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Disturbance Evolution in Rotating Boundary Layers

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Abstract

The global linear stability of the family of rotating boundary layers (that includes discs and cones) is reviewed. Using a velocity-vorticity form of the linearised Navier-Stokes equations, disturbance evolution is impulsively excited for a variety of flow geometries and perturbation parameter settings. For azimuthal mode numbers below a fixed threshold value, disturbance development is dominated by convectively unstable characteristics, even though the flow might be locally absolutely unstable. As the azimuthal mode number is increased to larger values a form of global linear instability emerges that is characterised by a faster than exponential temporal growth. However, this is only observed when the azimuthal mode number is taken to be significantly greater than the conditions necessary for the onset of absolute instability to occur.

Introduction

The flow that develops above an infinite rotating-disc has long been used as a model to study three-dimensional stability and transition mechanisms. A boundary layer forms when a rigid disc of infinite radius rotates at a constant angular velocity beneath an otherwise still incompressible fluid. This particular rotating flow was first modelled by von Kármán [1] and has since been studied using theoretical, numerical and experimental methods [2]. Gregory *et al* [3] undertook physical experiments on the rotating-disc and observed co-rotating vortices on the disc surface, which are better known as stationary crossflow. Similar observations were made by Kobayashi and co-workers [4, 5, 6] on the surfaces of rotating-cones.

More recently, studies have focused on the connection between local and global linear stability properties and the behaviour of disturbances as they develop in the spatial-temporal plane. The reason for this is largely due to the discovery by Lingwood [7] that for large enough parameter settings the rotating-disc is locally unstable to an absolute form of disturbance; a critical Reynolds number $Re_a = 507.3$ and an azimuthal mode number $n_a = 68$. Absolute instability was identified following the coalescence of the crossflow instability and a spatially damped mode [8], whereby the basic state was simplified using a parallel flow approximation that neglects radial variations. Since Lingwood's discovery, absolute instability has been identified in many other rotating flow systems using local stability theory that applies a similar parallel flow assumption. These include the boundary layers that develop on rotating-spheres and rotating-cones [9, 10].

Davies and Carpenter [11] undertook the first study on the global linear stability characteristics of the rotating-disc boundary layer, where the radial dependence of the von Kármán flow was retained. Disturbance development was numerically simulated using a velocity-vorticity formulation [12] that is fully consistent with the linearised Navier-Stokes equations. Perturbations were impulsively excited through a linearisation of the no-slip condition and formed wavepackets in the spatial-temporal plane. It was shown, for parameter settings near the onset of absolute instability, that the radial dependence of the basic state brought about a stabilisation effect. Temporal decay was observed about all radial locations and azimuthal mode numbers considered. Thus, local absolute instability was not found to give rise to any globally unstable modes. Instead disturbances were characterised by a convectively unstable response.

The global stability of rotating boundary layers was extended by Thomas and Davies [13, 14, 15, 16] to include the effects of flow control mechanisms that were locally stabilising; mass suction through the disc surface and an axial magnetic field. Remarkably the introduction of the control mechanisms established a form of global instability that was identified by a faster than exponential temporal growth. Disturbances were found to grow about all radial positions and for all azimuthal mode numbers modelled. It was originally speculated that the locally stabilising control device was somehow responsible for generating globally unstable behaviour. However, this theory was eventually dismissed by Thomas and Davies [17] following a more thorough investigation of the infinite rotating-disc boundary layer.

Due to the availability of greater computational resources, Thomas and Davies [17] were able to undertake a more thorough global linear stability investigation of the infinite rotating-disc boundary layer than what was feasible in the earlier study by Davies and Carpenter [11]. Disturbance evolution was numerically simulated for an extensive range of perturbation parameter settings. For large enough azimuthal mode numbers, the same form of strong temporal growth was found to develop in the unmodified rotating-disc as that observed in the earlier studies that included flow control [14, 15]. However, globally unstable characteristics were only realised for azimuthal mode numbers greater than the critical conditions for absolute instability. Thus, local absolute instability in the infinite rotating-disc boundary layer [7] can excite a form of global linear instability (albeit one without a fixed global frequency) but only when the azimuthal mode number is taken to be sufficiently large.

More recently, Thomas and Davies [18] extended their global linear stability study to encompass the family of rotating-cone boundary layers that develop in an otherwise still fluid. Results were qualitatively similar to those obtained on the rotating-disc and a change in global behaviour (stable to unstable) was again realised for azimuthal mode numbers greater than those conditions necessary for the onset of absolute instability.

Davies and co-workers [13, 14, 17] were able to provide an explanation for the change in global behaviour by coupling solutions of the linearised Ginzburg-Landau equation [19] with numerical simulation results. Depending on the precise balance in the radial variations in the temporal growth rate and matching shifts in frequency, it was possible for the flow to remain globally stable or become unstable. Using this approach, Thomas and Davies [17, 18] were able to predict, to a reasonable degree of accuracy, the azimuthal mode number needed to bring about a change in the global response.

Hence, the local-global stability of the infinite rotating-disc and the system of rotating-cone boundary layers can be described using the scenario outlined by Huerre and Monkewitz [20]; local absolute instability is a necessary but not sufficient condition for globally unstable behaviour to occur. Much like many other studies on global instability in spatially varying flows [21], self sustained oscillations are possible once the extent of local absolute instability achieves a threshold size.

In the subsequent section, the governing equations for generating the undisturbed flow and simulating perturbation evolution are described in the context of a rotating-cone in a still fluid. Results are then presented that compare the differences in global behaviour as the azimuthal mode number is increased to larger values.

Formulation

Basic State

A rigid cone with a half-angle ψ and cross-sectional radius $r^* = x^* \sin\psi$, rotates with a constant angular frequency Λ^* in an otherwise still fluid. In orthogonal curvilinear coordinates $\{x^*, \theta, z^*\}$, the streamwise and wall-normal directions are assumed to be semi-infinite; $0 \le x^* < \infty$ and $0 \le z^* < \infty$. Further, as the cone rotates within a fixed frame of reference, the Coriolis force terms are omitted from all governing equations. The non-dimensional representation of the undisturbed flow is defined as

$$\mathbf{U_B}(x,z,\psi) = \left(\frac{x\sin\psi}{Re}F(z), \frac{x\sin\psi}{Re}G(z), \frac{1}{Re}H(z)\right), \qquad (1)$$

where the Reynolds number

$$Re = \frac{x_o^*\Lambda^*\delta^*\sin\psi}{\nu^*} = \frac{x_o^*\sin\psi}{\delta^*} = x_o\sin\psi, \qquad (2)$$

for some reference streamwise position x_o. The wall-normal functions F, G, H are obtained by solving the following system of differential equations

$$(F^2 - G^2)\sin\psi + F'H = F'', \qquad (3a)$$

$$2FG\sin\psi + G'H = G'', \qquad (3b)$$

$$2F\sin\psi + H' = 0, \qquad (3c)$$

where a prime denotes differentiation with respect to z. This particular system of equations is similar to that derived by von Kármán [1] for the rotating-disc boundary layer, in the instance that $\psi = 90^o$. Equations (3a-3c) are solved subject to the boundary conditions on the cone surface

$$F = G - 1 = H = 0 \quad \text{on} \quad z = 0 \qquad (3d)$$

and the freestream conditions

$$F \to 0, \quad G \to 0 \quad \text{as} \quad z \to \infty. \qquad (3e)$$

Perturbation Equations

Total velocity and vorticity fields are respectively defined as

$$\mathbf{U} = \mathbf{U_B} + \mathbf{u}, \quad \Omega = \Omega_B + \omega,$$

for $\mathbf{U_B}$ given by (1), $\Omega_B = \nabla \times \mathbf{U_B}$ and linearised perturbations of the form

$$\{\mathbf{u}, \omega\} = \{\hat{\mathbf{u}}, \hat{\omega}\}e^{in\theta}, \qquad (4)$$

for an integer valued azimuthal mode numbers n. Disturbance development is then numerically simulated using a velocity-vorticity formulation that is fully equivalent to the linearised Navier-Stokes equations [12, 18]. The system of governing equations comprises the streamwise and azimuthal components of the vorticity transport equation in curvilinear coordinates and the matching wall-normal component of the Poisson equation. Perturbations are then impulsively excited through a linearisation of the no-slip boundary condition, which establishes disturbances that take the form of wavepackets that initially comprise a wide range of frequencies.

Disturbance Development

The change in global behaviour as the azimuthal mode number is raised to larger values is illustrated by numerically simulating disturbance evolution in a rotating-cone boundary layer with a half-angle $\psi = 40^o$. Two perturbations are generated using an impulse centred about $x_f = 675$ ($Re = 434$), for azimuthal mode numbers $n = 40$ and 60. These conditions correspond to disturbances that are respectively absolutely unstable ($x_a \approx 622$ for $n = 40$) and marginally absolutely unstable ($x_f \approx 668$ for $n = 60$) when the parallel flow approximation is enforced.

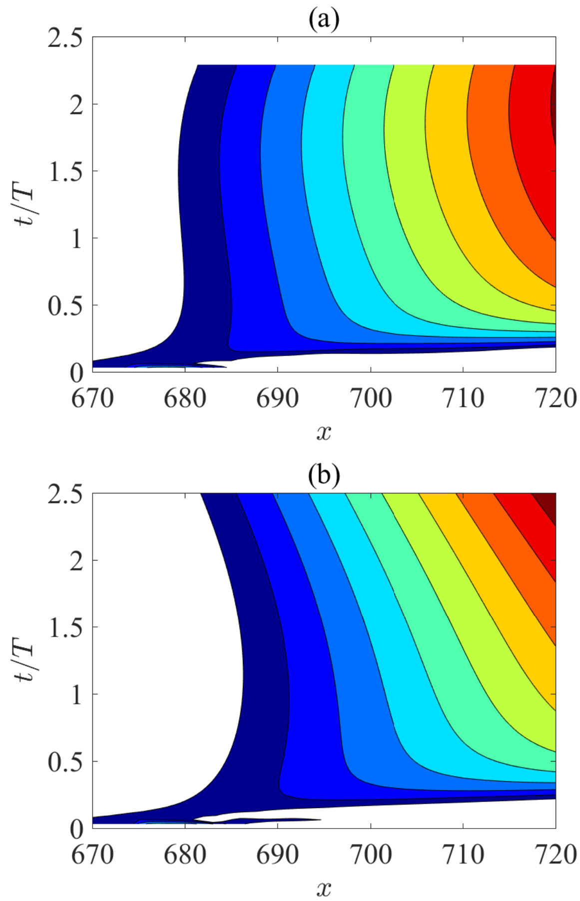

Figure 1: Spatial-temporal disturbance development on a rotating-cone with a half-angle $\psi = 40^o$, for an impulse centred about $x_f = 675$ ($Re = 434$). (a) Azimuthal mode number $n = 40$; (b) $n = 60$.

Figure 1 displays the development of the corresponding disturbances to the genuine non-parallel flow as wavepackets in the spatial-temporal plane. Contours of the azimuthal vorticity at the wall have been drawn using a natural logarithm scaling and normalised about time $t/T = 0.2$, where $T = 2\pi Re$. The leading edges of the two wavepackets convect downstream (to the right) with approximately the same positive non-zero velocity. However, the trailing edges display very contrasting features. The trailing edge associated with $n = 40$ convects upstream (to the left) for a very small period of time, but eventually reverses direction and propagates downstream. Meanwhile, for $n = 60$, the trailing-edge propagates upstream for the entire time period shown, with what appears to be an increasing velocity. Thus, a change in the global response is realised as the azimuthal mode number is increased; global instability develops for the larger azimuthal mode number n, while convective processes and global stability prevails for the lower valued n.

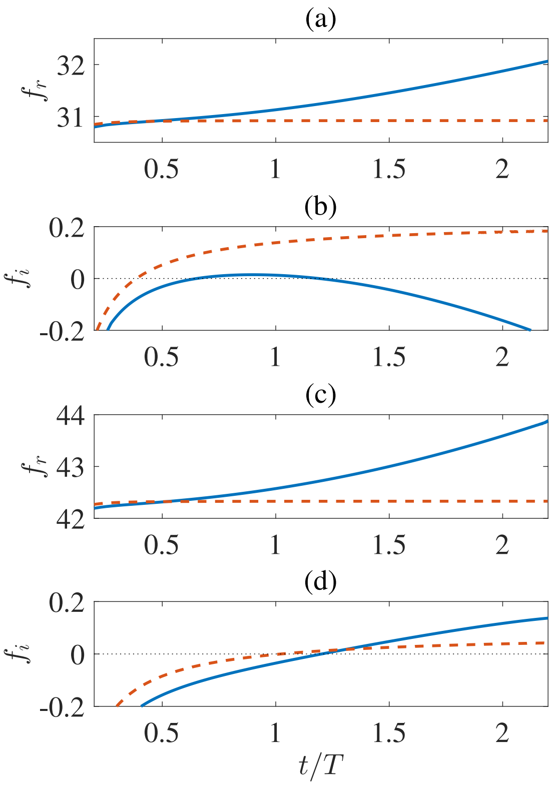

Figure 2: Temporal frequencies f_r and growth rates f_i as functions of time for those disturbances plotted in figure 1. (a,b) $n = 40$; (c,d) $n = 60$. The temporal development is plotted about the impulse centre for calculations based on the non-parallel (solid) and parallel flows (dashed).

Temporal frequencies f_r and growth rates f_i matching to those disturbances illustrated in figure 1 are depicted in figure 2. Solutions were obtained via the complex-valued quantity

$$f = \frac{iRe}{A}\frac{\partial A}{\partial t}, \tag{5}$$

where A is some measure of the disturbance evolution. Results are plotted about the impulse centre $x_f = 675$ for solutions based on the parallel (dashed lines) and non-parallel flows (solid lines). The two sets of frequencies and growth rates corresponding to the non-parallel flow are found to vary with time, with little to suggest that a fixed global mode will be achieved at very large time. Moreover, temporal growth rates (figures 2(b,d)) further emphasize that there is a change in the global behaviour as the azimuthal mode number is raised to larger values. For $n = 40$, the growth rate f_i is only positive over the time period $0.5 \leq t/T \leq 1$, and for larger time a negative and decreasing growth rate is observed. This is despite the fact that this particular mode number is strongly locally absolutely unstable for the impulse centred about $x_f = 675$ ($f_i \approx 0.2$ for the parallel flow). For the larger azimuthal mode number $n = 60$, the growth rate increases for the entire time duration shown, achieving a positive value shortly after the first period of rotation. Thus, a form of global instability is generated that is characterised by a faster than exponential temporal growth. Hence, non-parallelism brings about a stabilising effect for lower valued n and a destabilising effect for sufficiently large n.

Figure 3 plots neutral stability curves for local absolute instability in the $\{Re, n\}$-plane. Neutral curves are plotted for cone half-angles $\psi = 20^o$ through to $\psi = 90^o$ at ten unit intervals. Additionally, circular markers indicate the critical azimuthal mode numbers n_c for the emergence of global instability; for smaller valued n globally stable behaviour prevails, while for larger valued n disturbances are expected to display globally unstable characteristics. These calculations were obtained by Thomas and Davies [17, 18] following an extensive numerical study and by utilising solutions of the linearised Ginzburg-Landau equation [19]. In all instances considered, the size of n_c suggested as being necessary for global linear instability to occur is greater than the conditions for critical absolute instability. However, it is worth noting that the high n modes only appear about very large streamwise positions (Reynolds numbers) that are typically greater than the experimental observations for the onset of laminar-turbulent transition. Thus, it is expected that the transition process in the rotating-disc and -cone boundary layers is still governed by the lower valued n convective instabilities [17].

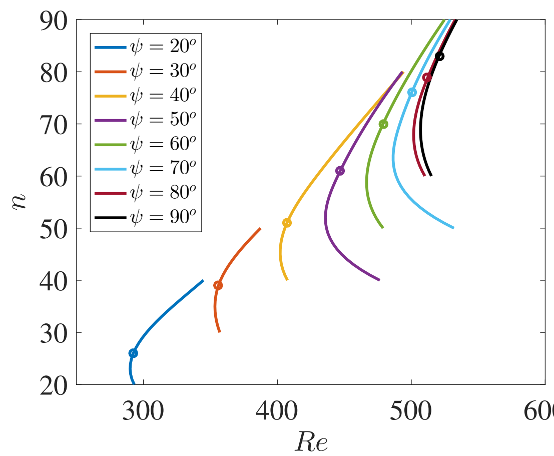

Figure 3: Neutral stability curves for local absolute instability, for cone half angles $\psi \in [20^0 : 10^o : 90^o]$ [7, 10]. Circular markers indicate the azimuthal mode number required to establish globally unstable behaviour as predicted by Thomas and Davies [17, 18].

Concluding Remarks

The local-global stability of rotating boundary layers has been reviewed. Recent studies by Thomas and Davies [17, 18] have shown that linear disturbances can become globally unstable, but only when the azimuthal mode number is taken to be significantly greater than the conditions for the onset of local absolute instability.

Furthermore, we anticipate that similar characteristics will be found in many other rotating boundary layers. Including, but not limited to, Ekman layers, B odewadt layers and rotating-spheres.

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