Separation of variables in PDEs using nonlinear transformations: Applications to reaction-diffusion type equations

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Abstract

The paper describes a new approach to constructing exact solutions of nonlinear partial differential equations that employs separation of variables using special (nonlinear integral) transformations and the splitting principle. To illustrate its effectiveness, the method is applied to nonlinear reaction-diffusion type equations that involve variable coefficients and arbitrary functions. New exact functional separable solutions as well as generalized traveling wave solutions are obtained.

Keywords: functional separation of variables, generalized separation of variables, exact solutions, nonlinear PDEs, reaction-diffusion equations

1. Brief introduction

By relying on the methods of generalized and functional separation of variables, the studies [1–12] obtained a large number of exact solutions to equations arising in the theory of heat and mass transfer, wave theory, optics, and fluid dynamics as well as to other nonlinear equations of mathematical physics.

The methods of generalized and functional separation of variables suggest an a priori setting of the structural form of the unknown variable, u, so that it de-

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pends on a few free functions; the specific expressions of these functions are determined in a subsequent analysis of the arising functional differential equations. The present study employs a more general approach that allows one to determine, rather than set a priori, the structural form of the unknown variable in the course of the solution.

2. General description of the method. Some remarks

For definiteness, we will consider nonlinear equations of mathematical physics in two independent variables, x and t, and one dependent (unknown) variable, $u = u(x,t)$:

$$F(x, t, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \dots) = 0. \tag{1}$$

To find exact solutions of equation (1), we first apply the nonlinear transformation

$$\vartheta = \int h(u)\,du, \tag{2}$$

with the functions $\vartheta = \vartheta(x,t)$ and $h = h(u)$ to be determined in the subsequent analysis. Once ϑ and h are known, the integral relation (2) defines an exact solution to the original equation in implicit form.

By differentiating (2) with respect to the independent variables, we get the partial derivatives

$$u_x = \frac{\vartheta_x}{h}, \quad u_t = \frac{\vartheta_t}{h}, \quad u_{xx} = \frac{\vartheta_{xx}}{h} - \frac{\vartheta_x^2 h'_u}{h^3}, \quad u_{xt} = \frac{\vartheta_{xt}}{h} - \frac{\vartheta_x \vartheta_t h'_u}{h^3}, \quad \dots \tag{3}$$

We assume that after inserting the expressions of (3) into (1) and rearranging, the resulting equation can be rewritten in a bilinear form with N terms

$$\sum_{n=1}^{N} \Phi_n \Psi_n = 0, \tag{4}$$

where

$$\Phi_n = \Phi_n(x, t, \vartheta_x, \vartheta_t, \vartheta_{xx}, \dots), \qquad \Psi_n = \Psi_n(u, h, h'_u, h''_{uu}, \dots). \tag{5}$$

Solutions to equation (4) with Φ_n and Ψ_n defined by (5) will be sought using the splitting principle stated below.

Splitting principle. Consider the sets of elements $\{\Phi_n\}$ and $\{\Psi_n\}$ appearing in (4) that meet two or more linear relations

$$\sum_{n=1}^{N} \alpha_{in}\Phi_n = 0, \quad i = 1, \ldots, l; \quad \sum_{n=1}^{N} \beta_{jn}\Psi_n = 0, \quad j = 1, \ldots, s, \tag{6}$$

with $1 \leq l \leq N-1$ and $1 \leq s \leq N-1$. The constants α_{in} and β_{jn} in (6) are chosen so that the bilinear equation (4) is satisfied identically (this can always be done). Importantly, the linear relations (6) are purely algebraic and they hold regardless of the specific expressions of the differential forms (5).

Once a set of relations (6) is available, the differential forms (5) are inserted into them. This leads to systems of differential equations, usually overdetermined, for the unknown functions $\vartheta = \vartheta(x,t)$ and $h = h(u)$, which appear in (2).

Remark 1. Apart from the linear relations (6), it is also necessary to treat the degenerate cases where some of the differential forms Φ_n and/or Ψ_n vanish.

Remark 2. For even N, the easiest way to satisfy equation (4) is to set

$$\Phi_i - \gamma_{ij}\Phi_j = 0, \quad \gamma_{ij}\Psi_i + \Psi_j = 0 \quad (i \neq j),$$

where γ_{ij} are arbitrary constants; the subscripts i and j do not repeat and together assume all values from 1 to N.

For $N \geq 3$, equation (4) can also be satisfied identically by using the set of linear relations

$$\begin{aligned} &\Phi_n - A_n\Phi_{N-1} - B_n\Phi_N = 0, \quad n = 1, \ldots, N-2; \\ &\Psi_{N-1} + A_1\Psi_1 + \cdots + A_{N-2}\Psi_{N-2} = 0, \\ &\Psi_N + B_1\Psi_1 + \cdots + B_{N-2}\Psi_{N-2} = 0, \end{aligned} \tag{7}$$

where A_i and B_i are arbitrary constants. In formulas (7), the Φ and Ψ forms can all be swapped, $\Phi_n \rightleftarrows \Psi_n$; also the simultaneous transpositions $\Phi_i \rightleftarrows \Phi_j$ and $\Psi_i \rightleftarrows \Psi_j$ can be used. This results in other suitable sets of linear relations (6). For more formulas, see [6].

Remark 3. The approach outlined above is a generalization of the method for constructing functional separable solutions employed in [10, 11], which is based on presetting the structure of solution in implicit form $\int h(u)\,du = \xi(x)\omega(t) + \eta(x)$. The method adopted in the present study allows one to find solutions without presetting their structure.

3. Exact solutions to nonlinear reaction-diffusion type equations

We consider the class of nonlinear partial differential equations with variable coefficients

$$u_t = [a(x)f(u)u_x^m]_x + b(x)g(u), \tag{8}$$

which includes reaction-diffusion type equations (with $m = 1$) and generalized porous medium equations with a nonlinear source ($m > 0$). Numerous exact solutions to various equations of the form (8) with $m = 1$ as well as other related equations can be found in [3–7, 9, 10, 12–22]. Also some solutions with $m \neq 1$ can be found in [6, 23–25].

In what follows, we will only consider nondegenerate cases of equation (8) where $a(x) \not\equiv 0$, $f(u) \not\equiv 0$, $b(x) \not\equiv 0$, and $g(u) \not\equiv 0$.

By employing the approach described in Section 2, we will obtain a few new simple exact solutions to equations of the form (8) where two functional coefficients, $a(x)$ and $f(u)$, are set arbitrarily and the others are expressed in terms of them. For brevity, the arguments of the functions appearing in the transformation (2) and equation (8) will often be omitted.

After making the change of variable (2), we substitute the derivatives (3) into (8) and rearrange the terms to obtain

$$-\vartheta_t + (a\vartheta_x^m)_x fh^{1-m} + a\vartheta_x^{1+m}(fh^{-m})'_u + bgh = 0. \qquad (9)$$

For $h = 1$, equation (9) coincides with the original equation (8) where $u = \vartheta$. Therefore, no solution has been lost in this stage.

Equation (9) can be represented in the bilinear form (4) with $N = 4$ if we set

$$\begin{array}{llll} \Phi_1 = -\vartheta_t, & \Phi_2 = (a\vartheta_x^m)_x, & \Phi_3 = a\vartheta_x^{1+m}, & \Phi_4 = b; \\ \Psi_1 = 1, & \Psi_2 = fh^{1-m}, & \Psi_3 = (fh^{-m})'_u, & \Psi_4 = gh. \end{array} \qquad (10)$$

Example 1. Equation (4) with $N = 4$ can be satisfied identically by using the linear relations

$$\Phi_1 = -A\Phi_4, \quad \Phi_2 = B\Phi_4; \quad \Psi_3 = 0; \quad \Psi_4 = A\Psi_1 - B\Psi_2, \qquad (11)$$

where A and B are arbitrary constants. By inserting (10) into (11), we arrive at the equations

$$\vartheta_t = Ab, \quad (a\vartheta_x^m)_x = Bb; \quad (fh^{-m})'_u = 0, \quad gh = A - Bfh^{1-m}. \qquad (12)$$

The general solution to the overdetermined system consisting of the first two equations in (12) with $m \neq 0, 1$ is given by

$$b(x) = \frac{k}{A}, \quad \vartheta(x,t) = kt + \int \left(\frac{k}{A} \frac{Bx + C_1}{a(x)} \right)^{1/m} dx + C_2, \qquad (13)$$

where $a(x)$ is an arbitrary function, while C_1, C_2, and k are arbitrary constants. The solutions to the other two equations of (12) can be written as

$$g = Af^{-1/m} - B, \quad h = f^{1/m}, \tag{14}$$

where $f = f(u)$ is an arbitrary function. By setting $A = k$ in (13) and (14), we get the equation

$$u_t = [a(x)f(u)u_x^m]_x + \frac{k}{f^{1/m}(u)} - B, \tag{15}$$

which admits the exact generalized traveling wave solution in implicit form

$$\int f^{1/m}(u)\,du = kt + \int \left(\frac{Bx + C_1}{a(x)}\right)^{1/m} dx + C_2. \tag{16}$$

Example 2. Equation (4) with $N = 4$ also holds if we set

$$\Phi_1 = -A\Phi_2, \quad \Phi_3 = -\Phi_4; \quad \Psi_2 = A\Psi_1, \quad \Psi_3 = \Psi_4, \tag{17}$$

where A is an arbitrary constant. Substituting (10) into (17) yields

$$\vartheta_t = A(a\vartheta_x^m)_x, \quad a\vartheta_x^{1+m} = -b; \quad fh^{1-m} = A, \quad (fh^{-m})'_u = gh. \tag{18}$$

Solutions to the overdetermined system consisting of the first two equations in (18) will be sought in the form $\vartheta = kt + r(x)$. This results in

$$b(x) = -a(x)\left(\frac{kx + C_1}{Aa(x)}\right)^{\frac{m+1}{m}}, \quad \vartheta(x,t) = kt + \int \left(\frac{kx + C_1}{Aa(x)}\right)^{\frac{1}{m}} dx + C_2, \tag{19}$$

where $a(x)$ is an arbitrary function, while C_1, C_2, and k are arbitrary constants. The solutions to the last two equations of (18) are given by

$$g = \frac{1}{1-m}\left(\frac{f}{A}\right)^{\frac{1+m}{1-m}} f'_u, \quad h = \left(\frac{f}{A}\right)^{\frac{1}{m-1}} \quad (m \neq 1). \tag{20}$$

By setting $C_1 = 0$, $A = k = 1$, and $m \neq 1$ in (19) and (20), we arrive at the equation

$$u_t = [a(x)f(u)u_x^m]_x + \frac{1}{m-1}\left(\frac{x^{m+1}}{a(x)}\right)^{\frac{1}{m}} f^{\frac{1+m}{1-m}}(u)f'_u(u), \tag{21}$$

which admits the exact solution in implicit form

$$\int f^{\frac{1}{m-1}}(u)\,du = t + \int \left(\frac{x}{a(x)}\right)^{\frac{1}{m}} dx + C_2. \tag{22}$$

4. Some generalizations and modifications

Some other exact solutions to equation (1) can be obtained if instead of (4) and (5) we consider equivalent equations reducible to (4) and (5) on the set of functions satisfying relation (2).

Example 3. Let us return to the class of reaction-diffusion equations (8). After making the change of variable (2), we consider instead of (9) the more complex equation

$$-e^{\lambda\vartheta}e^{-\lambda H}\vartheta_t+(a\vartheta_x^m)_x fh^{1-m}+a\vartheta_x^{1+m}(fh^{-m})'_u+bgh=0, \tag{23}$$

where $H=\int h\,du$ and λ is an arbitrary constant. Equations (9) and (23) are equivalent, since $\vartheta=H$ by virtue of the transformation (2).

Equation (23) can be represented in the bilinear form (4) with $N=4$ where

$$\begin{aligned}&\Phi_1=-e^{\lambda\vartheta}\vartheta_t, \quad \Phi_2=(a\vartheta_x^m)_x, \quad \Phi_3=a\vartheta_x^{1+m}, \qquad \Phi_4=b;\\ &\Psi_1=e^{-\lambda H}, \qquad \Psi_2=fh^{1-m}, \quad \Psi_3=(fh^{-m})'_u, \quad \Psi_4=gh.\end{aligned} \tag{24}$$

Just as previously, equation (4) with $N=4$ can be satisfied by using relations (11). Inserting (24) into (11) gives

$$e^{\lambda\vartheta}\vartheta_t=Ab, \quad (a\vartheta_x^m)_x=Bb; \quad (fh^{-m})'_u=0, \quad gh=Ae^{-\lambda H}-Bfh^{1-m}. \tag{25}$$

These equations coincide with (12) at $\lambda=0$. A solution to system (25) is given by

$$\begin{aligned}&b(x)=\frac{C_1}{A}\exp[\lambda r(x)], \quad \vartheta(x,t)=\frac{1}{\lambda}\ln(C_1\lambda t+C_2)+r(x),\\ &g=Af^{-1/m}\exp\left(-\lambda\int f^{1/m}du\right)-B, \quad h=f^{1/m},\end{aligned} \tag{26}$$

where $f=f(u)$ is an arbitrary function, and the function $r=r(x)$ satisfies the ordinary differential equation

$$[a(r'_x)^m]'_x=\frac{BC_1}{A}e^{\lambda r}. \tag{27}$$

Formulas (26) and equation (27) define the functional coefficients of equation (8) and its solution (2). It is noteworthy that for $a(x)=a_0x^k$, equation (27) admits the exact solution

$$r(x)=\sigma\ln x+\mu, \quad \sigma=\frac{k-m-1}{\lambda}, \quad \mu=\frac{1}{\lambda}\ln\frac{Aa_0\sigma^m(k-m)}{BC_1}.$$

Example 4. Now we look at the equation

$$-(H/\vartheta)\vartheta_t + (H/\vartheta)^m(a\vartheta_x^m)_x fh^{1-m} + a\vartheta_x^{1+m}(fh^{-m})'_u + bgh = 0 \tag{28}$$

with $H = \int h(u)\,du$, which is equivalent to (9) by virtue of the transformation (2). Equation (28) can be rewritten in the bilinear form (4) with $N = 4$ where

$$\begin{array}{llll} \Phi_1 = -\vartheta_t/\vartheta, & \Phi_2 = \vartheta^{-m}(a\vartheta_x^m)_x, & \Phi_3 = a\vartheta_x^{1+m}, & \Phi_4 = b; \\ \Psi_1 = H, & \Psi_2 = fh^{1-m}H^m, & \Psi_3 = (fh^{-m})'_u, & \Psi_4 = gh. \end{array} \tag{29}$$

Taking into account that equation (4), $N = 4$, can be satisfied with relations (11). Substituting the expressions (29) into (11), we arrive at the system of equations

$$\vartheta_t/\vartheta = Ab, \quad \vartheta^{-m}(a\vartheta_x^m)_x = Bb; \quad (fh^{-m})'_u = 0, \quad gh = AH - Bfh^{1-m}H^m.$$

It admits the following solution:

$$b(x) = \lambda, \quad \vartheta(x,t) = Ce^{A\lambda t}r(x); \quad g = Af^{-1/m}H - BH^m, \quad h = f^{1/m}, \tag{30}$$

where $f = f(u)$ is an arbitrary function, C and λ are arbitrary constants, $H = \int h\,du$, and $r = r(x)$ is a function satisfying the ordinary differential equation

$$[a(r'_x)^m]'_x = B\lambda r^m. \tag{31}$$

Using (30), we find that the equation

$$u_t = [a(x)f(u)u_x^m]_x + A\lambda f^{-1/m}(u)\int f^{1/m}(u)\,du - B\lambda\Big(\int f^{1/m}(u)\,du\Big)^m$$

admits the exact solution in implicit form

$$\int f^{1/m}(u)\,du = Ce^{A\lambda t}r(x),$$

where $a(x)$ and $f(u)$ are arbitrary functions and $r(x)$ is a function satisfying the ordinary differential equation (31).

Remark 4. The presented method of functional separation of variables for nonlinear partial differential equation on the basis the transformation (2) is also applicable to other classes of PDEs as well as those in three or more independent variables. In particular, these include nonlinear wave equations, nonlinear Klein–Gordon type equations, nonlinear telegraph type equations, and others; these also include some higher-order equations. The solutions and examples given in this paper can be used to test various numerical methods for nonlinear equations as well as relevant approximate analytical methods.

Remark 5. The proposed method is a generalization of the direct method of functional separation of variables [10, 11]. Hence, in certain cases, it may be more effective than the nonclassical method of symmetry reductions based on an invariant surface condition. For details, see [26].

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