta)\Psi_E) \end{pmatrix}.$$

Note that the operator $\tilde{V}_{\omega,0}$ is independent of δ . Further, the operator $\tilde{V}_{\omega,c}$ is compact and the operator $\tilde{V}_{\omega,\delta}$ is $O(\delta)$ -small in a suitable norm, as we show in the following two lemmata.

Lemma 23. *The operator $\tilde{V}_{\omega,c} : \mathcal{H}_2 \rightarrow \mathcal{H}_3$ is compact.*

Proof. By a direct calculation it is easy to see that $\operatorname{div}((\omega\varepsilon_0 + i\sigma_0)\Psi_E) = 0$ and $h(\mu_0\Psi_H) = 0$, using that ε_0 , σ_0 and μ_0 are scalar. Since the operators

$$\begin{aligned} \operatorname{div} : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}), & T_T : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{X}_T(\Omega)/K_T(\Omega), \\ h : L^2(\Omega; \mathbb{C}^3) &\rightarrow (\nabla \dot{H}^1(\Omega; \mathbb{C}))', & T_N : L^2(\Omega; \mathbb{C}^3) &\rightarrow \dot{X}_N(\Omega)/K_N(\Omega), \end{aligned} \quad (24)$$

are bounded, it is enough to show that the operators

$$\begin{aligned} F_T : \mathcal{H}_0(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) &\rightarrow L^2(\Omega; \mathbb{C}^3), & \Psi_E &\mapsto (\omega\varepsilon_c + i\sigma_c)\Psi_E, \\ F_N : \mathcal{H}(\operatorname{curl}, \Omega) \cap \operatorname{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) &\rightarrow L^2(\Omega; \mathbb{C}^3), & \Psi_H &\mapsto \mu_c\Psi_H, \end{aligned}$$

are compact. We now prove that F_T is compact, the other proof is completely analogous. Let $R > 0$ be big enough so that $K := \operatorname{supp}(\omega\varepsilon_c + i\sigma_c) \subseteq B(0, R) \cap \bar{\Omega}$ and $\chi \in C^\infty(\Omega)$ be a cutoff function such that $\chi \equiv 1$ in K and $\operatorname{supp} \chi \subseteq$

$B(0, R) \cap \bar{\Omega}$. Setting $\Omega_R = B(0, R) \cap \Omega$, the operator F_T may be expressed via the following compositions

$$\begin{array}{ccc}
 \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) & & \Psi_E \\
 \downarrow & & \downarrow \\
 \mathcal{H}_0(\text{curl}, \Omega_R) \cap \mathcal{H}(\text{div}, \Omega_R) & & (\chi\Psi_E)|_{\Omega_R} \\
 \downarrow & & \downarrow \\
 L^2(\Omega_R; \mathbb{C}^3) & & (\chi\Psi_E)|_{\Omega_R} \\
 \downarrow & & \downarrow \\
 L^2(\Omega_R; \mathbb{C}^3) & & ((\omega\varepsilon_c + i\sigma_c)\Psi_E)|_{\Omega_R} \\
 \downarrow & & \downarrow \\
 L^2(\Omega; \mathbb{C}^3) & & (\omega\varepsilon_c + i\sigma_c)\Psi_E
 \end{array} ,$$

where the third operator is the multiplication by $\omega\varepsilon_c + i\sigma_c$ and the fourth operator is simply the extension by zero. Therefore, since the embedding $\mathcal{H}_0(\text{curl}, \Omega_R) \cap \mathcal{H}(\text{div}, \Omega_R) \hookrightarrow L^2(\Omega_R; \mathbb{C}^3)$ is compact [19] (see also [4, Theorem 2.8]), the operator F_T is compact. \square

Lemma 24. *There exists a constant $C > 0$ depending only on Ω and on the coefficients μ , ε and σ , such that for each $\delta > 0$ we have*

$$\|\tilde{V}_{\omega, \delta}\|_{\mathcal{H}_2 \rightarrow \mathcal{H}_3} \leq C(1 + |\omega|)\delta.$$

Proof. Note that by (16) we have

$$\|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{L^2(\Omega; \mathbb{C}^3)^4} \leq \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_2}.$$

Thus, since the four operators in (24) are bounded, there exists a constant $C > 0$ depending only on Ω and on the coefficients μ , ε and σ , such that

$$\begin{aligned}
 \|\tilde{V}_{\omega, \delta}(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_3} &\leq C(1 + |\omega|)m_\delta \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{L^2(\Omega; \mathbb{C}^3)^4} \\
 &\leq C(1 + |\omega|)\delta \|(\nabla q_E, \nabla q_H, \Psi_E, \Psi_H)\|_{\mathcal{H}_2},
 \end{aligned}$$

where the second inequality follows from (22). This concludes the proof. \square

It is helpful to recall that $\tilde{V}_\omega : \mathcal{H}_2 \rightarrow \mathcal{H}_3$, where

$$\begin{aligned}
 \mathcal{H}_2 &= \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \times \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \\
 &\quad \times \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega))
 \end{aligned}$$

and

$$\mathcal{H}_3 = \dot{H}^{-1}(\Omega; \mathbb{C}) \times \left(\nabla \dot{H}^1(\Omega; \mathbb{C})\right)' \times (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega)).$$

Proposition 25. *The $\sigma_{e,2}$ essential spectrum of \tilde{V}_ω is the union of the $\sigma_{e,2}$ essential spectra of the two block operator pencils*

$$\begin{aligned}
 \mathcal{A}_\omega &: \nabla \dot{H}_0^1(\Omega) \times \nabla \dot{H}^1(\Omega) \rightarrow \dot{H}^{-1}(\Omega; \mathbb{C}) \times \left(\nabla \dot{H}^1(\Omega; \mathbb{C})\right)' \\
 \mathcal{D}_\omega &: \mathcal{H}_0(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_T(\Omega)/K_T(\Omega)) \times \mathcal{H}(\text{curl}, \Omega) \cap \text{curl}(\dot{X}_N(\Omega)/K_N(\Omega)) \\
 &\quad \rightarrow (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega))
 \end{aligned}$$

determined by the expressions

$$\mathcal{A}_\omega = \begin{pmatrix} i \operatorname{div}((\omega\varepsilon + i\sigma)\cdot) & 0 \\ 0 & -i\omega h(\mu\cdot) \end{pmatrix}, \quad \mathcal{D}_\omega = \begin{pmatrix} I_N & -i\omega\mu_0 T_N \\ i(\omega\varepsilon_0 + i\sigma_0) T_T & I_T \end{pmatrix}. \quad (25)$$

Here the operators I_N and I_T are the canonical mappings from $\dot{X}_N(\Omega)$ and $\dot{X}_T(\Omega)$ to the quotient spaces $\dot{X}_N(\Omega)/K_N(\Omega)$ and $\dot{X}_T(\Omega)/K_T(\Omega)$, respectively.

Proof. By inspection of (23), the operator pencil $\tilde{V}_{\omega,0}$ may be written as the block lower triangular operator matrix pencil

$$\tilde{V}_{\omega,0} = \begin{pmatrix} \mathcal{A}_\omega & 0 \\ \mathcal{C}_\omega & \mathcal{D}_\omega \end{pmatrix}, \quad (26)$$

in which \mathcal{A}_ω and \mathcal{D}_ω are as in Eq. (25) and the off-diagonal component $\mathcal{C}_\omega: \nabla\dot{H}_0^1(\Omega) \times \nabla\dot{H}^1(\Omega) \rightarrow (\dot{X}_N(\Omega)/K_N(\Omega)) \times (\dot{X}_T(\Omega)/K_T(\Omega))$ is given by

$$\mathcal{C}_\omega = \begin{pmatrix} 0 & -i\omega T_N(\mu\cdot) \\ iT_T((\omega\varepsilon + i\sigma)\cdot) & 0 \end{pmatrix}.$$

The proof is divided into several steps.

1. $\sigma_{e,2}(\tilde{V}_{\omega,0}) \subseteq \sigma_{e,2}(\mathcal{A}_\omega) \cup \sigma_{e,2}(\mathcal{D}_\omega)$.

Let us prove the claim. Take $\omega \in \sigma_{e,2}(\tilde{V}_{\omega,0})$. Keeping in mind the block structure (26) of $\tilde{V}_{\omega,0}$, let $(u_n, v_n)_n$ be a singular sequence for $\tilde{V}_{\omega,0}$. If $u_n \rightarrow 0$, then at least some subsequence of $v_n/\|v_n\|$ is a singular sequence for \mathcal{D}_ω , and so $\omega \in \sigma_{e,2}(\mathcal{D}_\omega)$. Otherwise, at least some subsequence of $u_n/\|u_n\|$ is a singular sequence for \mathcal{A}_ω , and so $\omega \in \sigma_{e,2}(\mathcal{A}_\omega)$.

2. $\sigma_{e,2}(\mathcal{D}_\omega) \subseteq \sigma_{e,2}(\tilde{V}_{\omega,0})$.

In order to prove the claim, it is enough to observe that if $(v_n)_n$ is a singular sequence for \mathcal{D}_ω , then $(0, v_n)$ is a singular sequence for $\tilde{V}_{\omega,0}$.

3. $\sigma_{e,2}^*(\tilde{V}_\omega) \subseteq \sigma_{e,2}(\tilde{V}_\omega)$.

Recall that $\sigma_{e,2}^*(\tilde{V}_\omega)$ is the set of ω for which \tilde{V}_ω is not in the class \mathcal{F}_- of semi-Fredholm operators with finite-dimensional cokernel. By [9, Chapter IX, section 1] (the relevant argument is valid also if the domain and the codomain of the operators are different, as in our case), we have

$$\sigma_{e,2}^*(\tilde{V}_\omega) = \overline{\sigma_{e,2}(\tilde{V}_\omega^*)}, \quad \overline{\sigma_{e,4}(\tilde{V}_\omega^*)} = \sigma_{e,4}(\tilde{V}_\omega).$$

Further, combining Lemmata 2, 18, 20 and 22 we have

$$\sigma_{e,2}(\tilde{V}_\omega) = \sigma_{e,4}(\tilde{V}_\omega).$$

Thus, the claim follows using the inclusion $\sigma_{e,2}(\tilde{V}_\omega^*) \subseteq \sigma_{e,4}(\tilde{V}_\omega^*)$.

4. $\sigma_{e,2}(\tilde{V}_\omega) = \sigma_{e,2}(\tilde{V}_{\omega,0})$.

Take $\omega \in \mathbb{C}$. Recall that $\tilde{V}_\omega = \tilde{V}_{\omega,0} + \tilde{V}_{\omega,c} + \tilde{V}_{\omega,\delta}$, in which \tilde{V}_ω and $\tilde{V}_{\omega,0}$ are independent of δ . By Lemma 24 we have

$$\lim_{\delta \rightarrow 0} \|(\tilde{V}_\omega - \tilde{V}_{\omega,0}) - \tilde{V}_{\omega,c}\|_{\mathcal{H}_2 \rightarrow \mathcal{H}_3} = 0.$$

In other words, $\tilde{V}_\omega - \tilde{V}_{\omega,0}$ is the operator norm limit of the compact operators $\tilde{V}_{\omega,c}$, and is therefore compact. Thus $\tilde{V}_\omega = \tilde{V}_{\omega,0} + K_\omega$ for some compact operator K_ω . Hence, the claim follows by the invariance of $\sigma_{e,2}$ under compact perturbations.

$$5. \sigma_{e,2}^*(\tilde{V}_\omega) = \sigma_{e,2}^*(\tilde{V}_{\omega,0}).$$

The proof is completely analogous to that of the previous claim.

$$6. \sigma_{e,2}^*(\mathcal{A}_\omega) = \sigma_{e,2}(\mathcal{A}_\omega).$$

For $u, u' \in \dot{H}_0^1(\Omega)$, by integrating by parts we obtain

$$\langle i \operatorname{div}((\omega\varepsilon + i\sigma)\nabla u), u' \rangle = \langle i \operatorname{div}((\omega\varepsilon + i\sigma)\nabla u'), u \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the duality product between $\dot{H}^{-1}(\Omega; \mathbb{C})$ and $\dot{H}_0^1(\Omega; \mathbb{C})$. This shows that u' belongs to the cokernel of the operator $i \operatorname{div}((\omega\varepsilon + i\sigma) \cdot)$ if and only if $\nabla u'$ belongs to its kernel, hence

$$\sigma_{e,2}^*(i \operatorname{div}((\omega\varepsilon + i\sigma) \cdot)) = \sigma_{e,2}(i \operatorname{div}((\omega\varepsilon + i\sigma) \cdot)).$$

By a similar argument based on the definition of h given in (19), we obtain

$$\sigma_{e,2}^*(-i\omega h(\mu \cdot)) = \sigma_{e,2}(-i\omega h(\mu \cdot)).$$

Combining these two identities yields the claim.

$$7. \sigma_{e,2}^*(\mathcal{A}_\omega) \subseteq \sigma_{e,2}^*(\tilde{V}_{\omega,0}).$$

Take $\bar{\omega} \in \sigma_{e,2}^*(\mathcal{A}_\omega) = \overline{\sigma_{e,2}(\mathcal{A}_\omega^*)}$. Thus there exists a singular sequence $(u_n)_n$ for \mathcal{A}_ω^* . By (26), we have

$$\tilde{V}_{\omega,0}^* = \begin{pmatrix} \mathcal{A}_\omega^* & \mathcal{C}_\omega^* \\ 0 & \mathcal{D}_\omega^* \end{pmatrix},$$

and so $(u_n, 0)_n$ is a singular sequence for $\tilde{V}_{\omega,0}^*$. Thus $\bar{\omega} \in \sigma_{e,2}(\tilde{V}_{\omega,0}^*) = \sigma_{e,2}^*(\tilde{V}_{\omega,0})$.

Let us now conclude the proof. By items 1. and 4. we obtain

$$\sigma_{e,2}(\tilde{V}_\omega) \subseteq \sigma_{e,2}(\mathcal{A}_\omega) \cup \sigma_{e,2}(\mathcal{D}_\omega).$$

By items 2. and 4. we have

$$\sigma_{e,2}(\mathcal{D}_\omega) \subseteq \sigma_{e,2}(\tilde{V}_\omega).$$

By items 3., 5., 6. and 7. we have

$$\sigma_{e,2}(\mathcal{A}_\omega) \subseteq \sigma_{e,2}(\tilde{V}_\omega).$$

This concludes the proof. \square

Remark 26. The text [18] contains many interesting results on essential spectra of block-operator matrices and pencils; Theorem 2.4.1 is very close to what we would need, but our pencil $\tilde{V}_{\omega,0}$ is lower triangular rather than diagonally dominant.

We are now ready to prove our main result.

Proof of Theorem 6. We commence the proof by observing the following identity:

$$\sigma_{e,2}(V_\omega) = \sigma_{e,2}(\mathcal{A}_\omega) \cup \sigma_{e,2}(\mathcal{D}_\omega). \quad (27)$$

This is an immediate consequence of Lemmas 18 and 20 and of Proposition 25. We now consider $\sigma_{e,2}(\mathcal{A}_\omega)$ and $\sigma_{e,2}(\mathcal{D}_\omega)$ in more detail.

The essential spectrum of \mathcal{A}_ω consists of the point $\{0\}$, arising from the $(2, 2)$ diagonal entry of \mathcal{A}_ω , which has $\omega = 0$ as an eigenvalue of infinite multiplicity and is otherwise invertible; and of the essential spectrum of the pencil in the $(1, 1)$ entry, which is as stated in the theorem, namely

$$\sigma_{e,2}(\mathcal{A}_\omega) = \{0\} \cup \sigma_{e,2}(\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)). \quad (28)$$

In order to deal with the essential spectrum of \mathcal{D}_ω we observe that if we replace V_ω by a new pencil V_ω^0 in which the coefficients have the constant values ε_0 , μ_0 and σ_0 , then \mathcal{D}_ω will be unchanged while \mathcal{A}_ω will be replaced by a pencil $\mathcal{A}_{\omega,0}$ in which all the coefficients are constant. For the constant coefficient pencil $\mathcal{A}_{\omega,0}$ we see that 0 lies in the $\sigma_{e,2}$ essential spectrum as we reasoned before, while the $(1, 1)$ term is invertible and Fredholm precisely when $\omega\varepsilon_0 + i\sigma_0 \neq 0$, by the Babuška–Lax–Milgram theorem; hence $\sigma_{e,2}(\mathcal{A}_{\omega,0}) = \{0, -i\sigma_0/\varepsilon_0\}$. Using (27) for the constant coefficient pencil, we now have

$$\sigma_{e,2}(V_\omega^0) = \{0, -i\sigma_0/\varepsilon_0\} \cup \sigma_{e,2}(\mathcal{D}_\omega). \quad (29)$$

We now prove that the $\sigma_{e,2}$ essential spectrum of \mathcal{A}_ω already contains the set $\{0, -i\sigma_0/\varepsilon_0\}$. The $(2, 2)$ component has 0 as an eigenvalue of infinite multiplicity. If Ω is bounded, we have $\sigma_0 = 0$ and so the claim is proven. Otherwise, for the point $-i\sigma_0/\varepsilon_0$ we observe that by the Hypothesis (3), given $n > 0$ there exists $R_n > 0$ such that if $\omega_0 := -i\sigma_0/\varepsilon_0$ then

$$\sup_{|x| \geq R_n} \|\omega_0\varepsilon(x) + i\sigma(x)\|_2 < \frac{1}{n}.$$

Choosing any function $\phi_n \in C_0^\infty(\Omega)$ with support in $\{x \in \Omega : |x| > R_n\}$, with $\|\nabla\phi_n\|_{L^2(\Omega)} = 1$, we see that

$$\|\operatorname{div}((\omega_0\varepsilon + i\sigma)\nabla\phi_n)\|_{\dot{H}^{-1}(\Omega)} \leq \frac{1}{n}.$$

Since the supports of the sequence $(\nabla\phi_n)_{n \in \mathbb{N}}$ move off to infinity, the sequence converges weakly to zero; it is therefore a singular sequence in $\nabla\dot{H}_0^1(\Omega)$ for the $(1, 1)$ element of \mathcal{A}_{ω_0} . Thus ω_0 lies in the $\sigma_{e,2}$ essential spectrum of \mathcal{A}_ω . Combining the observations (27), (28) and (29) with the fact that $\sigma_{e,2}(\mathcal{A}_\omega) \supseteq \{0, -i\sigma_0/\varepsilon_0\}$ completes the proof. \square

We conclude this section with a more explicit description of the essential spectrum of the divergence form operator $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$ in the case of continuous coefficients.

Proposition 27. *When the coefficients ε and σ are continuous in $\bar{\Omega}$, the $\sigma_{e,2}$ essential spectrum of $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$, acting from $\dot{H}_0^1(\Omega; \mathbb{C})$ to $\dot{H}^{-1}(\Omega; \mathbb{C})$, consists of the closure of the set of all $\omega = i\nu$, $\nu \in \mathbb{R}$, such that $\nu\varepsilon + \sigma$ is indefinite at some point in Ω . Equivalently, when Ω is bounded, it is the set of $\omega = i\nu$, $\nu \in \mathbb{R}$, such that $\nu\varepsilon + \sigma$ is indefinite at some point in $\bar{\Omega}$.*

Proof. If $\Re(\omega) \neq 0$ then the real part of $\omega\varepsilon + i\sigma$ is definite, and the result follows by the Lax–Milgram theorem. If $\omega = i\nu$ is purely imaginary, this reasoning still works if $\nu\varepsilon + \sigma$ is uniformly definite in Ω . It remains only to show that if $\nu\varepsilon + \sigma$ is indefinite at some point $x_0 \in \Omega$, then 0 lies in the essential spectrum of $\operatorname{div}((\omega\varepsilon + i\sigma)\nabla \cdot)$.

We prove the result by constructing a Weyl singular sequence. Define $a := \nu\varepsilon + \sigma$ and $a_0 := a(x_0)$. Let $\chi : [0, \infty) \mapsto [0, 1]$ be a smooth cutoff function such that $\chi(t) = 1$ for $0 \leq t \leq 1$ and $\chi(t) = 0$ for all $t \geq 2$. Let $\theta \in \mathbb{R}^3$ be a unit vector chosen such that $\theta^T a(x_0) \theta = 0$. For each sufficiently small $\delta > 0$ and large $r > 0$ let

$$\chi_\delta(x) := \frac{1}{\delta^{3/2}} \chi\left(\frac{|x - x_0|}{\delta}\right), \quad u_{r,\delta}(x) := \chi_\delta(x) r^{-1} \exp(ir\theta \cdot x). \quad (30)$$

A direct calculation shows that $\nabla u_{r,\delta}$ in sup-norm is $O(r^{-1}\delta^{-5/2}) + O(\delta^{-3/2})$. We suppose that $r\delta^{5/2} \gg 1$, so that the $\delta^{-3/2}$ term dominates; we have $\|\nabla u_{r,\delta}\|_{L^\infty(B_{2\delta}(x_0))} = O(\delta^{-3/2})$ and $\|u_{r,\delta}\|_{\dot{H}_0^1(\Omega)} \geq c$ for some $c > 0$ independent of r and δ . If v is any smooth test function then

$$\begin{aligned} |\langle \nabla u_{r,\delta}, \nabla v \rangle| &\leq C\delta^3 \|\nabla u_{r,\delta}\|_{L^\infty(B_{2\delta}(x_0))} \|\nabla v\|_{L^\infty(B_{2\delta}(x_0))} \\ &\leq C\|\nabla v\|_\infty \delta^3 \delta^{-3/2} = O(\delta^{3/2}), \end{aligned}$$

so that the $u_{r,\delta}$ tend to zero weakly in $H_0^1(\Omega)$ as $r \nearrow +\infty$ and $\delta \searrow 0$, with $r \geq \delta^{-5/2}$.

To complete the proof that 0 lies in the essential spectrum of our operator we show that $\|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)}$ can be made arbitrarily small. It is easy to see that

$$\|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)} \leq \|a - a_0\|_{L^\infty(B_{2\delta}(x_0))} + \sup_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle|}{\|v\|_{H_0^1(\Omega)}}. \quad (31)$$

We compute $\nabla u_{r,\delta}$ by direct differentiation of eqn. (30) and deduce that for each $v \in H_0^1(\Omega)$,

$$\begin{aligned} |\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| &\leq \left| \left\langle a_0 \frac{1}{r\delta^{5/2}} \frac{x - x_0}{|x - x_0|} \chi' \left(\frac{|x - x_0|}{\delta} \right) \exp(ir\theta \cdot x), \nabla v \right\rangle \right| \\ &\quad + |\langle \chi_\delta a_0 \nabla (r^{-1} \exp(ir\theta \cdot x)), \nabla v \rangle| \\ &\leq C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + |\langle \operatorname{div}(\chi_\delta a_0 \nabla (r^{-1} \exp(ir\theta \cdot x)), v \rangle| \\ &= C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + |\langle (\nabla \chi_\delta) \cdot a_0 \nabla (r^{-1} \exp(ir\theta \cdot x)), v \rangle|; \end{aligned}$$

in the last step we have used the fact that $\operatorname{div}(a_0 \nabla(r^{-1} \exp(ir\theta \cdot x))) = 0$, which follows immediately from $\theta^T a_0 \theta = 0$. Integration by parts yields

$$|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| \leq C \frac{|a_0|}{r\delta} \|v\|_{H_0^1(\Omega)} + |\langle r^{-1} \exp(ir\theta \cdot x), \operatorname{div}(va_0^T \nabla \chi_\delta) \rangle|.$$

We estimate the final inner product by observing that χ_δ is $O(\delta^{-3/2})$, its gradient is $O(\delta^{-5/2})$ and its second derivatives $O(\delta^{-7/2})$, while its support is a ball whose volume is $O(\delta^3)$: thus

$$|\langle a_0 \nabla u_{r,\delta}, \nabla v \rangle| \leq C \frac{|a_0|}{r} \left\{ \frac{1}{\delta} + \frac{1}{\delta^2} \right\} \|v\|_{H_0^1(\Omega)},$$

for some constant $C > 0$. Substituting this back into (31) we obtain

$$\|\operatorname{div}(a \nabla u_{r,\delta})\|_{H^{-1}(\Omega)} \leq \|a - a_0\|_{L^\infty(B_{2\delta}(x_0))} + C \frac{|a_0|}{r} \left\{ \frac{1}{\delta} + \frac{1}{\delta^2} \right\}.$$

Letting $r \nearrow \infty$ and then letting $\delta \searrow 0$ we obtain the required result. \square

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Appendix A: The Helmholtz Decomposition for Cylinders

This appendix is devoted to the study of the decompositions (12a) and (12b) for a large class of cylinders of the form $\Omega = \mathbb{R} \times \Omega'$, with $\Omega' \subseteq \mathbb{R}^2$. We will then show that this class includes the full space, the half space, the slab, and the cylinders with bounded sections as in Proposition 15 part(6), thereby providing a proof to the corresponding parts of Proposition 15.

We denote coordinates in Ω by (x_1, x') where $x' = (x_2, x_3) \in \Omega'$, with similar conventions for components of vectors and operators, such as gradient

and Laplacian. For simplicity of notation, we shall write $a \lesssim b$ to mean $a \leq Cb$ for some positive constant C depending only on Ω' . We assume that the cross-section Ω' satisfies the following additional hypothesis.

Assumption 28. Let $g, h \in L^2(\Omega')$. If $\psi' \in \mathcal{D}'(\Omega')$ satisfies

$$\begin{cases} \operatorname{curl}' \psi' = g & \text{in } \Omega', \\ \operatorname{div}' \psi' = h & \text{in } \Omega', \\ \psi' \cdot \nu' = 0 \text{ or } \psi' \cdot \tau' = 0 & \text{on } \partial\Omega', \end{cases} \quad (32)$$

where $\nu' = (\nu_2, \nu_3)$ and $\tau' = (-\nu_3, \nu_2)$ denote the unit normal and tangent vectors to $\partial\Omega'$, respectively, then

$$\|\nabla' \psi'\|_{L^2(\Omega')} \lesssim \|g\|_{L^2(\Omega')} + \|h\|_{L^2(\Omega')}.$$

This assumption guarantees the existence of the decompositions (12a) and (12b) with the spaces $K_T(\Omega)$ and $K_N(\Omega)$ (Definition 12) both trivial.

Proposition 29. Let $\Omega = \mathbb{R} \times \Omega'$, where $\Omega' \subseteq \mathbb{R}^2$ is a Lipschitz domain satisfying Assumption 28. Then $K_N(\Omega) = K_T(\Omega) = \{0\}$ and

- (a) $\mathcal{H}(\operatorname{div} 0, \Omega) = \operatorname{curl}\{\psi \in \dot{H}^1(\Omega) : \operatorname{div} \psi = 0 \text{ in } \Omega, \psi \cdot \nu = 0 \text{ on } \partial\Omega\}$,
- (b) $\mathcal{H}_0(\operatorname{div} 0, \Omega) = \operatorname{curl}\{\psi \in \dot{H}^1(\Omega) : \operatorname{div} \psi = 0 \text{ in } \Omega, \psi \times \nu = 0 \text{ on } \partial\Omega\}$.

Proof. We divide the proof into three steps.

1. First, we prove that every function f in $\mathcal{H}_0(\operatorname{div} 0, \Omega)$ may be written as the curl of a unique divergence-free function ψ such that $\psi \times \nu = 0$ on $\partial\Omega$. In particular, this implies that the space $K_N(\Omega)$ is trivial.
2. Second, we prove that every function f in $\mathcal{H}(\operatorname{div} 0, \Omega)$ may be written as the curl of a unique divergence-free function ψ such that $\psi \cdot \nu = 0$ on $\partial\Omega$. In particular, this implies that the space $K_T(\Omega)$ is trivial.
3. Third, we prove that the potentials ψ constructed in steps (1) and (2) belong to $\dot{H}^1(\Omega)$.

Step (1) Given $f \in \mathcal{H}_0(\operatorname{div} 0, \Omega)$, we look for ψ such that

$$\operatorname{curl} \psi = f \text{ in } \Omega, \quad (33)$$

$$\operatorname{div} \psi = 0 \text{ in } \Omega, \quad (34)$$

$$\psi \times \nu = 0 \text{ on } \partial\Omega. \quad (35)$$

Since $\nu_1 = 0$, the second and third components of (35) yield $\psi_1 \nu_3 = 0$ and $\psi_1 \nu_2 = 0$, giving $\psi_1 = 0$ on $\partial\Omega$. Taking the curl of Eq. (33) we obtain

$$-\Delta \psi_1 = \partial_2 f_3 - \partial_3 f_2 \text{ in } \Omega;$$

upon taking the Fourier transform with respect to the first coordinate x_1 we obtain the boundary value problem

$$\begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \partial_2 \hat{f}_3 - \partial_3 \hat{f}_2 & \text{in } \Omega', \\ \hat{\psi}_1 = 0 & \text{on } \partial\Omega', \end{cases} \quad (36)$$

in which Δ' denotes the Laplacian with respect to $x' \in \Omega'$ and $\xi \in \mathbb{R}$ is the dual variable of x_1 under Fourier transformation. For almost every $\xi \in \mathbb{R}$, this Dirichlet boundary value problem admits a unique solution $\hat{\psi}_1(\xi) \in \dot{H}_0^1(\Omega')$

by the Lax Milgram theorem, and so ψ_1 is uniquely determined. To obtain the remaining components of ψ we rewrite (33) and (34) as

$$\begin{aligned}\partial_2\psi_3 - \partial_3\psi_2 &= f_1, & \partial_3\psi_1 - \partial_1\psi_3 &= f_2, & \partial_1\psi_2 - \partial_2\psi_1 &= f_3, \\ \partial_1\psi_1 + \partial_2\psi_2 + \partial_3\psi_3 &= 0.\end{aligned}$$

Again take the Fourier transform with respect to x_1 and obtain

$$\begin{aligned}\partial_2\hat{\psi}_3 - \partial_3\hat{\psi}_2 &= \hat{f}_1, & \partial_3\hat{\psi}_1 - i\xi\hat{\psi}_3 &= \hat{f}_2, & i\xi\hat{\psi}_2 - \partial_2\hat{\psi}_1 &= \hat{f}_3, \\ i\xi\hat{\psi}_1 + \partial_2\hat{\psi}_2 + \partial_3\hat{\psi}_3 &= 0.\end{aligned}\tag{37}$$

Using the second and third identities in (37) yields

$$\hat{\psi}_2 = -i\frac{\partial_2\hat{\psi}_1 + \hat{f}_3}{\xi}, \quad \hat{\psi}_3 = i\frac{\hat{f}_2 - \partial_3\hat{\psi}_1}{\xi}, \quad \text{for a.e. } \xi \in \mathbb{R}.$$

It remains to check the first and fourth identities in (37) and the first component of (35). For the first identity in (37) we observe that

$$\partial_2\hat{\psi}_3 - \partial_3\hat{\psi}_2 = \frac{i}{\xi} \left(\partial_2\hat{f}_2 - \partial_{23}\hat{\psi}_1 + \partial_{32}\hat{\psi}_1 + \partial_3\hat{f}_3 \right) = \frac{i}{\xi} (-i\xi\hat{f}_1) = \hat{f}_1.$$

Here we have used, for the second equality, the fact that $\operatorname{div} f = 0$. For the fourth identity in (37), by (36) we have

$$\begin{aligned}i\xi\hat{\psi}_1 + \partial_2\hat{\psi}_2 + \partial_3\hat{\psi}_3 &= i\xi\hat{\psi}_1 + \frac{i}{\xi} \left(-\partial_2^2\hat{\psi}_1 - \partial_2\hat{f}_3 + \partial_3\hat{f}_2 - \partial_3^2\hat{\psi}_1 \right) \\ &= \frac{i}{\xi} \left((\xi^2\hat{\psi}_1 - \partial_2^2\hat{\psi}_1 - \partial_3^2\hat{\psi}_1) - (\partial_2\hat{f}_3 - \partial_3\hat{f}_2) \right) \\ &= 0.\end{aligned}$$

Finally, for the first component of (35), using \mathcal{F} to denote the Fourier transform,

$$\nu_3\hat{\psi}_2 - \nu_2\hat{\psi}_3 = \frac{i}{\xi} \left(-\partial_2\hat{\psi}_1\nu_3 - \hat{f}_3\nu_3 - \hat{f}_2\nu_2 + \partial_3\hat{\psi}_1\nu_2 \right) = \frac{i}{\xi} \mathcal{F}(\nabla' \psi_1 \cdot \tau - f \cdot \nu) = 0,$$

where we have used the fact that $\psi_1 = 0$ and $f \cdot \nu = 0$ on $\partial\Omega$ in the last step.

Step (2) The only difference between this case and the one above lies in the boundary conditions. We no longer have $f \cdot \nu = 0$ on the boundary. This time $f \in \mathcal{H}(\operatorname{div} 0, \Omega)$ and we seek ψ such that

$$\operatorname{curl} \psi = f \quad \text{in } \Omega, \tag{38}$$

$$\operatorname{div} \psi = 0 \quad \text{in } \Omega, \tag{39}$$

$$\psi \cdot \nu = 0 \quad \text{on } \partial\Omega. \tag{40}$$

The calculations follow as above except that problem (36) is replaced by

$$\begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \operatorname{div}'(\hat{f}_3, -\hat{f}_2) & \text{in } \Omega', \\ -\nabla' \hat{\psi}_1 \cdot \nu' = (\hat{f}_3, -\hat{f}_2) \cdot \nu' & \text{on } \partial\Omega', \end{cases}\tag{41}$$

in which the reason for the slightly curious Neumann boundary condition will become clear shortly. As above, for almost every $\xi \in \mathbb{R}$, this problem admits a unique solution $\hat{\psi}_1(\xi) \in \dot{H}^1(\Omega')$ by the Lax Milgram theorem. Having found $\hat{\psi}_1$, we construct $\hat{\psi}_2$ and $\hat{\psi}_3$ as before, and the verification of (38) and (39) is

similar to the calculations for (33) and (34). This leaves the boundary condition (40): since $\hat{\psi} \cdot \nu = \hat{\psi}_2 \nu_2 + \hat{\psi}_3 \nu_3$, we have

$$\hat{\psi} \cdot \nu = \frac{-i}{\xi} \left\{ (\partial_2 \hat{\psi}_1 + \hat{f}_3) \nu_2 - (\hat{f}_2 - \partial_3 \hat{\psi}_1) \nu_3 \right\} = \frac{-i}{\xi} \left\{ \nabla' \hat{\psi}_1 \cdot \nu' + (\hat{f}_3, -\hat{f}_2) \cdot \nu' \right\} = 0,$$

the equality at the last step coming from the boundary equation in (41).

Step (3) We now verify that ψ lies in $\dot{H}^1(\Omega)$ in both cases. From (36, 41) we have

$$\begin{cases} -\Delta' \hat{\psi}_1 + \xi^2 \hat{\psi}_1 = \operatorname{div}'(\hat{f}_3, -\hat{f}_2) & \text{in } \Omega', \\ \hat{\psi}_1 = 0 \text{ or } -\nabla' \hat{\psi}_1 \cdot \nu' = (\hat{f}_3, -\hat{f}_2) \cdot \nu' & \text{on } \partial\Omega'. \end{cases} \quad (42)$$

An integration against $\hat{\psi}_1$ gives

$$\|\nabla' \hat{\psi}_1\|_{L^2(\Omega')}^2 + \xi^2 \|\hat{\psi}_1\|_{L^2(\Omega')}^2 = \left((-\hat{f}_3, \hat{f}_2), \nabla' \hat{\psi}_1 \right)_{L^2(\Omega')}$$

whence

$$\|\nabla' \hat{\psi}_1(\xi)\|_{L^2(\Omega')} \leq \|\hat{f}'(\xi)\|_{L^2(\Omega')}, \quad \|\hat{\psi}_1(\xi)\|_{L^2(\Omega')} \leq \frac{\|\hat{f}'(\xi)\|_{L^2(\Omega')}}{|\xi|}. \quad (43)$$

From the first and fourth identities of (37) we get, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned} \operatorname{curl}' \hat{\psi}' &= \hat{f}_1 & \text{in } \Omega', \\ \operatorname{div}' \hat{\psi}' &= -i\xi \hat{\psi}_1 & \text{in } \Omega'. \end{aligned}$$

We also have the desired boundary conditions:

$$\hat{\psi}' \cdot \tau' = 0 \text{ on } \partial\Omega' \text{ for (b), or } \hat{\psi}' \cdot \nu' = 0 \text{ on } \partial\Omega' \text{ for (a).}$$

By Assumption 28 we have

$$\|\nabla' \hat{\psi}'(\xi)\|_{L^2(\Omega')} \lesssim \|\hat{f}_1(\xi)\|_{L^2(\Omega')} + \|\xi \hat{\psi}_1(\xi)\|_{L^2(\Omega')} \lesssim \|\hat{f}(\xi)\|_{L^2(\Omega')}, \quad (44)$$

the last inequality following from the second inequality in (43).

We now regularize $\hat{\psi}_i$ for $\xi \rightarrow 0$. Define, for $\varepsilon > 0$,

$$\hat{\psi}_{i,\varepsilon}(\xi, x') = \frac{|\xi|}{|\xi| + \varepsilon} \hat{\psi}_i(\xi, x'), \quad i = 1, 2, 3.$$

By a direct calculation we have $\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon} = \frac{\varepsilon}{|\xi| + \varepsilon} \nabla' \hat{\psi}_i$, and so we get

$$\|\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} = \frac{\varepsilon}{|\xi| + \varepsilon} \|\nabla' \hat{\psi}_i\|_{L^2(\Omega')} \leq \|\nabla' \hat{\psi}_i\|_{L^2(\Omega')} \lesssim \|\hat{f}(\xi)\|_{L^2(\Omega')},$$

where the last inequality follows from (43) for $i = 1$ and (44) for $i = 2, 3$. By the Dominated Convergence Theorem, therefore,

$$\lim_{\varepsilon \rightarrow 0} \|\nabla' \hat{\psi}_i - \nabla' \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} = 0.$$

By direct calculation,

$$\|\xi \hat{\psi}_i - \xi \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} = \frac{|\xi| \varepsilon}{|\xi| + \varepsilon} \|\hat{\psi}_i\|_{L^2(\Omega')}. \quad (45)$$

Now, using (43) for $i = 1$ and the two identities

$$\xi \hat{\psi}_2 = -i(\partial_2 \hat{\psi}_1 + \hat{f}_3), \quad \xi \hat{\psi}_3 = i(\hat{f}_2 - \partial_2 \hat{\psi}_1),$$

together with (43) for $i = 2, 3$, we have $|\xi| \cdot \|\hat{\psi}_i\|_{L^2(\Omega')} \lesssim \|\hat{f}\|_{L^2(\Omega')}$, whence $\|\xi \hat{\psi}_i - \xi \hat{\psi}_{i,\varepsilon}\|_{L^2(\Omega')} \lesssim \|\hat{f}\|_{L^2(\Omega')}$. Taking inverse Fourier transforms in (45), dominated convergence yields

$$\lim_{\varepsilon \rightarrow 0} \|\partial_1 \psi_{i,\varepsilon} - \partial_1 \psi_i\|_{L^2(\Omega)} = 0.$$

Altogether we have that $\|\nabla \psi_{i,\varepsilon} - \nabla \psi_i\|_{L^2(\Omega)} \rightarrow 0$ as $\varepsilon \rightarrow 0$, namely

$$\lim_{\varepsilon \rightarrow 0} \|\psi_{i,\varepsilon} - \psi_i\|_{\dot{H}^1(\Omega)} = 0. \tag{46}$$

Since $|\hat{\psi}_{i,\varepsilon}(\xi, x')| \leq \varepsilon^{-1} |\xi \hat{\psi}_i(\xi, x')|$ we have that $\|\psi_{i,\varepsilon}\|_{L^2(\Omega)} \leq \varepsilon^{-1} \|\partial_1 \psi_i\|_{L^2(\Omega)} < +\infty$, and so $\psi_{i,\varepsilon} \in H^1(\Omega)$. Since $H^1(\Omega)$ is dense in $\dot{H}^1(\Omega)$, by (46) we conclude that $\psi \in \dot{H}^1(\Omega)$. \square

We now observe that Assumption 28 is verified in many situations of interest.

Lemma 30. *Assumption 28 is verified in each of the following cases:*

1. Ω' is the full space \mathbb{R}^2 ;
2. Ω' is the half space $\{(x_2, x_3) \in \mathbb{R}^2 : x_3 > 0\}$;
3. Ω' is a strip $\{(x_2, x_3) \in \mathbb{R}^2 : 0 < x_3 < L\}$ for some $L > 0$;
4. Ω' is a simply connected bounded domain of class $C^{1,1}$ or piecewise smooth with no re-entrant corners.

Proof. (1) Taking Fourier transforms in (32) we obtain

$$i\xi_2 \hat{\psi}'_3 - i\xi_3 \hat{\psi}'_2 = \hat{g}, \quad i\xi_2 \hat{\psi}'_2 + i\xi_3 \hat{\psi}'_3 = \hat{h},$$

with unique solution

$$\hat{\psi}'_2 = \frac{-i\xi_2 \hat{h} + i\xi_3 \hat{g}}{|\xi'|^2}, \quad \hat{\psi}'_3 = \frac{-i\xi_3 \hat{h} - i\xi_2 \hat{g}}{|\xi'|^2}.$$

Hence $|\xi_2 \hat{\psi}'_2| = \frac{|\xi_2^2 \hat{h}|}{|\xi'|^2} + \frac{|\xi_2 \xi_3 \hat{g}|}{|\xi'|^2} \leq |\hat{h}| + |\hat{g}|$, so that

$$\|\partial_2 \psi'_2\|_{L^2(\mathbb{R}^2)} \leq \|\hat{g}\|_{L^2(\mathbb{R}^2)} + \|\hat{h}\|_{L^2(\mathbb{R}^2)},$$

and similarly for the other conditions.

(2) In this case the boundary condition is either $\psi'_2 = 0$ or $\psi'_3 = 0$ on $\{x_3 = 0\}$. We study the case $\psi'_2 = 0$; the other is similar.

Taking Fourier transforms we get

$$\begin{aligned} \psi'_2(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \hat{\psi}'_2(\xi_2, \xi_3) e^{i\xi_2 x_2} \sin(\xi_3 x_3) d\xi_3 d\xi_2, \\ \psi'_3(x_2, x_3) &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} \hat{\psi}'_3(\xi_2, \xi_3) e^{i\xi_2 x_2} \cos(\xi_3 x_3) d\xi_3 d\xi_2, \end{aligned}$$

$$g(x_2, x_3) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} g(\xi_2, \xi_3) e^{i\xi_2 x_2} \cos(\xi_3 x_3) d\xi_3 d\xi_2,$$

$$h(x_2, x_3) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} h(\xi_2, \xi_3) e^{i\xi_2 x_2} \sin(\xi_3 x_3) d\xi_3 d\xi_2.$$

The equations in (32) become

$$i\xi_2 \hat{\psi}'_3 - \xi_3 \hat{\psi}'_2 = \hat{g}, \quad i\xi_2 \hat{\psi}'_2 - \xi_3 \hat{\psi}'_3 = \hat{h},$$

and then everything proceeds as for the case $\Omega' = \mathbb{R}^2$.

(3) This follows by using the Fourier transform with respect to the variable x_2 and the Fourier series in the variable x_3 , as in cases (1) and (2) above; the calculations are completely analogous.

(4) This part was proven in [10, Chapter 1, Remark 3.5]. □

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Essential Spectrum for Maxwell's Equations

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