Supplementary Material for
“Computational Design of Steady 3D Dissection Puzzles”

1 Preliminaries

Mathematical Notation. A puzzle with $n + 1$ pieces are denoted as $\{P_1, P_2, \ldots, P_n, R_n\}$, with $d_i (1 \leq i \leq n)$ represents the extraction direction of the piece $P_i$. Note that $d_i \in \mathbb{D}$, where $\mathbb{D}$ denotes the six axial directions, i.e., $\mathbb{D} = \{-x, +x, -y, +y, -z, +z\}$.

We call a group of puzzle pieces as a piece group $G$ by considering only the blocking relationship between pieces within $G$ and pieces out of $G$ while ignoring the blocking relationship among pieces within $G$. In particular, we denote $R_i$ as a piece group that contains all successive pieces of $P_i$, i.e., $R_i = \{P_{i+1}, P_{i+2}, \ldots, P_n, R_n\}$. Note that a piece group is allowed to contain a single piece only.

Lemma 1. If a piece group $G_i$ is blocked by a piece group $G_j$ in direction $D_i$, then $G_i \cup G_k$ is blocked by $G_j$ in $D_i$ if $G_j \cap G_k = \emptyset$, and $G_i$ is blocked by $G_j \cup G_k$ in $D_i$ if $G_i \cap G_k = \emptyset$.

Lemma 2. If a piece group $G_i$ is blocked by a piece group $G_j$ in $l$ (1 $\leq l \leq 6$) axial directions, then $G_j$ is blocked by $G_i$ in the $l$ opposite axial directions.

These lemmas appear to be straightforward but as we shall soon see, they are helpful in deriving the formal models below.

2 Proof of Basic Formal Model

2.1 Basic Formal Model

Requirements for $P_1$. When constructing $P_1$, it should be immobilized by $R_1$ such that it is only movable along $d_1$.

Requirements for $P_i$ (2 $\leq i \leq n$). When constructing $P_i$, it should satisfy the following requirements, where $S_i$ denotes the set of all neighboring pieces of $P_i$ that have been extracted before $P_i$:

1) $P_i$ should be immobilized by $P_{i-1}$ and $R_i$ such that it is movable only along $d_i$.

2) $R_i$ should block $P_i$ from moving along $d_{i-1}$, if $d_{i-1} \neq d_i$.

3) For each $P_j \in S_i$ and each direction $d' \in \mathbb{D} \setminus \{d_i, d_j\}$, if $R_i$ does not block $P_j$ while $P_i$ blocks $P_j$ from moving along $d'$, then $R_i$ should block $P_i$ from moving along $d'$.

2.2 Proof of Basic Formal Model

Here we prove puzzle pieces constructed by iteratively partitioning $R_{i-1}$ into $P_i$ and $R_i$ according to the requirements of the Basic Formal Model are guaranteed to be generalized interlocking. In other words, we are going to prove that an arbitrary group of pieces in the puzzle $\{P_1, P_2, \ldots, P_n, R_n\}$ is movable at most along one axial direction (i.e., either immobilized or only movable along one axial direction).

To achieve this, we first prove an arbitrary group of pieces in $\{P_1, P_2, \ldots, P_n\}$ is movable at most along one axial direction. It is realized by proving that for each $k (1 \leq k \leq n)$, an arbitrary piece group in $\{P_1, \ldots, P_k\}$ with $P_k$ is movable at most along $d_k$ using the method of mathematical induction (see Statement #1). This is doable because the set of all piece groups in $\{P_1, P_2, \ldots, P_n\}$ is equivalent to the union of the sets of all piece groups in $\{P_1\}$ with $P_2$, $\{P_1, P_2\}$ with $P_3$, ..., $\{P_1, P_2, \ldots, P_n\}$ with $P_{n+1}$, if we ignore duplicated elements in the union of the sets. Next, we generalize the proof to an arbitrary piece group in $\{P_1, P_2, \ldots, P_n, R_n\}$ (see Statement #2).

Statement #1: For each $k$, an arbitrary piece group in $\{P_1, \ldots, P_k\}$ with $P_k$ is movable at most along $d_k$.

Proof:

1) When $k = 1$, as $R_1 (R_1 = \{P_2, \ldots, P_n, R_n\})$ should block $P_1$ in all the other five directions except $d_1$, thus $P_1$ is movable at most along $d_1$.

2) Suppose when $k = i - 1 (2 \leq i \leq n)$, an arbitrary piece group in $\{P_1, \ldots, P_{i-1}\}$ with $P_{i-1}$ is movable at most along $d_{i-1}$. We are going to prove when $k = i$, an arbitrary piece group $G$ in $\{P_1, \ldots, P_i\}$ with $P_i$ is movable at most along $d_i$. To prove this, we classify the statement into two cases:

2.1) $P_{i-1} \notin G$.

According to Requirement 1, $P_i$ should be blocked by $P_{i-1}$ and $R_i$ ($R_i = \{P_{i+1}, \ldots, P_n, R_n\}$) in all the other five directions except $d_i$, thus $G$ is blocked by $P_{i-1}$ and $R_i$ in all the other five directions except $d_i$ due to Lemma 1, namely $G$ is movable at most along $d_i$.

2.2) $P_{i-1} \in G$.

As supposed, piece group $G' = G \setminus \{P_i\}$ with $P_{i-1} \in G'$ is movable at most along $d_{i-1}$. For each other direction
where

\[ D_a \in D \setminus \{d_{i-1}\} \] of group \( G' \), it could also be classified into two cases:

Case 1: \( D_a \) of group \( G' \) is blocked by \( R_i \) or \( \{P_1, ..., P_{i-1}\} \setminus G' \), then \( G' \cup \{P_i\} = G \) is blocked by \( R_i \) or \( \{P_1, ..., P_{i-1}\} \setminus G' \) in \( D_a \) according to Lemma 1.

Case 2: \( D_a \) is enforced by \( P_i \) on its neighbour piece in \( G' \) such as \( P_b \in G' \), and \( R_i \) do not block \( P_b \) in \( D_a \). Note that \( d_b \neq D_a \) as \( P_b \) is extracted before \( P_t \), so \( P_i \) is impossible to block \( P_b \) in \( d_b \). According to Requirement 3, \( R_i \) should block \( P_i \) in \( D_a \) if \( D_a \neq d_i \). Therefore, \( G' \cup \{P_i\} = G \) is blocked by \( R_i \) in \( D_a \) if \( D_a \neq d_i \) according to Lemma 1.

Since case 1 and case 2 cover all possible values of \( D_a \), we induce \( G \) is blocked in \( D \setminus \{d_{i-1}, d_i\} \).

According to Requirement 2, \( R_i \) should block \( P_i \) in \( d_{i-1} \) if \( d_{i-1} \neq d_i \), thus \( G \) is movable at most along \( d_i \) according to Lemma 1.

As 1) and 2) are satisfied, we can infer that for each \( k \), an arbitrary piece group in \( \{P_1, ..., P_k\} \) with \( P_k \) is movable at most along \( d_k \).

\[ \text{Statement } #2: \text{An arbitrary piece group in } \{P_1, P_2, ..., P_n, R_n\} \text{ is movable at most along one axis direction.} \]

\[ \text{Proof:} \]

According to Statement #1, an arbitrary group of pieces \( G \in \{P_1, P_2, ..., P_n\} \) is movable at most along one axis direction. Therefore, the group \( G' \) with all the other pieces should block \( G \) in at least five directions, where \( G' = \{P_1, P_2, ..., P_n, R_n\} \setminus G \), \( G \cap G' = \emptyset \) and \( G \cup G' = \{P_1, P_2, ..., P_n, R_n\} \). According to Lemma 2, \( G \) should also block \( G' \) in at least five directions, namely \( G' \) is movable at most along one axis direction. As \( G \) and \( G' \) cover all the subset of groups in \( \{P_1, P_2, ..., P_n, R_n\} \), we can infer that an arbitrary piece group in \( \{P_1, P_2, ..., P_n, R_n\} \) is movable at most along one axis direction.

\[ \text{Statement } #3: \text{For each } k, \text{ an arbitrary piece group } G \in \{P_1, ..., P_k\} \text{ with } P_k \text{ is movable at most along } D^G, \text{ where } D^G \text{ is the extraction direction of one piece in } G. \]

\[ \text{Proof:} \]

1) When \( k = 1 \), as \( R_1 \) \( (R_1 = \{P_2, ..., P_n, R_n\}) \) should block \( P_1 \) in all the other five directions except \( d_1 \), thus \( P_1 \) is movable at most along \( d_1 \).

2) Suppose when \( k \leq i - 1 \) (\( 2 \leq i \leq n \)), arbitrary piece group \( G'' \in \{P_1, ..., P_k\} \) with \( P_k \) is movable at most along \( D^{G''} \), where \( D^{G''} \) is the extraction direction of one piece in \( G'' \). We are going to prove when \( k = i \), an arbitrary piece group \( G \in \{P_1, ..., P_i\} \) with \( P_i \) is movable at most along \( D^G \), where \( D^G \) is the extraction direction of one piece in \( G \). To prove this, we classify the statement into two cases:

2.1) If \( P_i \) does not adopt any \( P_j \) to stabilize it, then \( R_i \) \( (R_i = \{P_{i+1}, ..., P_n, R_n\}) \) should block \( P_i \) in all the other five directions except \( d_i \), thus \( G \) is blocked by \( R_i \) in all the other five directions except \( d_i \) due to Lemma 1, namely \( G \) is movable at most along \( d_i \).

2.2) \( P_i \) is constructed by adopting \( P_t \) (\( t \leq i - 1 \)) and \( R_t \), which could be classified into two cases:

2.2.1) \( P_t \notin G \).

According to Requirement 1, \( P_t \) should be blocked by \( P_i \) and \( R_i \) in all the other five directions except \( d_i \), thus \( G \) is blocked by \( P_i \) and \( R_i \) in all the other five directions except \( d_i \) due to Lemma 1, namely \( G \) is movable at most along \( d_i \).

2.2.2) \( P_t \in G \).

As supposed, piece group \( G' = G \setminus \{P_i\} \) with \( P_i \in G' \) is movable at most along one direction \( D^{G'} \). We denote \( P_m \) \( (P_m \in G') \) as an arbitrary piece whose extraction direction is \( d_m \) and \( d_m = D^{G'} \).
For each other direction $D_a \in \mathbf{D} \setminus \{d_m\}$ of group $G'$, it could be classified into two cases:

Case 1: $D_a$ is blocked by $R_i$ or $\{P_1, \ldots, P_{i-1}\} \setminus G'$, then $G' \cup \{P_i\} = G$ is blocked by $R_i$ or $\{P_1, \ldots, P_{i-1}\} \setminus G'$ in $D_a$ according to Lemma 1.

Case 2: $D_a$ is applied by $P_i$ on its neighbour piece in $G'$ such as $P_b \in G'$, and $R_i$ do not block $P_b$ in $D_a$. Note that $d_b \neq D_a$ as $P_b$ is extracted before $P_i$, so $P_i$ is impossible to block $P_b$ in $d_b$. According to Requirement 3, $R_i$ should block $P_i$ in $D_a$ if $D_a \neq d_i$. Therefore, $G' \cup \{P_i\} = G$ is blocked by $R_i$ in $D_a$ if $D_a \neq d_i$ according to Lemma 1.

Since case 1 and case 2 cover all possible values of $D_a$, we induce $G$ is movable at most along $\{d_i, d_m\}$.

For the direction $d_i$ and $d_m$, it could also be classified into two cases:

Case 3: $d_i = d_m$, then $G$ is movable at most along $d_i$ or $d_m$.

Case 4: $d_i \neq d_m$, it could be also classified into two subcases:

- Subcase 4.1: $R_i$ block $P_i$ or $P_m$ in $d_i$.
- Subcase 4.2: $R_i$ do not block $P_m$ or $P_i$ in $d_i$. Then according to Requirement 2, $R_i$ should block $P_i$ in $d_m$.

In the above two subcases, $G' \cup \{P_i\} = G$ is blocked by $R_i$ in $d_i$ or $d_m$ according to Lemma 1, namely $G$ is movable at most in $d_i$ or $d_m$.

Since case 3 and case 4 cover all possible values of $d_i$ and $d_m$, we induce $G$ is movable at most along $d_i$ or $d_m$.

As 1) and 2) are satisfied, we can infer that for each $k$, an arbitrary piece group $G$ in $\{P_1, \ldots, P_k\}$ with $P_k$ is movable at most along $D^G$, where $D^G$ is the extraction direction of one piece in $G$.

Statement #4: An arbitrary piece group in $\{P_1, P_2, \ldots, P_n, R_n\}$ is movable at most along one direction.

Proof:

Similar to the proof of Statement #2.