Multiple-location matched approximation for Bessel function J_0 and its derivatives

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Abstract

I present an approximation of Bessel function $J_0(r)$ of the first kind for small arguments near the origin. The approximation comprises a simple cosine function that is matched with $J_0(r)$ at $r = \pi/\mathrm{e}$. A second matching is then carried out with the standard, but slightly modified, far-field approximation for $J_0(r)$, such that zeroth, first and second derivatives are also considered. Finally, a third matching is made with the standard far-field approximation of J_0 but at multiple locations, to guarantee matching all concerned derivatives. The proposed approximation is practical when nonlinear dynamics come into play, in particular in the case of nonlinear interactions that involve higher order differential equations.

1. INTRODUCTION

The boundary-value problem of the generation of gravity and/or acoustic waves in fluids due to a localised disturbance at the surface is associated with the three dimensional wave equation in cylindrical coordinates. Under standard conditions, the solution is given by Bessel functions which may become cumbersome for many problems, in particular when nonlinearity comes into play, such as in acoustic–gravity wave triad resonance [1–5]. Here we are concerned with an approximated solution valid within a distance of a few wavelengths from the disturbance origin, after which the waves are effectively damped or dissipated. Yet, for brevity, we do not consider damping or dissipation which can be easily added or treated numerically once a closed form solution is structured. Fast and accurate Bessel function computations were presented in the literature, e.g. Ref. [6], though the nature of terms collected in various nonlinear problems, e.g. as in nonlinear acoustic–gravity wave theory, require a much more simplified approach. Thus, we present a simple cosine approximation for the near-field that is matched with $J(r)$ at $r = \pi/\mathrm{e}$. The near-field approximation is matched with a modified far-field approximation at various locations, in r, that take into account the zeroth, first, and second derivatives. This allows no singularities at the origin with incremental deviation from the exact solution in the near-field, and only small errors in the far-field. Additional approximations can be made as required, to insure smoothness as $r \to \infty$.

2. BACKGROUND

When studying weakly nonlinear interactions for radial symmetric waves, we are often concerned with the differential equation

$$\mathbb{R}_{rr} + \frac{1}{r}\mathbb{R}_r + \mathbb{R} = 0. \tag{.1}$$

The solution of (.1) is well known and it is given by Ref. [7]:

$$\mathbb{R} = C_1 J_0(r), \tag{.2}$$

where J_0 is a Bessel function of the first kind, and C_1 is a coefficient that can be found from the space-time boundary conditions, though it would be omitted here for simplicity.

Specifically, J_0 takes the form

$$J_0(r) = \sum_{n=0}^{\infty} \frac{(-r^2/4)^n}{(n!)^2}. \quad (.3)$$

In the far field J_0 oscillates and behaves like a damped cosine function, namely,

$$J_0(r) \approx \sqrt{\frac{2}{\pi r}} \cos\left(r - \frac{\pi}{4}\right). \quad (.4)$$

The solution of (.1) is accurately described by (.3) and (.4), though studying resonance via nonlinear wave interaction could become a real challenge. In particular, when the interaction involves quadratic and cubic terms that comprise $\mathbb{R}_r$ and $\mathbb{R}_{rr}$, as in the problem of triad resonance interaction of acoustic-gravity waves (e.g. see eqs. (2.2) and (2.3a) of Ref. [5]). In such cases, an approximation is required to allow collection of secular terms associated with resonance. To this end, we propose the approximation described in the following section.

3. MATCHED APPROXIMATION

The matched approximation comprises three steps. The **first step** of the matching is approximating the near-field using a simple cosine function of the form,

$$J_{0,near}(r) = \cos(\mathcal{M}r), \quad (.5)$$

where $\mathcal{M}$ is a parameter calculated by matching the approximated near-field $J_{0,near}$ with the exact Bessel function J_0 at $r = \pi/\mathrm{e}$. Thus, $\mathcal{M} = (\mathrm{e}/\pi)\cos^{-1} J_0(\pi/\mathrm{e}) = 0.6967398$.

In the **second step** we match the near-field with a modified far-field approximation of the form,

$$J_{0,far}(r) = \sqrt{\frac{2}{\pi b_1 r}} \cos(b_2 r - b_3) \quad (.6)$$

where b_1, b_2 and b_3 are matching parameters. Obviously, when $b_1 = 1$, $b_2 = 1$ and $b_3 = \pi/4$ the standard far-field approximation is met. However, these would result in a discontinuity with the near-field solution. Thus, instead of using the standard approximation we introduce a three-point matching, at $r = r_{1j}$ $(j = 0, 1, 2)$, such that the zeroth, first and second derivates, presented by subscript j, are all matched. Note that subscript 1 indicates that this is the first three-point matching being carried out; a second three-point matching (at $r = r_{2j}$) will be required as discussed in the following section. Thus, combining eqs.(.5) and (.6), we require that

$$-\cos(\mathcal{M}r) + \sqrt{\frac{2}{\pi b_1 r}} \cos(b_2 r - b_3) = 0, \qquad (r = r_{10}) \quad (.7)$$

for the zeroth derivative. Similarly, for the first and second derivatives we obtain

$$\mathcal{M}\sin(\mathcal{M}r) - b_2\sqrt{\frac{2}{\pi b_1 r}}\sin(b_2 r - b_3) - \frac{1}{2}\sqrt{\frac{2}{\pi b_1 r^3}}\cos(b_2 r - b_3) = 0, \qquad (r = r_{11}), \tag{.8}$$

$$\mathcal{M}^2\cos(\mathcal{M}r) - \left(\frac{b_2^2}{\sqrt{r}} - \frac{3}{4\sqrt{r^5}}\right)\sqrt{\frac{2}{\pi b_1}}\cos(b_2 r - b_3) = 0, \qquad (r = r_{12}). \tag{.9}$$

Thus, the complete matched approximation can be compactly described by

$$\frac{\partial^j \mathbb{R}(r)}{\partial r^j} = \begin{cases} \cos(\mathcal{M}r) & (0 \le r \le r_{1j}) \\ \sqrt{\frac{2}{\pi b_1 r}}\cos\left(b_2 r - b_3\right) & (r_{1j} \le r \le r_{2j}) \\ \sqrt{\frac{2}{\pi r}}\cos\left(r - \frac{\pi}{4}\right) & (r \ge r_{2j}) \end{cases} \tag{.10}$$

The choice of r_{10}, r_{11}, r_{12} is made where the near-field solution is in the neighbourhood of the exact solution of J_0, though not necessarily identical, e.g. $r_{10} = 0.698132$, $r_{11} = 0.926421$, $r_{12} = 3.47390$. With these numerical values, we have a set of three equations with three unknowns, that give a solution $b_1 = 1.09713$, $b_2 = 1.03946$, $b_3 = 0.972672$. Note that any $b_3 = 0.972672 \pm 2n\pi$ $(n = 0, 1, 2, ...)$ is also a solution. Moreover, other choices of the matching locations r_{1j} would result in a new set of b_j, and in that sense the solution is not unique. The choice of the matching location is described below, along with the third step of the approximation.

4. NUMERICAL VALIDATION

The standard far-field approximation with $b_1 = 1$, $b_2 = 1$, and $b_3 = \pi/4$ cannot match the near-field, and at least one of the matching parameters should be made free to allow the matching. Here, we solve for three different cases (see figure 1): (I) $b_3 = \pi/4$ and $b_1 = b_2 = b$ which gives $b = 0.99837$; (II) $b_1 = 1$, $b_3 = \pi/4$, and b_2 is unknown resulting in $b_2 = 0.99654$; and (III) $b_1 = 1$, $b_3 = \pi/4$, and b_2 is unknown but with a slightly different cosine argument, i.e. $\cos[b_2(r - \pi/4)]$, which gives $b_2 = 0.989228$. It is easy to see that the proposed approximation is exact at the origin and at the matching, $r = \pi/\mathrm{e}$, whereas (.1) is satisfied with an error negligible in the near-field and below %5 for all cases illustrated in figure 1. The matching at π/e allows almost an exact approximation of the near-field and a smooth transition to the far-field as illustrated in figure 1. However, this result can only be useful if the derivatives are not concerned, since they cannot be smooth. To obtain a proper approximation for the derivatives as well, we employ the multiple-location matching described above. Considering r_{10}, r_{11}, r_{12}, b_1, b_2, and b_3 presented in section 3 we are able to design a solution whose first and second derivatives are also matched, see figure 2.

Since matching the far-field *deforms* the standard far-field solution of $\mathbb{R}$, the solution diverges as $r \to \infty$, and a clear discrepancy is noticed after a few periods only. To overcome this difficulty, we impose a **third step** with a second three-location far-field matching at r_{20}, r_{21}, r_{22}, where the standard far-field parameters, $b_1 = b_2 = 1$, $b_3 = \pi/4$, and $\mathbb{R}$ and its derivatives are matched exactly. Here, r_{2j} $(j = 0, 1, 2)$ are chosen points where $J_{0,far}$ with r_{1j} match (or tangent to) the exact solution. This guarantees that the second far-field approximation of $\mathbb{R}$, $\mathbb{R}_r$, and $\mathbb{R}_{rr}$, for large r, tend to the exact solution, as illustrated in figure 3.

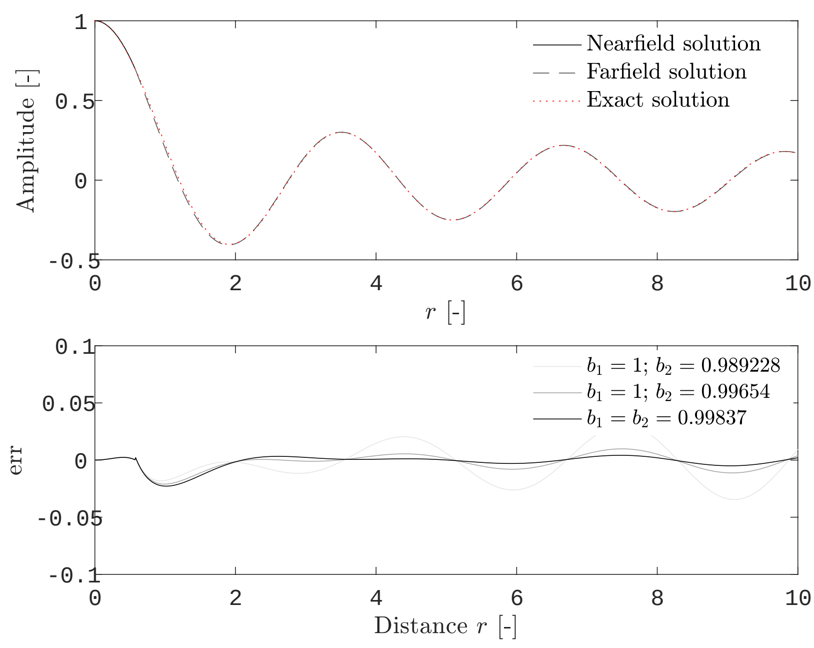

FIG. 1. Top: Exact (dotted), near-field (solid) and far-field (dashed) solutions of $\mathbb{R}$. Bottom: error in the approximate solution normalised by J_0, with a standard far-field approximation ($b_3 = \pi/4$) but with a slightly different values of the matching parameters b_1 and b_2 to allow exact matching.

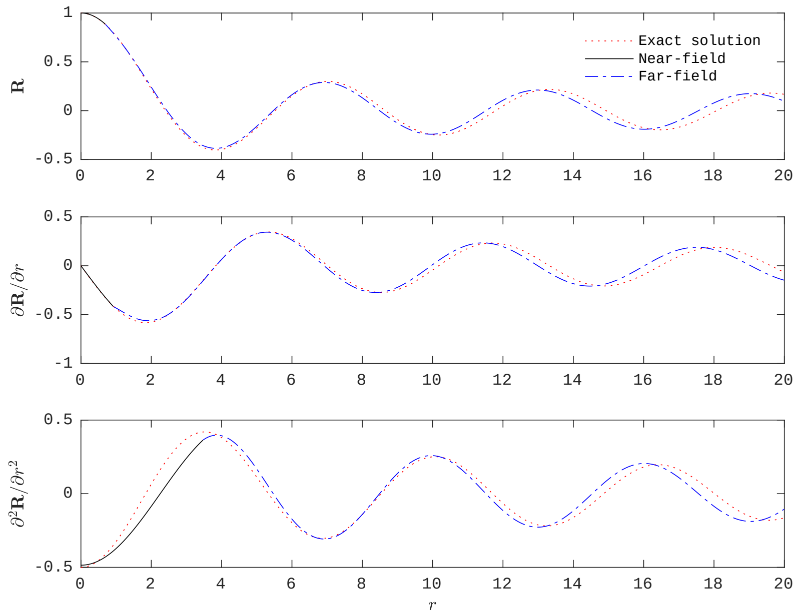

FIG. 2. Exact (dotted), near-field (solid) and far-field (dashdotted) solutions of $\mathbb{R}$ and its first two derivatives. Here, $r_{10} = 0.69813$, $r_{11} = 0.92642$, $r_{12} = 3.4739$.

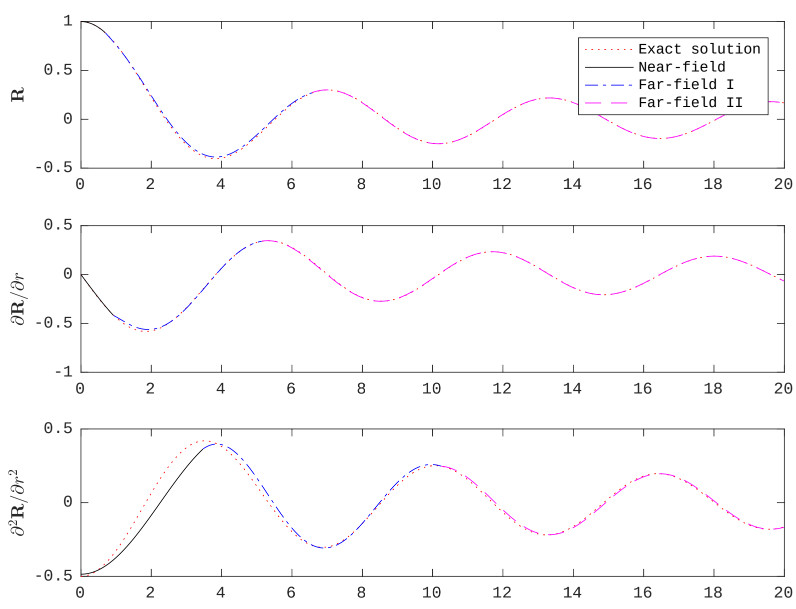

FIG. 3. Exact (dotted), near-field (solid), first far-field (dashdotted), and second far-field (dashed) solutions of $\mathbb{R}$ and its first two derivatives. Here, $r_{20} = 6.5845$, $r_{21} = 5.1403$, $r_{22} = 10.241$.

DISCUSSION

A simple near-field approximation for Bessel function of the first kind, J_0, is presented. The approximation comprises a cosine function with an argument that is matched with $J_0(r)$ at exactly $r = \pi/\mathrm{e}$ resulting in a highly accurate approximation. This leads to two main straight forward questions: what makes a cosine a good approximation function? and what is special about π/e? The answer to the first question is rather straight forward, as expanding $\cos(\mathcal{M}r)$ into a series one can conclude that for small arguments the leading terms are very similar to their counterpart in the Bessel function of the first kind. The second question remains open, though a numerical coincidence is not an excluded possibility.

For the far-field, a standard far-field approximation is employed. However, this cannot be smooth with near-field approximation solution, or its derivatives. Therefore, an *intermediate zone*, referred to as *far-field I*, is introduced, whose standard approximation is modified with three more parameters, that are matched with the Bessel corresponding derivatives, from one hand, and three different locations that allow matching with the near-field, on the other hand. Since, such modification creates some discrepancy in the actual far-field solution, referred to as *far-field II*, a second matching is allowed with the standard far-field, at three proper locations in space.

The developed matched approximation was originally motivated by the problem of non-linear interaction of acoustic–gravity wave triad in cylindrical coordinates. For resonance to be obeyed, the wave numbers k_m and frequencies ω_m ($m = 1, 2, 3$), should satisfy the conditions $k_1 + k_2 = k_3$, and frequencies $\omega_1 + \omega_2 = \omega_3$, among others. Thus, by rewriting (.10) in complex exponential form, with terms $\propto \exp[\mathrm{i}(k_m r - \omega_m t)]$, collecting resonance terms becomes rather straight forward avoiding a cumbersome work that involves Bessel function derivatives and their quadratic and cubic forms.

Obviously, the matched approximation results in uncertainties in the amplitude evolution equations. If the physical problem at hand concerns a standard quality (Q) factor resonance then small deviations about the peak energy (exact resonance) act as a tuning parameter, which will cause small differences in the amplitude, e.g. in the acoustic–gravity wave triad problem a 2.5% error leads to 0.5% difference in the maximum amplitude. On the other hand, if the physical problem involves a Q factor that is extremely high, then the maximum amplitude might become sensitive to slight tunings, and thus the proposed approximation might introduce large uncertainties. These might be minimised if the matching locations are chosen carefully.

Last but not least, the proposed approximation deals with first two derivatives only, though in principle one can introduce a Polynomial, in b_{ij}, of enough orders to match as many derivatives as required. Moreover, the work can be extended for other kinds of Bessel functions, towards a general approximation that can be valid for a wider range of physical problems.

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