Adhesive contact problems for a thin elastic layer: Asymptotic analysis and the JKR theory

By Feodor M. Borodich[a,1], Boris A. Galanov[b], Nikolay V. Perepelkin[a,d], and Danila A. Prikazchikov[c]

[a] *School of Engineering, Cardiff University, Cardiff, CF24 3AA, UK*

[b] *Institute for Problems in Materials Science, National Academy of Sciences of Ukraine, 3 Krzhyzhanovsky St., Kiev, 03142, Ukraine*

[c] *School of Computing and Mathematics, Keele University, Staffordshire, ST5 5BG, UK*

[d] *Department of Applied Mathematics, National Technical University "Kharkiv Polytechnic Institute", 2 Kyrpychova Str., Kharkiv, 61002, Ukraine*

Abstract

Contact problems for a thin compressible elastic layer attached to a rigid support are studied. Assuming that the thickness of the layer is much less than characteristic dimension of the contact area, a direct derivation of asymptotic relations for displacements and stress is presented. The proposed approach is compared with other published approaches. The cases are established when the leading order approximation to the non-adhesive contact problems is equivalent to contact problem for a Winkler-Fuss elastic foundation. For this elastic foundation, the axisymmetric adhesive contact is studied in the framework of the JKR (Johnson, Kendall, and Roberts) theory. The JKR approach has been generalized to the case of the punch shape being described by an arbitrary blunt axisymmetric indenter. Connections of the results obtained to problems of nanoindentation in the case of the indenter shape near the tip has some deviation from its nominal shape are discussed. For indenters whose shape is described by power-law functions, the explicit expressions are derived for the values of the pull-off force and for the corresponding critical contact radius.

Keywords: thin elastic layer, asymptotics, JKR theory, adhesive contact, Winkler-Fuss foundation

1 Introduction

The original Hertz contact theory studied the problem of contact between two infinite isotropic elastic solids whose shapes are represented by two elliptic paraboloids. The contact problem was studied in geometrically linear formulation as a mixed boundary value problem for an elastic isotropic half-space loaded by a rigid elliptic paraboloid. However, the contact problems for layered or coated solids are also very important for many practical applications. These problems were studied in a number of papers using various assumptions that are additional to the assumptions of the Hertz formulation. Here some popular approximate and asymptotic approaches to the problem for an elastic layer are examined (see literature reviews in [1–3]). It is noted that some mathematically correct asymptotic approaches are rather sophisticated. Currently one of the best asymptotic approachs to contact problems for layered solids is the direct approach that has been recently developed by Argatov and Mishuris (the AM approach) [3]. Here we show that the method of direct asymptotic integration or the GKN method [4], that was successfuly applied in problems of theory of shells and plates, can be also used in applications to layered solids. Using the GKN approach, a simple and clear asymptotic

[1] Corresponding author. Tel.: +44 29 2087 5909; fax: +44 29 2087 4716. E-mail address: BorodichFM@cardiff.ac.uk

solution to the problem is derived in the case when the size of contact region is much greater than the thickness of the elastic compressible layer. The expressions for displacements and stresses acting within the layer are presented. We demonstrate that although the AM approach has clear similarities with the GKN approach, the latter has some advantages.

Using the GKN method, a simple and clear asymptotic solution to the contact problem is derived in the case when the size of contact region is much greater than the thickness of the elastic compressible layer. The expressions for displacements and stresses acting within the layer are presented. It is shown that if the layer is isotropic or transversely isotropic then the leading term asymptotic approximation may be considered as a layer of springs (the Winkler-Fuss elastic foundation or the Fuss-Winkler-Zimmermann foundation). We compare our approach based on the GKN method with other approaches, including approaches used by Argatov and his co-authors [3, 5, 6].

Usually contact problems are considered without taking into account adhesive forces. Indeed, adhesion between contacting bodies has usually a negligible effect on surface interactions at the macro-scale, whereas it becomes increasingly significant as the contact size decreases. Because modern materials science deals often with very thin coatings and nano-world is the 'Sticky Universe' [7], we consider here the main ideas of the mechanics of adhesive contact between two surfaces that are attracted to each other by intermolecular forces, i.e. the solids can interact with each other even if the external load is not applied. Here we need to use the notion of the specific work of adhesion (w) that is equal to the energy needed to separate two dissimilar surfaces from contact to infinity. This is the crucial material parameter for application of theories of adhesive contact.

The modern theory of adhesive contact for elastic solids may be traced back to the pioneering papers by Derjaguin [8] and Johnson [9] (see also a discussion by Kendall [7]). Sperling [10] derived formulae for the force – displacement diagram for a sticky sphere using Derjaguin's results and a rather sophisticated solution of the contact problems for a sphere obtained by Jung [11]. Later Johnson et al. [12] presented independently the JKR (Johnson, Kendall and Roberts) approach to the same problem. The JKR approach is mathematically more elegant than the Sperling one and the former is applied to the Winkler-Fuss foundation in the present paper.

The classical JKR or JKRS (Johnson-Kendall-Roberts-Sperling) theory of adhesive contact proposes methodologies to predict the adhesion force between spherical indenters (paraboloids of revolution) [10, 12]. Recently it has been shown that the theory can be extended to arbitrary blunt axisymmetric indenters and to materials having rotational symmetry like transversely isotropic or homogeneously prestressed materials [13–15]. Similar approach is also valid for probing of stretched two-dimensional (2D) membranes [16]. Here we use the leading order approximation for an elastic isotropic or transversely isotropic layer considered as the Winkler-Fuss elastic foundation to solve the adhesive contact problem for a rigid axisymmetric blunt indenter of arbitrary shape. The problem is studied employing the main assumptions of the JKR theory of adhesive contact. The governing equations of the JKR theory are formulated and transformed using the slope of the non-adhesive force-displacement curve in general form, which simplifies transformations and makes them clearer. Because the shapes of non-ideal shaped indenters may be well approximated as monomial functions of radius [17], the particular case of power-law shaped probes is studied in detail.

2 Preliminaries: Problem formulation and state-of-the-art results

The classic formulation of the Hertz-type contact problems was independently introduced by Hertz (1882) and Boussinesq (1895) (see references in [15]). This formulation assumes that the shape of the bodies and the compressing force P are given and molecular adhesion can be ignored. Hence, the fields of displacements and stresses appear in the solids only after the external load is applied. In addition, it is assumed that the contact region is small in comparison with the main radii of curvature of contacting solids and, therefore the boundary value problem for contacting solids may be formulated as a boundary value problem for an isotropic elastic half-space. Here we consider action of a smooth, convex rigid indenter on an elastic layer.

2.1 Formulation of non-adhesive contact problem

Let us use both the Cartesian and cylindrical coordinate frames, namely $x_1 = x, x_2 = y, x_3 = z$ and r, θ, z, where $r = \sqrt{x^2 + y^2}$ and $x = r\cos\theta, y = r\sin\theta$. Let us place the origin (O) of Cartesian coordinates at the point of initial contact between the indenter and the layer. Let us direct the axis of x_3 along the normal to the layer towards the inside of the layer (see Fig. 1).

Usually the Hertz-type contact problems assume that a rigid indenter, for which the equation of the surface is given by a function f, i.e., $x_3 = -f(x_1, x_2)$, $\quad f \geqslant 0$, is pressed by the force P to a boundary of the contacting solid. After the indenter contacts with the layer, the displacements u_i and stresses σ_{ij} are generated. It is supposed that the shape of the indenter f and the external parameter of the problem $\mathcal{P}$ are given and one has to find the bounded region G on the boundary plane $x_3 = 0$ of the layer at the points where the punch and the layer are in mutual contact, the displacements u_i, and the stresses σ_{ij}. If the pressing force P is taken as the external parameter $\mathcal{P}$ then one has to find the depth of indentation δ (the relative approach between the indenter and the boundary) and the the contact region G. If δ is taken as $\mathcal{P}$ then P and G are the sought values.

The formulation of the original Hertz contact problem, i.e. the contact between two blunt elastic solids, contains several assumptions (see e.g., [15,18]). These assumptions are accepted here, though we do not assume that the contact region is an ellipse. In general case of Hertz type contact problems, the boundary value problem may be formulated for a positive half-space. However, here we consider an elastic isotropic layer of thickness h occupying the area $0 \leq x_3 \leq h$, bounded to a rigid half-space $x_3 \geq h$ assuming that the thickness of the layer h is small compared to the characteristic size a of the contact region (see Fig. 1). Evidently, for an axisymmetric problem a is the radius of the contact region. The indenter is blunt, hence the contact problem can be considered in a geometrically linear formulation.

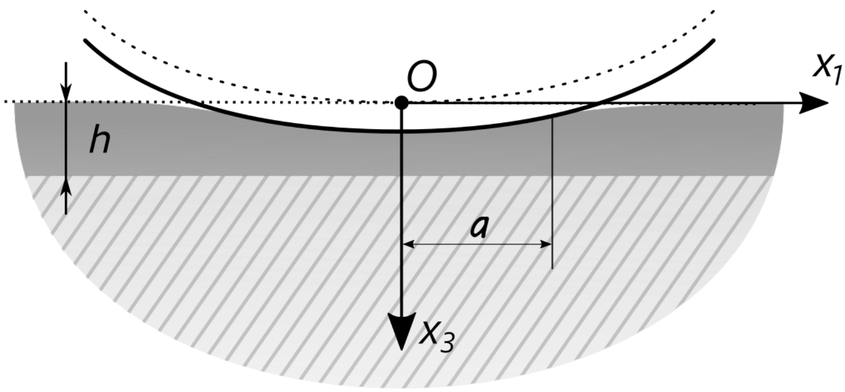

Figure 1: Problem formulation

Let us denote as a comma differentiation with respect to the associated spatial coordinate x_j, $j = 1, 2, 3$. The sought quantities must satisfy the following equations.

The equations of equilibrium

$$\sigma_{i1,1} + \sigma_{i2,2} + \sigma_{i3,3} = 0. \tag{1}$$

The constitutive relations for the linear elastic materials are represented by the Hooke's law. For an isotropic solids, the equations can be written as

$$\sigma_{ij} = \lambda\,\delta_{ij}\,(u_{1,1} + u_{2,2} + u_{3,3}) + \mu\,(u_{i,j} + u_{j,i})\,, \quad \lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)}. \tag{2}$$

Here λ and μ are the Lamé constants, E and ν are the Young modulus and Poisson's ratio, respectively, u_i are components of the displacement field, and δ_{ij} is the Kronecker delta.

The boundary conditions should describe the problem within and out the contact region G, and conditions describing interactions between the layer and the support. Because the problem is frictionless, we have

$$\sigma_{3i}(x_1, x_2, 0) = 0, \quad (x_1, x_2) \in \mathbb{R}^2 \quad i = 1, 2. \tag{3}$$

The contact region G is defined as an open region such that if $\mathbf{x} \in G$ then the gap $(u_3 - g)$ between the punch and the half-space is equal to zero and surface stresses are non-positive, while outside the contact region, i.e. for $\mathbf{x} \in \mathbb{R}^2 \setminus G$, the gap is positive and the stresses are equal to zero. Hence, the boundary conditions that describe the action of an indenter on the layer within the contact region G and outside the region $(\mathbb{R}^2 \setminus G)$ can be written as

$$\begin{array}{lll} u_3(x_1, x_2, 0) = g(x_1, x_2), & \sigma_{33}(x_1, x_2, 0) \leq 0, & (x_1, x_2) \in G, \\ u_3(x_1, x_2, 0) > g(x_1, x_2), & \sigma_{3i}(x_1, x_2, 0) = 0, & (x_1, x_2) \in \mathbb{R}^2 \setminus G. \end{array} \tag{4}$$

For the general case of the problem of vertical frictionless pressing, we have

$$g(\mathbf{x}) = \delta - f(x_1, x_2) = \delta - \varphi\left(\frac{x_1}{a}, \frac{x_2}{a}\right), \quad f(x_1, x_2) = \varphi\left(\frac{x_1}{a}, \frac{x_2}{a}\right). \tag{5}$$

If the layer is bonded to the support then the interactions between the layer and the support can be written as

$$u_1(x_1, x_2, h) = u_2(x_1, x_2, h) = u_3(x_1, x_2, h) = 0. \tag{6}$$

In addition, there is the integral condition

$$\int_{R^2} \sigma_{33}(x_1, x_2, 0) dx_1 dx_2 = -P. \tag{7}$$

The above formulation is also valid for transversely isotropic linear or linearized materials having rotational isotropy of their mechanical properties. For such materials, one should change the form of the operator of constitutive relations, i.e. to change (2).

2.2 Approximate and asymptotic solutions for non-adhesive problems

The problems of contact between a rigid punch and an elastic layer attracted attentions of many authors. These approaches to these problems with and without adhesion may be quite roughly divided in two groups: (i) approximate solutions and (ii) asymptotic solutions to the contact problems. Here we present a brief review of main papers in the groups. However, it is difficult to split the papers devoted to elastic foundation models [19, 20] and the papers devoted to an elastic layer. Indeed, it has been known for a long time that the simplest approximation of the contact problem for a punch and a thin elastic layer (thickness is much less than characteristic dimension of the contact area) can be reduced to a contact problem between the punch and a Winkler-Fuss elastic foundation with stiffness coefficient K. Many authors repeatedly obtained this result using different approaches (see, e.g. [1–3] and references therein).

The Winkler-Fuss elastic foundation can be imagined as a spring layer of thickness h, which rests on a rigid base $x_3 \geqslant h$. There is no interaction between the springs, i.e., shear between adjacent springs is ignored. Hence, $u_1 = u_2 = 0$ at any point of the foundation, i.e. the displacement vector $\mathbf{u} \equiv u_3$. The stresses are $\sigma_{ij} = \delta_{i3}\delta_{3j}\sigma_{ij}$. The contact pressure $p(x_1, x_2) = -\sigma_{33}(x_1, x_2, 0)$ and the corresponding component of the strain tensor ϵ_{33} at any point depend only on the displacement at the point, hence they can be calculated as

$$\begin{aligned} \epsilon_{33}(x_1, x_2, 0) &= -u_3(x_1, x_2, 0)/h; \\ \sigma_{33}(x_1, x_2, 0) &= -K u_3(x_1, x_2, 0), \end{aligned} \tag{8}$$

where K is the elastic modulus of the foundation.

Because the depth of indentation δ is taken as the external parameter of the contact problem, the load P and the contact region G are unknown. It follows from (8) and (7) that the compressing force is

$$P(\delta) = KV(\delta), \tag{9}$$

where $V(\delta)$ is the volume of the body under the cross–section of height $z = \delta$. The contact problem for Winkler foundation is discussed further in the present paper.

One of the first to introduce this structural model were Winkler [21] and Zimmermann [22]. The model is also known as the spring bed mattress elastic foundation [18]. The foundation was considered in a number of papers and books (see, e.g., [18, 19]) because the simplicity of this model makes it very helpful for modelling various engineering problems.

The foundation was used to model contact problems for rough surfaces; it was shown that if the surface roughness is represented by a linear or non-linear Winkler-Fuss foundation and the foundation is attached to a linear elastic half-space then the problem of contact between a rigid indenter and the rough solid may be reduced to Hammerstein-type integral equations that in turn can be solved numerically [23, 24]. The foundation was also used to estimate validness of some fractal ideas using three different models of fractal surfaces: the Cantor-Borodich profile [25], the hierarchical multilevel Borodich-Onishchenko profile [26] and indenters whose profiles described by parametric-homogeneous (PH) function [27]. The PH functions were introduced by Borodich and they are discussed in detail in several papers [28, 29]. Recently the foundation was employed to model adhesive contact between non-smooth surfaces whose roughness statistics is described by a Gaussian distribution [30]. It will be discussed later that the contact problem for an isotropic elastic coating bonded to a substrate in the leading term of asymptotic expansion reduces to a problem of contact for a Winkler-Fuss spring layer whose stiffness is (see, e.g. [1, 31, 32])

$$K = \frac{E\,(1-\nu)}{h\,(1+\nu)\,(1-2\nu)} \tag{10}$$

where E and ν denote the elastic modulus and Poisson's ratio of the layer respectively.

In his famous book Johnson [18] examined the indentation by a rigid frictionless cylinder (the plane-strain conditions of line contact) of an elastic layer which is supported on a rigid plane surface in both bonded and unbonded formulations. He mentioned both the integral form of a solution to the problem and some asymptotic approaches to the problem, e.g. [33]. Then he noted that in the limit when $\varepsilon = h/a < 1$, the state of affairs can be analysed in an elementary way by assuming that the deformation through the layer is homogeneous, i.e. plane sections remain plane after compression, so that the stress is uniform through the thickness.

Mathematically Johnson's model is equivalent to the Winkler-Fuss model with K defined by (10). Jaffar [34] applied the Johnson approach to problems of axisymmetric contacts involving thin layers bonded and unbonded to a rigid foundation and indented by a frictionless rigid sphere.

Approximate solutions to axisymmetric contact problems for an elastic coating bonded to rigid substrate were considered by many authors (see, e.g. [35, 36]). Among computational methods, the Finite Element Method is quite often used to model contact problems. Some authors engage finite elements to study adhesive contact problems (e.g., [37–39]). However, in the present paper we deal with analytical approaches and will not consider the papers devoted to numerical simulations.

One of the analytical approaches most commonly used to study contact problems is the asymptotic approach. The asymptotic expansions in the problems under consideration may use various parameters: (i) the parameter $\varepsilon = h/a$ is small; (ii) $\varepsilon = h/a$ is large; and (iii) for transversely isotropic materials $\varepsilon = E_1/E_2$ where E_1 and E_2 are the elastic moduli of the material along the layer and in transverse direction respectively (see, e.g. [40]). Many books and papers are devoted to asymptotic approaches to these problems see e.g. [1–3]. Here we are concentrated mainly on the case when $\varepsilon = h/a$ is small.

It is well known [23, 24, 29, 41–47] that the class of mixed boundary value problems (with free (unknown) boundary between the different types of boundary conditions) for a thin layer can be formulated using integral equations relating the unknowns: contact pressure $p(x_1, x_2)$, contact area G and its boundary ∂G, the approach of contacting bodies δ (in case the applied force P is known). We agree with Johnson's remark (see page 138 in [18]) that the integrand in integral formulation equation for contact problem of the layer has the awkward form and this has led to serious difficulties in the analysis of contact stresses in strips and layers. Nevertheless the asymptotic approaches were mainly developed for the integral equation or equations of similar structure (see, e.g. [1, 3, 5, 6, 31, 32, 48]).

Aghalovyan [2] has developed an alternative asymptotic approach to the elasticity equations of anisotropic plates and shells in both $2D$ and $3D$ formulations. In isotropic limit his calculations give the known expression of K for plane strain (10) and

$$K_{PS} = \frac{E}{h\,(1-\nu^2)} \tag{11}$$

for plane stress case (see page 76 in [2]). In addition, Aghalovyan derives stiffness of a multilayered foundation by addition of compliances of individual Winkler-Fuss-type layers in the following way

$$K_M = \left(\sum_{j=1}^{N} \frac{1}{K_j} \right)^{-1}, \tag{12}$$

where K_M is the resulting stiffness coefficient of the multilayered structure, K_j individual stiffness of j-th layer and N is the total number of layers. Although the methods developed by Aghalovyan [2] are very powerful, they are rather sophisticated.

Using much simpler transformations than those used in [2], Argatov and Mishuris (see page 14 in [3]), demonstrate that the contact problem for a thin transversely isotropic layer in leading order asymptotic approximation is reduced to the problem for a Winkler-Fuss layer. If Hooke's law for a transversely isotropic layer is written in matrix form as

$$\{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}^T = [A]\,\{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\}^T,$$

where $[A]$ is the matrix of elastic constants, then in leading order approximation the transversely-isotropic thin layer is reduced to the Winkler-Fuss foundation with stiffness coefficient

$$K_{TI} = \frac{A_{33}}{h}. \tag{13}$$

Note that alternative asymptotic approaches were also considered in application to the problem for a thin coating. Alexandrov [49] considered $2D$ contact problem for a thin coating covering elastic half-space (plane strain). However, this solution had several additional assumptions. Further we will study only direct asymptotic approaches to the contact problems.

3 The GKN asymptotic approach to contact problem for an elastic compressible layer

As it has been discussed above, contact problem for a rigid indenter and an elastic layer bonded to rigid substrate was studied in a number of publications. Nevertheless, for the sake of completeness, we derive leading order asymptotic solution of indentation problem for an elastic layer bonded to rigid substrate using the GKN (Goldenveizer-Kaplunov-Nolde) method. This quite simple asymptotic method was initially introduced as "the method of direct asymptotic integration" by Goldenveizer in his papers and the second edition of his classic book on shell theory [50, 51], then the method was enhanced and clearly explained in [4]. Although the GKN method was originally developed for applications in theory of plates and shells, it was already applied to two-dimensional contact problems [52]. Here the GKN method is applied directly to variables of the spatial contact problem formulation. This allows us to obtain the expressions for displacements and stresses acting in the elastic layer.

According to the ideas of the GKN asymptotic integration procedure, we first dilate the scale of the independent variables and assume that differentiation with respect to the scaled variables does not change the asymptotic order of the quantities to be found. We assume also that the layer thickness is small compared to the radius of contact, i.e. the parameter

$$\varepsilon = h/a \tag{14}$$

is small. Following the GKN asymptotic procedure, we set

$$\begin{aligned}
&u_i\left(x_1, x_2, x_3\right) = h\varepsilon u_i^*\left(x_1, x_2, x_3\right), \qquad i = 1, 2,\\
&u_3\left(x_1, x_2, x_3\right) = hu_3^*\left(x_1, x_2, x_3\right),\\
&\sigma_{jj}\left(x_1, x_2, x_3\right) = \mu\,\sigma_{jj}^*\left(x_1, x_2, x_3\right), \qquad j = 1, 2, 3,\\
&\sigma_{k3}\left(x_1, x_2, x_3\right) = \sigma_{3k}\left(x_1, x_2, x_3\right) = \mu\,\varepsilon\sigma_{k3}^*\left(x_1, x_2, x_3\right), \qquad k = 1, 2,\\
&\sigma_{12}\left(x_1, x_2, x_3\right) = \sigma_{21}\left(x_1, x_2, x_3\right) = \mu\,\varepsilon^2\sigma_{12}^*\left(x_1, x_2, x_3\right),
\end{aligned} \tag{15}$$

where quantities with asterisks are assumed of the same asymptotic order. Note that repeated subscripts *do not* imply summation in the latter expression and throughout the whole paper.

According to the GKN asymptotic method, we introduce the following dimensionless variables

$$\xi_1 = \frac{x_1}{a}, \qquad \xi_2 = \frac{x_2}{a}, \qquad \xi_3 = \frac{x_3}{h}. \tag{16}$$

Substituting (16) into (15), we can write the governing equations as

$$\begin{aligned}
&\sigma_{11,1}^* + \varepsilon^2\sigma_{12,2}^* + \sigma_{13,3}^* = 0,\\
&\varepsilon^2\sigma_{12,1}^* + \sigma_{22,2}^* + \sigma_{23,3}^* = 0,\\
&\varepsilon^2\left(\sigma_{13,1}^* + \sigma_{23,2}^*\right) + \sigma_{33,3}^* = 0,\\
&\sigma_{11}^* = \varepsilon^2\left(\kappa^2 u_{1,1}^* + \left(\kappa^2 - 2\right)u_{2,2}^*\right) + \left(\kappa^2 - 2\right)u_{3,3}^*,\\
&\sigma_{22}^* = \varepsilon^2\left(\left(\kappa^2 - 2\right)u_{1,1}^* + \kappa^2 u_{2,2}^*\right) + \left(\kappa^2 - 2\right)u_{3,3}^*,\\
&\sigma_{33}^* = \varepsilon^2\left(\kappa^2 - 2\right)\left(u_{1,1}^* + u_{2,2}^*\right) + \kappa^2 u_{3,3}^*,\\
&\sigma_{ij}^* = u_{i,j}^* + u_{j,i}^*, \qquad i, j = 1, 2, 3\,(i \neq j),
\end{aligned} \tag{17}$$

with comma in the subscript denoting differentiation with respect to the corresponding dimensionless variable ξ_j and

$$k^2 = \frac{2 - 2\nu}{1 - 2\nu}.$$

The boundary conditions (4) can be represented as

$$\begin{aligned}
&u_3^*\left(\xi_1, \xi_2, 0\right) = \frac{1}{h}\left(\delta - \varphi\left(\xi_1, \xi_2\right)\right), \qquad \left(\xi_1, \xi_2\right) \in G^*,\\
&\sigma_{13}^*\left(\xi_1, \xi_2, 0\right) = \sigma_{23}^*\left(\xi_1, \xi_2, 0\right) = 0,\\
&u_1^*\left(\xi_1, \xi_2, 1\right) = u_2^*\left(\xi_1, \xi_2, 1\right) = u_3^*\left(\xi_1, \xi_2, 1\right) = 0,
\end{aligned} \tag{18}$$

where G^* is the region of contact in the (ξ_1, ξ_2) plane.

In order to reduce the number of unknowns let us transform (17) into the form of Lamé equations, keeping only unknown displacements.

$$\begin{aligned}
&\left(k^2 - 1\right)u_{3,13}^* + u_{1,33}^* + \varepsilon^2\left(k^2 u_{1,11}^* + u_{1,22}^* + \left(k^2 - 1\right)u_{2,12}^*\right) = 0,\\
&\left(k^2 - 1\right)u_{3,23}^* + u_{2,33}^* + \varepsilon^2\left(u_{2,11}^* + k^2 u_{2,22}^* + \left(k^2 - 1\right)u_{1,12}^*\right) = 0,\\
&k^2 u_{3,33}^* + \varepsilon^2\left(u_{3,11}^* + u_{3,22}^* + \left(k^2 - 1\right)\left(u_{1,13}^* + u_{2,23}^*\right)\right) = 0.
\end{aligned} \tag{19}$$

After eliminating stresses, the boundary conditions (18) can be transformed to the following form

$$\begin{aligned}
&u_3^*(\xi_1,\xi_2,0) = \frac{1}{h}(\delta - \varphi(\xi_1,\xi_2)), \qquad (\xi_1,\xi_2) \in G^*, \\
&u_{i,3}^*(\xi_1,\xi_2,0) + u_{3,i}^*(\xi_1,\xi_2,0) = 0, \qquad i = 1,2, \\
&u_1^*(\xi_1,\xi_2,1) = u_2^*(\xi_1,\xi_2,1) = u_3^*(\xi_1,\xi_2,1) = 0,
\end{aligned} \tag{20}$$

Equations (19) contain only ε^2 terms, therefore the following asymptotic expansion may be used

$$u_i{}^* = u_i{}^{(0)} + \varepsilon^2 u_i{}^{(1)} + \varepsilon^4 u_i{}^{(2)} + ... \qquad i = 1,2,3. \tag{21}$$

(Note that $u_i{}^{(k)}$ are *dimensionless* asymptotic approximations of k-th order for i-th displacement).

Thus, the leading order approximation to the non-adhesive contact problems is reduced to the following boundary value problem:

(i) the governing equations

$$\left(k^2 - 1\right) u_{3,i3}^{(0)} + u_{i,33}^{(0)} = 0, \qquad i = 1,2, \tag{22a}$$

$$k^2 u_{3,33}^{(0)} = 0, \tag{22b}$$

and (ii) the boundary conditions

$$u_3^{(0)} = \frac{1}{h}(\delta - \varphi(\xi_1,\xi_2)), \quad \xi_3 = 0, \tag{23a}$$

$$u_{i,3}^{(0)} + u_{3,i}^{(0)} = 0, \qquad \xi_3 = 0, \qquad i = 1,2, \tag{23b}$$

$$u_1^{(0)} = u_2^{(0)} = u_3^{(0)} = 0, \qquad \xi_3 = 1. \tag{23c}$$

Further we consider and solve only the above leading order approximation of the problem (the zero-order problem).

3.1 Leading order solution

It follows from (22b) that

$$u_3^{(0)} = \xi_3 F_1(\xi_1,\xi_2) + F_2(\xi_1,\xi_2). \tag{24}$$

On satisfying boundary conditions (23a) and (23c) one obtains that $F_2 = \frac{1}{h}(\delta - \varphi)$, $F_1 = -F_2$. Hence, we have

$$u_3^{(0)}(\xi_1,\xi_2,\xi_3) = \frac{1}{h}[\delta - \varphi(\xi_1,\xi_2)](1 - \xi_3). \tag{25}$$

Substitution of the latter into (22a) yields

$$u_{i,33}^{(0)} = \frac{1}{h}\left(1 - k^2\right)\varphi_{,i} \qquad i = 1,2. \tag{26}$$

The solutions of the above equations have the following structure

$$u_i^{(0)} = \frac{1}{2h}\left(1 - k^2\right)\varphi_{,i}\xi_3^2 + \xi_3 F_{3i}(\xi_1,\xi_2) + F_{4i}(\xi_1,\xi_2), \qquad i = 1,2. \tag{27}$$

Substituting the above expressions into (23b), one obtains

$$\left[\frac{1}{h}\left(\xi_3 - k^2\xi_3 - 1 + \xi_3\right)\varphi_{,i} + F_{3i}\right]\Bigg|_{\xi_3=0} = 0 \quad \text{or} \quad F_{3i} = \frac{1}{h}\varphi_{,i} \qquad i = 1,2.$$

Taking the latter expressions into account and applying the boundary condition (23c) we arrive at

$$F_{4i} = -\frac{1}{h}\left(1 + \frac{1-k^2}{2}\right)\varphi_{,i} = \frac{1}{2h}\left(k^2 - 3\right)\varphi_{,i} \qquad i = 1, 2.$$

Finally the expressions for $u_i^{(0)}$ are the following:

$$u_i^{(0)} = \frac{1}{2h}\varphi_{,i}\left[\left(1-k^2\right)\xi_3^2 + 2\xi_3 + k^2 - 3\right] \qquad i = 1, 2. \tag{28}$$

Thus, the displacements of the leading order approximation are

$$\begin{aligned} u_i^{(0)} &= \frac{1}{2h}\varphi_{,i}\left[\left(1-k^2\right)\xi_3^2 + 2\xi_3 + k^2 - 3\right], \qquad i = 1, 2, \\ u_3^{(0)} &= \frac{1}{h}\left(\delta - \varphi\right)\left(1 - \xi_3\right). \end{aligned} \tag{29}$$

Because in the leading term approximation of the problem it is assumed that $u_1^* \approx u_1^{(0)}$, $u_2^* \approx u_2^{(0)}$, $u_3^* \approx u_3^{(0)}$, we obtain from (17) that $\sigma_{ii}^* \approx \left(k^2 - 2\right)u_{3,3}^* \, (i = 1, 2)$, $\sigma_{33}^* \approx k^2 u_{3,3}^*$, and $\sigma_{ij}^* = u_{i,j}^* + u_{j,i}^*$ $(i, j = 1, 2, 3;\, i \neq j)$.

Therefore, the normalized stresses of the leading order approximation are

$$\begin{aligned} \sigma_{ii}^* &\approx \frac{1}{h}\left(2 - k^2\right)\left(\delta - \varphi\right), \qquad i = 1, 2, \\ \sigma_{33}^* &\approx -\frac{k^2}{h}\left(\delta - \varphi\right), \\ \sigma_{12}^* &\approx \frac{1}{h}\left(\left(1-k^2\right)\xi_3^2 + 2\xi_3 + k^2 - 3\right)\varphi_{,12}, \\ \sigma_{i3}^* &\approx \frac{1}{h}\varphi_{,i}\left(2 - k^2\right)\xi_3, \qquad i = 1, 2. \end{aligned} \tag{30}$$

3.2 Solution in dimensional variables

Let us write the above solution for the leading order approximation in dimensional variables. Using the scaling expressions (15), we obtain for the displacements

$$\begin{aligned} u_i &= \varepsilon h u_i^* \approx \frac{h}{2a}\varphi_{,i}\left(\left(1-k^2\right)\xi_3^2 + 2\xi_3 + k^2 - 3\right), \qquad i = 1, 2, \\ u_3 &= h u_3^* \approx \left(\delta - \varphi\right)\left(1 - \xi_3\right), \end{aligned} \tag{31}$$

and the stresses

$$\begin{aligned} \sigma_{ii} &= \mu\sigma_{ii}^* \approx \frac{\mu}{h}\left(2 - k^2\right)\left(\delta - \varphi\right), \qquad i = 1, 2, \\ \sigma_{33} &= \mu\sigma_{33}^* \approx -\frac{\mu}{h}k^2\left(\delta - \varphi\right), \\ \sigma_{12} &= \varepsilon^2\mu\sigma_{12}^* \approx \frac{\mu h}{a^2}\left(\left(1-k^2\right)\xi_3^2 + 2\xi_3 + k^2 - 3\right)\varphi_{,12}, \\ \sigma_{i3} &= \varepsilon\mu\sigma_{i3}^* \approx \frac{\mu}{a}\left(2 - k^2\right)\varphi_{,i}\xi_3, \qquad i = 1, 2. \end{aligned} \tag{32}$$

respectively.

Consider the second formula from (32). Since the contact pressure can be expressed as $p = -\left.\sigma_{33}\right|_{\xi_3=0}$, in leading order approximation we have

$$p \approx \frac{\mu}{h}k^2\left(\delta - \varphi\right). \tag{33}$$

The external force may be found from (7).

It is clear from (33) that pressure value at any given point of contact area is proportional to the vertical displacement of that point (which is equal to $\delta - \varphi$). In fact, this means that the thin layer attached to rigid substrate behaves exactly as Winkler-Fuss elastic foundation (compare (33) with (8)). The stiffness coefficient of the foundation K is described by (10).

Without doubt the transformations of the GKN approach presented in the current Section are much more straightforward than these of the integral equations approach. The above direct asymptotic decomposition is based on the GKN asymptotic approach, i.e. the direct asymptotic integration of the boundary value problem using ideas described in [4, 50]. However, as it has been mentioned above there is another direct asymptotic approach to the problem developed by Argatov and Mishuris in [3].

Let us compare the two direct asymptotic approaches: (i) the AM approach [3] and (ii) the GKN one [4]. It is clear that the proper scaling plays a key role in both approaches, namely the independent variables are scaled as $x_1 = a\xi_1$, $x_2 = a\xi_2$, $x_3 = h\xi_3$. However, the GKN approach scales additionally the unknown displacements and stresses using (15). Hence, working in the AM techniques, one has to deal with the terms of orders $O(1)$, $O(\varepsilon)$, $O(\varepsilon^2)$ in the governing equations, while the GKN approach results in twice less amount of transformations because the equations contain only terms of orders $O(1)$ and $O(\varepsilon^2)$ in this case. Indeed, the unknowns can be expanded in asymptotic series using only even powers of ε in the GKN case, while one has to consider the equations corresponding to all consecutive powers of the small parameter in the AM approach. In addition, as the result of proper scaling in each subsequent approximation, one has all the unknowns to be non-zero while in the AM case in each subsequent approximation some unknowns are always zero.

The further analysis shows that after all the back-substitutions (i.e. (31)-(32) in our case) the resulting expansions in dimensional variables have the same structure in both approaches. However, these expansions are obtained in the GKN case using much lesser number of operations and more reasonable scaling. This scaling indeed can be verified, for example, by consideration of a model problem for a specific indenter profile allowing separation of variables, say, of sinusoidal shape or using numerical simulations by the Finite Element Method.

Thus, the GKN approach is more favourable because it benefits from more reasonable scaling and twice lesser number of operations. It is evident that if the GKN approach is applied to a thin transversely isotropic elastic layer then one obtains (13) instead of (10).

4 Adhesive contact problems for a thin elastic layer

Now let us consider a simple Winkler-Fuss elastic foundation whose elastic properties are characterized by the foundation modulus K. First we derive an expression for the slope of the force -displacement curve for an axisymmetric non-adhesive contact problem for the Winkler-Fuss foundation. Then we consider an adhesive contact problem for the foundation in the framework of the JKR approach.

4.1 The slope of the force-displacement curve for an axisymmetric indenter pressing the Winkler-Fuss foundation.

It is known (e.g. [15–17]) that knowledge of the slope of the force-displacement curve can be used to solve problems of adhesive contact within the framework of the JKR ideas. In particular, it allowed the authors of [15–17] to provide compact solutions to the adhesive contact problems within the framework of the JKR contact theory.

Further in the present paper, we also use the expression of the slope of the force-displacement curve of the non-adhesive contact problem to simplify the transformations related to the JKR contact theory. The expression for the slope of the $P - \delta$ curve can be obtained as follows.

Let us consider a simple Winkler-Fuss elastic foundation whose elastic properties are characterized by the foundation modulus K. Consider an axisymmetric convex, smooth indenter of arbitrary shape $f(r)$ pressed into this elastic foundation (Fig. 2).

In this case the contact region is a circle of radius a and the boundary conditions for the foundation can be written as

$$
\begin{aligned}
&u_3(r,0) = \delta - f(r), \qquad \sigma_{33}(r,0) \le 0, \qquad r \le a, \\
&u_3(r,0) > \delta - f(r), \qquad \sigma_{33}(r,0) = 0, \qquad (x_1, x_2) \in \mathbb{R}^2 \setminus G,
\end{aligned}
\tag{34}
$$

Since the contact pressure becomes zero on the boundary of the contact region $r = a$, one can easily obtain from (33) that $\delta - f(a) = 0$ and therefore

$$
\delta = f(a). \tag{35}
$$

Because the foundation is bounded to the rigid support, the conditions at the bottom of the layer are

$$
u_3(r,h) = 0. \tag{36}
$$

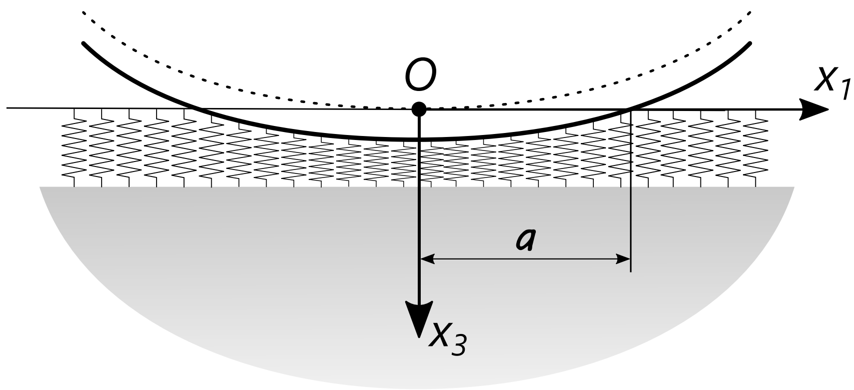

Figure 2: Problem formulation for Winkler-type foundation

The volume of the body under the cross–section of height δ is given by

$$
V = 2\pi \int_0^\infty [\delta - f(r)] H[\delta - f(r)] r dr,
$$

where H is the Heaviside step function. Using (35) this expression can be presented as

$$
V = 2\pi \left[0.5\delta a^2 - \int_0^a f(r) r dr \right] = \pi \left[f(a)a^2 - 2\int_0^a f(r) r dr \right].
$$

Hence, for the contact force according to (9), we have

$$
P = 2\pi K \left[0.5\delta a^2 - \int_0^a f(r) r dr \right]. \tag{37}
$$

By differentiating (37), we obtain

$$
\frac{dP}{da} = 2\pi K \left[0.5 a^2 \frac{d\delta}{da} + \delta a - f(a) a \right].
$$

Therefore, we have the final result for the slope

$$S(a) = \frac{dP/da}{d\delta/da} = \frac{dP}{d\delta} = \pi a^2 K. \tag{38}$$

The above expression is also valid for leading order approximations of the solutions for isotropic and transversely isotropic linear or linearized elastic layers. If the material is isotropic then K is defined by (10), if coating is multilayered or it is transversely isotropic then K is defined by (12) or (13) respectively.

Comment. Yang [36] considered three particular cases of both non-adhesive and adhesive contact problems for an isotropic layer within the framework of his approximate model: flat-ended cylindrical, spherical and conical indenters. For each of the cases of non-adhesive contact, the slope $\frac{dP}{d\delta}$ was evaluated individually. Evidently, these results are particular cases of (38). The solutions for each case of indenters were obtained individually disregarding that all the three shapes are particular cases of power law indenter $f(r) = Br^d$.

4.2 JKR-type adhesion problems for axisymmetric indenter and Winkler-Fuss foundation

Our approach was announced in [53], where we claimed that using the results presented in [1] we can solve problems of adhesive contact for the indenter and a layer represented as the Winkler-Fuss elastic foundation. However, no detail of our calculations were published. The JKR approach to adhesion contact problems for the Winkler-Fuss foundation were also studied earlier by other researchers. As it has been mentioned above, Yang [36] considered separately adhesive contact problems for three cases of indenters disregarding that all the three shapes are particular cases of power law indenter $f(r) = Br^d$. The approach developed in [54] was based on a conservation of energy, however the geometrically non-linear formulation was used, while here we develop approaches based on geometrically linear formulations of the contact problems.

Following the JKR idea, we can note that if there were no surface forces of attraction, the radius of the contact area under a punch subjected to the external load P_0 would be a_0 and it could be found by solving the Hertz-type contact problem. However, in the presence of the forces of molecular adhesion, the equilibrium contact radius a_1 is greater than a_0 under the same force P_0 (Fig. 3). Hence, the total energy of the contact system U_T is built up of three terms, the stored elastic energy U_E, the mechanical energy in the applied load U_M and the surface energy U_S, while U_E is calculated as the difference between the stored elastic energies $(U_E)_1$ and $(U_E)_2$. Here $(U_E)_1$ is the elastic energy of Hertz-type contact system without adhesive interactions; the system is loaded until the true contact radius a_1 is obtained, however the values of the corresponding external load P_1 and the displacement δ_1 are not correct. $(U_E)_2$ is the elastic energy of Boussinesq-type contact system for a flat ended indenter having constant radius; the system is keeping the true contact radius and it is unloaded from P_1 to the value of true external load P_0 and the true displacement value δ_2 is calculated assuming that the total energy has minimum at equilibrium. We will keep further in this subsection the original JKR notations a_1, δ_2 and P_0 of the true values for the contact radius, depth of indentation and the force respectively.

Therefore, we can write

$$(U_E)_1 = P_1\delta_1 - \int_0^{P_1} \delta dP. \tag{39}$$

Using an analogy to the Boussinesq solution, we obtain for the unloading branch

$$(U_E)_2 = \int_{P_0}^{P_1} \frac{P}{S(a_1)} dP = \frac{P_1^2 - P_0^2}{2S(a_1)}. \tag{40}$$

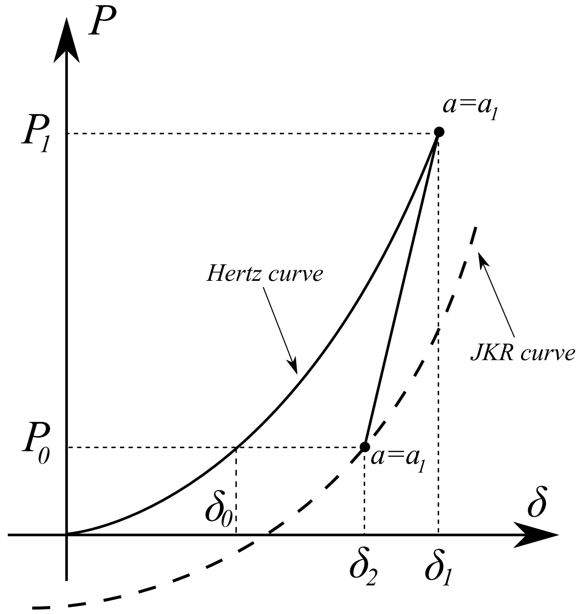

Figure 3: Construction of JKR load-displacement curve

Thus, the stored elastic energy U_E is

$$U_E = (U_E)_1 - (U_E)_2. \tag{41}$$

The mechanical energy in the applied load

$$U_M = -P_0\delta_2 = -P_0(\delta_1 - \Delta\delta), \tag{42}$$

where $\Delta\delta = \delta_1 - \delta_2$ is the change in the depth of penetration due to unloading.

Since only the surface adhesive interactions within the contact region are taken into account (one neglects the adhesive forces acting outside the contact region), the surface energy can be written as

$$U_S = -w\pi a_1^2. \tag{43}$$

The total energy U_T can be obtained by summation of (41), (42) and (43), i.e.

$$U_T = U_E + U_M + U_S. \tag{44}$$

It is assumed in the JKR model that the equilibrium at contact satisfies the equation

$$\frac{dU_T}{da_1} = 0, \quad \text{or} \quad \frac{dU_T}{dP_1} = 0. \tag{45}$$

Taking into account an analogy to the Boussinesq solution , one obtains for the unloading branch

$$\Delta\delta = \frac{P_1 - P_0}{S(a_1)},$$

and therefore, one has

$$U_M = -P_0\left(\delta_1 - P_0\frac{P_1 - P_0}{S(a_1)}\right) \quad \text{and} \quad (U_E)_2 = \frac{P_1^2 - P_0^2}{2S(a_1)}.$$

Hence, the total energy U_T can be written as

$$U_T = P_1\delta_1 - \int_0^{P_1} \delta(P)dP - \frac{(P_1^2 - P_0^2)}{2S(a_1)} - P_0\delta_1 + P_0\frac{(P_1 - P_0)}{S(a_1)} - w\pi a_1^2$$

or

$$U_T = P_1\delta_1 - \int_0^{P_1} \delta(P)dP - \frac{(P_1 - P_0)^2}{2S(a_1)} - P_0\delta_1 - w\pi a_1^2.$$

Finally, one obtains

$$U_T = (P_1 - P_0)\delta_1 - \int_0^{P_1} \delta(P)dP - \frac{(P_1 - P_0)^2}{2S(a_1)} - w\pi a_1^2. \tag{46}$$

Taking into account the following expressions

$$\frac{d}{dP_1}[(P_1 - P_0)\delta_1] = (P_1 - P_0)\frac{d\delta_1}{dP_1} + \delta_1 = \frac{(P_1 - P_0)}{S(a_1)} + \delta_1,$$
$$\frac{d}{dP_1}\int_0^{P_1} \delta(P)dP = \delta(P_1) = \delta_1,$$

and

$$\frac{d}{dP_1}\frac{(P_1 - P_0)^2}{2S(a_1)} = \frac{(P_1 - P_0)}{S(a_1)} - \frac{(P_1 - P_0)^2}{2S^2(a_1)}\frac{dS}{da_1}\frac{da_1}{dP_1},$$

and applying the equilibrium condition (45) to (46), one obtains

$$\frac{dU_T}{dP_1} = \frac{(P_1 - P_0)}{S(a_1)} + \delta_1 - \delta_1 - \frac{(P_1 - P_0)}{S(a_1)} + \left(\frac{(P_1 - P_0)^2}{2S^2(a_1)}\frac{dS}{da_1} - 2w\pi a_1\right)\left(\frac{da_1}{dP_1}\right) = 0. \tag{47}$$

Due to the expressions (38), it follows from (47) that the equilibrium condition for the general JKR model is

$$\frac{dU_T}{dP_1} = \left[\frac{(P_1 - P_0)^2}{\pi K a_1^3} - 2w\pi a_1\right]\frac{da_1}{dP_1} = 0 \tag{48}$$

or

$$(P_1 - P_0)^2 = 2\pi^2 wKa_1^4. \tag{49}$$

Further one has

$$P_1 - P_0 = \sqrt{2wK}\pi a_1^2 = \pi K a_1^2 \Delta\delta$$

and hence, the following expression is valid

$$\Delta\delta = \frac{\sqrt{2wK}\pi a_1^2}{\pi K a_1^2} = \sqrt{\frac{2w}{K}}.$$

Thus, for an arbitrary convex body of revolution $f(r)$, $f(0) = 0$, the JKR theory leads to the following expressions

$$P_1 = P_0 + \sqrt{2wK}\pi a_1^2, \quad \delta_2 = \delta_1 - \sqrt{\frac{2w}{K}}. \tag{50}$$

Taking into account formulae (35) and (37), the relations (50) can be written as

$$P_0 = P_1 - \sqrt{2wK}\pi a_1^2 = \pi K\left[f(a_1)a_1^2 - 2\int_0^{a_1} f(r)rdr\right] - \sqrt{2wK}\pi a_1^2 \tag{51}$$

and

$$\delta_2 = f(a_1) - \left(\frac{2w}{K}\right)^{1/2}. \tag{52}$$

Further we remove auxiliary subscripts and will use the notations P, δ and a instead of P_0, δ_2 and a_1 for the true values of the force, the approach of the indenter and the contact radius in adhesive contact problem.

Finally, after simple transformations the relations (51) and (52) between the true values of the force, the approach of the indenter and the contact radius can be written as

$$P = \pi K a^2 \left(f(a) - \sqrt{\frac{2w}{K}} \right) - 2\pi K \int_0^a f(r) r dr \tag{53}$$

and

$$\delta = f(a) - \sqrt{\frac{2w}{K}}. \tag{54}$$

Note that the above transformations of the JKR approach have been done in general form, without specifying the particular expression for the slope of the $P - \delta$ curve $S(a)$ until the last stage. This makes the transformations more compact and easier to read.

4.3 Adhesive indentation of an elastic layer by non-ideal shaped indenters

The depth-sensing indentation (DSI) is the continuously monitoring of the $P - \delta$ diagram where P is the applied load and δ is the displacement (the approach of the distant points of the indenter and the sample). DSI techniques are especially important when mechanical properties of materials are studied using very small volumes of materials such as thin films. Hence, the present asymptotic approach may be used in material testing (however, these questions are out the scope of the paper). It is usually assumed that the indenter is a sharp pyramid or a cone. However, the indenter shape near the tip has some deviation from its nominal shape. The shapes of these non-ideal shaped indenters may be well approximated as monomial functions of radius

$$f(r) = B_d r^d, \quad d \geq 1 \tag{55}$$

where B_d is the constant of the shape of the monomial function of degree d (see, [13, 17, 55] for details). For indenters, whose shape is described by (55), the general expressions (53) and (54) have the following form

$$P = \pi K \left(\frac{d}{d+2} B_d a^{d+2} - \sqrt{\frac{2w}{K}} a^2 \right) \tag{56}$$

and

$$\delta = B_d a^d - \sqrt{\frac{2w}{K}}. \tag{57}$$

It follows from (56) and (57) that the relation $P(\delta)$ can be expressed not only in a parametric form but also as an explicit relation

$$P = \frac{\pi K}{d+2} \left[\frac{1}{B_d} \left(\delta + \sqrt{\frac{2w}{K}} \right) \right]^{2/d} \left(\delta d - 2\sqrt{\frac{2w}{K}} \right). \tag{58}$$

In order to write the latter expressions in dimensionless form, let us follow the same procedure as it was used in [14, 15] for the generalized JKR theory for elastic solids.

It follows from (56) that at $P = 0$ the radius a of the contact region and the corresponding displacement $\delta_c = \delta[a(0)]$ are

$$a(0) = \left(\frac{d+2}{dB_d} \sqrt{\frac{2w}{K}} \right)^{1/d}, \qquad \delta_c = \delta[a(0)] = \frac{2}{d} \sqrt{\frac{2w}{K}}. \tag{59}$$

Further note that
$$\frac{dP}{d\delta} = \frac{dP}{da} \Big/ \frac{d\delta}{da}.$$
Therefore, the root (a_c) of the equation
$$\frac{dP}{da} = 0$$
is the critical radius of the contact region. It gives the maximum absolute value of the adherence force $P_c = -P(a_c)$. Taking the derivative of (56), we obtain
$$a_c = \left(\frac{2}{dB_d} \sqrt{\frac{2w}{K}} \right)^{1/d}. \tag{60}$$

Substituting (60) into (56), we obtain
$$P_c = -P(a_c) = \pi K \frac{d}{d+2} \left(\frac{2}{dB_d} \right)^{2/d} \left(\frac{2w}{K} \right)^{(2+d)/2d}. \tag{61}$$

As it was mentioned in [14], various variables may be used to write the dimensionless solutions. In particular, both values $a(0)$ and a_c can be used as a characteristic size of the contact region in order to write dimensionless parameters.

If one takes the characteristic parameters as a_c, P_c and δ_c, then substituting (59), (61) and (60) into the system (56) and (57), we obtain the following dimensionless equations
$$P/P_c = \frac{d+2}{d} \left[\frac{2}{d+2} (a/a_c)^{d+2} - (a/a_c)^2 \right], \tag{62}$$
$$\frac{\delta}{\delta_c} = \left(\frac{a}{a_c} \right)^d - \frac{d}{2}. \tag{63}$$
and
$$P/P_c = \left(\frac{2}{d} \right)^{1-2/d} \left(\frac{2}{d} \frac{\delta}{\delta_c} + 1 \right)^{2/d} \left(\frac{\delta}{\delta_c} - 1 \right). \tag{64}$$

Let us consider another variant of the characteristic parameters, namely take $a(0)$ as the characteristic size of the contact region in order to write dimensionless parameters. value. The two remaining parameters are chosen as follows:
$$a^* = a(0), \quad P^* = \pi K \left(\frac{d+2}{dB_d} \right)^{\frac{2}{d}} \left(\frac{2w}{K} \right)^{\frac{2+d}{2d}}, \quad \delta^* = \sqrt{\frac{2w}{K}}. \tag{65}$$

Then (56) and (57) have the following dimensionless form
$$P/P^* = (a/a^*)^{d+2} - (a/a^*)^2 \tag{66}$$
and
$$\frac{\delta}{\delta^*} = \frac{d+2}{d} \left(\frac{a}{a^*} \right)^d - 1. \tag{67}$$

4.4 Adhesive contact of a spherical indenter

For a spherical indenter of radius R, we can represent the shape of the indenter as a parabolid of revolution:

$$f(r) = \frac{r^2}{2R},$$

i.e.

$$d = 2 \quad \text{and} \quad B_2 = (2R)^{-1} \tag{68}$$

Now we substitute these values into the expressions of the general solution. From (56) and (57), we obtain

$$\begin{aligned} P &= \pi K \left(\tfrac{1}{4R} a^4 - \sqrt{\tfrac{2w}{K}} a^2 \right) \\ \delta &= a^2/(2R) - \sqrt{\tfrac{2w}{K}}. \end{aligned} \tag{69}$$

and from (58) the following explicit force-displacement relation is obtained:

$$P = \pi K R \left[\left(\delta + \sqrt{\frac{2w}{K}} \right) \left(\delta - \sqrt{\frac{2w}{K}} \right) \right] = \pi R K \left(\delta^2 - \frac{2w}{K} \right). \tag{70}$$

Substituting (68) into (56), (60) and (61), we obtain

$$a_c = \left(\frac{8R^2 w}{K} \right)^{1/4}, \qquad a(0) = \left(\frac{32 R^2 w}{K} \right)^{1/4}, \tag{71}$$

$$P_c = -P(a_c) = 2\pi R w \quad \text{and} \quad \delta_c = \delta[a(0)] = \sqrt{\frac{2w}{K}}. \tag{72}$$

The force-displacement relation in the contact problem without adhesion is a parabola $P = \pi R K \delta^2$, while in the adhesive contact problem, the relation is described by (70), i.e. it is the same parabola shifted by the value P_c in the negative direction of P-axis.

The contact pressure can be represented as

$$p(r) = K \left(\delta - \frac{r^2}{2R} \right) = \frac{K}{2R} \left(a^2 - r^2 \right) - \sqrt{2Kw}. \tag{73}$$

Zero contact pressure is achieved at the radius value

$$r_0 = \sqrt{a^2 - 2R\sqrt{\frac{w}{K}}}. \tag{74}$$

Maximum contact pressure is at $r = 0$

$$p_{max} = \frac{K a^2}{2R} - \sqrt{2Kw}. \tag{75}$$

The graph of the function $p(r)$ is represented in Fig. 4.

Substituting $d = 2$ into (62), we obtain the following dimensionless equation

$$\frac{P}{P_c} = \left(\frac{a}{a_c} \right)^4 - 2 \left(\frac{a}{a_c} \right)^2, \tag{76}$$

The graph of dimensionless dependency (76) is represented in Fig. 5.

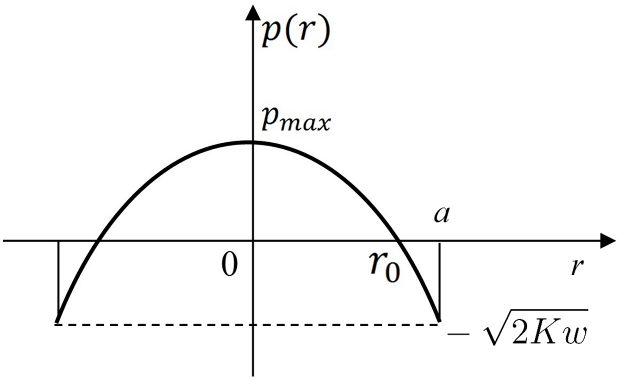

Figure 4: Contact pressure distribution under a spherical indenter.

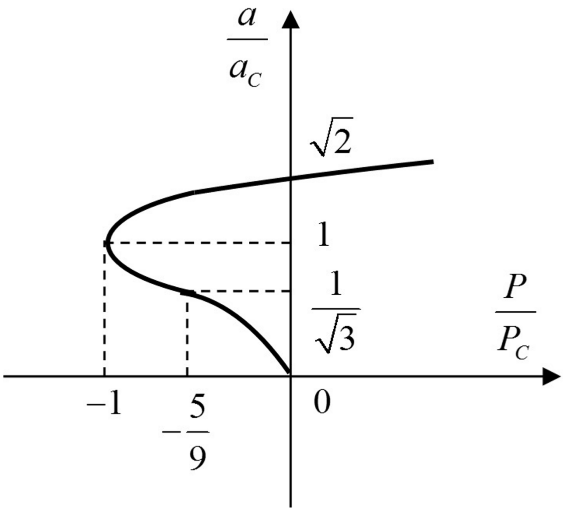

Figure 5: Dependency between indentation force and contact radius in dimensionless form

In the same way the equations (63) and (64) can be transformed in the following dimensionless equations

$$\frac{\delta}{\delta_c} = \left(\frac{a}{a_c}\right)^2 - 1 \tag{77}$$

and

$$\frac{P}{P_c} = \left(\frac{\delta}{\delta_c}\right)^2 - 1. \tag{78}$$

The graph of dimensionless dependency (78) is represented in Fig. 6.

The above graphs can be used to obtain the leading order asymptotic approximation of the problem of adhesive contact between a spherical indenter and a thin isotropic or transversely isotropic elastic layer. One just needs to use either (10) or (13) for the elastic modulus of the elastic foundation.

Conclusions

Problems of contact between a rigid convex indenter and an elastic thin compressible layer bonded to rigid substrate were studied in a number of publications. We have reviewed and examined some approaches to the problems. It has been shown that many approximate solutions are in essence the solution to the problem of contact between the indenter and a Winkler-Fuss elastic foundation. On the other hand, asymptotic approaches to the problems provide mathematical justification to the use of the Winkler-Fuss elastic foundation. However, most of the asymptotic approaches are rather

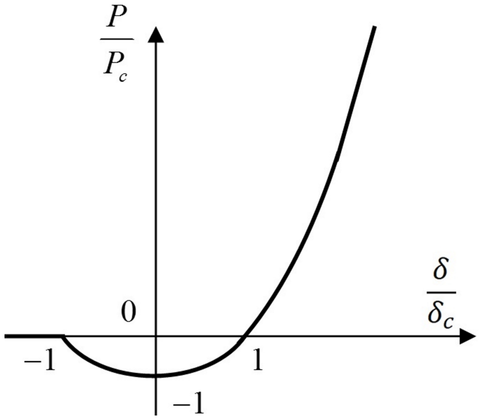

Figure 6: The dimensionless force-displacement curve for a spherical indenter

sophisticated. In particular, approaches based on integral formulation of the problem. As it was mentioned by Johnson [18] the kernel of this formulation may have an awkward form and this may lead to serious difficulties in the analysis of contact stresses in strips and layers.

Only relatively recently simple direct asymptotic approaches have been developed and applied to the contact problems. Assuming that the thickness of the layer is much less than characteristic dimension of the contact area, it has been shown that the GKN (Goldenveizer-Kaplunov-Nolde) method [4] that was originally developed for applications in theory of plates and shells, may be applied directly to variables of the contact problem formulation. It is easy to follow the method and it has been naturally shown that the leading order asymptotic approximation of the problem for a thin isotropic or transversely isotropic layer is actually the problem for a layer of springs (the Winkler-Fuss elastic foundation). The GKN approach has been compared with another direct asymptotic approach, the AM (Argatov-Mishuris) one [3]. We argue that although the GKN and AM approaches are mathematically equivalent, the GKN approach has several advantages in producing series, formulation of the boundary conditions and writing expressions for displacements and stresses acting in the elastic layer.

For this leading order asymptotic approximation, i.e. for the Winkler-Fuss elastic foundation, the axisymmetric adhesive contact is studied in the framework of the JKR theory. The governing equations of the JKR theory are formulated and transformed using the slope of the non-adhesive force-displacement curve in general form, which simplifies transformations and makes them clearer. The JKR approach has been generalized to the case of the punch shape being described by an arbitrary blunt axisymmetric indenter. Connections of the results obtained to problems of nanoindentation in the case of the indenter shape near the tip has some deviation from its nominal shape are discussed. The explicit expressions are derived for the values of the pull-off force and for the corresponding critical contact radius for indenters whose shape is described by power-law functions. The solution to the particular case of spherical indenter has been also discussed in detail.

Acknowledgements

The work was initiated as a part of activities of the CARBTRIB International Network supported by the Leverhulme Trust. The results of the work were presented at the second CARBTRIB International Network, Seville 21-23 April, 2017. The authors are grateful to the Leverhulme Trust for the support of their collaboration.

Dr. Nikolay Perepelkin gratefully acknowledges that his participation in this project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 663830.

References

[1] Alexandrov VM and Mkhitaryan SM. *Contact Problems for Bodies with Thin Coatings and Interlayers (in Russian)*. Nauka, Moscow, 1983.

[2] Aghalovyan L. *Asymptotic theory of anisotropic plates and shells*. World Scientific, 2015.

[3] Argatov I and Mishuris G. *Contact Mechanics of Articular Cartilage Layers. Asymptotic Models*. Springer, 2015.

[4] Goldenveizer AL, Kaplunov JD and Nolde EV. On Timoshenko-Reissner type theories of plates and shells. *Int J Solids Struct* 1993; 30(5): 675–694.

[5] Argatov II, Mishuris GS and Popov VL. Asymptotic modelling of the JKR adhesion contact for a thin elastic layer. *The Quarterly Journal of Mechanics and Applied Mathematics* 2016; 69(2): 161–179.

[6] Argatov II, Borodich FM and Popov VL. JKR adhesive contact for a transversely isotropic layer of finite thickness. *J Phys D: Appl Phys* 2016; 49(4): 045307.

[7] Kendall K. *Molecular Adhesion and Its Applications*. Kluwer Academic/Plenum Publishers, New York, 2001.

[8] Derjaguin BV. Untersuchungen über die reibung und adhäsion, IV. Theorie des anhaftens kleiner teilchen. *Kolloid Zeitschrift* 1934; 69: 155–164.

[9] Johnson KL. A note on the adhesion of elastic solids. *Brit J Appl Phys* 1958; 9: 199–200.

[10] Sperling G. *Eine Theorie der Haftung von Feststoffteilchen an festen Körpern*. PhD Thesis, Technische Hochschule Karlsruhe, 1964.

[11] Jung H. Ein beitrag zum loveschen verschiebungsfunktion. *Ingenieur-Archiv* 1950; 18: 178–190.

[12] Johnson KL, Kendall K and Roberts AD. Surface energy and the contact of elastic solids. *Proc R Soc Lond A* 1971; 324: 301–313.

[13] Borodich FM, Galanov BA and Suarez-Alvarez MM. The JKR-type adhesive contact problems for power-law shaped axisymmetric punches. *J Mech Phys Solids* 2014; 75: 14–32.

[14] Borodich FM, Galanov BA, Keer LM et al. The JKR-type adhesive contact problems for transversely isotropic elastic solids. *Mech Mater* 2014; 75: 34–44.

[15] Borodich FM. The Hertz-type and adhesive contact problems for depth-sensing indentation. *Adv App Mech* 2014; 47: 225–366.

[16] Borodich FM and Galanov BA. Contact probing of stretched membranes and adhesive interactions: Graphene and other two-dimensional materials. *Proc R Soc Ser A* 2016; 464: 2759–2776.

[17] Borodich FM and Keer LM. Contact problems and depth-sensing nanoindentation for frictionless and frictional boundary conditions. *Int J Solids Struct* 2004; 41: 2479–2499.

[18] Johnson KL. *Contact Mechanics*. Cambridge University Press, Cambridge, 1985.

[19] Kuznetsov VI. *Elastic Foundation (in Russian)*. Literature on Construction and Architecture Publishing, Moscow, 1952.

[20] Kerr AD. Elastic and viscoelastic foundationmodels. *J Appl Mech Trans ASME* 1964; 31(3): 491–498.

[21] Winkler E. *Die Lehre von der Elastizitätund Festigkeit, mit Besonderer Rücksicht auf ihre Anwendung in der Technik, für Polytechnische Schulen, Bauakademien, Ingenieure, Maschinenbauer, Architekten, etc.* Verlag H. Dominicus, Prague, 1867.

[22] Zimmermann H. *Die Berechnung des Eisenbahnoberbaues.* Wilhelm Ernst and Sohn, Berlin, 1888.

[23] Galanov BA. Spatial contact problems for rough elastic bodies under elastoplastic deformations of the unevenness. *PMM J Appl Math Mech* 1984; 48(6): 750–757.

[24] Galanov BA. The method of boundary equations of the Hammerstein-type for contact problems of the theory of elasticity when the regions of contact are not known. *PMM J Appl Math Mech* 1985; 49(5): 634–640.

[25] Borodich FM and Mosolov AB. Fractal roughness in contact problems. *PMM J Appl Math Mech* 1992; 56: 681–690.

[26] Borodich FM and Onishchenko DA. Fractal roughness for problem of contact and friction (the simplest models). *J Friction Wear* 1993; 14: 452–459.

[27] Borodich FM. Parametric homogeneity and non-classical self-similarity. II. Some applications. *Acta Mech* 1998; 131: 47–67.

[28] Borodich FM. Parametric homogeneity and non-classical self-similarity. I. Mathematical background. *Acta Mech* 1998; 131: 27–45.

[29] Borodich FM and Galanov BA. Self-similar problems of elastic contact for non-convex punches. *J Mech Phys Solids* 2002; 50: 2441–2461.

[30] Galanov BA. Models of adhesive contact between rough elastic bodies. *Int J Mech Sci* 2011; 53: 968–977.

[31] Alexandrov VM. On the solution of certain contact problems of the theory of elasticity. *PMM J Appl Math Mech* 1963; 27(5): 1490–1494.

[32] Alexandrov VM and Vorovich II. Contact problems for the elastic layer of small thickness. *PMM J Appl Math Mech* 1964; 28(2): 425–427.

[33] Meijers P. The contact problems for the rigid cylinder on an elastic layer. *Appl Sci Res* 1968; 18: 353–383.

[34] Jaffar MJ. Asymptotic behaviour of thin elastic layers bonded and unbonded to a rigid foundation. *Int J Mech Sci* 1989; 31(3): 229–235.

[35] Matthewson MJ. Axi-symmetric contact on thincompliant coatings. *J Mech Phys Solids* 1981; 29(2): 89–113.

[36] Yang F. Asymptotic solution to axisymmetricindentation of a compressible thin film. *Thin Solid Films* 2006; 515: 2274–2283.

[37] Sridhar I, Johnson K and Fleck NA. Adhesion mechanics of the surface force apparatus. *J Phys D: Appl Phys* 1997; 30: 1710–1719.

[38] Johnson KL and Sridhar I. Adhesion between a spherical indenter and an elastic solid with a compliant elastic coating. *J Phys D: Appl Phys* 2001; 34: 683.

[39] Sridhar I, Zheng ZW and Johnson KL. Adetailed analysis of adhesion mechanics between a compliant elastic coating and a spherical probe. *J Phys D: Appl Phys* 2004; 37: 2886–2895.

[40] Manevich LI and Vorobeva NI. On approximate equations of axisymmetrical problem of elasticity theory for transversely isotropic foundation (in Russian). *Soviet Applied Mechanics* 1972; 8(10): 33–38.

[41] Vorovich II, Alexandrov VM and Babeshko VA. *Non-classical Mixed Problems of the Theory of Elasticity (in Russian).* Nauka, Moscow, 1974.

[42] Alexandrov VM and Pozharskiy DA. *Non-classical Spatial Problems of Mechanics of Contact Interaction of Elastic Bodies (in Russian).* Faktorial, Moscow, 1998.

[43] Galanov BA. Nonlinear boundary equations of one-sided boundary value problems in the mechanics of the contact of elastic bodies (in Russian). *Izvestiya Vysshikh Uchebnykh Zavedeniy: Mathematika* 1989; 3: 82–85.

[44] Strel'nitskij VE, Galanov BA, Grigir'ev ON et al. Elastic characteristics of diamond-like carbon coatings. *Diamond and Related Materials* 1993; 2: 869–872.

[45] Kindrachuk VM, Galanov BA, Kartuzov VV et al. On elastic nanoindentation of coated half-spaces by point indenters of non-ideal shapes. *Nanotechnology* 2006; 17: 1104–1111.

[46] Galanov BA and Valeeva IK. Sliding adhesive contact of elastic solids with stochastic roughness. *Int J Eng Sci* 2016; 101: 64–80.

[47] Galanov BA. *Sbornik Nauchnykh Trudov AN USSR, Institut Matematiki*, chapter "On resolvability of boundary equations of one-sided boundary value problems of contact mechanics of coated elastic bodies. Contemporary analysis and applications"(in Russian). Naukova Dumka, Kiev, 1989. pp. 23–30.

[48] Aleksandrova GP. Contact problems of bending of plates resting on elastic foundation (in Russian). *Izvestiya Academii Nauk Mekhanika Tverdogo Tela* 1973; 1: 97–106.

[49] Alexandrov VM. Contact problems on soft and rigid coatings of an elastic half-plane. *Mech Solids* 2010; 45: 34–40.

[50] Goldenveizer AL. *Theory of Elastic Thin Shells (in Russian), 2nd Edn.* Nauka, Moscow, 1976.

[51] Goldenveizer AL. Derivation of an approximate theory of bending of a plate by the method of asymptotic integration of the equations of the theory of elasticity. *PMM J Appl Math Mech* 1962; 26(4): 1000–1025.

[52] Erbaş B, Yusufoğlu E and Kaplunov J. A plane contact problem for an elastic orthotropic strip. *J Eng Math* 2011; 70(4): 399–409.

[53] Borodich FM, Galanov BA, Gorb SN et al. Adhesive contact problems at macro and nano scales. In *Proc. 3rd Europ. Conf. Tribology CD-ROM (ISBN 978-3-901657-39-9)*, volume 1. pp. 15–20.

[54] Hill IJ and Sawyer WG. Energy, adhesion, and the elastic foundation. *Tribol Lett* 2010; 37: 453–461.

[55] Borodich FM, Keer LM and Korach CS. Analytical study of fundamental nanoindentation test relations for indenters of non-ideal shapes. *Nanotechnology* 2003; 14: 803–808.