Stability of Solutions of a One-Dimensional p-Laplace Equation with Periodic Potential

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Abstract

From a problem in elasticity that uses a nonlinear stress-strain relation, we derive an equation featuring the one-dimensional p-Laplacian operator with a periodic potential term. This equation is nonlinear, but homogeneous, and we derive a modified Prüfer transform to convert this second-order equation into a two-dimensional first-order system, with radial and angular components. The homogeneity of the original equation is reflected in the complete independence of the angular equation on any radial terms. This allows us to restate conditions of periodic behaviour in terms of the angular component only.

Using these techniques, we compare the nonlinear equation with its linear counterpart, the equation featuring the standard Laplacian operator. This linear equation can also be converted into a first-order system, the linearity of which allows the effect of the equation acting over one period to be written as a constant-valued matrix. This gives a certain structure to the linear equation, which is almost completely absent from the nonlinear case.

The p-Laplace equation with a constant potential has solutions that behave analogously to the trigonometric functions. We detail methods of approximating these functions and their inverses, along with proving accuracy bounds. In turn, we use these to approximate an asymptotic average of the increase in the angular component as $t \to \infty$. This function, called the *rotation number*, is dependent only on a spectral parameter in our equation, and gives information about the stability of the solutions of the equation at that spectral value.

In the linear case, the spectral values that give periodic and anti-periodic behaviour can be characterised exactly as the boundary points on intervals over which this function is constant. These values also separate values of the spectral parameter that give bounded and unbounded behaviour. We shall show that this characterisation is no longer true in the nonlinear case, specifically that periodic behaviour can stem from spectral values inside these intervals, and that the intervals can occur outside of the bounds of the (anti-)periodic spectral values.

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Contents

1 Introduction

We consider the movement of elastic solids, let $\mathbf{u}(t, \mathbf{x})$ be the displacement of the particle initially at the point $\mathbf{x} \in \mathbb{R}^n$ at time $t \geq 0$. Let U be the domain of the solid, then for any $W \subset U$, the force due to the acceleration of the solid over this subdomain is given by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_W \mathbf{p} \; ,$$

where the momentum of the solid, $\mathbf{p} \in \mathbb{R}^3$, is of the form

$$\mathbf{p} := \varrho \frac{\partial u}{\partial t} \; .$$

The constant $\varrho \in \mathbb{R}$ is a density term. The contact force due to the elasticity of the solid is given by

$$\int_{\partial W} \sigma \nu \; \mathrm{dS} \; ,$$

where $\sigma \in \mathbb{R}^{n \times n}$ is the stress acting on the solid. The vector ν is the normal vector to each point on the surface ∂W being summed over. The stress-strain relation of certain materials can be approximated by the linear Hooke's law, that is, the deformation of the solid at any point is a linear transformation of the stress forces acting on it. This means that we can write the stress forces as $\sigma = A \nabla \mathbf{u}$, for some constant tensor, A.

We also consider a potential term that models the external forces acting on the solid, we assume this term is dependent on the spatial variable $\mathbf{x}$ and is linear in the displacement $\mathbf{u}$. Therefore, we take this term to be of the form $-Q(\mathbf{x})\mathbf{u}$, for some matrix-valued function, $Q \in L^1_{\mathrm{loc}}(\mathbb{R}^{n \times n})$, with locally integrable elements. The resultant force due to the external forces acting on the volume W is therefore of the form

$$-\int_W Q\mathbf{u} \; .$$

These forces must be balanced in every volume W over the domain U, therefore, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_W \mathbf{p} = -\int_{\partial W} \sigma\nu \,\mathrm{dS} - \int_W Q\mathbf{u} \,. \tag{1}$$

The negative sign on the stress integral reflects the fact that the stress force is acting in the opposite direction to the acceleration. Next, we transform this integral using the Gauss Divergence Theorem,

$$\int_{\partial W} \sigma\nu \,\mathrm{dS} = \int_W \operatorname{div}_{\mathrm{x}} \sigma \,.$$

Implementing this in (1), we get

$$\int_W \frac{\partial \mathbf{p}}{\partial t} + \operatorname{div}_{\mathrm{x}} \sigma + Q\mathbf{u} = 0 \,,$$

and since this equality holds over every subset W on the domain U, the following equation holds,

$$\frac{\partial \mathbf{p}}{\partial t} = -\operatorname{div}_{\mathrm{x}} \sigma - Q\mathbf{u} \,.$$

Combining this with the expressions for $\mathbf{p}$ and σ results in the linear, second-order PDE

$$\varrho\frac{\partial^2 \mathbf{u}}{\partial t^2} = -\operatorname{div}_{\mathrm{x}}(A\nabla\mathbf{u}) - Q\mathbf{u} \,.$$

Taking $A = -I$, this equation becomes

$$\varrho\frac{\partial^2 \mathbf{u}}{\partial t^2} = \Delta\mathbf{u} - Q\mathbf{u} \,, \tag{2}$$

where the operator $\Delta := \operatorname{div}_{\mathrm{x}} \cdot \nabla$ is the *Laplacian*, an elliptic, linear operator that measures the rate at which the average value of $\mathbf{u}$ in a sphere surrounding the point $\mathbf{x}$ varies with respect to the radius. This operator is well-studied, and has been used in electromagnetics and quantum mechanics, as well as here in elasticity problems. Equations of the form (2) are referred to as *Wave Equations*, and their use as an elasticity model is found in the context of waves in acoustics, optics and fluids. See [12] for a further discussion of the derivation and analysis of the equation (2).

On a one-dimensional spatial domain, i.e. $n = 1$, $x, u \in \mathbb{R}$, this equation models the vibration of a string. Let $\varrho \equiv 1$, the equation (2) can be written as

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - Qu \ ,$$

for $t \geq 0$, $x \in [0, \alpha)$, and some fixed $\alpha > 0$. The classical technique for solving equations of this form is the *separation of variables*, that is, assuming the solution can be written in the form $u(x,t) = v(x)w(t)$. Substituting this into the Wave Equation, we have

$$v(x)w''(t) = v''(x)w(t) - Q(x)v(x)w(t) \ ,$$

and dividing through by $w(t)$,

$$v(x)\frac{w''(t)}{w(t)} = v''(x) - Q(x)v(x) \ .$$

Given that the term ${}^{w''(t)}/_{w(t)}$ does not depend on x, we replace it with the constant $-\lambda \in \mathbb{R}$, and derive the equation

$$v''(x) + (\lambda - Q(x))v(x) = 0 \ . \tag{3}$$

As the potential Q is not necessarily continuous, this equation is understood in the Carathéodory sense, meaning that it is satisfied on the entirety of the domain except on a set of measure zero (see [8][Chapter 2] for a detailed description of existence results regarding this case). We call a function v a *solution* of (3) if it is in the set

$$V = \{f : \mathbb{R} \to \mathbb{R} \,|\, f, f' \in AC_{\text{loc}}(\mathbb{R})\} \ , \tag{4}$$

and satisfies (3) almost everywhere on $\mathbb{R}$. Note that the set $AC_{\text{loc}}(\mathbb{R})$ is the set of locally absolutely continuous functions on $\mathbb{R}$, that is, the set of functions that are absolutely continuous on all compact subsets of $\mathbb{R}$.

The next question is that of boundary conditions. Take the one-dimensional Wave Equation, we have the initial profile of the string at time $t = 0$,

$$u(x, 0) = g(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x, 0) = h(x) \ ,$$

for some $g, h \in L^1_{\text{loc}}(\mathbb{R})$. We can also give conditions on the spatial boundary, that is, at the points $x = 0$ and $x = \alpha$. One of the most common examples are the *Dirichlet Boundary Conditions*, given by

$$u(0, t) = 0 \qquad \text{and} \qquad u(\alpha, t) = 0 \ ,$$

for all $t \geq 0$. These constraints model a situation in which the string is pinned-down to fixed points at both ends. However, in this thesis we consider the *Periodic Boundary Conditions*,

$$u(0, t) = u(\alpha, t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\alpha, t) \ , \tag{5}$$

for all $t \geq 0$. These are used to model the movement of a material over a large scale, that has an internal structure that is periodic on a small scale. We divide the whole space into small *unit cells* and assume that the behaviour of each unit cell is identical throughout the space. Therefore, the solution u must be periodic throughout the space variable x, and given that the Wave Equation is second-order, it is sufficient to impose periodicity on the function and its first derivative. The equation (2), combined with these periodic boundary conditions, is referred to as the *Periodic Problem*.

Naturally, as we are now assuming the string has a periodic structure on the small scale, the initial profiles $g(x)$ and $h(x)$ must also be α-periodic. Similarly, the potential term $Q(x)$ must also be α-periodic for this model to be valid. Any interval $[i\alpha, (i+1)\alpha)$, for any $i \in \mathbb{N}_0$, is referred to as a *period* of our problem.

With regards to the separation of variables detailed above, it suffices to impose the following conditions on v,

$$v(0) = v(\alpha) \qquad \text{and} \qquad v'(0) = v'(\alpha) \ .$$

We now must deduce which values of $\lambda \in \mathbb{R}$ admit non-trivial solutions of the equation (3) with these boundary conditions, such values are called the *periodic eigenvalues* of this periodic problem, and the resulting solutions of (3) that satisfy these boundary conditions are the *periodic eigenfunctions*.

This type of problem was first studied by Sturm and Liouville in 1836, although the problem they studied had separable boundary conditions, rather than periodic boundary conditions, (see [19] for the historical development of these problems). The earliest research focused on the existence of such eigenvalues, the properties of the resulting eigenfunctions, and the possibility of the set of eigenfunctions forming a basis of the periodic functions inside the set $L^2_{\text{loc}}(\mathbb{R})$. The same results have been shown for the periodic problem, and we list the main results below.

Theorem 1.0.1. *Let $Q \in L^1_{\text{loc}}(\mathbb{R})$, consider a second-order linear equation of the form*

$$v'' + (\lambda - Q)v = 0 \ .$$

There exist countably infinitely many periodic eigenvalues,

$$\lambda_0 < \lambda_1 \leq \lambda_2 < \lambda_3 \leq \lambda_4 < \dots \ ,$$

that is, values $\lambda \in \mathbb{R}$ for which there exists a non-trivial solution of the equation with periodic boundary conditions,

$$v(0) = v(\alpha) \qquad \textit{and} \qquad v'(0) = v'(\alpha) \ .$$

Furthermore, the eigenfunction corresponding to the eigenvalue λ_i, for any $i \in \mathbb{N}_0$, has $\lceil i/2 \rceil$ zeros on the interval $[0, \alpha)$.

The details of the proof of this theorem can be found in [7, Chapter 2]. We also state the following result regarding completeness of the periodic eigenfunctions.

Theorem 1.0.2. *Consider the countably infinite set $\{\lambda_0, \lambda_1, \lambda_2, \ldots\}$ of periodic eigenvalues of the equation*

$$v'' + (\lambda - Q)v = 0 \, .$$

For each eigenvalue λ_i , there either exists a one-dimensional eigenspace of α-periodic solutions, or the whole solution space is α-periodic.

For each eigenvalue, take an orthonormal basis of eigenvectors for each corresponding eigenspace, (orthonormal with respect to the standard inner product,

$$\langle \, f, g \, \rangle = \int_0^\alpha f \, g \quad ,$$

for any $f, g \in L^2_{\mathrm{loc}}(\mathbb{R})$). The union of all of these bases forms an orthonormal basis of $L^2_{\mathrm{loc,per}}(\mathbb{R})$; the space of locally integrable, α-periodic functions on $\mathbb{R}$.

Returning to the separation of variables argument, consider the periodic eigenvalues $\lambda_i \in \mathbb{R}$ of the equation (3), for each $i \in \mathbb{N}_0$. We took $w''(t)/w(t) = -\lambda$, therefore for each eigenvalue we have

$$w''(t) + \lambda_i w(t) = 0 \, .$$

Note that the solution space of this equation is spanned by the functions $\sin(\sqrt{\lambda_i}\, t)$ and $\cos(\sqrt{\lambda_i}\, t)$. (We use the informal notation that for any $\lambda_i < 0$, $\sin(\sqrt{\lambda_i}\, t) = \sinh(\sqrt{-\lambda_i}\, t)$ and $\cos(\sqrt{\lambda_i}\, t) = \cosh(\sqrt{-\lambda_i}\, t)$)

As per Theorem 1.0.2, we define $v_i(x)$ as eigenfunction corresponding to λ_i such that the set $\{v_i \mid i \in \mathbb{N}_0\}$ forms an orthonormal basis of $C^1_{\mathrm{per}}(\mathbb{R})$; the space of continuously differentiable, α-periodic functions on $\mathbb{R}$. Then we have

$$u(x,t) = \sum_{i=0}^{\infty} \alpha_i v_i(x) \sin(\sqrt{\lambda_i}\ t) + \beta_i v_i(x) \cos(\sqrt{\lambda_i}\ t)\ ,$$

with

$$\alpha_i = \frac{1}{\sqrt{\lambda_i}} \int_0^{\alpha} v_i(x)\ h(x)\,\mathrm{d}x \qquad \text{and} \qquad \beta_i = \int_0^{\alpha} v_i(x)\ g(x)\,\mathrm{d}x\ .$$

Therefore, every solution to the Wave Equation (2) can be expanded using the periodic eigenfunctions of (3), and an analysis of any solution can be done by considering these functions alone.

We now change the model to consider materials that do not obey Hooke's law, and instead rely on a nonlinear stress-strain relation. Specifically, *Ludwick Solids*, which have a stress-strain relation given by

$$\sigma = K\phi_p(\epsilon)\ ,$$

for some $K > 0$ and $p > 1$, where $\phi_p(\mathbf{x}) = |\mathbf{x}|^{p-2}\mathbf{x}$. For example, High Speed Steel Alloy (HSSA) obeys Ludwick's law with exponent $p = 1.11$, similarly, 304 Stainless Steel has exponent $p = 1.45$. Note that when $p = 2$, we have $\sigma = K\epsilon$, and we return to the linear case of Hooke's law. The paper by Wei [27] gives a review of several types of nonlinear adaptions of the wave equation that stems from this nonlinear relation.

We now adapt our previous model to fit this nonlinear case, we fix some value $p > 1$. The stress matrix $\sigma \in \mathbb{R}^{n\times n}$ will now be defined by

$$\sigma = \phi_p(\nabla\mathbf{u})\ .$$

In the context of relativistic models, particles moving at speeds approaching the speed of light have a nonlinear velocity-momentum relation, we therefore take the momentum to be of the form

$$\mathbf{p} = \varrho \phi_p \left(\frac{\partial \mathbf{u}}{\partial t} \right) \ .$$

Finally, we once again include a potential term, given by $Q(\mathbf{x})\phi_p(\mathbf{u})$.

We previously derived the following equation, relating the momentum, stress, and displacement,

$$\frac{\partial \mathbf{p}}{\partial t} = -\operatorname{div}_{\mathrm{x}} \sigma - Q\mathbf{u} \ .$$

Substituting in the definitions for this new case, we get

$$\varrho \frac{\partial}{\partial t} \left(\phi_p \left(\frac{\partial \mathbf{u}}{\partial t} \right) \right) = -\Delta_p \mathbf{u} - Q\phi_p(\mathbf{u}) \ , \tag{6}$$

where the operator $\Delta_p := \operatorname{div}_{\mathrm{x}}(\phi_p(\nabla \mathbf{u}))$ is the *p-Laplacian*. The monograph by Lindqvist [18] gives a detailed analysis of inequalities and regularity theory associated with this operator over multiple dimensions. We however, return to the case $n = 1$, for which the equation (6) can be written as

$$\varrho \frac{\partial}{\partial t} \left(\phi_p \left(\frac{\partial u}{\partial t} \right) \right) = -\frac{\partial}{\partial x} \left(\phi_p \left(\frac{\partial u}{\partial x} \right) \right) - Q\phi_p(u) \ , \tag{7}$$

for $t \geq 0$, $x \in [0, \alpha)$, where $\alpha > 0$. This equation is effectively a model for the vibration of strings made of certain metal alloys. If these alloys have a periodic structure on the small scale, then once again, we implement periodic boundary conditions

$$u(0,t) = u(\alpha,t) \qquad \text{and} \qquad \frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\alpha,t) \ ,$$

for all $t \geq 0$, as well as the initial string profile conditions

$$u(x,0) = g(x) \qquad \text{and} \qquad \frac{\partial u}{\partial t}(x,0) = h(x) \ ,$$

for some $g, h \in L^1_{\mathrm{loc}}(\mathbb{R})$. Note that we once again assume the functions g, h, and Q are all α-periodic.

Similar to the separation of variables used in the linear model, (2), we now assume that u is of the form $u(x,t) = e^{\omega\, t} v(x)$. Substituting this into our nonlinear wave equation (7), we get

$$\varrho(p-1)|\omega|^p e^{(p-1)\omega\, t}\phi_p(v(x)) = -e^{(p-1)\omega\, t}(\phi_p(v'(x))' - e^{(p-1)\omega\, t}Q(x)\phi_p(v(x)) \;.$$

Dividing through by $e^{(p-1)\omega\, t}$ and simplifying, the equation becomes

$$(\phi_p(v'(x))' + (\lambda - Q(x))\phi_p(v(x)) = 0 \;, \tag{8}$$

where $\lambda := \varrho(p-1)|\omega|^p$. Thus, we have a nonlinear, homogeneous, second-order equation in x, with corresponding boundary conditions

$$v(0) = v(\alpha) \qquad \text{and} \qquad v'(0) = v'(\alpha) \;.$$

Again, this equation is understood in the Carathéodory sense. A function v is a solution if it is in the set

$$V_p = \{f : \mathbb{R} \to \mathbb{R} \,|\, f,\ \phi_p(f') \in AC_{\mathrm{loc}}(\mathbb{R})\} \;,$$

and satisfies (8).

This gives us a nonlinear eigenvalue problem; finding the values of $\lambda \in \mathbb{R}$ for which there exist non-trivial solutions of (8), satisfying the periodic boundary conditions. For any such value, $\tilde{\lambda} \in \mathbb{R}$, let $\tilde{v}$ be a corresponding eigenfunction, then there exists a solution u of (7) of the form

$$u(x,t) = \exp\left(\left(\frac{\tilde{\lambda}}{\varrho(p-1)}\right)^{1/p} t\right)\tilde{v}(x) \;.$$

Unfortunately, as the equation (7) is nonlinear, there is no superposition principle that will allow us to deconstruct any general solution as some combination of all of the solutions that stem from the eigenvalues, as was done in the linear case above. We can however use methods from the Calculus of Variations to recharacterise the periodic eigenvalues of this problem as the set of permissible energies of a Ludwick solid undergoing periodic movement. By the Principal of Minimal Energy, it is valid to say that after energy is continuously applied to such a solid, the movement of the solid in the long term occurs in such a way as to minimise the total remaining energy. Therefore, excluding solutions with eigenfunctions that are orthogonal to the induced movement, the solution of (2) corresponding to the minimal eigenvalue gives the long term behaviour of the solid.

The Dirichlet Energy functional, given by

$$\frac{1}{2}\int_{\Omega} |\nabla u(\mathbf{x})|^2 \, \mathrm{d}\mathbf{x} \ ,$$

for any function $u \in \mathbb{R}^n$, models the kinetic energy of certain elastic materials, see [13, Chapter 10]. For $n = 1$, consider the model of Hookean strings moving periodically, with a unit cell given by the period $[0, \alpha)$. We include an α-periodic potential energy term, Q. The energy model becomes

$$E(v) = \frac{1}{2}\int_0^{\alpha} |v'(x)|^2 - Q(x)\ |v(x)|^2 \, \mathrm{d}x \ ,$$

where $v \in V$ is an α-periodic function. We further subject this functional to the constraint

$$\frac{1}{2}\int_0^{\alpha} |v(x)|^2 \, \mathrm{d}x = 1 \ , \tag{9}$$

to eliminate the trivial solution, which is always a minimiser of the functional $E(v)$. By the method of Lagrangian multipliers, minimising this functional subject to the constraint (9) is equivalent to minimising the functional

$$E^{\lambda}(v) = \frac{1}{2}\int_0^{\alpha} |v'(x)|^2 + (\lambda - Q(x))\ |v(x)|^2 \, \mathrm{d}x \ ,$$

over the space of α-periodic functions, $v \in V$. The Euler-Lagrange equation resulting from this functional is the linear, second-order ODE (3). Therefore, the minimal eigenvalue λ_0 of the resulting periodic problem gives an eigenfunction that minimises the total energy of the string, and this eigenfunction models the movement of the string in x.

Minimising again, after adding the additional constraint,

$$\int_0^{\alpha} v_0(x) v(x) \, \mathrm{d}x = 0 \,,$$

gives the next smallest possible energy of the string, corresponding to the next smallest eigenvalue.

We now adapt this for Ludwick solids. Consider the functional

$$\frac{1}{p} \int_{\Omega} |\nabla u(\mathbf{x})|^p \, \mathrm{d}\mathbf{x} \,,$$

for any function $u \in \mathbb{R}^n$, sometimes referred to as the *p-Dirichlet Energy* functional. This functional has been used to model tug-of-war games in Game Theory [9], the movement of sandpiles [2], and is used in the process of denoising images [16]. Just as the Laplace operator can be derived from the Euler-Lagrange equation stemming from the Dirichlet Energy functional, the Euler-Lagrange equation resulting from this functional is

$$\Delta_p(v) = 0 \,,$$

where Δ_p is the p-Laplace operator that we first saw in equation (6). The model for the energy of a Ludwick string moving periodically, with unit cell given by the period $[0, \alpha)$, is

$$E_p(v) = \frac{1}{p} \int_0^{\alpha} |v'(x)|^p - Q(x) \, |v(x)|^p \, \mathrm{d}x \,,$$

where $v \in V$ is an α-periodic function. As before, we subject this functional to the constraint

$$\frac{1}{p}\int_0^\alpha |v(x)|^p \,\mathrm{d}x = 1 \ ,$$

to eliminate the trivial solution. The minimisers of this constrained functional are the periodic eigenfunctions of (7), and the corresponding eigenvalues are the total energies of the system.

The focus of this thesis is the stability of solutions of the equation

$$(\phi_p(u'(t)))' + (\lambda - Q(t))\phi_p(u(t)) = 0 \ , \tag{10}$$

for $t \geq 0$, $\lambda \in \mathbb{R}$, and $Q \in L^1_{\mathrm{loc}}(\mathbb{R})$, and how this stability is affected as the value λ is changed. The concept of stability here, is defined as the boundedness of a solution over the whole real line. The value λ is the *spectral parameter* of (10), and Q is the *potential.* We choose all potentials, Q, to be α-periodic.

The existence and uniqueness of solutions to Dirichlet problems resulting from such equations was first studied by Ôtani [21], and for a more general class of equations involving the one-dimensional p-Laplacian, by Manásevich and Zanolin [20]. Eigenvalue problems relating to the one-dimensional p-Laplacian have been studied extensively over the last thirty years. Drábek and Robinson [10] showed the existence of eigenvalues of Dirichlet problems relating to equations of the form (10), with extra terms individually dependent on u and t. Walter [26] proved the existence of eigenvalues of the Dirichlet problem relating to the radial p-Laplacian.

Existence results regarding equations with a more general weight term on the spectral parameter were studied by Agarwal, Lü and O'Regan [1], and Huy and Thanh [15]. Multiplicity results of such Dirichlet problems were then proved by Ubilla [25], and Tanaka and Naito [24], who showed the (non-)existence of non-trivial solutions to the Dirichlet problems with a prescribed number of zeros in the domain. Eigenvalue problems of one-dimensional p-Laplace operators with separable boundary conditions were studied by Reichel and Walter [22], [23]. More recently, the spectrum of the periodic problem has been considered by Zhang [30], and Binding and Rynne [4], [5], [6].

We wish to derive results connecting the stability of solutions of (10) to the periodic eigenvalues. When $p = 2$, this equation is linear, and there are many well-known results regarding this connection. We state them briefly here, but [7, Chapters 1 and 2] and [29, Chapter 2, Section 7] covers the topic in more detail.

The first step in the analysis of the equation

$$u''(t) + (\lambda - Q(t))u(t) = 0 \ , \tag{11}$$

is to convert it to a two-dimensional first-order system. Define

$$x := u \qquad \text{and} \qquad y := -u' \ ,$$

then the second-order equation (11) is equivalent to the system $\mathbf{x}' = A\mathbf{x}$, where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \qquad \text{and} \qquad A = \begin{pmatrix} 0 & -1 \\ (\lambda - Q) & 0 \end{pmatrix} .$$

The first result shows how the α-periodicity of A gives structure to certain solutions of the system.

Theorem 1.0.3. *Let Φ be a matrix-valued solution of the system $\boldsymbol{x}' = A\boldsymbol{x}$, with initial value $\Phi(0) = I$. Then for any $t \geq 0$,*

$$\Phi(t + \alpha) = \Phi(t)\Phi(\alpha) \ .$$

The value $\mu \in \mathbb{R}$ is an eigenvalue of the matrix $\Phi(\alpha)$ if and only if there is a non-trivial solution, u of (11), with

$$u(t + \alpha) = \mu \, u(t) \ ,$$

for all $t \in \mathbb{R}$.

The matrix $\Phi(\alpha)$ is called the *Monodromy matrix* of the system. An analysis of the characteristic equation of $\Phi(\alpha)$ allows the stability of solutions of (11) to be characterised by the trace of this matrix. As the equation (11) is often referred to as *Hill's equation*, we call the trace of the Monodromy matrix, the *Hill's discriminant* of the system, and denote it $D(\lambda)$.

Theorem 1.0.4. *Let $D(\lambda)$ be the Hill's discriminant of the equation* (11)*.*

- *If $|D(\lambda)| > 2$, there exist two solutions of the equation* (11) *with*

$$u_1(t) = e^{C\,t}\, g_1(t) \qquad \textit{and} \qquad u_2(t) = e^{-C\,t}\, g_2(t)\ ,$$

  for all $t \geq 0$, for some $C \in \mathbb{R}$, and α-periodic functions $g_1, g_2 \in V$ (for V defined in (4)*).*

- *If $|D(\lambda| < 2$, there exist two solutions of the equation* (11) *with*

$$\begin{aligned} & u_1(t) = \cos(C\,t)\; g_1(t) - \sin(C\,t)\; g_2(t)\ , \\ \textit{and} \qquad & u_2(t) = \cos(C\,t)\; g_2(t) + \sin(C\,t)\; g_1(t)\ , \end{aligned}$$

  for all $t \geq 0$, for some $C \in \mathbb{R}$, and α-periodic functions $g_1, g_2 \in V$.

- *If $D(\lambda) = 2$, either all solutions of* (11) *are α-periodic, or there exists an α-periodic solution, u_1 and a second solution of the form*

$$u_2(t) = g(t) + C\; t\; u_1(t)\ ,$$

  for all $t \geq 0$, for some $C \in \mathbb{R}$, and α-periodic function $g \in V$.

- *If $D(\lambda) = -2$, then the solutions behave the same as in the previous case, except we have α-antiperiodicity of at least one solution, instead of α-periodicity.*

Solutions of (11) with $|D(\lambda)| > 2$ are called *Floquet solutions*. By Theorem 1.0.3, they have the property that

$$u(t+\alpha) = \mu\, u(t) \ ,$$

for some $\mu \in \mathbb{R}$. This value μ is the *Floquet multiplier* of the solution. In Chapter 4, we re-derive this type of solution for the nonlinear equation (10). In the linear case, $p = 2$, we know that for each $\lambda \in \mathbb{R}$ such that $|D(\lambda)| > 2$, there exists only a single pair of Floquet solutions. However, for $p \neq 2$, given any $n \in \mathbb{N}$, we prove conditions under which it is possible to construct potentials for which there are n pairs of Floquet solutions, each pair with its own set of distinct multipliers.

Theorem 1.0.4 tells us that the periodic eigenvalues of the equation (11) can be characterised as values λ such that $D(\lambda) = 2$. Similarly, the anti-periodic eigenvalues can be characterised as values λ with $D(\lambda) = -2$. Solutions of (11) in intervals in λ for which $|D(\lambda)| < 2$ are bounded on $\mathbb{R}$, and the solutions in intervals of λ for which $|D(\lambda)| > 2$ are unbounded, hence, we refer to such intervals in λ as *stability intervals* and *instability intervals*, respectively. By Theorem 1.0.4, stability and instability intervals must be separated by (anti-)periodic eigenvalues.

We next consider the functional properties of the Hill's discriminant, to determine whether eigenvalues can fall in between two instability intervals.

Theorem 1.0.5. *Let $D(\lambda)$ be the Hill's discriminant of the equation* (11)*. At any value λ such that $D(\lambda) < 2$, D is strictly monotonic. For any λ such that $|D(\lambda)| = 2$, if $D'(\lambda) = 0$ then $D(\lambda)D''(\lambda) < 0$, and all solutions of* (11) *for this λ are α-periodic.*

This theorem tells us that periodic eigenvalues cannot exist in between two instability intervals, and can only separate instability and stability intervals, or two stability intervals. If a periodic eigenvalue existed between two instability intervals, then for this value λ, we have $D(\lambda) = 2$, and $D'(\lambda) = 0$. So by Theorem 1.0.5, $D''(\lambda) < 0$ and so in a neighbourhood to the right of λ, $D < 2$, and so this neighbourhood is in a stability interval.

Given that our p-Laplace equation, (10), is non-linear, we cannot apply any of the above results. It is possible to convert this equation into a two-dimensional first-order system, but as this system is nonlinear, there does not exist a Monodromy matrix, and as a result, there is no Hill's discriminant. However, further analysis of equations of the type (11) has also been derived using Oscillation Theory. This uses the *Prüfer Transform*, given by

$$x = r\cos\theta \qquad \text{and} \qquad y = r\sin\theta \, . \tag{12}$$

If we consider the original solution components x, y as being coordinates in the phase-plane, this transform effectively gives the equivalent solution in terms of a radial component r, and and angular component, θ. We substitute the transformation (12) into the system $\mathbf{x}' = A\mathbf{x}$, and derive the equations

$$\begin{cases} r' = r(\lambda - Q - 1)\cos\theta \, \sin\theta \\ \theta' = 1 + (\lambda - Q - 1)\cos^2\theta \, . \end{cases}$$

Although this transformed system is nonlinear, it has the advantage that the latter equation is not dependent on the radial term, r. This reflects the homogeneity present in the original equation (11), that is, for any given solution u of (11), the function $C\,u$ (for any $C \in \mathbb{R}$) is also a solution. Given that $(C\,u)' = C\,u'$, we also have

$$C\,u(0) = C\,u(\alpha) \qquad \text{and} \qquad C\,u'(0) = C\,u'(\alpha) \, ,$$

and so rescaling the solution by a scalar factor, C, does not affect whether the solution satisfies the periodic boundary conditions.

Therefore, the periodicity (or lack thereof) of any solution cannot be affected by a simple rescaling. Note that this technique can be applied to the nonlinear equation (10), as nothing about this approach relies on linearity, only homogeneity, and the equation (10) is also homogeneous.

After removing any consideration of the radial term r, an analysis of the angular term, θ, leads to the definition of a function called the *rotation number*. This is defined using θ, and only depends on the spectral parameter, λ.

Definition 1.0.1. *For any angular component θ to a solution of* (11)*, the rotation number of the solution is given by*

$$\rho(\lambda) := \lim_{t\to\infty} \frac{\theta(t,\lambda)}{t} \ ,$$

for all $\lambda \in \mathbb{R}$.

The rotation number gives an normalised, asymptotic average of the increase in θ over each period. Note that whenever $\theta = {}^{\pi}\!/_{2}$, the corresponding solution, $u = r\cos({}^{\pi}\!/_{2}) = 0$. Therefore, an increase in θ by π over an interval, shows the existence of a zero of the solution u inside the interval. Thus, the rotation number can also be viewed as an normalised average of the number of zeros in all periods of a given length. We now state a result linking the stability of solutions of (11) with the values of ρ.

Theorem 1.0.6. *Let ρ be the rotation number resulting from the angular component, θ, of the system $\boldsymbol{x}' = A\boldsymbol{x}$ after a Prüfer transform. On any interval of λ in which $|D(\lambda)| < 2$, the function ρ is strictly monotonically increasing. On any interval of λ in which $|D(\lambda)| > 2$, the function ρ is constant.*

The appearance of a graph of this function is therefore that of a function, increasing without bound, with intervals in λ over which ρ is constant. The maximal intervals of constancy of ρ are referred to as *plateaus* of the rotation number. We have already seen that the point that separates stability and instability intervals is an (anti-)periodic eigenvalue, therefore, the end-points of these plateaus are (anti-)periodic eigenvalues. We have also shown that no periodic eigenvalues can occur in between two instability intervals, and as such, there are no periodic eigenvalues in the interior of the plateaus. It is however possible that there are periodic eigenvalues that are found between two stability intervals, the rotation number can also be used to characterise these.

Theorem 1.0.7. *Let ρ be the rotation number resulting from the angular component, θ, of the system $\boldsymbol{x}' = A\boldsymbol{x}$ after a Prüfer transform. For any periodic eigenvalue, λ, of this system,*

$$\rho(\lambda) = \frac{2n\pi}{\alpha} ,$$

for some $n \in \mathbb{N}_0$, where n is the number of zeros that the first component of $\boldsymbol{x}$ attains over each period of length α

By Theorem 1.0.6, if a periodic eigenvalue is found between two stability intervals, the image of the function ρ is locally strictly monotonic. Hence, the eigenvalue can be evaluated as the boundary points on λ, of the preimage of the rotation number at the value $2n\pi\alpha^{-1}$, the aforementioned plateaus. The generalisation of this theory to the nonlinear equation (10) was first derived by Zhang [30]. In that paper, it was shown that the properties of $\rho(\lambda)$ shown in Theorems 1.0.6 and 1.0.7 also hold for the nonlinear case.

Zhang conjectured however, that just as in the linear case, there could be no periodic eigenvalues in the interior of the plateaus. This was disproven by Binding and Rynne [4] who showed that for any $m \in \mathbb{N}$, there exists a potential Q for which a plateau at any given multiple of $2\pi\alpha^{-1}$ has m periodic eigenvalues in its interior. In Chapter 2.5, we re-derive this result by analysing the effect of a perturbation on the potential.

We briefly mentioned earlier that as well as plateaus at multiples of $2\pi\alpha^{-1}$, the end-points of which are the periodic eigenvalues, there can also exist plateaus at odd multiples of $\pi\alpha^{-1}$, and the end-points of these plateaus correspond to the anti-periodic eigenvalues. The remaining question is whether there can exist plateaus at levels outside of integer multiples of $\pi\alpha^{-1}$.

We start by noting the connection between the rotation number, and the periodic eigenvalue problem over several periods, the so-called *iterated periodic problem.* A function u is a solution to the iterated periodic problem over m periods, for some $m \in \mathbb{N}$, if u solves (10) and satisfies the conditions

$$u(0) = u(m\alpha) \qquad \text{and} \qquad u'(0) = u'(m\alpha)\ .$$

Theorem 1.0.8. *Let ρ be the rotation number resulting from the angular component, θ, of the system $\boldsymbol{x}' = A\boldsymbol{x}$ after a Prüfer transform. For any $m \in \mathbb{N}$, all values, λ of the spectral parameter, for which there exists a solution, u, that satisfies the iterated periodic problem over m periods has rotation number*

$$\rho(\lambda) = \frac{2n\pi}{m\alpha}\ ,$$

for some $n \in \mathbb{N}_0$, where n is the number of zeros that the first component of $\boldsymbol{x}$ attains over each period of length $m\alpha$.

From this theorem, we deduce that for a plateau to exist at a rational, non-integer, multiple of $\pi\alpha^{-1}$, there must exist two distinct spectral values λ, such that the resulting solutions are $m\alpha$-periodic, for some $m > 2$, each with n zeros in the interval $[0, m\alpha)$. We now introduce a result that connects the periodic problem over m periods, to the periodic problem over a single period.

Theorem 1.0.9. *For any $m \in \mathbb{N}$, the value $\lambda \in \mathbb{R}$ is a periodic eigenvalue of the iterated periodic problem over m periods (i.e., there exists a solution, u, that satisfies*

$$u(0) = u(m\alpha) \qquad \textit{and} \qquad u'(0) = u'(m\alpha) \quad)$$

if and only if there exists an $\omega \in \mathbb{C}$ such that $\omega^m = 1$, with

$$u(0) = \omega\, u(\alpha) \qquad \textit{and} \qquad u'(0) = \omega\, u'(\alpha)\ .$$

The case $m = 1$ simply gives the periodic eigenvalues, and so $\omega = 1$. For the case $m = 2$, we have $\omega^2 = 1$, and so $\omega = \pm 1$. This shows that for any λ that gives periodicity after two periods, either $\omega = 1$, and λ is a periodic eigenvalue, or $\omega = -1$, and λ is an anti-periodic eigenvalue. We know that for the cases $m \leq 2$, the eigenvalues can exist in pairs, each with geometric multiplicity one. However, for any $m > 2$, there exist values ω with non-zero imaginary parts.

Let u be a periodic eigenfunction corresponding to one such ω, then as there are no complex terms in the equation (11), the conjugate of this solution, $\bar{u}$ is also a solution. As the ratios $\omega = u(\alpha)(u(0))^{-1}$ and $\bar{\omega} = \bar{u}(\alpha)(\bar{u}(0))^{-1}$ are distinct, the two solutions are linearly independent. By Theorem 1.0.9,

$$\bar{u}(0) = \bar{u}(m\alpha) \qquad \text{and} \qquad \bar{u}'(0) = \bar{u}'(m\alpha)\ .$$

Taking the linear combinations

$$\frac{u+\bar{u}}{2} \qquad \text{and} \qquad \frac{u-\bar{u}}{2i}\ ,$$

results in two linearly independent, real-valued, $m\alpha$-periodic solutions of the equation (3). Therefore, there exist two linearly independent periodic eigenfunctions of the problem over m periods. This means that the eigenvalue corresponding to these eigenfunctions has geometric multiplicity two, and by Theorem 1.0.5, this eigenvalue exists between two stability intervals. By Theorem 1.0.6, ρ is strictly increasing in a neighbourhood of this eigenvalue, and therefore, there is no plateau at this level.

This technique however, does not work for the nonlinear equation, (10). For example, the linear combinations taken above to produce the two real-valued solutions would not produce a solution in the nonlinear case, as the superposition principle becomes invalid. In fact, in Chapter 5 we show that for any rational value ${}^{n}/_{m}$, there exist potentials Q, such that there exists an 'extra' plateau at the level ${}^{Cn}/_{m}$, for some fixed value C. This highlights the many qualitative differences in the structure of the spectra of the linear problem and the nonlinear problem.

2 Background and Auxiliary Results

As mentioned in Chapter 1, a key tool for analysing the spectrum (and hence the stability of solutions) of the equation (11) is the Prüfer Transform. This transform is defined using the sin and cos functions. However, in the context of the equation (10), such a transform is unsatisfactory. The goal is to find solutions in terms of a radial and an angular component, where the angular component reflects the terms inside the argument of the sin and cos functions, themselves solutions in the case $\lambda - Q \equiv 1$. But sin and cos are no longer solutions to the equation (10) when $p \neq 2$, so we must redefine the Prüfer transform in terms of some analagous functions.

2.1 The $\sin_p$ and $\cos_p$ Functions

To derive this *p-Prüfer Transform* for the nonlinear equation (10), we first define the $\sin_p$ and $\cos_p$ functions. They are defined as solutions to (10) with $\lambda - Q \equiv 1$, and become the standard sin and cos functions for the linear case, $p = 2$. The existence of such solutions was first proved by Ôtani [21], and Lindqvist [17] derived several results regarding their inverses, derivatives, and algebraic connections with other solutions that parallel the properties of the standard trigonometric functions.

Definition 2.1.1. *The solution of the equation* (10), *with* $\lambda = 1$ *and* $Q \equiv 0$, *and initial conditions* $u(0) = 0,\ u'(0) = 1$, *is called the* $\sin_p$ *function. The solution with initial conditions* $u(0) = (p-1)^{1/p},\ u'(0) = 0$, *is called the* $\cos_p$ *function.*

For the derivations below, we introduce a value

$$q = \frac{p}{p-1}\ ,$$

called the *conjugate exponent* of p. Such a construction is familiar from the duals of L^p spaces, and throughout this thesis, we will use the properties

$$\frac{1}{p} + \frac{1}{q} = 1 \qquad , \qquad (p-1)(q-1) = 1 \ ,$$

without comment. One such consequence of the latter identity is that

$$\phi_p \circ \phi_q = \mathrm{id} \ ,$$

that is, ϕ_p and ϕ_q are inverses. It is this property that causes the value q to appear so frequently in the following derivations.

It is worth mentioning that the concept of generalised p-trigonometric functions has been extensively studied, and that several variants of the $\sin_p$ and $\cos_p$ functions have appeared in the literature, each with different results. The papers [28] and [11] give a good overview of these variants.

The $\sin_p$ and $\cos_p$ functions are periodic. For the linear case, $p = 2$, there exists an integral formulation of the inverses of sin and cos, from which the value of π can be derived. We generalise this here, to define values π_p, and show that for any fixed $p > 1$, the solutions $\sin_p$ and $\cos_p$ are $2\pi_p$-periodic.

Definition 2.1.2. *For any $p > 1$, we define the value π_p as*

$$\pi_p := 2 \int_0^{(p-1)^{1/p}} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \ .$$

In Chapter 3, we derive numerical schemes for certain p-trigonometric functions. Thus, it will be necessary to approximate the values π_p, for any given p. Fortunately, there exists a closed form expression for such values.

Lemma 2.1.1. *For any $p > 1$,*

$$\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin(\pi/p)} \ .$$

Proof. We make the substitution $r = s^p/(p-1)$, from which we get

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \frac{p\, s^{p-1}}{p-1} \; .$$

This transforms the integral expression in the definition of π_p as follows,

$$\pi_p := 2 \int_0^{(p-1)^{1/p}} \frac{\mathrm{d}r}{\left(1 - \left({|r|^p}/{(p-1)}\right)\right)^{1/p}} = \frac{2(p-1)}{p\,(p-1)^{1/q}} \int_0^1 \frac{\mathrm{d}r}{r^{1/q}\,(1-r)^{1/p}} \; .$$

By the definition of the Beta function,

$$B(x,y) := \int_0^1 t^{x-1}\,(1-t)^{y-1}\,\mathrm{d}t \; ,$$

we have,

$$\pi_p = \frac{2(p-1)^{1/p}}{p}\, B({1}/{p}, {1}/{q}) \; .$$

We use the following well known relation between the Beta function and the Gamma function,

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)} \; ,$$

to get

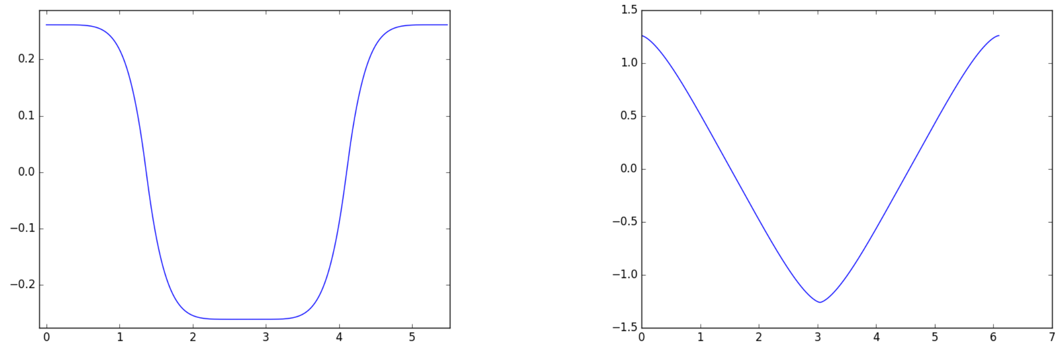
$$\pi_p = \frac{2(p-1)^{1/p}}{p} \frac{\Gamma({1}/{p})\,\Gamma({1}/{q})}{\Gamma({1}/{p} + {1}/{q})} = \frac{2(p-1)^{1/p}}{p} \frac{\Gamma({1}/{p})\,\Gamma(1 - {1}/{p})}{\Gamma(1)} \; .$$

Finally, by the Euler Reflection Formula, and the fact that $\Gamma(1) = 0! = 1$, we have

$$\pi_p = \frac{2(p-1)^{1/p}}{p} \frac{\pi}{\sin({\pi}/{p})} \; .$$

$\square$

Using the symmetry of the Beta function, we also note the equality



(a) $\cos_{1.2}(t)$ on the domain $[0, 2\pi_{1.2}]$.

(b) $\cos_3(t)$ on the domain $[0, 2\pi_3]$.

Figure 1: Examples of $\cos_p$ functions.

$$\pi_p = \frac{2(p-1)^{1/p}}{p} B(1/p, 1/q) = \frac{2(p-1)^{1/p-1}}{q} B(1/p, 1/q) = \frac{2(q-1)^{1/q}}{p} B(1/q, 1/p) = \pi_q \ ,$$

where q is the conjugate exponent of p. In order to show that the functions $\sin_p$ and $\cos_p$ are $2\pi_p$-periodic, we observe the following result regarding the inverses of these functions.

Theorem 2.1.1. *Let $t \in [0, \pi_p/2)$. The function $\sin_p$ satisfies the equation*

$$t = \int_0^{\sin_p(t)} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \ ,$$

and $\cos_p$ satisfies,

$$t = \int_{\cos_p(t)}^{(p-1)^{1/p}} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \ .$$

Proof. Consider the equation (10), with $Q \equiv 0$ and $\lambda = 1$,

$$(\phi_p(u'))' + \phi_p(u) = 0 \ .$$

Multiplying through by u', we get

$$(|u'|^{p-2}u')'u' + |u|^{p-2}uu' = 0 \ ,$$

and by the product rule,

$$(|u'|^{p-2}(u')^2)' - |u'|^{p-2}u'u'' + |u|^{p-2}uu' = 0 \ .$$

Next, we integrate between both sides of this equation between 0 and t,

$$\int_0^t (|u'|^{p-2}(u')^2)' \ - \int_0^t |u'|^{p-2}u'u'' \ + \int_0^t |u|^{p-2}uu' = 0 \ , \tag{13}$$

and we can evaluate each of these individual integrals as:

$$\int_0^t (|u'|^{p-2}(u')^2)' = \int_0^t (|u'|^p)' = |u'(t)|^p - |u'(0)|^p \ ,$$

$$\int_0^t |u'|^{p-2}u'u'' = \int_{u'(0)}^{u'(t)} |s|^{p-2}s \ \mathrm{d}s = \frac{|u'(t)|^p}{p} - \frac{|u'(0)|^p}{p} \ ,$$

$$\int_0^t |u|^{p-2}uu' = \int_{u(0)}^{u(t)} |s|^{p-2}s \ \mathrm{d}s = \frac{|u(t)|^p}{p} - \frac{|u(0)|}{p} \ .$$

Substituting all of these into the equation (13), we get

$$|u'(t)|^p \left(1 - \frac{1}{p}\right) + \frac{|u(t)|^p}{p} = C \ ,$$

with $C \in \mathbb{R}$, given by

$$C = |u'(0)|^p - \frac{|u'(0)|^p}{p} + \frac{|u(0)|}{p} \ .$$

We consider first the $\sin_p$ function, substituting in $u(0) = 0$, $u'(0) = 1$, it follows that

$$C = 1 - \frac{1}{p} \ ,$$

and therefore,

$$|\sin_p'(t)|^p + \frac{|\sin_p(t)|^p}{p-1} = 1 \ . \tag{14}$$

Similarly, for the $\cos_p$ function, if we substitute the initial conditions $u(0) = (p-1)^{1/p}$, $u'(0) = 0$, we get

$$C = \frac{p-1}{p} = 1 - \frac{1}{p} \ ,$$

and similarly to (14), we have

$$|\cos_p'(t)|^p + \frac{|\cos_p(t)|^p}{p-1} = 1 \ . \tag{15}$$

When $u = \sin_p$, we have $u(0) = 0$ and $u'(0) = 1$, hence $u > 0$ and $u' > 0$ for some neighbourhood to the right of zero. Therefore, the equation for the $\sin_p$ function can be written as

$$\sin_p' = \left(1 - \frac{|\sin_p|^p}{(p-1)}\right)^{1/p} \ ,$$

and dividing through by the right-hand side gives us,

$$1 = \frac{\sin_p'}{\left(1 - \left(|\sin_p|^p/p-1\right)\right)^{1/p}} \ .$$

Finally, integrating this between 0 and t, we get

$$\begin{aligned} t = \int_0^t 1 &= \int_0^t \frac{\sin_p'}{\left(1 - \left(|\sin_p|^p/(p-1)\right)\right)^{1/p}} \\ &= \int_{\sin_p(0)}^{\sin_p(t)} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \\ &= \int_0^{\sin_p(t)} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \ , \end{aligned}$$

and this equality holds for all t between 0 and the value $\bar{t}$ that gives $\sin_p(\bar{t}) = (p-1)^{1/p}$, at which point the integrand becomes singular. By Definition 2.1.2, this point is $\bar{t} = \pi_p/2$.

Similarly, for $u = \cos_p$, we have $u(0) = (p-1)^{1/p}$, hence $u > 0$ for some neighbourhood of zero. Therefore, the equation (15) can be written as

$$\cos_p' = -\left(1 - \frac{|\cos_p|^p}{p-1}\right)^{1/p} ,$$

and dividing through by $\cos_p'$ gives us,

$$1 = \frac{-\cos_p'}{\left(1 - \left(|\cos_p|^p/(p-1)\right)\right)^{1/p}} .$$

and integrating between 0 and t, we get

$$\begin{aligned} t = \int_0^t \frac{-\cos_p'}{\left(1 - \left(|\cos_p|^p/(p-1)\right)\right)^{1/p}} &= \int_{\cos_p(0)}^{\cos_p(t)} \frac{-1}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} \, \mathrm{d}s \\ &= \int_{\cos_p(t)}^{(p-1)^{1/p}} \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} , \end{aligned}$$

and again, this equality holds for all values $t \in [0, \pi_p/2]$, at which point $\cos_p = 0$. If we let $s \in [0, (p-1)^{1/p})$, then $|s|^p/(p-1) < 1$ and the above integrand,

$$\frac{1}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} ,$$

is a positive and continuous function, therefore

$$\int_0^u \frac{\mathrm{d}s}{\left(1 - \left(|s|^p/(p-1)\right)\right)^{1/p}} ,$$

defines a continuous, strictly increasing function for $u \in [0, (p-1)^{1/p}]$, with range $[0, \pi_p/2]$, and this gives us the implicit formulas for $\sin_p(t)$ and $\cos_p(t)$, for all $t \in [0, \pi_p/2]$.

□

The reason for generalising the Prüfer transform is to derive a first-order system from the equation (10). To do this, we will need to differentiate $\sin_p$ and $\cos_p$ functions. Ôtani [21] proved the following result regarding the differentiability of these functions.

Theorem 2.1.2. *For* $1 < p < 2$*;* $\sin_p$, $\cos_p \in C^2(\mathbb{R})$*. For* $p > 2$*;* $\sin_p$, $\cos_p \in C^1(\mathbb{R})$.

This can be proved through explicit computation of the derivatives. In fact, it can be shown that the function $\sin_p$ is real-analytic outside of the set $\{{}^{k\pi_p}/_2 \mid k \in \mathbb{Z}\}$. For $1 < p < 2$, it fails to have a continuous third derivative at multiples of π_p, and for $p > 2$, it fails to have a continuous second derivative at odd multiples of ${}^{\pi_p}/_2$. We now find explicit formulas for the derivatives of $\sin_p$ and $\cos_p$, for use in our generalised Prüfer transform.

Theorem 2.1.3. *For any* $t \in \mathbb{R}$*,*

$$\sin_p'(t) = (p-1)^{1/p}\phi_q(\cos_q(t)) \qquad \textit{and} \qquad \cos_p'(t) = -(p-1)^{1/p}\phi_q(\sin_q(t)) \ .$$

Proof. We show that $\sin_p' = (p-1)^{1/p}\phi_q(\cos_q)$, or equivalently, $(q-1)^{1/q}\phi_p(\sin_p') = \cos_q$. To do this, we show that both sides of the equality satisfy the same IVP. By definition, the function $\cos_q$ satisfies

$$(\phi_q(\cos_q'))' + \phi_q(\cos_q) = 0 \ ,$$

and similarly,

$$\begin{aligned}(\phi_q((q-1)^{1/q}(\phi_p(\sin_p'))'))' + \phi_q((q-1)^{1/q}\phi_p(\sin_p')) &= (q-1)^{1/p}((\phi_q((\phi_p(\sin_p'))'))' + \phi_q(\phi_p(\sin_p'))) \\ &= (q-1)^{1/p}(\phi_q(-\phi_p(\sin_p))' + \sin_p') \\ &= (q-1)^{1/p}(-\sin_p' + \sin_p') = 0 \ ,\end{aligned}$$

where we have used the fact that the function $\sin_p$ satisfies

$$(\phi_p(\sin_p'))' + \phi_p(\sin_p) = 0 \ .$$

Also, for the initial values; by definition, $\sin_p'(0) = 1$, and we have

$$(p-1)^{1/p}\phi_q(\cos_q(0)) = (p-1)^{1/p} \cdot (q-1)^{1/q} = 1 \ ,$$

so both functions have equal initial values. Also, for the initial values of the derivatives, from the definition of $\sin_p$,

$$(\phi_p(\sin_p'))'(0) = -\phi_p(\sin_p(0)) = -\phi_p(0) = 0 \ ,$$

and similarly,

$$\begin{aligned}((\phi_p((p-1)^{1/p}\phi_q(\cos_q))'(0) &= (p-1)^{1/q}(\phi_p(\phi_q(\cos_q)))'(0) \\ &= (p-1)^{1/q}\cos_q'(0) \\ &= (p-1)^{1/q} \cdot 0 = 0 \ .\end{aligned}$$

Thus, both $\sin_p'$ and $(p-1)^{1/p}\phi_q(\cos_q)$ satisfy the same IVP, and therefore must coincide everywhere. The proof is similar for the derivative of $\cos_p$.

□

Using this result, we can show that the $\sin_p$ and $\cos_p$ functions are both $2\pi_p$-periodic. Theorem 2.1.1 gives an implicit formula for the $\sin_p$ and $\cos_p$ functions on $[0, \pi_p/2]$. At the point $\hat{t} = \pi_p/2$, we have $\sin_p(\hat{t}) = (p-1)^{1/p}$, and by Theorem 2.1.3,

$$\sin_p'(\hat{t}) = (p-1)^{1/p} \cos_q(\hat{t}) = 0 \ .$$

So the zeroth and first derivatives of $\sin_p$ at the point $t = \pi_p/2$ are equal to the zeroth and first derivatives of $\cos_p$ at the point $t = 0$. Alternatively, the functions $\sin_p(t + \pi_p/2)$ and $\cos_p(t)$ have the same initial conditions, and as the equation (10) is autonomous when $\lambda - Q \equiv 1$, both satisfy (10). Therefore,

$$\sin_p(t + {}^{\pi_p}\!/\!_2) = \cos_p(t) \ , \tag{16}$$

for all $t \in \mathbb{R}$.

Similarly, for $\cos_p$, at the point $\hat{t} = {}^{\pi_p}\!/\!_2$, we have $\cos_p(\hat{t}) = 0$, and

$$\cos'_p(\hat{t}) = -(p-1)^{1/p}\,\phi_q(\sin_q(\hat{t})) = -(p-1)^{1/p}\,(q-1)^{1/q} = -1 \ .$$

Therefore, the zeroth and first derivatives of the $\cos_p$ function at the point $t = {}^{\pi_p}\!/\!_2$ are equal to the zeroth and first derivatives of the function $\sin_p$ at the point $t = 0$, multiplied by minus one. Or alternatively, the functions $\cos_p(t + {}^{\pi_p}\!/\!_2)$ and $-\sin_p(t)$ have the same initial conditions, and as the equation (10) is autonomous and homogeneous, both satisfy (10). Therefore,

$$\cos_p(t + {}^{\pi_p}\!/\!_2) = -\sin_p(t) \ ,$$

for all $t \in \mathbb{R}$. Combining these last two equalities together, we see that for all $t \in \mathbb{R}$,

$$\sin_p(t + \pi_p) = \sin_p((t + {}^{\pi_p}\!/\!_2) + {}^{\pi_p}\!/\!_2) = \cos_p(t + {}^{\pi_p}\!/\!_2) = -\sin_p(t) \ ,$$

which is similarly valid for the function $\cos_p$,

$$\cos_p(t + \pi_p) = -\cos_p(t) \ .$$

Finally, we use this last result to derive the $2\pi_p$-periodicity of both of these functions,

$$\sin_p(t + 2\pi_p) = \sin_p((t + \pi_p) + \pi_p) = -\sin_p(t + \pi_p) = \sin_p(t) \ .$$

Which, again, is also valid for $\cos_p$,

$$\cos_p(t + 2\pi_p) = \cos_p(t) \ .$$

As a result of this, and Theorem 2.1.1, we can define the functions

$$\begin{aligned} &\arcsin_p : [-(p-1)^{1/p}, (p-1)^{1/p}] \to [-{}^{\pi_p}\!/2, {}^{\pi_p}\!/2] \\ \text{and} \qquad &\arccos_p : [-(p-1)^{1/p}, (p-1)^{1/p}] \to [0, \pi_p] \ , \end{aligned}$$

by the formulas

$$\arcsin_p(t) = \int_0^t \frac{\mathrm{d}s}{\left(1 - \left({}^{|s|^p}\!/_{(p-1)}\right)\right)^{1/p}} \qquad , \qquad \arccos_p(t) = \int_t^{(p-1)^{1/p}} \frac{\mathrm{d}s}{\left(1 - \left({}^{|s|^p}\!/_{(p-1)}\right)\right)^{1/p}} \ .$$

We also have the following analogue of the Pythagorean identity, $\sin^2 + \cos^2 = 1$, for this nonlinear case.

Theorem 2.1.4. *For any $t \in \mathbb{R}$, we have:*

$$\frac{|\sin_p(t)|^p}{p-1} + \frac{|\cos_q(t)|^q}{q-1} = 1 \ . \tag{17}$$

Proof. Differentiating the left-hand side of (17) with respect to t, we get

$$\begin{aligned} \left(\frac{|\sin_p(t)|^p}{p-1} + \frac{|\cos_q(t)|^q}{q-1}\right)' &= \frac{p \cdot \phi_p(\sin_p(t))}{p-1} \cdot (p-1)^{1/p} \phi_q(\cos_q(t)) \\ &\quad + \frac{q \cdot \phi_q(\cos_q(t))}{q-1} \cdot \left(-(q-1)^{1/q} \phi_p(\sin_p(t))\right) \\ &= p \cdot \frac{(p-1)^{1/p}}{p-1} \phi_p(\sin_p(t)) \phi_q(\cos_q(t)) \\ &\quad - \frac{q}{q-1} \cdot (q-1)^{1/q} \phi_q(\cos_q(t)) \phi_p(\sin_p(t)) \\ &= p \cdot (p-1)^{1/p - 1} \phi_p(\sin_p(t)) \phi_q(\cos_q(t)) \\ &\quad - p \cdot (p-1)^{-1/q} \phi_p(\sin_p(t)) \phi_q(\cos_q(t)) \ , \end{aligned}$$

as ${}^{q}\!/_{(q-1)} = p$. We then use the identity ${}^{1}\!/_{p} + {}^{1}\!/_{q} = 1$ to show,

$$
\begin{aligned}
\left(\frac{|\sin_p(t)|^p}{p-1}+\frac{|\cos_q(t)|^q}{q-1}\right)' &= p\cdot(p-1)^{1/p-1}\phi_p(\sin_p(t))\phi_q(\cos_q(t)) \\
&\qquad - p\cdot(p-1)^{-1/q}\phi_p(\sin_p(t))\phi_q(\cos_q(t)) \\
&= p\cdot(p-1)^{-1/q}\phi_p(\sin_p(t))\phi_q(\cos_q(t)) \\
&\qquad - p\cdot(p-1)^{-1/q}\phi_p(\sin_p(t))\phi_q(\cos_q(t)) \\
&= 0\ .
\end{aligned}
$$

Therefore, the derivative of right-hand side of (17) is identically zero, and so its value is constant. We can use the initial values of these functions to determine this constant,

$$
\frac{|\sin_p(0)|^p}{p-1}+\frac{|\cos_q(0)|^q}{q-1}=\frac{0}{p-1}+\frac{((q-1)^{1/q})^q}{q-1}=1\ ,
$$

and thus, we have the result.

□

For the purposes of the numerical schemes derived in Chapter 3, we define the analogue to the cot function.

Definition 2.1.3. *For any $t\in\mathbb{R}$, the function $\cot_p:\mathbb{R}\to\mathbb{R}$ is given by*

$$
\cot_p(t):=\frac{\cos_p(t)}{(p-1)^{1/p}\phi_q(\sin_q(t))}\ .
$$

We also have a formulation for the derivative of $\cot_p$.

Theorem 2.1.5. *For any $t\in\mathbb{R}$, the function $\cot_p$ is differentiable and*

$$
\cot_p'(t)=-\left(1+\frac{|\cot_p(t)|^p}{p-1}\right)\ .
$$

Proof. By the definition of $\cot_p$,

$$
\cot_p'=\left(\frac{\cos_p}{(p-1)^{1/p}\phi_q(\sin_q)}\right)'\ ,
$$

and by the Quotient Rule,

$$
\begin{aligned}
\cot_p' &= \frac{-(p-1)^{2/p}|\sin_q|^{2q-2} - (p-1)^{2/q}|\cos_p|^p\,|\sin_q|^{q-2}}{(p-1)^{2/p}} \\
&= -1 - \frac{1}{p-1}\frac{|\cos_p|^p}{|\sin_q|^q} \\
&= -\left(1 + \frac{|\cot_p|^p}{p-1}\right) .
\end{aligned}
$$

□

We will also need the inverse function, a function analogous to arccot when $p = 2$. We give this definition below.

Definition 2.1.4. *For any $t \in \mathbb{R}$, the function* $\operatorname{arccot}_p : \mathbb{R} \to (-{}^{\pi_p}/2, {}^{\pi_p}/2)$, *is given by the relation*

$$\operatorname{arccot}_p(\cot_p(t)) = \cot_p(\operatorname{arccot}_p(t)) = t .$$

In Chapter 3, the function arccot_p will have to be approximated, so we use the following integral formulation.

Theorem 2.1.6. *For any $t \geq 0$,*

$$\operatorname{arccot}_p(t) = \int_t^\infty \frac{1}{1 + {}^{|s|^p}/_{(p-1)}}\,\mathrm{d}s .$$

Proof. By the definition of arccot_p, and the Chain Rule,

$$(\operatorname{arccot}_p(\cot_p))' = \operatorname{arccot}_p'(\cot_p)\ \cot_p' = 1 ,$$

and by Theorem 2.1.5,

$$\operatorname{arccot}_p'(\cot_p) = \frac{1}{\cot_p'} = \frac{-1}{1 + {}^{|\cot_p|^p}/_{p-1}} .$$

Integrating both sides, we have

$$\operatorname{arccot}_p(t) = \int_a^t \frac{-\,\mathrm{d}s}{1 + {|s|^p}/{(p-1)}} = \int_t^a \frac{\mathrm{d}s}{1 + {|s|^p}/{(p-1)}} \ ,$$

for any $t > 0$, and some fixed $a \in \mathbb{R}$. By the definition of $\cot_p$, we know that

$$\lim_{t\to 0} \cot_p(t) = \lim_{t\to 0} \frac{\cos_p(t)}{(p-1)^{1/p}\phi_q(\sin_q(t))} = +\infty \ ,$$

hence

$$\lim_{t\to\infty} \operatorname{arccot}_p(t) = 0 \ ,$$

therefore, $a = +\infty$.

□

This gives us a way of approximating the arccot_p function. In Chapter 3, we will expand the integrand using a geometric series, integrate term-by-term, and prove convergence. Note that as $\cos_p$ is an even function, and $\sin_q$ an odd function, $\cot_p$ is also an odd function. Similarly, the inverse, arccot_p is odd, which allows us to focus on finding approximations for the positive values of t only.

We also briefly mention the p-hyperbolic functions, which we will use in the approximation of the rotation number function in Chapter 3.

Definition 2.1.5. *The solution of the equation* (10)*, with $\lambda = -1$ and $Q \equiv 0$, and initial conditions $u(0) = 0$, $u'(0) = 1$, is called the* $\sinh_p$ *function. The solution with initial conditions $u(0) = (p-1)^{1/p}$, $u'(0) = 0$, is called the* $\cosh_p$ *function.*

For any $t \in \mathbb{R}$, the inverses of these functions,

$$\operatorname{arcsinh}_p(t) = \int_0^t \frac{\mathrm{d}s}{\left(1 + \left({|s|^p}/{(p-1)}\right)\right)^{1/p}} \qquad \text{and} \qquad \operatorname{arccosh}_p(t) = \int_{(p-1)^{1/p}}^t \frac{\mathrm{d}s}{\left(\left({|s|^p}/{(p-1)}\right) - 1\right)^{1/p}} \ ,$$

and the derivatives,

$$\sinh'(t) = (p-1)^{1/p}\, \phi_q(\cosh_q(t)) \qquad \text{and} \qquad \cosh'(t) = (p-1)^{1/p}\, \phi_q(\sinh_q(t))\ ,$$

can be computed similarly to the derivations of the p-trigonometric counterparts shown in Theorems 2.1.1 and 2.1.3. We have the analogous Pythagorean-type identity

$$\frac{|\cosh_p(t)|^p}{p-1} - \frac{|\sinh_q(t)|^q}{q-1} = 1\ .$$

There is also a corresponding $\coth_p$, given by

$$\coth_p = \frac{\cosh_p}{(p-1)^{1/p}\, \phi_q(\sinh_q)}\ ,$$

and an inverse, $\operatorname{arccoth}_p$, that can be expressed as

$$\operatorname{arccoth}_p(t) = \int_t^{(p-1)^{1/p}} \frac{\mathrm{d}s}{{}^{|s|^p}\!/_{(p-1)} - 1}\ ,$$

for all $t \geq 0$.

From the $\cos_p$ and $\sin_p$ functions given in Definition 2.1.1, we can now define a more suitable transformation for the system than the standard Prüfer transform, a transform that we refer to as the p-Prüfer transform. This transform will allow us to convert the equation (10) into a first-order system dependent on a radial component, r, and an angular component, θ. The advantage of this is the independence of the θ equation (23) from the radial component, a result of the homogeneity of the equation (10). This will allow us to restate the periodic boundary conditions in terms of the angular component, θ only.

In the rest of this chapter, we introduce the aforementioned p-Prüfer transform, with which we convert the equation (10) into a first-order system dependent on a radial component r, and an angular component θ. By a consideration of the relation between r and θ, we can rewrite the periodic boundary conditions in terms of θ only. We then introduce the *renormalised Poincaré map*, a function dependent on the initial angle of the system θ_0 and the spectral parameter λ. This function gives the change in the angular component of the solution to the problem (10) over one period. The periodic boundary conditions given in this problem can be restated in terms of this function, and therefore an analysis of the properties of this function will reveal properties of the spectrum of the problem.

2.2 The p-Prüfer Transform, the Renormalised Poincaré Map and Periodic Eigenvalues

In this chapter, we introduce the p-Prüfer Transform, first derived by Zhang [30]. After converting (10) to a first-order system, we apply this transform to change the system into *radial* and *angular* parts. The advantage here, is that due to the homogeneity of equation (10), the angular equation (23) does not depend on the radial term, and the radial term can be calculated directly from the angular component θ. This will allow us to reduce the periodic boundary conditions to conditions in terms of θ only.

We define

$$x := u \qquad \text{and} \qquad y := -\phi_p(u') \; ,$$

and from this, we have a (nonlinear) first-order system equivalent to (10),

$$\begin{cases} x' = -\phi_q(y) \\ y' = (\lambda - Q)\phi_p(x) \; . \end{cases} \tag{18}$$

To define the transform, we evaluate x , y in the case $\lambda - Q \equiv 1$, with initial values $x(0) = (p-1)^{1/p}$, $y(0) = 0$. By Definition 2.1.1, the solution to the system with these initial conditions is given by,

$$\begin{aligned} x(t) &= u(t) & \text{and} \qquad y(t) &= -\phi_p(u'(t)) \\ &= \cos_p(t) & &= -\phi_p(-(p-1)^{1/p}\phi_q(\sin_q(t))) \\ & & &= (p-1)^{1/q}\sin_q(t) \; , \end{aligned}$$

for all $t \in \mathbb{R}$. Following this, we define the p-Prüfer transform as

$$x = r^{2/p}\cos_p\theta \qquad \text{and} \qquad y = (p-1)^{1/q} r^{2/q}\sin_q\theta \; . \tag{19}$$

Note that for the case $(\lambda - Q) \equiv 1$, the angular component, $\theta(t) = \theta_0 + t$, for some initial angle, $\theta_0 \in \mathbb{R}$, and the radial component $r(t) \equiv r_0$, for some initial radius $r_0 > 0$. We can invert this transform by utilising the properties of the $\cos_p$ and $\sin_q$ functions, most notably, the Pythagorean-type identity given in Theorem 2.1.4.

Lemma 2.2.1. *Given variables x, y and r, θ, that satisfy the p-Prüfer relation* (19), *we have*

$$\begin{aligned} r &= \left((q-1)|x|^p + |y|^q\right)^{1/2} , \\ \theta &= \operatorname{arccot}_p\left(\frac{x}{\phi_q(y)}\right) . \end{aligned}$$

Proof. For the radial variable, r, we have

$$\begin{aligned} (q-1)|x|^p + |y|^q &= (q-1)|r^{2/p}\cos_p\theta|^p + |(p-1)^{1/q}r^{2/q}\sin_q\theta|^q \\ &= r^2\left(\frac{|\cos_p\theta|^p}{p-1} + \frac{|\sin_q\theta}{q-1}\right) \\ &= r^2 , \end{aligned}$$

by the Pythagorean-type identity given in Theorem 2.1.4. For the angular variable θ, we have

$$\begin{aligned} \operatorname{arccot}_p\left(\frac{x}{\phi_q(y)}\right) &= \operatorname{arccot}_p\left(\frac{r^{2/p}\cos_p\theta}{\phi_q((p-1)^{1/q}r^{2/q}\phi_q(\sin_q\theta))}\right) \\ &= \operatorname{arccot}_p\left(\frac{r^{2/p}\cos_p\theta}{(p-1)^{1/p}r^{2/p}\phi_q(\sin_q\theta)}\right) \\ &= \operatorname{arccot}_p(\cot_p\theta) = \theta . \end{aligned}$$

$\square$

We now use this transform to convert the system (18) into an equivalent system dependent on r and θ.

Theorem 2.2.1. *The equation* (10) *under the p-Prüfer transform is equivalent to the first-order system*

$$\begin{cases} r' = ({}^{q}\!/_{2})(p-1)^{1/p}\, r(\lambda - Q - 1)\phi_p(\cos_p\theta)\phi_q(\sin_q\theta) \\ \theta' = 1 + (q-1)(\lambda - Q - 1)|\cos_p\theta|^p \; . \end{cases}$$

Proof. We substitute the transform (19) into the system (18), resulting in

$$({}^{2}\!/_{p})r^{(2/p)-1}r'\cos_p\theta - (q-1)^{1/p}r^{2/p}\phi_q(\sin_q\theta))\theta' = -(p-1)^{1/p}r^{2/p}\phi_q(\sin_q\theta)\;, \tag{20}$$

$$(p-1)^{1/q}({}^{2}\!/_{q})r^{(2/q)-1}r'\sin_q\theta + r^{2/q}\phi_p(\cos_p\theta))\theta' = (\lambda - Q)r^{2/q}\phi_p(\cos_p\theta) \tag{21}$$

Taking the combination ${}^{q}\!/_{2}r^{2/q}\phi_p(\cos_p\theta)\cdot(20) + {}^{q}\!/_{2}(p-1)^{1/p}r^{2/p}\phi_q(\sin_q\theta)\cdot(21)$, we get

$$r' = (({}^{q}\!/_{2})(p-1)^{1/p}r(\lambda - Q - 1)\phi_p(\cos_p\theta)\phi(\sin_q\theta)) \tag{22}$$

Similarly, we take the combination $-(p-1)^{1/q}r^{2/q-1}\sin_q\theta\cdot(20) + (q-1)r^{2/p-1}\cos_p\theta\cdot(21)$ to get

$$\theta' = 1 + (q-1)(\lambda - Q - 1)|\cos_p\theta|^p \tag{23}$$

□

The latter of these equations is a first-order equation depending only on θ. Therefore, if this single equation is solved, the whole system can be solved too. This allows us to characterise the eigenvalue problem in terms of θ only. We can now rewrite the initial values in terms of the new Prüfer variables, with r_0 as the initial radius, and θ_0 as the initial angle.

Theorem 2.2.2. *A value $\lambda \in \mathbb{R}$ is a periodic eigenvalue of the problem* (10) *if and only if there exists an initial angle $\theta_0 \in \mathbb{R}$ and some $n \in \mathbb{N}_0$ such that*

$$\begin{cases} \theta(\alpha, r_0, \theta_0, \lambda) = \theta_0 + 2n\pi_p \\ r(\alpha, r_0, \theta_0, \lambda) = r_0 \ . \end{cases}$$

Proof. Let u be a solution of the equation (10), satisfying the above hypotheses. By the definition of the p-Prüfer Transform,

$$u = x = r^{2/p} \cos_p \theta \ ,$$

hence at $t = \alpha$, we have,

$$\begin{aligned} u(\alpha, r_0, \theta_0, \lambda) &= (r(\alpha, r_0, \theta_0, \lambda))^{2/p} \cos_p(\theta(\alpha, r_0, \theta_0, \lambda)) \\ &= r_0^{2/p} \cos_p(\theta_0 + 2n\pi_p) \\ &= r_0^{2/p} \cos_p(\theta_0) \\ &= u(0, r_0, \theta_0, \lambda) \ , \end{aligned}$$

using the first hypothesis, and the $2\pi_p$-periodicity of $\cos_p$. Similarly, by the definition of the p-Prüfer Transform,

$$-\phi_p(u') = y = (p-1)^{1/q} r^{2/q} \sin_q \theta \ ,$$

and so, at $t = \alpha$,

$$\begin{aligned} -\phi_p(u'(\alpha, r_0, \theta_0, \lambda)) &= (p-1)^{1/q} (r(\alpha, r_0, \theta_0, \lambda))^{2/q} \sin_q(\theta(\alpha, r_0, \theta_0, \lambda)) \\ &= (p-1)^{1/q} r_0^{2/q} \sin_q(\theta_0 + 2n\pi_p) \\ &= (p-1)^{1/q} r_0^{2/q} \sin_q(\theta_0) \\ &= -\phi_p(u'(0, r_0, \theta_0, \lambda)) \ . \end{aligned}$$

Hence,

$$u(\alpha, r_0, \theta_0, \lambda) = u(0, r_0, \theta_0, \lambda)$$
$$u'(\alpha, r_0, \theta_0, \lambda) = u'(0, r_0, \theta_0, \lambda) ,$$

and so the periodic boundary conditions are satisfied. Conversely, if these two conditions are satisfied, we can derive the two conditions given in the hypothesis using Lemma 2.2.1. For the radius, we have

$$\begin{aligned} r(\alpha, r_0, \theta_0, \lambda) &= ((q-1)|x(\alpha, r_0, \theta_0, \lambda)|^p + |y(\alpha, r_0, \theta_0, \lambda)|^q)^{1/2} \\ &= ((q-1)|x(0, r_0, \theta_0, \lambda)|^p + |y(0, r_0, \theta_0, \lambda)|^q)^{1/2} \\ &= r_0 , \end{aligned}$$

and for the angle,

$$\begin{aligned} \theta(\alpha, r_0, \theta_0, \lambda) &= \operatorname{arccot}_p \left(\frac{x(\alpha, r_0, \theta_0, \lambda)}{\phi_q(y(\alpha, r_0, \theta_0, \lambda))} \right) \\ &= \operatorname{arccot}_p \left(\frac{x(0, r_0, \theta_0, \lambda)}{\phi_q(y(0, r_0, \theta_0, \lambda))} \right) \\ &= \theta_0 + 2n\pi_p , \end{aligned}$$

for some $n \in \mathbb{N}_0$.

$\square$

This still requires us to fulfill a constraint on the radial variable, so we use the following connection between the angle and radius. Throughout this thesis, we will use the notation $\partial_n f$ to refer to the partial derivative of f with respect to its n^{th} argument.

Lemma 2.2.2. *For any $t, \theta_0, \lambda \in \mathbb{R}$ and $r_0 > 0$, we have*

$$\partial_3 \theta(t, r_0, \theta_0, \lambda) = \left(\frac{r_0}{r(t, r_0, \theta_0, \lambda)} \right)^2 .$$

Proof. Differentiating (23) with respect to the initial angle θ_0, we have

$$\begin{aligned}
\partial_3\theta'(t,r_0,\theta_0,\lambda) &= \frac{\partial}{\partial\theta}(1+(q-1)(\lambda-Q-1)|\cos_p\theta|^p)\cdot\partial_3\theta(t,r_0,\theta_0,\lambda)\\
&= -p(q-1)(p-1)^{1/p}(\lambda-Q-1)\phi_p(\cos_p\theta)\phi_q(\sin_q\theta)\cdot\partial_3\theta(t,r_0,\theta_0,\lambda)\\
&= -q(p-1)^{1/p}(\lambda-Q-1)\phi_p(\cos_p\theta)\phi_q(\sin_q\theta)\cdot\partial_3\theta(t,r_0,\theta_0,\lambda)\\
&= -2(\log r(t,r_0,\theta_0,\lambda))'\cdot\partial_3\theta(t,r_0,\theta_0,\lambda)\ ,
\end{aligned}$$

by (22). Therefore,

$$(\log\partial_3\theta(t,r_0,\theta_0,\lambda))' = -2(\log r(t,r_0,\theta_0,\lambda))'\ .$$

Note that as $\theta(0,r_0,\theta_0,\lambda)=\theta_0$, we have $\partial_3\theta(0,r_0,\theta_0,\lambda)=1$.
Also, since $r(0,r_0,\theta_0,\lambda)=r_0$, we get

$$\log\partial_3\theta(t;\theta_0,\lambda) = -2\log\left(\frac{r(t,r_0,\theta_0,\lambda)}{r_0}\right)\ ,$$

and taking the exponential of both sides,

$$\partial_3\theta(t,r_0,\theta_0,\lambda) = \left(\frac{r_0}{r(t;\theta_0,\lambda)}\right)^2\ ,$$

giving us the result.

□

Lemma 2.2.2 shows that

$$\partial_3\theta(t,r_0,\theta_0,\lambda) = \exp\left(\int_0^t -q(p-1)^{1/p}(\lambda-Q-1)\phi_p(\cos_p\theta)\phi_q(\sin_q\theta)\right)\ .$$

Therefore, the angular component θ is strictly monotonically increasing with respect to the initial angle, θ_0.

Using this fact, we can rewrite the periodicity conditions given in Theorem 2.2.2 in terms of θ and λ only. We introduce the *renormalised Poincaré map* of the equation (10). This quantity gives the increase in θ for each initial angle θ_0, and each spectral value λ. Note that as the equation (23) is independent of the initial radius r_0, the renormalised Poincaré map is equal for all initial radial values, and we therefore suppress any dependence on r_0 in the notation.

Definition 2.2.1. *The renormalised Poincaré map of the equation* (10) *is given by,*

$$\Psi(\theta_0, \lambda) := \theta(\alpha, r_0, \theta_0, \lambda) - \theta_0 \ ,$$

for all $\theta_0, \lambda \in \mathbb{R}$.

We can now express the periodic boundary conditions in terms of the renormalised Poincaré map.

Theorem 2.2.3. *A value* $\lambda \in \mathbb{R}$ *is a periodic eigenvalue of the problem* (10) *if and only if there exists an initial angle* $\theta_0 \in \mathbb{R}$ *and some* $n \in \mathbb{N}_0$ *such that*

$$\begin{aligned} \Psi(\theta_0, \lambda) &= 2n\pi_p \\ \partial_1 \Psi(\theta_0, \lambda) &= 0 \ . \end{aligned}$$

Proof. We assume that both of there exist a λ, $\theta_0 \in \mathbb{R}$ such that the two conditions are satisfied. Take the first condition,

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

then by the definition of Ψ, the corresponding angular component, θ, satisfies

$$\theta(\alpha, r_0, \theta_0, \lambda) = \theta_0 + 2n\pi_p \ .$$

If the second condition is satisfied, then

$$\partial_1\Psi(\theta_0,\lambda) = \frac{\partial}{\partial\theta_0}\Big(\theta(\alpha,r_0,\theta_0,\lambda) - \theta_0\Big) = \partial_3\theta(\alpha,r_0,\theta_0,\lambda) - 1 = 0 ,$$

and hence, using Lemma 2.2.2,

$$\left(\frac{r_0}{r(\alpha,r_0,\theta_0,\lambda)}\right)^2 = \partial_3\theta(\alpha,r_0,\theta_0,\lambda) = 1 .$$

Therefore, we have

$$\begin{aligned} \theta(\alpha,r_0,\theta_0,\lambda) &= \theta_0 + 2n\pi_p \\ r(\alpha,r_0,\theta_0,\lambda) &= r_0 , \end{aligned}$$

which by Theorem 2.2.2, is equivalent to the corresponding solution, u, satisfying the periodic boundary conditions.

$\square$

We also state three results regarding the renormalised Poincaré Map, $\Psi(\theta_0,\lambda)$, that will be used in the next chapter to find a characterisation for the periodic eigenvalues.

Lemma 2.2.3. *For any $\theta_0,\lambda \in \mathbb{R}$, and $n \in \mathbb{N}_0$,*

$$\Psi(\theta_0 + n\pi_p,\lambda) = \Psi(\theta_0,\lambda) .$$

Proof. Fix $r_0 > 0$, $\theta_0,\lambda \in \mathbb{R}$. By definition, the angular solution $\theta(\cdot,r_0,\theta_0+n\pi_p,\lambda)$ solves the equation

$$z'(t) = 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p z(t)|^p ,$$

with initial value $\theta_0 + n\pi_p$. Now consider $z(\cdot) = \theta(\cdot,r_0,\theta_0,\lambda) + n\pi_p$, we have

$$
\begin{aligned}
z'(t) &= (\theta(t, r_0, \theta_0, \lambda) + n\pi_p)' \\
&= \theta'(t, r_0, \theta_0, \lambda) \\
&= 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p \theta(t, r_0, \theta_0, \lambda)|^p \\
&= 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p(z(t) - n\pi_p)|^p \\
&= 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p z(t)|^p \ .
\end{aligned}
$$

Therefore, the functions $\theta(t, r_0, \theta_0 + n\pi_p, \lambda)$ and $\theta(t, r_0, \theta_0, \lambda) + n\pi_p$ both satisfy the same ODE, and both have the initial value $\theta_0 + n\pi_p$. Therefore, by the uniqueness of solutions, they are identically equal.

□

By Rolle's Theorem and Lemma 2.2.3, we know for each $\lambda \in \mathbb{R}$, there must exist a point θ_0 that satisfies the second condition in Theorem 2.2.3. Next, we state a property of the renormalised Poincaré map Ψ, on its second variable.

Theorem 2.2.4. *For any* $\theta_0, \lambda \in \mathbb{R}$

$$
\partial_2 \Psi(\theta_0, \lambda) = \frac{(q-1)}{r(\alpha, r_0, \theta_0, \lambda)} \int_0^\alpha |u(s, r_0, \theta_0, \lambda)|^p \, \mathrm{d}s \ .
$$

Proof. Fix $r_0 > 0$. From (23), for any $t \in \mathbb{R}$, we have

$$
\begin{aligned}
\partial_4 \theta'(t, r_0, \theta_0, \lambda) &= \frac{\partial}{\partial \lambda}\Big(1 + (q-1)(\lambda - Q(t) - 1)|\cos_p \theta(t, r_0, \theta_0, \lambda)|^p\Big) \\
&= (q-1)|\cos_p \theta|^p - q(p-1)^{1/p}(\lambda - Q(t) - 1)\phi_p(\cos_p \theta(t, r_0, \theta_0, \lambda)) \\
&\qquad \cdot \phi_q(\sin_q \theta(t, r_0, \theta_0, \lambda))\partial_4 \theta(t, r_0, \theta_0, \lambda) \\
&= (q-1)|\cos_p \theta(t, r_0, \theta_0, \lambda)|^p - 2\,\frac{r'(t, r_0, \theta_0, \lambda)}{r(t, r_0, \theta_0, \lambda)}\,\partial_4 \theta(t, r_0, \theta_0, \lambda)
\end{aligned}
$$

Multiplying through by r^2, we get

$$r^2\partial_4\theta' + 2rr'\partial_3\theta = (q-1)|\cos_p\theta|^p r^2 \ ,$$

and since $(r^2\partial_4\theta)' = r^2\partial_4\theta' + 2rr'\partial_4\theta$, we have

$$\left(r^2\partial_4\theta\right)' = (q-1)|\cos_p\theta|^p r^2 \ .$$

Integrating both sides, we get

$$r^2\partial_4\theta = \int_0^t (q-1)|\cos_p\theta(s,\theta_0,\lambda)|^p\, r(s,\theta_0,\lambda)^2\,\mathrm{d}s \ .$$

Note that as the initial angle, θ_0, does not depend on λ, $\partial_4\theta(0,r_0,\theta_0,\lambda) = 0$. Finally, we get

$$\begin{aligned}
\partial_4\theta(t,r_0,\theta_0,\lambda) &= \frac{(q-1)}{(r(t,r_0,\theta_0,\lambda))^2}\int_0^t |\cos_p\theta(s,r_0,\theta_0,\lambda)|^p (r(s,r_0,\theta_0,\lambda))^2\,\mathrm{d}s \\
&= \frac{(q-1)}{(r(t,r_0,\theta_0,\lambda))^2}\int_0^t |(r(s,r_0,\theta_0,\lambda))^{2/p}\cos_p\theta(s,r_0,\theta_0,\lambda)|^p\,\mathrm{d}s \\
&= \frac{(q-1)}{(r(t,r_0,\theta_0,\lambda))^2}\int_0^t |u(s,r_0,\theta_0,\lambda)|^p\,\mathrm{d}s
\end{aligned}$$

Letting $t = \alpha$, we get

$$\partial_2\Psi(\theta_0,\lambda) = \partial_4\theta(\alpha,r_0,\theta_0,\lambda) = \frac{(q-1)}{(r(\alpha,r_0,\theta_0,\lambda))^2}\int_0^\alpha |u(s,r_0,\theta_0,\lambda)|^p\,\mathrm{d}s \ ,$$

which gives us the result.

□

We can combine Lemma 2.2.3 and Theorem 2.2.4 to state the following result of the dependence of $\Psi(\theta_0,\lambda)$ on both of its variables.

Corollary 2.2.1. *The map $\Psi(\theta_0,\lambda)$ is π_p-periodic in its first variable, and strictly monotonically increasing its second variable.*

By the monotonicity property of Ψ on its second variable, we know that for any $\theta_0 \in \mathbb{R}$, and $n \in \mathbb{N}$, there must exist a value λ_n such that $\Psi(\theta_0, \lambda_n) = 2n\pi_p$, which is the first eigenvalue condition stated in Theorem 2.2.3. As above, the periodicity property of Ψ in its first variable gives the existence of a point $\theta_0 \in \mathbb{R}$ that satisfies the second eigenvalue condition in Theorem 2.2.3. We combine these two properties in the next chapter, to analyse the periodic eigenvalues of the equation (10).

2.3 The Rotation Number

We now explore more properties of the Prüfer angle, and how it can be used to characterise the periodic eigenvalues of the problem (10). We will use these properties to define a function called the *rotation number*, dependent only on the spectral parameter λ. We will show that this function encodes information about the spectrum of the problem, namely that eigenvalues exist only over intervals of λ for which the image of this rotation number function attains certain values.

The first result derives from the equation for θ in Theorem 2.2.1. As the potential Q, is α-periodic, we have the following semigroup property for the Prüfer angle.

Lemma 2.3.1. *For any* $t, \theta_0, \lambda \in \mathbb{R}$, $r_0 > 0$*, and any* $m \in \mathbb{N}$*,*

$$\theta(t + m\alpha, r_0, \theta_0, \lambda) = \theta(t, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda) \ .$$

Proof. Fix $r_0 > 0$, $\theta_0, \lambda \in \mathbb{R}$. Consider the shifted angular solution $\theta(\cdot + m\alpha, r_0, \theta_0, \lambda)$, by definition, this function solves

$$\begin{aligned} z'(t) &= 1 + (q-1)(\lambda - Q(t + m\alpha) - 1)|\cos_p z(t)|^p \\ &= 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p z(t)|^p \ , \end{aligned}$$

as Q is α-periodic. The initial value of this shifted solution is

$$\theta(m\alpha, r_0, \theta_0, \lambda) \ .$$

Now consider the solution $\theta(\cdot, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda)$, this function solves

$$z'(t) = 1 + (q-1)(\lambda - Q(t) - 1)|\cos_p z(t)|^p \ .$$

By definition, the initial value of this solution is

$$\theta(m\alpha, r_0, \theta_0, \lambda) \; .$$

Therefore, the two functions satisfy the same equation, and have the same initial value. By the uniqueness of solutions, they must coincide.

□

Following directly from this, we can rewrite this result in terms of an iterated generalisation of the renormalised Poincaré Map. For any $m \in \mathbb{N}$, the *iterated renormalised Poincaré map* over m periods gives the increase in the angular component over m periods for all initial angles $\theta_0 \in \mathbb{R}$ at all spectral values, $\lambda \in \mathbb{R}$.

Definition 2.3.1. *Fix some $m \in \mathbb{N}$, for any $\theta_0, \lambda \in \mathbb{R}$, the iterated renormalised Poincaré map over m periods is defined by,*

$$\Psi^m(\theta_0, \lambda) := \theta(m\alpha, r_0, \theta_0, \lambda) - \theta_0 \; .$$

Using Lemma 2.3.1, the iterated renormalised Poincaré map can be expressed as a summation of the standard renormalised Poincaré map, evaluated at several points in the orbit of θ.

Corollary 2.3.1. *For any $\theta_0, \lambda \in \mathbb{R}$ and $m \in \mathbb{N}$,*

$$\Psi^m(\theta_0, \lambda) = \sum_{i=0}^{m-1} \Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda) \; .$$

Proof. For $m = 1$,

$$\Psi^1(\theta_0, \lambda) = \Psi(\theta_0, \lambda) = \sum_{i=0}^{0} \Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda) \; ,$$

and so the result holds. Assume the result holds for some fixed $m \in \mathbb{N}$, then for $m + 1$, we have

$$
\begin{aligned}
\Psi^{m+1}(\theta_0,\lambda) &= \theta((m+1)\alpha, r_0, \theta_0, \lambda) - \theta_0 \\
&= \theta(\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda)\theta_0, \lambda) - \theta_0 \qquad \text{(by Lemma 2.3.1)} \\
&= \Big(\theta(\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda)\theta_0, \lambda) - \theta(m\alpha, r_0, \theta_0, \lambda)\Big) + \Big(\theta(m\alpha, r_0, \theta_0, \lambda) - \theta_0\Big) \\
&= \Psi(\theta(m\alpha, r_0, \theta_0, \lambda), \lambda) + \Psi^m(\theta_0, \lambda) \\
&= \Psi(\theta(m\alpha, r_0, \theta_0, \lambda), \lambda) + \sum_{i=0}^{m-1} \Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda) \qquad \text{(by the inductive hypothesis)} \\
&= \sum_{i=0}^{m} \Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda)\ ,
\end{aligned}
$$

and so by induction, the argument holds for all $m \in \mathbb{N}$.

□

Following the semigroup property shown in Lemma 2.3.1, we also have the following bounds on the map $\Psi(\theta_0, \lambda)$.

Lemma 2.3.2. *Fix $\lambda \in \mathbb{R}$, if*

$$\Psi(\tilde{\theta_0}, \lambda) = n\pi_p\ ,$$

for some $\tilde{\theta_0} \in \mathbb{R}$ and $n \in \mathbb{N}$, then

$$(n-1)\pi_p < \Psi(\theta_0, \lambda) < (n+1)\pi_p\ ,$$

for all $\theta_0 \in \mathbb{R}$.

Proof. Fix $r_0 > 0$, for any $\theta_0 \in [\tilde{\theta_0}\, ,\, \tilde{\theta_0} + \pi_p)$, we have

$$
\begin{aligned}
\Psi(\theta_0, \lambda) &= \theta(\alpha, r_0, \theta_0, \lambda) - \theta_0 \\
&< \theta(\alpha, r_0, \tilde{\theta_0} + \pi_p, \lambda) - \tilde{\theta_0}\ ,
\end{aligned}
$$

due to the monotonicity of θ in its third variable (a result of Lemma 2.2.2), and the fact that $\tilde{\theta_0} \leq \theta_0$. By the proof of Lemma 2.2.3,

$$\theta(\alpha, r_0, \tilde{\theta_0} + \pi_p, \lambda) = \theta(\alpha, r_0, \tilde{\theta_0}, \lambda) + \pi_p \ ,$$

and so

$$\begin{aligned}\Psi(\theta_0, \lambda) &< \theta(\alpha, r_0, \tilde{\theta_0}, \lambda) + \pi_p - \tilde{\theta_0} \\ &= \Psi(\tilde{\theta_0}, \lambda) + \pi_p \\ &= (n+1)\pi_p \ ,\end{aligned}$$

by the hypothesis, $\Psi(\tilde{\theta_0}, \lambda) = n\pi_p$. We have proven this inequality on the interval $\theta_0 \in [\tilde{\theta_0} \, , \, \tilde{\theta_0} + \pi_p)$, by Lemma 2.2.3, it therefore holds for all $\theta_0 \in \mathbb{R}$.

For the lower bound, we can take any $\theta_0 \in [\tilde{\theta_0} - \pi_p \, , \, \tilde{\theta_0})$, and similarly

$$\begin{aligned}\Psi(\theta_0, \lambda) &= \theta(\alpha, r_0, \theta_0, \lambda) - \theta_0 \\ &> \theta(\alpha, r_0, \tilde{\theta_0} - \pi_p, \lambda) - \tilde{\theta_0} \ ,\end{aligned}$$

again, due to the monotonicity of θ in its third variable, and the fact that $\theta_0 < \tilde{\theta_0}$. Once again, by the proof of Lemma 2.2.3, we have

$$\begin{aligned}\Psi(\theta_0, \lambda) &> \theta(\alpha, r_0, \tilde{\theta_0}, \lambda) - \pi_p - \tilde{\theta_0} \\ &= \Psi(\tilde{\theta_0}, \lambda) - \pi_p \\ &= (n-1)\pi_p \, .\end{aligned}$$

This inequality holds on the interval $\theta_0 \in [\tilde{\theta_0} - \pi_p \, , \, \tilde{\theta_0})$, and by Lemma 2.2.3, it holds for all $\theta_0 \in \mathbb{R}$.

□

Lemma 2.3.2 tells us that the range of the renormalised Poincaré Map is bounded between the values $(n-1)\pi_p$ and $(n+1)\pi_p$ for some value $n \in \mathbb{N}$. The final property relates the monotonicity argument in Corollary 2.2.1 with the bounds on the range of this map.

Lemma 2.3.3. *The functions* $\max_{\theta_0 \in \mathbb{R}} \Psi(\theta_0, \lambda)$*, and* $\min_{\theta_0 \in \mathbb{R}} \Psi(\theta_0, \lambda)$*, are continuous and strictly monotonically increasing in* λ*.*

Proof. We first show that $\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda))$ is strictly monontone increasing in λ. By Corollary 2.2.1, for any fixed $\tilde{\theta_0} \in \mathbb{R}$, the function $\Psi(\theta_0, \lambda)$ is strictly increasing in λ. For $\lambda_1 < \lambda_2$, we therefore have

$$\Psi(\tilde{\theta_0}, \lambda_1) < \Psi(\tilde{\theta_0}, \lambda_2) \leq \max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda_2)) .$$

Taking a value of $\tilde{\theta_0}$ that maximises the function $\Psi(\cdot, \lambda_1)$ on $\mathbb{R}$, we find that $\max_{\theta_0 \in \mathbb{R}} \Psi(\theta_0, \lambda)$ is a strictly increasing function in λ.

Next, we show the continuity of $\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda))$ in λ. Let $\hat{\theta_0}$ be a value of $\theta_0 \in \mathbb{R}$ such that $\Psi(\theta_0, \lambda_2)$ attains its maximum, then using the strict monotonicity in the second variable, we have

$$\Psi(\hat{\theta_0}, \lambda_1) \leq \max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda_1)) < \max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda_2)) = \Psi(\hat{\theta_0}, \lambda_2) .$$

Therefore,

$$\begin{aligned}
\left| \max_{\tilde{\theta} \in \mathbb{R}}(\Psi(\tilde{\theta}, \lambda_1)) - \max_{\tilde{\theta} \in \mathbb{R}}(\Psi(\tilde{\theta}, \lambda_2)) \right| &\leq |\Psi(\hat{\theta_0}, \lambda_1) - \Psi(\hat{\theta_0}, \lambda_2)| \\
&= |(\theta(\alpha, r_0, \hat{\theta_0}, \lambda_1) - \hat{\theta_0}) - (\theta(\alpha, r_0, \hat{\theta_0}, \lambda_2) - \hat{\theta_0})| \\
&= |\theta(\alpha, r_0, \hat{\theta_0}, \lambda_1) - \theta(\alpha, r_0, \hat{\theta_0}, \lambda_2)| ,
\end{aligned}$$

and as the function $\theta(\alpha, r_0, \hat{\theta_0}, \lambda)$ is continuous in λ (a consequence of Theorem 2.2.4), the function $\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda))$ is also continuous in λ.

The proof for the continuity and strict monotonicity of $\min_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda))$ in λ is similar.

$\square$

We now define the rotation number, $\rho(\lambda)$.

Definition 2.3.2. *The rotation number of the Prüfer angle θ, is given by:*

$$\rho(\lambda) := \lim_{t\to\infty} \frac{\theta(t, r_0, \theta_0, \lambda) - \theta_0}{t} \, .$$

By the definition of Ψ^m, we can equivalently write the rotation number as

$$\rho(\lambda) = \lim_{m\to\infty} \frac{\Psi^m(\theta_0, \lambda)}{m\alpha} \, ,$$

for which the limit in the definition of ρ is now taken over a countably infinite subset of evenly-spaced points in t. The properties of Ψ proven in the Lemmas 2.2.2, 2.2.4, and 2.3.1 are used to prove the following results regarding the existence and properties of ρ.

Theorem 2.3.1. *The rotation number, $\rho(\lambda)$, exists for all $\lambda \in \mathbb{R}$, and its value is independent of the initial Prüfer angle, θ_0.*

Proof. First, we show that if $\rho(\lambda)$ exists for some $\tilde{\theta_0} \in \mathbb{R}$, then it exists for all $\theta_0 \in \mathbb{R}$. Consider $\theta_0 \in [\tilde{\theta_0} - \pi_p \, , \, \tilde{\theta_0} + \pi_p)$, by Lemmas 2.2.2 and 2.2.3, we have

$$\begin{aligned}
\theta(t, r_0, \tilde{\theta_0}, \lambda) - \pi_p &= \theta(t, r_0, \tilde{\theta_0} - \pi_p, \lambda) \\
&\leq \theta(t, r_0, \theta_0, \lambda) \\
&\leq \theta(t, r_0, \tilde{\theta_0} + \pi_p, \lambda) \\
&= \theta(t, r_0, \tilde{\theta_0}, \lambda) + \pi_p \, .
\end{aligned}$$

If we divide by t, and taking the limit as $t \to \infty$, the Squeeze Theorem shows that the rotation number exists for all θ_0, and the value is independent of θ_0.

Next, we show that there always exists at least one value, θ_0, such that the rotation number ρ exists, without loss of generality, we take $\theta_0 \in [0 \, , \, \pi_p)$.

For any $\theta_1 \in \mathbb{R}$, there exists an $m \in \mathbb{Z}$ such that $\theta_0 \leq \theta_1 - m\pi_p \leq \theta_0 + \pi_p$, and by Lemma 2.2.2,

$$\theta(t, r_0, \theta_0, \lambda) \leq \theta(t, r_0, \theta_1 - m\pi_p, \lambda) \leq \theta(t, r_0, \theta_0 + \pi_p, \lambda) \ .$$

By the proof of Lemma 2.2.3, we have,

$$\theta(t, r_0, \theta_0, \lambda) \leq \theta(t, r_0, \theta_1, \lambda) - m\pi_p \leq \theta(t, r_0, \theta_0, \lambda) + \pi_p \ .$$

Let $\gamma = \theta_1 - m\pi_p$,

$$\theta(t, r_0, \theta_0, \lambda) - \gamma \leq \theta(t, r_0, \theta_1, \lambda) - \theta_1 \leq \theta(t, r_0, \theta_0, \lambda) + \pi_p - \gamma \ .$$

Since $\theta_0 \leq \gamma \leq \theta_0 + \pi_p$, we have

$$\theta(t, r_0, \theta_0, \lambda) - \theta_0 - \pi_p \leq \theta(t, r_0, \theta_1, \lambda) - \theta_1 \leq \theta(t, r_0, \theta_0, \lambda) - \theta_0 + \pi_p \ .$$

Given that $0 \leq \theta_0 < \pi_p$, for any $m \in \mathbb{Z}$,

$$\theta(m\alpha, r_0, \theta_0, \lambda) - 2\pi_p \leq \theta(m\alpha, r_0, \theta_1, \lambda) - \theta_1 \leq \theta(m\alpha, r_0, \theta_0, \lambda) + 2\pi_p \ . \tag{24}$$

By Lemma 2.3.1,

$$\theta(2m\alpha, r_0, \theta_0, \lambda) = \theta(m\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda) \ ,$$

and therefore

$$\begin{aligned}\theta(2m\alpha, r_0, \theta_0, \lambda) &= \theta(m\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda) \\ &= \theta(m\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda) - \theta(m\alpha, r_0, \theta_0, \lambda) + \theta(m\alpha, r_0, \theta_0, \lambda) \ .\end{aligned}$$

We use (24) with $\theta_1 = \theta(m\alpha, r_0, \theta_0, \lambda)$ to get

$$2\theta(m\alpha, r_0, \theta_0, \lambda) - 2\pi_p \leq \theta(2m\alpha, r_0, \theta_0, \lambda) \leq 2\theta(m\alpha, r_0, \theta_0, \lambda) + 2\pi_p \ ,$$

as $\theta_0 \in [0, \pi_p)$. Successively applying this, we get

$$n\,\theta(m\alpha, r_0, \theta_0, \lambda) - n\pi_p \leq \ \theta(nm\alpha, r_0, \theta_0, \lambda) \leq n\,\theta(m\alpha, r_0, \theta_0, \lambda) + n\pi_p \ ,$$

for all $n \geq 0$, and therefore

$$\left| \frac{\theta(nm\alpha, r_0, \theta_0, \lambda)}{nm} - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m} \right| \leq \frac{2\pi_p}{m} \ ,$$

for all $n, m > 0$.

Using the triangle inequality, we get

$$\begin{aligned}
\left| \frac{\theta(n\alpha, r_0, \theta_0, \lambda)}{n} - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m} \right| &= \left| \frac{\theta(n\alpha, r_0, \theta_0, \lambda)}{n} - \frac{\theta(nm\alpha, r_0, \theta_0, \lambda)}{nm} \right. \\
&\qquad \left. + \frac{\theta(nm\alpha, r_0, \theta_0, \lambda)}{nm} - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m} \right| \\
&\leq \left| \frac{\theta(nm\alpha, r_0, \theta_0, \lambda)}{nm} - \frac{\theta(n\alpha, r_0, \theta_0, \lambda)}{n} \right| \\
&\qquad + \left| \frac{\theta(nm\alpha, r_0, \theta_0, \lambda)}{nm} - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m} \right| \\
&\leq \frac{2\pi_p}{n} + \frac{2\pi_p}{m} \\
&< \varepsilon \ ,
\end{aligned}$$

for some $\varepsilon > 0$, for large enough values of n, m. Therefore, elements of the sequence $\theta(n\alpha, r_0, \theta_0, \lambda)\, n^{-1}$ form a Cauchy sequence, and the sequence therefore converges to a limit. $\square$

This theorem explains one of the reasons why this function is useful in characterising the spectrum of our operator, it is independent of any initial conditions that can be imposed on the solutions.

Theorem 2.3.2. *The rotation number, $\rho(\lambda)$, is continuous in λ.*

Proof. Fix some values r_0 , $\theta_0 \in \mathbb{R}$, and consider the sequence of functions in λ given by

$$\frac{\theta(n\alpha, r_0, \theta_0, \lambda) - \theta_0}{n\alpha} \;, \tag{25}$$

for all $n \in \mathbb{N}$. By Theorem 2.2.4, each of these functions is differentiable with respect to λ, and therefore continuous in λ. The proof of Theorem 2.3.1 shows that the sequence (25) converges to $\rho(\lambda)$ at each $\lambda \in \mathbb{R}$, and that this convergence is uniform in λ. Therefore, by the Uniform Limit Theorem, the limiting function

$$\rho(\lambda) = \lim_{n\to\infty} \frac{\theta(n\alpha, r_0, \theta_0, \lambda) - \theta_0}{n\alpha} \;,$$

is continuous in λ on $\mathbb{R}$.

□

The characterisation of the spectrum requires local properties of the range of the function $\rho(\lambda)$, one of these is continuity, another is the monotonicity in λ, a property that reflects the strict monotonicity shown in Lemmas 2.2.4 and 2.3.3. Note that as the rotation number is a limit, this monotonicity is no longer strict.

Theorem 2.3.3. *The rotation number, $\rho(\lambda)$, is monotonically increasing in λ.*

Proof. Let $\lambda_1 < \lambda_2$, by definition of $\rho(\lambda)$,

$$\rho(\lambda_1) := \lim_{t\to\infty} \frac{\theta(t; \theta_0, \lambda_1) - \theta_0}{t}$$

By Corollary 2.2.1, $\theta(t, r_0, \theta_0, \lambda_1) < \theta(t, r_0, \theta_0, \lambda_2)$,
for all $t \geq \alpha$. Therefore

$$\begin{aligned}\rho(\lambda_1) &= \lim_{t\to\infty} \frac{\theta(t;\theta_0,\lambda_1)-\theta_0}{t} \\ &\leq \lim_{t\to\infty} \frac{\theta(t;\theta_0,\lambda_2)-\theta_0}{t} \\ &= \rho(\lambda_2)\ .\end{aligned}$$

□

Finally, we state some global properties of the range of the rotation number.

Theorem 2.3.4. *The rotation number, $\rho(\lambda)$, is non-negative for all $\lambda \in \mathbb{R}$. There exists a value $\hat{\lambda} \in \mathbb{R}$ such that $\rho(\lambda) = 0$ for all $\lambda < \hat{\lambda}$, and $\lim_{\lambda\to\infty} \rho(\lambda) = +\infty$.*

Proof. Fix $\lambda \in \mathbb{R}$, then suppose for some $t_0 \in \mathbb{R}$, we have

$$\theta(t_0, r_0, \theta_0, \lambda) = \frac{(2m-1)\pi_p}{2},$$

for some $m \in \mathbb{N}$, this gives

$$\cos_p(\theta(t_0, r_0, \theta_0, \lambda)) = \cos_p({}^{(2m-1)\pi_p}/_2) = 0\ ,$$

then by (23),

$$\theta'(t_0, r_0, \theta_0, \lambda) = 1 + (q-1)(\lambda - Q(t_0) - 1) \cdot 0 = 1 > 0\ .$$

Therefore, if

$$\theta(t_0, r_0, \theta_0, \lambda) \geq {}^{(2m-1)\pi_p}/_2\ ,$$

for some $t_0 \in \mathbb{R}$, then

$$\theta(t, r_0, \theta_0, \lambda) \geq {}^{(2m-1)\pi_p}/_2\ , \tag{26}$$

for all $t > t_0$.

Now, by Theorem 2.3.1, the value of $\rho(\lambda)$ is independent of the choice of θ_0, therefore without loss of generality, choose $\theta_0 = 0$.

Finally, as $\theta(0, r_0, \theta_0, \lambda) = \theta_0 = 0 > {}^{-\pi_p}\!/_2$, then by the bound (26), we have

$$\rho(\lambda) = \lim_{t\to\infty} \frac{\theta(t, r_0, 0, \lambda)}{t} \geq \lim_{t\to\infty} \frac{-\pi_p}{2t} = 0 \ .$$

$\square$

2.4 Rotation Number Plateaus and Periodic Eigenvalues

Given the characterisation of periodic eigenvalues derived in Theorem 2.2.3, we now find connections between the renormalised Poincaré Map and the rotation number defined in Chapter 2.3, and this will allow us to find a characterisation in terms of the rotation number alone. This will prove useful as the rotation number is independent of any initial conditions, and so would require fewer objects to be analysed.

As per Theorem 2.3.3, the rotation number is monotone increasing. This results in intervals of the domain of this function over which the function is constant. Once again, maximal intervals of constancy of ρ are referred to as *plateaus* of the rotation number. The first connection between the renormalised Poincaré map, Ψ, and the rotation number, ρ, is a direct calculation that shows the equivalence between these plateaus and values of λ for which the renormalised Poincaré Map crosses a multiple of $2\pi_p$.

Theorem 2.4.1. *Let $\lambda \in \mathbb{R}$ be such that there exists a $\theta_0 \in \mathbb{R}$, and $n \in \mathbb{N}$, with*

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

then the rotation number

$$\rho(\lambda) = \frac{2n\pi_p}{\alpha} \ .$$

Proof.

$$\begin{aligned}
\rho(\lambda) &= \lim_{k\to\infty} \frac{\theta(k\alpha, r_0, \theta_0, \lambda) - \theta_0}{k\alpha} \\
&= \lim_{k\to\infty} \frac{\theta_0 + 2kn\pi_p - \theta_0}{k\alpha} \\
&= \lim_{k\to\infty} \frac{2kn\pi_p}{k\alpha} \\
&= \frac{2n\pi_p}{\alpha}
\end{aligned}$$

□

Theorem 2.4.1 shows us that any value of λ such that there exists a value θ_0 at which point the renormalised Poincaré Map, Ψ, crosses a multiple of $2\pi_p$ corresponds to the rotation number, ρ, being a multiple of $2\pi_p\,\alpha^{-1}$. For the non-degenerate case, in which the map Ψ is non-constant, this results in the rotation number having a plateau at this level. If Ψ is constant however, these plateaus degenerate to a single point.

The following results are used to prove the converse, that any value of λ that gives a rotation number that is a multiple of $2\pi_p\,\alpha^{-1}$, corresponds to a renormalised Poincaré Map that crosses a multiple of $2\pi_p$.

Lemma 2.4.1. *For any $\lambda \in \mathbb{R}$, we have*

$$\min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha} \le \rho(\lambda) \le \max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}$$

Proof. For all $\theta_0 \in \mathbb{R}$ and $i \in \mathbb{N}_0$, we have

$$\min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha} < \frac{\Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda)}{\alpha} < \max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\ .$$

Using this bound over the sum of $\Psi(\cdot, \lambda)$ evaluated in the orbit of θ over $n-1$ periods, we get,

$$n \min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha} < \sum_{i=0}^{n-1} \frac{\Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda)}{\alpha} < n \max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\ ,$$

and dividing through by n,

$$\min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha} < \frac{1}{n}\sum_{i=0}^{n-1} \frac{\Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda)}{\alpha} < \max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\ , \tag{27}$$

for any $n \in \mathbb{N}_0$. From Corollary 2.3.1, we have

$$\rho(\lambda) = \lim_{n\to\infty} \frac{\theta(n\alpha, r_0, \theta_0, \lambda) - \theta_0}{n\alpha} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \frac{\Psi(\theta(i\alpha, r_0, \theta_0, \lambda), \lambda)}{\alpha}\ ,$$

and from the bounds in (27), we have

$$
\begin{aligned}
\min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha} &= \lim_{n\to\infty}\left(\min_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\right) \\
&\le \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n-1}\frac{\Psi(\theta(i\alpha,r_0,\theta_0,\lambda),\lambda)}{\alpha}\right) \\
&= \rho(\lambda) \\
&\le \lim_{n\to\infty}\left(\max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\right) \\
&= \max_{\theta_0\in\mathbb{R}} \frac{\Psi(\theta_0,\lambda)}{\alpha}\ .
\end{aligned}
$$

□

The following corollary shows the converse of Theorem 2.4.1.

Corollary 2.4.1. *Let $\lambda\in\mathbb{R}$, and $n\in\mathbb{N}_0$, then*

$$
\begin{aligned}
\rho(\lambda) \ge \frac{2n\pi_p}{\alpha} &\qquad\Longleftrightarrow\qquad \max_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\lambda)) \ge 2n\pi_p \\
\rho(\lambda) \le \frac{2n\pi_p}{\alpha} &\qquad\Longleftrightarrow\qquad \min_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\lambda)) \le 2n\pi_p
\end{aligned}
$$

Proof. First, we assume that

$$
\rho(\lambda) \ge \frac{2n\pi_p}{\alpha}\ ,
$$

then by Lemma 2.4.1,

$$
\max_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\lambda)) \ge \alpha\rho(\lambda) \ge 2n\pi_p\ .
$$

Similarly, if we assume

$$
\rho(\lambda) \le \frac{2n\pi_p}{\alpha}\ ,
$$

then by Lemma 2.4.1,

$$\min_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) \leq \alpha\rho(\lambda) \leq 2n\pi_p \ .$$

Conversely, if we take

$$\max_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) \geq 2n\pi_p \ ,$$

then we are left with two cases. Either there exists a $\theta_0 \in \mathbb{R}$ such that $\Psi(\theta_0, \lambda) = 2n\pi_p$, in which case, by Theorem 2.4.1,

$$\rho(\lambda) = \frac{2n\pi_p}{\alpha} \ ,$$

or alternatively, we have the case $\min_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda)) > 2n\pi_p$, for which we can use the negation of the statement

$$\rho(\lambda) \leq \frac{2n\pi_p}{\alpha} \qquad \Longrightarrow \qquad \min_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) \leq 2n\pi_p \ ,$$

which is proven above. This shows that

$$\max_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) \geq 2n\pi_p \qquad \Longrightarrow \qquad \rho(\lambda) \geq \frac{2n\pi_p}{\alpha} \ ,$$

the statement

$$\min_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) \leq 2n\pi_p \qquad \Longrightarrow \qquad \rho(\lambda) \leq \frac{2n\pi_p}{\alpha} \ ,$$

can be proved similarly.

□

The next key result shows the characterisation of the periodic eigenvalues in terms of the rotation number function. The proof uses Theorem 2.4.1 and Corollary 2.4.1.

Theorem 2.4.2. *Let $\lambda \in \mathbb{R}$ be such that $\rho(\lambda) = 2n\pi_p \, \alpha^{-1}$ (for some $n \in \mathbb{N}$), then there exists a $\theta_0 \in \mathbb{R}$, such that*

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ .$$

Proof. We can divide the values $\lambda \in \mathbb{R}$ into three distinct cases:

- *Case 1:* All values of $\theta_0 \in \mathbb{R}$ give $\Psi(\theta_0, \lambda) < 2n\pi_p$.

  As $\Psi(\theta_,\lambda) < 2n\pi_p$, for all $\theta_0 \in \mathbb{R}$, we have $\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda) < 2n\pi_p$. Combining this with Corollary 2.4.1, we have,

$$\rho(\lambda) \leq \max_{\theta_0 \in \mathbb{R}} \left(\frac{\Psi(\theta_0, \lambda)}{\alpha} \right) < \frac{2n\pi_p}{\alpha} \ .$$

- *Case 2:* There exists a $\theta_0 \in \mathbb{R}$ such that $\Psi(\theta_0, \lambda) = 2n\pi_p$.

  By Theorem 2.4.1, this means that

$$\rho(\lambda) = \frac{2n\pi_p}{\alpha} \ .$$

- *Case 3:* All values of $\theta_0 \in \mathbb{R}$ give $\Psi(\theta_0, \lambda) > 2n\pi_p$.

  As $\Psi(\theta_,\lambda) > 2n\pi_p$, for all $\theta_0 \in \mathbb{R}$, we have $\min_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda) > 2n\pi_p$. Combining this with Corollary 2.4.1, we have,

$$\rho(\lambda) \geq \min_{\theta_0 \in \mathbb{R}} \left(\frac{\Psi(\theta_0, \lambda)}{\alpha} \right) > \frac{2n\pi_p}{\alpha} \ .$$

Therefore, we have a trichotomy, and if $\rho(\lambda) = 2n\pi_p \, \alpha^{-1}$, then there must exist a $\theta_0 \in \mathbb{R}$ such that

$$\Psi(\theta_0, \lambda) = \frac{2n\pi_p}{\alpha} \ .$$

$\square$

This theorem can be restated in the following form, showing that the end-points of the plateaus at multiples of $2\pi_p\,\alpha^{-1}$ of the rotation number function are periodic eigenvalues.

Corollary 2.4.2. *Let* $n \in \mathbb{N}$, *then*

$$\underline{\lambda} := \min\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p\,\alpha^{-1}\}\ ,$$
$$\text{and} \qquad \overline{\lambda} := \max\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p\,\alpha^{-1}\}\ ,$$

are periodic eigenvalues of (10).

Proof. By the monotonicity of ρ in λ, shown in Theorem 2.3.3,

$$\lambda < \underline{\lambda} \qquad \Longrightarrow \qquad \rho(\lambda) \le \rho(\underline{\lambda}) = \frac{2n\pi_p}{\alpha}\ .$$

Also, by the assumption $\underline{\lambda} = \min\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p\,\alpha^{-1}\}$, we have

$$\lambda < \underline{\lambda} \qquad \Longrightarrow \qquad \rho(\lambda) \neq 2n\pi_p\,\alpha^{-1}\ ,$$

and so, for any $\lambda < \underline{\lambda}$,

$$\rho(\lambda) < \rho(\underline{\lambda}) = 2n\pi_p\,\alpha^{-1}\ . \tag{28}$$

By Corollary 2.3.3, as $\rho(\lambda) < 2n\pi_p\,\alpha^{-1}$,

$$\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda)) \le 2n\pi_p\ .$$

If there existed a $\theta_0 \in \mathbb{R}$ such that $\Psi(\theta_0, \lambda) = 2n\pi_p$, then by Theorem 2.4.1,

$$\rho(\lambda) = \frac{2n\pi_p}{\alpha}\ ,$$

which would contradict (28). Therefore

$$\max_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\lambda)) < 2n\pi_p \ , \tag{29}$$

for all $\theta_0 \in \mathbb{R}$.

At $\underline{\lambda}$, $\rho = 2n\pi_p\,\alpha^{-1}$, and so by Theorem 2.4.2, there exists a $\tilde{\theta}_0 \in \mathbb{R}$ such that

$$\Psi(\tilde{\theta}_0,\underline{\lambda}) = 2n\pi_p \ .$$

From the bound (29), for any $\lambda < \underline{\lambda}$,

$$\Psi(\hat{\theta}_0,\lambda) < \max_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\lambda)) < 2n\pi_p \ ,$$

for all $\hat{\theta}_0 \in \mathbb{R}$. Therefore, this initial value $\tilde{\theta}_0$ must attain the maximum value for the function $\Psi(\cdot,\underline{\lambda})$, i.e.

$$\max_{\theta_0\in\mathbb{R}}(\Psi(\theta_0,\underline{\lambda})) = \Psi(\tilde{\theta}_0,\underline{\lambda}) = 2n\pi_p \ ,$$

and since $\Psi(\cdot,\underline{\lambda})$ is a continuously differentiable function that attains a local maximum at $\tilde{\theta}_0$,

$$\partial_1\Psi(\tilde{\theta}_0,\lambda) = 0 \ .$$

Thus, both conditions of Theorem 2.2.3 are satisfied, and $\underline{\lambda}$ is therefore a periodic eigenvalue of the problem (10). The proof that $\overline{\lambda}$ also satisfies these conditions is similar.

□

Finally, we state a result regarding the case where the plateaus degenerate to a single point.

Corollary 2.4.3. *Fix $n \in \mathbb{N}$ and consider the values $\underline{\lambda}$, $\overline{\lambda}$ as defined in Corollary 2.4.2. The interval $[\underline{\lambda},\overline{\lambda}]$ degenerates to a single point, i.e. $\underline{\lambda} = \overline{\lambda}$, if and only if all solutions of* (10) *for the spectral value $\lambda = \underline{\lambda} = \overline{\lambda}$ are α-periodic.*

Proof. If we have

$$\underline{\lambda} = \min\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p\,\alpha^{-1}\} = \max\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p\,\alpha^{-1}\} = \overline{\lambda}\ ,$$

then by the montonicity of ρ, for all $\lambda < \underline{\lambda}$,

$$\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda)) < 2n\pi_p\ ,$$

and for all $\lambda > \underline{\lambda}$,

$$\min_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda)) > 2n\pi_p\ .$$

By the continuity of Ψ in both variables (a result of Lemma 2.2.2 and Theorem 2.2.4),

$$\Psi(\theta_0, \underline{\lambda}) = 2n\pi_p\ ,$$

for all $\theta_0 \in \mathbb{R}$. As $\Psi(\cdot, \underline{\lambda})$ is constant,

$$\partial_1\Psi(\theta_0, \underline{\lambda}) = 2n\pi_p\ ,$$

for all $\theta_0 \in \mathbb{R}$. Hence, by Theorem 2.2.3, all solutions to (10) for this spectral value, $\underline{\lambda}$, are α-periodic.

Conversely, if for any $\lambda \in \mathbb{R}$, all solutions of (10) are α-periodic, then all initial values $\theta_0 \in \mathbb{R}$ must give

$$\Psi(\theta_0, \lambda) = 2n\pi_p\ .$$

Therefore, for any $\tilde{\lambda} < \lambda$,

$$\max_{\theta_0 \in \mathbb{R}}(\Psi(\theta_0, \lambda)) < 2n\pi_p\ ,$$

by the monotonicity of Ψ in its second variable. Similarly, for any $\tilde{\lambda} > \lambda$,

$$\min_{\theta_0 \in \mathbb{R}} (\Psi(\theta_0, \lambda)) > 2n\pi_p \ .$$

By Corollary 2.4.1, this means that for any $\tilde{\lambda} < \lambda$,

$$\rho(\tilde{\lambda}) < \frac{2n\pi_p}{\alpha} = \rho(\lambda) \ ,$$

and for any $\tilde{\lambda} > \lambda$,

$$\rho(\tilde{\lambda}) > \frac{2n\pi_p}{\alpha} = \rho(\lambda) \ .$$

Therefore,

$$\underline{\lambda} = \min\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p \alpha^{-1}\} = \max\{\lambda \in \mathbb{R} : \rho(\lambda) = 2n\pi_p \alpha^{-1}\} = \overline{\lambda} \ .$$

□

For each fixed $\lambda \in \mathbb{R}$, the set of solutions of the problem (10) is a two-dimensional manifold in $L^1_{\text{loc}}(\mathbb{R})$. We note that by Corollary 2.4.3, if the pair of eigenvalues that occur at the end-points of any rotation number plateau, $\underline{\lambda}$, $\overline{\lambda}$, are equal, then all eigenfunctions are α-periodic. Thus, this solution manifold is completely included in the space of α-periodic functions.

We call the case for which there exists a value $\underline{\lambda}$, with

$$\Psi(\theta_0, \underline{\lambda}) = 2n\pi_p$$

for all $\theta_0 \in \mathbb{R}$, the *degenerate case*, and say that the renormalised Poincaré map is degenerate for this given spectral value $\underline{\lambda}$. We also refer to the periodic eigenvalue $\underline{\lambda}$ as a *degenerate eigenvalue*, note that for this value $n \in \mathbb{N}$, this is the unique eigenvalue for which the renormalised Poincaré map, $\Psi(\theta_0, \underline{\lambda}) = 2n\pi_p$, for any $\theta_0 \in \mathbb{R}$. In the next chapter, we derive conditions for which a perturbation of the function can cause a degenerate Poincaré map to become non-degenerate.

2.5 Effect of Perturbing the Potential on the Spectrum

In this chapter, we further examine the case of the renormalised Poincaré map being constant-valued in θ_0, for some periodic eigenvalue $\lambda \in \mathbb{R}$; the *degenerate case.* Given a potential for which there exists a degenerate eigenvalue λ, we consider the possibility of perturbing this potential by some other potential function, the effect of which is to make the map $\Psi(\cdot, \lambda)$ non-degenerate. We derive conditions on the perturbation that ensure this non-degeneracy. We also derive conditions that cause the resulting renormalised Poincaré map to oscillate about the levels $2n\pi_p$, in such a way as to guarantee the existence of extra eigenvalues that occur in the interior of the rotation number plateaus.

We start by introducing a perturbation to the potential in the equation (10), and quantify its effect on Ψ. Consider the modified equation

$$(\phi_p(u'))' + (\lambda - Q_1 - \varepsilon Q_2)\phi_p(u) = 0 \ , \tag{30}$$

for some perturbing potential $Q_2 \in L^1_{\text{loc}}(\mathbb{R})$, and some $\varepsilon > 0$. We now show how this perturbation affects the renormalised Poincaré map.

Lemma 2.5.1. *Consider the equation with potential $Q_1 \in L^1_{\text{loc}}(\mathbb{R})$, and a perturbation $Q_2 \in L^1_{\text{loc}}(\mathbb{R})$, we have*

$$\partial_3 \Psi(\theta_0, \lambda, \varepsilon) = \frac{-(q-1)}{(r(\alpha, r_0, \theta_0, \lambda, \varepsilon))^2} \int_0^\alpha Q_2(s) \ |u(s, r_0, \theta_0, \lambda, \varepsilon)|^p \, \mathrm{d}s \ .$$

Proof. We have the equation for the evolution of the Prüfer angle of the system (10), adapted for the εQ_2 term,

$$\theta'(t, r_0, \theta_0, \lambda, \varepsilon) = 1 + (q-1)(\lambda - Q_1(t) - \varepsilon Q_2(t)|\cos_p \theta(t, r_0, \theta_0, \lambda, \varepsilon)|^p \ ,$$

then differentiating with respect to the parameter ε, we get

$$
\begin{aligned}
\partial_5\theta'(t,r_0,\theta_0,\lambda,\varepsilon) &= \frac{\partial}{\partial\varepsilon}\Big(1+(q-1)(\lambda-Q_1-\varepsilon Q_2-1)|\cos_p\theta|^p\Big) \\
&= -(q-1)|\cos_p\theta|^p Q_2 - q(p-1)^{1/p}(\lambda-Q_1-\varepsilon Q_2-1)\phi_p(\cos_p\theta)\phi_q(\sin_q\theta)\partial_5\theta \\
&= -(q-1)|\cos_p\theta|^p Q_2 - 2\,\frac{r'}{r}\,\partial_5\theta\ .
\end{aligned}
$$

If we then multiply through by r^2, we have

$$
r^2\partial_5\theta' + 2rr'\partial_5\theta = -(q-1)Q_2\; r^2|\cos_p\theta|^p\ ,
$$

and since $\left(r^2\partial_5\theta\right)' = r^2\partial_5\theta' + 2rr'\partial_5\theta$,

$$
\begin{aligned}
\big(r^2\partial_5\theta\big)' &= -(q-1)Q_2\; r^2|\cos_p\theta|^p \\
&= -(q-1)(r^{2/p})^p|\cos_p\theta|^p \\
&= -(q-1)|r^{2/p}\cos_p\theta|^p \\
&= -(q-1)|u|^p\ .
\end{aligned}
$$

The initial angle, θ_0, does not depend on the parameter, ε, so

$$
\partial_5\theta(0,r_0,\theta_0,\lambda,\varepsilon) = \frac{\partial}{\partial\varepsilon}\Big(\theta_0\Big) = 0\ .
$$

Therefore integrating both sides of

$$
(r^2\partial_5\theta_0)' = -(q-1)|u|^p\ ,
$$

we get

$$
r^2(t,r_0,\theta_0,\lambda,\varepsilon)\,\partial_5\theta(t,r_0,\theta_0,\lambda,\varepsilon) = -(q-1)\int_0^t Q_2(s)\;|u(s,r_0,\theta_0,\lambda,\varepsilon)|^p\;\mathrm{d}s\ .
$$

Finally, letting $t=\alpha$,

$$\begin{aligned}\partial_3\Psi(\theta_0,\lambda,\varepsilon) &= \frac{\partial}{\partial\varepsilon}\Big(\theta(\alpha,r_0,\theta_0,\lambda,\varepsilon)-\theta_0\Big)\\ &= \partial_5\theta(\alpha,r_0,\theta_0,\lambda,\varepsilon)\\ &= \frac{-(q-1)}{(r(\alpha,r_0,\theta_0,\lambda,\varepsilon))^2}\int_0^\alpha Q_2(s)\;|u(s,r_0,\theta_0,\lambda,\varepsilon)|^p\;\mathrm{d}s\;.\end{aligned}$$

□

If we assume that we are perturbing away from a potential Q_1, and a spectral value λ, that gives a degenerate renormalised Poincaré map, then by Theorem 2.4.2 and Corollary 2.4.3, the resulting solution manifold of (10) will be entirely embedded within the space of α-periodic functions. As such, we will have $r(\alpha,r_0,\theta_0,\lambda,\varepsilon)=r_0$, at the point $\varepsilon=0$. This leads us to the following result.

Theorem 2.5.1. *Let $\lambda\in\mathbb{R}$ be such that*

$$\Psi(\theta_0,\lambda,0)=2n\pi_p\;,$$

for all $\theta_0\in\mathbb{R}$. Then if there exists a perturbing potential, Q_2, such that

$$\begin{aligned}&\int_0^\alpha Q_2(s)|u(s,r_0,\theta_1,\lambda,0)|^p\;\mathrm{d}s>0\\ &\int_0^\alpha Q_2(s)|u(s,r_0,\theta_2,\lambda,0)|^p\;\mathrm{d}s<0\\ &\qquad\qquad\vdots\\ &\int_0^\alpha Q_2(s)|u(s,r_0,\theta_{2j},\lambda,0)|^p\;\mathrm{d}s<0\end{aligned}$$

for some distinct $\theta_1<\theta_2<\ldots<\theta_{2j}\in[0,\pi_p)$, with $j\in\mathbb{N}$; then there exists some $\varepsilon>0$ such that:

a) There are $2j$ points, $\vartheta_1,\ldots,\vartheta_{2j}$, with

$$\Psi(\vartheta_i, \lambda, \varepsilon) = 2n\pi_p \ , \qquad \textit{for all} \quad i \in \{1, \ldots, 2j\} \ ,$$

$$\begin{aligned} \partial_1 \Psi(\vartheta_{2i}, \lambda, \varepsilon) &\geq 0 \ , \qquad\qquad \textit{for all} \quad i \in \{1, \ldots, j\} \ , \\ \partial_1 \Psi(\vartheta_{2i-1}, \lambda, \varepsilon) &\leq 0 \ , \end{aligned}$$

b) There are $2j$ periodic eigenvalues $\lambda_1, \ldots, \lambda_{2j}$, with

$$\Psi(\tilde{\theta}_i, \lambda_i, \varepsilon) = 2n\pi_p \ ,$$

for some corresponding $\tilde{\theta}_1, \ldots, \tilde{\theta}_{2j} \in \mathbb{R}$.

Proof. a) At $\varepsilon = 0$, all solutions are α-periodic. hence, $r(\alpha, r_0, \theta_0, \lambda, 0) = r_0$, for all $\theta_0 \in \mathbb{R}$. By Lemma 2.5.1,

$$\partial_3 \Psi(\theta_i, \lambda, 0) = -\frac{(q-1)}{r_0^2} \int_0^\alpha Q_2(s) \, |u(s, r_0, \theta_i, \lambda, 0)|^p \, \mathrm{d}s \ ,$$

for each value θ_i . Hence,

$$\begin{aligned} \partial_3 \Psi(\theta_1, \lambda, 0) &< 0 \\ \partial_3 \Psi(\theta_2, \lambda, 0) &> 0 \\ &\vdots \\ \partial_3 \Psi(\theta_{2j}, \lambda, 0) &> 0 \ . \end{aligned}$$

At $\varepsilon = 0$, $\Psi(\theta_0, \lambda) = 2n\pi_p$ for all $\theta_0 \in \mathbb{R}$, therefore there exists some $\tilde{\varepsilon} > 0$ such that

$$\Psi(\theta_1, \lambda, \tilde{\varepsilon}) < 2n\pi_p$$
$$\Psi(\theta_2, \lambda, \tilde{\varepsilon}) > 2n\pi_p$$
$$\vdots$$
$$\Psi(\theta_{2j}, \lambda, \tilde{\varepsilon}) > 2n\pi_p \; .$$

By Lemma 2.5.1, the renormalised Poincaré map is continuous in ε. So by the Intermediate Value Theorem, there must exist $2j$ points ϑ_i, for $i \in \{1, \dots, 2j\}$, with $\theta_i < \vartheta_i < \theta_{i+1}$, for $i \in \{1, \dots, 2j-1\}$, and $\theta_{2j} < \vartheta_{2j} < \theta_1 + \pi_p$, such that

$$\Psi(\vartheta_i, \lambda, \tilde{\varepsilon}) = 2n\pi_p \; , \qquad \text{for all} \quad i \in \{1, \dots, 2j\} \; ,$$

$$\partial_1 \Psi(\vartheta_{2i}, \lambda, \tilde{\varepsilon}) \geq 0 \; , \qquad \text{for all} \quad i \in \{1, \dots, j\} \; ,$$
$$\partial_1 \Psi(\vartheta_{2i-1}, \lambda, \tilde{\varepsilon}) \leq 0 \; .$$

b) By Theorem 2.2.3, λ is a periodic eigenvalue of (10) if and only if

$$\Psi(\theta_0, \lambda) = 2n\pi_p \qquad \text{and} \qquad \partial_1 \Psi(\theta_0, \lambda) = 0 \; ,$$

for some $n \in \mathbb{N}$. Consider the values ϑ_i defined in part a). We have

$$\Psi(\vartheta_i, \lambda, \tilde{\varepsilon}) = 2n\pi_p \; ,$$

for all $i \in \{1, \dots, 2j\}$. By Rolle's Theorem, and the periodicity of Ψ in the θ_0 variable, there must exist $2j$ points $\tilde{\vartheta}_i$ that are all distinct (mod π_p) such that

$$\partial_1 \Psi(\tilde{\vartheta}_i, \lambda, \tilde{\varepsilon}) = 0 \; .$$

for $i \in \{1, \ldots, 2j\}$, with

$$\begin{aligned} \Psi(\tilde{\vartheta}_{2i}, \lambda, \tilde{\varepsilon}) &> 2n\pi_p \;, \qquad \text{for all} \quad i \in \{1, \ldots, j\} \;, \\ \Psi(\tilde{\vartheta}_{2i-1}, \lambda, \tilde{\varepsilon}) &< 2n\pi_p \;. \end{aligned}$$

Therefore, by the monotonicity of Ψ in the λ variable, there exist $2j$ values $\lambda_1, \ldots, \lambda_{2j}$ such that

$$\Psi(\tilde{\theta}_i, \lambda_i) = 2n\pi_p \qquad \text{and} \qquad \partial_1 \Psi(\tilde{\theta}_i, \lambda_i) = 0 \;,$$

for some $\tilde{\theta}_1, \ldots, \tilde{\theta}_{2j} \in \mathbb{R}$.

$\square$

Out of these $2j$ eigenvalues, the smallest and largest form the end-points of a plateau in the rotation number function, and the remaining $2j-2$ are 'extra' eigenvalues in the interior.

The only remaining question is whether or not there exist perturbing potentials Q_2 that satisfy the properties stated in Theorem 2.5.1 . We can consider perturbing by functions of the form

$$Q_2 = \sum_{i=1}^{k} \alpha_i |u(t, r_0, \theta_i, \lambda, 0)|^p \;,$$

and then, by the conditions in Theorem 2.5.1 , it suffices to find values $\alpha_1, \ldots, \alpha_k$ that satisfy the system

$$\left(\int_0^\alpha |u(s, r_0, \theta_i, \lambda, 0)|^p |u(s, r_0, \theta_j, \lambda, 0)|^p \, \mathrm{d}s \right)_{i,j=1,\ldots,k} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_k \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ \vdots \\ -1 \end{pmatrix} . \tag{31}$$

Given an inner product space, I, with inner product $\langle \cdot \ , \ \cdot \rangle$, a square matrix A is *Gramian* in I, if it can be written as

$$A = (\langle v_i \ , \ v_j \rangle)_{i,j=1,\ldots,k} \ ,$$

for vectors $v_1, \ldots, v_k \in I$. It has been shown, [14, Theorem 7.2.10], that Gramian matrices are invertible if and only if the vectors $v_1, \ldots, v_k$ are linearly independent.

The matrix in the system (31) is Gramian in $L^2_{\mathrm{loc,per}}(\mathbb{R})$, with inner product

$$\langle f, g \rangle = \int_0^\alpha f \, g \quad .$$

Therefore, to show that there exists a Q_2 that satisfies the hypotheses of Theorem 2.5.1, it would suffice to show that the functions $|u(t, r_0, \theta_i, 0)|^p$ are linearly independent for all θ_i (for $i \in \{1, \ldots, 2j\}$).

We conclude this chapter with a result regarding the genericity of non-degenerate eigenvalues with respect to potentials $Q \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Theorem 2.5.2. *All periodic eigenvalues* λ *for the equation* (10) *are non-degenerate for Baire-almost all* $Q \in L^1_{\mathrm{loc}}(\mathbb{R})$.

Proof. For any fixed $n \in \mathbb{N}_0$, define $X_n \subset L^1_{\mathrm{loc}}(\mathbb{R})$ to be the set of potentials for which any periodic eigenvalue, $\lambda \in \mathbb{R}$, with

$$\Psi(\theta_1, \lambda) = 2n\pi_p \ ,$$

for some $\theta_1 \in \mathbb{R}$, is non-degenerate. That is, there exists a value θ_2 such that

$$\Psi(\theta_2, \lambda) \neq 2n\pi_p \ .$$

We first show that X_n is open in $L^1_{\mathrm{loc}}(\mathbb{R})$. Let

$$\delta := \sup_{\theta_0 \in \mathbb{R}} |\Psi(\theta_0, \lambda) - 2n\pi_p| \ .$$

Consider a perturbation on the potential, with some function Q_2, as in (30). By Lemma 2.5.1, Ψ is continuous in the ε variable. Hence, we can choose a value $\tilde{\varepsilon}(\delta) > 0$ such that

$$|\Psi(\theta_0, \lambda, \tilde{\varepsilon}(\delta)) - \Psi(\theta_0, \lambda, 0)| < \delta \ ,$$

for all $\theta_0 \in \mathbb{R}$. Therefore,

$$\sup_{\theta_0 \in \mathbb{R}} |\Psi(\theta_0, \lambda, \tilde{\varepsilon}) - 2n\pi_p| > 0 \ .$$

So for any perturbation, there exists a small enough $\varepsilon > 0$ such that the resulting periodic eigenvalues with $\Psi = 2n\pi_p$ remain non-degenerate. Therefore X_n is open in $L^1_{\text{loc}}(\mathbb{R})$.

Next, we prove that X_n is dense in $L^1_{\text{loc}}(\mathbb{R})$. To show this, we prove that any potential $Q_1 \in L^1_{\text{loc}}(\mathbb{R}) \setminus X_n$ is the limit of elements in X_n. Let Q_1 be a potential, such that there exists a periodic eigenvalue, λ with

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$. Again, consider a perturbation with some function Q_2. Let $\theta_1, \theta_2 \in \mathbb{R}$ be two initial angles that are distinct (mod π_p), and fix some $r_0 > 0$. From the initial values alone, we know that the resulting functions

$$|u(t, r_0, \theta_1, \lambda, 0)|^p \qquad \text{and} \qquad |u(t, r_0, \theta_2, \lambda, 0)|^p \ ,$$

(where $u(t, r_0, \theta_0, \lambda, 0)$ is the solution of (10) for initial values r_0, θ_0 at the spectral value λ), are linearly independent. Hence, there exist values α_1 , $\alpha_2 \in \mathbb{R}$ such that

$$Q_2 = \alpha_1 \, |u(t, r_0, \theta_1, \lambda, 0)|^p + \alpha_2 \, |u(t, r_0, \theta_2, \lambda, 0)|^p \ ,$$

such that

$$\Psi(\theta_1, \lambda, \tilde{\varepsilon}) > 2n\pi_p \qquad \text{and} \qquad \Psi(\theta_2, \lambda, \tilde{\varepsilon}) < 2n\pi_p \ ,$$

for some $\tilde{\varepsilon} > 0$. Therefore, a perturbation resulting from this Q_2 causes the degeneracy of the periodic eigenvalue to collapse. So X_n is dense in $L^1_{\text{loc}}(\mathbb{R})$.
Thus, each set $L^1_{\text{loc}}(\mathbb{R}) \setminus X_n$ for $n \in \mathbb{N}$ is nowhere dense in $L^1_{\text{loc}}(\mathbb{R})$, and the union

$$\bigcup_{n=1}^{\infty} L^1_{\text{loc}}(\mathbb{R}) \setminus X_n \ ,$$

is a set of the first Baire category in $L^1_{\text{loc}}(\mathbb{R})$, which gives us the result.

$\square$

We will use these perturbation arguments in Chapter 5, to prove the existence of ‘extra’ plateaus at levels of the rotation number that are not multiples of $\pi_p \, \alpha^{-1}$. These plateaus are linked with the non-degeneracy of eigenvalues of the iterated periodic problem over m periods for $m > 2$. As a consequence of Theorem 1.0.9, such plateaus do not exist for $p = 2$. We link the existence of a perturbing function Q_2 capable of creating such non-degeneracy, with the linear independence of sums of certain solutions, and show that such linear independence only occurs for $p \neq 2$.

3 Numerical Schemes and Stability Analysis

In Chapter 2.4, we introduced the concept of a periodic eigenvalue being *degenerate*, in the sense that the renormalised Poincaré map for that value of λ is constant in $\theta_0 \in \mathbb{R}$, i.e. for some $n \in \mathbb{N}$,

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$. An example we will use throughout this thesis is the case where the potential Q is constant, for which all periodic eigenvalues are degenerate.

Theorem 3.0.1. *If the potential Q in the equation* (10) *is a constant, then given an initial radius $r_0 > 0$, initial angle $\theta_0 \in \mathbb{R}$ and spectral value $\lambda \in \mathbb{R}$, the solution of* (10) *takes one of the following forms:*

- *Case 1:* $\quad \lambda - Q > 0$

  The solution u is a p-trigonometric function, given by:

  $$u(t) = A \cos_p((\lambda - Q)^{1/p}(t - t_0)) \ ,$$

  for some $A, t_0 \in \mathbb{R}$.

- *Case 2:* $\quad \lambda - Q = 0$

  The solution u is a linear function, given by:

  $$u(t) = At + B \ ,$$

  for some $A, B \in \mathbb{R}$.

- *Case 3:* $\quad \lambda - Q < 0$ *and* $|(Q - \lambda)^{1/p} \cot_p \theta_0| > (p-1)^{1/p}$

  The solution u is a p-hyperbolic function, given by:

  $$u(t) = A \cosh_p((Q - \lambda)^{1/p}(t - t_0)) \ ,$$

for some $A, t_0 \in \mathbb{R}$.

- *Case 4:* $\lambda - Q < 0$ *and* $|(Q-\lambda)^{1/p} \cot_p \theta_0| < (p-1)^{1/p}$

  The solution u *is a* p*-hyperbolic function, given by:*

$$u(t) = A \sinh_p((Q-\lambda)^{1/p}(t-t_0)) \, ,$$

  for some $A, t_0 \in \mathbb{R}$.

- *Case 5:* $\lambda - Q < 0$ *and* $(Q-\lambda)^{1/p} \cot_p \theta_0 = (p-1)^{1/p}$

  The solution u *is an exponential function, given by:*

$$u(t) = A \, \exp(-(q-1)^{1/p} \, (Q-\lambda)^{1/p} \, t) \, ,$$

  for some $A \in \mathbb{R}$.

- *Case 6:* $\lambda - Q < 0$ *and* $(Q-\lambda)^{1/p} \cot_p \theta_0 = -(p-1)^{1/p}$

  The solution u *is an exponential function, given by:*

$$u(t) = A \, \exp((q-1)^{1/p} \, (Q-\lambda)^{1/p} \, t) \, ,$$

  for some $A \in \mathbb{R}$.

Furthermore, for all periodic eigenvalues, $\lambda - Q > 0$*, and the periodic eigenvalues are given by*

$$\lambda_n = Q + \left(\frac{2n\pi_p}{\alpha}\right)^p \, ,$$

for all $n \in \mathbb{N}$*. For each* n*, the periodic eigenvalue is degenerate.*

For the case of a constant potential Q, by Corollary 2.4.3, each of the periodic eigenvalues results in an α-periodic solution for all initial values. Thus, the solution manifold for each eigenvalue is completely embedded in the space of α-periodic functions.

We now consider the iterated problem, that is, the question of which values $\lambda \in \mathbb{R}$ give periodic solutions over m periods, with $m > 2$. Trivially, all periodic eigenvalues are also periodic eigenvalues of the iterated problem, as α-periodic solutions are $m\alpha$-periodic for all $m \in \mathbb{N}$. As a result of Theorem 1.0.9, in the linear case $p = 2$, any value $\lambda \in \mathbb{R}$ that admits a solution u that is periodic after m intervals, but is not (anti-)periodic on $[0, \alpha)$ has a solution space that is entirely embedded in the space of $m\alpha$-periodic functions.

Therefore, by Corollary 2.4.3, any periodic eigenvalue of the iterated periodic problem over m periods, with $m > 2$, that is not an (anti-)periodic eigenvalue of the problem over a single period, is a degenerate eigenvalue. In this Chapter, we use numerical evidence to illustrate that this is not true for the nonlinear case, $p \neq 2$.

For any value λ that is an eigenvalue of the iterated periodic problem over m periods, for which the corresponding eigenfunction has n zeros, we have

$$\rho(\lambda) = \frac{2n\pi_p}{m\alpha} \,,$$

(see Theorem 5.1.2 for details). By Corollary 2.4.2, the existence of a pair of non-degenerate periodic eigenvalues of the iterated problem over m periods (for $m > 2$) is equivalent to the existence of a rotation number plateau, over which ρ does not equal an integer multiple of $\pi_p\,\alpha^{-1}$.

To show the existence of non-degenerate eigenvalues of the iterated problem (for $m > 2$) in the nonlinear case, we introduce a numerical scheme to approximate the rotation number function of (10) for certain potentials, and use it to find these extra plateaus, should they exist. For this scheme, we take all potentials Q to be piecewise-constant, and use the analytic results derived in Chapter 2, as well as the form of the solution given in Theorem 3.0.1, to find the number of zeros the solution has over each subinterval over which Q is constant. We then prove bounds for the difference between a normalisation of this enumeration of zeros and the rotation number function.

3.1 Approximations of the p-Trigonometric Functions

Our numerical scheme for the approximation of the rotation number function relies on the p-trigonometric functions, and we will therefore need reliable numerical schemes to approximate them. For the $\sin_p$ function, we observe the following result, due to Biezuner, Ercole and Martins [3].

Theorem 3.1.1. *Let $t \in [0, {}^{\pi_p}/_2]$, and define the functions $\psi_i : [0, {}^{\pi_p}/_2] \to \mathbb{R}$, for any $i \in \mathbb{N}_0$ by the recurrence relation*

$$\begin{cases} (\phi_p(\psi'_{i+1}))' + \phi_p(\psi_i) = 0 \\ \psi_{i+1}(0) = \psi'_{i+1}({}^{\pi_p}/_2) = 0 \ , \end{cases}$$

with initial function $\psi_0 \equiv 1$. Then the sequence of functions

$$\frac{(p-1)^{1/p}\, \psi_i}{||\psi_i||_\infty}$$

converges uniformly to $\sin_p$.

The above recurrence relation can be written as

$$\psi_{i+1}(t) = \int_0^t \phi_q \left(\int_s^{\pi_p/2} \phi_p(\psi_i(r))\, \mathrm{d}r \right) \mathrm{d}s \ ,$$

and we can approximate both integrals using a quadrature rule. In Chapter 6.1, we give the pseudocode for an implementation of this using the trapezium rule. We calculate a table of values of the $\sin_p$ function on the interval $[0, {}^{\pi_p}/_2)$, and then use the periodicity properties to evaluate the function at any other values.

By (16), the $\cos_p$ function is simply a shift of the $\sin_p$ function, and so this table of values will suffice for the purpose of approximating values of $\cos_p$ as well. Similarly, given that $\cot_p$ is the ratio of $\cos_p$ and $(p-1)^{1/p}\phi_q(\sin_q)$, this table is also sufficient to compute $\cot_p$. However, the algorithm also requires an approximation of the arccot_p function. One could, in principle, derive the values from the approximation of $\cot_p$, but we detail a more efficient way below.

To approximate the arccot_p function, we use the formulation given in Theorem 2.1.6, combined with a geometric expansion of the integrand. This expansion must be done about different values of t depending on the value of t being approximated.

3.1.1 Expanding the arccot_p Function for $|t| < (p-1)^{1/p}$

For $|t| < (p-1)^{1/p}$, we expand about $t = 0$, this can be achieved by using the identity

$$\operatorname{arccot}_p(t) + \arctan_p(t) = \frac{\pi_p}{2} \ ,$$

for all $t \in \mathbb{R}$, which can itself be derived from the definition of $\arctan_p$

$$\arctan_p(t) := \int_0^t \frac{1}{1 + \frac{s^p}{(p-1)}} \, \mathrm{d}s \ .$$

Expanding this integrand using a geometric series, for $|t| < (p-1)^{1/p}$, we get

$$\begin{aligned} \arctan_p(t) = \int_0^t \frac{1}{1 + \frac{s^p}{(p-1)}} \, \mathrm{d}s &= \int_0^t \sum_{j=0}^{\infty} \frac{s^{pj}}{(1-p)^j} \, \mathrm{d}s \\ &= \sum_{j=0}^{\infty} \frac{1}{(1-p)^j} \int_0^t s^{pj} \, \mathrm{d}s \\ &= \sum_{j=0}^{\infty} \frac{t^{pj+1}}{(1-p)^j (pj+1)} \ . \end{aligned}$$

Therefore,

$$\begin{aligned} \operatorname{arccot}_p(t) &= \frac{\pi_p}{2} - \arctan_p(t) \\ &= \frac{\pi_p}{2} - \sum_{j=0}^{\infty} \frac{t^{pj+1}}{(1-p)^j (pj+1)} \ . \end{aligned}$$

We can then find error bounds on this expansion, let $N \in \mathbb{N}$,

$$
\begin{aligned}
\left|\operatorname{arccot}_p(t) - \sum_{j=0}^{N} \frac{t^{pj+1}}{(1-p)^j(pj+1)}\right| &= \left|\sum_{j=0}^{\infty} \frac{t^{pj+1}}{(1-p)^j(pj+1)} - \sum_{j=0}^{N} \frac{t^{pj+1}}{(1-p)^j(pj+1)}\right| \\
&= \left|\sum_{j=N+1}^{\infty} \frac{t^{pj+1}}{(1-p)^j(pj+1)}\right| \\
&\leq \left|\frac{t^{p(N+1)+1}}{(1-p)^{N+1}(p(N+1)+1)}\right| \quad ,
\end{aligned}
$$

as the term in the summation has an alternating sign. If we then use the restriction that $|t| < (p-1)^{1/p}$, we have

$$
\begin{aligned}
\left|\operatorname{arccot}_p(t) - \sum_{j=0}^{N} \frac{t^{pj+1}}{(1-p)^j(pj+1)}\right| &\leq \left|\frac{t^{p(N+1)+1}}{(1-p)^{N+1}(p(N+1)+1)}\right| \\
&< \left|\frac{(p-1)^{N+1}(p-1)^{1/p}}{(1-p)^{N+1}(p(N+1)+1)}\right| \\
&= \left|\frac{(p-1)^{1/p}}{p(N+1)+1}\right| \quad .
\end{aligned}
$$

3.1.2 Expanding the arccot_p Function for $|t| > (p-1)^{1/p}$

For $|t| > (p-1)^{1/p}$, we derive an expansion about $t = +\infty$, we start with the substitution $r = s^{1-p}$, then

$$
\begin{aligned}
\operatorname{arccot}_p(t) &= \int_t^{\infty} \frac{1}{1 + \frac{s^p}{(p-1)}} \,\mathrm{d}s \\
&= \frac{1}{p-1} \int_0^{t^{1-p}} \frac{\mathrm{d}r}{s^{-p} + \frac{1}{(p-1)}} \\
&= \int_0^{t^{1-p}} \frac{\mathrm{d}r}{(p-1)r^q + 1} \quad .
\end{aligned}
$$

As before, we apply a geometric series, to get

$$
\begin{aligned}
\int_0^{t^{1-p}} \frac{\mathrm{d}r}{(p-1)r^q+1} &= \int_0^{t^{1-p}} \sum_{j=0}^{\infty} (1-p)^j r^{qj} \mathrm{d}r \\
&= \sum_{j=0}^{\infty} (1-p)^j \int_0^{t^{1-p}} r^{qj} \mathrm{d}r \\
&= \sum_{j=0}^{\infty} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} .
\end{aligned}
$$

This geometric series will converge for all $|(1-p)t^{(1-p)q}| < 1$, or equivalently, $|t| > (p-1)^{1/p}$.

We can also find error bounds for this expansion, again let $N \in \mathbb{N}$, then

$$
\begin{aligned}
\left| \operatorname{arccot}_p(t) - \sum_{j=0}^{N} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} \right| &= \left| \sum_{j=0}^{\infty} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} - \sum_{j=0}^{N} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} \right| \\
&= \left| \sum_{j=N+1}^{\infty} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} \right| \\
&\leq \left| \frac{(1-p)^{N+1} t^{(1-p)(q(N+1)+1)}}{q(N+1)+1} \right| .
\end{aligned}
$$

Where we once again use the fact that the term in the summation has an alternating sign. If we now take $|t| > (p-1)^{1/p}$, we have

$$
\begin{aligned}
\left| \operatorname{arccot}_p(t) - \sum_{j=0}^{N} \frac{(1-p)^j t^{(1-p)(qj+1)}}{qj+1} \right| &\leq \left| \frac{(1-p)^{N+1} t^{(1-p)(q(N+1)+1)}}{q(N+1)+1} \right| \\
&< \left| \frac{(1-p)^{N+1} (p-1)^{1/p(1-p)(q(N+1)+1)}}{q(N+1)+1} \right| \\
&= \left| \frac{(1-p)^{N+1} (p-1)^{-1/q} (p-1)^{-(N+1)}}{q(N+1)+1} \right| \\
&= \left| \frac{(p-1)^{-1/q}}{q(N+1)+1} \right| \\
&= \left| \frac{(q-1)^{1/q}}{q(N+1)+1} \right| .
\end{aligned}
$$

3.1.3 Expanding the arccot_p Function for $t = (p-1)^{1/p}$

As the asymptotic expansion about $t = 0$ converges for all $|t| < (p-1)^{1/p}$, and the asymptotic expansion about $t = +\infty$ converges for all $|t| > (p-1)^{1/p}$, our final problem is to find an expansion about $t = (p-1)^{1/p}$.

Again, using the identity

$$\arctan_p(t) + \operatorname{arccot}_p(t) = {}^{\pi_p}/_2 \ ,$$

for all $t \in \mathbb{R}$, we can reduce the problem to approximating the integral

$$\int_0^{(p-1)^{1/p}} \frac{\mathrm{d}s}{1 + {}^{s^p}/_{(p-1)}} \ .$$

Then using the substitution $s = (p-1)^{1/p} r$, we have

$$\int_0^{(p-1)^{1/p}} \frac{\mathrm{d}s}{1 + {}^{s^p}/_{(p-1)}} = (p-1)^{1/p} \int_0^1 \frac{\mathrm{d}r}{1 + r^p} \ .$$

Next, we repeatedly perform integration-by-parts to derive an expansion for this integral.

$$\begin{aligned}
\int_0^1 \frac{\mathrm{d}r}{1+r^p} &= \left[\frac{r}{1+r^p}\right]_0^1 + p\int_0^1 \frac{r^p}{(1+r^p)^2}\mathrm{d}r \\
&= \frac{1}{2} + \frac{p}{p+1}\left[\frac{r^{p+1}}{(1+r^p)^2}\right]_0^1 + \frac{2p^2}{p+1}\int_0^1 \frac{r^{2p}}{(1+r^p)^3}\mathrm{d}r \\
&= \frac{1}{2} + \frac{p}{4(p+1)} + \frac{2p^2}{(p+1)(2p+1)}\left[\frac{r^{2p+1}}{(1+r^p)^3}\right]_0^1 \\
&\qquad + \frac{6p^3}{(p+1)(2p+1)}\int_0^1 \frac{r^{3p}}{(1+r^p)^4}\mathrm{d}r \ .
\end{aligned}$$

Inductively, we can show that

$$\int_0^1 \frac{\mathrm{d}r}{1+r^p} = \sum_{i=0}^{N} \frac{i!\, p^i}{2^{i+1}(p+1)\dots(ip+1)} + \frac{(N+1)!\, p^{N+1}}{(p+1)\dots(Np+1)}\int_0^1 \frac{r^{(N+1)p}}{(1+r^p)^{N+2}}\mathrm{d}r \ . \quad (32)$$

We now consider a bound for the remainder term, first, using the Binomial theorem, we have

$$(1+r^p)^{N+2} = \sum_{i=0}^{N+2} \binom{N+2}{i} r^i p; ,$$

and since $r \geq 0$ for the domain of integration,

$$(1+r^p)^{N+2} > (N+2)r^p \ .$$

The integrand in the left-hand side of (32) is therefore bounded by

$$\frac{r^{(N+1)p}}{(1+r^p)^{N+2}} < \frac{r^{Np}}{N+2} \ .$$

Hence, the remainder term of the expansion (32) is bounded by

$$\begin{aligned} \frac{(N+1)!\, p^{N+1}}{(p+1)\ldots(Np+1)} \int_0^1 \frac{r^{(N+1)p}}{(1+r^p)^{N+2}} \mathrm{d}r &< \frac{(N+1)!\, p^{N+1}}{(p+1)\ldots(Np+1)} \left[\frac{r^{Np+1}}{(N+2)(Np+1)} \right]_0^1 \\ &= \frac{(N+1)!\, p^{N+1}}{(p+1)\ldots(Np+1)} \, \frac{1}{(N+2)(Np+1)} \ . \end{aligned}$$

Finally, as

$$\prod_{i=1}^{N} (ip+1) > N!\, p^N \ ,$$

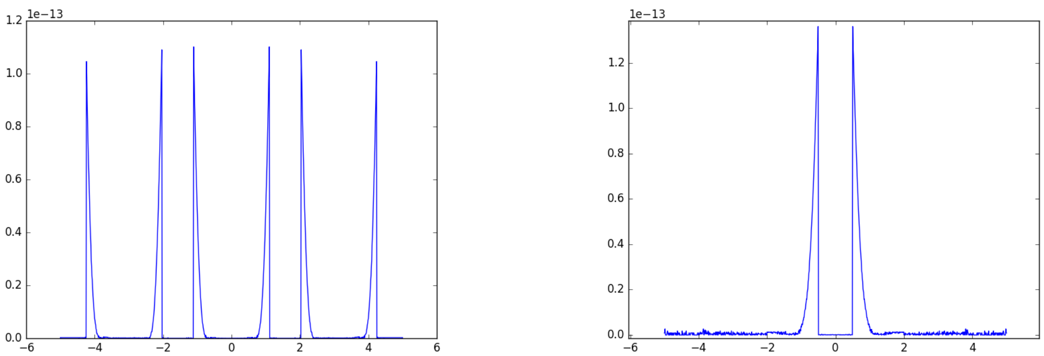
we have the bound

$$\left| \int_0^1 \frac{\mathrm{d}r}{1+r^p} - \sum_{i=0}^{N} \frac{i!\, p^i}{2^{i+1}(p+1)\dots(ip+1)} \right| = \frac{(N+1)!\, p^{N+1}}{(p+1)\dots(Np+1)} \int_0^1 \frac{r^{(N+1)p}}{(1+r^p)^{N+2}} \mathrm{d}r$$
$$< \frac{(N+1)\, p}{(N+2)(Np+1)}$$
$$< \frac{p}{Np+1}$$
$$< \frac{1}{N} \, .$$

Therefore,

$$\left| \operatorname{arccot}_p(1) - \left(\frac{\pi_p}{2} - (p-1)^{1/p} \sum_{i=0}^{N} \frac{i!\, p^i}{2^{i+1}(p+1)\dots(2ip+1)} \right) \right| < \frac{(p-1)^{1/p}}{N} \, .$$

To gauge the accuracy of our scheme for approximating arccot_p, we choose $p = 2$ and use a built-in implementation of cot to plot the values of the functions $\operatorname{arccot}_p(\cot(t)) - t$ and $\cot(\operatorname{arccot}_p(t)) - t$. Figure 2 shows that in both cases, the errors are of the order 10^{-13}. The plots appear to spike at several points in the domain, which can be accounted for by the different methods used to approximate the function for different values of $t \in \mathbb{R}$.



(a) The values of $|\operatorname{arccot}_p(\cot(t)) - t|$ for $p = 2$, using our implementation of arccot_p.

(b) The values of $|\cot(\operatorname{arccot}_p(t)) - t|$ for $p = 2$, using our implementation of arccot_p.

Figure 2: Error plots of the approximation of arccot_p.

We give the pseudocode for an implementation of this approximation in Chapter 6.2.

3.2 Approximations of the p-Hyperbolic Functions

To approximate the $\sinh_p$ function, we use the recurrence relation

$$\begin{cases} (\phi_p(\psi'_{i+1}))' - \phi_p(\psi_i) = 0 \\ \psi_{i+1}(0) = 0 \ , \ \psi'_{i+1}(0) = 1 \ , \end{cases}$$

with $\psi_0 \equiv 1$, and derive the equivalent integral relation

$$\psi_{i+1}(t) = t + \int_0^t \phi_q \left(\int_0^s \phi_p(\psi_i(r)) \, \mathrm{d}r \right) \mathrm{d}s \ .$$

Then the resulting functional sequence

$$\frac{(p-1)^{1/p} \, \psi_i}{\psi'_i(0)}$$

can be shown to converge uniformly to $\sinh_p$. A sequence can similarly be derived for $\cosh_p$. Unlike $\sin_p$ and $\cos_p$, the functions $\sinh_p$ and $\cosh_p$ are not related by a simple shift, and so both must be approximated. However, there is an asymptotic relation that connects the $\sinh_p$ and $\cosh_p$ functions with an exponential function, and this can be used to approximate these functions for large values of $t \in \mathbb{R}$.

First, we observe the different solutions to the ODE

$$(\phi_p(u'(t)))' - \phi_p(u(t)) = 0 \ ,$$

with varying initial conditions. Obviously, any variation in the initial Prüfer radius r, would only result in the solution being changed by a constant factor. Therefore, we consider solutions with varying initial Prüfer angles θ_0, on the domain $(-{}^{\pi_p}/_2, {}^{\pi_p}/_2]$. By Lemma 2.2.3, this is sufficient for all initial angles, as the angular component θ, is π_p-periodic with respect to its initial value.

Consider the Pythagorean-type identity for the p-hyperbolic functions,

$$\frac{|\cosh_p|^p}{p-1} - \frac{|\sin_q|^q}{q-1} = 1 \ ,$$

dividing through by $|\sin_q|^q$ gives

$$|\coth_p|^p - (p-1) = \frac{1}{|\sinh_p|^p} > 0 \ ,$$

and hence

$$|\coth_p(t)| > (p-1)^{1/p} \ ,$$

for all $t \in \mathbb{R}$. Similarly, we can derive $|\tanh_p(t)| < (p-1)^{1/p}$ for all $t \in \mathbb{R}$.

Therefore, any solution $u(t) = \sinh_p(t - t_0)$, for some $t_0 \in \mathbb{R}$ has

$$|\cot_p(\theta_0)| = \left|\frac{u(0)}{u'(0)}\right| = \left|\frac{\sinh_p(t_0)}{(p-1)^{1/p}\phi_q(\cosh_q(t_0))}\right| = |\coth_p(t_0)| > (p-1)^{1/p} \ .$$

Similarly, for equations of the form $u(t) = \cosh_p(t - t_0)$ have

$$|\cot_p(\theta_0)| = \left|\frac{u(0)}{u'(0)}\right| = \left|\frac{\cosh_p(t_0)}{(p-1)^{1/p}\phi_q(\sinh_q(t_0))}\right| = |\tanh_p(t_0)| < (p-1)^{1/p} \ .$$

We have two remaining cases. The first being solutions that are multiples of $u(t) = \exp((q-1)^{1/p}\, t)$, for which

$$-\cot_p(\theta_0) = \frac{u(0)}{u'(0)} = \frac{\exp((q-1)^{1/p}\, 0)}{(q-1)^{1/p}\exp((q-1)^{1/p}\, 0)} = (p-1)^{1/p} \ ,$$

and finally constant multiples of $u(t) = \exp(-(q-1)^{1/p}\, t)$, which result in $-\cot_p(\theta_0) = -(p-1)^{1/p}$.

The set of initial Prüfer angles for these four cases, after being mapped through the $\cot_p$ function, comprises the entirety of $\mathbb{R}$. The image of $\mathbb{R}$ through the arccot_p is the set $(-\pi_p/2, \pi_p/2]$. As before, the set of solutions with these initial values consists of the whole solution manifold, up to constant multiples. Thus, all of the solutions of the ODE

$$(\phi_p(u'(t)))' - \phi_p(u(t)) = 0$$

can be written as one of the four cases considered.

Lemma 3.2.1. *The functions* $\sinh_p$ *and* $\cosh_p$ *are connected to the function* $\exp((q-1)^{1/p}\cdot)$ *through the following asymptotic relation,*

$$\sinh_p(t) = \exp((q-1)^{1/p}t) + O(t) \qquad \text{and} \qquad \cosh_p(t) = \exp((q-1)^{1/p}\,t) + O(t)\ ,$$

as $t \to \infty$.

Proof. We have the following formula for the inverse of $\sinh_p$,

$$\operatorname{arcsinh}_p(t) = \int_0^t \frac{\mathrm{d}s}{(1 + s^p/(p-1))^{1/p}}\ , \tag{33}$$

for all $t > 0$. As this integrand is bounded above by 1, and $\sinh_p(0) = 0$, we have the inequality

$$\operatorname{arcsinh}_p(t) < \int_0^t \mathrm{d}s = t\ . \tag{34}$$

As the integrand in (33) is positive, the function $\operatorname{arcsinh}_p$ is strictly monotonically increasing, and therefore, $\sinh_p$ is also a strictly increasing function. Therefore, mapping both sides of the inequality (34) through the $\sinh_p$ function preserves the inequality, and we get

$$\sinh_p(t) > t\ ,$$

for all $t > 0$. From Lemma 2.2.1, the Prüfer radius of this solution is given by

$$r = \left(\frac{|x|^p}{p-1} + |y|^q\right)^{1/2} = \left(\frac{|\sinh_p|^p}{p-1} + \frac{|\cosh_q|^q}{q-1}\right)^{1/2}\ ,$$

and using the Pythagorean-type identity for p-hyperbolic functions,

$$r = \left(1 + 2\frac{|\sinh_p|^p}{p-1}\right)^{1/2} .$$

Therefore, from our lower bound on sinh. we have

$$r(t) > (1 + 2(q-1)t^p)^{1/2} > \sqrt{2(q-1)}\, t^{p/2} ,$$

for all $t \geq 0$. The solution $\cosh_p$ has radius

$$r = \left(\frac{|\cosh_p|^p}{p-1} + \frac{|\sinh_q|^q}{q-1}\right)^{1/2} = \left(1 + 2\frac{|\sinh_q|^q}{q-1}\right)^{1/2} ,$$

hence we have the bound

$$r(t) > (1 + 2(p-1)t^q)^{1/2} > \sqrt{2(p-1)}\, t^{q/2} , \tag{35}$$

for all $t \geq 0$. The solution $\exp((q-1)^{1/p} \cdot)$ has radius

$$r(t) = \left(\frac{\exp((q-1)^{1/p}\, t)}{p-1} + (q-1)^{q-1} \exp((q-1)^{1/p}\, t)\right)^{1/2} .$$

Therefore, for all $\theta_0 \in A := (-\pi_p/2, \pi_p/2] \setminus \{\mathrm{arccot}_p((p-1)^{1/p})\}$, the Prüfer radius is divergent as $t \to \infty$. By Lemma 2.2.2, the effect of varying the initial Prüfer angle on the angular component, θ, is given by

$$\frac{\partial\theta}{\partial\theta_0}(t) = \left(\frac{r_0}{r(t)}\right)^2 ,$$

Using the radial bound (35), for any $\theta_0 \in A$,

$$\frac{\partial\theta}{\partial\theta_0}(t) \to 0 ,$$

as $t \to \infty$. Therefore, for all Prüfer angles of solutions to this equation,

$$\theta(t,\theta_1) = \theta(t,\theta_2) + o(1)\ .$$

as $t \to \infty$, for any $\theta_1, \theta_2 \in A$. The function $\cos_p$ is continuously differentiable, and so

$$\cos_p \theta(t,\theta_1) = \cos_p \theta(t,\theta_2) + O(1)\ .$$

Similarly, $\phi_p(\cos_p)\phi_q(\sin_q)$ is continuously differentiable in a neighbourhood of $-\operatorname{arccot}_p((p-1)^{1/p})$, hence

$$\exp(\phi_p(\cos_p \theta(t,\theta_1))\phi_p(\cos_p \theta(t,\theta_1))) = \exp(\phi_p(\cos_p \theta(t,\theta_2))\phi_p(\cos_p \theta(t,\theta_2))) + O(1)\ ,$$

and so, by the radial equation, (22) (noting that $\lambda - Q \equiv -1$ here),

$$r(t,\theta_1) = r(t,\theta_2) + O(1)\,.$$

Therefore, $u(t,\theta_1) = u(t,\theta_2) + O(1)$, for any θ_1 , $\theta_2 \in A$.

□

So for large values of $t \in \mathbb{R}$, we can approximate the $\sinh_p$ and $\cosh_p$ functions, and their shifts, using the function $\exp((q-1)^{1/p}\,\cdot)$. Fix $k \in \mathbb{N}$, then by radial bound (35) and Lemma 2.2.2,

$$\frac{\partial \theta}{\partial \theta_0}(t) < \max\left\{ \frac{1}{\sqrt{2(p-1)}\,(k\alpha)^{p/2}}\,, \frac{1}{\sqrt{2(q-1)}\,(k\alpha)^{q/2}} \right\}$$

for all $t \geq k\alpha$.

Therefore, the error resulting from truncating the tables for $\sinh_p$ and $\cosh_p$ after k periods is bounded by

$$\pi_p \max\left\{ \frac{1}{\sqrt{2(p-1)}\,(k\alpha)^{p/2}}\,, \frac{1}{\sqrt{2(q-1)}\,(k\alpha)^{q/2}} \right\}\ .$$

Finally, we briefly note that approximations for the $\operatorname{arccoth}_p$ function can be derived similarly to the approximations for the arccot_p function derived in Chapter 3.1. We use the formulation,

$$\operatorname{arccoth}_p(t) = \int_t^\infty \frac{\mathrm{d}s}{{}^{s^p}\!/_{(p\,-\,1)} - 1}\ ,$$

and once again expand the integrand using a geometric series.

3.3 Approximations of the Rotation Number

The rotation number gives an asymptotic average for how often solutions of our the equation (10) rotate about the phaseplane, however, we note that the system completing a single rotation about the phaseplane results in the solution attaining two zeros. We can therefore derive an equivalent definition of the rotation number, in terms of an enumeration of the zeros of the solution over the interval $[0, +\infty)$, and this definition can then be implemented into our numerical scheme.

Theorem 3.3.1. *Fix some $r_0 > 0$, $\theta_0, \lambda \in \mathbb{R}$, let z be the function defined as*

$$\begin{aligned} z &: [0, +\infty) \to \mathbb{N}_0 \\ t &\mapsto \#\{s \in (0,t) \mid u(s, r_0, \theta_0, \lambda) = 0\} \ , \end{aligned}$$

then the value of the rotation number for this solution at this value of λ is given by

$$\rho(\lambda) = \lim_{t \to \infty} \frac{\pi_p \, z(t)}{t} \ .$$

Proof. By the definition of the p-Prüfer transform, we have $u = r^{2/p} \cos_p \theta$. Therefore, any solution $u(\cdot, r_0, \theta_0, \lambda)$ has a zero at the point $t > 0$ if and only if $\theta(t, r_0, \theta_0, \lambda) = {}^{\pi_p}\!/_2 + k\pi_p$, for some $k \in \mathbb{Z}$.

Given this, we observe that by equation (23),

$$\begin{aligned} \theta'(t, r_0, \theta_0, \lambda) &= 1 + (q-1)(\lambda - Q - 1)|\cos_p \theta(t, r_0, \theta_0, \lambda)|^p \\ &= 1 \\ &> 0 \ . \end{aligned}$$

Hence all of the zeros of the solution u occur discretely, and have total measure zero on $\mathbb{R}$.

Let $R(t) := \theta(t, r_0, \theta_0, \lambda) - z(t)$. Let

$$k\pi_p - {}^{\pi_p}\!/_2 < \theta_0 \leq k\pi_p + {}^{\pi_p}\!/_2 \ ,$$

for some $k \in \mathbb{Z}$. We now show that

$$k\pi_p - {}^{\pi_p}\!/_2 < R(t) \leq k\pi_p + {}^{\pi_p}\!/_2 \ ,$$

for all $t \geq 0$.

If we assume that there exists a point $t_1 > 0$ such that $R(t_1) < k\pi_p + {}^{\pi_p}\!/_2$, and some $t_3 > t_1$ such that $R(t_1) > k\pi_p + {}^{\pi_p}\!/_2$, then by the Intermediate Value Theorem, there also exists a point $t_1 < t_2 < t_3$ such that $R(t_2) = k\pi_p + {}^{\pi_p}\!/_2$. Therefore,

$$\begin{aligned}\theta(t_2) &= R(t_2) + \pi_p\, z(t_2) \\ &= {}^{\pi_p}\!/_2 + (k + z(t_2))\pi_p \ ,\end{aligned}$$

and since the image of z is the nonnegative integers, we have $u(t_2, r_0, \theta_0, \lambda) = 0$. Hence, there exists some $\varepsilon > 0$ such that $z(t) = z(t_2) + 1$, for all $t \in (t_2, t_2 + \varepsilon)$.

By the continuity of θ in the fist variable, there exists a $0 < \tilde{\varepsilon} < \varepsilon$ such that for all $t \in (t_2, t_2 + \tilde{\varepsilon})$, $\theta(t, r_0, \theta_0, \lambda) < (k + z(t_2) + 1)\pi_p$. Then, for all $t \in (t_2, t_2 + \tilde{\varepsilon})$, we have

$$\begin{aligned}R(t) &= \theta(t, r_0, \theta_0, \lambda) - \pi_p\, z(t) \\ &< (k + z(t_2) + 1)\pi_p - \pi_p\, z(t_2) - \pi_p \\ &= k\pi_p \ .\end{aligned}$$

Therefore, for any value $t > t_1$ such that $R(t) = k\pi_p + {}^{\pi_p}\!/_2$, we have $R(t) < k\pi_p$ on some right-neighbourhood of t. Hence $R(t) \leq k\pi_p + {}^{\pi_p}\!/_2$ for all $t > t_1$.

Next, we consider the lower bound. If we assume there exists some $t_1 > 0$ such that $R(t_1) > k\pi_p - {}^{\pi_p}\!/_2$, and some $t_3 > t_1$ such that $R(t_1) < k\pi_p - {}^{\pi_p}\!/_2$, then again there exists some $t_1 < t_2 < t_3$, such that $R(t_2) = k\pi_p - {}^{\pi_p}\!/_2$. Then,

$$\begin{aligned}\theta(t_2, r_0, \theta_0, \lambda) &= R(t_2) + \pi_p\, z(t_2) \\ &= -{}^{\pi_p}\!/_2 + (k + z(t_2))\pi_p \ ,\end{aligned}$$

and again, the image of z is the nonnegative integers, so therefore $\theta'(t_2, r_0, \theta_0, \lambda) > 0$.

Hence,

$$\theta(t, r_0, \theta_0, \lambda) > -{}^{\pi_p}\!/_2 + (k + z(t_2))\pi_p \ ,$$

for all $t > t_2$. Therefore, any value $t > t_1$ that is a zero of the solution u, must have a p-Prüfer angle $\theta(t, r_0, \theta_0, \lambda) > (k + z(t_2))\pi_p$. Therefore, any increase in z corresponds to the p-Prüfer angle increasing past a value ${}^{\pi_p}\!/_2 \pmod{\pi_p}$, and this increase is balanced in the difference $R(t)$. So for any $t > t_1$,

$$\begin{aligned}R(t) &= \theta(t, r_0, \theta_0, \lambda) - \pi_p\, z(t) \\ &> -{}^{\pi_p}\!/_2 + (k + z(t_2))\pi_p - \pi_p\, z(t_2) \\ &= -{}^{\pi_p}\!/_2 + k\pi_p \ .\end{aligned}$$

Finally, we have

$$R(0) = \theta(0, r_0, \theta_0, \lambda) - z(0) = \theta_0 \ ,$$

and given our initial bounds on θ_0, combined with the the bounds on the variation of the function R, we have

$$-{}^{\pi_p}/2 + k\pi_p < R(t) < {}^{\pi_p}/2 + k\pi_p \ ,$$

for all $t > 0$. Given that the function R is bounded, we have $R(t) = o(t)$, and therefore,

$$\rho(\lambda) := \lim_{t\to\infty} \frac{\theta(t, r_0, \theta_0, \lambda) - \theta_0}{t} = \lim_{t\to\infty} \frac{\pi_p, z(t) + R(t)}{t} = \lim_{t\to\infty} \frac{\pi_p \, z(t)}{t} \ .$$

□

This recharacterisation of the rotation number requires the evaluation of solutions of (10) at large values of t, causing large inaccuracies in numerical schemes. In order to mitigate this, we instead restrict our consideration to piecewise-constant potentials. Using Theorem 3.0.1, these solutions can be evaluated in a closed form on each interval of t over which the potential is constant. The closed form expressions are however, dependent on the p-trigonometric and p-hyperbolic functions. Thus, using the approximations we derived in Chapters 3.1 and 3.2, we may increase the accuracy of any approximation of ρ.

The algorithm for the approximation of the rotation number starts off with some arbitrary initial conditions (arbitrary, as per the independence of ρ on θ_0, shown in Lemma 2.3.1). We use the forms of the solutions shown in Theorem 3.0.1 to find the value of the solution and its derivative at the end of the first subinterval over which Q is constant. We also count the number of zeros attained by the solution on this subinterval.

From the values of the solution and its derivative at the end of the first subinterval, we may evaluate the form of the solution on the second subinterval, along with any necessary constants. This allows us once more to find the value of the solution and its derivative at the end of the second subinterval, along with a count of the number of zeros the solution attained throughout. We can continue in this way to enumerate the zeros over all subintervals of constant potential within the period $[0, \alpha)$, as well as finding the values of u and u' at the boundary points between them.

Once this has been completed, we use the values of $u(\alpha)$ and $u'(\alpha)$ as the new initial values and repeat this until we have found an approximation of the number of zeros that the solution attains over m periods, for some $m \in \mathbb{N}$. Finally, we sum the number of zeros that occurred over each period, to find the number of zeros of the function on the whole of $[0, m\alpha)$, and we multiply this number by a factor of $\pi_p \, (m\alpha)^{-1}$, to give an approximation for the value of the rotation number (as per Theorem 3.3.1).

The greater the value m, the more accurate this approximation becomes. If we consider the bounds on the difference between the rotation number, and an approximation of the ratio $t^{-1}\theta(t)$ over the m periods, we get,

$$\left|\rho(\lambda) - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m\alpha}\right| < \frac{2\pi_p}{m} \, ,$$

(first seen in the proof of Theorem 2.3.1). Combining this with the bounds in Theorem 3.3.1, and using the triangle inequality, we have

$$\begin{aligned}
\left|\rho(\lambda) - \frac{\pi_p \, z(m\alpha)}{m\alpha}\right| &= \left|\rho(\lambda) - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m\alpha} + \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m\alpha} - \frac{\pi_p \, z(m\alpha)}{m\alpha}\right| \\
&\leq \left|\rho(\lambda) - \frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m\alpha}\right| + \left|\frac{\theta(m\alpha, r_0, \theta_0, \lambda)}{m\alpha} - \frac{\pi_p \, z(m\alpha)}{m\alpha}\right| \\
&< \frac{2\pi_p}{m} + \frac{\pi_p}{m\alpha} \, ,
\end{aligned}$$

giving us bounds on the accuracy of the function $\pi_p \, (m\alpha)^{-1} z(m\alpha)$ as an approximation of ρ.

We now detail how the values of the solutions and their derivatives are calculated. Consider the subinterval $[a, b)$ on which the potential Q is identically equal to some $C \in \mathbb{R}$, we can use Theorem 3.0.1 to find a form for the solution, and thus find the number of zeros on $[a, b)$. For all possible values of $\lambda - C$, and the ratio $u(a)(u'(a))^{-1}$, we state the corresponding solution form, a method to evaluate any necessary constants, and a brief statement on how this form is used to enumerate the zeros.

Case 1: $\lambda - C > 0$

The solution is of the form

$$u(t) = A\,\cos_p((\lambda - C)^{1/p}\,(t - t_0))\;,$$

for some $A, t_0 \in \mathbb{R}$. In order to evaluate t_0, we take the derivative

$$u'(t) = -A((\lambda - C)^{1/p}(p-1)^{1/p}\phi_q(\sin_q((\lambda - C)^{1/p}(t - t_0))\;.$$

From these two expressions, we have

$$\frac{-\,u(t)}{(\lambda - C)^{1/p}u'(t)} = \cot_p((\lambda - C)^{1/p}(t - t_0))\;.$$

Rearranging to get an expression for t_0, this gives

$$t_0 = a + \frac{1}{(\lambda - C)^{1/p}}\,\mathrm{arccot}_p\left(\frac{u(a)}{(\lambda - C)^{1/p}\,u'(a)}\right)\;.$$

We can also evaluate A, but it is unnecessary here as A is effectively the radial component, i.e., the effect of changing A is a mere rescaling of the solution, which will not affect the number of zeros it attains.

For any $\beta, C, t_0 \in \mathbb{R}$, the maximum number of zeros that the function $\cos_p(C\,t{+}t_0)$ can attain over the interval $[0, \beta)$ is bounded above by $\lceil C\pi_p^{-1}\,\beta\rceil$. Therefore, on the interval $[a, b)$, the solution can have a maximum of $\lceil(\lambda - C)^{1/p}\,\pi_p^{-1}\,(b-a)\rceil$ zeros. Thus, we can partition the subinterval $[a, b)$ into smaller subintervals each with length less than $(\lceil(\lambda - C)^{1/p}\,\pi_p^{-1}\,(b-a)\rceil)^{-1}\,(b-a)$, and count how many times the sign of the solution changes over all of them.

Case 2: $\lambda - C = 0$

The solution is of the form

$$u(t) = At + B\;,$$

for some $A, B \in \mathbb{R}$. The values A and B can be calculated immediately from the initial values. Non-trivial linear functions can attain a maximum of one zero, therefore to count the number of zeros of the solution inside $[a, b)$, it suffices to check whether the sign of the solution has changed over $[a, b)$.

Case 3: $\lambda - Q < 0$ **and** $\left|\frac{(C-\lambda)^{1/p}u(a)}{u'(a)}\right| > (p-1)^{1/p}$

The solution is of the form

$$u(t) = A\cosh_p((C-\lambda)^{1/p}(t-t_0)) \ ,$$

for some $A, t_0 \in \mathbb{R}$. Differentiating, we get

$$u'(t) = A(C-\lambda)^{1/p}(p-1)^{1/p}\phi_q(\sinh_q((C-\lambda)^{1/p}(t-t_0))) \ .$$

From these two expressions, we have

$$\frac{(C-\lambda)^{1/p}u(t)}{u'(t)} = \coth_p(((C-\lambda)^{1/p}(t-t_0))) \ .$$

Rearranging to get an expression for t_0, this gives

$$t_0 = a - \frac{1}{(C-\lambda)^{1/p}}\operatorname{arccoth}_p\left(\frac{(C-\lambda)^{1/p}u(a)}{u'(a)}\right) \ ,$$

which is well-defined, as $\operatorname{arccoth}_p(t)$ is defined for all $|t| > (p-1)^{1/p}$. Once again, it is unnecessary to calculate the value A, as that just gives a rescaling of the solution that does not affect the number of zeros attained. The $\cosh_p$ function has no zeros on the real line, so we do not check for zeros on $[a, b)$.

Case 4: $\lambda - Q < 0$ **and** $\left|\frac{(C-\lambda)^{1/p}u(a)}{u'(a)}\right| < (p-1)^{1/p}$

The solution is of the form

$$u(t) = A \sinh_p((C-\lambda)^{1/p}(t-t_0)) \ .$$

for some $A, t_0 \in \mathbb{R}$. Differentiating, we get

$$u'(t) = A(C-\lambda)^{1/p}(p-1)^{1/p}\phi_q(\cosh_q((C-\lambda)^{1/p}(t-t_0))) \ .$$

From these two expressions, we have

$$\frac{(C-\lambda)^{1/p}u(t)}{u'(t)} = \tanh_p(((C-\lambda)^{1/p}(t-t_0))) \ .$$

Rearranging to get an expression for t_0, this gives

$$t_0 = a - \frac{1}{(C-\lambda)^{1/p}} \operatorname{arctanh}_p \left(\frac{(C-\lambda)^{1/p}u(a)}{u'(a)} \right) \ ,$$

which is well-defined, as $\operatorname{arctanh}_p(t)$ is defined for all $|t| < (p-1)^{1/p}$. Again, we do not calculate A. The $\sinh_p$ function attains only one zero on the real line, therefore to count the number of zeros of the solution inside $[a, b)$, it suffices to check whether the sign of the solution has changed over $[a, b)$.

Cases 5 and 6: $\lambda - Q < 0$ and $\frac{(C-\lambda)^{1/p}u(a)}{u'(a)} = \pm(p-1)^{1/p}$

The solution is of the form

$$u(t) = A \exp(\mp(q-1)^{1/p}(C-\lambda)^{1/p}\, t) \ ,$$

for some $A \in \mathbb{R}$. We do not calculate A, in fact, as the angle depends explicitly on the ratio $u(t)(u'(t))^{-1}$, and this ratio is constant for exponentials, we just take the value for the solution and its derivative at the end of the subinterval to be equal to the initial values. The exponential function has no zeros on the real line, so we do not check for zeros on $[a, b)$.

Note that even though the function $\operatorname{arctanh}_p$ occurs in the above calculations, there is no need to derive a separate approximation for it, as

$$\operatorname{arctanh}_p(1/\phi_q(t)) = (p-1)\operatorname{arccoth}_q(t) \ .$$

The pseudocode for this whole scheme is listed in Chapter 6.3. An implementation of this code was used to plot Figure 3 below. The rotation number here was calculated for $p = 3$, period $\alpha = 2\pi_3$, and the potential is piecewise-constant, given by

$$Q(t) = \begin{cases} -1 \ , & \text{if } \ t \in [0, 2) \\ -4 \ , & \text{if } t \in [2, 3) \\ 1 \ , & \text{if } \ t \in [3, 2\pi_3) \ . \end{cases} \tag{36}$$

For these values, the periodic eigenvalues for which the corresponding solution has n zeros in $[0, 2\pi_3)$ have,

$$\rho = \frac{2n\pi_p}{\alpha} = \frac{2n\pi_3}{2\pi_3} = n \ ,$$

and for the anti-periodic eigenvalues,

$$\rho = \frac{2n\pi_p}{2\alpha} = \frac{2n\pi_3}{2 \cdot 2\pi_3} = \frac{n}{2} \ .$$

Therefore, we anticipate plateaus for which the rotation number is a half-integer. There are clear plateaus at the levels $\rho = 3/2$, the end-points of which correspond to a pair of anti-periodic eigenvalues, and $\rho = 2$, the end-points corresponding to a pair of periodic eigenvalues. However, we notice a small extra plateau at the level $\rho = 7/4$. This would imply the existence of a non-degenerate pair of eigenvalues of the periodic problem over four periods. Non-degenerate eigenvalues for such a problem are precluded from existing in the linear case, $p = 2$, by Theorem 1.0.9. We now verify this observation by using further numerical schemes to evaluate the properties of solutions of the equation (10) for values of the spectral parameter that fall inside such extra plateaus.

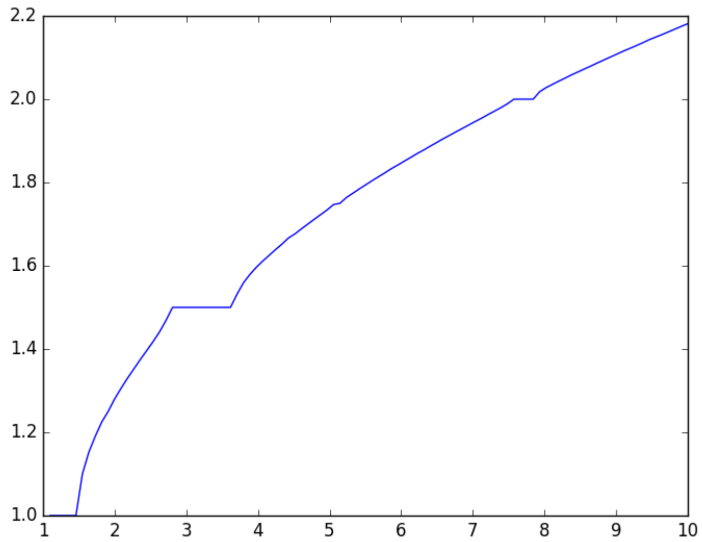


Figure 3: Values of the rotation number, $\rho(\lambda)$, on the domain $1 \leq \lambda \leq 10$, with $p = 3$ and Q defined in (36).

3.4 Study of Stability and Instability in the Phase Plane

We next remove the constraint of considering only piecewise constant potentials, and return to the general case $Q \in L^1_{\mathrm{loc}}(\mathbb{R})$. We consider the analysis of the stability of solutions to the equation (10) for fixed values of $\lambda \in \mathbb{R}$. In Chapter 1, the linear ODE (11) was converted to a first-order system, with matrix A. Theorem 1.0.3 shows that the solution of this first-order system has the property that the effect of evaluating the solution after an additional period is equivalent to multiplying the solution by the Monodromy matrix.

The behaviour of the solutions of (10) is therefore characterised by the eigenvalues of the corresponding Monodromy matrix. Given that $\mathrm{trace}(A) \equiv 0$, an application of Liouville's formula to the matrix A shows that any matrix solution of $\mathbf{x}' = A\mathbf{x}$ has a constant determinant. For the solution Φ, $\Phi(0) = I$, the determinant is identically equal to one, therefore the product of the two eigenvalues of the Monodromy matrix is one. By Theorem 1.0.4, we once can once again characterise the solutions of the equation (11) as falling into one of four distinct cases:

- *Case 1: Stability* The eigenvalues of the system are complex conjugate pairs, $\mu, \bar{\mu}$.

- *Case 2: Instability* The eigenvalues are a pair of real numbers, μ, μ^{-1}.

- *Case 3: Periodicity* The eigenvalues are both equal to 1.

- *Case 4: Anti-periodicity* The eigenvalues are both equal to -1.

Furthermore, Theorem 1.0.6 shows that the case of stability exactly corresponds to the values of λ such that the rotation number is strictly increasing, and the case of instability corresponds to the values of λ inside the plateaus. As a consequence of Theorem 1.0.8, the case of periodicity corresponds to the end-points of the plateaus at levels that are even multiples of $\pi_p\,\alpha^{-1}$, and the case of anti-periodicity corresponds to the end-points of the plateaus at levels that are odd multiples of $\pi_p\,\alpha^{-1}$.

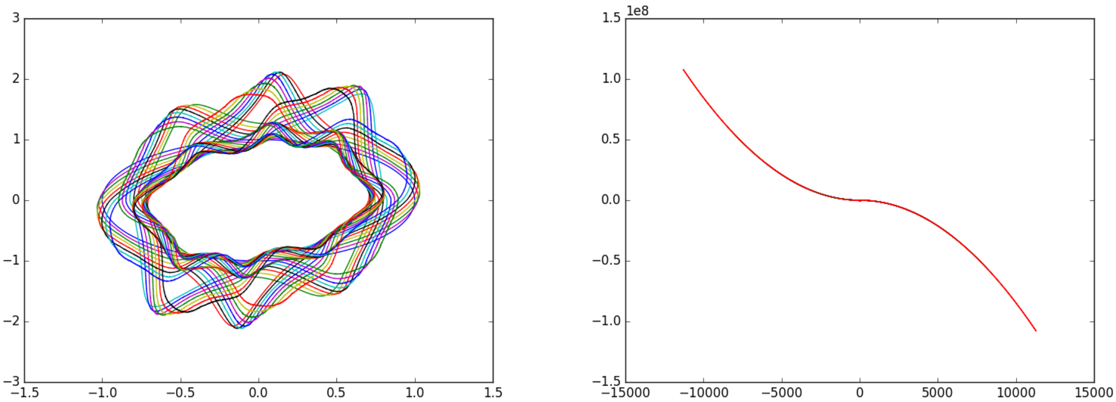
This theory gives a full account of the stability of solutions of the equation (11) and how they relate to the spectrum of the periodic problem. However, given that we have seen qualitative differences in the behaviour of the spectrum for the nonlinear equation (10), it is worth analysing the stability of solutions numerically to find any possible analogous connection between the stability of solutions and the spectrum of the periodic problem. For example, given that our numerical approximation of the rotation number suggested the existence of non-degenerate eigenvalues of the iterated problem (for $m > 2$), we will show if the newly formed plateaus also give rise to unstable solutions, and describe any differences in the structure of the solution manifold.

As we noted in Chapter 1, Floquet Theory fails in the nonlinear case $p \neq 2$, so to determine stability, we will need to consider a different method. To do this, we start by plotting a locus of points of Prüfer radius one. By Lemma 2.2.1, this is the set of points $(x, y) \in \mathbb{R}^2$ satisfying

$$(q-1)|x|^p + |y|^q = 1\ .$$

We call this the unit p-*circle*. We plot a unit p-circle in the phaseplane, each point of which gives a pair of initial values $u(0), u'(0)$ of the solution. For each pair of initial conditions, we use a standard numerical scheme (e.g. Runge-Kutta) to determine the values, $u(\alpha), u'(\alpha)$, of the solution after one period. We then plot how each point on the circle has been affected after each additional period in t. We continue this for several more iterations, which then gives us a picture for how each of the solutions behaves in the long term. Importantly, as (22) shows that the behaviour of the solutions doesn't change with rescaling, the solutions that stem from this one circle give a complete picture of the stability.

We now compare the rotation number plots with these phaseplane plots, to see if the concept of the plateaus being regarded as instability intervals is valid in the case $p \neq 2$. Consider the example in Figure 3, the value $\lambda = 6.3$ clearly falls outside of any plateau. In the linear case, Theorem 1.0.6 shows that we can expect the resulting solutions to be stable. That is, they must be bounded over $\mathbb{R}$. Furthermore, the image of $\lambda = 3.1$ does not appear to be a integer/half-integer, and we can therefore claim it is not a periodic or anti-periodic eigenvalue. From this, we would anticipate that the phaseplane plots would similarly be bounded, and indeed, Figure 4a shows stable behaviour.



(a) Circle in phaseplane after each of the first 30 periods, with $p = 3$, $\lambda = 6.3$.

(b) Circle in phaseplane after each of the first 30 periods, with $p = 3$, $\lambda = 3.1$.

Figure 4: Phaseplane Plots for Q defined in (36), at varying values of $\lambda \in \mathbb{R}$.

Compare this with the value $\lambda = 3.1$. This seems to fall inside the plateau at the level $\rho = {}^{3}/_{2}$. Corollary 2.4.2 shows that even in the nonlinear case, the end-points of this plateau are anti-periodic eigenvalues. However, we do not yet know whether all solutions with spectral parameter, λ, inside this plateau produce divergent behaviour, as in the linear case. If this were true, then we would expect the corresponding phaseplane plots to diverge, and the plot in Figure 4b does indeed show unstable behaviour.

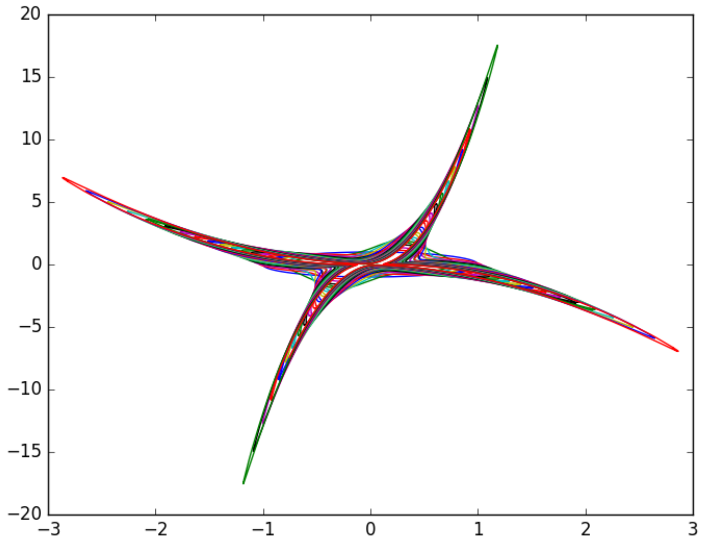


Figure 5: Circle in phaseplane after each of the first 30 periods, with $p = 3$, $\lambda = 5.1$ and Q defined in (36). Note that as the p-radius is increasing after each period, we have unstable behaviour.

Theorem 1.0.9 shows that in the linear case $p = 2$, plateaus can only exist at multiples of $\pi_p\,\alpha^{-1}$. The potential Q in the system used in the Figure 3 therefore seems to have an extra plateau at the level $\rho(\lambda) = 1.75$. Reading off the rotation number plot, we can see that $\lambda = 5.1$ is a value that falls in the interior of this plateau, therefore we would expect unstable behaviour of the phaseplane plot. Indeed, Figure 5 shows that the effect of this operator on the unit p-circle includes initial conditions that produce divergent behaviour.

We observe that this extra plateau is significantly smaller than the other standard plateaus, and the behaviour of the solution for $\lambda = 5.1$ of the system is somewhat 'less unstable' than the system for $\lambda = 3.1$, which falls inside the more sizeable plateau at the level $\rho = {}^{3}/_{2}$. Note that in the latter case, the phaseplane plot expands out to the order

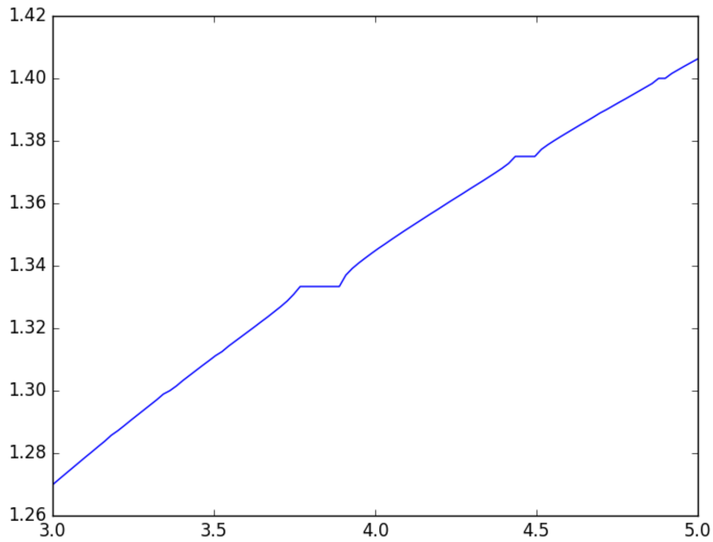


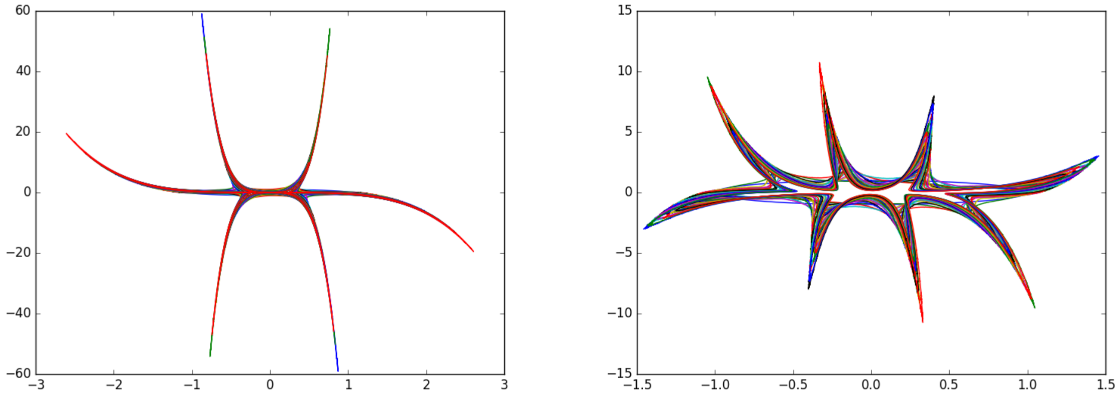
Figure 6: Values of the rotation number, $\rho(\lambda)$, on the domain $3 \leq \lambda \leq 5$, with $p = 5$ and Q defined in (37) .

of 10^8, whereas in the former case, it merely reaches an order of 10^1.

Also note that as the end-points of the plateau at $\rho = {}^3/_2$ are anti-periodic eigenvalues, all values of λ between these points there exist initial conditions for which the angular component is anti-periodic. That is, the solution components in the phaseplane rotate by an odd multiple of π_3. This results in the phaseplane plot, Figure 4b, having two 'horns', because points that lie on one horn at a certain iteration are mapped to the other horn at the next iteration. We can then check for consistency for values of λ inside extra plateaus. For example, the case $p = 5$, with potential Q given by

$$Q(t) = \begin{cases} -1\,, & \text{if } \; t \in [0, 2) \\ -4\,, & \text{if } \; t \in [2, 3) \\ 1\,, & \text{if } \; t \in [3, 2\pi_5)\,. \end{cases} \tag{37}$$

In Figure 6, we observe extra plateaus at levels $\rho = {}^4/_3$, (e.g at $\lambda = 3.8$); and at $\rho = {}^{11}/_8$, (e.g. at $\lambda = 4.5$). The resulting phaseplane plots are given in Figures 7a and 7b, respectively.



(a) Circle in phaseplane after each of the first 30 periods, with $p = 5$, $\lambda = 3.8$. Note that we have 6 horns.

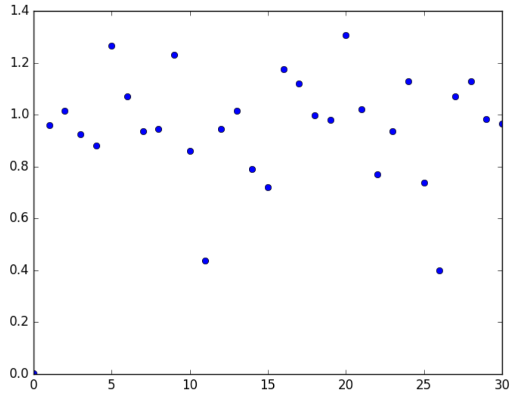
(b) Circle in phaseplane after each of the first 30 periods, with $p = 5$, $\lambda = 4.5$. Note that we have 8 horns.

Figure 7: Phaseplane Plots for a piecewise-constant potential, Q, at varying values of $\lambda \in \mathbb{R}$.

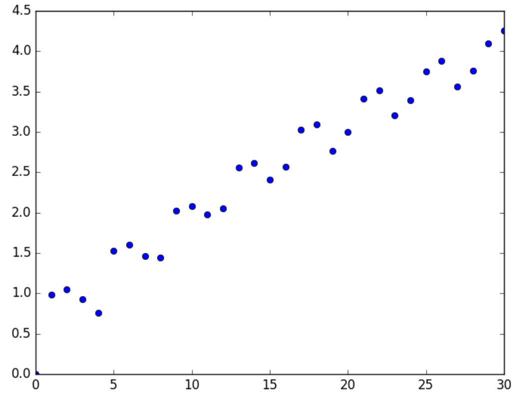
For the value $\lambda = 3.8$, as $\rho(\lambda) = {}^{4}/_{3}$, we expect each solution to rotate, on average, by about ${}^{8\pi_5}/_{3}$. This results in the phaseplane plot having six horns, three due to the average rotation per period, and another three due to the same property and the symmetry of the phaseplane plots. Similarly, the value $\lambda = 4.5$, as $\rho(\lambda) = {}^{11}/_{8}$, we expect each solution to rotate, on average, by about ${}^{11\pi_5}/_{4}$. This results in the phaseplane plot having eight horns. Chapter 4 discusses the number of horns expected in each plot in more detail, as well as proving the validity of this argument.

We can quantify stability more precisely by plotting the log of the ratio of the radii of the inner circle of the plot (that is, the largest circle, centred at the origin, that fits inside the curve) and the outer circle (the smallest circle, centred at the origin, that fits outside the curve) after each period. In the linear case, by Theorem 1.0.4, every unstable solution u has corresponding radius $r(t) \sim A \exp(C\,t)$, for some $A, C \in \mathbb{R}$. Therefore, the resulting plots would have a linear asymptotic.

We now check to see if the plots are similar for the nonlinear case. We calculate the radius for each set of initial values x, y after each iteration, and find the maximum and minimum values. The results for different values of λ with the potential Q defined in (37) are given in Figure 8.



(a) $\lambda = 4.3$, this is a case of stability, so there is no overall increase.



(b) $\lambda = 4.5$, this is a case of instability, so there is an overall increase.

Figure 8: The logarithm of the ratio of the radii after each of the first 30 periods, for $p = 5$, and Q defined in (37).

Note that in the case of instability in Figure 7b, as $\rho(\lambda) = {}^{11}\!/_{8}$, we have these ratios increasing in patterns of four. These patterns correspond to the four pairs of horns that are given in the corresponding phaseplane plot, each of which corresponds to a jump of the solution after one period by approximately ${}^{11\pi_5}\!/_{4}$. Note that in the linear case, the gradient between the points of each of these four separate cases are just the Floquet multipliers (the eigenvalues of the Monodromy matrix). However, in the linear case we could never have unstable behaviour at the level $\rho(\lambda) = {}^{11}\!/_{8}$, so we would in general only find patterns of two points, corresponding to one pair of horns.

In Chapter 4, we validate the claim that the horns in the phaseplane plots are the result of values of λ inside the instability intervals that result in initial angles for which the Prüfer angle is periodic, but the Prüfer radius is not. These are analogous to the Floquet solutions in the linear case; solutions of (11) for which $D(\lambda) > 2$ that have the form $u(t) = \exp(C\,t)\,g(t)$, for some periodic $g \in V$. In Chapter 5, we prove the existence of extra plateaus for certain potentials, using a perturbation argument similar to the one in Chapter 2, proving that the extra plateaus appearing in Figures 3 and 6 are artifacts of the nonlinearity of the problem, and not a by-product of any numerical inaccuracies.

4 Floquet Type Solutions

4.1 A Generalisation of Floquet Solutions and Multipliers to the p-Laplace Equation

In Theorem 2.4.2, we characterised the plateaus of the rotation number as values of $\lambda \in \mathbb{R}$ for which there exists an initial angle $\theta_0 \in \mathbb{R}$ with angular periodicity, i.e.,

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ , \tag{38}$$

for some $n \in \mathbb{N}$. In the linear case, $p = 2$, the rotation number plateaus coincide with the so-called *instability intervals* of the spectral parameter; values of λ such that there exist solutions that increase without bound. In particular, these solutions have the property that the effect of shifting the function argument by one period is to multiply the value by some fixed, positive constant. Consider a spectral value λ that lies in the interior of a rotation number plateau, then there exists a $\theta_0 \in \mathbb{R}$ such that (38) holds (with $\pi_p = \pi_2 = \pi$). By the definition of the renormalised Poincaré map Ψ,

$$\theta(\alpha, r_0, \theta_0, \lambda) = \theta_0 + 2n\pi \ ,$$

for some initial values $r_0, \theta_0 \in \mathbb{R}$. Given this, by the definition of the p-Prüfer Transform for $p = 2$, we have

$$\begin{aligned}
u(\alpha, r_0, \theta_0, \lambda) &= r(\alpha, r_0, \theta_0, \lambda) \ \cos(\theta(\alpha, r_0, \theta_0, \lambda)) \\
&= r_0 \ \exp\left(\int_0^\alpha (\lambda - Q - 1) \cos(\theta(s, r_0, \theta_0, \lambda)) \sin(\theta(s, r_0, \theta_0, \lambda)) \, \mathrm{d}s \right) \cos(\theta_0 + 2n\pi) \\
&= \exp\left(\int_0^\alpha (\lambda - Q - 1) \cos(\theta(s, r_0, \theta_0, \lambda)) \sin(\theta(s, r_0, \theta_0, \lambda)) \, \mathrm{d}s \right) r_0 \cos(\theta_0) \\
&= \mu \, u(0, r_0, \theta_0, \lambda) \ ,
\end{aligned}$$

where

$$\mu = \exp\left(\int_0^\alpha (\lambda - Q - 1)\cos(\theta(s, r_0, \theta_0, \lambda))\sin(\theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \right) .$$

In general, for the solutions of (10) at any value $t \in \mathbb{R}$, and any initial values $r_0, \theta_0 \in \mathbb{R}$,

$$u(t + \alpha, r_0, \theta_0, \lambda) = \mu u(t, r_0, \theta_0, \lambda) .$$

In the linear case, this type of solution is referred to as a *Floquet solution*, and the value μ as its *Floquet multiplier*. We show now that even in the nonlinear case, the rotation number plateaus comprise spectral values that always result in unstable solutions.

Lemma 4.1.1. *Let $\lambda \in \mathbb{R}$ be such that there exists an initial angle $\theta_0 \in \mathbb{R}$ with the renormalised Poincaré map*

$$\Psi(\theta_0, \lambda) = 2n\pi_p ,$$

then solutions to the equation (10) *with this initial angle have the form*

$$u(t + \alpha, r_0, \theta_0, \lambda) = \mu\, u(t, r_0, \theta_0, \lambda) ,$$

where

$$\mu = \exp\left(\int_0^\alpha (p-1)^{-1/q}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \right) .$$

Proof. By the assumption, we have

$$\theta(\alpha, r_0, \theta_0, \lambda) = \theta_0 + 2n\pi_p ,$$

and from the radial equation (22),

$$
\begin{aligned}
r(t+\alpha, r_0, \theta_0, \lambda) &= r_0 \exp\Bigg(\int_0^{t+\alpha} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda)) \\
&\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg) \\
&= r_0 \exp\Bigg(\int_0^{\alpha} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda)) \\
&\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg) \cdot \exp\Bigg(\int_\alpha^{t+\alpha} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1) \\
&\qquad \cdot \phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg) \\
&= r_0 \exp\Bigg(\int_0^{\alpha} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda)) \\
&\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg) \cdot \exp\Bigg(\int_0^{t} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1) \\
&\qquad \cdot \phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg)\,,
\end{aligned}
$$

using the substitution $t \mapsto t - \alpha$ in the latter integral, and using the fact that the period Q is α-periodic, and that by our assumption, the angle θ is α-periodic, modulo $2\pi_p$. We therefore have,

$$
\begin{aligned}
r(t+\alpha, r_0, \theta_0, \lambda) &= \mu^{p/2}\ r_0\ \exp\Bigg(\int_0^{t} ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_0, \lambda)) \\
&\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_0, \lambda))\,\mathrm{d}s \Bigg) \\
&= \mu^{p/2}\ r(t, r_0, \theta_0, \lambda)\ .
\end{aligned}
$$

Therefore, by definition

$$\begin{aligned}
u(t+\alpha, r_0, \theta_0, \lambda) &= (r(t+\alpha, r_0, \theta_0, \lambda))^{2/p} \cos_p \theta(t+\alpha, r_0, \theta_0, \lambda) \\
&= (\mu^{p/2}\, r(t, r_0, \theta_0, \lambda))^{2/p} \cos_p(\theta(t, r_0, \theta_0, \lambda) + 2n\pi_p) \\
&= \mu\, (r(t, r_0, \theta_0, \lambda))^{2/p} \cos_p \theta(t, r_0, \theta_0, \lambda) \\
&= \mu\, u(t, r_0, \theta_0, \lambda)\ .
\end{aligned}$$

□

This type of behaviour is analogous to the Floquet solutions seen in the classical, linear case $p = 2$. Motivated by this observation, we now generalise this definition.

Definition 4.1.1. *Fix $\lambda \in \mathbb{R}$, if there exists a $\theta_0 \in \mathbb{R}$ such that the corresponding solution to the equation* (10) *has the form*

$$u(t+\alpha, r_0, \theta_0, \lambda) = \mu u(t, r_0, \theta_0, \lambda)\ ,$$

for some $\mu \in \mathbb{R}$, then we call it a Floquet Type solution.

For any value λ such that there exists a solution of (10) with

$$\Psi(\theta_0, \lambda) = 2n\pi_p\ ,$$

for some $\theta_0 \in \mathbb{R}$, $n \in \mathbb{N}$, the ratio of the increase in radius,

$$\frac{r(\alpha, r_0, \theta_0, \lambda)}{r_0}\ ,$$

is equal to $\mu^{p/2}$. By Lemma 2.2.2,

$$\mu = \left(\frac{r(\alpha, r_0, \theta_0, \lambda)}{r_0}\right)^{2/p} = (\partial_3 \theta(\alpha, r_0, \theta_0, \lambda))^{-1/p} = (\partial_1 \Psi(\theta_0, \lambda) + 1)^{-1/p}\ ,$$

and therefore,

$$\mu < 1 \qquad \Longleftrightarrow \qquad \partial_1 \Psi(\theta_0, \lambda) < 0 \ ,$$

$$\mu = 1 \qquad \Longleftrightarrow \qquad \partial_1 \Psi(\theta_0, \lambda) = 0 \ ,$$

$$\mu > 1 \qquad \Longleftrightarrow \qquad \partial_1 \Psi(\theta_0, \lambda) > 0 \ .$$

This shows that at any $\lambda \in \mathbb{R}$, there exist unbounded solutions of (10) if and only if there exists an initial angle $\theta_0 \in \mathbb{R}$ such that

$$\Psi(\theta_0, \lambda) = 2n\pi_p \qquad \text{and} \qquad \partial_1 \Psi(\theta_0, \lambda) \neq 0 \ .$$

By Theorem 2.4.2, for any value of λ in the interior of a rotation number plateau, there exist initial angles θ_1 , θ_2 such that

$$\Psi(\theta_1, \lambda) < 2n\pi_p \qquad \text{and} \qquad \Psi(\theta_2, \lambda) > 2n\pi_p \ ,$$

for the corresponding value $n \in \mathbb{N}$. By the continuity of $\Psi(\cdot, \lambda)$, and the Intermediate Value Theorem, there exists a value $\theta_0 \in [\theta_1, \theta_2]$ such that

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ .$$

Through the monotonicity of Ψ in λ, the set of points (θ_0, λ) that satisfy this equality is a connected curve. By Sard's Theorem, the set of points on this curve that also satisfy

$$\partial_1 \Psi(\theta_0, \lambda) = 0 \ ,$$

has measure zero. Therefore, for almost all values λ in the interior of a rotation number plateau, there exists a solution of (10) that is unbounded in t. This validates the characterisation of rotation number plateaus as being *instability intervals*, even in the nonlinear case.

The next question we consider is, for any value $\lambda \in \mathbb{R}$ that results in the existence of Floquet Type solutions, the number of other initial angles, $\theta_0 \in \mathbb{R}$, that also result in Floquet Type solutions.

4.2 Existence and Multiplicity of Floquet Type Solutions

In the linear problem, Floquet theory demonstrates that for any value of λ inside an instability interval, there exist two Floquet solutions with Floquet multipliers $\mu_1, \mu_2 \in \mathbb{R}$ such that $\mu_i\mu_2 = 1$. In the nonlinear case, we can also show that such solutions occur in pairs.

Lemma 4.2.1. *Fix some initial radius $r_0 > 0$ and let $\lambda \in \mathbb{R}$ be such that the problem* (10) *has a Floquet Type solution, u, i.e.*

$$u(t+\alpha, r_0, \theta_1, \lambda) = \mu_1 u(t, r_0, \theta_1, \lambda) \ ,$$

for some initial angle $\theta_1 \in \mathbb{R}$ and $\mu_1 \geq 1$, then there exists an initial angle $\theta_2 \in \mathbb{R}$ such that the corresponding solution has the form

$$u(t+\alpha, r_0, \theta_2, \lambda) = \mu_2 u(t, r_0, \theta_2, \lambda) \ ,$$

with $\mu_2 \leq 1$.

Proof. In Lemma 4.1.1, we showed that any $\lambda \in \mathbb{R}$ that resulted in the existence of a value $\theta_0 \in \mathbb{R}$ such that

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

for some $n \in \mathbb{N}$, corresponds to the problem (10) having a Floquet Type solution. Conversely, if the problem (10) has such a solution

$$u(t+\alpha, r_0, \theta_0, \lambda) = \mu_1 u(t, r_0, \theta_0, \lambda) \ ,$$

we can show that the renormalised Poincaré map at this spectral value, $\Psi(\cdot, \lambda)$, similarly attains the value $2n\pi_p$ (for some $n \in \mathbb{N}$) at the point θ_0.

By Lemma 2.2.1, we have

$$\theta = -\operatorname{arccot}_p({}^{u}\!/\!_{u'}). \tag{39}$$

Given that

$$u(t+\alpha, r_0, \theta_1, \lambda) = \mu_1 u(t, r_0, \theta_0, \lambda)$$
$$u'(t+\alpha, r_0, \theta_1, \lambda) = \mu_1 u'(t, r_0, \theta_0, \lambda) \ ,$$

then by (39),

$$\begin{aligned}
\theta(t+\alpha, r_0, \theta_1, \lambda) &= -\operatorname{arccot}_p\left(\frac{u(t+\alpha, r_0, \theta_1, \lambda)}{u'(t+\alpha, r_0, \theta_1, \lambda)}\right) \\
&= -\operatorname{arccot}_p\left(\frac{\mu_1 u(t, r_0, \theta_1, \lambda)}{\mu_1 u'(t, r_0, \theta_1, \lambda)}\right) \\
&= -\operatorname{arccot}_p\left(\frac{u(t, r_0, \theta_1, \lambda)}{u'(t, r_0, \theta_1, \lambda)}\right) \\
&= \theta(t, r_0, \theta_1, \lambda) + 2n\pi_p \ ,
\end{aligned}$$

for some $n \in \mathbb{N}$.

From this, at the point $t = 0$, we can deduce

$$\Psi(\theta_1, \lambda) = \theta(\alpha, r_0, \theta_1, \lambda) - \theta_1 = 2n\pi_p \ .$$

Similarly, for the Prüfer radius,

$$r = \left(\frac{|u|^p}{p-1} + |u'|^p\right)^{1/2} \ ,$$

we have

$$
\begin{aligned}
r(t+\alpha, r_0, \theta_1, \lambda) &= \Big(\frac{|u(t+\alpha, r_0, \theta_0, \lambda)|^p}{p-1} + |u'(t+\alpha, r_0, \theta_0, \lambda)|^p\Big)^{1/2} \\
&= \Big(\frac{|\mu_1 u(t, r_0, \theta_0, \lambda)|^p}{p-1} + |\mu_1 u'(t, r_0, \theta_0, \lambda)|^p\Big)^{1/2} \\
&= (\mu_1)^{2/p}\Big(\frac{|u(t, r_0, \theta_0, \lambda)|^p}{p-1} + |u'(t, r_0, \theta_0, \lambda)|^p\Big)^{1/2} \\
&= \mu_1^{2/p} r(t, r_0, \theta_1, \lambda) \ ,
\end{aligned}
$$

and by Lemma 2.2.2, at $t = 0$,

$$
\partial_3 \theta(\alpha, r_0, \theta_1, \lambda) = \Big(\frac{r_0}{r(\alpha, r_0, \theta_1, \lambda)}\Big)^2 = \Big(\frac{r_0}{\mu_1^{2/p} r_0}\Big)^2 = \frac{1}{\mu_1^{4/p}} \leq 1 \ ,
$$

and so,

$$
\partial_1 \Psi(\theta_1, \lambda) = \frac{\partial}{\partial \theta_1}\left(\theta(\alpha, r_0, \theta_1, \lambda) - \theta_1\right) = \partial_3 \theta(\alpha, r_0, \theta_1, \lambda) - 1 \leq 0 \ .
$$

By Corollary 2.2.1, the map $\Psi(\cdot, \lambda)$ is continuous and π_p-periodic. Therefore, by the Intermediate Value Theorem, there exists a point θ_2 that is distinct from θ_1, modulo π_p, such that

$$
\Psi(\theta_2, \lambda) = 2n\pi_p \qquad \text{and} \qquad \partial_1 \Psi(\theta_2, \lambda) \geq 0 \ .
$$

Again, by Lemma 2.2.2, we have

$$
r(\alpha, r_0, \theta_2, \lambda) = \mu_2^{2/p} r_0 \ ,
$$

for some $\mu_2 \leq 1$. Also, since $\Psi(\theta_2, \lambda) = 2n\pi_p$, then by Lemma 4.1.1,

$$
u(t+\alpha, r_0, \theta_2, \lambda) = \mu_2 u(t, r_0, \theta_2, \lambda) \ ,
$$

for any $t \in \mathbb{R}$.

$\square$

For the linear case with any such $\lambda \in \mathbb{R}$, constant multiples of either one of this pair form the set of all Floquet solutions of the problem. However, in the nonlinear case, this is no longer true. In Chapter 2.5, we have shown that rather than the renormalised Poincaré map oscillating just twice per period, for any $n \in \mathbb{N}$ there exist potentials for which the $\Psi(\cdot, \lambda)$ oscillates about a value $2n\pi_p$ at least n times. Any angle θ_0 such that $\Psi(\theta_0, \lambda) = 2n\pi_p$ gives the existence of another Floquet Type solution, and Theorem 2.5.1 shows us that any potential that has a degenerate eigenvalue λ, can be perturbed to give arbitrarily many such $\theta_0 \in \mathbb{R}$.

Theorem 4.2.1. *Let $\lambda \in \mathbb{R}$ be such that*

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$. Let Q_2 be an α-periodic function such that

$$\begin{gathered}
\int_0^\alpha Q_2(s)|u(s, r_0, \theta_1, \lambda, 0)|^p \,\mathrm{d}s > 0 \\
\int_0^\alpha Q_2(s)|u(s, r_0, \theta_2, \lambda, 0)|^p \,\mathrm{d}s < 0 \\
\vdots \\
\int_0^\alpha Q_2(s)|u(s, r_0, \theta_{2j}, \lambda, 0)|^p \,\mathrm{d}s < 0
\end{gathered}$$

for some distinct $\theta_1 < \theta_2 < \ldots < \theta_{2j} \in [0, \pi_p)$, with $j \in \mathbb{N}$; then there exists some $\varepsilon > 0$ such that the equation (30) *has n Floquet Type solutions given by*

$$u(t + \alpha, r_0, \vartheta_i, \lambda, \varepsilon) = \mu_i \, u(t, r_0, \vartheta_i, \lambda, \varepsilon) \ ,$$

for values $0 \leq \vartheta_1 < \ldots < \vartheta_i < \ldots < \vartheta_j < \pi_p$ such that

$$\mu_i \leq 1 \;, \qquad \textit{if } i \textit{ even,}$$

$$\mu_i \geq 1 \;, \qquad \textit{if } i \textit{ odd.}$$

Proof. By Theorem 2.5.1, there exist values $\vartheta_1 \in \mathbb{R}, \ldots, \theta_{2j}$ such that

$$\Psi(\vartheta_i, \lambda, \varepsilon) = 2n\pi_p \;,$$

$$\partial_1 \Psi(\vartheta_i, \lambda, \varepsilon) \geq 0 \;, \qquad \text{if } i \text{ even,}$$

$$\partial_1 \Psi(\vartheta_i, \lambda, \varepsilon) \leq 0 \;, \qquad \text{if } i \text{ odd.}$$

Therefore, by Lemma 4.1.1, each solution corresponding to any of these initial angles ϑ_i are Floquet Type solutions; and by Lemma 2.2.2

$$\mu_i = \left(\frac{r_0}{r(\alpha, r_0, \vartheta_i, \lambda, \varepsilon)}\right)^{p/2} = \left(\partial_1 \Psi(\vartheta_i, \lambda, \varepsilon)\right)^{p/2} \;,$$

and the values μ_i alternate above and below the value 1 accordingly.

□

We now consider the enumeration of the number of distinct initials angles, $\theta_0 \in \mathbb{R}$, that result in Floquet Type solutions. We introduce a function, λ^*, that allows this quantity to be determined by the analysis of the image of this function over one interval $[0, \pi_p)$.

4.3 Quantitative Measure of Multiplicity of Floquet Type Solutions

Determining the number of linearly independent Floquet Type solutions that the equation (10) has for each $\lambda \in \mathbb{R}$ can be difficult if one only considers the renormalised Poincaré maps, $\Psi(\theta_0, \lambda)$. This is because a new renormalised Poincaré map must be analysed for each λ. We instead introduce a function that allows us to consider how the number of linearly independent Floquet Type solutions varies as we move through an instability interval. This function depends only on the parameter θ_0, so all the analysis for each instability interval can be done from this single function only.

Lemma 4.3.1. *For each $\theta_0 \in \mathbb{R}$ and $n \in \mathbb{N}$, there exists a unique value $\lambda \in \mathbb{R}$ such that:*

$$\Psi(\theta_0, \lambda) = 2n\pi_p \ . \tag{40}$$

Proof. First, we show that there exists some $\underline{\lambda} \in \mathbb{R}$ such that $\Psi(\theta_0, \underline{\lambda}) < 2\pi_p$.
For any $\lambda \in \mathbb{R}$, and an initial angle θ_0 (without loss of generality, take $\theta_0 \in [-{}^{\pi_p}\!/2\,, {}^{\pi_p}\!/2)$.
For any $t_0 > 0$, if $\theta(t_0, r_0, \theta_0, \lambda) = -{}^{\pi_p}\!/2$, then by (10),

$$\begin{aligned}
\theta'(t_0, r_0, \theta_0, \lambda) &= 1 + (q-1)(\lambda - Q - 1)|\cos_p(\theta(t_0, r_0, \theta_0, \lambda)|^p \\
&= 1 + (q-1)(\lambda - Q - 1)|\cos_p(-{}^{\pi_p}\!/2)|^p \\
&= 1 > 0 \ ,
\end{aligned}$$

and as $\theta(0, r_0, \theta_0, \lambda) = \theta_0 > -{}^{\pi_p}\!/2$,

$$\theta(t, r_0, \theta_0, \lambda) > -{}^{\pi_p}\!/2 \ ,$$

for all $t > 0$. Given this bound, we have

$$
\begin{aligned}
\Psi(\theta_0, \lambda) &= \theta(\alpha, r_0, \theta_0, \lambda) - \theta_0 \\
&> -{}^{\pi_p}\!/_2 - {}^{\pi_p}\!/_2 \\
&= -\pi_p \ .
\end{aligned}
$$

By the π_p-periodicity of $\Psi(\cdot, \lambda)$, shown in Lemma 2.2.3, this inequality holds for all $\theta_0, \lambda \in \mathbb{R}$.

Fixing some value $\theta_0 \in \mathbb{R}$, by Theorem 2.2.4, $\Psi(\theta_0, \cdot)$ is a strictly monotonically increasing function. Therefore, any sequence of values of λ taken as $\lambda \to -\infty$ gives a sequence of values $\Psi(\theta_0, \lambda)$ that are strictly decreasing and bounded below. Therefore,

$$
\lim_{\lambda \to -\infty} \frac{\Psi(\theta_0, \lambda)}{\lambda} = 0 \ , \tag{41}
$$

and we may also define the limiting function

$$
\theta_{-\infty}(t) := \lim_{\lambda \to -\infty} \theta(t, r_0, \theta_0, \lambda) \ .
$$

Let $\lambda < 0$, then

$$
\begin{aligned}
\frac{\Psi(\theta_0, \lambda)}{\lambda} &= \frac{\theta(\alpha, r_0, \theta_0, \lambda) - \theta_0}{\lambda} \\
&= \frac{1}{\lambda} \int_0^\alpha \theta'(s, r_0, \theta_0, \lambda) \, \mathrm{d}s \\
&= \frac{1}{\lambda} \int_0^\alpha \left(1 + (q-1)(\lambda - Q - 1) |\cos_p \theta|^p \right) \mathrm{d}s \ .
\end{aligned}
$$

As $\lambda \to -\infty$, this integrand tends to $(q-1)|\cos_p \theta_{-\infty}|^p$ almost everywhere, and is bounded by $3 + |Q|$. Therefore, by the Lebesgue Dominated Convergence Theorem, and the limit (41), we have

$$0 = \int_0^\alpha |\cos_p \theta_{-\infty}|^p \ .$$

This implies that $|\cos_p \theta_{-\infty}|^p = 0$ almost everywhere. In fact, we can show that $\theta_{-\infty} = {}^{\pi_p}\!/2 + k\pi_p$ almost everywhere. Let $\lambda < 0$, then

$$\begin{aligned}\Psi(\theta_0, \lambda) &= \theta(\alpha, r_0, \theta_0, \lambda) - \theta_0 \\ &= \int_0^\alpha 1 + (q-1)(\lambda - Q - 1)|\cos_p \theta|^p \\ &\leq \int_0^\alpha 1 - (q-1)(Q+1)|\cos_p \theta|^p \ .\end{aligned}$$

Hence, once again using the Lebesgue Dominated Convergence Theorem, as $\lambda \to -\infty$ we have

$$\theta_{-\infty}(t) - \theta_0 \leq \int_0^t 1 - (q-1)(Q+1)|\cos_p \theta_{-\infty}|^p \ .$$

Given that $|\cos_p \theta_{-\infty}|^p = 0$ almost everywhere,

$$\theta_{-\infty}(t) - \theta_0 \leq \int_0^t 1 = t \ ,$$

which is absolutely continuous in t. Now since $\theta_0 < {}^{\pi_p}\!/2 + (k+1)\pi_p$,

$$\theta_{-\infty}(t) < {}^{\pi_p}\!/2 + (k+1)\pi_p \ ,$$

for all $t \in \mathbb{R}$. Hence $\theta_{-\infty}(t) = {}^{\pi_p}\!/2 + k\pi_p$ for any $t > 0$. Similarly, $\theta_{-\infty} = {}^{\pi_p}\!/2 + k\pi_p$ in any right-neighbourhood of any point where $\theta_{-\infty} = {}^{\pi_p}\!/2 + k\pi_p$, and so,

$$\theta_{-\infty}(t) = \begin{cases} \theta_0 & , \quad t = 0 \\ {}^{\pi_p}\!/2 + k\pi_p & , \quad t > 0 \ . \end{cases}$$

Therefore, since $\theta_{-\infty}$ is the limiting function, there must exist a $\underline{\lambda} \in \mathbb{R}$ such that

$$
\begin{aligned}
\Psi(\theta_0, \underline{\lambda}) &< {}^{\pi_p}\!/_2 + (k+1)\pi_p - \theta_0 \\
&< {}^{\pi_p}\!/_2 + (k+1)\pi_p + {}^{\pi_p}\!/_2 - k\pi_p \\
&= 2\pi_p \ .
\end{aligned}
$$

Next, we show that for any $n \in \mathbb{N}$, there exists some $\overline{\lambda} \in \mathbb{R}$ such that $\Psi(\theta_0, \overline{\lambda}) > 2n\pi_p$. Consider $\theta(\alpha, r_0, \theta_0, \lambda)$, as $\lambda \to \infty$. Assume that this function is bounded in λ, then define

$$
\theta_\infty(t) := \lim_{\lambda \to \infty} \theta(t, r_0, \theta_0, \lambda) \ .
$$

Take $\lambda > 1$, then we have

$$
\theta(t, r_0, \theta_0, \lambda) - \theta_0 \geq \int_0^t 1 + (q-1)Q|\cos_p \theta|^p \ . \tag{42}
$$

Since we assumed that the function $\theta(\alpha, r_0, \theta_0, \lambda)$ is bounded in λ, it follows that

$$
\lim_{\lambda \to \infty} \frac{\Psi(\theta_0, \lambda)}{\lambda} = \lim_{\lambda \to \infty} \frac{\theta(\alpha, r_0, \theta_0, \lambda) - \theta_0}{\lambda} = 0 \ .
$$

So again, as

$$
\frac{\Psi(\theta_0, \lambda)}{\lambda} = \frac{1}{\lambda} \int_0^\alpha 1 + (q-1)(\lambda - Q - 1)|\cos_p \theta|^p \ ,
$$

if we take the limit $\lambda \to \infty$, we have

$$
\int_0^\alpha |\cos_p \theta|^p = 0 \ ,
$$

and hence, $|\cos_p \theta|^p = 0$ almost everywhere. Therefore, taking (42) and passing to the limit $\lambda \to \infty$,

$$
\theta_\infty(t) - \theta_0 \geq \int_0^t 1 + (q-1)Q|\cos_p \theta_\infty|^p = \int_0^t 1 = t \ ,
$$

once again using the Lebesgue Dominated Convergence Theorem. Therefore, for all $t > 2n\pi_p$, $\theta_\infty(t) - \theta_0 \geq 2n\pi_p$, which implies that

$$\theta_\infty(t) \geq {}^{\pi_p}\!/_2 + (k+2n)\pi_p \ ,$$

Hence, there exists some $\overline{\lambda} \in \mathbb{R}$ such that

$$\begin{aligned}\Psi(\theta_0, \overline{\lambda}) &> {}^{\pi_p}\!/_2 + k\pi_p - \theta_0 \\ &> {}^{\pi_p}\!/_2 + (k+2n)\pi_p - {}^{\pi_p}\!/_2 - k\pi_p \\ &= 2n\pi_p \ .\end{aligned}$$

□

Using this lemma, we now know that the following function is well-defined.

Definition 4.3.1. *Fix $n \in \mathbb{N}$, then the function $\lambda_n^* : \mathbb{R} \to \mathbb{R}$ is given by the relation*

$$\Psi(\theta_0, \lambda_n^*(\theta_0)) = 2n\pi_p \ ,$$

where such a value $\lambda_n^*(\theta_0)$ exists, and is unique, as a result of Lemma 4.3.1.

We now list several properties of this λ_n^* function, starting with periodicity.

Lemma 4.3.2. *For any fixed $n \in \mathbb{N}$, the function λ_n^* is π_p-periodic.*

Proof. If $\Psi(\theta_0, \lambda) = 2n\pi_p$ for some $\lambda \in \mathbb{R}$, then Corollary 2.2.1, $\Psi(\theta_0 + m\pi_p, \lambda) = 2n\pi_p$, for all $m \in \mathbb{N}$.

□

As we have seen in Theorem 2.2.3, the conditions for $\lambda \in \mathbb{R}$ to be a periodic eigenvalue can be expressed in terms of the value of the function Ψ, and its derivative in the θ_0 variable. As a result of this, we shall see that these conditions can be rewritten with respect to this λ^* function. First however, we must show that this function is indeed differentiable.

Lemma 4.3.3. *For any fixed $n \in \mathbb{N}$, the function λ_n^* is differentiable, and for all $\theta_0 \in \mathbb{R}$,*

$$(\lambda_n^*)'(\theta_0) = \frac{1 - \exp\left(-q(p-1)^{1/p} \int_0^\alpha (\lambda_n^* - Q - 1)\phi_p(\cos_p \theta)\phi_q(\sin_q \theta)\right)}{(q-1)\int_0^\alpha |u|^p}$$

Proof. $\Psi(\theta_0, \lambda)$ is differentiable in θ_0 (by Lemma 4.3.1), and in λ (by Theorem 4.3.1). Also, by Theorem 4.3.1, $\partial_2\Psi(\theta_0, \lambda) > 0$, so applying the Implicit Function Theorem to (40), we see that $\lambda_n^*(\theta_0)$ is differentiable in θ_0 and:

$$\begin{aligned} 0 &= \frac{d\Psi}{d\theta_0}(\theta_0, \lambda_n^*(\theta_0)) \\ &= \partial_1\Psi(\theta_0, \lambda_n^*(\theta_0)) + \partial_2\Psi(\theta_0, \lambda^*(\theta_0)) \cdot (\lambda_n^*)'(\theta_0) \end{aligned}$$

Therefore, we have

$$\begin{aligned} (\lambda_n^*)'(\theta_0) &= \frac{-\partial_1\Psi(\theta_0, \lambda_n^*(\theta_0))}{\partial_2\Psi(\theta_0, \lambda_n^*(\theta_0))} \\ &= \frac{1 - \exp\left(-q(p-1)^{1/p} \int_0^\alpha (\lambda_n^* - Q - 1)\phi_p(\cos_p \theta)\phi_q(\sin_q \theta)\right)}{(q-1)\int_0^\alpha |u|^p} \end{aligned}$$

□

We now rewrite the conditions in Theorem 2.2.3 in terms of this λ_n^* function, and show that any $\lambda \in \mathbb{R}$ is a periodic eigenvalue if it is in the image of this function, for some initial angle $\theta_0 \in \mathbb{R}$, and has a zero derivative at this point.

Theorem 4.3.1. *For any fixed $n \in \mathbb{N}$, and any angle $\theta_0 \in \mathbb{R}$, the value $\lambda_n^*(\theta_0)$ is an eigenvalue of the problem* (10) *if and only if $(\lambda_n^*)'(\theta_0) = 0$.*

Proof. By definition, $\Psi(\theta_0, \lambda_n^*(\theta_0)) = 2n\pi_p$, therefore for $\lambda_n^*(\theta_0)$ to be an eigenvalue, we only require $\partial_1\Psi(\theta_0, \lambda_n^*(\theta_0)) = 0$. By Lemma 4.3.3, this is true if and only if $(\lambda_n^*)'(\theta_0) = 0$.

□

The main use of the λ_n^* function is as a way of enumerating the Floquet Type solutions, for any value λ of the spectral parameter. For this purpose, we characterise all Floquet Type solutions in terms of the image of the λ_n^* function, and note that the condition for the existence of such a solution is exactly the first of the two conditions in Theorem 2.2.3, that are required for any such λ to be a periodic eigenvalue.

Theorem 4.3.2. *For any $\hat{\lambda} \in \mathbb{R}$, a solution with initial Prüfer angle θ_0 is a Floquet Type solution if and only if $\lambda_n^*(\theta_0) = \hat{\lambda}$, for some $n \in \mathbb{N}$.*

Proof. By definition, if $\lambda_n^*(\theta_0) = \hat{\lambda}$ for some $n \in \mathbb{N}_0$, then $\Psi(\theta_0, \hat{\lambda}) = 2n\pi_p$, and by Lemma 4.1.1, the corresponding solution is of the Floquet type:

$$u(t+\alpha, r_0, \theta_0, \hat{\lambda}) = \mu\, u(t, r_0, \theta_0, \hat{\lambda}) \ ,$$

for some $\mu > 0$.

□

Now that we have demonstrated the properties of the function λ_n^*, we determine how the image of this function can be used to enumerate the number of initial angles, $\theta_0 \in \mathbb{R}$, that result in Floquet Type solutions.

4.4 Using the λ_n^* Function for the Analysis of the Structure of the Spectrum

Theorem 4.3.2 tells us how many Floquet Type solutions (with distinct initial angles, modulo π_p) we have for a given $\hat{\lambda}$, and that this number is equal to the number of $\theta_0 \in [0, \pi_p)$ such that $\lambda_n^*(\theta_0) = \hat{\lambda}$, for some $n \in \mathbb{N}_0$. Note that by Lemma 2.3.2, if there exists a $\tilde{\theta_0} \in \mathbb{R}$ with

$$\Psi(\tilde{\theta_0}, \hat{\lambda}) = 2n\pi_p \ ,$$

then for all $\theta_0 \in \mathbb{R}$,

$$(n-1)\pi_p < \Psi(\theta_0, \hat{\lambda}) < (n+1)\pi_p \ .$$

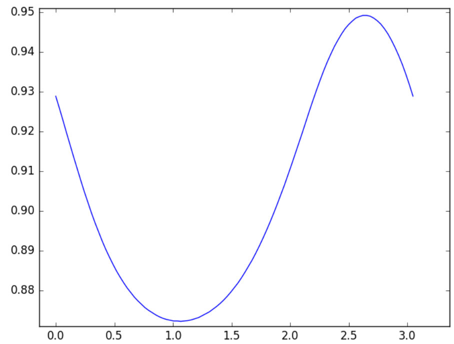
Therefore, if there exists an $n \in \mathbb{N}_0$ with $\lambda_n^*(\theta_0) = \hat{\lambda}$, for some $\theta_0 \in \mathbb{R}$, then for all other $j \in \mathbb{N}_0$,

$$\lambda_n^*(\theta_0) \neq \hat{\lambda} \ ,$$

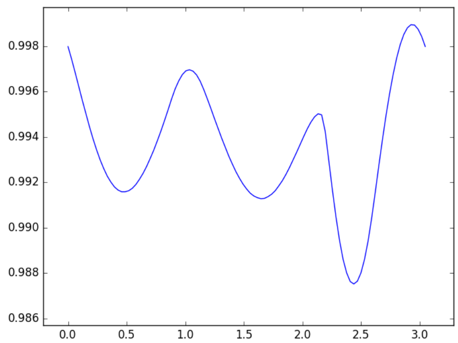
for all $\theta_0 \in \mathbb{R}$.

From this, we deduce that the number of Floquet Type solutions with distinct initial angles remains constant as we vary the value $\hat{\lambda}$, unless we come across a point with $(\lambda_n^*)'(\theta_0) = 0$ for some θ_0. By Theorem 4.3.1, this is true if and only if this $\hat{\lambda}$ is an eigenvalue of the problem (10). So the number of distinct Floquet Type solutions can only change when we vary λ through an eigenvalue of the problem.

We also note that if the 'creation' of new distinct Floquet Type solutions corresponds to a peak on these graphs, the matching 'annihilation' corresponds to a trough. So the Floquet Type solutions that are created between two adjacent peaks in these graphs are cancelled out by the resulting troughs in between them. We can therefore consider cycles of eigenvalues that stem from one eigenvalue that creates a Floquet Type solution, to the next eigenvalue that annihilates it, continuing by periodicity (Lemma 4.3.2) until we return to the first eigenvalue. Also, the fact that the λ_n^* function is well-defined (Lemma 4.3.1) also shows that there cannot be any separate closed cycles other than the main cycle, as this would result in a closed contour on our graph separate from the main graph, which would mean that λ^* is not a function.



(a) The function $\lambda_1^*(\theta_0)$, on the domain $0 \leq \theta_0 \leq \pi_3$, for $p = 3$, and some piecewise constant Q. Note that over a single π_p-period, there is a single oscillation, showing that there is only a single pair of eigenvalues that give a rotation number $\rho = 1$.



(b) The function $\lambda_1^*(\theta_0)$, on the domain $0 \leq \theta_0 \leq \pi_3$, for $p = 3$, and another piecewise constant Q. Note that over a single π_p-period, there are three oscillations, showing that there are three pairs of eigenvalues that give a rotation number $\rho = 1$; the end-points of the plateau, and four others in the interior.

Figure 9: The function $\lambda_1^*(\theta_0)$ for differing piecewise-constant potentials.

In this section, we discuss the properties of solutions of the iterated periodic problem over m periods. We once again use the *iterated renormalised Poincaré map* over m periods, for some $m \in \mathbb{N}$, given by

$$\Psi^m(\theta_0, \lambda) := \theta(m\alpha, r_0, \theta_0, \lambda) - \theta_0 \; .$$

We consider solutions of (10) for which the iterated renormalised Poincaré map is non-degenerate. Note that in the linear case, $p = 2$, by Theorem 1.0.9, all iterated renormalised Poincaré maps over m periods (for any $m > 2$) are degenerate. However, this is not necessarily true for $p \neq 2$, and indeed, in Chapter 5, we will derive conditions on a perturbation in the potential such that the iterated renormalised Poincaré map becomes non-degenerate.

We have shown in Theorem 2.4.2 that for any value of λ inside an instability intervals, say at the level $\rho(\lambda) = {}^{2n\pi_p}/_{m\alpha}$, there exists at least one value $\theta_0 \in \mathbb{R}$ such that

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ .$$

By the equation (22), we know that the effect of rescaling the initial Prüfer radius is to rescale the whole solution at any $t \in \mathbb{R}$ by the same ratio. Therefore, each initial angle θ_0 gives a solution that increases or decreases the radius by a certain factor after one period, and this factor is independent of the initial radius.

The following lemma shows that the given $m\alpha$-periodicity in the angular component, the Prüfer angle forms a cycle of m values after each period. We will then show that each of these angles can be associated with a corresponding radial increase/decrease.

Lemma 4.4.1. *Let $\lambda \in \mathbb{R}$ be such that there exists a value $\theta_0 \in [0, \pi_p)$ with*

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ ,$$

for some $m, n \in \mathbb{N}$ with m, n coprime. Then there exist $m-1$ other distinct values $\theta_1, \ldots, \theta_{m-1} \in [0, \pi_p)$ such that

$$\Psi^m(\theta_i, \lambda) = 2n\pi_p \ ,$$

for $i \in \{1, \ldots, m-1\}$.

Proof. Let $\theta_i = \theta(i\alpha, r_0, \theta_0, \lambda) + h\pi_p$, with $h \in \mathbb{Z}$ the unique value of h such that $\theta_i \in [0, \pi_p)$. Then for each θ_i, we have

$$
\begin{aligned}
\theta(m\alpha, r_0, \theta_i, \lambda) &= \theta(m\alpha, r_0, \theta(i\alpha, r_0, \theta_0, \lambda) + h\pi_p, \lambda) \\
&= \theta(m\alpha, r_0, \theta(i\alpha, r_0, \theta_0, \lambda), \lambda) + h\pi_p \qquad \text{(by Theorem 2.2.3)} \\
&= \theta((m+i)\alpha, r_0, \theta_0, \lambda) + h\pi_p \qquad \text{(by Theorem 2.3.1)} \\
&= \theta(i\alpha, r_0, \theta(m\alpha, r_0, \theta_0, \lambda), \lambda) + h\pi_p \\
&= \theta(i\alpha, r_0, \theta_0 + 2n\pi_p, \lambda) + h\pi_p \qquad \text{(by the hypothesis on } \theta_0\text{)} \\
&= \theta(i\alpha, r_0, \theta_0, \lambda) + (2n+h)\pi_p \\
&= \theta_i + 2n\pi_p \ ,
\end{aligned}
$$

and so it follows that

$$
\begin{aligned}
\Psi^m(\theta_i, \lambda) &= \theta(m\alpha, r_0, \theta_i, \lambda) - \theta_i \\
&= \theta_i + 2n\pi_p - \theta_i \\
&= 2n\pi_p \ ,
\end{aligned}
$$

for all the θ_i. These values are all distinct, else there would exist two values θ_j, θ_k in this set, such that

$$\theta_k = \theta_j + 2l\pi_p \ ,$$

for some $l \in \mathbb{N}$. This is equivalent to the statement

$$\Psi^{k-j}(\theta_j, \lambda) = 2l\pi_p \ ,$$

but since we already know that

$$\Psi^m(\theta_j, \lambda) = 2n\pi_p \ ,$$

then $k-j$ divides m, and more precisely, $ml = (k-j)n$. But we assumed that m and n were coprime, so this is a contradiction. Therefore all of the θ_i are distinct.

□

Following from this, we formalise the aforementioned concept that after each iteration of this type, each of the angles θ_i has an associated ratio of radial increase/decrease, and this translates to the solution u being rescaled through one of m distinct factors after each iteration.

Theorem 4.4.1. *Let $\lambda \in \mathbb{R}$ be such that there exists a $\theta_0 \in \mathbb{R}$ with*

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ ,$$

for some coprime $m, n \in \mathbb{N}$. By Lemma 4.4.1, there exist $m-1$ other values θ_i (with $i \in \{1, \ldots, m-1\}$) that along with θ_0, are all distinct modulo π_p, and have the property that

$$\Psi^m(\theta_i, \lambda) = 2n\pi_p \ .$$

For each of the values θ_i, with $i \in \{0, \ldots, m-1\}$, we have

$$u(t+\alpha, r_0, \theta_i, \lambda) = \mu_i u(t, r_0, \theta_{i+1}, \lambda) \ ,$$

where

$$\mu_i = \exp\left(\int_0^\alpha (p-1)^{-1/q}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\, \mathrm{d}s\right) \ .$$

Note that the values θ_i form a cycle modulo π_p, that is $\theta_m \equiv \theta_0 \ (mod\ \pi_p)$, and we therefore take $\theta_{m+i} \equiv \theta_i$ for all $i \in \{0, \ldots, m-1\}$.

Proof. The existence of the θ_i that are all distinct modulo π_p, is shown by Lemma 4.4.1. As before, we take $\theta_i = \theta(i\alpha, r_0, \theta_0, \lambda) + h\pi_p$, with $h \in \mathbb{Z}$ such that $\theta_i \in [0, \pi_p)$. As such, we have

$$\begin{aligned}\theta_m &= \theta(m\alpha, r_0, \theta_0, \lambda) + h\pi_p \\ &= \theta_0 + 2n\pi_p + h\pi_p\ ,\end{aligned}$$

We assumed that $\theta_0 \in [0, \pi_p)$, and so $h = -2n$, and $\theta_m = \theta_0$. The values θ_i therefore form a cycle, and $\theta_{m+i} = \theta_i$ for all i.

We have

$$\begin{aligned}r(t+\alpha, r_0, \theta_i, \lambda) &= r_0 \exp\Bigg(\int_0^{t+\alpha} ({}^q\!/\!_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda)) \\ &\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s\Bigg) \\ &= r_0 \exp\Bigg(\int_0^{\alpha} ({}^q\!/\!_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda)) \\ &\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s\Bigg) \cdot \exp\Bigg(\int_\alpha^{t+\alpha} ({}^q\!/\!_2)(p-1)^{1/p}(\lambda - Q(s) - 1) \\ &\qquad \cdot \phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s\Bigg) \\ &= r_0 \exp\Bigg(\int_0^{\alpha} ({}^q\!/\!_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda)) \\ &\qquad \cdot \phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s\Bigg) \cdot \exp\Bigg(\int_0^{t} ({}^q\!/\!_2)(p-1)^{1/p}(\lambda - Q(s\alpha) - 1) \\ &\qquad \cdot \phi_p(\cos_p \theta(s+\alpha, r_0, \theta_i, \lambda))\phi_q(\sin_q \theta(s+\alpha, r_0, \theta_i, \lambda))\,\mathrm{d}s\Bigg)\ ,\end{aligned}$$

by a substitution. We note that the function Q is α-periodic, and by the Theorem 2.3.1, we know that $\theta(s+\alpha, r_0, \theta_i, \lambda) = \theta(s, r_0, \theta(\alpha, r_0, \theta_i, \lambda), \lambda)$. Also, by Lemma 4.4.1, we have $\theta_{i+1} + 2k\pi_p = \theta(\alpha, r_0, \theta_i, \lambda)$, for some $k \in \mathbb{N}_0$, therefore

$$\theta(s+\alpha, r_0, \theta_i, \lambda) \equiv \theta_{i+1} \pmod{2\pi_p}\ .$$

Combining this with the expression for $r(t+\alpha, r_0, \theta_i, \lambda)$ above, we have

$$\begin{aligned}
r(t+\alpha, r_0, \theta_i, \lambda) = r_0 \exp \Bigg(& \int_0^\alpha ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda)) \\
& \cdot \phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s \Bigg) \exp \Bigg(\int_0^t ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1) \\
& \cdot \phi_p(\cos_p \theta(s, r_0, \theta_{i+1}, \lambda))\phi_q(\sin_q \theta(s, r_0, \theta_{i+1}, \lambda))\,\mathrm{d}s \Bigg) \\
= \exp \Bigg(& \int_0^\alpha ({}^q\!/\!{}_2)(p-1)^{1/p}(\lambda - Q(s) - 1)\phi_p(\cos_p \theta(s, r_0, \theta_i, \lambda)) \\
& \cdot \phi_q(\sin_q \theta(s, r_0, \theta_i, \lambda))\,\mathrm{d}s \Bigg) r(t, r_0, \theta_{i+1}, \lambda) \\
= \mu_i^{p/2}\, & r(t, r_0, \theta_{i+1}, \lambda)
\end{aligned}$$

then by the definition of r and θ,

$$\begin{aligned}
u(t+\alpha, r_0, \theta_i, \lambda) &= (r(t+\alpha, r_0, \theta_i, \lambda))^{2/p} \cos_p \theta(t+\alpha, r_0, \theta_i, \lambda) \\
&= (\mu_i^{p/2}\, r(t, r_0, \theta_i, \lambda))^{2/p} \cos_p \theta(t, r_0, \theta(\alpha, r_0, \theta_i, \lambda), \lambda) \\
&= \mu_i\, (r(t, r_0, \theta_{i+1}, \lambda))^{2/p} \cos_p \theta(t, r_0, \theta_{i+1}, \lambda) \\
&= \mu_i\, u(t, r_0, \theta_{i+1}, \lambda)
\end{aligned}$$

□

As before, we have the following corollary, which validates these solutions being referred to as Floquet Type solutions of the iterated problem.

Corollary 4.4.1. *Let $\lambda \in \mathbb{R}$ be such that there exists a value $\theta_i \in \mathbb{R}$ with*

$$\Psi^m(\theta_i, \lambda) = 2n\pi_p\ ,$$

for some $m, n \in \mathbb{N}$. Let

$$\mu := \prod_{i=0}^{m-1} \mu_i$$

where each μ_i *is defined as in Theorem 4.4.1, then*

$$u(t + km\alpha, r_0, \theta_i, \lambda) = \mu^k u(t, r_0, \theta_i, \lambda) \ ,$$

for any $k \in \mathbb{N}_0$.

These results provide an analytic basis for the behaviour we have seen on the phase-plane plots in Chapter 3, namely that there are values of λ for which each iteration deforms the unit circle outwards along some arcs, the greatest deformation occurring at angles to form horns. Likewise, other arcs are deformed inwards, forming horns that point towards the origin. The existence of these horns can be deduced from these initial angles θ_i that are periodic after m iterations.

If we take some fixed value $\lambda \in \mathbb{R}$, such that there exists a value θ_0 with

$$\Psi^m(\theta_0, \lambda) = n\pi_p \ ,$$

for some coprime $m, n \in \mathbb{N}$, and such that this iterated renormalised Poincaré map is non-degenerate, then by Lemma 4.4.1, there are m values θ_i with angular periodicity. By Corollary 4.4.1, the radial rescaling after m periods is equal for all m of these values, and Lemma 2.2.2 shows that this radial rescale is equal to the inverse of the square root of the gradient of the iterated renormalised Poincaré map at each of these θ_i. As a result, the gradient of $\Psi^m(\cdot, \lambda)$ is equal at each θ_i. The assumption that this map is non-degenerate means that $\Psi^m(\cdot, \lambda)$ is not constant, as such every point θ_i with angular periodicity and a positive gradient implies the existence of some ϑ_i that also has angular periodicity, and a negative gradient, and vice versa. The values θ_i such that

$$\Psi^m(\theta_i, \lambda) = 2n\pi_p \qquad \text{and} \qquad \partial_1 \Psi^m(\theta_0, \lambda) > 0 \ ,$$

each give a chain of m values of θ with an overall radial decrease after m periods.

Regarding the phaseplane, this tells us that at each θ_i, the radius will decrease after m periods, and so there exists a region of decreasing radius in the phaseplane with the extreme point being a inward-pointing horn. The same is true of values ϑ_i with $\partial_1\Psi^m(\vartheta_i, \lambda) < 0$, corresponding instead to a chain of points with an overall increasing radius. These show the existence of horns pointing outwards. By continuity of the normalised Poincaré map in θ_0, the points of increasing/decreasing gradient are interlacing, and as such, the horns of increasing and decreasing radius must also alternate.

As demonstrated in Lemma 4.4.1, if n is even, then there are m distinct values θ_i in each chain, and this simply gives m horns in the phaseplane. If however, n is odd, then after the first m iterations, we have anti-periodicity, and there are a further m values that are distinct, modulo $2\pi_p$. Therefore, if n is even, the phaseplane plot resulting from the system with this λ will have m horns, and if n is odd, the plot will have $2m$ horns.

We can also generalise the λ_n^* function, defined in Definition 4.3.1, to the iterated problem, by modifying the definition of $\lambda_n^*(\theta_0)$ to map each $\theta_0 \in \mathbb{R}$ to the unique λ such that:

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ ,$$

for some fixed $m, n \in \mathbb{N}$.

Definition 4.4.1. *Fix $m, n \in \mathbb{N}$, then the function $\lambda_n^{*,m} : \mathbb{R} \to \mathbb{R}$ is given by the relation*

$$\Psi^m(\theta_0, \lambda_n^{*,m}(\theta_0)) = 2n\pi_p \ .$$

Lemmas 4.3.1 through 4.3.3 are similarly true here, as the iterated renormalised Poincaré map, $\Psi^m(\theta_0, \lambda)$, has many of the same properties as $\Psi(\theta_0, \lambda)$ (e.g. differentiability, periodicity). So analogously to Theorem 4.3.1, we have:

$$
\begin{aligned}
(\lambda_n^{*,m})'(\theta_0) &= \frac{-\partial_1 \Psi^m(\theta_0, \lambda_n^{*,m}(\theta_0))}{\partial_2 \Psi(\theta_0, \lambda_n^{*,m}(\theta_0))} \\
&= \frac{1 - \exp\left(-q(p-1)^{1/p} \int_0^{m\alpha} (\lambda_n^{*,m} - Q - 1)\varphi_p(\cos_p \theta)\varphi_q(\sin_q \theta)\right)}{(q-1)\int_0^{m\alpha} |u|^p} ,
\end{aligned}
$$

and similarly $\lambda_n^{*,m}(\theta_0)$ is an eigenvalue if and only if $(\lambda_n^{*,m}(\theta_0))' = 0$. We have an analogue for Theorem 4.3.2.

Lemma 4.4.2. *Fix $m \in \mathbb{N}$. For any $\hat{\lambda} \in \mathbb{R}$, a solution with initial Prüfer angle θ_0 is a Floquet Type solution after m periods if and only if $\lambda_n^{*,m}(\theta_0) = \hat{\lambda}$, for some $n \in \mathbb{N}$.*

Proof. By definition , if $\lambda_n^{*,m}(\theta_0) = \hat{\lambda}$, then $\Psi^m(\theta_0, \hat{\lambda}) = 2n\pi_p$, and by Theorem 4.4.1, the corresponding solution is of the Floquet type:

$$
u(t + m\alpha, r_0, \theta_0, \lambda) = \mu\, u(t, r_0, \theta_0, \lambda) ,
$$

for some $\mu > 0$.

□

In conclusion, we have shown that the characterisation of the plateaus of the rotation number function as *instability intervals* is valid even in the nonlinear case, and that the existence of the so-called Floquet solutions can be generalised for solutions here too. We do however, note several key differences. Given that this problem is nonlinear, for any value of λ inside an instability interval, not all solutions of the problem are linear combinations of the Floquet Type solutions.

Also, as a result of the perturbation arguments in Chapter 2.5, there may be more than two Floquet Type Solutions, as opposed to the simple pairs that occur when $p = 2$. However, even in the nonlinear case, for every Floquet Type solution with Floquet multiplier $\mu_1 > 1$, there exists a Floquet Type solution with Floquet multiplier $\mu_2 < 1$, and vice-versa. Therefore, the Floquet Type solutions are still paired, in this sense.

5 The Periodic Problem on Multiple Periods

The results regarding the stability of solutions in Chapters 3 and 4, particularly the results regarding the existence of Floquet Type solutions, can now be extended to the behaviour of solutions of the equation (10) over m periods, for any $m \in \mathbb{N}$. As before, we have the equation

$$(\phi_p(u'))' + (\lambda - Q)\phi_p(u) = 0 ,$$

but we now modify the periodic boundary conditions to

$$\begin{aligned} u(m\alpha) &= u(0) \\ u'(m\alpha) &= u'(0) , \end{aligned}$$

to define the *Iterated Periodic Problem* on m periods.

5.1 The Iterated Renormalised Poincaré Map and the Rotation Number

Once again, we consider the iterated renormalised Poincaré map over m periods Ψ^m, given in Definition 2.3.1. As demonstrated for the case $m = 1$ in Theorem 2.2.3, we have a characterisation of periodic eigenvalues of the iterated problem in terms of the map Ψ^m.

Theorem 5.1.1. *A value $\lambda \in \mathbb{R}$ is a periodic eigenvalue of the iterated problem if and only if there exists a $\theta_0 \in \mathbb{R}$ such that*

$$\begin{aligned} \Psi^m(\theta_0, \lambda) &= 2n\pi_p \\ \partial_1 \Psi^m(\theta_0, \lambda) &= 0 , \end{aligned}$$

for some $m, n \in \mathbb{N}$.

We can also link this characterisation of periodic eigenvalues of the iterated periodic problem with conditions on the rotation number.

Theorem 5.1.2. *Given a spectral value* $\lambda \in \mathbb{R}$ *such that there exists a* $\theta_0 \in \mathbb{R}$, *and* $l, m \in \mathbb{N}$, *with*

$$\Psi^m(\theta_0, \lambda) = l\pi_p \ ,$$

then the rotation number

$$\rho(\lambda) = \frac{l\pi_p}{m\alpha} \ .$$

Proof.

$$\begin{aligned}\rho(\lambda) &= \lim_{k\to\infty} \frac{\Psi^{km}(\theta_0, \lambda)}{km\alpha} \\ &= \lim_{k\to\infty} \frac{\theta_0 + kl\pi_p - \theta_0}{km\alpha} \\ &= \lim_{k\to\infty} \frac{kl\pi_p}{km\alpha} \\ &= \frac{l\pi_p}{m\alpha}\end{aligned}$$

□

The converse is also proved below, and from this we can show that the periodic eigenvalues of the problem after m periods exactly correspond to the end-points of the intervals at which $\rho = 2n\pi_p(m\alpha)^{-1}$, for some $n \in \mathbb{N}$. We start by considering the following lemma.

Lemma 5.1.1. *For any* $m \in \mathbb{N}$, *and any* $\lambda \in \mathbb{R}$, *we have*

$$\min_{\theta_0\in\mathbb{R}} \left(\frac{\Psi^m(\theta_0, \lambda)}{m\alpha} \right) \le \rho(\lambda) \le \max_{\theta_0\in\mathbb{R}} \left(\frac{\Psi^m(\theta_0, \lambda)}{m\alpha} \right)$$

Proof. We note that for any $k \in \mathbb{N}$, $\theta_0 \in \mathbb{R}$, we have

$$\Psi^{km}(\theta_0,\lambda) = \sum_{i=1}^{k} \theta(im\alpha, r_0, \theta_0, \lambda) - \theta((i-1)m\alpha, r_0, \theta_0, \lambda)\ ,$$

and therefore

$$\begin{aligned}
\min_{\theta_0\in\mathbb{R}} \Psi^{km}(\theta_0,\lambda) &= \min_{\theta_0\in\mathbb{R}} \left(\sum_{i=1}^{k} \theta(im\alpha, r_0, \theta_0, \lambda) - \theta((i-1)m\alpha, r_0, \theta_0, \lambda) \right) \\
&= \min_{\theta_0\in\mathbb{R}} \left(\sum_{i=1}^{k} \theta(m\alpha, r_0, \theta(i\alpha, r_0, \theta_0, \lambda), \lambda) - \theta(m\alpha, r_0, \theta((i-1)\alpha, r_0, \theta_0, \lambda), \lambda) \right) \\
&\geq \sum_{i=1}^{k} \min_{\theta_0\in\mathbb{R}} \left(\theta(m\alpha, r_0, \theta_0, \lambda) - \theta_0\right) \\
&= k \min_{\theta_0\in\mathbb{R}} \left(\theta(m\alpha, r_0, \theta_0, \lambda) - \theta_0\right) \\
&= k \min_{\theta_0\in\mathbb{R}} \Psi^{m}(\theta_0,\lambda)\ ,
\end{aligned}$$

giving us

$$\min_{\theta_0\in\mathbb{R}} \Psi^{m}(\theta_0,\lambda) \leq \min_{\theta_0\in\mathbb{R}} \frac{\Psi^{km}(\theta_0,\lambda)}{k}$$

Dividing both sides of this inequality by $m\alpha$, and taking the limit as $k \to \infty$, we get

$$\min_{\theta_0\in\mathbb{R}} \left(\frac{\Psi^{m}(\theta_0,\lambda)}{m\alpha} \right) \leq \min_{\theta_0\in\mathbb{R}} \left(\lim_{k\to\infty} \frac{\Psi^{km}(\theta_0,\lambda)}{km\alpha} \right) = \rho(\lambda)$$

Similarly;

$$\max_{\theta_0\in\mathbb{R}} \left(\frac{\Psi^{m}(\theta_0,\lambda)}{m\alpha} \right) \geq \max_{\theta_0\in\mathbb{R}} \left(\lim_{k\to\infty} \frac{\Psi^{km}(\theta_0,\lambda)}{km\alpha} \right) = \rho(\lambda)$$

□

We now prove the converse of Theorem 5.1.2.

Theorem 5.1.3. *Given a spectral value $\lambda \in \mathbb{R}$ such that*

$$\rho(\lambda) = \frac{l\pi_p}{m\alpha}\ ,$$

(for some $l, m \in \mathbb{N}$), there exists a $\theta_0 \in \mathbb{R}$, such that

$$\Psi^m(\theta_0, \lambda) = l\pi_p \ .$$

Proof. We can divide the values $\lambda \in \mathbb{R}$ into three distinct cases:

- *Case 1:* All values of $\theta_0 \in \mathbb{R}$ give $\Psi^m(\theta_0, \lambda) < l\pi_p$

  By Lemma 5.1.1, this means that:

$$\rho(\lambda) \leq \max_{\theta_0 \in \mathbb{R}} \left(\frac{\Psi^m(\theta_0, \lambda)}{m\alpha} \right) < \frac{l\pi_p}{m\alpha}$$

- *Case 2:* There exists a $\theta_0 \in \mathbb{R}$ such that $\Psi^m(\theta_0, \lambda) = l\pi_p$

  By Theorem 5.1.2, this means that:

$$\rho(\lambda) = \frac{l\pi_p}{m\alpha}$$

- *Case 3:* All values of $\theta_0 \in \mathbb{R}$ give $\Psi^m(\theta_0, \lambda) > l\pi_p$

  By Lemma 5.1.1, this means that:

$$\rho(\lambda) \geq \min_{\theta_0 \in \mathbb{R}} \left(\frac{\Psi^m(\theta_0, \lambda)}{m\alpha} \right) > \frac{l\pi_p}{m\alpha}$$

Therefore, we have a trichotomy, and if $\rho(\lambda) = \frac{l\pi_p}{m\alpha}$, then there exists a $\theta_0 \in \mathbb{R}$ such that:

$$\Psi^m(\theta_0, \lambda) = l\pi_p \ .$$

$\square$

From this theorem, we derive the following characterisation of periodic eigenvalues of the problem (10) after m periods.

Corollary 5.1.1. *The end-points of the interval for which the rotation number takes the value $2n\pi_p(m\alpha)^{-1}$ are eigenvalues of the periodic problem over m periods, the eigenfunctions of which each have $2n$ zeros over m periods.*

Proof. By Theorem 5.1.3, for any value of λ inside this interval, there exists some $\theta_0 \in \mathbb{R}$ such that

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ ,$$

and also that for any point to the left of this interval, no such θ_0 exists, in fact

$$\max_{\theta_0 \in \mathbb{R}} \left(\Psi^m(\theta_0, \underline{\lambda})\right) < 2n\pi_p \ .$$

Therefore, at the left end-point of the interval, we have $\underline{\lambda}$ such that

$$\max_{\theta_0 \in \mathbb{R}} \left(\Psi^m(\theta_0, \underline{\lambda})\right) = 2n\pi_p \ .$$

As the function Ψ^m is differentiable in its first variable, and the point at which the maximum is attained is a stationary point, this is equivalent to saying that there exists a $\tilde{\theta_0} \in \mathbb{R}$ such that

$$\Psi^m(\tilde{\theta_0}, \underline{\lambda}) = 2n\pi_p \qquad \text{and} \qquad \partial_1(\Psi^m(\tilde{\theta_0}, \underline{\lambda})) = 0 \ .$$

By Theorem 5.1.1, these conditions are equivalent to $\underline{\lambda}$ being an eigenvalue of the periodic problem over m periods.

Similarly, for the right end-point, $\overline{\lambda}$,

$$\min_{\theta_0 \in \mathbb{R}} \left(\Psi^m(\theta_0, \overline{\lambda})\right) = 2n\pi_p \ ,$$

and again, this is equivalent to saying that there exists a $\hat{\theta_0} \in \mathbb{R}$ such that

$$\Psi^m(\hat{\theta}_0, \overline{\lambda}) = 2n\pi_p \qquad \text{and} \qquad \partial_1(\Psi^m(\hat{\theta}_0, \overline{\lambda})) = 0 \ ,$$

which tells us that $\overline{\lambda}$ is also a periodic eigenvalue of this iterated problem.

□

We now return to an analysis of perturbation arguments, and introduce a perturbation in the potential. Let $\lambda \in \mathbb{R}$ be a periodic eigenvalue over several periods, but not a periodic eigenvalue over the problem on a single period. In the linear case, $p = 2$, such an eigenvalue must be degenerate, that is, the entire resulting solution space is $m\alpha$-periodic. However, we will show through these perturbation arguments, that there exist potentials that result in non-degenerate eigenvalues of the iterated problem, that are similarly not periodic eigenvalues of the problem on a single period.

5.2 Existence of Potentials Generating Non-Standard Rotation Number Plateaus

We continue our consideration of the problem extended over several periods, and apply the results from the perturbation arguments discussed in Chapter 2.5. We once again have the equation (30),

$$(\phi_p(u'))' + (\lambda - Q_1 - \varepsilon Q_2)\phi_p(u) = 0 \ ,$$

with some α-periodic perturbation $Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})$, and we require the boundary conditions

$$\begin{aligned} u(m\alpha) &= u(0) \\ u'(m\alpha) &= u'(0) \ , \end{aligned}$$

to be satisfied for some $m \in \mathbb{N}$. Any value $\lambda \in \mathbb{R}$ for which the equation has a non-trivial solution that satisfies these boundary conditions is a *periodic eigenvalue after m periods.*

For $\varepsilon = 0$, there are certain potentials $Q_1 \in L^1_{\mathrm{loc}}(\mathbb{R})$ for which an eigenvalue λ results in a solution manifold that lies entirely within the space of $m\alpha$-periodic functions. As before, we refer to these eigenvalues as *degenerate*. For example, $Q_1 \equiv 0$, has $m\alpha$-periodic eigenvalues of the form

$$\lambda_n = \left(\frac{2n\pi_p}{m\alpha}\right)^p \ ,$$

and all of these eigenvalues have a resulting space of solutions given by

$$A \cos_p\left(\frac{2n\pi_p}{m\alpha}(t - t_0)\right) \ ,$$

for some $A, t_0 \in \mathbb{R}$. Since all solutions of this form are $m\alpha$-periodic, these eigenvalues are degenerate.

From Chapter 2, we know that the end-points of plateaus of the resulting rotation number are periodic eigenvalues, but unlike the linear case, it has been shown that there may also exist periodic eigenvalues in the interior of the plateaus, [4]. Each plateau corresponds to a set of values, λ, such that for the remormalised Poincaré map, there exits an initial Prüfer angle that increases a multiple of $2\pi_p$ over one period. For degenerate eigenvalues, this map is constant, as all initial angles map to the same multiple of $2\pi_p$, the resulting rotation number plateau therefore degenerates to a single point.

The alternative to the degeneracy of an eigenvalue is a set of values λ_i that all result in the same value of the rotation number, two of these are the end-points of the plateau, and all others are in the interior. We now consider whether any α-periodic potential, Q_1, that has a degenerate periodic eigenvalue λ (after m periods) can be perturbed by some α-periodic Q_2 such that the degenerate $m\alpha$-periodic eigenvalue, λ, is perturbed to become several non-degenerate $m\alpha$-periodic eigenvalues, λ_i. This is akin to 'opening up' a plateau of the resulting rotation numbers, as ε diverges from zero. Our first step is to state a generalisation of Lemma 2.5.1 for the iterated problem.

Lemma 5.2.1. *Consider the equation* (30) *with potential* $Q_1 \in L^1_{\text{loc}}(\mathbb{R})$*, and a perturbation* $Q_2 \in L^1_{\text{loc}}(\mathbb{R})$*, we have*

$$\partial_3 \Psi^m(\theta_0, \lambda, \varepsilon) = \frac{-(q-1)}{(r(m\alpha, r_0, \theta_0, \lambda, \varepsilon))^2} \int_0^{m\alpha} Q_2(s)\, |u(s, r_0, \theta_0, \lambda, \varepsilon)|^p \, \mathrm{d}s \, .$$

The proof of this Lemma can be adapted from the proof of Lemma 2.5.1. We can apply this lemma to the perturbation of degenerate renormalised Poincaré maps. We know that degenerate renormalised Poincaré maps are constant in θ_0, so for any α-periodic potential Q_1 and degenerate eigenvalue λ, it suffices to find a perturbation of the potential such that the renormalised Poincaré map becomes non-constant.

Theorem 5.2.1. *Let* $\lambda \in \mathbb{R}$ *be such that*

$$\Psi^m(\theta_0, \lambda, 0) = 2n\pi_p \, ,$$

for all $\theta_0 \in \mathbb{R}$*, for some coprime* $m, n \in \mathbb{N}$*. If there exists an* α*-periodic function* Q_2 *such* *that*

$$\int_0^{m\alpha} Q_2(s)|u(s,r_0,\theta_1,\lambda,0)|^p \,\mathrm{d}s > 0$$
$$\int_0^{m\alpha} Q_2(s)|u(s,r_0,\theta_2,\lambda,0)|^p \,\mathrm{d}s < 0$$

for some distinct $\theta_1,\theta_2 \in [0,\pi_p)$ *, then the* m^{th}*-iterated Poincaré map,* $\Psi^m(\cdot,\lambda)$ *will become non-constant for some* $\varepsilon > 0$*.*

Proof. As the periodic eigenvalue λ is degenerate, we know that all resulting solutions are α-periodic, and so, for $\varepsilon = 0$, we have $r(m\alpha, r_0, \theta_0, \lambda, 0) = r_0$ for any $r_0 \in \mathbb{R}$. Also, from Lemma 5.2.1 , we have

$$\begin{aligned}
\partial_3\Psi^m(\theta_1,\lambda,0) &= \frac{-(q-1)}{(r(m\alpha,r_0,\theta_1,\lambda,0))^2}\int_0^{m\alpha} Q_2(s)\ |u(s,r_0,\theta_1,\lambda,0)|^p \,\mathrm{d}s \\
&= -\frac{(q-1)}{r_0^2}\int_0^{m\alpha} Q_2(s)\ |u(s,r_0,\theta_1,\lambda,0)|^p \,\mathrm{d}s \\
&< 0\ ,
\end{aligned}$$

by the hypotheses on Q_2. Similarly,

$$\partial_3\Psi^m(\theta_2,\lambda,0) > 0\ .$$

Therefore, since $\Psi^m(\theta_1,\lambda,0) = \Psi^m(\theta_2,\lambda,0) = 2n\pi_p$, there exists some $\varepsilon > 0$ such that

$$\Psi^m(\theta_1,\lambda,\varepsilon) < 2n\pi_p$$
$$\Psi^m(\theta_2,\lambda,\varepsilon) > 2n\pi_p$$

Therefore, the map $\Psi(\cdot,\lambda,\varepsilon)$ is non-constant.

□

The effect of this perturbation is that a single degenerate eigenvalue will become (at least) two separate eigenvalues, each of which has a one-dimensional solution manifold. As mentioned above, this can be stated in terms of 'opening up' the plateau of the rotation number functions.

Corollary 5.2.1. *Let $\lambda \in \mathbb{R}$ be such that*

$$\Psi^m(\theta_0, \lambda, 0) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$, for some $m, n \in \mathbb{N}$. Then the corresponding rotation number

$$\rho(\lambda, 0) = \frac{2n\pi_p}{m\alpha} \ .$$

We also have

$$\rho(\underline{\lambda}, 0) < \frac{2n\pi_p}{m\alpha}$$

for all $\underline{\lambda} < \lambda$, and

$$\rho(\overline{\lambda}, 0) > \frac{2n\pi_p}{m\alpha}$$

for all $\overline{\lambda} > \lambda$.

Furthermore, if there exists an α-periodic function Q_2, such that

$$\int_0^{m\alpha} Q_2(s)|u(s, r_0, \theta_1, \lambda, 0)|^p \ \mathrm{d}s > 0$$
$$\int_0^{m\alpha} Q_2(s)|u(s, r_0, \theta_2, \lambda, 0)|^p \ \mathrm{d}s < 0; ,$$

for some distinct $\theta_1, \theta_2 \in [0, \pi_p)$, then there exists some $\varepsilon > 0$ such that for the resulting perturbed problem (30) *with this value ε, there exist two values $\underline{\lambda}, \overline{\lambda}$ such that $\underline{\lambda} < \lambda < \overline{\lambda}$, and*

$$\rho(\hat{\lambda}, \varepsilon) = \frac{2n\pi_p}{m\alpha} \ ,$$

for all $\hat{\lambda} \in [\underline{\lambda}, \overline{\lambda}]$.

Proof. For $\varepsilon = 0$, at the periodic eigenvalue λ, we have

$$\begin{aligned}
\rho(\lambda, 0) &= \lim_{t\to\infty} \frac{\theta(t, r_0, \theta_0, \lambda, 0) - \theta_0}{t} \\
&= \lim_{k\to\infty} \frac{\theta(km\alpha, r_0, \theta_0, \lambda, 0) - \theta_0}{km\alpha} \\
&= \lim_{k\to\infty} \frac{\theta_0 + 2kn\pi_p - \theta_0}{km\alpha} \\
&= \frac{2n\pi_p}{m\alpha} \ .
\end{aligned}$$

By our assumption,

$$\Psi^m(\theta_0, \lambda, 0) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$. For any fixed value θ_0, we have monotonicity in the second variable, and taking $\underline{\lambda} < \lambda$, thus

$$\Psi^m(\theta_0, \underline{\lambda}, 0) < 2n\pi_p \ .$$

This holds for any value of θ_0, therefore

$$\max_{\theta_0 \in \mathbb{R}} \Psi^m(\theta_0, \lambda, 0) < 2n\pi_p \ ,$$

and by Theorem 5.1.3,

$$\rho(\underline{\lambda}, 0) < \frac{2n\pi_p}{m\alpha} \ .$$

By the same argument,

$$\rho(\overline{\lambda}, 0) > \frac{2n\pi_p}{m\alpha} \ ,$$

for any $\overline{\lambda} > \lambda$.

By Lemma 5.2.1, we know that there exists an $\varepsilon > 0$ such that

$$\Psi(\theta_1, \lambda, \varepsilon) < 2n\pi_p$$
$$\Psi(\theta_2, \lambda, \varepsilon) > 2n\pi_p \ .$$

By the continuity and monotonicity of the map $\Psi(\theta_0, \lambda, \varepsilon)$ in the second variable, there exist values $\underline{\lambda} < \lambda < \overline{\lambda}$ such that

$$\Psi(\theta_1, \underline{\lambda}, \varepsilon) = 2n\pi_p$$
$$\Psi(\theta_2, \overline{\lambda}, \varepsilon) = 2n\pi_p \ .$$

Therefore, we have

$$\rho(\underline{\lambda}, \varepsilon) = \rho(\lambda, \varepsilon) = \rho(\overline{\lambda}, \varepsilon) = \frac{2n\pi_p}{m\alpha} \ ,$$

$\square$

We now turn our attention to whether or not such a perturbing potential $Q_2 \in L^1_{\text{loc}}(\mathbb{R})$, that satisfies the conditions in Theorem 5.2.1, can actually exist. As in Chapter 2.5, the conditions can be phrased in terms of linear independence, but given that we are looking for a Q_2 that is α-periodic, we must first rewrite the integral conditions into conditions on the domain $[0, \alpha)$.

Lemma 5.2.2. *Let $m \in \mathbb{N}$, for any α-periodic function $Q_2 \in L^1_{\text{loc}}(\mathbb{R})$ and $m\alpha$-periodic function $f \in L^\infty_{\text{loc}}(\mathbb{R})$, we have*

$$\int_0^{m\alpha} Q_2(s)\ f(s)\ \mathrm{d}s = \int_0^{\alpha} Q_2(s) \left(\sum_{i=0}^{m-1} f(s+i\alpha) \right)\ \mathrm{d}s$$

Proof. First, we note that the integrand is integrable by Hölder's Inequality, combined with the fact that Q_2 is absolutely locally integrable, and f is bounded. Next, divide the domain of integration into subintervals of length α,

$$\int_0^{m\alpha} Q_2(s)\ f(s)\ \mathrm{d}s = \sum_{i=0}^{m-1} \int_{i\alpha}^{(i+1)\alpha} Q_2(s) f(s)\ \mathrm{d}s\ .$$

For each integral in this summation, we have the substitution

$$\int_{i\alpha}^{(i+1)\alpha} Q_2(s)\ f(s)\ \mathrm{d}s = \int_0^{\alpha} Q_2(s)\ f(s+i\alpha)\ \mathrm{d}s\ ,$$

for all $i \in \{0, \dots, m-1\}$. Therefore, we have

$$\int_0^{m\alpha} Q_2(s)\ f(s)\ \mathrm{d}s = \sum_{i=0}^{m-1} \int_0^{\alpha} Q_2(s) f(s+i\alpha)\ \mathrm{d}s = \int_0^{\alpha} Q_2(s) \left(\sum_{i=0}^{m-1} f(s+i\alpha) \right)\ \mathrm{d}s\ .$$

□

Next, we apply this Lemma to the integrals in the conditions stated in Theorem 5.2.1 to create equivalent conditions that give sufficient conditions for the existence of a Q_2.

Lemma 5.2.3. *The integral conditions stated in Theorem 5.2.1 are equivalent to finding a function $Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that*

$$\begin{aligned}
&\int_0^{\alpha} Q_2(s) \left(\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_1, \lambda, 0))^p |u(s, r_0, \theta(i\alpha, r_0, \theta_1, \lambda, 0), \lambda, 0)|^p \right)\ \mathrm{d}s > 0 \\
&\int_0^{\alpha} Q_2(s) \left(\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_2, \lambda, 0))^p |u(s, r_0, \theta(i\alpha, r_0, \theta_2, \lambda, 0), \lambda, 0)|^p \right)\ \mathrm{d}s < 0\ ,
\end{aligned}$$

for some distinct $\theta_1, \theta_2 \in [0, {}^{\pi_p}\!/{}_2)$.

Proof. Use Lemma 5.2.2, with $f(t) = |u(t, r_0, \theta_0, \lambda, 0)|^p$, for any $\theta_0 \in \mathbb{R}$. Note that the solutions are continuous, and at $\varepsilon = 0$, the solutions are $m\alpha$-periodic. So the solutions are all bounded, and we can apply Lemma 5.2.2. Then we have

$$\begin{aligned} f(t + i\alpha) &= |u(t + i\alpha, r_0, \theta_0, \lambda, 0)|^p \\ &= |u(t, r(i\alpha, r_0, \theta_0, \lambda, 0), \theta(i\alpha, r_0, \theta_0, \lambda, 0), \lambda, 0)|^p \\ &= (r(i\alpha, 1, \theta_0, \lambda, 0))^2 \, |u(t, r_0, \theta(i\alpha, r_0, \theta_0, \lambda, 0), \lambda, 0)|^p \, . \end{aligned}$$

□

We can now use this to formulate conditions for the existence of a perturbing potential, Q_2.

Corollary 5.2.2. *For any two fixed initial angles, $\theta_1, \theta_2 \in \mathbb{R}$, there exists a $Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})$ that satisfies the conditions in Theorem 5.2.1 if the functions*

$$\begin{aligned} &\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_1, \lambda, 0))^2 |u(t, r_0, \theta(i\alpha, r_0, \theta_1, \lambda, 0), \lambda, 0)|^p \qquad \textit{and} \\ &\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_2, \lambda, 0))^2 |u(t, r_0, \theta(i\alpha, r_0, \theta_2, \lambda, 0), \lambda, 0)|^p \, , \end{aligned}$$

are linearly independent.

Proof. Let

$$f(t, \theta_0) := \sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_0, \lambda, 0))^2 |u(t, r_0, \theta(i\alpha, r_0, \theta_0, \lambda, 0), \lambda, 0)|^p \, ,$$

for any $\theta_0 \in \mathbb{R}$. If we choose $Q_2(t) = \alpha_1 \, f(t, \theta_1) + \alpha_2 \, f(t, \theta_2)$, then for Q_2 to satisfy the conditions in Theorem 5.2.1, the values $\alpha_1, \alpha_2 \in \mathbb{R}$ satisfy the system

$$\left(\int_0^\alpha f(s, \theta_i) \, f(s, \theta_j) \, \mathrm{d}s \right)_{i,j=1,2} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \, .$$

This matrix is Gramian, and therefore is invertible if and only if the functions in the inner products are all linearly independent, [14, Theorem 7.2.10]. Therefore, for such a Q_2 to exist, it is sufficient for the functions $f(t, \theta_1)$ and $f(t, \theta_2)$ are linearly independent.

$\square$

For the unperturbed problem, $\varepsilon = 0$, we assumed that the solutions we were considering were $m\alpha$-periodic, and therefore, the functions

$$\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_0, \lambda, 0))^2 |u(t, r_0, \theta(i\alpha, r_0, \theta_0, \lambda, 0), \lambda, 0)|^p$$

are also $m\alpha$-periodic, for all $\theta_0 \in \mathbb{R}$. We now state a result that can be used to prove linear independence of these functions, for distinct values $\theta_0 \in \mathbb{R}$.

Lemma 5.2.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be a real-valued function which is differentiable, non-constant and α-periodic for some period $\alpha > 0$, then there exists some $\gamma \in (0, \alpha)$ such that the shifted function $f(\cdot + \gamma)$ is linearly independent of $f(\cdot)$.*

Proof. Assume that every shift of f is linearly dependent of f, then there exists a function $m : \mathbb{R} \to \mathbb{R}$ such that

$$f(t + c) = m(c) f(t) \ ,$$

for all $c \in \mathbb{R}$. At $t = 0$, we have

$$f(c) = m(c) f(0) \ .$$

Hence if $f(0) = 0$, it follows that $f \equiv 0$, contradicting the assumption that f is non-constant. Since $f(0) \neq 0$, we can rewrite the first equality as

$$f(t + c) = \frac{f(c)}{f(0)} f(t) \ ,$$

Differentiating this w.r.t. c, we have

$$f'(t+c) = \frac{f'(c)}{f(0)} f(t) ,$$

and so if we take, for example, $c = 0$, then we have

$$f'(t) = C f(t) ,$$

where $C = {}^{f'(0)}/_{f(0)}$. The general solution of this ODE is of the form

$$f(t) = A \exp(Ct) ,$$

with $A \in \mathbb{R}$. But since we assumed f to be periodic, the only possibility is $C = 0$, which would mean f is constant, but this also gives a contradiction. Therefore, there exists a linearly independent shift of f.

□

We use this lemma to prove the existence of such a perturbing potential Q_2 in the case $Q_1 \equiv 0$. The solutions of the unperturbed problem ($\varepsilon = 0$) are of the form

$$A \cos_p(\lambda^{1/p} (t - t_0)) ,$$

for some $A, t_0 \in \mathbb{R}$. We fix $m \in \mathbb{N}$, then using Lemma 5.2.3, we show that it is sufficient to prove that the function

$$\sum_{i=0}^{m-1} \left| \cos_p \left(t + \frac{2i\pi_p}{m} \right) \right|^p \qquad (t \in \mathbb{R}) ,$$

is non-constant.

Lemma 5.2.5. *Let $m \in \mathbb{N}$, and $p \neq 2$. Then the function*

$$|\cos_p(t)|^p + \sum_{i=1}^{m-1} \left| \cos_p \left(t + \frac{2i\pi_p}{m} \right) \right|^p \qquad (t \in \mathbb{R}) ,$$

is non-constant.

Proof. Suppose that

$$|\cos_p(t)|^p + \sum_{i=1}^{m-1} \left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p \equiv C \qquad (t \in \mathbb{R}) \ , \tag{43}$$

for some $C \in \mathbb{R}$. The function $|\cos_p|^p$ is positive on the interval $(0, {}^{\pi_p}\!/_2)$, and so from (43), we would have

$$\cos_p(t) = \left(C - \sum_{i=1}^{m-1} \left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)^{1/p} \qquad (t \in (0, {}^{\pi_p}\!/_2)) \ . \tag{44}$$

By definition, the function $\cos_p$ solves the ODE

$$(\phi_p(u'))' + \phi_p(u) = 0 \ ,$$

and so if (44) is true, then v, defined by

$$v(t) := \left(C - \sum_{i=1}^{m-1} \left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)^{1/p} \qquad (t \in (0, {}^{\pi_p}\!/_2)) \ ,$$

also solves the same ODE. Note that, by Theorem 2.1.3, we have the following expressions for the derivatives of $\cos_p$ and $\sin_p$,

$$\cos_p' = -(p-1)^{1/p} \ \phi_q(\sin_q) \ ,$$
$$\sin_p' = (p-1)^{1/p} \ \phi_q(\cos_q) \ .$$

From our expression for v, we have

$$
\begin{aligned}
v'(t) &= \frac{1}{p}\left(C - \sum_{i=1}^{m-1}\left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)^{1/p-1}\left(-\sum_{i=1}^{m-1}\left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)' \\
&= (p-1)^{1/p}\left(\sum_{i=1}^{m-1}\phi_p\left(\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t + \frac{2i\pi_p}{m}\right)\right)\right) \\
&\quad \times \left(C - \sum_{i=1}^{m-1}\left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)^{-1/q},
\end{aligned}
$$

and so

$$
\begin{aligned}
\phi_p(v')(t) &= (p-1)^{1/q}\phi_p\left(\sum_{i=1}^{m-1}\phi_p\left(\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t + \frac{2i\pi_p}{m}\right)\right)\right) \\
&\quad \times \left(C - \sum_{i=1}^{m-1}\left|\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right|^p\right)^{-(p-1)/q}.
\end{aligned}
$$

Differentiating again, we have

$$
\begin{aligned}
(\phi_p(v'))'(t) = (p-1)^{1+1/q} & \left|\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right|^{p-2} \\
& \times \left(\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right)' \\
& \times \left(C-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)^{-(p-1)/q} \\
& - \frac{(p-1)^{2+1/q}}{p}\cdot\phi_p\left(\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right) \\
& \times \left(C-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)^{-((p-1)/q)-1} \\
& \times \left(-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)' \\
= & \left|\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right|^{p-2} \\
& \times \left[(p-1)^{1+1/q}\cdot\left(\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right)'\right. \\
& \times \left(C-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)^{-(p-1)/q} \\
& - \frac{(p-1)^{2+1/q}}{p}\left(\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t+\frac{2i\pi_p}{m}\right)\right)\right) \\
& \times \left(C-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)^{-((p-1)/q)-1} \\
& \left.\times \left(-\sum_{i=1}^{m-1}\left|\cos_p\left(t+\frac{2i\pi_p}{m}\right)\right|^p\right)'\right].
\end{aligned}
$$

We know that all of the derivatives in this expression exist in a neighbourhood of zero, as the functions $|\cos_p(t + {}^{2j\pi_p}/_m)|^p$ are twice differentiable at all $t \notin \{{}^{j\pi_p}/_2 : j \in \mathbb{N}\}$, for $1 < p \leq 2$. If $p > 2$, then we can use the Pythagorean type identity,

$$
\frac{|\cos_p|^p}{p-1} + \frac{|\sin_q|^q}{q-1} = 1
$$

to rephrase the condition (43) in terms of functions $|\sin_q|^q$. A simple shift in t again rewrites the condition as a sum of functions $|\cos_q|^q$,the same argument can then be repeated with the exponent $1 < q < 2$, removing the problem of differentiability in a neighbourhood of zero. So without loss of generality, we can take $1 < p \leq 2$.

Next, differentiating (43), we have

$$\phi_p(\cos_p(t))\phi_q(\sin_q(t)) + \sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t + \frac{2i\pi_p}{m}\right)\right) \equiv 0 ,$$

and since, at $t = 0$,

$$\phi_p(\cos_p(0))\phi_q(\sin_q(0)) = 0 ,$$

we have

$$\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t + \frac{2i\pi_p}{m}\right)\right) = 0 ,$$

at $t = 0$. Therefore, for $p \neq 2$, by the continuity of $\cos_p$ and $\sin_p$,

$$\left|\sum_{i=1}^{m-1} \phi_p\left(\cos_p\left(t + \frac{2i\pi_p}{m}\right)\right)\phi_q\left(\sin_q\left(t + \frac{2i\pi_p}{m}\right)\right)\right|^{p-2} \to 0 , \qquad (45)$$

as $t \to 0$, and by the above expression for $(\phi_p(v'))'$, $(\phi_p(v'))'(t) \to 0$, as $t \to 0$. Note that if $p = 2$, the exponent of the summation in (45) is zero, and thus the term is identically one for all $t \in \mathbb{R}$. Therefore, for $p = 2$, the limit (45) is one instead of zero.

However, by (44), $\phi_p(v(t)) = \phi_p(\cos_p(t)) \to (p-1)^{1/q}$, as $t \to 0$. Therefore, v does not satisfy the equation

$$(\phi_p(u'))' + \phi_p(u) = 0 ,$$

and so $v \neq \cos_p$.

$\square$

We can combine the results of Corollary 5.2.2, and Lemmas 5.2.4 and 5.2.5, to prove the existence of a perturbing potential, $Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})$, that matches the conditions in Theorem 5.2.1. Thus, for the potential $Q_1 \equiv 0$, there exists a Q_2 such that the (degenerate) periodic eigenvalues of this problem over m periods, can be perturbed to form sets of non-degenerate eigenvalues.

Theorem 5.2.2. *Let $p \neq 2$ and $m \in \mathbb{N}$, consider the unperturbed problem with $Q_1 \equiv 0$, the periodic eigenvalues of the problem over m periods are given by*

$$\lambda_n = \left(\frac{2n\pi_p}{m\alpha}\right)^p ,$$

for any $n \in \mathbb{N}$. All of these eigenvalues are degenerate, and the associated space of solutions for each eigenvalue is given by

$$\begin{aligned} u(t, r_0, \theta_0, \lambda_n, 0) &= A\cos_p(\lambda_n^{1/p}(t - t_0)) \\ &= A\cos_p\left(\frac{2n\pi_p}{m\alpha}(t - t_0)\right) , \end{aligned}$$

where the constants $A, t_0 \in \mathbb{R}$ are dependent on the initial values $r_0, \theta \in \mathbb{R}$.

For any fixed value n that is coprime to m, there exists an α-periodic function $Q_2 \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that the eigenvalue λ_n becomes a pair of non-degenerate eigenvalues $\underline{\mu_n}, \overline{\mu_n}$.

Proof. For the unperturbed problem, we have solutions of the form

$$A\cos_p(\lambda^{1/p}(t - t_0)) , \tag{46}$$

for some $A, t_0 \in \mathbb{R}$. By Lemma 2.2.1,

$$\begin{aligned} \theta(t, r_0, \theta_0, \lambda, 0) &= -\operatorname{arccot}_p\left(\frac{u(t, r_0, \theta_0, \lambda, 0)}{u'(t, r_0, \theta_0, \lambda, 0)}\right) \\ &= -\operatorname{arccot}_p\left(\frac{A\cos_p(\lambda^{1/p}(t - t_0))}{-\lambda^{1/p}A\,(p-1)^{1/p}\,\phi_q(\sin_q(\lambda^{1/p}(t - t_0)))}\right) \\ &= -\operatorname{arccot}_p(\lambda^{-1/p}\cot_p(\lambda^{1/p}(t - t_0))) , \end{aligned}$$

and at the periodic eigenvalue λ_n ,

$$\theta(t, r_0, \theta_0, \lambda_n, 0) = -\operatorname{arccot}_p\left(\frac{m\alpha}{2n\pi_p}\cot_p\left(\frac{2n\pi_p(t-t_0)}{m\alpha}\right)\right) . \tag{47}$$

Therefore, by the definition of Ψ^m ,

$$\begin{aligned}
\Psi^m(\theta_0, \lambda_n, 0) &= \theta(m\alpha, r_0, \theta_0, \lambda_n, 0) - \theta_0 \\
&= -\operatorname{arccot}_p\left(\frac{m\alpha}{2n\pi_p}\cot_p\left(2n\pi_p - \frac{2n\pi_p\, t_0}{m\alpha}\right)\right) - \operatorname{arccot}_p\left(\frac{m\alpha}{2n\pi_p}\cot_p\left(\frac{2n\pi_p\, t_0}{m\alpha}\right)\right) \\
&= \operatorname{arccot}_p\left(\frac{m\alpha}{2n\pi_p}\cot_p\left(\frac{2n\pi_p\, t_0}{m\alpha}\right)\right) - \operatorname{arccot}_p\left(\frac{m\alpha}{2n\pi_p}\cot_p\left(\frac{2n\pi_p\, t_0}{m\alpha}\right)\right) \\
&= k\pi_p ,
\end{aligned}$$

for some $k \in \mathbb{N}$. The iterated renormalised Poincaré map is constant in θ_0, and is therefore degenerate. So each λ_n is a degenerate periodic eigenvalue. Consider the function f , given by

$$f(t, t_0) := \sum_{i=0}^{m-1}\left|A\cos_p\left(\frac{2n\pi_p}{m\alpha}(t + i\alpha - t_0)\right)\right|^p \qquad (t \in \mathbb{R}) .$$

The function $f(t, t_0)$ is constant in $t \in \mathbb{R}$, if and only if the function

$$\tilde{f}(t, t_0) := \frac{1}{|A|^p} f\left(\frac{m\alpha}{2n\pi_p}(t + t_0), t_0\right)$$

is constant in t, as $\tilde{f}$ just results in a shift, and a horizontal and vertical rescaling of f. The function $\tilde{f}$ is given by,

$$\tilde{f}(t, t_0) = \sum_{i=0}^{m-1}\left|\cos_p\left(t + \frac{2ni\pi_p}{m}\right)\right|^p .$$

By Lemma 5.2.5, for all $p \neq 2$, the function $\tilde{f}$ is non-constant in t on $\mathbb{R}$, and thus, so is the function f. Note that we require n, m to be coprime here in order to ensure that the terms in the shifts $2nm^{-1}\pi_p\, i$ follow the same sequence (modulo $2\pi_p$) as $2m^{-1}\pi_p\, i$, which is the sequence of shifts in the hypothesis of Lemma 5.2.5. Therefore, by Lemma 5.2.4, for any $t_1 \in \mathbb{R}$, there exists some value $\gamma \in \mathbb{R}$ such that $f(t+\gamma, \theta_1)$ is linearly independent of $f(t, \theta_1)$.

However, we have

$$f(t+\gamma, t_1) = f(t, t_1 - \gamma) ,$$

and if we define $t_2 := t_1 - \gamma$, then there exist two values t_1, t_2 such that the functions $f(t, t_1)$ and $f(t, t_2)$ are linearly independent. By the equation (47), for any t_0 in (46),

$$\cot_p \theta_0 = -\frac{m\alpha}{2n\pi_p} \cot_p \left(\frac{2n\pi_p\, t_0}{m\alpha} \right) ,$$

and so the initial Prüfer angles, θ_0, are surjective with respect to the values t_0. The values t_1, t_2 must give distinct values $\theta_1, \theta_2 \in \mathbb{R}$, or else they would differ by a radial rescaling, and thus be linearly dependent, which would be a contradiction.

By Corollary 5.2.2, there exist two values $\alpha_1, \alpha_2 \in \mathbb{R}$ such that the function

$$Q_2(t) := \alpha_1 f(t, t_1) + \alpha_2 f(t, t_2) ,$$

satisfies the conditions in Theorem 5.2.1.

Therefore, by Corollary 2.4.2, there exists some value $\varepsilon > 0$ such that for this perturbed problem, there are two values $\underline{\lambda} < \lambda < \overline{\lambda}$, with

$$\rho(\underline{\lambda}, \varepsilon) = \rho(\lambda, \varepsilon) = \rho(\overline{\lambda}, \varepsilon) = \frac{2n\pi_p}{m\alpha} .$$

Thus, there exists a non-trivial plateau at the level $2n\pi_p(m\alpha)^{-1}$, and we can label the end-points of this plateau $\underline{\mu_n}$, $\overline{\mu_n}$. From Theorem 2.4.2, the values $\underline{\mu_n}$, $\overline{\mu_n}$ are periodic eigenvalues of the problem over m periods, and we have $\underline{\mu_n} < \lambda_n < \overline{\mu_n}$.

$\square$

We conclude by stating an open problem that is relevant to the perturbation arguments used above. The question is whether for any $\lambda \in \mathbb{R}$ that is a degenerate periodic eigenvalue of the problem over m periods, the functions

$$\sum_{i=0}^{m-1} (r(i\alpha, 1, \theta_0, \lambda))^2 |u(t, r_0, \theta(i\alpha, r_0, \theta_0, \lambda)|^p \ ,$$

are linearly independent for any two initial Prüfer angles, θ_1 , $\theta_2 \in \mathbb{R}$ such that

$$\theta(i\alpha, r_0, \theta_1, \lambda) \neq \theta_2 + 2n\pi_p \ ,$$

for any $i, n \in \mathbb{N}$. If this were true for the solutions resulting from any potential $Q \in L^1_{\text{loc}}(\mathbb{R})$, we could derive a genericity result similar to Theorem 2.5.2. Specifically, we could prove that the set of potentials $X_{n,m}$ (for any $n, m \in \mathbb{N}$) , for which there exists a value $\lambda \in \mathbb{R}$ such that,

$$\Psi^m(\theta_0, \lambda) = 2n\pi_p \ ,$$

for all $\theta_0 \in \mathbb{R}$, is open and dense in $L^1_{\text{loc}}(\mathbb{R})$. As a result, all periodic eigenvalues of the problem over any number of periods would be non-degenerate, Baire-almost all $Q \in L^1_{\text{loc}}(\mathbb{R})$. This would show that for $p \neq 2$, general rotation number functions have plateaus at every rational level, potentially having fractal properties akin to the Cantor function.

6 Appendix

In this appendix, we list the codes used throughout the thesis. These codes are implementations of the algorithms described in Chapter 3.

6.1 Table of Values of $\sin_p$

```
def sinp_vals(p,n,m):   # n is the number of steps
                        # m is the number of iterations of the recurrence relation
        l=[1]*(n+1)     # This list is the set of values of the approximation
                        after each iteration.
        for i in range(0,m):
                l_int1=[0]      # This list is the set of values
                                of the inner integral, with limits 0 to i
                for j in range(1,n+1):
                        l_int1.append(l_int[j-1]+0.5*(l[j-1]**(p-1) \
                        +l[j]**(p-1))*(pi_p(p)/(2*n)))      # Implementation of
                                                            the Trapezium Rule
                l_int2=[l_int1[n]-l_int1[i] for i in range(0,n+1)]      # This
                                                      list is the set of values of the
                                                      integrand of the inner integral,
                                                      with limits i to pi_p(p)/2
                l=[0]       # We redefine l to generate the next
                            approximation by applying the Trapezium Rule
                            to the list l_int2
                for j in range(1,n+1):
                      l.append(l[j-1]+0.5*(((l_int2[j-1])**(1/(p-1))) \
                      +((l_int2[j])**(1/(p-1))))*(pi_p(p)/(2*n)))   # Trapezium Rule
        return [(l_[i]*((p-1)**(1/p)))/l[n] for i in range(1,n+1)]  # Return the
                                                  normalised list of values
```

6.2 The arccot_p Function

```
def arccotp(t,p,n):     # n is the number of terms in the sum

        q=p/(p-1)       # Define useful constants

        a=(p-1)**(1/p)

        b=(q-1)**(1/q)

        if t>=0:

                if t<a:     # Case 1

                        s=0

                        j=0

                        while a/(p*(j+1)+1)>1/n:    # From accuracy bounds

                                                          in Chapter 3.1

                                s=s+(t**(p*j+1))/((p*j+1)*((1-p)**j))

                                j=j+1

                        return pi_p(p)/2-s

                if t<a:     # Case 2

                        s=0

                        j=0

                        while b/(q*(j+1)+1)>1/n:    # From accuracy bounds

                                                          in Chapter 3.1

                                s=s+((1-p)**j)*(t**((q*j+1)*(1-p)))/(q*j+1)

                                j=j+1

                        return s

                if t=a:     # Case 3

                        s=0

                        j=0

                        while a/j>1/n:      # From accuracy bounds

                                                          in Chapter 3.1

                                l=[i*p+1 for i in range(1,j)]
```

```
                s=s+(factorial(j)*(p**j))/((2**(j+1))*product(l))
                j=j+1
            return pi_p(p)/2-a*s
    else:
        return -arcotp(-t,p,n)     # As arccotp is an odd function
```

6.3 The Rotation Number

```
def rotation_number(lambda,C,pts,p,a,n,m,k):
                # lambda is the spectral parameter
                # C is the list of values of the potential at
                each subinterval over one period
                # pts is the list of end-points of each
                subinterval over one period (first point 0, final point alpha)
                # a is the number of periods over which the rotation
                number is approximated
                # n is the number of steps in p-trig approximations
                # m is the number of iterations of p-trig approximations
                # k is the number of periods over which p-hyperbolics
                are calculated in the table

        f=(p-1)**(1/p)      # f is the value of the solution
        d=0     # d is the value of the derivative of the solution
        t_0=0   # t is the value of the shift in the argument
                  of the p-trig functions
        z=0     # z is a count of the number of zeros of the solution
        C=[lambda-j for j in C]    # Redefine C to make it the set of
                                    coefficients of the p-trig and p-hyperbolic
                                    functions, to the power of p
        l=len(C)
        def map_1(f,d,z,C_j):     # For the case lambda-C>0
                C_j=C_j**(1/p)    # Redefine C_j as the coefficient of
                                    the p-trig function on the
                                    subinterval being analysed
                if d!=0:
```

```
            t_0=arccotp(C_j*f/d,p,n)/C_j
        else:
            t_0=0
        b=ceil(C_j/pi_p(p))
        sub_pts=[(pts[j+1]-pts[j])*k/b for k in range(0,b+1)]
        for k in range(0,b):            # Evaluation of total number
                                          of zeros in whole subinterval
                                          by considering whether sign
                                          changes on several smaller intervals
            if cosp(C_j*sub_pts[k]+t_0,p,n,m) \
            *cosp(C_j*sub_pts[k+1]+t_0,p,n,m)<0:
                z=z+1
        f=cosp(C_j*sub_pts[b]+t_0,p,n,m)
        d=C_j*((p-1)**(1/p))*phi(sinp(C_j*sub_pts[b]+ \
        t_0,p/(p-1),n,m),p/(p-1))
        return f,d,z
    def map_2(f,d,z,C_j):     # For the case lambda-C<0
        C_j=(-C_j)**(1/p)               # Redefine C_j as the coefficient
                                          of the p-hyperbolic function on the
                                          subinterval being analysed
        if abs(C_j*f/d)>(p-1)**(1/p):          # Initial values resulting
                                                 in coshp, we do not enumerate
                                                 zeros because coshp has no zeros
            if d!=0:
                t_0=arccothp(C_j*f/d,p,n)/C_j
            else:
                t_0=0
            f=coshp(C_j*(pts[j+1]-pts[j])+t_0,p,n,m)
```

```
            d=C_j*((p-1)**(1/p))*phi(sinhp(C_j*(pts[j+1]-pts[j])+ \
            t_0,p/(p-1),n,m),p/(p-1))
            return f,d,z
        if abs(C_j*f/d)<(p-1)**(1/p):       # Initial values resulting
                                            in sinhp, we only check for
                                            one zero on the subinterval,
                                            as sinhp is zero at one
                                            point only
            if d!=0:
                t_0=(p-1)*arccothp(\phi(C_j*f/d,p),p/(p-1),n)/C_j
            else:
                t_0=0
            if sinhp(C_j*(pts[j+1]-pts[j])+t_0,p,n,m)* \
            coshp(t_0,p,n,m)<0:
                    z=z+1
            f=sinhp(C_j*(pts[j+1]-pts[j])+t_0,p,n,m)
            d=C_j*((p-1)**(1/p))*phi(coshp(C_j*(pts[j+1]-pts[j])+ \
            t_0,p/(p-1),n,m),p/(p-1))
            return f,d,z
        if abs(C_j*f/d)=(p-1)**(1/p):       # Initial values resulting
                                            in exponentials, exponentials
                                            do not have zeros, and the ratio
                                            f/d is constant, so nothing
                                            needs to be done in this case
            return f,d,z
def map_3(f,d,z):     # For the case lambda-C=0, resulting in linear
                        solutions, we only check for one zero, as non-trivial
                        linear functions are zero at one point only
```

```
        if f*(f+d*(pts[j+1]-pts[j]))<0:
                z=z+1
        f=f+d*(pts[j+1]-pts[j])
        return f,d,z
for i in range(0,a*l):      # Iterate over all l subintervals for a periods
        if C[j%l]>0:
                f,d,z=map_1(f,d,z,C[j%l])
        if C[j%l]<0:
                f,d,z=map_2(f,d,z,C[j%l])
        if C[j%l]=0:
                f,d,z=map_3(f,d,z)
return pi_p(p)*z/a      # Return normalised count of zeros
```

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