Subfactors and unitary R-matrices

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November 11, 2019

Abstract

This is an extended abstract from a talk at the Oberwolfach workshop "Subfactors and Applications" in October 2019. It summarizes some results from [2] (joint work with Roberto Conti) and [5, 4].

The Yang-Baxter equation is a cubic equation for a linear map $R \in V \otimes V \to V \otimes V$ on the tensor square of a vector space V, namely

$$(R \otimes 1)(1 \otimes R)(R \otimes 1) = (1 \otimes R)(R \otimes 1)(1 \otimes R), \qquad \text{(YBE)}$$

where 1 is the identity on V. This equation and its variants come from quantum physics, but also play a central role in various branches of mathematics, for instance in knot theory, quantum groups/Hopf algebras, and braid groups. Further recent interest in the solutions of the YBE stems from topological quantum computing [6].

Despite this widespread interest in the YBE, no satisfactory understanding of its solutions has been reached. In this talk, a new approach to the YBE was presented, based on operator algebras and subfactors [2]. We restrict to the case of most interest in applications, namely the case where V is a finite-dimensional Hilbert space and R is unitary. Such "R-matrices" exist in any dimension $d = \dim V$, simple examples being the identity 1 on $V \otimes V$, the tensor flip $F(v \otimes w) = w \otimes v$, diagonal R-matrices, and Gaussian R-matrices. The (unknown) set of *all* R-matrices of dimension d is denoted $\mathcal{R}(d)$.

The general strategy of our approach is to start from an arbitrary R-matrix $R \in \mathcal{R}(d)$ with base space V and derive operator-algebraic data (such as endomorphisms, subfactors, indices) from it that inform us about R. The main structural elements of our approach can be summarized in the following diagram:

$$\begin{array}{ccccc} \varphi(\mathcal{N}) & \subset & \mathcal{N} & & \\ \cup & & \cup & & \\ \varphi(\mathcal{L}_R) & \subset & \mathcal{L}_R & \xleftarrow{\rho_R} & B_\infty \\ \cap & & \cap & & \\ \lambda_R(\mathcal{N}) & \subset & \mathcal{N} & & \end{array} \qquad (*)$$

Starting at the top of the diagram, $\mathcal{N}$ is the hyperfinite II_1 factor realised as an infinite tensor product $\mathcal{N} = \bigotimes_{n \geq 1} \mathrm{End}V$, weakly closed w.r.t. the normalised trace $\tau = \bigotimes_{n \geq 1} \frac{\mathrm{Tr}_V}{d}$,

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and equipped with the shift $\varphi : \mathcal{N} \to \mathcal{N}$, $\varphi(x) = 1 \otimes x$. We identify finite tensor powers $\operatorname{End} V^{\otimes n}$ with their natural embeddings into $\mathcal{N}$, so that $R \in \mathcal{N}$ and the YBE reads $\varphi(R)R\varphi(R) = R\varphi(R)R$.

The second line of the diagram is about the braid group structure: As is well known, any $R \in \mathcal{R}(d)$ defines a group homomorphism ρ_R from the infinite braid group B_∞ into the unitary group of $\mathcal{N}$ by mapping the standard generators b_n, $n \in \mathbb{N}$, of B_∞ to $\varphi^{n-1}(R) \in \mathcal{N}$. The von Neumann algebra generated by this representation is denoted $\mathcal{L}_R$.

The third line of the diagram introduces the *Yang-Baxter endomorphism* $\lambda_R \in \operatorname{End} \mathcal{N}$. It is defined in such a way that it restricts to the shift φ on $\mathcal{N}$. Explicitely,

$$\lambda_R : \mathcal{N} \to \mathcal{N}, \qquad \lambda_R(x) := \operatorname*{w-lim}_{n\to\infty} R \cdots \varphi^n(R) x \varphi^n(R^*) \cdots R^*. \tag{$**$}$$

This definition is natural also from the point of view of the Cuntz algebra[1]. As particular examples, we note that the identity R-matrix gives the identity endomorphism, $\lambda_1 = \mathrm{id}_\mathcal{N}$, and the flip F gives the canonical endomorphism, $\lambda_F = \varphi$.

Let us list a few results from [2] (joint work with Roberto Conti):

(1) $\mathcal{L}_R$ is a factor (II_1 for non-trivial R). This provides us with three subfactors (I) $\lambda_R(\mathcal{N}) \subset \mathcal{N}$, (II) $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, and (III) $\mathcal{L}_R \subset \mathcal{N}$ derived from R.

(2) Subfactors (I),(II) have always finite index $\leq d^2$, but (III) may have infinite index. Its relative commutant coincides with the fixed point algebra $\mathcal{N}^{\lambda_R}$.

(3) The subfactors (I), (II) can be iterated by taking powers of λ_R and φ, respectively. One has $R \in \varphi^2(\mathcal{L}_R)' \cap \mathcal{L}_R \subset \lambda_R^2(\mathcal{N})' \cap \mathcal{N}$. Hence, for any non-trivial R-matrix, λ_R^2 is reducible and λ_R is not an automorphism [1].

(4) Both squares in $(*)$ are commuting squares. Denoting the τ-preserving conditional expectation $\mathcal{N} \to \lambda_R(\mathcal{N})$ by E_R, and the associated left inverse of λ_R by $\phi_R := \lambda_R^{-1} \circ E_R$, this implies $\phi_R(x) = \phi_F(x)$, $x \in \mathcal{L}_R$.

An interesting object to consider is $\phi_R(R)$. This is an element of $\varphi(\mathcal{L}_R)' \cap \mathcal{L}_R$, which thanks to (4) coincides with the (normalised) left partial trace $\phi_F(R)$ of R. We therefore have explicit elements of the relative commutant, and a connection from operator-algebraic structures to concrete properties of R. One finds [2]:

(5) Let $R \in \mathcal{R}$. Then the left and right partial traces of R coincide and are normal elements of $\operatorname{End} V$.

(6) Define the *character* τ_R of an R-matrix as the map $\tau_R : B_\infty \to \mathbb{C}$, $\tau_R := \tau \circ \rho_R$. If two R-matrices $R, S \in \mathcal{R}(d)$ have the same character, then $\phi_R(R)$ and $\phi_S(S)$ are unitarily equivalent.

(7) Any R-matrix with spectrum contained in a disc of radius less than $1 - 2^{-1/4}$ is trivial[2].

[1] Viewing $R \in \mathcal{R}(d)$ as a unitary in $\mathcal{O}_d$ yields a canonically associated endomorphism λ_R of $\mathcal{O}_d$. This endomorphism gives $(**)$ by extension to a type $\mathrm{III}_{1/d}$ factor $\mathcal{M} \supset \mathcal{N}$ and restriction.

[2] This result has its origin in an estimate on the Jones index $[\mathcal{N} : \lambda_R(\mathcal{N})]$ in terms of $\phi_R(R)$.

Item (6) suggests to consider R-matrices up to the natural equivalence relation $R \sim S$ given by coinciding characters and dimensions of R-matrices. Then $\phi_R(R)$ is an invariant for $\sim$, and in the involutive case ($R^2 = 1$), it is even a *complete* invariant: $R \sim S \iff \phi_R(R) \cong \phi_S(S)$ [4]. In the general non-involutive case, the partial trace is not a complete invariant.

As the last section in this overview, let us consider the problem of *classifying* all R-matrices up to the equivalence $\sim$ and announce some results from the upcoming article [5]. We consider here the case that the spectrum of R has cardinality 2, and normalise it to $\sigma(R) = \{-1, q\}$, $|q| = 1$, $q \neq -1$. In this situation, the representation ρ_R factors through the Hecke algebra $H_\infty(q)$, and we moreover have [5]:

(8) If $R \in \mathcal{R}(d)$ has no two opposite eigenvalues $\mu, -\mu$ in its spectrum, then $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is irreducible and τ_R is a (positive) Markov trace.

Hence for $q \neq 1$, any R-matrix gives a positive Markov trace on $H_\infty(q)$. We may therefore use Wenzl's classification of positive Markov traces on $H_\infty(q)$ [7]. Recall that his results state in particular that for a positive Markov trace to exist, one must have $q \in \{1, e^{2\pi i/\ell} : \ell \in \{4, 5, \ldots\}\}$, and at fixed ℓ, there exist finitely many possible Markov traces. In our Yang-Baxter setting, these possibilities are severely restricted [5]:

(9) Let R be an R-matrix with spectrum $\{-1, q\}$, $q \neq 1$, and eigen projection P for the eigenvalue -1. Then $q \in \{\pm i, e^{i\pi/3}\}$. If $q = \pm i$, then $\tau(P) = \frac{1}{2}$, and if $q = e^{\pm i\pi/3}$, then $\tau(P) = \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\}$. Two such R-matrices R, S are equivalent (in the sense of $\sim$) iff they have the same spectrum (q), dimension (d), and trace ($\tau(P)$).

The above result does *not* imply that all the possible combinations of eigenvalues q and traces $\tau(P)$ are indeed realised. We have found explicit R-matrices realising the combinations ($q = \pm i$, $\tau(P) = \frac{1}{2}$), ($q = e^{i\pi/3}$, $\tau(P) = \frac{1}{3}$), ($q = e^{i\pi/3}$, $\tau(P) = \frac{2}{3}$) and conjecture that the last possibility, ($q = e^{i\pi/3}$, $\tau(P) = \frac{1}{2}$), is not realised by any R-matrix. This is in line with observations made by Galindo, Hong, and Rowell [3], but so far no proof of this conjecture exists.

It is instructive to compare these findings with the situation at $q = 1$, which is completely different. For $q \neq 1$, we always have irreducible $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$, and the equivalence takes a simple form (it is given by the three parameters d, q, $\tau(P)$). For $q = 1$, on the other hand, $\varphi(\mathcal{L}_R) \subset \mathcal{L}_R$ is reducible except for the special cases $R \sim \pm 1, \pm F$, and the equivalence is more involved (it is given by the unitary equivalence class of $\phi_R(R)$). The case $q = 1$ corresponds to R being involutive and ρ_R factoring through the infinite *symmetric* group. In that case, a complete and explicit classification of R-matrices up to equivalence exists: R-matrices are parameterised by pairs of Young diagrams with d boxes in total, corresponding to the positive and negative eigenvalues of $\phi_R(R)$ [4]. We also mention that in this case, the index $[\mathcal{L}_R : \varphi(\mathcal{L}_R)]$ is a rational typically non-integer number.

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