Author's latest version published in *Doklady Mathematics*, 2019, Vol. 99, No. 3, pp. 321–324.

MATHEMATICAL PHYSICS

Functional Separable Solutions for Two Classes of Nonlinear Equations of Mathematical Physics

A. D. Polyanin [a,b,*] **and A. I. Zhurov** [a,b,**]
Translated by I. Ruzanova

[a] *Ishlinsky Institute for Problems in Mechanics, Russian Academy of Sciences, Moscow, 119526 Russia*
[b] *National Research Nuclear University "MEPhI," Moscow, 115409 Russia*
** e-mail: polyanin@ipmnet.ru*
*** e-mail: zhurov@ipmnet.ru*

Presented by Academician of the RAS F.L. Chernousko December 27, 2018
Received January 10, 2019

Abstract—This study describes a new modification of the method of functional separation of variables for nonlinear equations of mathematical physics. Solutions are sought in an implicit form that involves several free functions (specific expressions for these functions are determined by analyzing the arising functional differential equations). The effectiveness of the method is illustrated by examples of nonlinear reaction–diffusion equations and Klein–Gordon type equations with variable coefficients that depend on one or more arbitrary functions. A number of new exact functional separable solutions and generalized traveling-wave solutions are obtained.

1. PRELIMINARY REMARKS

For definiteness, we consider nonlinear equations of mathematical physics with the unknown function dependent on two variables, $u = u(x,t)$.

The idea of the method proposed below for finding exact functional separable solutions given in implicit form is based on a generalization of traveling-wave solutions to nonlinear heat and wave equations. Before describing the method, we give two simple examples illustrating the existence of solutions in implicit form.

Example 1. Consider the nonlinear heat equation

$$u_t = [f(u)u_x]_x, \tag{1}$$

where $f(u)$ is an arbitrary function. Equation (1) has an exact traveling-wave solution $u = u(z)$, $z = \lambda t + kx$, where k and λ are arbitrary constants and the function $u(z)$ is described by the equation $\lambda u'_z = k^2[f(u)u'_z]'_z$. The general solution of this equation can be represented in the implicit form

$$k^2 \int \frac{f(u)du}{\lambda u + C_1} = \lambda t + kx + C_2, \tag{2}$$

where C_1 and C_2 are arbitrary constants. On the right-hand side of (2), the invariant variable z has been replaced by the original variables x and t.

Example 2. The nonlinear wave equation

$$u_{tt} = [f(u)u_x]_x, \tag{3}$$

where $f(u)$ is an arbitrary function, also has a traveling-wave solution $u = u(z)$, $z = \lambda t + kx$, which can be represented in implicit form

$$\int [k^2 f(u) - \lambda^2] du = C_1(\lambda t + kx) + C_2. \tag{4}$$

Examples 1 and 2 show that nonlinear equations (1) and (3), which involve an arbitrary function, have traveling-wave solutions that can be represented in implicit form. This fact will be used in Sections 2–4 below for finding solutions of more complicated form (see formula (6) and Remark 2).

Remark 1. Functional separable solutions written in explicit form are given by $u = U_1(z_1)$, $z_1 = \phi(x) + \psi(t)$ (or $z_1 = \phi(x)\psi(t)$), where the functions $\phi(x)$, $\psi(t)$, and $U_1(z_1)$ are determined in the subsequent analysis [1–9]. Generalized traveling-wave solutions written in explicit form are given by $u = U_2(z_2)$, $z_2 = \zeta(x)t + \theta(x)$, where $\zeta(x)$, $\theta(x)$, and $U_2(z_2)$ are undetermined functions [7, 9, 10].

2. SHORT DESCRIPTION OF THE METHOD

We look at the class of nonlinear partial differential equations of sufficiently general form

$$F(x, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0. \tag{5}$$

Their exact solutions will be sought in the implicit form

$$\int h(u) du = \xi(x)\omega(t) + \eta(x). \tag{6}$$

Here, the functions $h = h(u)$, $\xi = \xi(x)$, $\eta = \eta(x)$, and $\omega = \omega(t)$ are to be determined in the course of the further analysis, which can be described as follows. First, using (6), we calculate the derivatives u_x, u_t, u_{xx}, …, which are expressed in terms of the functions h, ξ, η, ω, and their derivatives. Next, the resulting expressions for the derivatives are substituted into Eq. (5) and the variable t is eliminated with the help of (6). As a result (with a suitable choice of the function ω), we obtain bilinear functional differential equations

$$\sum_{j=1}^{N} \Phi_j[x]\Psi_j[u] = 0,$$
$$\Phi_j[x] \equiv \phi_j(x, \xi, \eta, \xi'_x, \eta'_x, \xi''_{xx}, \eta''_{xx} \ldots), \quad \Psi_j[u] \equiv \psi_j(u, h, h'_u, h''_{uu}, \ldots). \tag{7}$$

Here, $\Phi_j[x]$ and $\Psi_j[u]$ depend only on x and u, respectively. Finally, the functional differential equations (7) are solved, for example, by applying differentiation or a splitting method [7, 9]. As a result, exact solutions of the original equation (5) are found.

Remark 2. The representation of solutions in the form (6) is based on a natural generalization of solution (2) performed according to the scheme

$$\frac{k^2 f(u)}{\lambda u + C_1} \Rightarrow h(u), \quad \lambda \Rightarrow \xi(x), \quad t \Rightarrow \omega(t), \quad kx + C_2 \Rightarrow \eta(x).$$

Remark 3. Looking for a solution in the implicit form (6) with an integral term on the left-hand side often leads to equations for h that have a lower order than in the case of looking for exact solutions in an explicit form. Additionally, a solution in implicit form usually leads to simpler explicit representations of the functions g and f in terms of h (when exact solutions are sought in explicit form, g and f are often expressed in parametric form [9]).

3. REACTION–DIFFUSION EQUATIONS AND THEIR SOLUTIONS

Now we will use the method described in Section 2 to construct exact solutions of nonlinear reaction–diffusion equations with variable coefficients

$$u_t = [a(x)f(u)u_x]_x + b(x)g(u). \tag{8}$$

Note that a number of exact solutions to Eq. (8) with polynomial or exponential functions $f(u)$ and $g(u)$ were obtained in [3, 9–12].

In this paper, primary attention is given to fairly general equations involving one or two arbitrary functions. Note that exact solutions to nonlinear equations of mathematical physics that contain arbitrary functions and, hence, have a considerable degree of generality are of particular practical interest for testing and estimating the accuracy of approximate analytical and numerical methods for integrating related initial-boundary value problems. In what follows, the arguments of the functions $f = f(u)$, $g = g(u)$, $h = h(u)$, $a = a(x)$, $b = b(x)$, and $\omega = \omega(t)$ involved in Eq. (8) and solution (6) will often be omitted.

Differentiating (6) with respect to t and x, we obtain the partial derivatives u_t, u_x, and u_{xx}. Substituting them into (8) yields the functional differential equation

$$\omega'_t = \Theta_1(x,u)\omega^2 + \Theta_2(x,u)\omega + \Theta_3(x,u), \tag{9}$$

where the functions Θ_n do not depend explicitly on t and are given by the formulas

$$\begin{aligned} &\Theta_1(x,u) = \frac{a(\xi'_x)^2}{\xi}\Big(\frac{f}{h}\Big)'_u, \quad \Theta_2(x,u) = \frac{1}{\xi}\Big[(a\xi'_x)'_x f + 2a\xi'_x\eta'_x\Big(\frac{f}{h}\Big)'_u\Big],\\ &\Theta_3(x,u) = \frac{1}{\xi}\Big[(a\eta'_x)'_x f + a(\eta'_x)^2\Big(\frac{f}{h}\Big)'_u + bgh\Big]. \end{aligned} \tag{10}$$

The functional differential equation (9)–(10) depends on three variables t, x, u, which are connected by the single relation (6), and involves unknown functions and their derivatives dependent on different arguments.

An analysis shows that, for $\eta(x) \neq \text{const}$, Eq. (9)–(10) can be reduced to a bilinear functional differential equation of the form (7) if $\omega(t) = kt$, $\omega(t) = ke^{\lambda t}$, or $\omega(t) = k\ln t$. In what follows, we omit the intermediate calculations and present some exact solutions of the form (6) for Eq. (8).

Solution 1. The equation

$$u_t = [a(x)u_x]_x - \frac{x^2}{a(x)}g(u), \tag{11}$$

which contains two arbitrary functions $a(x)$ and $g(u)$, has the following traveling-wave solution in implicit form:

$$\int h(u)\,du = t + \int \frac{x\,dx}{a(x)} + C_1, \quad h(u) = \Big(2\int g(u)\,du + C_2\Big)^{-1/2}, \tag{12}$$

where C_1 and C_2 are arbitrary constants.

Solution 2. The equation

$$u_t = [a(x)f(u)u_x]_x + \frac{a'_x(x)}{\sqrt{a(x)}}u, \tag{13}$$

which contains two arbitrary functions $a(x)$ and $f(u)$, has a traveling-wave solution wave in implicit form:

$$\int \frac{f(u)}{u}\,du = 4t - 2\int \frac{dx}{\sqrt{a(x)}} + C. \tag{14}$$

Solution 3. The equation

$$u_t = [a(x) f(u) u_x]_x + m + \frac{k}{f(u)}, \tag{15}$$

which contains two arbitrary functions, $a(x)$ and $f(u)$, and two arbitrary constants, k and m, has a generalized traveling-wave solution in implicit form

$$\int f(u) du = kt - m \int \frac{x dx}{a(x)} + C_1 \int \frac{dx}{a(x)} + C_2, \tag{16}$$

where C_1 and C_2 are arbitrary constants.

Solution 4. The equation

$$u_t = [f(u) u_x]_x + x[k + \frac{1}{f(u)}], \tag{17}$$

which contains an arbitrary function $f(u)$ and an arbitrary constant k, has a generalized traveling-wave solution in implicit form

$$\int f(u) du = xt - \frac{1}{6} k x^3 + C.$$

Solution 5. The equation

$$u_t = [x^n f(u) u_x]_x - u + \frac{(n-2)u}{f(u)} \int \frac{f(u)}{u} du,$$

where $f(u)$ is an arbitrary function and $n \neq 2$ is an arbitrary constant, has a generalized separable solution in implicit form

$$\int \frac{f(u)}{u} du = k e^{(n-2)t} + \frac{x^{2-n}}{2-n},$$

where k is an arbitrary constant.

4. KLEIN–GORDON TYPE EQUATIONS AND THEIR SOLUTIONS

Consider nonlinear Klein–Gordon type equations with variable coefficients:

$$u_{tt} = [a(x) f(u) u_x]_x + b(x) g(u). \tag{18}$$

Several exact solutions to equations of this form were described in [8, 9].

Below, for fairly general equations (18) involving one or two arbitrary functions, we present some exact solutions obtained by the method described in Section 2 (the intermediate calculations are omitted).

Solution 6. The equation

$$u_{tt} = [a(x) u_x]_x + \frac{a'_x(x)}{\sqrt{a(x)}} g(u), \tag{19}$$

which contains two arbitrary functions $a(x)$ and $g(u)$, admits the following generalized traveling-wave solutions in implicit form:

$$\int \frac{du}{g(u)} = \pm 2t - 2 \int \frac{dx}{\sqrt{a(x)}} + C. \tag{20}$$

Solution 7. The equation

$$u_{tt} = [a(x) f(u) u_x]_x - m - \frac{f'_u(u)}{f^3(u)}, \tag{21}$$

which contains two arbitrary functions $a(x)$ and $f(u)$ and an arbitrary constant m, has exact generalized traveling-wave solutions in implicit form

$$\int f(u)du = \pm t + m\int \frac{xdx}{a(x)} + C_1\int \frac{dx}{a(x)} + C_2, \tag{22}$$

where C_1 and C_2 are arbitrary constants.

Solution 8. The equation

$$u_{tt} = [f(u)u_x]_x - x^2[k + \frac{f_u'(u)}{f^3(u)}], \tag{23}$$

which contains an arbitrary function $f(u)$ and an arbitrary constant k, has a generalized traveling-wave solution in implicit form

$$\int f(u)du = xt + \frac{1}{12}kx^4 + C.$$

Solution 9. The equation

$$u_{tt} = [a(x)u^{-1/2}u_x]_x + b(x), \tag{24}$$

which depends on two arbitrary functions $a(x)$ and $b(x)$, admits a generalized traveling-wave solution, which can be written in explicit form

$$u = \frac{1}{4}[\xi(x)t + \eta(x)]^2.$$

Here, the function $\xi = \xi(x)$ is defined by the formula

$$\xi = C_1\int \frac{dx}{a(x)} + C_2, \tag{25}$$

where C_1 and C_2 are arbitrary constants and the function $\eta = \eta(x)$ satisfies the linear ordinary differential equation

$$[a(x)\eta_x']_x' = \frac{1}{2}\xi^2 - b(x).$$

Since its right-hand side is known, η can be found by integration.

Solution 10. The equation

$$u_{tt} = (e^x u_x)_x - \frac{1}{2h} + \frac{h'_u}{h^3}\int hdu\,,$$

where $h = h(u)$ is an arbitrary function, has a generalized separable solution in implicit form:

$$\int hdu = e^{-x} - \frac{1}{4}(t + C)^2,$$

where C is an arbitrary constant.

5. BRIEF CONCLUSIONS

A new modification of the method of functional separation of variables for constructing exact solutions in implicit form was described. The effectiveness of the method was illustrated by examples of nonlinear reaction–diffusion equations and Klein–Gordon type equations with variable coefficients that depend on one or more arbitrary functions. A number of new functional separable solutions and generalized traveling-wave solutions were obtained. Importantly, the solutions constructed are not invariant, that is, they cannot be obtained using group analysis of differential equations.

FUNDING

This work was performed within the framework of the State Assignment (state registration no. AAAA-A17-117021310385-6) and was supported in part by the Russian Foundation for Basic Research (project no. 18-29-10025).

REFERENCES

1. A. M. Grundland and E. Infeld, J. Math. Phys. **33**, 2498–2503 (1992).
2. R. Z. Zhdanov, J. Phys. A **27**, L291–L297 (1994).
3. Ph. W. Doyle and P. J. Vassiliou, Int. J. Non-Linear Mech. **33** (2), 315–326 (1998).
4. V. K. Andreev, O. V. Kaptsov, V. V. Pukhnachov, and A. A. Rodionov, *Applications of Group-Theoretical Methods in Hydrodynamics* (Kluwer, Dordrecht, 1998).
5. P. G. Estevez, C. Qu, and S. Zhang, J. Math. Anal. Appl. **275**, 44–59 (2002).
6. P. G. Estevez and C. Z. Qu, Theor. Math. Phys. **133**, 1490–1497 (2002).
7. A. D. Polyanin, V. F. Zaitsev, and A. I. Zhurov, *Methods for Solving Nonlinear Equations of Mathematical Physics and Mechanics* (Fizmatlit, Moscow, 2005) [in Russian].
8. J. Hu and C. Qu, J. Math. Anal. Appl. **330**, 298–311 (2007).
9. A. D. Polyanin and V. F. Zaitsev, *Handbook of Nonlinear Partial Differential Equations*, 2nd ed. (CRC, Boca Raton, 2012).
10. A. D. Polyanin, Appl. Math. Comput. **347**, 282–292 (2019).
11. O. O. Vaneeva, A. G. Johnpillai, R. O. Popovych, and C. Sophocleous, J. Math. Anal. Appl. **330** (2), 1363–1386 (2007).
12. O. O. Vaneeva, R. O. Popovych, and C. Sophocleous, J. Math. Anal. Appl. **396**, 225–242 (2012).