# GENERALISED TRANSLATIONS, PERIODICITY AND PERFORATED DOMAINS WITH APPLICATIONS TO GRUSHIN SPACES 

Ahmed Jama<br>First Supervisor: Dr. Federica Dragoni<br>Second Supervisor: Prof. Nicolas Dirr

# CARDIFF 

UNIVERSITY
PRIFYSGOL CAERDYB

A thesis presented for the degree of
Doctor of Philosophy

Cardiff University
Mathematics Department

## Contents

I Preliminaries ..... 7
1 Manifolds ..... 9
1.1 Topological Spaces ..... 9
1.2 Vectors, Co-Vectors and the Tangent Space ..... 18
2 Riemannian and Sub-Riemannian Geometries ..... 25
2.1 Riemannian Manifolds ..... 25
2.2 Sub-Riemannian Manifolds ..... 31
2.3 Geodesics ..... 38
2.4 Carnot Groups ..... 42
2.5 Grushin spaces ..... 53
II Generalised translations, Periodic sets and Perforated domains ..... 59
3 Generalised translations and generalised periodic sets ..... 61
3.1 Translations and periodicity along vector fields. ..... 61
3.1.1 Main definition and examples ..... 62
3.1.2 Periodic sets ..... 67
4 Translations in Grushin spaces, perforated domains and tilings ..... 75
4.1 The case of the Heisenberg group. ..... 75
4.1.1 The case of Carnot groups ..... 82
4.2 Generalised translation in Grushin spaces ..... 84
4.2.1 Translations and rescaling in the Grushin plane ..... 86
4.3 Construction of perforated domains with non overlap- ping holes in the Grushin plane ..... 89
4.4 Construction of Grushin perforated sets by translations of (Euclidean) balls ..... 89
4.5 Translating diamonds ..... 104
4.6 Tilings in the Grushin plane ..... 106
5 Poincaré Inequality ..... 121
6 Open problems and Applications to Homogenization ..... 125
Bibliography ..... 128

## Introduction

The focus of the thesis is to develop a notion of translations in the setting where usually translations are not defined and used that to build interesting geometrical constructions, coherent with the geometry underlying Grushin spaces. The aim being to use these constructions for future applications to homogenisation problems in perforated domains in Grushin spaces. The first step towards the study of homogenisation problems will be to prove a Poincaré inequality for those perforated domains in the Grushin plane.

Translations are usually associated to geometrical structures where one can define a group structure. Here we introduce a new idea, translating along vector fields to construct perforated domains for homogenisation in the setting of Grushin spaces. This idea can be applied to very general geometries where neither a vector space nor an algebraic structure are defined. In particular this notion can be applied to the case of Hörmander vector fields (sub-Riemannian manifolds).

Briefly the idea is the following: given a family of vector fields on $\mathbb{R}^{n}$ and any point in $x \in \mathbb{R}^{n}$ we translate the point $x$ along the integral curves associated to the vector fields for a time $t=1$.

This can be applied in many different settings. We will give several examples of translating along one ore more vector fields in the Euclidean $\mathbb{R}^{n}$ which will lead to interesting periodic structures which are not periodic in the usual sense. Still the main focus of the thesis is to apply generalised translations to vector
fields associated to sub-Riemannian structures (i.e. satisfying the Hörmander condition) and in particular to the Grushin spaces. Grushin spaces are very important sub-Riemannian geometries induced by specific polynomial vector fields defined on $\mathbb{R}^{n}$, which satisfy the Hörmander condition with step 2.

Thus we will first study in details this notion of translations in the Heisenberg group and in general Carnot groups, comparing that with the group translations. We will show that in this setting translating along the vector fields coincides with group translations horizontally. Therefore this idea is in the direction of the approach to homogenisation problems given in [11] and [48].

Then we will concentrate to the case of the Grushin plane where so far, to our knowledge, none has been previously done in this setting of problems.

The Thesis is structured as follows:

Part I of the thesis will outline some of the preliminaries to understand the geometries we work in throughout the main part of the thesis. We will give a flavour of the main theorems and definitions showing examples where necessary. This will allow the reader not familiar to the field a chance to catch up before delving into the main thrust of the project.

In particular in Chapter 1 we will introduce notions regarding manifolds. For further work see [1].

In Chapter 2 we will in depth also at Riemannian and Sub-Riemannian geometries, expanding on what definitions related to geodesics and the Chow's Theorem. Some important books and papers which can help in the under-
standing of these geometries include [23], [24], [10], [18], [20], [49], [47], [50], [44], [5], [16], [29], [33], [34], [38], [39] and [26].

In Part II of the thesis we will explore our generalised translations, outlining how these translations can be applied to construct interesting geometrical sets as perforated domains and tilings. We will write down generalised translations explicitly for both the Heisenberg group and the Grushin plane, iterating these to find out whether or not they satisfy some conditions related to the construction of periodic perforated domains.

In particular in Chapter 3 we will explore generalised translations and periodic sets in depth. We look at the case in different spaces including the Grushin space and the Heisenberg group.

In Chapter 4 instead we will look at perforated domains and tilings in our setting.

Most of the results in Chapter 3 and Chapter 4 are contained in the preprint [28].

In Chapter 5 we will give a general geometric approach to prove a Poincaré inequality for perforated domains as the ones constructed in Chapter 3, by using the partition of $\mathbb{R}^{2}$ given in Chapter 4. These results are contained in a preprint in preparation [43].

Finally in Chapter 6 we will briefly discussed the homogenisation problems which are the focus of our future research and the motivations for the geometrical constructions and the Poincaré inequality proved in the Thesis.

## Part I

## Preliminaries

## Chapter 1

## Manifolds

In this first chapter we introduce notions associated to manifolds and some important definitions that are linked to them. Manifolds form the basis to understanding Riemannian and Sub-Riemannian structures (which we will see later) and are used in many fields of mathematics including topology and differential geometry. For further details on smooth manifolds see [36].

### 1.1 Topological Spaces

Definition 1.1.1. Let $X$ be a non-empty set. A topology on $X$ is a collection $\tau$ of subsets of $X$ such that:
(1) The empty set $\emptyset \in \tau$ and the space $X \in \tau$.
(2) If $U_{\alpha} \in \tau$ for all $\alpha \in A$ ( $A$ is a generic family of indices), then

$$
\bigcup_{\alpha \in A} U_{\alpha} \in \tau
$$

(the union of any number of open sets is open).
(3) If $U_{\alpha} \in \tau$ for all $\alpha=1, \ldots, n$, then

$$
\bigcap_{\alpha=1}^{n} U_{\alpha} \in \tau
$$

(the finite intersection of open sets is open).

If $\tau$ is a topology on $X$ we say that $(X, \tau)$ is a topological space and the elements of $\tau$ are called open sets.

Definition 1.1.2. If $x \in X$, then an open set containing $x$ is said to be an (open) neighbourhood of $x$.

Example 1.1.1. If we have the set $X=\{1,2,3\}$ then the collections of subsets $\tau=\{\emptyset,\{1\},\{2,3\},\{1,2,3\}\}$ form a topology on $X$. Moreover $\{1\}$ and $\{1,2,3\}$ are a neighbourhood of 1 .

Example 1.1.2. If we have the set $X=\{1,2,3\}$ then the collections of subsets $\tau=\{\emptyset,\{1,2\},\{2,3\},\{1,2,3\}\}$ does not form a topology on $X$ as $\{2,3\} \cap$ $\{1,2\}=\{2\} \notin \tau$.

Definition 1.1.3. A set in a topological space is said to be disconnected if it is the union of two disjoint nonempty open sets. Otherwise, $X$ is said to be connected.

Definition 1.1.4. Given a topological space ( $X, \tau$ ) and a subset $S \subseteq X$ we can define a topology on $S$, called the subspace topology, as

$$
\tau_{S}=\{S \cap U \mid U \in \tau\}
$$

Example 1.1.3. Consider the standard topology of open sets on $\mathbb{R}$. The set $X=[1,5]$ is connected as there do not exist two nonempty open sets
whose union is equal to $X$. If we take for example we take $S=(0,3) \cup(3,5)$. Referring to Definition 1.1.4 we can see that the set $S$ is disconnected. In fact $U_{1}=(0,3) \cap S$ and $U_{1}=(3,4) \cap S$ are open w.r.t induced topology and $U_{1} \cup U_{2}=S$.

Definition 1.1.5. A topological space ( $X, \tau$ ) is called Hausdorff if, whenever $x, y \in X$ and $x \neq y$, we can find $U, V \in \tau$ such that $x \in U, y \in V$ and $U \cap V=\emptyset$.

Example 1.1.4. Let $\left(X, \tau_{X}\right)$ be a topological space with the trivial topology $\tau_{X}=\{\emptyset, X\}$. Then $\left(X, \tau_{X}\right)$ is not a Hausdorff space as there are no disjoint subsets other than the empty set which does not contain any elements. The set $(0,1)$ with its standard topology is a Hausdorff space as for all distinct points $x, y \in(0,1)$ you find open sets $(x-\varepsilon, x+\varepsilon),(y-\varepsilon, y+\varepsilon)$ (for some $\epsilon>0$ small enough) such that their intersection is the empty set.

Definition 1.1.6. A cover of a set $X$ is a collection of sets whose union contains $X$ as a subset. If

$$
\begin{equation*}
C=\left\{U_{\alpha}: \alpha \in A\right\} \tag{1.1}
\end{equation*}
$$

is an indexed family of sets $U_{\alpha}$, then $C$ is a cover of $X$ if

$$
\begin{equation*}
X \subseteq \bigcup_{\alpha \in A} U_{\alpha} . \tag{1.2}
\end{equation*}
$$

A subcover $V$ is a subset of $C$ that still covers $X$. A subset of a topological space is called compact if every open cover has a finite subcover. We say that C is an open cover if each of its elements is an open set (i.e. each $U_{\alpha}$ is contained in $\tau$, where $\tau$ is a given topology on $X$ ).

Example 1.1.5. Let $X=\{1,2,3\}$ with topology

$$
\tau=\{\emptyset, X,\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}
$$

$U_{1}=\{1\}, U_{2}=\{2\}$ and $U_{3}=\{3\}$. Then $C=\left\{U_{1}, U_{2}, U_{3}\right\}$ is an open cover of $X$ as we have that $X=\bigcup_{i=1}^{3} U_{i}$.

Looking at the set $(0,1)$ with its standard topology we see that the sets $(0+$ $\frac{1}{n}, 1-\frac{1}{n}$ ) (with $n \in \mathbb{N} /\{0\}$ ) form a open cover. As there does not exist a finite subcover, then $(0,1)$ is not compact.

To define continuity we first need to define the preimage of a set.

Definition 1.1.7. (Preimage) Let $f$ be defined as the function $f: X \rightarrow Y$. The preimage of a set $B \subseteq Y$ is the set

$$
f^{-1}(B)=\{x \in X \mid f(x) \in B\} .
$$

Definition 1.1.8. (Continuity) Let $\left(X, \tau_{X}\right),\left(Y, \tau_{Y}\right)$ be two topological spaces.

A function $f: X \rightarrow Y$ is said to be continuous if and only if for every open set $V$ (i.e. $\forall V \in \tau_{Y}$ ), the preimage $f^{-1}(V)$ is an open subset of $X, f^{-1}(V) \in \tau_{X}$.

Example 1.1.6. Let $X=\{1,2,3,4\}$,

$$
\tau_{X}=\{\emptyset,\{1\},\{2\},\{3\},\{2,3\},\{1,3\},\{1,2\},\{1,2,3\},\{1,2,3,4\}\}, Y=\{1,2,3\}
$$

and

$$
\tau_{Y}=\{\emptyset,\{2\},\{1,2,3\}\} .
$$

Let us define $f: X \rightarrow Y$ as,

$$
f(1)=2, f(2)=3, f(3)=f(4)=1 .
$$

Clearly $f$ is continuous as $\forall B \in \tau_{Y}$ we have that $f^{-1}(B) \in \tau_{X}$. In fact

$$
f^{-1}(\{2\})=\{1\} \in \tau_{X} \quad \text { and } \quad f^{-1}(\{1,2,3\})=\{1,2,3,4\} \in \tau_{X}
$$

Definition 1.1.9. A function $f: X \rightarrow Y$ defined between two topological spaces $\left(X, \tau_{X}\right)$ and $\left(Y, \tau_{Y}\right)$ is called a homeomorphism if $f$ has the following properties:
(i): $f$ is a bijection (i.e. surjective and injective),
(ii): $f$ is continuous,
(iii): $f^{-1}$ is continuous.

Example 1.1.7. Let us look at the function $f: X \rightarrow Y$ acting over the topological spaces $\left(X, \tau_{X}\right),\left(Y, \tau_{Y}\right)$ where $X=\{1,2\}, \tau_{X}=\{\emptyset,\{1\},\{2\},\{1,2\}\}$, $Y=\{3,4\}, \tau_{Y}=\{\emptyset,\{3\},\{4\},\{3,4\}\}$ with $f$ defined as,

$$
f(1)=4, f(2)=3 .
$$

Firstly, $f$ is injective as $f(x)=f(y)$ implies $x=y$. Secondly, $f$ is surjective as $f^{-1}(3)=2, f^{-1}(4)=1 \in X$. Thirdly, both $f$ and $f^{-1}$ are continuous. To see this for $f$ we have that

$$
\begin{equation*}
f^{-1}(\{3\})=\{2\} \in \tau_{X}, f^{-1}(\{4\})=\{1\} \in \tau_{X} \tag{1.3}
\end{equation*}
$$

where in this case $f^{-1}(B)$ is the preimage of the set $\{3\}$. For $f^{-1}$ we have that $\left(f^{-1}\right)^{-1}(\{1\})=\{1\},\left(f^{-1}\right)^{-1}(\{2\})=\{1\}, \in \tau_{Y}$. Actually all functions are continuous given how $\tau_{X}$ is defined.

We now need to define what we mean by a basis for a topological space.

Definition 1.1.10. (Basis) If $X$ is a set, a basis for a topology $\tau$ on $X$ is a collection of subsets of $X$ which belong to $\tau$ (called basis elements, that we indicate by B) satisfying the following properties:
(i): For each $x \in X$, there exists at least one basis element $B$ containing $x$.
(ii): If $x$ belongs to the intersection of two elements $B_{1}$ and $B_{2}$, then there exist a third element $B_{3}$ containing $x$ such that $B_{3} \subset B_{1} \cap B_{2}$.

Example 1.1.8. Consider $X$ and $\tau_{X}$ as given in Example 1.1.6 then $B=$ $\{\{1\},\{2\}\}$ forms a topological basis.

Definition 1.1.11. Let $(M, \tau)$ be a topological Hausdorff space with a countable basis. Then $M$ is called a manifold if there exists a positive integer $n$, such that for each $p \in M$ there exists an open neighborhood $U$ (i.e. $U$ belongs to $\tau$ and $x \in U)$ and a continuous map $f: U \rightarrow \mathbb{R}^{n}$ which is a homeomorphism onto its image $f(U)$. The natural number $n$ is called the dimension of the manifold. (Note that the dimension $n$ is always uniquely determined, even if we omit the proof).

A chart $(U, \varphi)$ is a bijective map $\varphi: U \rightarrow V$, where $V \subset \mathbb{R}^{n}$ is an open set in $\mathbb{R}^{n}$ and $U$ is an open set in a topological manifold $\left(X, \tau_{X}\right)$.

The inverse map $\varphi^{-1}: V \rightarrow X$ is an injection from an open domain $V$ into $X$. There is a one-to-one correspondence between points in $U \subset X$ and arrays
$\left(x_{1}, \ldots, x_{n}\right) \in V \subset \mathbb{R}^{n}$ given by the maps $\varphi$ and $\varphi^{-1}$.

$$
\mathbf{x} \in U \subset X, \varphi(\mathbf{x})=\left(x_{1}, \ldots, x_{n}\right) \in V \subset \mathbb{R}^{n}
$$

The numbers $\varphi(\mathbf{x})=\left(x_{1}, \ldots, x_{n}\right)$ are coordinates of a point $\mathbf{x} \in U \subset X$. A chart $\varphi$ on $U \subset X$ is a coordinate system on $X$.

Definition 1.1.12. An atlas $\mathcal{A}$ for a topological space $\left(X, \tau_{X}\right)$ is a collection

$$
\left\{\left(V_{\alpha}, \varphi_{\alpha}\right) \mid \alpha \in A\right\}
$$

indexed by a set A , of charts on $X$ such that $\bigcup_{\alpha \in A} V_{\alpha}=X$.

Definition 1.1.13. Consider two sets $U_{\alpha}$ and $U_{\beta}$ such that $U_{\alpha} \cap U_{\beta} \neq \emptyset$ and any point $\mathbf{x} \in U_{\alpha} \cap U_{\beta}$ with the two coordinate descriptions: $\varphi_{\alpha}(x)=\left(x_{1}{ }^{\alpha}, \ldots, x_{n}{ }^{\alpha}\right)$ and $\varphi_{\beta}(x)=\left(x_{1}{ }^{\beta}, \ldots, x_{n}{ }^{\beta}\right)$. The transition map is defined as

$$
\Psi_{\alpha \beta}=\varphi_{\alpha} \circ \varphi_{\beta}^{-1}\left(x_{1}{ }^{\beta}, \ldots, x_{n}{ }^{\beta}\right): \varphi_{\beta}\left(U_{\alpha} \cap U_{\beta}\right) \rightarrow \varphi_{\alpha}\left(U_{\alpha} \cap U_{\beta}\right),
$$

we call the function

$$
\left(x_{1}{ }^{\beta}, \ldots, x_{n}{ }^{\beta}\right) \mapsto\left(x_{1}{ }^{\alpha}, \ldots, x_{n}{ }^{\alpha}\right)
$$

the change of coordinates between charts $\varphi_{\alpha}$ and $\varphi_{\beta}$, or transition functions form coordinates $\left(x_{1}{ }^{\beta}, \ldots, x_{n}{ }^{\beta}\right)$ to coordinates $\left(x_{1}{ }^{\alpha}, \ldots, x_{n}{ }^{\alpha}\right)$. Since transition go from a subset of $\mathbb{R}^{n}$ to another subset of $\mathbb{R}^{n}$, by smooth we mean the standard regularity in $\mathbb{R}^{n}$.

If each transition function is a smooth map, then the atlas is called a smooth atlas, and the manifold itself is called smooth. Alternatively, one could require
that the transition maps have only $r$ continuous derivatives in which case the atlas is said to be $C^{r}$.

Definition 1.1.14. A differentiable or smooth manifold $X$ is a topological set endowed with a smooth atlas.

From the above definition we understand that a $n$-dimensional manifold $X$ is locally homeomorphic to the space $\mathbb{R}^{n}$ for some $n$ with the standard Euclidean topology even if it could be not globally homeomorphic to $\mathbb{R}^{n}$.

Example 1.1.9. (Stereographic projection)

Consider the unit sphere $S^{1}$.


We have that $U_{1}=S^{1} \backslash\{N\}$ and $\varphi_{1}: U_{1} \rightarrow \mathbb{R}, P \mapsto x$ where $x$ is the first component of the point of intersection between the $x$-axis with the line passing from $N$ and $P \in S^{1} \backslash\{N\}$ (represented by $Q_{i}:=\left(x_{i}, 0\right)$ in the above picture). Similarly let us define $U_{2}=S^{1} \backslash\{S\}$ and $\varphi_{2}: U_{2} \rightarrow \mathbb{R}, P \mapsto x$. Thus we have that $U_{1} \cup U_{2}=S^{1}$ and the atlas is $\mathcal{A}=\left\{\left(U_{1}, \varphi_{1}\right),\left(U_{2}, \varphi_{2}\right)\right\} . S^{1}$ is locally homeomorphic to $\mathbb{R}^{2}$, but not globally homeomorphic (in fact there does not exist any chart $\varphi: S^{1} \rightarrow \mathbb{R}$ ).

Definition 1.1.15. A chart $(U, f)$ on $X$ is said to be compatible with a $C^{r}$ atlas $\mathcal{A}$ if the union $\mathcal{A} \cup(U, f)$ is a $C^{r}$-atlas. A $C^{r}$-atlas is said to be maximal if it contains all the charts that are compatible with it. It is important to note also that charts in the same atlas cannot have different dimensions.

Example 1.1.10. Let us consider $M=\mathbb{R}$ with the standard topology of open sets and define $f: \mathbb{R} \rightarrow \mathbb{R}, f(x)=-x^{3}$.

The function $f$ clearly is bijective, however inspecting the derivatives of the inverse we see that, $f^{-1}(x)=-x^{1 / 3}$. Computing the derivatives of the inverse we need that the inverse is differentiable everywhere. Let us choose to compute the differentiation at 0

$$
\begin{equation*}
f^{-1}(0)^{\prime}=\lim _{h \rightarrow 0} \frac{f^{-1}(0+h)-f^{-1}(0)}{h}=\lim _{h \rightarrow 0} \frac{-h^{\frac{1}{3}}}{h}=\lim _{h \rightarrow 0}\left(-\frac{1}{h^{\frac{2}{3}}}\right) . \tag{1.4}
\end{equation*}
$$

Since the limit is not finite, the inverse is not differentiable on $\mathbb{R}$, thus $f$ is not a diffeomorphism. But the bijective function $g(x)=x$ defined on $\mathbb{R}$ is a diffeomorphism as its inverse, which is $g$, is $C^{\infty}$.

Example 1.1.11. The Euclidean $\mathbb{R}^{n}$ is made into a $n$-dimensional smooth manifold using the identity chart. The complex coordinate space $\mathbb{C}^{n}$ becomes a $2 n$-dimensional smooth manifold via the chart $\mathbb{C}^{n} \rightarrow \mathbb{R}^{2 n}$ replacing every complex coordinate $z_{j}$ by the pair of real coordinates $\left(\operatorname{Re} z_{j}, \operatorname{Im} z_{j}\right)$.

Example 1.1.12. (Open subset of $\left.\mathbb{R}^{n}\right)$ Any open subset $\mathcal{O}$ of $\mathbb{R}^{n}$ is a smooth manifold of dimension $n$. One possible atlas is $\mathcal{A}=(\mathcal{O}, \varphi)$, where $\varphi$ is the identity map. Of course one possible choice of $\mathcal{O}$ is $\mathbb{R}^{n}$ itself.

### 1.2 Vectors, Co-Vectors and the Tangent Space

In this section we will expand on Vectors, Co-vectors and the Tangent Spaces which are essential to defining structures in the study of Riemannian and SubRiemannian geometries that we will encounter later on.

Definition 1.2.1. If $M$ is a smooth manifold, we define a curve in $M$ to be a continuous map $\gamma: I \rightarrow M$, where $I \subset \mathbb{R}$ is an interval.

Example 1.2.1. Let the unit circle $S^{1}=\left\{(x, y) \in \mathbb{R}^{2} \mid x^{2}+y^{2}=1\right\}$ be our manifold. Then the map $\gamma:(0,2 \pi) \rightarrow S^{1}$ defined as $\gamma(t)=(\sin t, \cos t)$ is a curve.

Definition 1.2.2 (Tangent vector at a point $p$ ). Let $M$ be a differentiable manifold and for a point $p \in M$ we denote by $\varepsilon(p)$ the set of differentiable real-valued functions defined locally around $p$. Let $p \in M$, then a tangent vector $\gamma_{p}$ at $p$ is a map $\gamma_{p}: \varepsilon(p) \rightarrow \mathbb{R}$ such that
(i) $\gamma_{p}(\lambda \cdot f+\mu \cdot g)=\lambda \gamma_{p}(f)+\mu \gamma_{p}(g)$,
(ii) $\gamma_{p}(f \cdot g)=\gamma_{p}(f) g(p)+\gamma_{p}(g) f(p)$,
for all $\lambda, \mu \in \mathbb{R}$ and $f, g \in \varepsilon(p)$. The set of tangent vectors at $p$ is called the tangent space at $p$ and denoted by $T_{p} M$. The Tangent space itself has the structure of a $n$-dimensional vector space (same as the dimension of $M$ ).

Definition 1.2.3. (Differential) Let $\varphi: M \rightarrow \tilde{M}$ be a smooth map of smooth manifolds. Given some $p \in M$, the differential of $\varphi$ at $p$ is a linear map

$$
d \varphi_{p}: T_{p} M \rightarrow T_{\varphi(p)} \tilde{M}
$$

from the tangent space of $M$ at $p$ to the tangent space of $\tilde{M}$ at $\varphi(p)$ defined
as

$$
d_{\varphi(p)}\left(\gamma^{\prime}(0)\right)=(\varphi \circ \gamma)^{\prime}(0) .
$$

Here $\gamma$ is a curve in $M$ with $\gamma(0)=p$.

For manifolds in $\mathbb{R}^{n}$ we can compute the tangent vector as follows.

Definition 1.2.4. Let $M$ be a $n$-smooth manifold and fix any point $p$ in $M$. Assume $\gamma: I \rightarrow M$ is a $C^{1}$-curve such that $\gamma(0)=p$ with $\epsilon(p)$ the set of differentiable functions at $p$. The tangent vector to the curve $\gamma$ at $t=0$ is defined as the function:

$$
\begin{equation*}
\dot{\gamma}(0): \epsilon(p) \rightarrow \mathbb{R} ; \dot{\gamma}(0) f:=\left.\frac{d(f \circ \gamma)}{d t}\right|_{t=0}, f \in \epsilon(p) \tag{1.5}
\end{equation*}
$$

The tangent vector acts on smooth functions by

$$
\begin{equation*}
(f \circ \gamma)^{\prime}\left(t_{0}\right)=\left.\frac{d}{d t}\right|_{t_{0}}(f \circ \gamma)=d f \gamma^{\prime}\left(t_{0}\right), \tag{1.6}
\end{equation*}
$$

where $d f$ is the differential defined Definition 1.2.3. So we see that $\gamma^{\prime}\left(t_{0}\right)$ is the derivation at $\gamma\left(t_{0}\right)$ obtained by taking the derivative at a function along $\gamma$.

Definition 1.2.5. The tangent bundle on a $n$-dimensional differentiable manifold $M$ is defined as

$$
\begin{equation*}
T M=\left\{(p, v) \mid p \in M, v \in T_{p} M\right\} . \tag{1.7}
\end{equation*}
$$

$T M$ is a vector space with $\operatorname{dim} 2 n$. If we take the tangent bundle of $\mathbb{R}^{n}$ we get

$$
\begin{equation*}
T \mathbb{R}^{n}=\left\{(p, v) \mid p \in \mathbb{R}^{n}, v \in \mathbb{R}^{n}\right\} \cong \mathbb{R}^{2 n} \tag{1.8}
\end{equation*}
$$

Example 1.2.2. The tangent bundle on unit circle $S^{1}=\{p=(x, y) \in$ $\left.\mathbb{R}^{2} \mid x^{2}+y^{2}=1\right\}$ is,

$$
\begin{equation*}
T S^{1}=\left\{(p, v) \mid p \in S^{1}, v \in T_{p} S^{1}\right\} \tag{1.9}
\end{equation*}
$$

As we can see the tangent space to unit circle at the $p$ is the set $T_{p} S^{1}=\mathbb{R}$. In fact $T_{p} S^{1}$ is the tangent line at $p$ and this can be identified as mathbb $R$ And so the tangent bundle can be seen as,

$$
\begin{equation*}
T S^{1}=\left\{(p, v) \mid p \in S^{1}, v \in \mathbb{R}\right\} \cong S^{1} \times \mathbb{R} \tag{1.10}
\end{equation*}
$$

which can be visualized as an infinite cylinder on $\mathbb{R}^{3}$. See [36] for details.

Definition 1.2.6. A co-vector at $p \in M$ (where $M$ is the manifold) is a linear map $\omega: T_{p} M \rightarrow \mathbb{R}$. The space of all co-vectors at $p$ is a vector space and is the dual space of $T_{p} M$ denoted as $T_{p}^{*} M$ and is known as the cotangent space.

Example 1.2.3. The cotangent space of the manifold $\mathbb{S}^{1}$ at $p$ is the set of linear maps $\omega: T_{p} \mathbb{S}^{1} \cong \mathbb{R} \rightarrow \mathbb{R}$.

Definition 1.2.7. A smooth vector field $X$ on a manifold $M$ is a linear map $X: C^{\infty}(M) \rightarrow C^{\infty}(M)$ such that $X$ is a derivation:

$$
X(f g)=f X(g)+X(f) g \quad \forall f, g \in C^{\infty}(M)
$$

Lemma 1.2.1. Let $X, Y$ be two smooth vector fields on a manifold $M$. Then
the map

$$
[X, Y]: C^{\infty}(M) \rightarrow C^{\infty}(M), f \mapsto X(Y(f))-Y(X(f))
$$

is a vector field.

Proof. Clearly the map $[X, Y]$ is $\mathbb{R}$-linear. We need to check that it has the correct derivation property. Pick two functions $f, g \in C^{\infty}(M)$. Then

$$
\begin{align*}
{[X, Y](f g) } & =X(Y(f g))-Y(X(f g))  \tag{1.11}\\
& =X(Y(f) g+f Y(g))-Y(X(f) g+f X(g))  \tag{1.12}\\
& =X(Y(f)) g+Y(f) X(g)+X(f) Y(g)+f X(Y(g))  \tag{1.13}\\
& -Y(X(f)) g-X(f) Y(g)-Y(f) X(g)-f Y(X(g))  \tag{1.14}\\
& =X(Y(f)) g-Y(X(f)) g+f X(Y(g))-f Y(X(g))  \tag{1.15}\\
& =([X, Y](f)) g+f([X, Y](g)) . \tag{1.16}
\end{align*}
$$

Example 1.2.4. An example of two vector fields $X$ and $Y$ in $\mathbb{R}^{2}$ are

$$
X(p)=\binom{y}{x}, Y(p)=\binom{-y}{x} \quad \forall p=(x, y) \in \mathbb{R}^{2}
$$

They act on smooth functions $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$ as

$$
\begin{equation*}
X(p)(f)=y \frac{\partial f}{\partial x}+x \frac{\partial f}{\partial y} \text { and } Y(p)(f)=-y \frac{\partial f}{\partial x}+x \frac{\partial f}{\partial y} \tag{1.17}
\end{equation*}
$$

Moreover they satisfy the chain rule defined in Definition 1.2.7 since

$$
\begin{equation*}
X(f g)=y \frac{\partial(f g)}{\partial x}+x \frac{\partial(f g)}{\partial y}=y \frac{\partial f}{\partial x} g+y \frac{\partial g}{\partial x} f+x \frac{\partial f}{\partial x} g+x \frac{\partial g}{\partial x} f=f X(g)+X(f) g \tag{1.18}
\end{equation*}
$$

and similarly this holds for $Y$.

Therefore the $X$ and $Y$ defined in (1.17) are vector fields according to Definition 1.2.7.

From now on we will always indicate the vector fields as in 1.17, omitting to recall every time how they act on smooth functions and omitting to check the validity of the chain rule.

The bracket $[X, Y]$ can be evaluated as following as the action on the smooth function $f: \mathbb{R}^{2} \rightarrow \mathbb{R}$, as

$$
\begin{equation*}
[X, Y] f=X(Y(f))-Y(X(f)) \tag{1.19}
\end{equation*}
$$

then,

$$
\begin{aligned}
X(Y f) & =\left(y \frac{\partial}{\partial x}+x \frac{\partial}{\partial y}\right)\left(-y \frac{\partial f}{\partial x}+x \frac{\partial f}{\partial y}\right) \\
& =-y^{2} f_{x x}+y f_{y}-x f_{x}+x^{2} f_{x_{2} x_{2}} \quad \text { and } \\
Y(X f) & =\left(-y \frac{\partial}{\partial y}+x \frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x}+x \frac{\partial}{\partial y}\right) \\
& =-y^{2} f_{x x}-y f_{y}+x f_{x}+x^{2} f_{y y}
\end{aligned}
$$

Now, we obtain:

$$
[X, Y] f=\left(-2 x \frac{\partial}{\partial x}+2 y \frac{\partial}{\partial y}\right) f
$$

which can be written as

$$
[X, Y]=\binom{-2 x}{2 y}
$$

Remark 1.2.1. Recall that the differential defined in Definition 1.2.3 can also be defined as

$$
d_{\varphi_{p}}(X)(f)=X(f \circ \varphi), \quad \forall f \in C^{\infty}(\tilde{M}) .
$$

## Chapter 2

## Riemannian and

## Sub-Riemannian Geometries

### 2.1 Riemannian Manifolds

In this section we introduce Riemannian geometry. These structures are widely studied in various areas of mathematics including topology, differential geometry and mechanics. They form the basis to understanding abstract surfaces hence their importance to analytical mathematicians. In this section various works we will be citing include [36] and [45]. For further details see [2] and [14].

Definition 2.1.1. Let $M$ be an $n$-smooth manifold. We define a Riemannian metric $g$ which associates to every point $P \in M$ an inner product $g_{P}: T_{P} M \times$ $T_{P} M \rightarrow \mathbb{R}$ by $\left(Q_{1}, Q_{2}\right) \mapsto\left\langle Q_{1}, Q_{2}\right\rangle_{P}$. In other words, for each $P \in M$, the metric $g_{P}$ satisfies the following conditions:

1. $g_{P}\left(a Q_{1}+b Q_{2}, Q\right)=a g_{P}\left(Q_{1}, Q\right)+b g_{P}\left(Q_{2}, Q\right), \quad \forall Q_{1}, Q_{2}, Q \in T_{P} M$ and
$a, b \in \mathbb{R}$,
2. $g_{P}\left(Q, a Q_{1}+b Q_{2}\right)=a g_{P}\left(Q, Q_{1}\right)+b g_{P}\left(Q, Q_{2}\right), \quad \forall Q, Q_{1}, Q_{2} \in T_{P} M$ and $a, b \in \mathbb{R}$,
3. $g_{P}\left(Q_{1}, Q_{2}\right)=g_{P}\left(Q_{2}, Q_{1}\right), \quad \forall Q_{1}, Q_{2} \in T_{P} M$,
4. $g_{P}(Q, Q) \geq 0 \quad \forall Q \in T_{P} M$ and
5. $g_{P}(Q, Q)=0 \Longleftrightarrow Q=0$.

Definition 2.1.2. A manifold with a given Riemannian metric $g$ is known as a Riemannian manifold. Riemannian manifolds are usually denoted as $(M, g)$.

Definition 2.1.3. Let $(M, g),(\tilde{M}, \tilde{g})$ be two Riemannian manifolds. An isometry between $M$ and $\tilde{M}$ is a diffeomorphism $\varphi: M \rightarrow \tilde{M}$ whose differential is a linear isometry between the corresponding tangent spaces, with inner product i.e.:

$$
g_{p}(v, w)=\tilde{g}_{p}\left(D \varphi_{p}(v), D \varphi_{p}(w)\right) \quad \forall p \in M \text { and } v, w \in T_{p} M
$$

where $D f$ is the usual differential defined by using charts in Definition 1.2.3. For a formal definition see [35].

If there exists an isometry between two Riemannian manifolds $(M, g),(\tilde{M}, \tilde{g})$ we say that that the two manifolds are isometric.

Definition 2.1.4. (Co-ordinate Representations) Given a vector $v \in V$ we can write this as a linear combination of basis vectors as follows,

$$
v=\sum_{i} v^{i} e_{i}=\left[e_{1} \cdots e_{n}\right]\left[\begin{array}{c}
v^{1} \\
\vdots \\
v^{n}
\end{array}\right] .
$$

Cartesian coordinates on $\mathbb{R}^{n}$ and coordinates on a manifold have superscripts $x^{j}$ as they are coordinate coefficients; coordinates vector fields can be the defined as,

$$
\partial_{i}=\frac{\partial}{\partial x^{i}} .
$$

They also form a basis for the tangent space. If we define a dual space $e^{i}$ for the dual space $V^{*}$ as $e^{i}\left(e^{j}\right)=\delta_{j}^{i}$. The dual 1-forms $d x^{i}$ satisfy $d x^{j}\left(\partial_{i}\right)=\delta_{j}^{i}$ and consequently form the natural dual basis for the cotangent space.

Given coordinates $x(p)=\left(x^{1}, \ldots, x^{n}\right)$ on an open set $U$ of $M$ we can construct bilinear forms $d x^{i} \cdot d x^{j}$. Also if $M$ a has a Riemannian metric $g$ then we can write

$$
g=g_{p}\left(\partial_{i}, \partial_{j}\right) d x_{i} \cdot d x_{j}
$$

as

$$
(v, w)=g\left(\partial_{i}(v) d x_{i}, \partial_{j}(w) d x_{j}\right)=g\left(\partial_{i}, \partial_{j}\right) d x^{i}(v) \cdot d x^{j}(w)
$$

The functions $g\left(\partial_{i}, \partial_{j}\right)$ are denoted by $g_{i j}$. This gives us a representation of $g$ in local coordinates as a positive definite symmetric matrix with entries parameterized over $U$.

Any finite dimensional vector space $M$ with an inner product becomes a Riemannian manifold by declaring, as with Euclidean space, that

$$
\begin{equation*}
g_{p}(v, w)=v \cdot w . \tag{2.1}
\end{equation*}
$$

Example 2.1.1. The canonical metric on $\mathbb{R}^{n}$ in the identity chart is

$$
g=\delta_{i}^{j} d x^{i} d x^{j}=\sum_{i}\left(d x^{i}\right)^{2} .
$$

In the next example we express the standard metric in $\mathbb{R}^{2}$ using polar coordinates.

Example 2.1.2. Using coordinate representations on $\mathbb{R}^{2}$ - (half line) we also have the polar coordinates $(r, \theta)$. In these coordinates the canonical metric looks like

$$
g=d r^{2}+r^{2} d \theta^{2}
$$

So we have that

$$
g_{r r}=1, g_{r \theta}=g_{\theta r}=0, g_{\theta \theta}=r^{2}
$$

To see this recall that

$$
x=r \cos (\theta), y=r \sin (\theta)
$$

Thus

$$
d x=\cos \theta d r+-r \sin \theta d \theta, d y=\sin \theta d r+r \cos (\theta) d \theta
$$

which gives

$$
\begin{align*}
g & =d x^{2}+d y^{2} \\
& =d(r \cos \theta)^{2}+d(r \sin \theta)^{2} \\
& =(\cos \theta d r-r \sin \theta d \theta)^{2}+(\sin \theta d r+r \cos \theta d \theta)^{2} \\
& =\left(\cos ^{2} \theta+\sin ^{2} \theta\right) d r^{2}+\left(r^{2} \sin ^{2} \theta+r^{2} \cos ^{2} \theta\right) d \theta^{2} \\
& +(-2 r \cos \theta \sin \theta+2 r \sin \theta \cos \theta) d r d \theta \\
& =d r^{2}+r^{2} d \theta^{2} \tag{2.2}
\end{align*}
$$

Remark 2.1.1. The Riemannian metric defined as above is a metric tensor. If we treat the metric tensor in any given coordinate system as a matrix, then it is positive definite if all its eigenvalues are positive, and invertible if all its eigenvalues are non-zero, and this property is independent of the coordinate system. The metric tensor is Riemannian whenever its representation as a matrix is positive definite.

Now we will go on to define what Riemannian distance means, however before this we need to go through some preliminary definitions.

Definition 2.1.5. (Absolute continuity) A function $f:[0, T] \rightarrow \mathbb{R}$ is absolutely continuous if and only if given $\epsilon>0$, there exists some $\delta>0$ such that

$$
\sum_{i=1}^{n}\left|f\left(y_{i}\right)-f\left(x_{i}\right)\right|<\epsilon
$$

where $\left\{\left[x_{i}, y_{i}\right]: i=1, \ldots, n\right\}$ is any finite set of mutually disjoint subintervals of $[0, T]$ with

$$
\sum_{i=1}^{n}\left|y_{i}-x_{i}\right|<\delta
$$

Alternatively this can stated by using an equivalent definition. For a real valued function $f$ on a compact interval $[a, b]$ has a derivative $\dot{f}$ almost everywhere, the derivative is Lebesgue integrable, and

$$
f(x)=f(a)+\int_{a}^{x} \dot{f}(t) d t
$$

for all $x \in[a, b]$

This property is very useful to define the length of absolutely continuous curves as follows.

Definition 2.1.6. Let $(M, g)$ be a Riemannian manifold and $\gamma:[0, T] \rightarrow M$ an absolutely continuous curve, we call length of the curve the following real functional

$$
\begin{equation*}
L(\gamma)=\int_{0}^{T} g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))^{1 / 2} d t \tag{2.3}
\end{equation*}
$$

Definition 2.1.7. Let $(M, g)$ be a Riemannian manifold and $x, y \in M$, the (Riemannian) distance $d: M \times M \rightarrow[0,+\infty]$ between these two points is defined as

$$
d(x, y)=\inf \{L(\gamma) \mid \gamma \text { absolutely continuous joining } x \text { to } y\} .
$$

Definition 2.1.8. (Geodesics) A curve $\gamma:[a, b] \rightarrow M$, where $M$ is a Riemannian manifold is called a (minimizing) geodesic if

$$
L(\gamma)=d(\gamma(a), \gamma(b))
$$

Example 2.1.3. The straight line $\gamma(t)=a t+b$ is a minimizing geodesic on the Euclidean space $\mathbb{R}^{n}$.

Definition 2.1.9. (Completeness) A Riemannian manifold $M$ is called complete if every Cauchy sequence in $M$ with respect to the Riemannian distance $d$ is convergent (with respect to $d$ ) to a point in the manifold $M$.

For instance $\mathbb{R}^{n}$ is complete, but also the union of two disjoint spheres is complete.

Theorem 2.1.1. Let $(M, g)$ be a Riemannian manifold and $p_{0} \in M$, for any $\varepsilon>0$, there exists a neighborhood $U$ of $p_{0}$ such that for any $p \in U$ there exists a unique minimizing geodesic, joining $p_{0}$ to $p$ with length less or equal to $\varepsilon$.

Moreover, if the Riemannian manifold $M$ is complete, then there always exists at least a geodesic joining any pair of points. [27]

We will look at how we compute the geodesics for certain spaces later on in the thesis.

### 2.2 Sub-Riemannian Manifolds

Sub-Riemannian geometries are a certain type of generalization of Riemannian geometries widely studied in Analysis.In this section we will look at some classical definitions from [40], [36], [27] and [19].

Definition 2.2.1. Let $M$ be a $n$-dimensional manifold and $r \leq n$, a $r$ dimensional distribution $\mathcal{H}$ is a sub-bundle of the tangent bundle, i.e.

$$
\mathcal{H}=\{(p, v) \mid p \in M, v \in \mathcal{H}(p)\}
$$

where $\mathcal{H}(p)$ is a $r$-dimensional subspace of the tangent space at the point $p$.

Definition 2.2.2. (Sub-Riemannian metric). Let $M$ be a manifold and $\mathcal{H} \subset$ $T M$ a distribution, a sub-Riemannian metric on $M$ is a Riemannian metric defined on the fibers of the subbundle $\mathcal{H}$.

Definition 2.2.3. (Sub-Riemannian geometry). A sub-Riemannian geometry is the triple $(M, \mathcal{H}, g)$ where $M$ is a smooth manifold, $\mathcal{H}$ is a distribution, and $g$ is a Riemannian metric defined on $\mathcal{H}$.

For more details see [8] and [9].

Definition 2.2.4. (Horizontal curves). Let $(M, \mathcal{H}, g)$ be a sub-Riemannian
geometry and $\gamma:[0, T] \rightarrow M$ an absolutely continuous curve, we say that $\gamma$ is a horizontal curve if and only if

$$
\gamma^{\prime}(t) \in \mathcal{H}_{\gamma(t)} \text {, for a.e. } t \in[0, T] .
$$

For more details see [3] and [15].
Definition 2.2.5. We call the Carnot-Carathéodory distance associated to the Sub-Riemannian geometry $(M, \mathcal{H}, g)$ the function $d: M \times M \rightarrow[0,+\infty]$, defined by

$$
d_{C C}(p, q)=\inf \{L(\gamma) \mid \gamma \text { a horizontal curve connecting } \mathrm{p} \text { to } \mathrm{q}\},
$$

where $L(\gamma)$ is the length defined in (2.3). Note that $L(\gamma)$ is well defined for all the horizontal curves and only for those (in fact $g$ is defined only on $\mathcal{H}$ ).

Let us consider a family of vector fields $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ spanning some distribution $\mathcal{H} \subset T M$, the associated Lie algebra is the set of all the brackets between the vector fields of the family.

Let us now introduce the notion of a step of a bracket-generating distribution. Let $\mathcal{L}$ be a family of vector fields, we write:

$$
\begin{align*}
& \mathcal{L}^{1}:=\operatorname{Span}(\{Z=X \mid X \in \mathcal{H}\}) \\
& \mathcal{L}^{2}:=\operatorname{Span}\left(\left\{Z=[X, Y] \mid X, Y \in \mathcal{L}^{1}\right\}\right)  \tag{2.4}\\
& \ldots \ldots \ldots \ldots \ldots \ldots \ldots \\
& \mathcal{L}^{i}:=\operatorname{Span}\left(\left\{Z=[X, Y] \mid X \in \mathcal{H}, Y \in \mathcal{L}^{i-1}\right\}\right),
\end{align*}
$$

Let us denote $\mathcal{L}^{i}(p)$ as the vector space to $\mathcal{L}^{i}$ evaluated at point $p \in M$. Note we call $k$ - length bracket all the vector field defined recursively as $[X, Y]$ with
$X \in \mathcal{H}$ and $Y(k-1)$-length bracket, where the 1-length bracket vector fields are the elements of the form $\left[X_{1}, X_{2}\right]$ for $X_{1}, X_{2} \in \mathcal{H}$.

Definition 2.2.6. Let $M$ be a smooth manifold and $\mathcal{H}$ a distribution defined on $M$. We say that the distribution is bracket-generating if and only if, at any point, the Lie algebra $\mathcal{L}(X)$ spans the whole tangent space.

In the following definition and theorem we see that there is a bracket generating condition which is key to our understanding of sub-Riemannian geometry.

Definition 2.2.7. (Hörmander condition). We say that a sub-Riemannian geometry satisfies the Hörmander condition if and only if the associated distribution is bracket generating.

The following theorem is the main theorem of the chapter and asserts that the condition of bracket generating being satisfied coupled with connectedness implies that the manifold is horizontally path connected. The theorem was first demonstrated by Rashevskii in [46] and later proved by Chow in [21].

Theorem 2.2.1. (Chow's Theorem) Let $M$ be a smooth manifold and $\mathcal{H} a$ bracket generating distribution defined on $M$. If $M$ is connected, then there exists a horizontal curve joining any two given points of $M$.

Proof. We omit the proof here. For a proof see [40].

Example 2.2.1. (Heisenberg group). We call 1-dimensional exponential Heisenberg group or also canonical Heisenberg group, the sub- Riemannian geometry induced by the vector fields,

$$
X(x, y, z)=\left(\begin{array}{c}
1  \tag{2.5}\\
0 \\
\frac{-y}{2}
\end{array}\right), \quad Y(x, y, z)=\left(\begin{array}{c}
0 \\
1 \\
\frac{x}{2}
\end{array}\right)
$$

Looking at the commutator we see that

$$
\begin{equation*}
[X, Y](f)=\left(\frac{\partial}{\partial x}-\frac{y}{2} \frac{\partial}{\partial t}\right)\left(\frac{\partial f}{\partial y}+\frac{x}{2} \frac{\partial f}{\partial z}\right)-\left(\frac{\partial}{\partial y}+\frac{x}{2} \frac{\partial}{\partial z}\right)\left(\frac{\partial f}{\partial x}-\frac{y}{2} \frac{\partial f}{\partial z}\right)=\frac{\partial f}{\partial z} \tag{2.6}
\end{equation*}
$$

so we can see that

$$
[X, Y](x, y, z)=\left(\begin{array}{l}
0  \tag{2.7}\\
0 \\
1
\end{array}\right)
$$

then $X(p)$ and $Y(p)$ taken together with the associated commutator $[X, Y](p)$ span the space $\mathbb{R}^{3}$ (i.e. $\operatorname{span}\left(X(p), Y(p),[X, Y](p)=\mathbb{R}^{3}\right)$ at any point $p=$ $(x, y, z)$. Chow's Theorem guarantees we can connect any point to any other by a horizontal path.

We learn from Chow's Theorem that if a sub-Riemannian geometry satisfies the Hörmander condition then this implies that the Carnot-Carathéodory distance is always finite. The reverse implication (i.e. that a finite distance implies the Hörmander conditon) requires a stronger regularity of the distribution. It is only satisfied in the case that the distribution is analytic, but not in general when it is smooth.

Example 2.2.2. Look at the sub-Riemannian metric generated by the 2dimensional vector fields,

$$
\begin{equation*}
X(x, y)=\binom{1}{0}, \quad Y(x, y)=\binom{0}{a(x)} \tag{2.8}
\end{equation*}
$$

with $a \in C^{\infty}(\mathbb{R})$ such that $a(x)=0$,if $x \leq 0$, and $a(x)>0, x>0$. The corresponding sub-Riemannian distance is finite. Nevertheless, the associated distribution is not bracket generating. In fact, if $x \leq 0$, then

$$
\begin{equation*}
Y(x, y)=\binom{0}{0} \tag{2.9}
\end{equation*}
$$

and so $\operatorname{Span}(\mathcal{L}(X, Y)(p))=\operatorname{Span}(X(p)) \neq \mathbb{R}^{2}$.

Example 2.2.3. Let $X$ and $Y$ be as in the Example 2.2.2, but with $a(x)=1$ if $x \geq 0$,and $a(x)=0$ if $x<0$. In this case $a \notin C^{\infty}(\mathbb{R})$, however we can use this example in order to investigate the previous one. On the half-plane $x<0$, as we can move only in one direction the spanned distribution is not bracket generating. Nevertheless, it is easy to write explicitly the associated Carnot-Caratheódory distance, that is

$$
d\left((x, y),\left(x^{\prime}, y^{\prime}\right)= \begin{cases}\sqrt{|x-y|^{2}+\left|x^{\prime}-y^{\prime}\right|^{2}}, & \text { if } x \geq 0, x^{\prime} \geq 0  \tag{2.10}\\ |x|+\left|x^{\prime}\right|+\left|y-y^{\prime}\right|, & \text { if } x<0, x^{\prime}<0 \\ |x| \sqrt{\left|x^{\prime}\right|^{2}+\left|y-y^{\prime}\right|^{2}}, & \text { if } x<0, x^{\prime} \geq 0 \\ \left|x^{\prime}\right| \sqrt{|x|^{2}+\left|y-y^{\prime}\right|^{2}}, & \text { if } x \geq 0, x^{\prime}<0\end{cases}\right.
$$

It is clear to see that $d(x, y)$ is a finite distance.

Definition 2.2.8. (Step of a distribution). Let $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ be a family of vector fields defined on a smooth manifold $M$ and $\mathcal{H}$ the distribution generated by $X_{1}, \ldots, X_{m}$. Given $p \in M$, we call the step of the distribution $\mathcal{H}$ at the point $p$, and we indicate by $k(p)$, the smallest natural number such that

$$
\begin{equation*}
\bigcup_{i=1}^{k(p)} \mathcal{L}^{i-1}(p)=T_{p} M . \tag{2.11}
\end{equation*}
$$

Example 2.2.4. The Heisenberg group (see Def 2.2.1) is associated to a bracket generating distribution with step equal to 2 at any point.

Example 2.2.5. The Grushin plane (see Example 2.5.1) is associated to a bracket generating with step 2 at the origin, and with step 1 otherwise .

We will now include a proof (from Gromov [32]) for a particular type of SubRiemannian geometry with reference to Chow's theorem. We will see the Heisenberg group in more detail later on Section 2.4. We now look at the 1-dimensional Heisenberg group given in Example 2.2.1, endowed with the standard Euclidean metric on $\mathbb{R}^{2}$.

If we look at the 1-form

$$
\eta:=d z-\frac{1}{2}(x d y-y d x)
$$

So we write the $\mathbb{H}^{n}$ as the n-dimensional Heisenberg group. Referring to Definition 2.2.4, we can see that a curve $\gamma:[0, T] \rightarrow \mathbb{R}^{3}$ is $\mathbb{H}^{1}$ if and only if

$$
\eta(\gamma(t))=0, \quad \text { for } t \text { a.e. } t \in[0, T] .
$$

In the proof of Gromov he uses an equivalent definition of the Hesienberg
group, however we will look at a proof incorporating the above definition. So we state a reconfiguration of Chow's Theorem in this particular case.

Theorem 2.2.2. Given two points in $\mathbb{R}^{3}$, there exists an absolutely continuous $\mathbb{H}^{1}$-horizontal curve joining them.

Proof. Let $p=\left(x_{1}, y_{1}, z_{1}\right)$ and $q=\left(x_{2}, y_{2}, z_{2}\right)$ be two given points of $\mathbb{R}^{3}$. Let $\tilde{\gamma}(t)=(x(t), y(t))$ be a plane curve joining $\left(x_{1}, y_{1}\right)$ to $\left(x_{2}, y_{2}\right)$. For the sake of simplicity assume that $T=1$.

Remark that we can look only at the absolutely continuous curves with constant curvature, i.e. we can assume that

$$
\int_{\tilde{\gamma}} x d y=\int_{0}^{1} x(t) \dot{y}(t) d t=\frac{1}{2} \int_{0}^{1}(x(t) \dot{y}(t)-\dot{x}(t) y(t)) d t=C,
$$

for some $C \in \mathbb{R}$.

Then we can define a curve in $\mathbb{R}^{3}$, setting $\gamma(t)=(x(t), y(t), z(t))$, where the third-coordinate is given by

$$
z(t)=z_{1}+\frac{1}{2} \int_{0}^{t}(x(s) \dot{y}(s)-\dot{x}(s) y(s)) d s
$$

Obviously $\gamma$ is an absolutely continuous curve in $\mathbb{R}^{3}$. Moreover, since $z(0)=z_{1}$ and $z(1)=z_{1} C$, choosing $C=z_{1} z_{2}$, then $\gamma$ joins $p$ to $q$. In order to conclude the proof, we need only to observe that, for a.e. $t \in[0,1]$, it holds $\eta(\gamma(t))=0$ and so $\gamma$ is a $\mathbb{H}^{1}$-horizontal curve.

### 2.3 Geodesics

We will take a deeper look at geodesics and how they are derived. In the Riemannian case the minimising geodesic must satisfy the Euler-Lagrange equations. Geodesics can be applied in solving many problems within calculus of variations. We recall the definition of geodesics (see Definition 2.1.8 in the Riemannian case).

Definition 2.3.1. Geodesics in Sub-Riemannian manifolds are curves $\gamma$ : $[a, b] \rightarrow M$ where the following are satisfied:

1. $\gamma$ is an horizontal curve, i.e. $\gamma$ is absolutely continuous and

$$
\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}, \text { a.e. } t \in[a, b],
$$

(see Definition 2.2.4).
2. $L(\gamma)=d(P, Q)$, where $\gamma(a)=P, \gamma(b)=Q$ and where $L(\gamma)$ is defined by (2.3).

To compute geodesics we use the so called geodesic equation and to define that we need to introduce the cometric associated to the Riemannian metric. From now on we denote our Sub-Riemannian geometry as $(M, \mathcal{H},\langle.,\rangle$.$) .$

Definition 2.3.2 ([40]). A cometric $\beta: T^{*} M \rightarrow T M$ on $(M, \mathcal{H},\langle.,\rangle$.$) a Sub-$ Riemannian manifold is uniquely defined by the following conditions:

1. $\operatorname{Im}\left(\beta_{P}\right)=\mathcal{H}_{P}$,
2. $p(Q)=\left\langle\beta_{P}(p), Q\right\rangle_{P}, \quad \forall p \in T_{P}^{*} M \forall Q \in \mathcal{H}_{P}$ where $P \in M$.

Definition 2.3.3. Given a cometric $(., .)_{P}$ on the cotangent bundle $T_{P}^{*} M$, we
can define a sub-Riemannian Hamiltonian as follows:

$$
\begin{equation*}
H(P, Q)=\frac{1}{2}\langle P, P\rangle_{Q}, \text { where } P \in M \text { and } Q \in T^{*} M \tag{2.12}
\end{equation*}
$$

Remark 2.3.1. Consider we have the admissible curve $\gamma:[a, b] \rightarrow M$, i.e.:

$$
\dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)} \quad \text { a.e. } t \in[a, b]
$$

then we can write:

$$
\begin{aligned}
\frac{1}{2}\|\dot{\gamma}(t)\|_{\gamma(t)}^{2} & :=\frac{1}{2}\langle\dot{\gamma}(t), \dot{\gamma}(t)\rangle_{\gamma(t)} \\
& =\frac{1}{2}\langle P, P\rangle_{\gamma(t)}, \quad \text { with } P=\dot{\gamma}(t) \\
& =H(Q, P)
\end{aligned}
$$

Proof. The proof of the previous relation is easily comes from the definition of the cometric as follows:

$$
\begin{aligned}
\gamma(t) \text { is admissible } & \Leftrightarrow \dot{\gamma}(t) \in \mathcal{H}_{\gamma(t)}=\operatorname{Im}\left(\beta_{\gamma(t)}\right) \text { (by Definition 2.3.2), } \\
& \Leftrightarrow \exists Q \in T_{\gamma(t)}^{*} M \text { s.t. } \beta_{\gamma(t)}(P)=\dot{\gamma}(t)
\end{aligned}
$$

From Definition 2.3.2 we know that:

$$
p(\widetilde{P})=\left\langle\beta_{\gamma(t)}(P), \widetilde{P}\right\rangle_{Q}, \forall Q \in T_{P}^{*} M \forall \widetilde{P} \in \mathcal{H}_{Q}
$$

Take $v:=\gamma(t)$, then:

$$
\begin{aligned}
p(\widetilde{P}) & =\left\langle\beta_{\gamma(t)}(Q), \dot{\gamma}(t)\right\rangle_{\gamma(t)} \\
& =\langle\dot{\gamma}(t)(Q), \dot{\gamma}(t)\rangle_{\gamma(t)} \\
& =\|\dot{\gamma}(t)\|^{2} .
\end{aligned}
$$

Thus, we have:

$$
\frac{1}{2}\|\dot{\gamma}(t)\|^{2}=H(P, Q)
$$

To get the Hamilton equations we first need to introduce momentum.

Definition 2.3.4. Let $M$ be $n$-smooth manifold and $X_{a}$ be a vector field on $M$, we define a linear function $P_{X_{a}}:=P_{a}$ on the cotangent bundle, where:

$$
\begin{equation*}
P_{a}: T^{*} M \rightarrow \mathbb{R} \text { with }(P, Q) \mapsto p\left(X_{a}(P)\right), \quad \forall P \in M, Q \in T_{P}^{*} M \tag{2.13}
\end{equation*}
$$

This function $P_{a}$ is called a momentum function.

If we have the expression for the vector field $X_{a}$ in coordinate as:

$$
X_{a}(P)=\sum_{i} X_{a}^{i}(P)\left(\frac{\partial}{\partial x_{i}}\right)
$$

then, we can write the following expression:

$$
p_{a}(x, p)=\sum_{i} X_{a}^{i}(P) p_{i},
$$

where $P_{i}=P_{\frac{\partial}{\partial x_{i}}}$ are the momentum functions for the coordinate vector fields. Note that $x_{i}$ and $p_{i}$ from the coordinate system on the tangent bundle $T^{*} M$
are called canonical coordinates.

Let us define

$$
\begin{equation*}
g^{a b}(P)=\left\langle X_{a}(P), X_{b}(P)\right\rangle_{P} \tag{2.14}
\end{equation*}
$$

to be the matrix of inner products defined by our distribution frame $\mathcal{H}$. Consider $g^{a b}(P)$ to be the inverse matrix of $g_{a b}$. We can see that $g^{a b}$ is a $n \times n$ matrix-valued function defined in some open set of $M$.

Propostion 2.3.1. Let $P_{a}$ and $g^{a b}$ be the functions on the cotangent bundle $T^{*} M$ defined respectively by (2.13) and (2.14), which are induced by the local distribution $\left\{X_{a}\right\}$, then we have:

$$
H(P, Q)=\frac{1}{2} \sum_{a, b} g^{a b}(P) P_{a}(P, Q) P_{b}(P, Q)
$$

In order for us to compute the geodesic equations associated with the Hamiltonian differential equations using the canonical coordinate $\left(x_{i}, p_{i}\right)$ we can write:

$$
\begin{equation*}
\dot{x}_{i}=\frac{\partial H}{\partial p_{i}} \text { and } \dot{p}_{i}=-\frac{\partial H}{\partial x_{i}} \tag{2.15}
\end{equation*}
$$

Definition 2.3.5. The Hamiltonian differential Equations (2.15) are called normal geodesic equations.

Theorem 2.3.1. (Normal geodesics) Let $\zeta(t)=(\gamma(t), p(t))$ be a solution of the Hamiltonian differential equations on the cotangent bundle $T^{*} M$ for a subRiemannian Hamiltonian $H$ and consider $\gamma(t)$ be its projection to $M$. Then, every sufficiently short length of $\gamma$ is a minimizing sub-Riemannian geodesic. Moreover, $\gamma$ can be considered as the unique minimizing geodesic that joins the endpoints.

For a detailed proof see [40].

Theorem 2.3.2. [40] Let $M$ be a smooth manifold and $\mathcal{H}$ a bracket generating distribution. Then

- local existence: for any $p \in M$ there exists a neighborhood $U$ of $p$ such that, for any $q \in U$, there exists a geodesic joining $p$ to $q$;
- global existence: if moreover $M$ is connected and complete w.r.t. the subRiemannian metric induced by $\mathcal{H}$, for any pair of points $p, q \in M$ there exists a geodesic joining $p$ to $q$.

For the Riemannian case it is known that the geodesics are locally unique. In the sub Riemannian this is generally not true (not even locally). We will see this later on in the case of the Grushin plane.

### 2.4 Carnot Groups

In this section we will elaborate on a particular class of Riemannian structures known as Carnot groups. They can be viewed however from a Lie algebra viewpoint which we will expand on. For further details see [19], [40] and [13]. To see more examples of Carnot group such as the Engel group see [17].

To define Carnot groups we first have to define what Lie algebras are.

Definition 2.4.1. A Lie group $\mathbb{G}$ is a smooth manifold $M$, that is also a group in the algebraic sense, with the property that the multiplication map $f: M \times M \rightarrow M$ and the inversion $g: M \rightarrow M$, given by

$$
f(x, y)=x y, \quad g(x)=x^{-1}
$$

are both continuous.

Definition 2.4.2. (Abstract Lie Algebra) A real vector space ( $V, \cdot$ ) equipped with an operation $[\cdot, \cdot]: V \times V \rightarrow V$ is said to be a Lie algebra if the following relations hold:
(i) $[\lambda X+\mu Y, Z]=\lambda[X, Z]+\mu[Y, Z]$, (Bi-linear)
(ii) $[X, Y]=-[Y, X]$
(iii) $[X,[Y, Z]]+[Z,[X, Y]]+[Y,[Z, X]]=0$ (cyclic permutation)
for all $X, Y, Z \in V$ and $\lambda, \mu \in \mathbb{R}$.
The equation (iii) is called the Jacobi identity.

Example 2.4.1. The vector space $\mathbb{R}^{3}$ equipped with the cross product,

$$
\begin{equation*}
[X, Y]=\left(x_{2} y_{3}-x_{3} y_{2}, x_{3} y_{1}-x_{1} y_{3}, x_{1} y_{2}-x_{2} y_{1}\right) \tag{2.16}
\end{equation*}
$$

is an example of Lie Algebra as it satisfies the conditions in Definition 2.4.2.

Proof. To prove the first condition let us take $X=\left(x_{1}, x_{2}, x_{3}\right), Y=\left(y_{1}, y_{2}, y_{3}\right)$ and $Z=\left(z_{1}, z_{2}, z_{3}\right)$

$$
\begin{gathered}
{[\lambda X+\mu Y, Z]=\left[\left(\lambda x_{1}+\mu y_{1}, \lambda x_{2}+\mu y_{2}, \lambda x_{3}+\mu y_{3}\right),\left(z_{1}, z_{2}, z_{3}\right)\right]} \\
=\left(\left(\lambda x_{2}+\mu y_{2}\right) z_{3}-\left(\lambda x_{3}+\mu y_{3}\right) z_{2},\left(\lambda x_{3}+\mu y_{3}\right) z_{1}-\left(\lambda x_{1}+\mu y_{1}\right) z_{3},\left(\lambda x_{1}+\mu y_{1}\right) z_{2}-\left(\lambda x_{2}+\mu y_{2}\right) z_{1}\right) \\
=\left(\lambda\left(x_{2} z_{3}-x_{3} z_{2}\right), \lambda\left(x_{3} z_{1}-x_{1} z_{3}\right), \lambda\left(x_{1} z_{2}-x_{2} z_{1}\right)\right) \\
+\left(\mu\left(y_{2} z_{3}-y_{3} z_{2}\right), \mu\left(y_{3} z_{1}-y_{1} z_{3}\right), \mu\left(y_{1} z_{2}-y_{2} z_{1}\right)\right)=\lambda[X, Z]+\mu[Y, Z]
\end{gathered}
$$

The proof of the second condition is fairly trivial since

$$
\begin{gathered}
{[X, Y]=\left(x_{2} y_{3}-x_{3} y_{2}, x_{3} y_{1}-x_{1} y_{3}, x_{1} y_{2}-x_{2} y_{1}\right)=} \\
-\left(-\left(x_{2} y_{3}-x_{3} y_{2}, x_{3} y_{1}-x_{1} y_{3}, x_{1} y_{2}-x_{2} y_{1}\right)\right)=-[Y, X] .
\end{gathered}
$$

To prove that the third condition holds we simply plug in our $X, Y$ and $Z$ and we see a cancellation of terms on the right hand side when we add the equations below together,

$$
\begin{gathered}
{[X,[Y, Z]]=\left(x_{2}\left(y_{1} z_{2}-y_{2} z_{1}\right)-x_{3}\left(y_{3} z_{1}-y_{1} z_{3}\right), x_{3}\left(y_{2} z_{3}-y_{3} z_{2}\right)-x_{1}\left(y_{1} z_{2}-y_{2} z_{1}\right),\right.} \\
\left.x_{1}\left(y_{3} z_{1}-y_{1} z_{3}\right)-x_{2}\left(y_{2} z_{3}-y_{3} z_{2}\right)\right)
\end{gathered}
$$

$$
\begin{gathered}
{[Z,[X, Y]]=\left(z_{2}\left(x_{1} y_{2}-x_{2} y_{1}\right)-z_{3}\left(x_{3} y_{1}-x_{1} y_{3}\right), z_{3}\left(x_{2} y_{3}-x_{3} y_{2}\right)-z_{1}\left(x_{1} y_{2}-x_{2} y_{1}\right)\right.} \\
\left.z_{1}\left(x_{3} y_{1}-x_{1} y_{3}\right)-z_{2}\left(x_{2} y_{3}-x_{3} y_{2}\right)\right)
\end{gathered}
$$

$$
\begin{gathered}
{[Y,[Z, X]]=\left(y_{2}\left(z_{1} x_{2}-z_{2} x_{1}\right)-y_{3}\left(z_{3} x_{1}-z_{1} x_{3}\right), y_{3}\left(z_{2} x_{3}-z_{3} x_{2}\right)-y_{1}\left(z_{1} x_{2}-z_{2} x_{1}\right)\right.} \\
\left.y_{1}\left(z_{3} x_{1}-z_{1} x_{3}\right)-y_{2}\left(z_{2} x_{3}-z_{3} x_{2}\right)\right)
\end{gathered}
$$

Definition 2.4.3. Let $M$ be a smooth manifold. For two vector fields $X, Y \in$
$C^{\infty}\left(T_{p} M\right)$ we define the Lie bracket $[X, Y]: C^{\infty}(M) \rightarrow \mathbb{R}$ of $X$ and $Y$ by

$$
\begin{equation*}
[X, Y](f)=X(Y(f))-Y(X(f)), \quad \forall f \in C^{\infty}(M) \tag{2.17}
\end{equation*}
$$

Note that the Lie bracket is an operation that satisfies the definition of the abstract Lie algebra where $V$ is equal to tangent space of $M$ at the point $p$.

Example 2.4.2. The vector space of $n \times n$ real matrices is a Lie algebra equipped with the Lie bracket defined in Definition 2.4.3. It's clear to see all $n \times n$ matrices $X, Y$ and $Z$ satisfy the first two conditions in Definition 2.4.2. We prove the Jacobi identity as follows

$$
\begin{aligned}
{[X,[Y, Z]]+[Y,[Z, X]]+[Z,[X, Y]]=} & {[X, Y Z-Z Y]+[Y, Z X-X Z] } \\
& +[Z, X Y-Y X] \\
= & X Y Z-X Z Y-Y Z X+Z Y X \\
& +Y Z A-Y X Z-Z X Y+X Z Y \\
& +Z X Y-Z Y X-X Y Z+Y X Z \\
= & 0
\end{aligned}
$$

Before we can highlight the connection between a Lie group and its Lie Algebra we first introduce the notion of left translations and invariant translations.

Definition 2.4.4. Let $\mathbb{G}$ be a Lie group. Then for $P \in \mathbb{G}$, the left translation, denoted by $L_{P}$, and the right translation, denoted by $R_{P}$, are respectively
given by:

$$
\begin{aligned}
& L_{P}: \mathbb{G} \rightarrow \mathbb{G} \text { with } L_{P}(Q):=P \circ Q, \\
& R_{P}: \mathbb{G} \rightarrow \mathbb{G} \text { with } R_{P}(Q):=Q \circ P,
\end{aligned}
$$

where $Q \in \mathbb{G}$.

Because we can write $L_{P}$ as the composition of smooth maps:

$$
\mathbb{G} \xrightarrow{\imath_{P}} \mathbb{G} \times \mathbb{G} \xrightarrow{m} \mathbb{G},
$$

where $\imath_{P}(Q)=(P, Q)$ and $m$ is the left multiplication, it implies that $L_{P}$ is smooth. In fact $L_{P}$ is a diffeomorphism of $\mathbb{G}$, since $L_{P^{-1}}$ is the smooth inverse of $L_{P}$. The same is true for the right translation $R_{P}: \mathbb{G} \rightarrow \mathbb{G}$. Note that if $\mathbb{G}$ is not abelian in general $L_{P} \neq R_{P}$ so we will usually use $L_{P}$.

Remark 2.4.1. Note that if $\mathbb{G}_{1}, \mathbb{G}_{2}$ are smooth manifolds, $f: \mathbb{G}_{1} \rightarrow \mathbb{G}_{2}$ is a diffeomorphism and $d f: T \mathbb{G}_{1} \rightarrow T \mathbb{G}_{2}$ is the differential of $f$, then

$$
\begin{equation*}
d f([X, Y])=[d f(X), d f(Y)] \tag{2.18}
\end{equation*}
$$

Definition 2.4.5. Let $\mathbb{G}$ be a Lie group, a vector field $X$ on $\mathbb{G}$ is called left-invariant if it is invariant under all left translations, i.e. for any $P, Q \in \mathbb{G}$ :

$$
\begin{equation*}
d L_{P}(X(Q))=X(P \circ Q) \tag{2.19}
\end{equation*}
$$

Similarly, a vector field $X$ is called right invariant if for any $P \in \mathbb{G}$ we have:

$$
d R_{P}(X(Q))=X(Q \circ P)
$$

Definition 2.4.6 (The Lie algebra of a Lie group). The space $\mathfrak{g}$ of all leftinvariant vector fields on a Lie group $\mathbb{G}$ endowed with the standard Lie bracket is called Lie algebra of the Lie group $\mathbb{G}$.

Before giving the definition of Carnot group we introduce the concept of nilpotent Lie algebra.

Definition 2.4.7 (Nilpotent Lie algebra). A Lie algebra $\mathfrak{g}$ of a Lie group $\mathbb{G}$ (see Definition 2.4.1) is called nilpotent of step $k$ if there exists $k \in \mathbb{N} \backslash\{0\}$ and a decomposition

$$
\begin{equation*}
\mathfrak{g}=\mathfrak{g}^{(1)} \oplus \cdots \oplus \mathfrak{g}^{(k)} \tag{2.20}
\end{equation*}
$$

where

$$
\begin{aligned}
\mathfrak{g}^{(1)} & =\mathfrak{g}_{1} \leq \mathfrak{g} \\
\mathfrak{g}^{(n+1)} & =\left[\mathfrak{g}_{1}, \mathfrak{g}^{(n)}\right], \quad n \in \mathbb{N} \backslash\{0\},
\end{aligned}
$$

and

$$
\left[\mathfrak{g}_{1}, \mathfrak{g}^{(n)}\right]:=\left\{[X, Y]: X \in \mathfrak{g}_{1}, Y \in \mathfrak{g}^{(n)}\right\}
$$

for $n=1, \ldots, k$ and $\mathfrak{g}^{(k+1)}=\{0\}$.

Definition 2.4.8. A group $\mathbb{G}$ is called Carnot group (also called stratified group of step $k$ ), if it is a connected Lie group whose Lie algebra is nilpotent of step $k$. The identity of the group is the vector $(0,0,0)$.

Example 2.4.3. (Heisenberg group) The Heisenberg group introduced in Example 2.2.1 is endowed with the group law for $\mathbb{H}^{1}$ is given by:

$$
P \circ Q=\left(x_{1}+\widetilde{x}_{1}, x_{2}+\widetilde{x}_{1}, x_{3}+\widetilde{x}_{3}+\frac{1}{2}\left(x_{1} \widetilde{x}_{2}-\widetilde{x}_{1} x_{2}\right)\right),
$$

for all $P=\left(x_{1}, x_{2}, x_{3}\right), Q=\left(\widetilde{x}_{1}, \widetilde{x}_{2}, \widetilde{x}_{3}\right) \in \mathbb{R}^{3}$. The Heisenberg group is an example of a Carnot group, a non-commutative nilpotent Lie group with stratified Lie algebra

$$
\begin{equation*}
\mathfrak{g}=\mathfrak{g}_{1} \oplus \mathfrak{g}_{2} \tag{2.21}
\end{equation*}
$$

where $\mathfrak{g}_{1}$ is 2-dimensional and is generated by the vectors $X$ and $Y$, and $\mathfrak{g}_{2}$ is 1-dimensional where $\mathfrak{g}_{2}=\operatorname{span}\{[X, Y]\}$, given in Example 2.2.1.

We next show how to check that $X$ and $Y$ are exactly the ones given in Example 2.2.1. Since a Lie group is also a manifold, one can always associate to the Lie algebra $\mathfrak{g}$ a tangent space at the origin (the identity of the group).

Definition 2.4.9 (Left-invariant translations). Using left-translations one can define the tangent space at any other point $p$. Then one can easily identify a basis of left invariant vector fields as

$$
X_{i}=d L_{p}\left(e_{i}\right),
$$

where $L_{p}$ is the left-translation given in Definition 2.4.4, $d f$ is the differential defined in Definition 1.2.3 and $e_{i}$ are the standard Euclidean basis vectors.

Theorem 2.4.1. Let $\mathbb{G}$ be a Lie group and $\mathfrak{g}$ be its Lie algebra. Then the following statements are satisfied:

1. $\mathfrak{g}$ is a vector space and the function

$$
\begin{aligned}
\phi: \mathfrak{g} & \rightarrow T_{e} \mathbb{G} \\
X & \rightarrow \phi(X):=X(e)
\end{aligned}
$$

is isomorphism between $\mathfrak{g}$ and the tangent space $T_{e} \mathbb{G}$ (see Definition 1.2.2) to $\mathbb{G}$ at the identity ef $\mathbb{G}$. As a consequence, $\operatorname{dim} \mathfrak{g}=\operatorname{dim} T_{e} \mathbb{G}=$
$\operatorname{dim} \mathbb{G}$.
2. $\mathfrak{g}$ with the commutation operation is a Lie algebra.

For a proof see [13].

Theorem 2.4.1 together with the function left-translation allows us to compute the left-invariant vector fields for any $X(P) \in \mathbb{G}$. Take $X(e) \in T_{e} \mathbb{G}$, then we can define a corresponding vector $X(P) \in T_{P} \mathbb{G}$ as

$$
\begin{equation*}
X(P):=\left(d L_{P}\right)(X(e)), \quad P \in \mathbb{G} \tag{2.22}
\end{equation*}
$$

where $L_{P}$ is the left-translations, see Definition 2.4.4.

Example 2.4.4 (Left-invariant vector fields in the Heisenberg group $\mathbb{H}^{1}$ ). As the left-invariant vector fields for $T_{e} \mathbb{H}^{1}$ we choose $e_{1}, e_{2}, e_{3}$ standard Euclidean 3 -dimensional basis. In order to compute the left-invariant vectors for $\mathbb{H}^{1}$, let us fix a point $P=\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{H}^{1}$ and consider the curve $\tilde{\gamma}:[0,1] \rightarrow \mathbb{H}^{1}$ defined as

$$
\widetilde{\gamma}(t)=f(\gamma(t)),
$$

where $\gamma:[0,1] \rightarrow \mathbb{H}^{1}$ satisfies

$$
\begin{aligned}
& \gamma(0)=e=(0,0,0) \\
& \dot{\gamma}(0)=X(e)
\end{aligned}
$$

with $X(e)$ respectively equal to $e_{1}, e_{2}$ and then $e_{3}$, given $f: \mathbb{H}^{1} \rightarrow \mathbb{H}^{1}$ then

$$
(d f)_{e}(X(e)):=\left.\frac{d}{d t} \widetilde{\gamma}(t)\right|_{t=0}
$$

Choose as function $f$ the left-translations $L_{P}: \mathbb{G} \rightarrow \mathbb{G}$ with $L_{P}(Q)=P \circ Q$
and consider $X(e)=e_{i}$, for $i=1,2,3$, we have

$$
\begin{aligned}
& \widetilde{\gamma}_{i}(t)=L_{P}\left(\gamma_{i}(t)\right), \\
& \dot{\tilde{\gamma}}_{i}(0)=\left(d L_{P}\right)\left(e_{i}\right),
\end{aligned}
$$

for $i=1,2,3$.
Now, let us find the left translations with $P=\left(x_{1}, x_{2}, x_{3}\right)$

$$
\begin{aligned}
L_{P}(\gamma(t)) & =\left(x_{1}, x_{2}, x_{3}\right) \circ\left(\gamma_{1}(t), \gamma_{2}(t), \gamma_{3}(t)\right) \\
& =\left(x_{1}+\gamma_{1}(t), x_{2}+\gamma_{2}(t), x_{3}+\gamma_{3}(t)+\frac{1}{2}\left(x_{1} \gamma_{2}(t)-x_{2} \gamma_{1}(t)\right)\right) .
\end{aligned}
$$

To obtain the left-invariant vector fields we need to differentiate the left translations, so fix $p=\left(x_{1}, x_{2}, x_{3}\right)$, then

$$
\left(d L_{P}(X(e))=\left(\dot{\gamma}_{1}(0), \dot{\gamma}_{2}(0), \dot{\gamma}_{3}(0)+\frac{1}{2}\left(x_{1} \dot{\gamma}_{2}(0)-x_{2} \dot{\gamma}_{1}(0)\right)\right)\right.
$$

where $\dot{\gamma}(0)=X$. Hence, the left-invariant vector field $X_{1}(P)$ corresponding to $e_{1}=(1,0,0)$ is

$$
\begin{aligned}
X_{1}(P) & =\left(d L_{P}\right)(1,0,0) \\
& =\left(1,0,-\frac{x_{2}}{2}\right) \\
& =\frac{\partial}{\partial x_{1}}-\frac{x_{2}}{2} \frac{\partial}{\partial x_{3}} .
\end{aligned}
$$

Similarly, the left-invariant vector field $X_{2}(P)$ corresponding to $e_{2}=(0,1,0)$
in $T_{e} \mathbb{H}^{1}$ is:

$$
\begin{aligned}
X_{2}(P) & =\left(d L_{P}\right)(0,1,0) \\
& =\left(0,1, \frac{x_{1}}{2}\right) \\
& =\frac{\partial}{\partial x_{2}}+\frac{x_{2}}{2} \frac{\partial}{\partial x_{3}} .
\end{aligned}
$$

Finally, the left-invariant vector field $X_{3}(P)$ corresponding to $e_{3}=(0,0,1)$ in $T_{e} \mathbb{H}^{1}$ is:

$$
\begin{aligned}
X_{3}(P) & =\left(d L_{P}\right)(0,0,1) \\
& =(0,0,1) \\
& =\frac{\partial}{\partial x_{3}} .
\end{aligned}
$$

which are the three vector fields we'd expect to compute from Example 2.2.1.

Definition 2.4.10. (Dilation) For $\lambda>0$ family of dilations on $\mathfrak{g}$ is a family of maps $\Delta_{\lambda}: \mathfrak{g} \rightarrow \mathfrak{g}$ defined as dilations,

$$
\begin{equation*}
\Delta_{\lambda}(X)=\lambda^{j} X \quad \forall X \in \mathfrak{g}^{(i)}, \tag{2.23}
\end{equation*}
$$

where $\mathfrak{g}^{(i)}$ is the $i$-layer of the stratification defined in (2.21) and the element of the Lie algebra are identified with vector fields by Theorem 2.4.1.

Example 2.4.5. (Heisenberg group) Dilations for the Heisenberg group in Example 1.23 can be represented as,
$\Delta_{\lambda}(X)=\lambda X=\left(\begin{array}{c}\lambda \\ 0 \\ \frac{-\lambda y}{2}\end{array}\right), \Delta_{\lambda}(Y)=\lambda Y=\left(\begin{array}{c}0 \\ \lambda \\ \frac{\lambda x}{2}\end{array}\right)$ and $\Delta_{\lambda}(Z)=\lambda^{2} Z=\left(\begin{array}{c}0 \\ 0 \\ \lambda^{2}\end{array}\right)$.

Once we have some dilations on the Lie algebra $\mathfrak{g}$ we can induce a natural rescaling also on the group $\mathbb{G}$ by using the exponential map (see [37] for definition of exponential map).

Definition 2.4.11. We define $\delta_{\lambda}: \mathbb{G} \rightarrow \mathbb{G}$ as

$$
\begin{equation*}
\delta_{\lambda}(x)=\exp \circ \Delta_{\lambda} \circ \exp ^{-1}(x), \tag{2.24}
\end{equation*}
$$

where exp: $\mathfrak{g} \rightarrow \mathbb{G}$ is the exponential map. This gives a rescaling that is coherent with the manifold structure defined on the Lie groups.

Arising from this we have that following lemma.

Lemma 2.4.1. Given a Carnot group and the family of dilations $\delta_{\lambda}$ (defined above) on $\mathbb{G}$ consider a horizontal curve $\gamma$ (see Definition 2.2.4), then the curve

$$
\eta:=\delta_{\lambda}(\gamma),
$$

is still horizontal and the horizontal velocity satisfies

$$
\alpha^{\eta}=\lambda \alpha^{\gamma} .
$$

For proof see [13].

Example 2.4.6. In the Heisenberg group in Example 2.4.5 we have that the
dilation defined in (2.24) are

$$
\delta_{\lambda}(x, y, z)=\left(\lambda x, \lambda y, \lambda^{2} z\right),
$$

one can easily check the validity of Lemma 2.4.1.

### 2.5 Grushin spaces

Grushin spaces are very important geometries associated to Hörmander vector fields. We refer to [41] for properties and to [27] for properties and applications of the Grushin spaces. Grushin spaces are not Carnot groups since they do not have any group structure as it is not possible to associate to them a Lie group structure, therefore "standard" Lie group translations are not defined. We are going to use our new notion of generalised translations to the specific case of Grushin spaces and in particular the Grushin plane.

A generalised Grushin space can be defined as follows.

Definition 2.5.1. Let $(x, y) \in \mathbb{G}_{\alpha}^{n} \simeq \mathbb{R}^{n}=\mathbb{R}^{g} \times \mathbb{R}^{h}$, where $g, h \geq 1$ are integers and $n=g+h$. For a given real number $\alpha>0$, let us define the vector fields in $\mathbb{R}^{n}$,

$$
X_{i}(x, y)=\frac{\partial}{\partial x_{i}}, i=1, \ldots, g, \quad Y_{j}(x, y)=|x|^{\alpha} \frac{\partial}{\partial y_{j}}, \quad j=1, \ldots, h
$$

In this thesis we concentrate on the specific case of the so called Grushin plane, where $\alpha=1$ and $g=1=h$. In the literature on Grushin spaces the modulus is used for spaces greater than 1 , however in the simplest first case $x$ is used and not the modulus. See the next example.

Example 2.5.1. (The Grushin plane $G^{2}$ ) We take as underlying manifold of $G^{2}$ the $\mathbb{R}^{2}$ plane, with coordinates $x, y$ and consider the sub-Riemannian metric defined by the vector fields

$$
\begin{equation*}
X(x, y)=\binom{1}{0}, \quad Y(x, y)=\binom{0}{x}, \quad \forall(x, y) \in \mathbb{R}^{2} \tag{2.25}
\end{equation*}
$$

These vector fields span the tangent space everywhere, except along the line $x=0$. Consider the commutator:

$$
\begin{equation*}
X_{3}(x, y):=[X, Y](x, y)=\binom{0}{1}, \quad \forall(x, y) \in \mathbb{R}^{2} \tag{2.26}
\end{equation*}
$$

Then $X$ and $Y$ taken together with the associated commutator $X_{3}=[X, Y]$ span the whole tangent space at any point, so the Hörmander condition holds and Chow's Theorem applies.

Even if Grushin spaces are not Carnot groups (as there exists no associated group law that satisfies the conditions necessary for it to be a Lie algebra), one can introduce a natural rescaling in these geometries that we call again dilations. We will look at this in greater details in Section 4.2.1, highlighting how the rescaling is coherent with the manifold structure in the Grushin plane.

Definition 2.5.2. In the Grushin space the dilations $\delta_{\lambda}: G^{2} \rightarrow G^{2}$ are defined as

$$
\delta_{\lambda}(x, y)=\left(\lambda x, \lambda^{2} y\right)
$$

This scaling defined above is natural in the geometry since it respects horizontal curves and the associated Carnot-Carathéodory distance. In fact, consider an
horizontal curve $\gamma=\left(\gamma_{1}, \gamma_{2}\right):[0, T] \rightarrow G^{2}$ with horizontal velocity $\alpha=$ $\left(\alpha_{1}, \alpha_{2}\right)$, that means

$$
\dot{\gamma}(t)=\left(\alpha_{1}(t), \alpha_{2}(t) \gamma_{1}(t)\right), \quad \text { a.e. } t \in[0, T] .
$$

If we now consider the Euclidean rescaled curve defined as $\eta(t):=\lambda \gamma(t)$ for $t \in[0, T]$, in general $\eta$ is not everywhere horizontal. If we instead consider the rescaled curve defined as $\eta(t):=\delta_{\lambda}(t)$ then $\eta$ is still horizontal and its horizontal velocity $\alpha^{\eta}$ rescales as one could expect, i.e.

$$
\begin{equation*}
\alpha^{\eta}=\lambda \alpha \tag{2.27}
\end{equation*}
$$

In fact one can explicitly compute the velocity of $\eta$, that is

$$
\dot{\eta}(t)=\left(\lambda \dot{\gamma}_{1}(t), \lambda^{2} \dot{\gamma}_{2}(t)\right)=\lambda\left(\alpha_{1}(t), \alpha_{2}(t)\left(\lambda \gamma_{1}(t)\right)\right)=\lambda\left(\alpha_{1}(t), \alpha_{2}(t) \eta_{1}(t)\right) .
$$

By recalling that the Carnot-Carathéodory distance is defined as the minimum length of horizontal curves joining two given points, the previous rescaling property implies the following result.

Lemma 2.5.1. Let us consider two points $p, q \in G^{2}$ and the corresponding Carnot-Carathéodory distance $d_{C C}(p, q)$ defined in the Grushin plane (see Definition 2.2.5). Assume that $\delta_{\lambda}$ are the anisotropic dilations introduced in Definition 2.5.2. Then

$$
d_{C C}\left(\delta_{\lambda}(p), \delta_{\lambda}(q)\right)=\lambda d_{C C}(p, q), \quad \forall p, q \in G^{2}
$$

Proof. To show the lemma it is enough to remark that any horizontal curve $\eta$ joining the points $\delta_{\lambda}(p)$ to $\delta_{\lambda}(q)$ can be written as $\eta=\delta_{\lambda}(\gamma)$ for some $\gamma$
horizontal curve joining $p$ to $q$ (in fact it is enough to choose $\gamma=\delta_{1 / \lambda}(\eta)$ ). Then (2.27) implies the lemma.

Roberto Monti in his PhD thesis [41] neatly proved a very important estimate for the Carnot-Carathédory distance in Grushin spaces, by introducing the so called box distance. Next we report the result but written directly in the case of the Grushin plane.

Theorem 2.5.1. Let $\lambda>0$ and $p, q \in \mathbb{R}^{2}$ and $d_{C C}(p, q)$ the Carnot-Carathéodory defined in Definition 2.2.5, then there exists $c \geq 1$ such that for all $p=(x, y)$ and $q=(\xi, \eta) \in \mathbb{R}^{2}$ with $|x| \geq|\xi|$

$$
\begin{equation*}
d_{C C}(p, q) \leq|x-\xi|+\frac{|y-\eta|}{|x|} \leq c d_{C C}(p, q), \quad \text { if }|x|^{2} \geq \lambda|y-\eta| \tag{2.28}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{C C}(p, q) \leq|x-\xi|+|y-\eta|^{\frac{1}{2}} \leq c d_{C C}(p, q), \quad \text { if }|x|^{2}<\lambda|y-\eta|, \tag{2.29}
\end{equation*}
$$

where $p=(x, y)$ and $q=(\xi, \eta)$

For proof see [41].
This estimate will be extremely useful in Section 4.6.

The previous result tells that the Carnot-Carathédory distance in Grushin spaces is globally equivalent to the so called box-distance that we indicate by $d_{b o x}(p, q)$, i.e.

$$
d_{b o x}(p, q)= \begin{cases}|x-\xi|+\frac{|y-\eta|}{|x|}, & \text { if }|x|^{2} \geq|y-\eta| \\ |x-\xi|+|y-\eta|^{\frac{1}{2}}, & \text { if }|x|^{2}<|y-\eta|\end{cases}
$$

It is easy to show that also the box-distance rescales well in the sense that

$$
d_{b o x}\left(\delta_{\lambda}(p), \delta_{\lambda}(q)\right)=\lambda d_{b o x}(p, q) .
$$

We conclude the section by the picture of the box-ball centred in the origin with radius 1 , i.e.

$$
B_{b o x}:=\left\{p \in \mathbb{R}^{2} \mid d_{b o x}(p, 0) \leq 1\right\} .
$$

That can be found easily as the points $(x, y)$ solving

$$
1 \geq \begin{cases}|x|+\frac{|y|}{|x|}, & x^{2} \geq|y| \\ |x|+|y|^{\frac{1}{2}}, & x^{2}<|y| .\end{cases}
$$

see the following pictures.


Figure 2.1: In the picture we represent the two different graphs appearing in the definition of the box-distance together with the constrain $|x|^{2}=|y|$.


Figure 2.2: The Grushin box-ball centred in the origin and with radius 1.

## Part II

## Generalised translations,

# Periodic sets and Perforated 

## domains

## Chapter 3

## Generalised translations and

## generalised periodic sets

### 3.1 Translations and periodicity along vector fields.

Translations are usually associated to geometrical structures where one can define a vector space structure or at least a group law. Here we introduce a new idea for translating along vector fields, that can be applied to very general geometries where nor a vector space nor an algebraic structure are defined. The possibility to define periodic structures on these spaces lead to many important application, e.g. the possibility to study homogenization problems in this setting (e.g. see Part III). In particular this notion can be applied to Riemannian and sub-Riemannian manifolds.

Wherever you have a group structure you can translate a point w.r.t another point (i.e. w.r.t another group element) trivially by using the group law.

Instead in our case we translate points along waves which are uniquely determined by a fixed (integer) constant velocity. Usually (e.g. in Carnot groups) this has an important dimensional consequence; in fact we define the translation in a $n$-dimensional space, but depend only on $m$ parameters with $m<n$ (see the equivalence between generalised translations and horizontal translations in Lemma 4.1.4). In the Grushin plane this idea also works better.

We introduce the main definitions of the thesis with many examples.
Our setting is now the geometry of vector fields, then we consider a family of $m$ vector fields $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ which, for sake of simplicity, we always assume defined on $\mathbb{R}^{n}$, with usually $m \leq n$. Note that one could introduce the same notion starting from vector fields defined on a generic $n$-dimensional manifold but in this thesis we do not consider this more generic case. The vector fields will be assumed to be at least locally Lipschitz, though in our examples they are usually smooth.

### 3.1.1 Main definition and examples.

First we need to recall the notion of $\mathcal{X}$-lines used in [7] (see also [6] for more properties), which are curves with velocity constant along the directions of the given vector fields. More precisely, recall that an absolutely continuous curve $x:[0, T] \rightarrow \mathbb{R}^{n}$ is called horizontal whenever the velocity at almost every time belongs to the span generated by the vector fields at the corresponding point, i.e.

$$
\begin{equation*}
\dot{x}(t)=\sum_{i=1}^{m} \alpha_{i}(t) X_{i}(x(t)), \quad \text { a.e. } t \in[0, T] . \tag{3.1}
\end{equation*}
$$

The $m$-valued measurable function $\alpha:[0, T] \rightarrow \mathbb{R}^{m}$ is called the horizontal velocity and represents the velocity of a horizontal curve w.r.t. the given
family of vector fields (See Definition 2.2.4).

Definition 3.1.1 ( $\mathcal{X}$-lines). An horizontal curve $l:[0, T] \rightarrow \mathbb{R}^{n}$ is called $\mathcal{X}$-line if the $m$-valued horizontal velocity $\alpha$ is constant, i.e.

$$
\begin{equation*}
\dot{i}(t)=\sum_{i=1}^{m} \alpha_{i} X_{i}(l(t)), \quad \text { a.e. } t \in[0, T] \tag{3.2}
\end{equation*}
$$

for some constants $\alpha_{1}, \ldots, \alpha_{m} \in \mathbb{R}$.

Given a constant horizontal velocity $\alpha$ and a starting point $x \in \mathbb{R}^{n}$ there exists a unique $\mathcal{X}$-line solving (3.2) since we have required for the vector fields to be at least locally Lipschitz. It is important to remark that,(recalling Chow's Theorem see Theorem 2.2.1), whenever the vector fields $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ satisfy the Hörmander condition, a horizontal curve between any two given points $x$ and $y$ always exist while this is not anymore true if we restrict our attention to $\mathcal{X}$-lines: in fact the set of points that one can reach starting from $x$ is a $m$-dimensional object in $\mathbb{R}^{n}$. We can now introduce the main notion of the paper.

Definition 3.1.2 (Generalised translations). Given $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ family of vector fields, assume that the vector fields are at least locally Lipschitz and a point $x \in \mathbb{R}^{n}$ and a constant $\alpha \in \mathbb{R}^{m}$ with $m \leq n$, the generalised translation (or translation along vector fields) of the point $x$ in the direction induced by $\alpha$ is defined as the following point:

$$
\begin{equation*}
\tau_{\alpha}(x)=l_{x}^{\alpha}(1) \tag{3.3}
\end{equation*}
$$

where $l_{x}^{\alpha}(\cdot)$ is the unique solution of (3.2) with initial condition $l_{x}^{\alpha}(0)=x$ and horizontal constant velocity $\alpha \in \mathbb{R}^{m}$.

In the Euclidean case, if assume $m=n$ and the vector fields $X_{i}=e_{i}$ are the standard Euclidean basis, then we get back the usual translations: in fact the Euclidean $\mathcal{X}$-lines are the usual straight lines $l(t)=\alpha t+x$, thus $\tau_{\alpha}(x)=x+\alpha$.

Remark 3.1.1. Note that, unlike the usual translations, we are not translating a point w.r.t. another point but a point along selected set of directions (i.e. a point belonging to the set of solutions to our (3.3)).

Lemma 3.1.1. Given some constant $\alpha \in \mathbb{R}^{m}$, the generalised translation $\tau_{\alpha}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is always bijective and the inverse function is $\tau_{\alpha}^{-1}=\tau_{-\alpha}$.

Proof. Consider the $\mathcal{X}$-line $x^{\alpha}:[0,1] \rightarrow \mathbb{R}^{n}$ and define the curve $\eta(t)=$ $x^{\alpha}(1-t)$. Trivially $\eta(0)=\tau_{\alpha}(x)$ while $\eta(1)=x^{\alpha}(0)=x$. Moreover

$$
\dot{\eta}(t)=-\dot{x}^{\alpha}(1-t)=-\sigma\left(x^{\alpha}(1-t)\right) \alpha=\sigma(\eta(t))(-\alpha),
$$

where $\sigma$ is the $n \times m$ matrix whose columns are the vector fields of the family $\mathcal{X}$. Thus $\eta=l_{y}^{-\alpha}$ where $y=\tau_{\alpha}(x)$. Recalling that $\eta(1)=x$, we have proved that $\tau_{-\alpha}\left(\tau_{\alpha}(x)\right)=x$. Swapping $\alpha$ and $-\alpha$, we prove that $\tau_{\alpha} \circ \tau_{-\alpha}=$ identity map $=\tau_{\alpha} \circ \tau_{-\alpha}$.

We next compute explicitly the generalised translations in some easy but still quite interesting cases.

Example 3.1.1 (Translations in one direction in the Euclidean $\mathbb{R}^{N}$ ). On $\mathbb{R}^{N}$ we consider the vector field $X=e_{i}$, for some fixed $i=1, \ldots, N$. Then the generalised translation of a point $x \in \mathbb{R}^{N}$ w.r.t. to the direction induced by $\alpha \in \mathbb{R}$ is

$$
\begin{equation*}
\tau_{\alpha}(x)=\left(0, \ldots, x_{i}+\alpha, 0 \ldots, 0\right) \tag{3.4}
\end{equation*}
$$

where $x_{i}$ indicates the $i$-components of the point $x$. Thus the generalised translations coincide with the standard translations in the fixed $i$-direction.

Example 3.1.2 (Rotational geometry). On $\mathbb{R}^{2}$ we consider the the vector field

$$
X(x, y)=\binom{y}{-x}, \text { for }(x, y) \in \mathbb{R}^{2}
$$

Solving the Equation (3.2) for the $\mathcal{X}$-lines with horizontal velocity $\alpha \in \mathbb{R}$ and initial condition $(x, y) \in \mathbb{R}^{2}$, we find

$$
l_{1}(t)=y \sin (\alpha t)+x \cos (\alpha t),
$$

and

$$
l_{2}(t)=-x \sin (\alpha t)+y \cos (\alpha t) .
$$

Set $t=1$, we deduce

$$
\begin{equation*}
\tau_{\alpha}(x, y)=(y \sin \alpha+x \cos \alpha,-x \sin \alpha+y \cos \alpha) \in \mathbb{R}^{2} \tag{3.5}
\end{equation*}
$$

This means that in this case we translate a point along the circle centred at the origin and passing from that point to another with an angle equal to $\alpha$.

The next two examples are the the main focus of the thesis: the Heisenberg group and the Grushin plane.

Example 3.1.3 (1-dimensional Heisenberg group). Recall the 1-dimensional Heisenberg group $\mathbb{H}^{1}$ given in Example 2.2.1. Solving (3.2) with initial conditions $l(0)=(x, y, z)$ and imposing

$$
\left\{\begin{array}{l}
i_{1}(t)=\alpha_{1}  \tag{3.6}\\
i_{2}(t)=\alpha_{2} \\
i_{3}(t)=-\alpha_{1} i_{2}(t)+\alpha_{2} i_{1}(t)
\end{array}\right.
$$

then we can write explicitly the generalised translation of a point $(x, y, z)$ w.r.t. $\alpha=\left(\alpha_{1}, \alpha_{2}\right) \in \mathbb{R}^{2}$ as

$$
\begin{equation*}
\tau_{\alpha}(x, y, z)=\left(\alpha_{1}+x, \alpha_{2}+y, z+\frac{\alpha_{1} y-\alpha_{2} x}{2}\right) \tag{3.7}
\end{equation*}
$$

Example 3.1.4 (Grushin plane). For all $(x, y) \in \mathbb{R}^{2}$, consider the two vector fields given in Example 2.5.1. In this case, the $\mathcal{X}$-lines can be found solving

$$
\dot{i}(t)=\binom{\dot{i}_{1}(t)}{i_{2}(t)}=\left(\begin{array}{cc}
1 & 0  \tag{3.8}\\
0 & l_{2}(t)
\end{array}\right)\binom{\alpha_{1}}{\alpha_{2}}
$$

with starting point $(x, y)$. Then the solution to Equation (3.8) is

$$
\left\{\begin{array}{l}
l_{1}(t)=\alpha_{1} t+x  \tag{3.9}\\
l_{2}(t)=\frac{\alpha_{1} \alpha_{2}}{2} t^{2}+\alpha_{2} t+y
\end{array}\right.
$$

Hence the generalised translations in the Grushin plane are

$$
\begin{equation*}
\tau_{\alpha}(x, y)=\left(\alpha_{1}+x, \frac{\alpha_{1} \alpha_{2}}{2}+\alpha_{2} x+y\right) \tag{3.10}
\end{equation*}
$$

Remark 3.1.2. Different from the case of the Heisenberg group, in the case of the Grushin plane the translation parameter $\alpha$ has the same dimension of the space i.e. they are both 2-dimensions.

### 3.1.2 Periodic sets

Using the previous translations along vector fields we introduce an associated notion of periodicity for sets.

Definition 3.1.3 (Periodic sets). Given $\mathcal{X}=\left\{X_{1}, \ldots, X_{m}\right\}$ family of vector fields, a positive real number $T$ and a set $\Omega \subseteq \mathbb{R}^{N}$, we say that $\Omega$ is generalised periodic (or simply $\mathcal{X}$-periodic) with period $T$ if and only if

$$
\begin{equation*}
\tau_{T k}(x) \in \Omega, \quad \forall x \in \Omega \text { and } \forall k \in \mathbb{Z}^{m} \tag{3.11}
\end{equation*}
$$

where $\tau_{T k}(x)$ are the generalised translation defined by (3.3) with $\alpha=T k$. Usually the period $T$ will be chosen as the smallest positive number such that property (3.11) holds true.

Note that, in the Euclidean case, $\mathcal{X}$ is the standard Euclidean basis on $\mathbb{R}^{N}$, then the previous notion coincides with the standard notion of periodicity with period $T$.

The following remark is useful later on the thesis

Remark 3.1.3. Note that (3.11) is true if and only if the following holds,

$$
\begin{equation*}
\tau_{T k_{1}} \circ \cdots \circ \tau_{T k_{N}}(x) \in \Omega, \quad \forall x \in \Omega \tag{3.12}
\end{equation*}
$$

In fact (3.12) implies (3.11) trivially by choosing $N=1$. To prove the reverse implication we prove using induction. For the case $N=1$ we have equality.

Assume true for $N$ now we have to prove for $N+1$. Take

$$
y=\tau_{T k_{1}} \circ \cdots \circ \tau_{T k_{N}}(x) \in \Omega .
$$

As $y \in \Omega$ and then by definition $\tau_{T k_{N+1}}(y) \in \Omega$.

Next we give some examples of generalised periodic sets in different geometries.

Example 3.1.5 (Periodic sets in $\mathbb{R}^{2}$ translating only in one direction). Look at the $\mathbb{R}^{2}$ with the vector field $X=\mathrm{e}_{1}$ and the corresponding generalised translation defined in Example 3.1.1. We define the cube $Q_{0}=\left[\frac{-1}{2}, \frac{1}{2}\right] \times\left[\frac{-1}{2}, \frac{1}{2}\right]$ and the translated sets:

$$
Q_{k}=\tau_{2 k}\left(Q_{0}\right)=\left[2 k+\frac{-1}{2}, 2 k+\frac{1}{2}\right] \times\left[\frac{-1}{2}, \frac{1}{2}\right], \quad \forall k \in \mathbb{Z}
$$

The set $\Omega=\bigcup_{k \in \mathbb{Z}} Q_{k}$ and its complementary set $\Omega^{c}$ are both $\Omega X_{1}$-periodic with period $T=2$. Note that $\Omega^{c}$ and $\Omega$ are not periodic in the standard sense in $\mathbb{R}^{2}$ since they are not periodic w.r.t. the vertical direction.


Figure 3.1: A set periodic only in the $x$-direction.

Example 3.1.6 (Periodic sets in the rotational geometry). We consider now the vector field on $\mathbb{R}^{2}$ defined in Example 3.1.2 and the corresponding generalised translation. Since the $\mathcal{X}$-lines are circle centred in the origin and passing trough the starting point, then the periodic set need to have some
radial symmetry. We next present several examples generalised periodic sets in this specific geometry.

Note that all the complements of the given sets are generalised periodic as well with the same period $T$.


Figure 3.2: A set which is generalised periodic for every period $T \in \mathbb{R}$.


Figure 3.3: Given $Q_{1}=\left\{r \leq 1, \frac{\pi}{3} \leq \theta \leq \frac{2 \pi}{3}\right\}$ and $Q_{2}=\left\{r \leq 1, \frac{4 \pi}{3} \leq \theta \leq \frac{5 \pi}{3}\right\}$ (defined in polar coordinates), then the set $\Omega=Q_{1} \cup Q_{2}$ is generalised periodic with period $T=\pi$.


Figure 3.4: A perforated domain with non-overlapping holes that is also generalised periodic with period $T=\frac{\pi}{2}$.

Next we give two lemmas which are very useful to easily construct (generalised) periodic sets as both union of holes or perforated domains (i.e. their complements) in many different geometries.

Lemma 3.1.2. Consider a simply connected bounded set $B \subset \mathbb{R}^{N}$ and define

$$
\Omega:=\bigcup_{k \in \mathbb{Z}^{m}} \tau_{T k}(B),
$$

where $\tau_{T k}(x)$ are the generalised translations with period $T$ given in (3.3).
Assume that

$$
\begin{equation*}
\forall k, h \in \mathbb{Z}^{m}, \quad \exists z \in \mathbb{Z}^{m} \text { such that } \tau_{T k}\left(\tau_{T h}(x)\right)=\tau_{T z}(x) . \tag{3.13}
\end{equation*}
$$

Then $\Omega$ is (generalised) periodic with period $T$.

Proof. Given any $x \in \Omega$ we need to show that $\tau_{T h}(x) \in \Omega \forall h \in \mathbb{Z}^{m}$. If $x \in \Omega$ then there exists some $\tilde{k} \in \mathbb{Z}^{m}$ and some $y \in B$ such that $x=\tau_{T \tilde{k}}(y)$. Then

$$
\tau_{T h}(x)=\tau_{T h}\left(\tau_{T \tilde{k}}(y)\right)=\tau_{T \tilde{z}}(y),
$$

for some $\tilde{z} \in \mathbb{Z}^{m}$ by (3.13). Then $\tau_{T h}(x) \in \tau_{T \tilde{z}}(B) \subseteq \Omega$.

Similarly one can prove that the complement is itself a periodic set.

Lemma 3.1.3. Under the assumptions of Lemma 3.1.2, and assuming in addition that $\tau_{T k}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ is bijective for all $k \in \mathbb{Z}^{m}$ and that for all $x \in \mathbb{R}^{n}$

$$
\begin{equation*}
\exists \widetilde{k} \in \mathbb{Z}^{m} \text { such that } \tau_{T k}^{-1}(x)=\tau_{T \widetilde{k}}(x) \tag{3.14}
\end{equation*}
$$

Then

$$
\Omega^{c}=\mathbb{R}^{n} \backslash \bigcup_{k \in \mathbb{Z}} B_{k},
$$

is (generalised) periodic with period $T$.

Proof. We need to show that for all $x \in \Omega^{c}$ and for all $k \in \mathbb{Z}^{m}, \tau_{T k}(x) \in \Omega^{c}$. By using the definition of the complement of a set and the negation of a logic implication, this is equivalent to showing that

$$
\tau_{T k}(x) \in \Omega \quad \Rightarrow x \in \Omega
$$

Now $\tau_{T k}(x) \in \Omega$ means that $\tau_{T k}(x)=\tau_{T h}(y)$ for some $y \in B$ and for some $h \in \mathbb{Z}^{m}$. Applying the inverse function $\tau_{T k}^{-1}$ to both the sides and combining conditions (3.13) and (4.2) we can conclude

$$
x=\tau_{T k}^{-1}\left(\tau_{T h}(x)\right)=\tau_{T \widetilde{k}}\left(\tau_{T h}(x)\right)=\tau_{T z}(x),
$$

for some $z \in \mathbb{Z}^{m}$; thus $x \in \Omega$.

Remark 3.1.4. The generalised translations given in Definition 3.1.2 always satisfy condition (3.14) (see Lemma 3.1.1) while in general condition (3.13) may not be satisfied (as we will see in some of the following examples).

For the generalised translations defined in Examples 3.1.1 and 3.1.2 both conditions (3.13) and (3.14) are satisfied and thus the lemmas apply. In fact, e.g. for Ex. 3.1.2 translating twice, we get:

$$
\begin{aligned}
& \tau_{T k_{1}}\left(\tau_{T k_{2}}(x, y)\right)=\tau_{T k_{1}}\left(y \sin \left(T k_{2}\right)+x \cos \left(T k_{2}\right),-x \sin \left(T k_{1}\right)+y \cos \left(T k_{2}\right)=\right) \\
= & \left(y \sin \left(T\left(k_{1}+k_{2}\right)\right)+x \cos \left(T\left(k_{1}+k_{2}\right)\right),-x \sin \left(T\left(k_{1}+k_{2}\right)\right)+y \cos \left(T\left(k_{1}+k_{2}\right)\right)\right) .
\end{aligned}
$$

In this case $\tau_{T k}^{-1}=\tau_{-T k}$. Then it is very easy to build periodic sets just translating any given compact and simply connected domain radially. Still most of the time one would get as domain a "fat" ring (see Fig 3.2). To get more interesting examples translating radially we need to select the period $T$ as an angle such that the translations do not overlap (the translations of the set are disjoint) and moreover

$$
\Omega=\bigcup_{k \in \mathbb{Z}} \tau_{T k}(B)=\tau_{T k_{1}}(B) \cup \cdots \cup \tau_{T k_{n}}(B),
$$

for some finite number $n \in \mathbb{N}$. In the above picture for Example 3.1.2 the period will be the angle between any two consecutive centres. (Any smaller translated ball $B$ would also give a periodic set $\Omega$ ).

As already remarked unfortunately the generalised translations given in Definition 3.1.2 do not always satisfy conditions (3.13). This is due to the fact that if we connect 3 points by $\mathcal{X}$-lines in general a $\mathcal{X}$-line connecting the first point to the third point may not exist. We will show this explicitly in the following


Figure 3.5: Here the ball $B$ centred at $(1,0)$ is translated using the rotational geometry through a period $T=\frac{\pi}{4}$.
two examples.

Example 3.1.7. In the case of the Heisenberg group (see Example 3.1.3) assumption (3.13) is not true. In fact, using formula (3.7), we can check

$$
\begin{aligned}
& \tau_{T\left(k_{1}, k_{2}\right)}\left(\tau_{T\left(h_{1}, h_{2}\right)}(x, y, z)\right)= \\
& \left(T\left(k_{1}+h_{1}\right)+x, T\left(k_{2}+h_{2}\right)+y, z+\frac{T\left(k_{1}+h_{1}\right) y-T\left(k_{2}+h_{2}\right) x}{2}+\frac{T^{2}\left(k_{1} h_{2}-k_{2} h_{1}\right)}{2}\right),
\end{aligned}
$$

which is not equal to $\tau_{T\left(z_{1}, z_{2}\right)}(x, y, z)$ for any $z_{1}, z_{2} \in \mathbb{Z}$ and for all $T>$ 0 , as $\frac{T^{2}\left(k_{1} h_{2}-k_{2} h_{1}\right)}{2} \neq 0$ for all $k_{1} h_{2} \neq k_{2} h_{1}$. Nevertheless we will show in Section 4.1 that we can still apply the previous lemmas to build periodic sets in the Heisenberg group by, either using Lie group's translations or applying the generalised translations twice. The two strategies will generate different periodic sets.

Example 3.1.8. In the case of the Grushin plane (see Example 3.1.4) assump-
tion (3.13) is not true. In fact:
$\tau_{T k}\left(\tau_{T h}(\tilde{x})\right)=\left(x+T\left(h_{1}+k_{1}\right), T h_{1} h_{2}+T k_{1} k_{2}+T\left(h_{2}+k_{2}\right) x+T^{2} h_{1} k_{2}+y\right) \neq \tau_{T(k+h)}(\tilde{x})$.
for all $T^{2} h_{1} k_{2} \neq 0$. Therefore there does not exist a $\tilde{k} \in \mathbb{Z}^{2}$ as required in assumption (3.13) due to the extra term $T^{2} h_{1} k_{2}$.

In the next two sections we will study in detail the cases of Hörmander vector fields. In particular we will first study the case of the Heisenberg group and more general Carnot groups, comparing our notion with the usual translation in Lie groups. Then we will look at the case of Grushin spaces which are the real focus of the thesis.

## Chapter 4

## Translations in Grushin spaces, <br> perforated domains and tilings

### 4.1 The case of the Heisenberg group.

The $n$-dimensional Heisenberg group $\mathbb{H}^{n}$ is a step 2 sub-Riemannian geometry defined on $\mathbb{R}^{2 n+1}$, where in addition to the manifold structure one can introduce a Lie group structure. For sake of simplicity we here concentrate on the 1 dimensional case which can be defined as $\mathbb{R}^{3}$ endowed with the group law

$$
\begin{equation*}
(x, y, z) \circ\left(x^{\prime}, y^{\prime}, z^{\prime}\right)=\left(x+x^{\prime}, y+y^{\prime}, z+z^{\prime}+\frac{x y^{\prime}-y x^{\prime}}{2}\right), \tag{4.1}
\end{equation*}
$$

for every $(x, y, z),\left(x^{\prime}, y^{\prime}, z^{\prime}\right) \in \mathbb{H}^{1} \equiv \mathbb{R}^{3}$ (see Example 2.4.3).
Note that due to the non-Euclidean term in the third component, the group law is non commutative. To this Lie group structure, one can associate the left-invariant vector fields already introduced in Example 3.1.3, which generate
the first layer of the stratified Lie algebra while the second layer is generated by the left-invariant vector fields $Z=\left[X_{1}, X_{2}\right]$ (see Example 2.2.1 and Definition 2.4.4). For a more intrinsic definition of the Heisenberg group starting from the Hausdorff-Campbell formula, the left-invariant vector fields and other properties, we refer the reader to [19] and Section 2.4. Using the group structure one can define the following group-translations.

Definition 4.1.1 (Group-translations and group periodicity). Given a point $g=(x, y, z) \in \mathbb{H}^{1}$ and integers $k=\left(k_{1}, k_{2}, k_{3}\right) \in \mathbb{Z}^{3}$, we call group-translations or simply $\mathbb{H}^{1}$-translations the following family of functions:

$$
\begin{equation*}
\tau_{g^{\prime}}^{\mathbb{H}^{1}}(g):=g^{\prime} \circ g \in \mathbb{H}^{1} . \tag{4.2}
\end{equation*}
$$

A set $\Omega \subset \mathbb{H}^{1}$ is said group-periodic or simply $\mathbb{H}^{1}$-periodic if and only if

$$
\begin{equation*}
\tau_{2 k}^{\mathbb{H}^{1}}(\Omega) \subset \Omega \quad \text { for all } k \in 2 \mathbb{Z}^{3} \tag{4.3}
\end{equation*}
$$

The choice of fixing the period $T=2$ is related to the condition (3.13). In fact, the smallest $T>0$ such that the composition of two group translations is still a group translation is 2 . The previous definition has been broadly used in this setting, e.g. see [31] [11].

Lemma 4.1.1 (Property group translations). Consider the group translations in the Heisenberg group defined in (4.2), then the following properties are trivially true:

1. $\tau_{g_{1}}^{\mathbb{H}^{1}} \circ \tau_{g_{2}}^{\mathbb{H}^{1}}=\tau_{g_{1} \circ g_{2}}^{\mathbb{H}^{1}}$,
2. $\tau_{g}^{\mathbb{H}^{1}}$ is a bijective function and $\left(\tau_{g}^{\mathbb{H}^{1}}\right)^{-1}=\tau_{g^{-1}}^{\mathbb{H}^{1}}$.

Then Lemmas 3.1.2 and 3.1.3 both hold true, since $2 k \circ 2 h \in \mathbb{Z}^{m}$ for all
$k, h \in \mathbb{Z}^{m}$ as one can easily check:

$$
2 k \circ 2 h=\left(2 k_{1}+2 h_{1}, 2 k_{2}+2 h_{2}, 2 k_{3}+2 h_{3}+2 k_{2} h_{1}-2 k_{1} h_{2}\right) .
$$

Recall also that $k^{-1}=-k$. Note that, taking $T=1$ we would have the problem that $k \circ h$ does not belong to $\mathbb{Z}^{3}$; thus the choice of $T=2$. We have that $2 k \circ 2 h$ is in $2 \mathbb{Z}^{3}$, not in $\mathbb{Z}^{3}$ due to the fact that $2 \mathbb{Z}^{3}$ is subgroup of the Heisenberg group, however $\mathbb{Z}^{3}$ is not.

Using Lemmas 3.1.2 and 3.1.3 one can build many periodic domains in the Heisenberg groups. For example, taking the simply connected and compact starting set

$$
B=\left\{(x, y, z) \in \mathbb{R}^{3} \left\lvert\, x^{2}+y^{2}+z^{2}=\left(\frac{1}{3}\right)^{2}\right.\right\}
$$

we get the periodic sets $\Omega=\underset{k \in \mathbb{Z}^{3}}{ } B$ and $\Omega^{c}$ shown in Figure 4.1.
As already briefly remarked in Example 3.1.3 in the 1-dimensional Heisenberg group one can also define generalised translations. We need now to investigate the relations between group-translations and generalised translations.

An easy computation shows that:

$$
\begin{equation*}
\tau_{\left(\alpha_{1}, \alpha_{2}\right)}(x, y, z)=\tau_{\left(\alpha_{1}, \alpha_{2}, 0\right)}^{\mathbb{H}^{1}}(x, y, z), \tag{4.4}
\end{equation*}
$$

where $\tau_{\left(\alpha_{1}, \alpha_{2}\right)}$ are the generalised translations defined in (3.7) while $\tau_{\left(\alpha_{1}, \alpha_{2}, 0\right)}^{\mathbb{H}^{1}}$ are the group-translation we define in (4.2).

To recover $\tau_{\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)}^{\mathbb{H}^{1}}$ one should consider the generalised translations w.r.t. to $X, Y$ and $Z=[X, Y]$, but the corresponding curve would not be in general an admissible curve in $\mathbb{H}^{1}$ since the direction $Z$ is not allowed (since it does not belong to the distribution $\mathcal{H}=\operatorname{span}(X, Y)$, see Example 2.2.1 and Definition


Figure 4.1: Both the union of the balls and the compliment set are $\mathbb{H}^{1}$-periodic sets (i.e. in the sense of group periodicity).

### 2.4.4).

Therefore one can see that group-periodicity implies periodicity w.r.t. generalised translations by simply taking $k_{3}=0$ in definition (4.3) while the reverse is obviously not true.

So how can we construct periodic sets in $\mathbb{H}^{1}$ without using the stronger assumption of group periodicity?

Remark 4.1.1. Note that the group translations translate w.r.t points and in $\mathbb{H}^{1}$ are 3 -dimensional. Thus the translations in $\mathbb{H}^{1}$ depend on 3 parameters. The translations along vector fields only depend upon 2 parameters since they are only 2 vector fields, (in fact the distribution in $\mathbb{H}^{1}$ is 2-dimensional only). Still the translation along vector fields coincide with the group translations
with $k_{3}=0$. These are called the "horizontal" group translations and have been used (iterated more than once) to replace the 3-dimensional group translation to study homogenization in [11] and [48] (respectively in $\mathbb{H}^{1}$ in the first paper and in more general carnot groups in the second paper)

Lemma 4.1.2. Let us consider $\alpha, \beta \in \mathbb{R}^{2}$ and the corresponding generalised translations defined in Equation (3.7) (or equivalently horizontal translations in the sense of Equation (4.4)), then there exists $\gamma \in \mathbb{R}^{3}$ such that.

$$
\begin{equation*}
\tau_{\alpha}\left(\tau_{\beta}(x, y, z)\right)=\tau_{\gamma} \mathbb{H}^{1}(x, y, z), \quad \forall(x, y, z) \in \mathbb{H}^{1} \equiv \mathbb{R}^{3} \tag{4.5}
\end{equation*}
$$

Proof. Let's compute the left hand side of (4.5), we get

$$
\begin{gathered}
\tau_{\alpha}\left(\tau_{\beta}(x, y, z)=\tau_{\alpha}\left(\beta_{1}+x, \beta_{2}+y, z+\frac{\beta_{1} y-\beta_{2} x}{2}\right)=\right. \\
=\left(\alpha_{1}+\beta_{1}+x, \alpha_{2}+\beta_{2}, z+\frac{\beta_{1} y-\beta_{2} x}{2}+\frac{\alpha_{1}\left(\beta_{2}+y\right)-\alpha_{2}\left(\beta_{1}+x\right)}{2}\right) \\
=\left(\alpha_{1}+\beta_{1}+x, \alpha_{2}+\beta_{2}+y, z+\frac{\left(\alpha_{1}+\beta_{1}\right) y-\left(\alpha_{2}+\beta_{2}\right) x}{2}+\frac{\alpha_{1} \beta_{2}-\alpha_{2} \beta_{1}}{2}\right) .
\end{gathered}
$$

Now take

$$
\alpha_{1}+\beta_{1}=\gamma_{1}, \alpha_{2}+\beta_{2}=\gamma_{2} \text { and } \frac{\alpha_{1} \beta_{2}-\alpha_{2} \beta_{1}}{2}=\gamma_{3}
$$

and (4.5) is proved.

Therefore the composition of any two generalised translations is a group translation. This holds as it satisfies conditions (3.13) and (3.14)(See Lemma 4.1.1).

So applying twice the generalised translations do we get the same periodic sets that using group-translations? and if not, can we use Lemmas 3.1.2 and 3.1.3 to build periodic sets? In the next two propositions we will show that actually the answer is negative to both the questions. To prove both we will use the same idea; the main problems arise when we restrict the translations to only integer directions. From now on we will consider integer translations of period $T=2$, i.e.

$$
\begin{equation*}
\tau_{2\left(k_{1}, k_{2}\right)}(x, y, z)=\left(2 k_{1}+x, 2 k_{2}+y, z+k_{1} y-k_{2} x\right) \tag{4.6}
\end{equation*}
$$

We now answer our first question.
Lemma 4.1.3. The group-translations defined by (4.2) and the horizontal (or generalised) translations applied twice are not equivalent. This means that it is possible to find $\widetilde{k} \in \mathbb{Z}^{3}$ such that for all $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{Z}$

$$
\tau_{2\left(k_{1}, k_{2}\right)} \circ \tau_{2\left(k_{3}, k_{4}\right)}(x, y, z) \neq \tau_{2 \widetilde{k}}^{\mathbb{H}^{1}}(x, y, z),
$$

where $\tau_{2 k}$ are defined in (4.6) and $\tau_{2 \tilde{k}}^{\mathbb{H}^{1}}$ are defined in (4.2) and for some $(x, y, z) \in \mathbb{H}^{1}$

To prove this Lemma we are going to use a famous Theorem from number theory, namely Diophantine's Theorem.

Proof. Using (4.4) we can deduce

$$
\begin{equation*}
\tau_{2\left(k_{1}, k_{2}\right)} \circ \tau_{2\left(k_{3}, k_{4}\right)}=\tau_{2\left(k_{3}, k_{4}\right) \circ 2\left(k_{1}, k_{2}\right)}^{\mathbb{H}^{1}}=\tau_{2\left(\tilde{k}_{1}, \tilde{k}_{2}, \tilde{k}_{3}\right)}^{\mathbb{H}^{1}} . \tag{4.7}
\end{equation*}
$$

We want to show that such, choosing in a suitable way $\tilde{k}_{1}, \tilde{k}_{2}, \tilde{k}_{3} \in \mathbb{Z}$, then such $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{Z}$ do not exist. For (4.7) to be true we need to be able to
solve the following system:

$$
\left\{\begin{array}{l}
2 \tilde{k}_{1}=2\left(k_{1}+k_{3}\right)  \tag{4.8}\\
2 \tilde{k}_{2}=2\left(k_{2}+k_{4}\right) \\
2 \tilde{k}_{3}=\left(\frac{2 k_{2} 2 k_{3}-2 k_{1} 2 k_{4}}{2}\right)=2\left(k_{2} k_{3}-k_{1} k_{4}\right)
\end{array}\right.
$$

which simplifies to

$$
\left\{\begin{array}{l}
\tilde{k}_{1}=k_{1}+k_{3}  \tag{4.9}\\
\tilde{k}_{2}=k_{2}+k_{4} \\
\tilde{k}_{3}=k_{2} k_{3}-k_{1} k_{4}
\end{array}\right.
$$

Then we can take

$$
k_{1}=k_{3}-\tilde{k_{1}} \quad \text { and } \quad k_{2}=k_{4}-\tilde{k}_{2}
$$

which, substituting these into the expression for $\tilde{k}_{3}$, we find that

$$
\tilde{k}_{3}=\left(k_{4}-\tilde{k}_{2}\right) k_{3}-\left(k_{3}-\tilde{k_{1}}\right) k_{4}=\tilde{k}_{1} k_{4}-\tilde{k}_{2} k_{3} .
$$

Now a solution may exist for all $k_{1}, k_{2}, k_{3}, k_{4} \in \mathbb{R}$, however we need something stronger (i.e. integer solutions). To prove that there are not solutions in $\mathbb{Z}$ we only need to find one counter example.

Since

$$
\begin{equation*}
\tilde{k}_{3}=k_{4} \tilde{k}_{1}-k_{3} \tilde{k}_{2}=k_{4} \tilde{k}_{1}+\tilde{k}_{2}\left(-k_{3}\right) \tag{4.10}
\end{equation*}
$$

we can use number theory to find a criterion for the existence of integer so-
lutions. In particular we will use the following result for the linear Diophantine equation. We recall that a linear Diophantine equation has the form $a x+b y=c$, where $a, b$ and $c$ are given integers.

Theorem 4.1.1 (Diophantine's equation[4]). Given the equation

$$
\begin{equation*}
a x+b y=c, \tag{4.11}
\end{equation*}
$$

let $a, b$ be integers and let $d=\operatorname{gcd}(a, b)$ (where $\operatorname{gcd}(\cdot, \cdot)$ denotes the greatest common denominator between any two integers). The Equation (4.11) has solutions if and only if d divides $c$.

In our case the variables are $x=k_{2}, y=k_{1}$ and $-k_{3}$ are the coefficients are the integers $a=\tilde{k}_{1}, b=\tilde{k}_{2}$ and $c=\tilde{k}_{3}$.

Then if for example we take $\tilde{k}_{1}=6, \tilde{k}_{2}=3$ and $\tilde{k}_{3}=7$, we see that $\operatorname{gcd}\left(\tilde{k}_{1}, \tilde{k}_{2}\right)=$ 3. However this does not divide $7=\tilde{k}_{3}$. Then Diophantine's theorem tells us that there does not exist integer solutions.

Since condition (3.13) is not satisfied, we cannot use Lemmas 3.1.2 and 3.1.3 to build generalised periodic sets in $\mathbb{H}^{1}$. We can still build interesting perforated domains not necessarily periodic as we will do later in the Grushin plane.

### 4.1.1 The case of Carnot groups

In Carnot groups, one can define periodicity by considering the left translations $L_{g}$ : $\mathbb{G} \rightarrow \mathbb{G}$ given by

$$
L_{g}(x)=g \circ x
$$

exactly as in the case of the Heisenberg group.

Definition 4.1.2 (Group-translation and group-periodicity). We indicate the group-translations of period $T$ by $\tau_{T k}^{\mathbb{G}}$ for $k \in \mathbb{Z}^{N}$ and $T \in \mathbb{R}$, i.e. $\tau_{T k}^{\mathbb{G}}: \mathbb{G} \rightarrow \mathbb{G}$ as $\tau_{T k}^{\mathbb{G}}(x)=(T k) \circ x$.

A set $\Omega \subseteq \mathbb{R}^{N} \equiv \mathbb{G}$ is called group-periodic (or simply $\mathbb{G}$-periodic) with period $T$, if and only if,

$$
\tau_{T k}^{\mathbb{G}}(x) \in \Omega, \quad \forall x \in \Omega \text { and } k \in \mathbb{Z}^{N} .
$$

We will fix the period $T=2$ for coherence with the Heisenberg case and the literature on the subject (see [13]) but this does not affect the conclusions.

We now consider the $m$ horizontal left-invariant vector fields which span the first layer of the Lie algebra $\mathfrak{g}_{1}$ (where $m=\operatorname{dim} \mathfrak{g}_{1}$ ). For more details on this point see Section 2.4.6. In canonical coordinate the left-invariant vector fields have the following structure, see [19]:

$$
X_{j}(x)=\frac{\partial}{\partial x_{j}}+\sum_{i=m+1}^{n} a_{i j}(x) \frac{\partial}{\partial x_{j}}, \quad j=1, \ldots, m
$$

In particular we report the following result from [48].

Lemma 4.1.4. Let $\Omega$ be an open subset of a Carnot group $\mathbb{G}$, then for every point $x \in \Omega$ there exists a point $x_{0} \in \mathbb{G}$ and a finite number of group actions generated by elements of the form $(\underline{k}, 0), \underline{k} \in \mathbb{Z}^{m}$ that are applied to $x_{0}$ give $x$. For proof see [48].

We conclude by remarking that to recover the group-translations by using the generalised translations one needs to consider as family of vector fields all the $N$ left-vector fields spanning the whole $\mathfrak{g}$ Lie algebra and not only the horizontal one.

### 4.2 Generalised translation in Grushin spaces

In Example 3.1.4 we have computed the generalised translations in the Grushin plane. Here we report the explicit formulation once more, for the convenience of the reader:

$$
\begin{equation*}
\tau_{\alpha}(x, y)=\left(\alpha_{1}+x, \frac{\alpha_{1} \alpha_{2}}{2}+\alpha_{2} x+y\right) \tag{4.12}
\end{equation*}
$$

Lemma 4.2.1. Let us consider the generalised translations in the Grushin plane defined in (4.12), then we can always find some $\beta, \gamma \in \mathbb{R}^{2}$ such that

$$
\begin{equation*}
\tau_{\beta}\left(\tau_{\gamma}(x, y)\right) \neq \tau_{\alpha}(x, y), \quad \forall \alpha, x, y \in \mathbb{R}^{2} \tag{4.13}
\end{equation*}
$$

Proof. Multiplying out the left hand side of the equation we see that

$$
\begin{gathered}
\tau_{\beta}\left(\gamma_{1}+x, \frac{\gamma_{1} \gamma_{2}}{2}+\gamma_{2} x+y\right)=\left(\beta_{1}+\gamma_{1}+x, \frac{\beta_{1} \beta_{2}}{2}+\beta_{2}\left(\gamma_{1}+x\right)+\frac{\gamma_{1} \gamma_{2}}{2}+\gamma_{2} x+y\right) \\
=\left(\beta_{1}+\gamma_{1}+x, \frac{\beta_{1} \beta_{2}}{2}+\frac{\gamma_{1} \gamma_{2}}{2}+\left(\beta_{2}+\gamma_{2}\right) x+\beta_{2} \gamma_{1}+y\right)
\end{gathered}
$$

Equating terms with the right hand side of (4.13) we see that

$$
\begin{equation*}
\left(\beta_{1}+\gamma_{1}+x, \frac{\beta_{1} \beta_{2}}{2}+\frac{\gamma_{1} \gamma_{2}}{2}+\left(\beta_{2}+\gamma_{2}\right) x+\beta_{2} \gamma_{1}+y\right)=\left(\alpha_{1}+x, \frac{\alpha_{1} \alpha_{2}}{2}+\alpha_{2} x+y\right) . \tag{4.14}
\end{equation*}
$$

For equality to hold $\forall x, y$ in the first component we require that $\beta_{1}+\gamma_{1}=\alpha_{1}$. Equating $x$ terms in the second component we require that $\beta_{2}+\gamma_{2}=\alpha_{2}$. However for equality to hold we also require that $\alpha_{1} \alpha_{2}=\beta_{1} \beta_{2}+\gamma_{1} \gamma_{2}+2 \beta_{2} \gamma_{1}$, which holds only in the case $\beta_{1} \gamma_{2}=\beta_{2} \gamma_{1}$ and thus (4.13) does not hold for all $\beta_{1} \gamma_{2} \neq \beta_{2} \gamma_{1}$.

We have already remarked that to construct generalised periodic sets in the Grushin space is very hard due to the fact that the composition of two generalised translations is not a generalised translation anymore (see Lemma 4.2.1). Still, similarly to the Heisenberg case, if we take the composition of 3 different translations this cannot written in the form of a generalised translation, as you can see in the following lemma.

As in the case for the Heisenberg translations we choose our period to be $T=2$ and only focus on integer translations $\alpha=k \in \mathbb{Z}^{2}$, i.e.
$\tau_{\alpha}(x, y)=\left(\alpha_{1}+x, \frac{\alpha_{1} \alpha_{2}}{2}+\alpha_{2} x+y\right)=\left(2 k_{1}+x, 2 k_{1} k_{2}+k_{2} x+y\right)=\tau_{2 k}(x, y)$.

Lemma 4.2.2. The translation defined in (4.15) has an inverse translation and the inverse translation satisfies property (3.14) .

Proof. Taking the translation along the negative of the vector $k$ we see that

$$
\tau_{-2 k}\left(\tau_{2 k}(x, y)\right)=\tau_{-2 k}\left(2 k_{1}+x, 2 k_{1} k_{2}+2 k_{2} x+y\right),
$$

which means

$$
\left(2 k_{1}-2 k_{1}+x, 2 k_{1} k_{2}-2 k_{2}\left(2 k_{1}+x\right)+2 k_{1} k_{2}+2 k_{2} x+y\right)=(x, y)
$$

Similarly $\tau_{2 k}\left(\tau_{-2 k}(x, y)\right)=(x, y)$. Thus the inverse of $\tau_{2 k}$ is simply $\tau_{-2 k}$.

Remark 4.2.1. From Lemma 4.2 .1 we can see that $\tau_{2 k}$ does not satisfy property (3.14).

### 4.2.1 Translations and rescaling in the Grushin plane

Even though the Grushin spaces are not Carnot groups, it is still possible to define a natural scaling. We show that our new notion of translations behaves well w.r.t natural rescaling defined in the geometry.

Definition 4.2.1. A rescaling $\delta_{\lambda}$ is a natural scaling in the geometry whenever the following property holds: $\forall$ horizontal curves $\gamma:[a, b] \rightarrow \mathbb{R}^{n}$ (see Definition 2.2.4) the rescaled curve defined by

$$
\eta:=\delta_{\lambda}(\gamma),
$$

with $\lambda \in \mathbb{R}$, is still horizontal and the horizontal velocity rescales according to $\lambda$, in the sense that

$$
\alpha^{\eta}=\lambda \alpha^{\gamma},
$$

where $\alpha^{\gamma}$ and $\alpha^{\eta}$ are the $\mathbb{R}^{m}$ valued (measurable) functions such that

$$
\dot{\gamma}(t)=\sum_{i=1}^{m} \alpha_{i}^{\gamma}(t) X_{i}(\gamma(t)), \text { a.e. } \mathrm{t} \in[a, b],
$$

and

$$
\dot{\eta}(t)=\sum_{i=1}^{m} \alpha_{i}^{\eta}(t) X_{i}(\eta(t)), \quad \text { a.e. } \mathrm{t} \in[a, b] .
$$

Note that in the case of Carnot groups a rescaling always exists. This natural rescaling also exists in the Grushin spaces (see Definition 2.5.2).

Example 4.2.1. It is very well known that in the Euclidean case

$$
\tau_{\lambda \alpha}(x)=x+\lambda \alpha .
$$

Now we show how the generalised translations behave well w.r.t the geometrical scaling.

Lemma 4.2.3 (Generalised translation under rescaling). Let us assume that there exists a $\delta_{\lambda}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ such that for all horizontal curves $\gamma$, if the curve $\eta:=\delta_{\lambda}(\gamma)$ satisfies the following rescaling property

$$
\alpha^{\eta}=\lambda \alpha^{\gamma},
$$

then

$$
\tau_{\alpha}\left(\delta_{\lambda}(x)\right)=\delta_{\lambda}\left(\tau_{\alpha}(x)\right)
$$

For proof see [28].

Recalling the Grushin dilations (see Definition 2.5.2) we see that

$$
\delta_{\lambda}(x, y)=\left(\lambda x, \lambda^{2} y\right)
$$

this scaling is natural since it respects horizontal curves and the associated Carnot-Carathéodory distance. In fact, consider an horizontal curve $\gamma$ with horizontal velocity $\alpha=\left(\alpha_{1}, \alpha_{2}\right)$ that means that

$$
\dot{\gamma}(t)=\left(\alpha_{1}(t), \alpha_{2}(t) \gamma_{1}(t)\right) .
$$

(Note that in this case the horizontal velocity is a $\mathbb{R}^{2}$-valued measurable function).

Remark 4.2.2. In the space $\mathbb{R}^{2}$ with the structure of the Grushin plane, if we consider the Euclidean rescaled curve defined as $\eta:=\lambda \gamma$, in general $\eta$ is
not anymore horizontal, while, if we consider the rescaled curve defined as $\xi=\delta_{\lambda}(\gamma)$ then $\xi$ is still horizontal and

$$
\dot{x}_{i}(t)=\left(\lambda \dot{\gamma}_{1}(t), \lambda^{2} \dot{\gamma}_{2}(t)\right)=\lambda\left(\alpha_{1}(t), \alpha_{2}(t) \xi_{1}(t)\right) .
$$

This means that the horizontal velocity of $\xi$ is equal to $\lambda \alpha$, and the horizontal velocity rescaling is exactly the same as the total velocity for the Euclidean case.

This implies that:

$$
d_{C C}\left(\delta_{\lambda}(x, y)\right)=\lambda d_{C C}((x, y))
$$

As seen earlier, even though the Grushin plane is not a Carnot group, it is possible to define a rescaling coherent with the underlying geometry

$$
\begin{equation*}
\delta_{\lambda}(x, y)=\left(\lambda x \cdot \lambda^{2} y\right) . \tag{4.16}
\end{equation*}
$$

Recalling the estimate for the Carnot-Carathéodory distance and the box distance, (see Definition 2.5.1), it is easy to see that the Carnot-Carathéodory distance and the box distance both scale according to dilations defined in the geometry.
4.3. Construction of perforated domains with non overlapping holes in the Grushin plane.

### 4.3 Construction of perforated domains with non overlapping holes in the Grushin plane.

In this section we will use generalised translations to build an interesting class of perforated domains with non overlapping holes in the Grushin plane. We will first show how to find a special class of starting balls which never overlap once translated. Then we will generalise this to diamonds, in this case finding such an optimal class of compact and simply connected sets that can be used to build perforated domains with non overlapping holes.

### 4.4 Construction of Grushin perforated sets by translations of (Euclidean) balls

The following theorem is one of our main results.

Theorem 4.4.1. Consider the set

$$
\Omega:=\mathbb{R}^{2} \backslash \bigcup_{k \in \mathbb{Z}^{2}} B_{k},
$$

where $B_{k}:=\tau_{2 k}(B)$ with $\tau_{2 k}$ integer generalised translation defined in Equation (4.14) and B is a 2-dimensional (Euclidean) closed ball of radius $r$ and centred at some point of the form

$$
\left(a+\frac{1}{2}, b\right), \quad \text { where } a, b \in \mathbb{Z}
$$

Moreover we assume the following condition on the radius:

$$
r<\frac{1}{2 \sqrt{2}}
$$

Then $\Omega$ is a Grushin perforated domain with non overlapping holes.

Proof. All balls considered here are closed (so that $\Omega$ is open). We want to prove that

$$
B_{k} \cap B_{h}=\tau_{2 k}(B) \cap \tau_{2 h}(B)=\emptyset, \quad \forall k, h \in \mathbb{Z}^{2} \quad \text { with } \quad k \neq h
$$

To apply the generalised translations we consider the following coordinates, as follows:

$$
\left\{\begin{array}{l}
x=\widetilde{x}+2 k_{1} \quad \Longleftrightarrow \widetilde{x}=x-2 k_{1}  \tag{4.17}\\
y=\widetilde{y}+2 k_{1} k_{2}+2 k_{2} \widetilde{x} \Longleftrightarrow \widetilde{y}=y+2 k_{1} k_{2}-2 k_{2} x
\end{array}\right.
$$

Instead of proving directly the result we want to give a more constructive proof to highlight why and how the condition on the centre and the critical radius are found. This will also give an idea on how we found the structure of the center and conditions for the radius.

Therefore we start translating the following ball (that does not satisfy the conditions on the centre and radius). In particular we start translating the ball

$$
B=\left\{(x, y) \in \mathbb{R}^{2}:(x-2)^{2}+y^{2} \leq \frac{1}{4}\right\}
$$

Then for $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$ we get

$$
\begin{equation*}
B_{k}=\tau_{2 k}(B)=\left\{(x, y) \in \mathbb{R}^{2}:\left(x-2 k_{1}-2\right)^{2}+\left(y+2 k_{1} k_{2}-2 k_{2} x\right)^{2} \leq \frac{1}{4}\right\} \tag{4.18}
\end{equation*}
$$

4.4. Construction of Grushin perforated sets by translations of (Euclidean) balls
and for $h=\left(h_{1}, h_{2}\right) \in \mathbb{Z}^{2}$

$$
\begin{equation*}
B_{h}=\tau_{2 h}(B)=\left\{(x, y) \in \mathbb{R}^{2}:\left(x-2 h_{1}-2\right)^{2}+\left(y+2 h_{1} h_{2}-2 h_{2} x\right)^{2} \leq \frac{1}{4}\right\} \tag{4.19}
\end{equation*}
$$

We want to show that for any $\left(h_{1}, h_{2}\right) \neq\left(k_{1}, k_{2}\right)$ the intersection is empty.
Note that for all $U, V \in \mathbb{R}$ :

$$
U^{2}+V^{2}=\frac{1}{4} \quad \Longrightarrow \quad U^{2} \leq \frac{1}{4}
$$

which gives the restriction $\frac{-1}{2} \leq U \leq \frac{1}{2}$; so from (4.18) taking $U=x-2 k_{1}-2$ and $V=y+2 k_{1} k_{2}-2 k_{2} x$ we see that

$$
\frac{-1}{2} \leq x-2 k_{1}-2 \leq \frac{1}{2}
$$

which implies

$$
\begin{equation*}
2 k_{1}+\frac{1}{2} \leq x \leq 2 k_{1}+\frac{3}{2} \tag{4.20}
\end{equation*}
$$

whenever $x \in B_{k}$. Similarly using (4.19) one can deduce that the same inequality holds for $h_{1}$, i.e.

$$
\begin{equation*}
2 h_{1}+\frac{1}{2} \leq x \leq 2 h_{1}+\frac{3}{2} . \tag{4.21}
\end{equation*}
$$

We now proceed in following steps.

Step1: We show that the regions defined by (4.20) and (4.21) overlap if and only if $k_{1}=h_{1}$. This implies that $B_{k} \cap B_{h} \neq \emptyset \Longrightarrow k_{1}=h_{1}$.

One implication is obvious, so we need only to show that, if the two regions overlap, that implies $k_{1}=h_{1}$ or, equivalently, that whenever $k_{1} \neq h_{1}$, the two
regions do not overlap.
Assume that $k_{1} \neq h_{1}$ then without loss of generality:

$$
k_{1}>h_{1} \quad \Longrightarrow \quad k_{1}-h_{1} \geq 1 \quad \Longrightarrow \quad 2 k_{1}-2 h_{1} \geq 2
$$

Assume there exist some $x$ satisfying both (4.18) and (4.19), then we have the following chain of inequalities,

$$
\begin{equation*}
x \leq 2 h_{1}+\frac{3}{2}<2 h_{1}+2 \leq 2 k_{1}<2 k_{1}+\frac{1}{2} \leq x \tag{4.22}
\end{equation*}
$$

then $x<x$ which is impossible. This implies the two regions defined in (4.18) and (4.19) intersect only if $k_{1}=h_{1}$, which implies $B_{k} \cap B_{h} \neq \emptyset \Longrightarrow k_{1}=h_{1}$.

Step 2: We now show that the translations of the ball initially centred at the point $\left(\frac{3}{2}, 0\right)$ with radius $r<\frac{1}{2 \sqrt{2}}$ do not intersect.

Consider the two translated balls $B_{k}$ and $B_{h}$ defined, respectively, in (4.18) and (4.19) with $k=\left(k_{1}, k_{2}\right), h=\left(h_{1}, h_{2}\right) \in \mathbb{Z}^{2}$ and lets look for the intersection points.

By Step 1 we can assume $k_{1}=h_{1}$. Then $B_{h}$ can be written as the set of points $(x, y)$ satisfying

$$
\left(x-2 k_{1}-2\right)^{2}+\left(y+2 k_{1} h_{2}-2 h_{2} x\right)^{2} \leq \frac{1}{4} .
$$

4.4. Construction of Grushin perforated sets by translations of (Euclidean)

Then every point $(x, y) \in B_{k} \cap B_{h}$ needs to satisfy the following system:

$$
\left\{\begin{array}{l}
\left(x-2 k_{1}-2\right)^{2}+\left(y+2 k_{1} k_{2}-2 k_{2} x\right)^{2} \leq \frac{1}{4}  \tag{4.23}\\
\left(x-2 k_{1}-2\right)^{2}+\left(y+2 k_{1} h_{2}-2 h_{2} x\right)^{2} \leq \frac{1}{4}
\end{array}\right.
$$

Subtracting the two inequalities we find that for an intersection to exist the following equality must hold

$$
\begin{align*}
& y^{2}+4 k_{1}^{2} k_{2}^{2}+4 k_{2}^{2} x^{2}+4 k_{1} k_{2} y-4 k_{2} x y-8 k_{1} k_{2}^{2} x y^{2} \\
&-y^{2}-4 k_{1}^{2} h_{2}^{2}-4 h_{2}^{2} x^{2}-4 k_{1} h_{2}+4 h_{2} x y+8 k_{1} h_{2}^{2} x=0 . \tag{4.24}
\end{align*}
$$

Rearranging similar terms, Equation (4.24) can be written as

$$
\begin{equation*}
k_{1}^{2}\left(k_{2}^{2}-h_{2}^{2}\right)+\left(k_{2}^{2}-h_{2}^{2}\right) x^{2}-k_{1}\left(k_{2}-h_{2}\right) y-\left(k_{2}-h_{2}\right) x y-2 k_{1}\left(k_{2}^{2}-h_{2}^{2}\right)=0 . \tag{4.25}
\end{equation*}
$$

Note that we can assume $k_{2} \neq h_{2}$ since we are already in the assumption $k_{1}=h_{1}$ (Otherwise we would be trivially comparing the case when we have $\left.\left(k_{1}, k_{2}\right)=\left(h_{1}, h_{2}\right)\right)$. Dividing (4.25) through $k_{2}-h_{2} \neq 0$ we get

$$
\begin{equation*}
\left(k_{2}+h_{2}\right) x^{2}-\left(2 k_{1}\left(k_{2}+h_{2}\right)+y\right) x+\left(k_{1}^{2}\left(k_{2}+h_{2}\right)+k_{1} y\right)=0 . \tag{4.26}
\end{equation*}
$$

To show that Equation (4.26) has real solutions, we consider the equation as a second order equation in the variable $x$ and we compute the discriminant. The discriminant is $\Delta=b^{2}-4 a c$ with $a=\left(k_{2}+h_{2}\right), b=-\left(2 k_{1}\left(k_{2}+h_{2}\right)+y\right)$ and $c=\left(k_{1}^{2}\left(k_{2}+h_{2}\right)+k_{1} y\right)$. Then
$\Delta=4 k_{1}^{2}\left(k_{2}+h_{2}\right)^{2}+4 k_{1} y\left(k_{2}+h_{2}\right)+y^{2}-4 k_{1}^{2}\left(k_{2}+h_{2}\right)^{2}-4 k_{1} y\left(k_{2}+h_{2}\right)=y^{2}$.

Since $\Delta=b^{2}-4 a c=y^{2} \geq 0$ this implies that in general there are real solutions.

Are there any restriction on the starting ball that we can make to prevent the existence of those solutions (and hence prevent a non-empty overlapping)?

To find those restriction we need to look at the exact solutions.

Solving Equation (4.26) we use the standard formula for quadratic equations, thus assuming $k_{2} \neq-h_{2}$ the solutions are:

$$
\begin{equation*}
x_{1,2}=\frac{-\left(-\left(2 k_{1}\left(k_{2}+h_{2}\right)+y\right)\right) \pm \sqrt{y^{2}}}{2\left(k_{2}+h_{2}\right)}=k_{1}+\frac{y \pm|y|}{2\left(k_{2}+h_{2}\right)} . \tag{4.27}
\end{equation*}
$$

We will explore later the case where we have $k_{2}=-h_{2}$. Note that for all $y \in \mathbb{R}$, the solutions expressed in (4.27) can be written as

$$
\left\{\begin{array}{l}
x_{1}=k_{1},  \tag{4.28}\\
x_{2}=k_{1}+\frac{2 y}{2\left(k_{2}+h_{2}\right)}
\end{array}\right.
$$

and the two solutions coincide whenever $y=0$ (in fact in that case we have that the discriminant vanishes).
4.4. Construction of Grushin perforated sets by translations of (Euclidean)

Step 3: We first consider the solution $x=k_{1}$. Remember we are in the case that $k_{1}=h_{1}$ and $r=\frac{1}{2}$.

Substituting $x_{1}=k_{1}$ into (4.23) we get

$$
\begin{equation*}
\left(k_{1}-2 k_{1}-2\right)^{2}+\left(y+2 k_{1} k_{2}-2 k_{2} k_{1}\right)^{2}=\left(k_{1}+2\right)^{2}+y^{2} \leq \frac{1}{4} . \tag{4.29}
\end{equation*}
$$

Now we need to understand for what values of $k_{1}$ do there exist solutions for equation (4.29). Taking the discriminant of (4.29) we find out that whenever $k_{1} \in \mathbb{Z} \backslash\{-2\}$ we have

$$
\begin{equation*}
\Delta=-4\left(\left(k_{1}+2\right)^{2}-\frac{1}{4}\right)<0 \tag{4.30}
\end{equation*}
$$

since $\left(k_{1}+2\right)^{2} \geq 1>\frac{1}{2}$ (which is true as $\left|k_{1}+2\right| \geq 1$ ) implies $\left(k_{1}+2\right)^{2}-\frac{1}{4}>$ $\frac{1}{4}>0$.

This means that $\forall k_{1} \in \mathbb{Z} \backslash\{-2\}$ we have no real solutions.
It remains for us to look at the case $k_{1}=-2$.

Regardless of $k_{2}, h_{2}$, when $k_{2}=-2$ we get

$$
(-2+4-2)^{2}+y^{2}=\frac{1}{4}
$$

which implies

$$
y= \pm \frac{1}{2} .
$$

Then in this case we have proved that all balls overlap and their boundaries intersect always in the same two points $(-2,1 / 2)$ and $(-2,-1 / 2)$ (see Figure 4.2). To prevent intersections occurring along this line we need that the radius


Figure 4.2: The intersections occur along the line $x=k_{1}=-2$ with starting ball $B=\left\{(x, y) \in \mathbb{R}^{2}:(x-2)^{2}+y^{2} \leq \frac{1}{4}\right\}$.
is strictly less than $\frac{1}{2}$. This proves that the radius $\frac{1}{2}$ is so far "optimal": in fact whenever $r>\frac{1}{2}$ there is an overlapping, when $r=\frac{1}{2}$ the overlapping lies only on the boundary and when $r<\frac{1}{2}$ this overlapping does not occur. It remains to prove that no other overlapping occurs in the case $r<\frac{1}{2}$.

Step 4: We now look at the other possible solution $x_{2}$ in (4.28), i.e.

$$
\begin{equation*}
x=k_{1}+\frac{y}{k_{2}+h_{2}}, \quad k_{2} \neq-h_{2} . \tag{4.31}
\end{equation*}
$$

Here we find that we need a further restriction on the radius $r$. Following the ideas in Step 3, we now substitute (4.31) in

$$
\begin{equation*}
\left(x-2 k_{1}-\frac{3}{2}\right)^{2}+\left(y+2 k_{1} k_{2}-2 k_{2} x\right)^{2}=r^{2} \tag{4.32}
\end{equation*}
$$

4.4. Construction of Grushin perforated sets by translations of (Euclidean) balls


Figure 4.3: The intersection along $x=-2$ no longer occurs.
where we have the restriction on $r$ given by $r<\frac{1}{2}$.

Using solution (4.31) into Equation (4.32) we get

$$
\left(k_{1}+\frac{y}{\left(k_{2}+h_{2}\right)}-2 k_{1}-\frac{3}{2}\right)^{2}+\left(y+2 k_{1} k_{2}-2 k_{2}\left(k_{1}+\frac{y}{\left(k_{2}+h_{2}\right)}\right)\right)^{2}=r^{2}
$$

which implies

$$
\left(\frac{y}{\left(k_{2}+h_{2}\right)}-\left(k_{1}+\frac{3}{2}\right)\right)^{2}+\left(y-\frac{2 k_{2} y}{\left(k_{2}+h_{2}\right)}\right)^{2}=r^{2}
$$

which implies

$$
\frac{y^{2}}{\left(k_{2}+h_{2}\right)^{2}}-\frac{2\left(k_{1}+\frac{3}{2}\right)}{\left(k_{2}+h_{2}\right)} y+\left(k_{1}+\frac{3}{2}\right)^{2}+y^{2} \frac{\left(h_{2}-k_{2}\right)^{2}}{\left(k_{2}+h_{2}\right)^{2}}=r^{2},
$$

which finally gives

$$
\begin{equation*}
\left(\frac{\left(h_{2}-k_{2}\right)^{2}+1}{\left(k_{2}+h_{2}\right)^{2}}\right) y^{2}-\frac{2\left(k_{1}+\frac{3}{2}\right)}{k_{2}+h_{2}} y+\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)=0 . \tag{4.33}
\end{equation*}
$$

We now look at Equation (4.33) as a quadratic equation in $y$ and compute the corresponding discriminant $\Delta=b^{2}-4 a c$ with

$$
a=\frac{\left(h_{2}-k_{2}\right)^{2}+1}{\left(k_{2}+h_{2}\right)^{2}}, \quad b=-\frac{2\left(k_{1}+\frac{3}{2}\right)}{\left(k_{2}+h_{2}\right)} \quad \text { and } \quad c=\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right) .
$$

Thus we obtain

$$
\begin{align*}
\Delta=b^{2}-4 a c & =\frac{4\left(k_{1}+\frac{3}{2}\right)^{2}}{\left(k_{2}+h_{2}\right)^{2}}-4\left(\frac{\left(h_{2}-k_{2}\right)^{2}+1}{\left(k_{2}+h_{2}\right)^{2}}\right)\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right) \\
& =-4 \frac{\left(h_{2}-k_{2}\right)^{2}}{\left(k_{2}+h_{2}\right)^{2}}\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)+4 \frac{r^{2}}{\left(k_{2}+h_{2}\right)^{2}}  \tag{4.34}\\
& =-\frac{4}{\left(k_{2}+h_{2}\right)^{2}}\left(\left(h_{2}-k_{2}\right)^{2}\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)-r^{2}\right) .
\end{align*}
$$

As we know that $\left(k_{1}+\frac{3}{2}\right)^{2} \geq \frac{1}{4}, \forall k_{1} \in \mathbb{Z}$, this implies

$$
\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2} \geq \frac{1}{4}-r^{2}>\frac{1}{8}, \quad \forall r^{2}<\frac{1}{8} .
$$

So we get that

$$
\begin{equation*}
\left(h_{2}-k_{2}\right)^{2}\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)-r^{2}>\frac{1}{8}\left(h_{2}-k_{2}\right)^{2}-r^{2}>0 \tag{4.35}
\end{equation*}
$$

$\forall r^{2}<\frac{1}{8}, \forall k_{2}, h_{2} \in \mathbb{Z}, h_{2} \neq k_{2}$.
(We are not interested in the case where $k_{2}=h_{2}$ as this is the trivial case.)

The second inequality in (4.34) holds as

$$
\left(h_{2}-k_{2}\right)^{2} \geq 1 \Longrightarrow \frac{1}{8}\left(h_{2}-k_{2}\right)^{2}>\frac{1}{8} \quad \forall h_{2}, k_{2} \in \mathbb{Z}, \text { with } h_{2} \neq k_{2},
$$

4.4. Construction of Grushin perforated sets by translations of (Euclidean) balls
which gives

$$
\frac{1}{8}\left(h_{2}-k_{2}\right)^{2}-r^{2}>\frac{1}{8}-r^{2}>0, \quad \forall r^{2}<\frac{1}{8} .
$$

Moreover, since

$$
\left(h_{2}-k_{2}\right)^{2}\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)-r^{2}>0
$$

then, for all $r<\frac{1}{4}$ and for all $k_{2}, h_{2} \in \mathbb{Z}$ with $h_{2} \neq k_{2}$, we deduce

$$
\begin{equation*}
-\frac{4}{\left(k_{2}+h_{2}\right)^{2}}\left(\left(h_{2}-k_{2}\right)^{2}\left(\left(k_{1}+\frac{3}{2}\right)^{2}-r^{2}\right)-r^{2}\right)<0 . \tag{4.36}
\end{equation*}
$$

So the discriminant of (4.34) is strictly negative, which means that there exist no real solutions as soon as we assume that

$$
r^{2}<\frac{1}{8} \quad \text { i.e. } \quad r<\frac{1}{2 \sqrt{2}} .
$$



Figure 4.4: We have an intersection along the line $x=-2+\frac{y}{k_{2}+h_{2}}$ where $k_{2}=0$ and $h_{2}=1,-1$, with the starting ball with $r^{2}=\frac{1}{5}$

Step 5: Finally we need to consider the case $k_{2}=-h_{2}$, always under the assumption $k_{1}=h_{1}$.

Looking back at (4.26) with $k_{2}=-h_{2}$, we get

$$
-x y+k_{1} y=\left(k_{1}-x\right) y=0 .
$$

which gives two solutions $x=k_{1}$ and $y=0$. We have already considered the case $x=k_{1}$ for all $k_{2}$ and $h_{2}$ (see Step 3). So we need to look only at the case $y=0$. Are there any two translated balls intersecting each other along the line $y=0$ under the assumption $k_{2}=-h_{2}$ ?

We can assume $k_{2} \neq 0$ since the case $k_{2}=-h_{2}=0$ is trivial (in fact in this case $k_{2}=h_{2}$ ).

Without loss of generality, we can assume $k_{2}<0$. In this case we can show that actually the translated ball never intersects on the line $y=0$.

Substituting $y=0$ in our translated ball with the restricted on the radius $r<\frac{1}{2 \sqrt{2}}$, we find

$$
\left(x-2 k_{1}-\frac{3}{2}\right)^{2}+\left(2 k_{1} k_{2}-2 k_{2} x\right)^{2}=r^{2},
$$

which implies

$$
\left(x-2 k_{1}-\frac{3}{2}\right)^{2}+4 k_{2}^{2}\left(k_{1}-x\right)^{2}=r^{2} .
$$

Using that $A^{2}+B^{2}=r^{2}$ implies always $A^{2} \leq r$ or equivalently $-r \leq A \leq r$, we can deduce, taking $A=x-2 k_{1}-\frac{3}{2}$, that

$$
\begin{equation*}
2 k_{1}+\frac{3}{2}-r \leq x \leq 2 k_{1}+\frac{3}{2}+r, \tag{4.37}
\end{equation*}
$$

4.4. Construction of Grushin perforated sets by translations of (Euclidean)
and using $-r \leq B \leq r$ with $B=2\left|k_{2}\right|\left(k_{1}-x\right)$

$$
\begin{equation*}
k_{1}-\frac{r}{2\left|k_{2}\right|} \leq x \leq k_{1}+\frac{r}{2\left|k_{2}\right|} . \tag{4.38}
\end{equation*}
$$

It remains to show that for all $k_{1} \in \mathbb{Z}$ the two regions identified by (4.37) and (4.38) do not overlap.

To prove the above claim we need again to look at two cases separately.
First we assume $k_{1} \geq 0$ :
In this case

$$
2 k_{1} \geq k_{1} \geq 0 \quad \text { and } \quad r<\frac{1}{2 \sqrt{2}},
$$

which implies

$$
\frac{3}{2}-r>\frac{3}{2}-\frac{1}{2 \sqrt{2}}>\frac{1}{2 \sqrt{2}}>r>\frac{r}{2\left|k_{2}\right|} .
$$

Thus we can conclude

$$
2 k_{1}+\frac{3}{2}-r>k_{1}+\frac{r}{2\left|k_{2}\right|} .
$$

Referring to equations (4.37) and (4.38), then the lower bound of one region is greater than the upper bound of the other region. Hence the two regions do not intersect.

Now it remains to consider the case $k_{1}<0$ that, since $k_{1} \in \mathbb{Z}$, means $k_{1} \leq-1$.
Consider first the case $k_{1}=-1$.

The two regions become the sets of points solving:

$$
\begin{equation*}
-\frac{1}{2}-r \leq x \leq-\frac{1}{2}+r \tag{4.39}
\end{equation*}
$$

and

$$
\begin{equation*}
-1-\frac{r}{2\left|k_{2}\right|} \leq x \leq-1+\frac{r}{2\left|k_{2}\right|} \tag{4.40}
\end{equation*}
$$

The two equations (4.39) and (4.40) imply

$$
-\frac{1}{2}-r>-\frac{1}{2}-\frac{1}{2 \sqrt{2}}>-1+\frac{r}{2\left|k_{2}\right|}
$$

Again the lower bound of one region is greater than the upper bound of the other region. Likewise there is no overlapping in this case.

Finally we consider $k_{1}<-1$.
In this case we have no overlapping for any value of $k_{1} \in \mathbb{Z}$. Hence we can conclude that in all the cases there is no intersection of two balls along the line $y=0$. Therefore we have proved that there exists no intersections for translated balls for any radius $r$ satisfying $r<\frac{1}{2 \sqrt{2}}$ where the centre of the starting ball is of the form

$$
\left(a+\frac{1}{2}, b\right), \quad \text { where } a \in \mathbb{Z}, b \in \mathbb{R}
$$

Note that the critical radius is independent of the values $a$ and $b$ in our starting ball. However the radius is dependent upon the fact that the $x$ component of the centre is at least $\frac{1}{2}$ away from the nearest integer i.e. $x=a+\frac{1}{2}$ with $a \in \mathbb{Z}$.
4.4. Construction of Grushin perforated sets by translations of (Euclidean) balls

Example 4.4.1. The set $\Omega=\mathbb{R}^{2} \backslash \bigcup_{k \in \mathbb{Z}^{2}} \tau_{2 k}(B)$ with

$$
B=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,\left(x-\frac{3}{2}\right)^{2}+y^{2} \leq \frac{1}{9}\right.\right\}
$$

is a perforated domain with non overlapping holes (see Figure 4.6)


Figure 4.5: In the picture we show that translating the ball $B$ the complement of these translations is a perforated domain with non overlapping holes.

Remark 4.4.1. [Optimality of the radius $\bar{r}=\frac{1}{2 \sqrt{2}}$ ] Fix a starting ball with center of the form $\left(a+\frac{1}{2}, b\right)$ where $a, b \in \mathbb{Z}$, then the radius $\bar{r}=\frac{1}{2 \sqrt{2}}$ is optimal to avoid overlapping. In fact, by repeating exactly the computations in the previous proof, one can show that whenever $r<\bar{r}$, the holes generated by translating that ball do not intersect. Whenever $r>\bar{r}$ the holes have nonempty intersection in some internal points and finally when $r=\bar{r}$ the holes intersect only at some points on the boundary.

### 4.5 Translating diamonds.

Next we show results similar to the ones in the previous section but translating diamonds instead of balls. This will allows us to generate a large class of perforated domains with non overlapping holes.


Figure 4.6: In the picture we show that translating the diamond $D$ the complement of these translations is a perforated domain with non overlapping holes, in the sense that the holes intersect only on their boundaries.

Theorem 4.5.1. Consider the set $D$ defined as

$$
D=\left\{(x, y) \in \mathbb{R}^{2}| | x-\frac{3}{2}\left|+|y| \leq \frac{1}{2}\right\} .\right.
$$

We define

$$
D_{k}:=\tau_{2 k}(D), \quad \forall k \in \mathbb{Z}^{2},
$$

where $\tau_{2 k}$ are the translations in the Grushin plane defined in Definition 4.2.1. Then the complementary set

$$
\Omega=\mathbb{R}^{2} \backslash\left\{\cup_{k \in \mathbb{Z}} D_{k}\right\}
$$

is a perforated domain with non-overlapping holes, in the sense that the holes intersect only on their boundaries.

Proof. The proof follows computations similar to the ones for the translated balls in the Theorem 4.4.1. The picture (Fig 4.6) shows the statement graphically, thus we omit the proof.

The previous theorem is very important since it allows us to determine an optimal family of compact and simply connected sets. In fact any set inside the optimal diamond determined above can be translated without the risk that the translations have intersections; while if the set is not enclosed in the diamond the translated sets will overlap (and to prove it formally one needs only to repeat the computation of the proof of Theorem 4.4.1). The diamonds intersect only along the lines defined in the second root of Equation (4.28). The ball defined in Theorem 4.4.1 is ball of maximum radius that can be enclosed within the diamond. This reinforce the optimality of the radius $\bar{r}=\frac{1}{2 \sqrt{2}}$ (see 4.4.1). We will sum up all these remarks in the following corollary.

Corollary 4.5.1. Given any simply connected set A enclosed within the diamond $D$ defined in Theorem 4.5.1, i.e. $\bar{A} \subset \perp$, ( $D$ represents the interior of the set) we define

$$
\Omega=\mathbb{R}^{2} \backslash\left\{\cup A_{k} \mid k \in \mathbb{Z}^{2}\right\},
$$

where $A_{k}=\tau_{2 k}(A)$ and $\tau_{2 k}$ are the translations in the Grushin plane defined in Definition 4.2.1.

Then $\Omega$ is a perforated domain with non overlapping holes.

Using the above result we can construct very interesting examples of perforated domains with non overlapping holes. For example one can always enclose in the
"optimal" diamond $D$ balls rescaling according to the Grushin dilations defined in Definition 2.5.2. An interesting choice is to create perforated domains by translating Grushin-balls w.r.t. the Carnot-Carathéodory distance enclose in $D$ or also box-balls (i.e. balls w.r.t. the box-distance defined in Definition 2.5.1).

### 4.6 Tilings in the Grushin plane.

Tilings are an important tool in many applications, they are in particular extremely useful when studying homogenization problems and integral estimates. (see [31],[11] but also in the proof of the Poincaré inequality in Theorem 5.0.1) First we recall the definition of tiling.

Definition 4.6.1. A tiling is a set of disjoint open subsets $Y_{i} \subseteq \mathbb{R}^{n}$ such that the union of the closure of these subsets cover the whole of $\mathbb{R}^{n}$, i.e.

$$
\bigcup_{i} \bar{Y}_{i}=\mathbb{R}^{n}
$$

A pavage simply satisfies the above definition, however it differs in the sense that the subsets $Y_{i}$ have to be disjoint and the closure of the union has to cover the whole set.

Example 4.6.1. In the Euclidean plane one can build a tiling in many different ways.

The easiest way is to just take the family of semi-closed squares of the form

$$
Q_{k}=\left[k_{1}, 1+k_{1}\right) \times\left[k_{2}, 1+k_{2}\right),
$$

$\forall k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$, i.e.

$$
\mathbb{R}^{2}=\bigcup_{k \in \mathbb{Z}^{2}} Q_{k}
$$



Figure 4.7: A standard Euclidean tiling.

The idea of translating squares (or hypercubes in $\mathbb{R}^{n}$ ) can be applied to build tilings for the Heisenberg group and for Carnot groups by using the grouptranslations. Unfortunately the same idea does not work in the case of the Grushin plane. In fact, as we have shown in the picture of translating diamonds, we are not going to cover the whole space by translating a square (or a diamond) in the Grushin space, w.r.t the translations defined in Definition 3.10 .

In the Euclidean space, tilings (or nets) can be defined also by using the holes of perforated domains. In fact in the case of periodic spherical holes, one can build a tiling by simply connecting the centres of adjacent holes (if the holes are not periodic or not spherical a similar construction can still be developed for example by considering the baricenters of adjacent holes). We show this in the following example.

Example 4.6.2. Let us consider the following perforated domain:

$$
\Omega=\mathbb{R} \backslash \bigcup_{k \in \mathbb{Z}^{2}} B_{k}
$$

where $B_{k}:=B+k$ and $B=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\, x^{2}+y^{2} \leq \frac{1}{2}\right.\right\}$.


Figure 4.8: Periodic spherical holes in the Euclidean plane.

Taking the centre of each sphere $B_{k}$, we can build a tiling by joining adjacent centres, as in the following picture:


Figure 4.9: A tiling for the Euclidean plane built using the centres of periodic spehrical holes.

Note that in this way we get a partition of $\mathbb{R}^{2}$ which is coherent with the geometry of the perforated set. This idea is very general and it can be applied to construct a pavage in the Heisenberg group (as in Figure 4.1). We can apply the same idea to the perforated domains with non-overlapping holes built in Section 2.5 to construct a tiling in the Grushin plane.

In the case of the Euclidean translated holes we can easily show that the distance between adjacent holes is uniformly bounded and so is the diameter
of each set of the tiling. Let us be more precise. By adjacent holes we mean the following.

Definition 4.6.2. Given a simply connected and compact hole $B \subset \mathbb{R}^{n}$ and consider the translated hole $B_{k}=\tau_{k}(B)$ for some $k \in \mathbb{Z}$, the adjacent holes to $B_{k}$ are the holes $B_{h}$ such that $h_{i}=k_{i}$ for all $i \in\{1, \ldots, n\} \backslash\{j\}, j \in\{1, \ldots, n\}$ and $h_{j}=k_{j+1}$ or $h_{j}=k_{j-1}$.

So in the case $n=2$ the adjacent holes for $B_{k}=B_{\left(k_{1}, k_{2}\right)}$ are 4 and they are given by the balls $B_{\left(k_{1}+1, k_{2}\right)}, B_{\left(k_{1}-1, k_{2}\right)}, B_{\left(k_{1}, k_{2}+1\right)}$ and $B_{\left(k_{1}, k_{2}-1\right)}$.


Figure 4.10: Euclidean example: the diagonal of the squares represent the diameter of the cell.

The size of the tiling is the maximum distance between two adjacent holes. The shortest length on which connects two holes (on the boundary) can be easily estimated by looking at the distance between the centres of the holes of the form $B_{\left(k_{1}, k_{2}\right)}$ and $B_{\left(k_{1}+2, k_{2}+2\right)}$. In both the tilings we have built in the Euclidean case it is trivial to see that the sets (the squares in this case) are all uniformly bounded since the shortest length connecting points on circles
$B_{\left(k_{1}, k_{2}\right)}$ and $B_{\left(k_{1}+2, k_{2}+2\right)}$ is denoted by $d$ and is always equal to

$$
d=\sqrt{8}-\frac{2}{\sqrt{2}}=2 \sqrt{2}-\sqrt{2}=\sqrt{2}
$$

where $\sqrt{8}$ is the length of the diagonal connecting the circles and we subtract to that twice the length of the radius. If we try to connect centres of the four adjacent holes of our Grushin perforated domain to create a tiling of the space $\mathbb{R}^{2}$ we come across a problem. Unfortunately the sets of the tiling are now quadrilaterals which do not have anymore a uniformly bounded diameter, neither w.r.t. the Euclidean distance nor w.r.t the Carnot-Carathéodory distance, as we will see next.


Figure 4.11: The tiling of $\mathbb{R}^{2}$ connecting 4 adjacent holes by using the Grushin translation.

Before giving the two results we need to introduce formally the following definitions.

Given a tiling $\left\{Y_{k}\right\}$ of $\mathbb{R}^{n}$ we call the $Y_{k}$, cell of the tiling.

Definition 4.6.3. The Euclidean diameter of the cells $Y_{k}$ of a tiling is the number

$$
d_{E}^{k}=\operatorname{diam}_{E}\left(Y_{k}\right)=\max \left\{\mid p-q \| p, q \in \bar{Y}_{k}\right\} .
$$

Definition 4.6.4. The Carnot-Carathéodory (or CC) diameter is of the cells $Y_{k}$ of a tiling is the number

$$
d_{C C}^{k}=\operatorname{diam}_{C C}\left(Y_{k}\right)=\max \left\{d_{C C}(p, q) \mid p, q \in \bar{Y}_{k}\right\} .
$$

Proposition 4.6.1. Define the set

$$
\begin{equation*}
B_{\frac{1}{3}}=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,\left(x-\frac{3}{2}\right)^{2}+y^{2} \leq\left(\frac{1}{3}\right)^{2}\right.\right\} \tag{4.41}
\end{equation*}
$$

and consider the translated balls $B_{k}:=\tau_{2 k}\left(B_{\frac{1}{3}}\right)$ by using the generalised Grushin translations defined in (4.15), for $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$.

Given the tiling in $\mathbb{R}^{2}$ (see Fig 4.11) built by connecting the centers of the 4 adjacent balls, the Euclidean diameter $d_{E}^{k}$ is not uniformly bounded w.r.t. $k$.

Proof. Let $P_{k}$ be the centre of the hole for $k=\left(k_{1}, k_{2}+1\right)$ and let $Q_{k}$ be the centre of the hole for $k=\left(k_{1}, k_{2}\right)$ (i.e. two vertically adjacent holes). This means

$$
P_{k}=\left(\frac{3}{2}+2 k_{1}, 2 k_{1}\left(k_{2}+1\right)+3\left(k_{2}+1\right)\right)
$$

and

$$
Q_{k}=\left(\frac{3}{2}+2 k_{1}, 2 k_{1} k_{2}+3 k_{2}\right)
$$

The Euclidean distance between $P_{k}$ and $Q_{k}$ is

$$
\left|2 k_{1}\left(k_{2}+1\right)+3\left(k_{2}+1\right)-\left(2 k_{1} k_{2}+3 k_{2}\right)\right|=\left|2 k_{1}+3\right|,
$$

which becomes unbounded as $k_{1} \rightarrow \infty$.

Proposition 4.6.2. Define the set

$$
B_{\frac{1}{3}}=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,\left(x-\frac{3}{2}\right)^{2}+y^{2} \leq\left(\frac{1}{3}\right)^{2}\right.\right\}
$$

and consider the translated $B_{k}:=\tau_{2 k}\left(B_{\frac{1}{3}}\right)$ by using the generalised Grushin translations defined in (4.15), for $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$.

Given the tiling in $\mathbb{R}^{2}$ (see Fig 4.11) by connecting the centers of the 4 adjacent balls, the CC diameter $d_{C C}^{k}$ is not uniformly bounded w.r.t. $k$.

Proof. Let $\tilde{P}_{k}$ be the centre of the hole for $k=\left(k_{1}+1, k_{2}\right)$ and let $\tilde{Q}_{k}$ be the centre of the hole for $k=\left(k_{1}, k_{2}\right)$ (i.e. two horizontally adjacent holes). This means

$$
\tilde{P}_{k}=\left(2\left(k_{1}+1\right)+\frac{3}{2}, 2\left(k_{1}+1\right) k_{2}+3 k_{2}\right)
$$

and

$$
\tilde{Q}_{k}=\left(2 k_{1}+\frac{3}{2}, 2 k_{1} k_{2}+3 k_{2}\right) .
$$

We now use Theorem 2.5.1 to estimate the Carnot-Carathéodory distance be-
tween $\tilde{P}_{k}$ and $\tilde{Q}_{k}$. Let $x=2\left(k_{1}+1\right)+\frac{3}{2}, \xi=2 k_{1}+\frac{3}{2}, y=2\left(k_{1}+1\right) k_{2}+3 k_{2}$ and $\eta=2 k_{1} k_{2}+3 k_{2}$. Then we have that $|x-\xi|=2$ and $|y-\eta|=2 k_{2}$. Substituting in the formula for Theorem 2.5.1 we get
$d_{C C}(p, q) \leq|2|+\frac{\left|2 k_{2}\right|}{\left|2\left(k_{1}+1\right)+\frac{3}{2}\right|} \leq c d_{C C}(p, q), \quad$ if $\left|2\left(k_{1}+1\right)+\frac{3}{2}\right|^{2} \geq\left|2 k_{2}\right|$,
and

$$
\begin{equation*}
d_{C C}(p, q) \leq|2|+\left|2 k_{2}\right|^{\frac{1}{2}} \leq c d_{C C}(p, q), \quad \text { if }\left|2\left(k_{1}+1\right)+\frac{3}{2}\right|^{2}<\left|2 k_{2}\right|, \tag{4.43}
\end{equation*}
$$

for fixed $k_{1}$ there exists sufficient enough large $k_{2}$ such that

$$
\left|2\left(k_{1}+1\right)+\frac{3}{2}\right|^{2}<\left|2 k_{2}\right|
$$

holds. Now from the second part of the inequality (4.43) the Box distance multiplied by a constant is a bound for $|2|+\left|2 k_{2}\right|^{\frac{1}{2}}$ from below. If $k_{2}$ tends towards infinity then $|2|+\left|2 k_{2}\right|^{\frac{1}{2}}$ also tends towards infinity, hence the box distance explodes and so $d_{C C}$.

Nevertheless we manage to create a tiling of the Grushin plane by using our perforated domains such that CC diameter of the cells is uniformly bounded. In particular we want a tiling with the property that always at least two adjacent holes are partially contained in each cell of the tiling. This will be extremely useful if we want to use this geometrical partition of $\mathbb{R}^{2}$ to prove the Poincaré inequality for the perforated domains with non overlapping holes built in Theorem 4.4.1.

In the next picture we illustrate our tiling.


Figure 4.12: A special rectangular tiling of the Grushin plane.

We can actually prove that, even if the Euclidean diameter of the cells of this tiling is still not uniformly bounded (see Proposition 4.6.3), the CarnotCarathéodory diameter is instead uniformly bounded (see Theorem 4.6 .1 below).

Proposition 4.6.3. Define the set

$$
B_{\frac{1}{3}}=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,\left(x-\frac{3}{2}\right)^{2}+y^{2} \leq\left(\frac{1}{3}\right)^{2}\right.\right\},
$$

and consider the translated $B_{k}:=\tau_{2 k}\left(B_{\frac{1}{3}}\right)$ by using the generalised Grushin translations defined in (4.15), for $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$.

Given the tiling in $\mathbb{R}^{2}$ (see Fig 4.12) by connecting the centers of the 2 vertically adjacent balls, the Euclidean diameter $d_{E}^{k}$ is unbounded in $k$.

Proof. The proof is exactly the same as for Proposition 4.6.1, in fact there we have used the Euclidean distance between vertically adjacent holes.

The following result is one of the main results of the thesis.

Theorem 4.6.1. Define the set

$$
B_{\frac{1}{3}}=\left\{(x, y) \in \mathbb{R}^{2} \left\lvert\,\left(x-\frac{3}{2}\right)^{2}+y^{2} \leq\left(\frac{1}{3}\right)^{2}\right.\right\}
$$

and consider the translated $B_{k}:=\tau_{2 k}\left(B_{\frac{1}{3}}\right)$ by using the generalised Grushin translations defined in (4.15), by for $k=\left(k_{1}, k_{2}\right) \in \mathbb{Z}^{2}$.

Given the tiling in $\mathbb{R}^{2}$ (see Fig 4.12) built by connecting the centers of 2 vertically adjacent balls, the Carnot-Carathéodory diameter $d_{E}^{k}$ is uniformly bounded w.r.t $k \in \mathbb{Z}^{2}$.

Proof. To bound the Carnot-Carathéodory diameter we consider the distance between centers of the balls $B_{\left(k_{1}, k_{2}\right)}$ and $\left.B_{\left(k_{1}, k_{2}+1\right)}\right)$ (recall $B_{k}=\tau_{k}(B)$. Let us consider

$$
P_{k}=\left(\frac{3}{2}+2 k_{1}, 2 k_{1}\left(k_{2}+1\right)+3\left(k_{2}+1\right)\right)
$$

and

$$
Q_{k}=\left(\frac{3}{2}+2 k_{1}, 2 k_{1} k_{2}+3 k_{2}\right)
$$

We now use Theorem 2.5.1 to estimate the CC-distance by the box distance. Take $\lambda=1$ and $x=\frac{3}{2}+2 k_{1}, \xi=\frac{3}{2}+2 k_{1}, y=2 k_{1}\left(k_{2}+1\right)+3\left(k_{2}+1\right)$ and $\eta=2 k_{1} k_{2}+3 k_{2}$, then

$$
\begin{equation*}
d_{C C}\left(P_{k}, Q_{k}\right) \leq \frac{\left|2 k_{1}+3\right|}{\left|2 k_{1}+\frac{3}{2}\right|}, \quad \text { if } \quad\left|2 k_{1}+\frac{3}{2}\right|^{2} \geq\left|2 k_{1}+3\right| \tag{4.44}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{C C}\left(P_{k}, Q_{k}\right) \leq\left|2 k_{1}+3\right|^{\frac{1}{2}}, \quad \text { if }\left|2 k_{1}+\frac{3}{2}\right|^{2}<\left|2 k_{1}+3\right| . \tag{4.45}
\end{equation*}
$$

Note that there is no dependence on $k_{2}$.
We now find the restriction on $k_{1}$ for (4.44) to be applied, i.e. we look for $k_{1} \in \mathbb{Z}$ such that

$$
\begin{equation*}
\left|2 k_{1}+\frac{3}{2}\right|^{2} \geq\left|2 k_{1}+3\right| \tag{4.46}
\end{equation*}
$$

to hold true. First consider the case $2 k_{1}+3 \geq 0$. Hence we have that $k_{1} \geq \frac{-3}{2}$. The inequality (4.46) becomes

$$
4 k_{1}^{2}+6 k_{1}+\frac{9}{4} \geq 2 k_{1}+3
$$

which implies

$$
4 k_{1}^{2}+4 k_{1}-\frac{3}{4} \geq 0
$$

The polynomial $4 k_{1}{ }^{2}+4 k_{1}-\frac{3}{4}$ has roots $-\frac{1}{2} \pm \frac{\sqrt{7}}{4}$ hence we have that $4 k_{1}{ }^{2}+$ $4 k_{1}-\frac{3}{4} \geq 0$ is true $\forall k_{1}<-\frac{1}{2}-\frac{\sqrt{7}}{4}$ and $\forall k_{1}>-\frac{1}{2}+\frac{\sqrt{7}}{4}$. This is considering the case $k_{1} \geq \frac{-3}{2}$.

We now consider the case that $2 k_{1}+3 \leq 0$. Hence we have that $k_{1} \leq \frac{-3}{2}$.

In this case the inequality (4.46) becomes

$$
4 k_{1}^{2}+6 k_{1}+\frac{9}{4} \geq-2 k_{1}-3
$$

which implies

$$
4 k_{1}^{2}+8 k_{1}+\frac{21}{4} \geq 0
$$

which is positive for all $k_{1}$, i.e. for all $k_{1} \leq \frac{-3}{2}$.


Figure 4.13: The graph of the polynomial $4 k_{1}^{2}+4 k_{1}-\frac{3}{4}$

The two integers that lie outside this region where (4.46) holds are $-1,0$. So for all $k_{1} \neq 0,-1$ we can use (4.44) to get

$$
d_{C C}\left(P_{k}, Q_{k}\right) \leq d_{b o x}\left(P_{k}, Q_{k}\right)=\frac{\left|2 k_{1}+3\right|}{\left|2 k_{1}+\frac{3}{2}\right|} .
$$

Note that

$$
\frac{\left|2 k_{1}+3\right|}{\left|2 k_{1}+\frac{3}{2}\right|} \leq \max _{k \in \mathbb{Z}} \frac{\left|2 k_{1}+3\right|}{\left|2 k_{1}+\frac{3}{2}\right|}=2
$$



Figure 4.14: The graph of $f(t)=\frac{|2 t+3|}{\left|2 t+\frac{3}{2}\right|}$.

Then Theorem 2.5.1 shows that

$$
d_{C C}\left(P_{k}, Q_{k}\right) \leq 2, \quad \forall k_{1} \in \mathbb{Z} \backslash\{-1,0\} \text { and } k_{2} \in \mathbb{Z}
$$

Then for $k_{1}=-1$ and $k_{2}=0$, we consider

$$
d_{C C}\left(P_{k}, Q_{k}\right) \leq\left|2 k_{1}+3\right|^{\frac{1}{2}}
$$

The right hand side of the inequality is equal to 1 for $k_{1}=-1$ and is equal to $\sqrt{3}$ for $k_{1}=0$.

Thus we sum up the following uniform bound:

$$
\begin{equation*}
d_{C C}\left(P_{k}, Q_{k}\right) \leq \max \{2, \sqrt{3}, 1\}=2, \quad \forall k_{1}, k_{2} \in \mathbb{Z} \tag{4.47}
\end{equation*}
$$

This implies that the Carnot-Carathéodory diameter of the cells of our tiling are uniformly bound by 2 .

From Theorem 4.6.1 it follows the result below, which will be key in the proof of the Poincaré inequality in Chapter 5.

Corollary 4.6.1. Consider the perforated domain constructed in the Grushin plane by translating the balls given in Theorem 4.4.1. We can prove that, given a point $(x, y) \in \mathbb{R}^{2}$, the Carnot-Carathéodory distance between the point $(x, y)$ and the boundary of the nearest hole is uniformly bounded.

Proof. This follows from the fact $\left\{Y_{k}\right\}$ constructed above is a tiling of $\mathbb{R}^{2}$, i.e. $\bigcup_{k} \bar{Y}_{k}=\mathbb{R}^{2}$. Then given $p \in \mathbb{R}^{2}, \exists \bar{k}$ such that $p \in Y_{\bar{k}}$ and the $C C$-distance between $p$ and the nearest hole $B_{k}$ is smaller than the $\operatorname{diam}_{C C}\left(Y_{k}\right)$. Thus by using Theorem 4.6 .1 it is uniformly bounded w.r.t. $k \in \mathbb{Z}^{2}$.

## Chapter 5

## Poincaré Inequality

Our plan for the future is to study homogenization for perforated domains in Grushin spaces (starting from the Grushin plane). These problems are very hard and challenging. The first step in this direction is to prove that there holds true some Poincaré inequality in perforated domains as the one constructed in Theorem 4.4.1.

Poincaré inequalities has been extensively studied in this setting. Franchi, Gutiérrez and Wheeden [30] comprehensively studied the case in bounded subset of the Grushin plane. The inequalities required for homogenization have been proven in the Heisenberg setting (see [42]). D'Ambrosio in his paper proves a Poincaré inequality for the a stronger class of functions (namely $C_{0}^{1}$ ) with Dirichlet boundary condition (see [25]).

The difficulty here lies on the fact that for our aimed applications to homogenization problems, we need a Poincaré inequality where the constant can be uniformly bounded independently on the holes.

The idea to obtain that is to use the tilings constructed in Theorem 4.6.1 to create a suitable partition of $\mathbb{R}^{2}$.

We here give an idea on how to prove the Poincaré inequality in our setting. The result will be published in [43].

Let $\hat{\Omega}=\mathbb{R}^{2} \backslash \bigcup_{k \in \mathbb{Z}^{2}} B_{k}$.

We call a foliation of $D_{k}$ (the rectangular partition in Fig.4.12) by curves admissible if it satisfied the following definition:

Definition 5.0.1. (Admissible foliation) A family of curves $\gamma_{\theta}(t)$ is called an admissible if and only if
1.

$$
\begin{aligned}
\gamma_{\theta}(0) & =p(\theta) \\
\dot{\gamma}_{\theta}(t) & =\alpha(\theta, t) X_{1}\left(\gamma_{\theta}(t)\right)+\beta(\theta, t) X_{2}\left(\gamma_{\theta}(t)\right)
\end{aligned}
$$

where $\alpha, \beta$ and $p$ are at least $C^{1}$.
2. $p(\theta)$ defines a bijection from $[0,1]$ to $\partial B_{k} \cap \overline{D_{k}}$.
3. $\varphi:[0,1] \times[0,1] \rightarrow \bar{D}_{k}:(\theta, t) \mapsto \gamma_{\theta}(t)$ is a $C^{1}$-diffeomorphism, i.e. a change of coordinates.
4. There exists $C_{1}>0$ such that

$$
\sup _{[0,1] \times[0,1]}\left(\alpha(\theta, t)^{2}+\beta(\theta, t)^{2}\right)<C_{1} .
$$

5. With $J(\varphi):=\operatorname{det}(D \varphi)$ there exists a constant $C_{2}>0$ such that the
diffeomorphism $\varphi$ satisfies

$$
\sup _{0 \leq s \leq t \leq, 1} \frac{|J(\varphi(\theta, t))|}{|J(\varphi(\theta, s))|}<C_{2} .
$$

We have the following result:

Theorem 5.0.1. Suppose that for each $D_{k}$ as pictured in Fig.4.12 there exists an admissible foliation such that the constants $C_{1}>0$ and $C_{2}>0$ in Definition 5.0.1 are bounded uniformly in $k$. Then there exists a constant $C>0$ which does depends only on $C_{1}$ and $C_{2}$ such that

$$
\int_{\hat{\Omega}} u^{2}(x) d x \leq C \int_{\hat{\Omega}}\left|D_{\mathcal{X}} u(x)\right|^{2} d x,
$$

for all Lebesgue-measurable functions $u: \Omega \rightarrow \mathbb{R}$ such that the right hand side is finite.

Remark 5.0.1. Note that this inequality is not trivial, as the Euclidean distance of a point from the nearest hole explodes and the domain $\Omega$ is unbounded.

To show that the required foliations exist: Consider the polar coordinates (i.e. ellipsoidal coordinates) for a suitable $0<\epsilon \ll 1$

$$
\varphi(\theta, t)=\binom{(\epsilon+t) \cos \theta}{(\epsilon+t) k_{1} \sin (\theta)}
$$

on $D_{k}$. As $k_{1} \leq x \leq k_{1}+1$, we can find bounded $\alpha$ and $\beta$ which realise this for some suitable $0<\epsilon \ll 1$. Alternatively, we can replace the holes in the definition of $\Omega$ by the $\epsilon$-ellipse related to this map, extend the function by zero up to the ellipse and show the claim for this modified geometry. More details will be given in [43].

Proof of the Theorem 5.0.1: We write $\Omega=\bigcup_{k \in \mathbb{Z}^{2}} B_{k}$ and show the estimate separately for each $D_{k}$. By a change of variables

$$
I:=\int_{D_{k}}(u(x))^{2} d x=\int_{0}^{1} \int_{0}^{1}(u(\theta, t))^{2}|J(\varphi(\theta, t))| d \theta d t,
$$

and as $u(\theta, 0)=0$ we get by the definition of $\varphi$ and the Fundamental Theorem of Calculus, that

$$
\begin{aligned}
I & =\int_{0}^{1} \int_{0}^{1}(u(\theta, t)-u(\theta, 0))^{2} \mid J(\varphi(\theta, t) \mid d \theta d t \\
& =\int_{0}^{1} \int_{0}^{1}\left(\int_{0}^{t}\left(\dot{\gamma}_{\theta}(s) \cdot D U(\theta, s)\right) d s\right)^{2}|J(\varphi(\theta, t))| d \theta d t \\
& \leq \int_{0}^{1} \int_{0}^{1}\left(\int_{0}^{t}\left[\left|\dot{\gamma}_{\theta}(s)\right|^{2}\right] d s\right)\left(\int_{0}^{t}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2} d s\right)|J(\varphi(\theta, t))| d \theta d t
\end{aligned}
$$

where for the last inequality we have used the Cauchy-Schwartz inequality.
Using the admissibility of the foliation and the fact that the integrands are nonnegative, we estimate

$$
\begin{aligned}
I & \leq \int_{0}^{1} \int_{0}^{1}(\int_{0}^{t}[\underbrace{(\alpha(s))^{2}+(\beta(s))^{2}}_{\leq C_{1}}] d s)\left(\int_{0}^{t}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2} d s\right)|J(\varphi(\theta, t))| d \theta d t \\
& \leq C_{1} \int_{0}^{1} \int_{0}^{1} \int_{0}^{1}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2} \frac{|J(\varphi(\theta, s))|}{|J(\varphi(\theta, s))|}|J(\varphi(\theta, t))| d s d \theta d t \\
& =C_{1} \int_{0}^{1}(\int_{0}^{1} \int_{0}^{1}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2}|J(\varphi(\theta, s))| \underbrace{\frac{|J(\varphi(\theta, t))|}{|J(\varphi(\theta, s))|}}_{\leq C_{2}} d s d \theta) d t \\
& \leq C_{1} C_{2}\left(\int_{0}^{1} d t\right) \int_{0}^{1} \int_{0}^{1}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2}|J(\varphi(\theta, s))| \\
& =C_{1} C_{2} \int_{0}^{1} \int_{0}^{1}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2}|J(\varphi(\theta, s))| d s d \theta=C_{1} C_{2} \int_{D_{k}}\left|D_{\mathcal{X}} u(\theta, s)\right|^{2} d x,
\end{aligned}
$$

where the last equality comes again from the change of variables defined by $\varphi$.

## Chapter 6

## Open problems and

## Applications to Homogenization

The work carried out so far has highlighted the difficulties of working in these very degenerate geometries where there is a lack of any Lie group structure. Our new idea to induce translations coherent with the manifold structure (by using a special class of admissible paths, namely $\mathcal{X}$-lines) has many possible applications.

It already allowed us to build interesting structures as perforated domains with non-overlapping holes (see Section 4.3) and non-trivial nets (or tilings) whose size is uniformly bounded w.r.t. the Carnot-Carathéodory distance associated to the geometry.

This leaves us with many interesting open problems where further investigations are required. In particular, at our knowledge, currently not a single homogenization result has been proved in Grushin-type geometries. We then used these geometrical structures to prove the Poincaré inequality

The main open problem we want to investigate in the near future is to derive homogenization results for subelliptic PDEs (e.g. the sub-Laplacian) in the perforated domains in this geometric setting, in line with the results known in the standard Euclidean setting and in Carnot groups. This is actually also the motivation behind the idea of constructing the sets illustrated in the previous chapters.

In particular we want to generalise Cioranenscu-Murat's celebrated paper $A$ Strange Term Coming From Nowhere [22], we here briefly report their result.

Let $\Omega$ be an open bounded set of $\mathbb{R}^{n}$. Consider for every $\varepsilon>0$, closed subsets $T_{i}^{\varepsilon}, 1 \leq i \leq n(\epsilon)$, which are the "hole". The domain $\Omega$ is defined by removing the holes $T_{i}^{\varepsilon}$ from $\Omega$, that is

$$
\begin{equation*}
\Omega_{\varepsilon}=\Omega \backslash \bigcup_{i=1}^{n(\varepsilon)} T_{i}^{\varepsilon} . \tag{6.1}
\end{equation*}
$$

Let $f \in L^{2}(\Omega)$ and consider the boundary value for the Poisson equation with homogeneous Dirichlet boundary condition in $\Omega_{\varepsilon}$ :

$$
\left\{\begin{array}{l}
\Delta u_{\varepsilon}=f \in L^{2}(\Omega), \quad x \in \Omega_{\varepsilon}, \\
u_{\varepsilon}=0 \text { on } \partial \Omega_{\varepsilon},
\end{array} \quad u \in H_{0}^{1}\left(\Omega_{\varepsilon}\right) . ~ \$\right.
$$

Cioranenscu-Murat use a Poincaré inequality and a suitable extension of the operator $u_{\varepsilon}$ to prove weak convergence results for solutions $u_{\varepsilon}$ of the effective equation set in $\Omega$ i.e. the domain without holes.

In the Heisenberg group similar questions have been answered by Franchi and Tesi in [31].

We next write the result from [31]. Consider $\Omega \subseteq \mathbb{H}^{n} \simeq \mathbb{R}^{2 n+1}$ a bounded open set, then we define a sequence $\Omega_{\varepsilon}$ of periodically perforated subdomains as follows:

$$
\Omega_{\varepsilon}=\left\{p \in \Omega: \chi\left(\delta_{\frac{1}{\varepsilon}}(p)\right)=1\right\},
$$

where $\chi$ is the standard characteristic function on the set $\Omega$

Franchi and Tesi consider the problem

$$
\begin{cases}\operatorname{div}_{\mathfrak{X}}\left(A\left(\delta_{\frac{1}{\varepsilon}}(x)\right) \nabla_{o} u_{\varepsilon}\right)=f \in L^{2}(\Omega), & x \in \Omega_{\varepsilon}, \\ \frac{\partial u_{\epsilon}}{\partial n}=0 \quad \text { on } \partial \Omega \backslash \partial \Omega_{\varepsilon}, & \\ u_{\varepsilon}=0, & \text { on } \partial \Omega \cap \partial \Omega_{\varepsilon}\end{cases}
$$

where $\nabla_{o}$ and $d i v_{\mathfrak{X}}$ are the horizontal gradient and divergence in $\mathbb{H}^{n}$ respectively, and $\delta_{\frac{1}{\varepsilon}}$ are the dilations in $\mathbb{H}^{n}$.

The solutions $u_{\varepsilon}$ of the problem above converges in a two-scale sense (for details see the paper) to a limit function $u$, that can be characterized as the unique weak solution of a limit problem.

There are other results in this direction in both the Heisenberg group and Carnot groups.

We want to mention that in [31] periodic perforated domains are built by using
the group translation $\tau^{\mathbb{H}^{1}}$. (see also Figure 4.1 for an example of a domain as the ones considered in [31])).

However in [11] and [48] the authors use iterated group translations but only horizontally, i.e. for terms of the form $\left(h_{1}, \ldots, h_{m}, 0, \ldots, 0\right) \in \mathbb{R}^{N}$. This means that their approach is in the direction of our ideas: in fact we recall that the generalised translations coincide with group translations horizontally (see Lemma 4.1.4) but it is different since, in order to prevent overlapping, we cannot iterate our translations in the Grushin case.

We would also like to recall the paper by M.Biroli and N. Tchou [12] that presents a more geometrical approach. Still the results therein fail to apply to our perforated domains since the covering assumption is not satisfied. That is a consequence of Proposition 4.6.1 where we have proved that the distance between centres of the holes are unbounded in the Euclidean sense.

To conclude we want to prove the result in [31] for our perforated domains in the Grushin plane. The main ingredient in [31] is the use of a Poincaré inequality as the one given in Chapter 5. Therefore we are cautiously optimistic that we are not far from proving the homogenization result.

## Bibliography

[1] Ralph Abraham, Jerrold E Marsden, and Tudor Ratiu. Manifolds, tensor analysis, and applications, volume 75. Springer Science \& Business Media, 2012.
[2] Andrei A Agrachev, Davide Barilari, and Ugo Boscain. Introduction to riemannian and sub-riemannian geometry. 2012.
[3] Andrei A Agrachev, Alessandro Gentile, and Antonio Lerario. Geodesics and horizontal-path spaces in carnot groups. Geometry $\mathcal{G}$ Topology, 19(3):1569-1630, 2015.
[4] Titu Andreescu and Dorin Andrica. Number theory: structures, examples, and problems. Springer Science \& Business Media, 2009.
[5] Martino Bardi and Federica Dragoni. Convexity along vector fields and applications to equations of monge-ampére type. In Progress in Analysis and Its Applications: Proceedings of the 7th International ISAAC Congress, Imperial College, London, UK, 13-18 July 2009, page 455. World Scientific, 2010.
[6] Martino Bardi and Federica Dragoni. Convexity and semiconvexity along vector fields. Calculus of Variations and Partial Differential Equations, 42(3-4):405-427, 2011.
[7] Martino Bardi and Federica Dragoni. Subdifferential and properties of convex functions with respect to vector fields. Journal of Convex Analysis, 21(3):785-810, 2014.
[8] Davide Barilari, Ugo Boscain, and Mario Sigalotti. Geometry, analysis and dynamics on sub-Riemannian manifolds. 2016.
[9] André Bellaïche. The tangent space in sub-riemannian geometry. In SubRiemannian geometry, pages 1-78. Springer, 1996.
[10] Costante Bellettini and Enrico Le Donne. Regularity of sets with constant horizontal normal in the engel group. arXiv preprint arXiv:1201.6399, 2012.
[11] Isabeau Birindelli and Jérôme Wigniolle. Homogenization of hamiltonjacobi equations in the heisenberg group. Commun. Pure Appl. Anal, 2(4):461-479, 2003.
[12] Marco Biroli and Nicoletta Tchou. Convergence for strongly local dirichlet forms in perforated domains with homogeneous neumann boundary conditions. Commun. Pure Appl. Anal, 2(3):1-31, 2012.
[13] Andrea Bonfiglioli, Ermanno Lanconelli, and Francesco Uguzzoni. Stratified Lie groups and potential theory for their sub-Laplacians. Springer Science \& Business Media, 2007.
[14] William M Boothby. An introduction to differentiable manifolds and Riemannian geometry, volume 120. Academic press, 1986.
[15] Roger W Brockett. Control theory and singular riemannian geometry. In New directions in applied mathematics, pages 11-27. Springer, 1982.
[16] Ovidiu Calin, Der-Chen Chang, and Peter Greiner. Geometric analysis on the Heisenberg group and its generalizations. American Mathematical Soc., 2008.
[17] Colin M Campbell, MR Quick, EF Robertson, CM Roney-Dougal, GC Smith, and G Traustason. Groups St Andrews 2009 in Bath, volume 1. Cambridge University Press, 2011.
[18] Piermarco Cannarsa and Carlo Sinestrari. Semiconcave functions, Hamilton-Jacobi equations, and optimal control, volume 58. Springer Science \& Business Media, 2004.
[19] Luca Capogna, Donatella Danielli, Scott D Pauls, and Jeremy Tyson. An introduction to the Heisenberg group and the sub-Riemannian isoperimetric problem, volume 259. Springer Science \& Business Media, 2007.
[20] Luca Capogna, Scott Pauls, and Jeremy Tyson. Convexity and horizontal second fundamental forms for hypersurfaces in carnot groups. Transactions of the American Mathematical Society, 362(8):4045-4062, 2010.
[21] Wei-Liang Chow. Über systeme von linearen partiellen differentialgleichungen erster ordnung. In The Collected Papers Of Wei-Liang Chow, pages 47-54. World Scientific, 2002.
[22] Doina Cioranescu and François Murat. A strange term coming from nowhere. In Topics in the mathematical modelling of composite materials, pages 45-93. Springer, 1997.
[23] Giovanna Citti and Alessandro Sarti. A cortical based model of perceptual completion in the roto-translation space. Journal of Mathematical Imaging and Vision, 24(3):307-326, 2006.
[24] Michael G Crandall, Hitoshi Ishii, and Pierre-Louis Lions. User's guide to viscosity solutions of second order partial differential equations. Bulletin of the American mathematical society, 27(1):1-67, 1992.
[25] Lorenzo D'Ambrosio. Hardy inequalities related to grushin type operators. Proceedings of the American Mathematical Society, pages 725-734, 2004.
[26] Donatella Danielli and Nicola Garofalo. Geometric properties of solutions to subelliptic equations in nilpotent lie groups. Lecture Notes in Pure and Applied Mathematics, pages 89-106, 1997.
[27] Federica Dragoni. Carnot-Carathéodory metrics and viscosity solutions. PhD thesis, Citeseer, 2006.
[28] A. Jama F. Dragoni. Generalised translations and periodicity in the geometry of vector fields with application grushin spaces. Preprint.
[29] F Flaherty and M do Carmo. Riemannian geometry. mathematics: Theory \& applications, 2013.
[30] Bruno Franchi, Cristian E Gutiérrez, and Richard L Wheeden. Weighted sobolev-poincaré inequalities for grushin type operators. Communications in Partial Differential Equations, 19(3-4):523-604, 1994.
[31] Bruno Franchi and Maria Carla Tesi. Two-scale homogenization in the heisenberg group. Journal de mathématiques pures et appliquées, 81(6):495-532, 2002.
[32] Mikhael Gromov. Carnot-carathéodory spaces seen from within. In SubRiemannian geometry, pages 79-323. Springer, 1996.
[33] Brian Hall. Lie groups, Lie algebras, and representations: an elementary introduction, volume 222. Springer, 2015.
[34] Melvin Hausner and Jacob T Schwartz. Lie groups, Lie algebras. CRC Press, 1968.
[35] Serge Lang. Fundamentals of differential geometry, volume 191. Springer Science \& Business Media, 2012.
[36] John M Lee. Introduction to Smooth manifolds. Springer Verlag, New York, 2001.
[37] John M Lee. Riemannian manifolds: an introduction to curvature, volume 176. Springer Science \& Business Media, 2006.
[38] Wensheng Liu and Hector J Sussmann. Shortest paths for sub-Riemannian metrics on rank-two distributions, volume 564. American Mathematical Soc., 1995.
[39] Guozhen Lu, Juan J Manfredi, and Bianca Stroffolini. Convex functions on the heisenberg group. Calculus of Variations and Partial Differential Equations, 19(1):1-22, 2003.
[40] Richard Montgomery. A tour of subriemannian geometries, their geodesics and applications. Number 91. American Mathematical Soc., 2006.
[41] Roberto Monti. Distances, boundaries and surface measures in CarnotCarathéodory spaces. PhD thesis, Citeseer, 2001.
[42] Roberto Monti and Daniele Morbidelli. Regular domains in homogeneous groups. Transactions of the American Mathematical Society, 357(8):29753011, 2005.
[43] A. Jama N. Dirr, F.Dragoni. Poincaré inequality for a class of perforated domains in the grushin plane. Preprint.
[44] Alexander Nagel, Elias M Stein, and Stephen Wainger. Balls and metrics defined by vector fields i: Basic properties. Acta Mathematica, 155(1):103147, 1985.
[45] Peter Petersen. Riemannian geometry, volume 171 of graduate texts in mathematics, 2006.
[46] PK Rashevsky. Any two points of a totally nonholonomic space may be connected by an admissible line. Uch. Zap. Ped. Inst. im. Liebknechta, Ser. Phys. Math, 2:83-94, 1938.
[47] Antonio Sánchez-Calle. Fundamental solutions and geometry of the sum of squares of vector fields. Inventiones mathematicae, 78(1):143-160, 1984.
[48] Bianca Stroffolini. Homogenization of hamilton-jacobi equations in carnot groups. ESAIM: Control, Optimisation and Calculus of Variations, 13(1):107-119, 2007.
[49] Veeravalli S Varadarajan. Lie groups, Lie algebras, and their representations, volume 102. Springer Science \& Business Media, 2013.
[50] Anatolii Moiseevich Vershik and Vladimir Yakovlevich Gershkovich. Nonholonomic dynamical systems. geometry of distributions and variational problems. Itogi Nauki i Tekhniki. Seriya" Sovremennye Problemy Matematiki. Fundamental'nye Napravleniya", 16:5-85, 1987.

