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# Tempered Fractional Poisson Processes and Fractional Equations with $Z$ -Transform

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## Abstract

In this article, we derive the state probabilities of different type of space- and time-fractional Poisson processes using  $z$ -transform. We work on tempered versions of time-fractional Poisson process and space-fractional Poisson processes. We also introduce Gegenbauer type fractional differential equations and their solutions using  $z$ -transform. Our results generalize and complement the results available on fractional Poisson processes in several directions.

*Key Words:* Fractional Poisson process,  $Z$ -transform, inverse tempered stable subordinator, fractional derivatives.

## 1 Introduction

In recent years fractional processes are getting increased attention due to their real life applications. For example fractional Brownian motion (FBM) overcome the limitations of Brownian motion in modeling of long-range dependent phenomena occurring in financial time series, Nile river data and fractal analysis etc (see e.g., [1]). Similarly time-fractional Poisson process is helpful in modeling of counting processes where the inter-arrival times are heavy tailed or arrivals are delayed (see e.g., [2, 3]). In time-fractional Poisson process the waiting times are Mittag-Leffler (ML) distributed see [3]. Recently, [4] introduced space-fractional Poisson process by taking a fractional shift operator in place of an integer shift operator in the governing differential-difference equation of standard Poisson process. Moreover, they have shown that space-fractional Poisson process can also be obtained by time-changing the standard Poisson process with a stable subordinator. Further, they argue that time-fractional Poisson process and the space-fractional Poisson process are specific cases of the same generalized complete model and hence might be useful in the study of transport of charge carriers in semiconductors [5] or applications related to fractional quantum mechanics [6]. In this article, we extend the space-fractional and time-fractional Poisson process by considering a

tempered time-space-fractional Poisson process. We feel a strong motivation to study these processes since tempering introduces a finite moment condition in space-fractional Poisson process. Further, it gives more flexibility in modeling of natural phenomena discussed in [6], due to extra parameters which can be picked based on the situation. Moreover, we suggest to use  $z$ -transform since the  $z$ -transform method is more general than method of probability generating functions, and hence it could be applied for solutions of fractional equations which are not probability distributions, see Section 3.6. The governing equations for marginal distributions of Poisson and Skellam processes time-changed by inverse subordinators are discussed in [7]. For properties of Poisson processes directed by compound Poisson-Gamma subordinators (see [8]).

The rest of the paper is organized as follows. In Section 2, we introduce the  $z$ -transform and the inverse  $z$ -transform by indicating their main characteristics. In this section Caputo-Djrbashian fractional derivative, Caputo tempered fractional derivative and the Riemann-Liouville tempered fractional derivative are also discussed. Further, main properties of Poisson process are discussed briefly. Section 3 is devoted to different kind of fractional Poisson processes. In this section first we revisit the time- and space-fractional Poisson processes with  $z$ -transform approach. Our main results are given in Sections 3.4, 3.5 and 3.6. The last section concludes.

## 2 Preliminaries

In this section, we provide some basic definitions and results to be used further in subsequent sections.

### 2.1 The $Z$ -Transform and Its Inverse

The  $z$ -transform is a linear transformation and can be considered as an operator, mapping sequence of scalars into functions of complex variable  $z$ . For a function  $f(k)$ ,  $k \in \mathbb{Z}$ , the bilateral  $z$ -transform is defined by (see e.g., [9])

$$F(z) = \mathcal{Z}f(k) = \sum_{k=-\infty}^{\infty} f(k)z^{-k}, \quad z \in \mathbb{C}. \quad (2.1)$$

We assume that there exists a region of convergence (ROC) such that the infinite-series (2.1) converges in ROC. Alternatively, in case where  $f(k)$  is defined only for  $k \geq 0$ , the (unilateral)  $z$ -transform is defined as

$$F(z) = \mathcal{Z}f(k) = \sum_{k=0}^{\infty} f(k)z^{-k}, \quad z \in \mathbb{C},$$

where the coefficient of  $z^{-k}$  in this expansion is an inverse given by

$$f(k) = \mathcal{Z}^{-1}(F(z)). \quad (2.2)$$

The inverse  $z$ -transform is also defined by the complex integral

$$\mathcal{Z}^{-1}\{F(z)\} = f(k) = \frac{1}{2\pi i} \oint_C F(z) z^{k-1} dz,$$

where  $C$  is simple closed contour enclosing the origin and lying outside the circle  $|z| = R$ . The existence of the inverse imposes restrictions on  $f(k)$  for the uniqueness. If  $f(k)$ ,  $k \in \mathbb{N} \cup \{0\}$ , is probability distribution, that is,  $f(k) \geq 0$  and

$$\sum_{k=0}^{\infty} f(k) = 1,$$

then the probability generating function (PGF) is defined by

$$G(u) = \sum_{k=0}^{\infty} u^k f(k), \quad |u| \leq 1,$$

and relates to unilateral  $z$ -transform as follows  $G(z^{-1}) = F(z)$ . The following operational properties of  $z$ -transform are used further for the solution of initial value problem involving difference equations

$$\mathcal{Z}f(k) = F(z),$$

$$\mathcal{Z}(f(k-m)) = z^{-m} [F(z) + \sum_{r=-m}^{-1} f(r) z^{-r}], \quad (2.3)$$

$$\mathcal{Z}(f(k+m)) = z^m [F(z) - \sum_{r=0}^{m-1} f(r) z^{-r}], \quad m \geq 0. \quad (2.4)$$

## 2.2 Fractional Derivatives

The Caputo-Djrbashain (CD) fractional derivative of order  $\beta \in (0, 1]$ , for a function  $g(t)$ ,  $t \geq 0$  is defined as

$$\frac{d^\beta}{dt^\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(\tau)}{d\tau} \frac{d\tau}{(t-\tau)^\beta}, \quad \beta \in (0, 1]. \quad (2.5)$$

Note that the classes of functions for which the CD derivative is well defined is discussed in ([10], Sections 2.2, 2.3). The Laplace transform (LT) of CD fractional derivative is given by (see e.g., [10], p.39)

$$\mathcal{L} \left( \frac{d^\beta}{dt^\beta} g(t) \right) = \int_0^\infty e^{-st} \frac{d^\beta}{dt^\beta} g(t) dt = s^\beta \tilde{g}(s) - s^{\beta-1} g(0^+), \quad 0 < \beta \leq 1,$$

where  $\tilde{g}(s)$  is the LT of the function  $g(t)$ ,  $t \geq 0$ , such that

$$\mathcal{L}(g(t)) = \tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt.$$

The Riemann-Liouville tempered fractional derivative is defined by (see [11])

$$\mathbb{D}_t^{\beta, \nu} g(t) = e^{-\nu t} \mathbb{D}_t^\beta [e^{\nu t} g(t)] - \nu^\beta g(t),$$

where

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{g(u) du}{(t-u)^\beta}$$

is the usual Riemann-Liouville fractional derivative of order  $\beta \in (0, 1)$ . Further, the LT of Riemann-Liouville tempered fractional derivative is

$$\mathcal{L} \left[ \mathbb{D}_t^{\beta, \nu} g(t) \right] (s) = ((s + \nu)^\beta - \nu^\beta) \tilde{g}(s). \quad (2.6)$$

The Caputo tempered fractional derivative is defined by (see [11])

$$\frac{d^{\beta, \nu}}{dt^{\beta, \nu}} g(t) = \mathbb{D}_t^{\beta, \nu} g(t) - \frac{g(0)}{\Gamma(1-\beta)} \int_t^\infty e^{-\nu r} \beta r^{-\beta-1} dr. \quad (2.7)$$

The Laplace transform for the Caputo tempered fractional derivative for a function  $g(t)$  satisfies

$$\mathcal{L} \left[ \frac{d^{\beta, \nu}}{dt^{\beta, \nu}} g \right] (s) = ((s + \nu)^\beta - \nu^\beta) \tilde{g}(s) - s^{-1} ((s + \nu)^\beta - \nu^\beta) g(0). \quad (2.8)$$

### 2.3 Poisson Process

The homogeneous Poisson process  $N(t)$ ,  $t \geq 0$ , with parameter  $\lambda > 0$  is defined as,

$$N(t) = \max\{n : T_1 + T_2 + \dots + T_n \leq t\}, \quad t \geq 0,$$

where the inter-arrival times  $T_1, T_2, \dots, T_n$  are non-negative iid exponential random variables with mean  $1/\lambda$ . The probability mass function (PMF)  $P(k, t) = \mathbb{P}(N(t) = k)$  is given by

$$P(k, t) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, \quad k = 0, 1, 2, \dots \quad (2.9)$$

The PMF of the Poisson process satisfies the differential-difference equation of the form

$$\frac{d}{dt} P(k, t) = -\lambda (P(k, t) - P(k-1, t)) = -\lambda \nabla P(k, t) \quad (2.10)$$

with initial conditions

$$P(k, 0) = \delta_{k,0} = \begin{cases} 0, & k \neq 0, \\ 1, & k = 0. \end{cases} \quad (2.11)$$

Further, by definition  $P(-l, t) = 0$ ,  $l > 0$  and  $\nabla \equiv (1 - B)$  with  $B$  as the backward shift operator, i.e.  $B\{P(k, t)\} = P(k-1, t)$ .

### 3 Fractional Poisson Processes

In this section, we revisit space- and time-fractional Poisson processes using the  $z$ -transform approach. Note that  $z$ -transform is more general than the probability generating function approach and can be used to solve the difference-differential equations where the solution may not be a probability distribution. Also, we introduce and study tempered space-time-fractional Poisson processes. Further, to show the importance of  $z$ -transform, we consider Gegenbauer type fractional difference equations.

#### 3.1 The Time-Fractional Poisson Process

The time-fractional Poisson process (TFPP) was first introduced by Laskin (2003) (see [3]). The renewal process representation of TFPP is given by

$$N_\beta(t) = \max\{n : T_1^\beta + \dots + T_n^\beta \leq t\}, \quad t \geq 0, \quad \beta \in (0, 1], \quad (3.12)$$

where the inter arrival times  $T_1^\beta, T_2^\beta, \dots, T_n^\beta$  are iid non-negative random variables with Mittag-Leffler distribution function, given by

$$\mathbb{P}(T_k^\beta \leq x) = 1 - M_\beta(-\lambda x^\beta), \quad x \geq 0, \quad \lambda > 0. \quad (3.13)$$

Further, the probability density function (PDF) of the inter arrival time  $T_k^\beta$  is given by

$$f(x) = \lambda x^{\beta-1} M_{\beta,\beta}(-\lambda x^\beta), \quad x \geq 0, \quad \lambda > 0, \quad \beta \in (0, 1], \quad (3.14)$$

where

$$M_{a,b}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak + b)}, \quad z \in \mathbb{C}, \quad a, b > 0, \quad (3.15)$$

is two parameter Mittag-Leffler function, and  $M_a(z) = M_{a,1}(z)$ ,  $z \in \mathbb{C}$  is the classical Mittag-Leffler function (see [12]). Let  $S_\beta(t)$ ,  $t \geq 0$  be a stable subordinator with Laplace transform

$$\mathbb{E}e^{-zS_\beta(t)} = e^{-tz^\beta}, \quad z > 0, \quad t \geq 0, \quad \beta \in (0, 1). \quad (3.16)$$

Let  $Y_\beta(t)$  be its right-continuous inverse process defined by

$$Y_\beta(t) = \inf\{w > 0 : S_\beta(w) > t\}, \quad t \geq 0. \quad (3.17)$$

The process  $Y_\beta(t)$  is non-Markovian with non-stationary increments [13] and also its marginals are not infinitely divisible [14]. Further, the marginals of TFPP are not infinitely divisible [14]. Recently, Aletti et al. (2018) established that TFPP is a martingale with respect to its

natural filtration [15]. Alternatively, Meerschaert et al. (2011) (see [2]), gave the following subordination representation of TFPP

$$N_\beta(t) = N(Y_\beta(t)), \quad t \geq 0, \quad \beta \in (0, 1), \quad (3.18)$$

where  $N(t)$  is the homogenous Poisson process with parameter  $\lambda > 0$  and  $Y_\beta(t)$  is independent of  $N(t)$ .  $N_\beta(t)$  has the PMF (see e.g. [3, 16, 17, 2])

$$P_\beta(k, t) = \mathbb{P}\{N_\beta(t) = k\} = \frac{(\lambda t^\beta)^k}{k!} \sum_{r=0}^{\infty} \frac{(k+r)!}{r!} \frac{(-\lambda t^\beta)^r}{\Gamma(\beta(k+r)+1)}, \quad k = 0, 1, 2, \dots \quad (3.19)$$

Further, it is the solution of the following fractional differential-difference equation with CD fractional derivative in time

$$\frac{d^\beta}{dt^\beta} P_\beta(k, t) = -\lambda^\beta (P_\beta(k, t) - P_\beta(k-1, t)) = -\lambda^\beta \nabla P_\beta(k, t), \quad (3.20)$$

$$P_\beta(k, t) = 0, \quad \text{where } k < 0, \quad (3.21)$$

$$P_\beta(k, 0) = \delta_{k,0}, \quad k = 0, 1, 2, \dots, \quad \beta \in (0, 1]. \quad (3.22)$$

For the particular case  $\beta = 1$ , the process reduces to the standard Poisson process.

### 3.2 The Space-Fractional Poisson Process

Let  $S_\alpha(t)$ ,  $t \geq 0$ ,  $\alpha \in (0, 1)$ , be a stable subordinator and  $N(t)$ ,  $t \geq 0$ , is homogenous Poisson process with parameter  $\lambda > 0$ , independent of  $S_\alpha(t)$ . The space-fractional Poisson process (SFPP)  $N^\alpha(t)$ ,  $t \geq 0$ ,  $0 < \alpha < 1$  was introduced by [4], as follows

$$N^\alpha(t) = \begin{cases} N(S_\alpha(t)), & t \geq 0, \quad 0 < \alpha < 1, \\ N(t), & t \geq 0, \quad \alpha = 1. \end{cases} \quad (3.23)$$

The density  $f(x, 1)$  of  $S_\alpha(1)$  is infinitely differentiable on  $(0, \infty)$ , with the asymptotics given by (see e.g., [18])

$$f(x, 1) \sim \frac{\left(\frac{\alpha}{x}\right)^{\frac{2-\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha(1-\alpha)}} e^{-(1-\alpha)\left(\frac{x}{\alpha}\right)^{-\frac{\alpha}{1-\alpha}}}, \quad \text{as } x \rightarrow 0; \quad (3.24)$$

$$f(x, 1) \sim \frac{\alpha}{\Gamma(1-\alpha)x^{1+\alpha}}, \quad \text{as } x \rightarrow \infty. \quad (3.25)$$

Exact forms of the density  $f(x, 1)$  in terms of infinite series or integral are discussed in (see e.g. [19], p. 583) and has the following infinite-series representation

$$f(x, 1) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\Gamma(k\alpha+1)}{k!} \frac{1}{x^{k\alpha+1}} \sin(\pi\alpha k), \quad x > 0. \quad (3.26)$$

Note that from (3.24) and (3.25), we have

$$\lim_{x \rightarrow 0} f(x, 1) = f(0, 1) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} f(x, 1) = f(\infty, 1) = 0. \quad (3.27)$$

The PGF of this process is

$$G^\alpha(u, t) = \mathbb{E}u^{N^\alpha(t)} = e^{-\lambda^\alpha(1-u)^\alpha t}, \quad |u| \leq 1, \quad \alpha \in (0, 1). \quad (3.28)$$

We introduce the fractional difference operator (see e.g., [1], p.60)

$$\nabla^\alpha = (1 - B)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k B^k, \quad \alpha \in (0, 1), \quad (3.29)$$

where

$$\binom{\alpha}{k} = \frac{(\alpha)(\alpha - 1) \cdots (\alpha - k + 1)}{k!} = \frac{(-1)^k (-\alpha)_k}{k!},$$

and Pochhammer symbol

$$(\lambda)_k = \begin{cases} \lambda(\lambda + 1) \cdots (\lambda + k - 1), & k = 1, 2, \dots \\ 1, & k = 0. \end{cases} \quad (3.30)$$

Let

$$P^\alpha(k, t) = \mathbb{P}\{N^\alpha(t) = k\}, \quad k = 0, 1, 2, \dots, \quad (3.31)$$

be the PMF of the SFPP, which satisfies the following fractional differential-difference equations (see [4])

$$\frac{d}{dt} P^\alpha(k, t) = -\lambda^\alpha (1 - B)^\alpha P^\alpha(k, t), \quad \alpha \in (0, 1], \quad k = 1, 2, \dots \quad (3.32)$$

$$\frac{d}{dt} P^\alpha(0, t) = -\lambda^\alpha P^\alpha(0, t), \quad (3.33)$$

with initial condition

$$P^\alpha(k, 0) = \delta_{k,0}. \quad (3.34)$$

Using the  $z$ -transform in both side, it follows

$$\frac{d}{dt} \{ \mathcal{Z} P^\alpha(k, t) \} = -\lambda^\alpha [ \mathcal{Z} \{ (1 - B)^\alpha P^\alpha(k, t) \} ].$$

Further, using (3.29), we have

$$\frac{d}{dt} \{ \mathcal{Z} P^\alpha(k, t) \} = -\lambda^\alpha \mathcal{Z} P^\alpha(k, t) \left[ 1 - \frac{\alpha}{z} + \frac{\alpha(\alpha - 1)}{(2!)(z^2)} - \dots \right].$$



Further,

$$\frac{d}{dt}\{\mathcal{Z}P^\alpha(k, t)\} = \left[-\lambda \left(1 - \frac{1}{z}\right)\right]^\alpha \mathcal{Z}P^\alpha(k, t). \quad (3.35)$$

Solving (3.35) for  $\mathcal{Z}P^\alpha(k, t)$  and using initial condition in (3.34), leads to

$$\begin{aligned} \mathcal{Z}P^\alpha(k, t) &= e^{-\lambda^\alpha(1-\frac{1}{z})^\alpha t} = \sum_{r=0}^{\infty} (-\lambda^\alpha)^r t^r (1 - z^{-1})^{\alpha r} = \sum_{r=0}^{\infty} (-\lambda^\alpha)^r t^r \sum_{k=0}^{\infty} (-1)^k \binom{\alpha r}{k} z^{-k} \\ &= \sum_{k=0}^{\infty} z^{-k} \left[ \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^r}{r!} \frac{\Gamma(r\alpha + 1)}{\Gamma(r\alpha - k + 1)} \right]. \end{aligned}$$

To find  $P^\alpha(k, t)$ , invert the  $z$ -transform that is equivalent to finding the coefficient of  $z^{-k}$ , which leads to

$$P^\alpha(k, t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^r}{r!} \frac{\Gamma(r\alpha + 1)}{\Gamma(r\alpha - k + 1)}. \quad (3.36)$$

Moreover, one can write

$$\left(1 - \frac{1}{z}\right)^{2\alpha} = \left(1 - \frac{1}{z} \left(2 - \frac{1}{z}\right)\right)^\alpha. \quad (3.37)$$

By comparing the coefficients of  $z^{-2p}$  in both sides, we get the following identity. For any  $\alpha > 0$  and  $p \in \mathbb{N} \cup \{0\}$ , the following identity holds

$$\begin{aligned} \binom{2\alpha}{2p} &= \binom{\alpha}{p} \cdot \binom{p}{0} + 2^2 \binom{\alpha}{p+1} \cdot \binom{p+1}{2} + \dots \\ &+ 2^{2p-2} \binom{\alpha}{2p-1} \cdot \binom{2p-1}{2p-2} + 2^{2p} \binom{\alpha}{2p} \cdot \binom{2p}{2p}, \end{aligned} \quad (3.38)$$

where  $\binom{\alpha}{p} = \frac{\Gamma(\alpha+1)}{\Gamma(p+1)\Gamma(\alpha-p+1)}$ . The identity (3.38) is used in subsequent section.

**Remark 3.1.** *The composition of  $n$  stable subordinators is also a stable subordinator. Let  $S_1, S_2, \dots, S_n$  be  $n$  independent stable subordinators with parameters  $\alpha_i$ ,  $i = 1, 2, \dots, n$ , respectively. Then the iterated composition is defined by  $S^{(2)}(t) = S_2(S_1(t))$  and  $S^{(j)}(t) = S_j(S^{(j-1)}(t))$ ,  $j = 2, 3, \dots, n$ . The iterated composition  $S^{(n)}(t)$  is also a stable subordinator with parameter  $\alpha_1\alpha_2\cdots\alpha_n$ . It is easy to show that the PMF  $\tilde{P}(k, t) = \mathbb{P}(N(S^{(n)}(t)) = k)$ , satisfies the following difference- differential equation*

$$\frac{d^{2^n}}{dt^{2^n}} \tilde{P}(k, t) = \lambda[\tilde{P}(k, t) - \tilde{P}(k-1, t)], \quad t > 0, \quad k > 0. \quad (3.39)$$

### 3.3 The Time-Space-Fractional Poisson Process

Note that [4] introduced the following time-space-fractional differential equations

$$\frac{d^\beta}{dt^\beta} P_\beta^\alpha(k, t) = -\lambda^\alpha (1 - B)^\alpha P_\beta^\alpha(k, t), \quad \alpha \in (0, 1] \quad \beta \in (0, 1), \quad k = 1, 2, \dots \quad (3.40)$$

$$\frac{d^\beta}{dt^\beta} P_\beta^\alpha(0, t) = -\lambda^\alpha P_\beta^\alpha(0, t), \quad (3.41)$$

with initial conditions

$$P_\beta^\alpha(k, 0) = \delta_{k,0} = \begin{cases} 0, & k > 0, \\ 1, & k = 0, \end{cases} \quad (3.42)$$

where  $\frac{d^\beta}{dt^\beta}$  is the CD fractional derivative defined in (2.5). They have shown that

$$P_\beta^\alpha(k, t) = \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^{r\beta}}{\Gamma(1+r\beta)} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-k+1)}, \quad k = 0, 1, \dots, \quad (3.43)$$

and its PGF

$$G_\beta^\alpha(u, t) = \sum_{k=0}^{\infty} u^k P_\beta^\alpha(k, t) = M_\beta(-\lambda^\alpha t^\beta (1-u)^\alpha), \quad |u| \leq 1. \quad (3.44)$$

Using  $z$ -transform, we present an alternative proof of the fact that (3.43) satisfies (3.40). To solve (3.40), take the  $z$ -transform in both hand sides, which leads to

$$\frac{d^\beta}{dt^\beta} \{ \mathcal{Z} P_\beta^\alpha(k, t) \} = -\lambda^\alpha (1 - z^{-1})^\alpha \{ \mathcal{Z} P_\beta^\alpha(k, t) \}.$$

Further, using the Laplace transform with respect to the time variable  $t$  and  $\mathcal{Z}\{P_\beta^\alpha(k, 0)\} = 1$ , it follows

$$s^\beta \mathcal{L}[\mathcal{Z}\{P_\beta^\alpha(k, t)\}] - s^{\beta-1} = -\lambda^\alpha (1 - z^{-1})^\alpha \mathcal{L}[\mathcal{Z}\{P_\beta^\alpha(k, t)\}].$$

By some manipulation, it follows

$$\mathcal{L}[\mathcal{Z}\{P_\beta^\alpha(k, t)\}] = \frac{s^{\beta-1}}{s^\beta + \lambda^\alpha (1 - z^{-1})^\alpha}.$$

Using the LT of Mittag-Leffler function  $\mathcal{L}(M_{\beta,1}(-ut^\beta)) = \frac{s^{\beta-1}}{u+s^\beta}$  (see e.g. [10], p.36), it follows

$$\begin{aligned} \mathcal{Z}\{P_\beta^\alpha(k, t)\} &= \mathcal{L}^{-1} \left\{ \frac{s^{\beta-1}}{s^\beta + \lambda^\alpha (1 - z^{-1})^\alpha} \right\} = M_{\beta,1}(-\lambda^\alpha (1 - z^{-1})^\alpha t^\beta) \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^{k\alpha} t^{k\beta} (1 - z^{-1})^{k\alpha}}{\Gamma(1 + k\beta)}. \end{aligned} \quad (3.45)$$

Inverting the  $z$ -transform gives

$$\begin{aligned} P_\beta^\alpha(k, t) &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^{r\beta}}{\Gamma(1+r\beta)} (r\alpha)(r\alpha-1)\cdots(r\alpha-k+1) \\ &= \frac{(-1)^k}{k!} \sum_{r=0}^{\infty} \frac{(-\lambda^\alpha)^r t^{r\beta}}{\Gamma(1+r\beta)} \frac{\Gamma(r\alpha+1)}{\Gamma(r\alpha-k+1)}, \quad k = 0, 1, 2, \dots \end{aligned}$$

Alternatively, one can define the time-space-fractional Poisson process (TSFPP) as follows

$$N_\beta^\alpha(t) = N(S_\alpha(Y_\beta(t))) = N^\alpha(Y_\beta(t)), \quad t \geq 0, \quad (3.46)$$

where TSFPP is obtained by subordinating the standard Poisson process  $N(t)$  by an independent  $\alpha$ -stable subordinator  $S_\alpha(t)$  and then by the inverse  $\beta$ -stable subordinator  $Y_\beta(t)$ .

**Proposition 3.1.** *The state probabilities of time-space-fractional Poisson process defined in (3.46) satisfies the equation (3.40).*

*Proof.* Let  $F(z, t)$  be the  $z$ -transform of  $N_\beta^\alpha(t)$ , then it follows

$$\begin{aligned} F(z, t) &= \sum_{k=0}^{\infty} z^{-k} \mathbb{P}(N_\beta^\alpha(t) = k) = \sum_{k=0}^{\infty} z^{-k} \mathbb{P}(N(S_\alpha(Y_\beta(t))) = k) \\ &= \sum_{k=0}^{\infty} z^{-k} \mathbb{P}(N(Y_\beta(t)) = k) = \mathbb{E} \left[ \sum_{k=0}^{\infty} z^{-k} \mathbb{P}(N(Y_\beta(t)) = k | Y_\beta(t)) \right] \\ &= \mathbb{E} \left[ e^{-\lambda^\alpha (1-z^{-1})^\alpha Y_\beta(t)} \right] = M_{\beta,1}(-\lambda^\alpha (1-z^{-1})^\alpha t^\beta), \end{aligned} \quad (3.47)$$

which follows using the result  $\mathbb{E}(e^{-sY_\beta(t)}) = M_{\beta,1}(-st^\beta)$ . Note that the two  $z$ -transforms given in (3.45) and (3.47) are same and hence two representations are equivalent by the uniqueness of  $z$ -transform.  $\square$

In next subsections, we generalize the above discussed processes to their tempered counterparts, which can give more flexibility in modeling of the natural phenomena suggested for the space- and time-fractional Poisson processes due the the extra parameter.

### 3.4 The Tempered Space-Fractional Poisson Process

One can also define tempered space-fractional Poisson process (TSFPP) by subordinating homogeneous Poisson process with the tempered stable subordinator. Note that tempered stable subordinators are obtained by exponential tempering in the distribution of stable subordinator, see [20] for more details on tempering stable processes. Let  $f(x, t)$ ,  $0 < \alpha < 1$  denotes the density of a stable subordinator  $S_\alpha(t)$  with LT

$$\int_0^\infty e^{-sx} f(x, t) dx = e^{-ts^\alpha}. \quad (3.48)$$

A tempered stable subordinator  $S_{\alpha,\mu}(t)$  has a density

$$f_{\mu}(x, t) = e^{-\mu x + \mu^{\alpha} t} f(x, t), \quad \mu > 0. \quad (3.49)$$

Using (3.49) and (3.27), it follows

$$\lim_{x \rightarrow 0} f_{\mu}(x, t) = f_{\mu}(0, t) = 0 \quad \text{and} \quad \lim_{x \rightarrow \infty} f_{\mu}(x, t) = f_{\mu}(\infty, t) = 0. \quad (3.50)$$

The sample paths of  $S_{\alpha,\mu}(t)$  are strictly increasing similar to the stable subordinator. Further it has the LT

$$\tilde{f}_{\mu}(s, t) = \int_0^{\infty} e^{-sx} f_{\mu}(x, t) dx = e^{-t((s+\mu)^{\alpha} - \mu^{\alpha})}. \quad (3.51)$$

For  $S_{\alpha,\mu}(t)$ , we have  $\mathbb{E}(S_{\alpha,\mu}(t)) = \alpha \mu^{\alpha-1} t$  and  $\text{Var}(S_{\alpha,\mu}(t)) = \alpha(1 - \alpha) \mu^{\alpha-2} t$ . The tempered space-fractional Poisson process is defined by

$$N^{\alpha,\mu}(t) = N(S_{\alpha,\mu}(t)), \quad \alpha \in (0, 1), \quad \mu \geq 0, \quad (3.52)$$

where homogeneous Poisson process  $N(t)$  is independent of the tempered stable subordinator  $S_{\alpha,\mu}(t)$ . The TSFPP is a Lévy process with finite integer order moments due to the finite moments of the tempered stable subordinators. However the integer order moments of SFPP are not finite. The TSFPP with marginal PMF  $P^{\alpha,\mu}(k, t)$  can also be defined by taking a tempered fractional shift operator instead of an ordinary fractional shift operator in (2.10) such that

$$\frac{d}{dt} P^{\alpha,\mu}(k, t) = -((\mu + \lambda(1 - B))^{\alpha} - \mu^{\alpha}) P^{\alpha,\mu}(k, t), \quad \alpha \in (0, 1], \quad \mu \geq 0, \quad (3.53)$$

and  $P^{\alpha,\mu}(k, 0) = \delta_{k,0}$ , which reduces to the SFPP by taking  $\mu = 0$ . We have following proposition for the state probabilities of TSFPP.

**Proposition 3.2.** *The state probabilities for TSFPP are given by*

$$P^{\alpha,\mu}(k, t) = (-1)^k e^{t\mu^{\alpha}} \sum_{m=0}^{\infty} \mu^m \lambda^{\alpha r - m} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \binom{\alpha r}{m} \binom{\alpha r - m}{k}, \quad k \geq 0, \quad \mu \geq 0, \quad t \geq 0. \quad (3.54)$$

*Proof.* Suppose  $F(z, t)$  is the  $z$ -transform of  $N^{\alpha, \mu}(t)$ , then

$$\begin{aligned}
F(z, t) &= \mathcal{Z}\{\mathbb{P}(N^{\alpha, \mu}(t) = k)\} = e^{-t((\mu + \lambda(1 - \frac{1}{z}))^\alpha - \mu^\alpha)} \\
&= e^{t\mu^\alpha} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \left( \mu + \lambda \left( 1 - \frac{1}{z} \right) \right)^{\alpha r} \\
&= e^{t\mu^\alpha} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{m=0}^{\infty} \binom{\alpha r}{m} \mu^m \lambda^{\alpha r - m} \left( 1 - \frac{1}{z} \right)^{\alpha r - m} \\
&= e^{t\mu^\alpha} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \sum_{m=0}^{\infty} \binom{\alpha r}{m} \mu^m \lambda^{\alpha r - m} \sum_{k=0}^{\infty} \binom{\alpha r - m}{k} (-1)^k \frac{1}{z^k} \\
&= \sum_{k=0}^{\infty} z^{-k} \left[ (-1)^k e^{t\mu^\alpha} \sum_{m=0}^{\infty} \mu^m \lambda^{\alpha r - m} \sum_{r=0}^{\infty} \frac{(-t)^r}{r!} \binom{\alpha r}{m} \binom{\alpha r - m}{k} \right],
\end{aligned}$$

the result follows by taking the coefficient of  $z^{-k}$ .  $\square$

By a standard conditioning argument, it follows that

$$\mathbb{E}(N^{\alpha, \mu}(t)) = \mathbb{E}(\mathbb{E}(N(S_{\alpha, \mu}(t)) | S_{\alpha, \mu}(t))) = \mathbb{E}(\lambda S_{\alpha, \mu}(t)) = \lambda \alpha \mu^{\alpha-1} t.$$

Further,

$$\begin{aligned}
\text{Var}(N^{\alpha, \mu}(t)) &= \mathbb{E}(\text{Var}(N(S_{\alpha, \mu}(t)) | S_{\alpha, \mu}(t))) + \text{Var}(\mathbb{E}(N(S_{\alpha, \mu}(t)) | S_{\alpha, \mu}(t))) \\
&= \mathbb{E}(\lambda S_{\alpha, \mu}(t)) + \text{Var}(\lambda S_{\alpha, \mu}(t)) = \lambda \alpha \mu^{\alpha-1} t + \lambda^2 \alpha (1 - \alpha) \mu^{\alpha-2} t.
\end{aligned}$$

**Remark 3.2.** Using a similar argument as in Prop. 3.1, one can show that the marginal PMF given in (3.54) of the TSFPP defined by the subordination representation in (3.52) satisfies (3.53).

### 3.5 The Tempered Time-Space-Fractional Poisson Process

In this section, we introduce and study tempered time-space-fractional Poisson process (TTSFPP). A time-change representation of TTSFPP can be written as

$$N_{\beta, \nu}^{\alpha, \mu}(t) = N(S_{\alpha, \mu}(Y_{\beta, \nu}(t))) = N^{\alpha, \mu}(Y_{\beta, \nu}(t)), \quad \alpha, \beta \in (0, 1], \quad \mu, \nu \geq 0, \quad (3.55)$$

where  $Y_{\beta, \nu}(t) = \inf\{r > 0 : S_{\beta, \nu}(r) > t\}$  is the right-continuous inverse of tempered stable subordinator. Note that this process is non-Markovian due to the time-change component of  $Y_{\beta, \nu}(t)$ , which is not a Lévy process. However, all the moments of this process are finite. Alternatively, taking a tempered fractional derivative in the left hand side and tempered fractional shift operator in the right hand side of the equation (3.32), we obtained the governing

fractional difference-differential equation of the PMF  $P_{\beta,\nu}^{\alpha,\mu}(k, t) = \mathbb{P}(N_{\beta,\nu}^{\alpha,\mu}(t) = k)$  of TTSFPP, such that

$$\frac{d^{\beta,\nu}}{dt^{\beta,\nu}} P_{\beta,\nu}^{\alpha,\mu}(k, t) = -((\mu + \lambda(1 - B))^\alpha - \mu^\alpha) P_{\beta,\nu}^{\alpha,\mu}(k, t), \quad (3.56)$$

$$\frac{d^{\beta,\nu}}{dt^{\beta,\nu}} P_{\beta,\nu}^{\alpha,\mu}(0, t) = -((\mu + \lambda)^\alpha - \mu^\alpha) P_{\beta,\nu}^{\alpha,\mu}(0, t), \quad (3.57)$$

with initial condition

$$P_{\beta,\nu}^{\alpha,\mu}(k, 0) = \delta_{k,0}, \quad (3.58)$$

where  $\frac{d^{\beta,\nu}}{dt^{\beta,\nu}}$  is the Caputo tempered fractional derivative of order  $\beta \in (0, 1)$  with tempering parameter  $\nu > 0$ , given in (2.7). The governing equation (3.56) reduces to the governing equation of SFPP by taking  $\mu = \nu = 0$  and  $\beta = 1$ . The process  $Y_{\beta,\nu}(t)$  is called inverse tempered stable (ITS) subordinator. A driftless subordinator  $D(t)$  with Lévy measure  $\pi_D$  and density function  $f$  has the Lévy-Khinchin representation (see e.g., [21])

$$\int_0^\infty e^{-ux} f_{D(t)}(x) dx = e^{-t\Psi_D(u)}, \quad (3.59)$$

where

$$\Psi_D(u) = \int_0^\infty (1 - e^{-uy}) \pi_D(dy), \quad u > 0, \quad (3.60)$$

is called the Laplace exponent. The Lévy measure density corresponding to a tempered stable subordinator is given by

$$\pi_{S_{\beta,\nu}}(u) = \frac{\beta}{\Gamma(1 - \beta)} \frac{e^{-\nu u}}{u^{\beta+1}}, \quad u > 0,$$

which satisfies the condition  $\int_0^\infty \pi_{S_{\beta,\nu}}(u) du = \infty$ . Let  $\mathcal{L}_{t \rightarrow s}(g(x, t)) = \tilde{g}(x, s)$  be the Laplace transform (LT) of  $g$  with respect to time variable  $t$ . Using Theorem 3.1 of [22], the LT of the density  $h_{\beta,\nu}(x, t)$  of  $Y_{\beta,\nu}(t)$  with respect to the time variable  $t$  is given by

$$\tilde{h}_{\beta,\nu}(x, s) = \frac{1}{s} \left( (s + \nu)^\beta - \nu^\beta \right) e^{-x((s + \nu)^\beta - \nu^\beta)}. \quad (3.61)$$

From (3.61), it follows

$$\frac{\partial}{\partial x} \tilde{h}_{\beta,\nu}(x, s) = -((s + \nu)^\beta - \nu^\beta) \tilde{h}_{\beta,\nu}(x, s).$$

Now by inverting the LT with the help of (2.6), it follows

$$\frac{\partial}{\partial x} h_{\beta,\nu}(x, t) = -\mathbb{D}_t^{\beta,\nu} h_{\beta,\nu}(x, t). \quad (3.62)$$

Further,

$$\begin{aligned} -\frac{\partial}{\partial x} \tilde{h}_{\beta,\nu}(x, s) &= \left[ ((s + \nu)^\beta - \nu^\beta) \tilde{h}_{\beta,\nu}(x, s) - s^{-1} ((s + \nu)^\beta - \nu^\beta) h(x, 0) \right] \\ &\quad + s^{-1} ((s + \nu)^\beta - \nu^\beta) h(x, 0). \end{aligned} \quad (3.63)$$

For inverting the LT in (3.63) we will use the generalized Mittag-Leffler function, therefore we introduce it here. The generalized Mittag-Leffler function, introduced by [23], is defined by

$$M_{a,b}^c(z) = \sum_{n=0}^{\infty} \frac{(c)_n}{\Gamma(an+b)} \frac{z^n}{n!}, \quad (3.64)$$

where  $a, b, c \in \mathbb{C}$  with  $\Re(b) > 0$  and  $(c)_n$  is Pochhammer symbol see (3.30). When  $c = 1$ , it reduces to Mittag-Leffler function. Further,

$$M_{a,b}^c(0) = \frac{(c)_0}{\Gamma(b)} = \frac{1}{\Gamma(b)}. \quad (3.65)$$

The function  $L(s) = \frac{s^{ac-b}}{(s^a+\eta)^c}$  has the inverse LT [24]

$$\mathcal{L}^{-1}[L(s)] = t^{b-1} M_{a,b}^c(-\eta t^a). \quad (3.66)$$

Moreover,

$$\mathcal{L}^{-1} \left[ \frac{1}{s(s+\nu)^{-\beta}} \right] = t^{-\beta} M_{1,1-\beta}^{-\beta}(-\nu t), \quad (3.67)$$

which follows by taking  $a = 1, b = 1 - \beta, c = -\beta$  and  $\eta = \nu$ . Now by inverting the LT in (3.63) with the help of (3.67), it follows

$$-\frac{\partial}{\partial x} h_{\beta,\nu}(x,t) = \frac{\partial^{\beta,\nu}}{\partial t^{\beta,\nu}} h_{\beta,\nu}(x,t) + \left( t^{-\beta} M_{1,1-\beta}^{-\beta}(-\nu t) - \nu^\beta \right) \delta(x), \quad (3.68)$$

where  $h_{\beta,\nu}(x,0) = \delta(x)$  is the Dirac delta function. Taking  $\nu = 0$  in (3.68) and using (3.65), it follows

$$-\frac{\partial}{\partial x} h_{\beta,0}(x,t) = \frac{\partial^\beta}{\partial t^\beta} h_{\beta,0}(x,t) + \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta(x), \quad (3.69)$$

which is the governing equation of the density function of inverse  $\beta$ -stable subordinator, which complements the result obtained in literature (see e.g., [25, 26]).

**Proposition 3.3.** *The PMF of TTSFPP defined in (3.55) satisfies (3.56).*

*Proof.* Note that,

$$\begin{aligned} P_{\beta,\nu}^{\alpha,\mu}(k,t) &= \mathbb{P}(N^{\alpha,\mu}(Y_{\beta,\nu}(t)) = k) = \mathbb{E}(\mathbb{P}(N^{\alpha,\mu}(Y_{\beta,\nu}(t)) = k | Y_{\beta,\nu}(t))) \\ &= \int_0^\infty P^{\alpha,\mu}(k,y) h_{\beta,\nu}(y,t) dy, \end{aligned} \quad (3.70)$$

where  $P_{\alpha,\mu}(k,t)$  is the PMF of TSFPP and  $h_{\beta,\nu}(x,t)$  is the probability density function of

inverse tempered stable subordinator. Using (3.70) and (3.68)

$$\begin{aligned}
\frac{d^{\beta,\nu}}{dt^{\beta,\nu}} P_{\beta,\nu}^{\alpha,\mu}(k,t) &= \int_0^\infty P^{\alpha,\mu}(k,y) \frac{d^{\beta,\nu}}{dt^{\beta,\nu}} h_{\beta,\nu}(y,t) dy \\
&= - \int_0^\infty P^{\alpha,\mu}(k,y) \frac{\partial}{\partial y} h_{\beta,\nu}(y,t) dy \\
&\quad - \left( t^{-\beta} M_{1,1-\beta}^{-\beta}(-\nu t) - \nu^\beta \right) \int_0^\infty P^{\alpha,\mu}(k,y) \delta(y) dy \\
&= -P^{\alpha,\mu}(k,y) h_{\beta,\nu}(y,t) \Big|_{y=0}^{y=\infty} + \int_0^\infty \frac{d}{dy} P^{\alpha,\mu}(k,y) h_{\beta,\nu}(y,t) dy \\
&\quad - \left( t^{-\beta} M_{1,1-\beta}^{-\beta}(-\nu t) - \nu^\beta \right) P^{\alpha,\mu}(k,0) \\
&= \int_0^\infty \frac{d}{dy} P^{\alpha,\mu}(k,y) h_{\beta,\nu}(y,t) dy \\
&= -((\mu + \lambda(1-B))^\alpha - \mu^\alpha) \int_0^\infty P^{\alpha,\mu}(k,y) h_{\beta,\nu}(y,t) dy \\
&= -((\mu + \lambda(1-B))^\alpha - \mu^\alpha) P_{\beta,\nu}^{\alpha,\mu}(k,t),
\end{aligned}$$

using (3.53) and the fact that  $P^{\alpha,\mu}(k,0) = 0$ ,  $k > 0$ . □

**Remark 3.3.** Using similar argument as in Prop. 3.3 with (3.62), it follows that

$$\mathbb{D}_t^{\beta,\nu} P_{\beta,\nu}^{\alpha,\mu}(k,t) = -((\mu + \lambda(1-B))^\alpha - \mu^\alpha) P_{\beta,\nu}^{\alpha,\mu}(k,t), \quad k > 0, t > 0.$$

**Proposition 3.4.** The state probabilities for TTSFPP are given by

$$P_{\beta,\nu}^{\alpha,\mu}(k,t) = (-1)^k e^{-t\nu} \sum_{m=0}^\infty t^m \nu^m \sum_{r=0}^\infty (-t^\beta)^r M_{\beta,\beta r+m+1}^r (t^\beta \nu^\beta) \sum_{h=0}^r \binom{r}{h} (-\mu^\alpha)^{r-h} \quad (3.71)$$

$$\times \sum_{l=0}^\infty \binom{\alpha h}{l} \binom{\alpha h - l}{k} \mu^l \lambda^{\alpha h - l}, \quad k = 0, 1, \dots, \mu \geq 0, \nu \geq 0, t \geq 0. \quad (3.72)$$

*Proof.* Suppose  $F(z,t)$  is the  $z$ -transform of  $P_{\beta,\nu}^{\alpha,\mu}(k,t)$ , then

$$\frac{d^{\beta,\nu}}{dt^{\beta,\nu}} F(z,t) = -((\mu + \lambda(1-z^{-1}))^\alpha - \mu^\alpha) F(z,t), \quad \beta \in (0,1), \alpha \in (0,1], \mu \geq 0, \nu \geq 0.$$

Using Laplace transform with respect to the time variable  $t$  and assuming  $|(\mu + \lambda(1-z^{-1}))^\alpha - \mu^\alpha| < |(s + \nu)^\beta - \nu^\beta|$ , it follows

$$\begin{aligned}
\mathcal{L}[F(z,t)] &= \frac{1}{s} \left( 1 + \frac{((\mu + \lambda(1-z^{-1}))^\alpha - \mu^\alpha)}{(s + \nu)^\beta - \nu^\beta} \right)^{-1} \\
&= \sum_{r=0}^\infty (-1)^r \frac{((\mu + \lambda(1-z^{-1}))^\alpha - \mu^\alpha)^r}{s((s + \nu)^\beta - \nu^\beta)^r}.
\end{aligned}$$



Suppose  $L(s) = \frac{1}{(s^\beta - \nu^\beta)^r}$ , then the inverse LT of  $L(s)$  from (3.66) and the shifting property of LT  $E(s) = L(s + \nu)$  leads to the inverse LT of  $E(s)$  equal to  $e^{-\nu t} t^{\beta r - 1} M_{\beta, \beta r}^r(\nu^\beta t^\beta)$ . Further,

$$\mathcal{L}^{-1} \left[ \frac{E(s)}{s} \right] = \int_0^t e^{-\nu y} y^{\beta r - 1} M_{\beta, \beta r}^r(\nu^\beta y^\beta) dy.$$

We use the integral from [27],

$$\int_0^t y^{\mu-1} M_{\rho, \mu}^\nu(wy^\rho)(t-y)^{\nu-1} dy = \Gamma(\nu) t^{\nu+\mu-1} M_{\rho, \mu+\nu}^\nu(wt^\rho). \quad (3.73)$$

Then  $\mathcal{L}^{-1} \left[ \frac{E(s)}{s} \right] = e^{-t\nu} \sum_{m=0}^{\infty} \nu^m t^{\beta r + m} M_{\beta, \beta r + m + 1}^r(\nu^\beta t^\beta)$ . For simplicity, we assume  $H(t) = \mathcal{L}^{-1} \left[ \frac{E(s)}{s} \right]$ . Next,

$$\begin{aligned} F(z, t) &= \sum_{r=0}^{\infty} (-1)^r \sum_{h=0}^r \binom{r}{h} (-\mu^\alpha)^{r-h} \sum_{l=0}^{\infty} \binom{\alpha h}{l} \mu^l (\lambda(1-z^{-1}))^{\alpha h - l} H(t) \\ &= \sum_{r=0}^{\infty} (-1)^r \sum_{h=0}^r \binom{r}{h} (-\mu^\alpha)^{r-h} \sum_{l=0}^{\infty} \binom{\alpha h}{l} \mu^l \lambda^{\alpha h - l} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha h - l}{k} (z)^{-k} H(t) \\ &= \sum_{k=0}^{\infty} z^{-k} \left[ (-1)^k e^{-t\nu} \sum_{m=0}^{\infty} t^m \nu^m \sum_{r=0}^{\infty} (-t^\beta)^r M_{\beta, \beta r + m + 1}^r(t^\beta \nu^\beta) \sum_{h=0}^r \binom{r}{h} (-\mu^\alpha)^{r-h} \right. \\ &\quad \left. \times \sum_{l=0}^{\infty} \binom{\alpha h}{l} \binom{\alpha h - l}{k} \mu^l \lambda^{\alpha h - l} \right], \end{aligned}$$

the result follows by taking the coefficient of  $z^{-k}$ .  $\square$

**Remark 3.4.** For  $\mu = 0$ ,  $\nu = 0$ , eq. (3.71) is equivalent to putting  $m = 0$ ,  $l = 0$  and  $r = h$ , which reduces to

$$P_{\beta, 0}^{\alpha, 0}(k, t) = (-1)^k \sum_{r=0}^{\infty} (-1)^r \binom{\alpha r}{k} \frac{\lambda^{\alpha r} t^{\beta r}}{\Gamma(\beta r + 1)}, \quad (3.74)$$

which is same as the PMF of TSFPP given in (3.43).

### 3.6 Fractional Equation with Gegenbauer Type Fractional Operator and Generalized Poisson Distributions

In this section, we introduce new class of fractional differential equations and their solutions. We consider the backward-shift fractional operator

$$\begin{aligned} \nabla_u^d &= (1 - 2uB + B^2)^d = (1 - 2\cos(\nu)B + B^2)^d \\ &= [(1 - e^{i\nu}B)(1 - e^{-i\nu}B)]^d \\ &= -\lambda^{2d} (1 - 2uB + B^2)^d P_d^u(k, t), \quad |u| \leq 1, \quad d \in (0, 1/2], \end{aligned}$$

which often appears in the study of the so-called Gegenbauer times series (see [1, 28, 29]). Note that for  $u = 1$  the fractional operator  $\nabla_u^d = (1 - B)^d$  reduces to (3.40) with  $\alpha = 2d \in (0, 1)$ . where  $u = \cos(\nu)$  or  $\nu = \cos^{-1}(u)$ . We introduce the following fractional equation for unknown function  $P_d^u(k, t)$ ,  $t \geq 0$ ,

$$\frac{d}{dt} P_d^u(k, t) = -\lambda^{2d} \nabla_u^d (P_d^u(k, t)) = -\lambda^{2d} (1 - 2uB + B^2)^d P_d^u(k, t), k > 0, d \in (0, 1/2], \quad (3.75)$$

$$\frac{d}{dt} P_d^u(0, t) = -\lambda^{2d} P_d^u(0, t), \quad (3.76)$$

with initial condition

$$P_d^u(k, 0) = \delta_{k,0}. \quad (3.77)$$

Using the  $z$ -transform in both sides, it follows

$$\frac{d}{dt} [\mathcal{Z} P_d^u(k, t)] = -\lambda^{2d} [\mathcal{Z} \{(1 - 2uB + B^2)^d P_d^u(k, t)\}].$$

Expanding the fractional difference operator as

$$(1 - 2uB + B^2)^d = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{d}{j} \binom{d}{k} (e^{i\nu})^j (e^{-i\nu})^k B^{j+k},$$

leads to

$$\frac{d}{dt} [\mathcal{Z} P_d^u(k, t)] = \left[ -\lambda^{2d} \left( 1 - \frac{2u}{z} + \frac{1}{z^2} \right)^d \right] \mathcal{Z} P_d^u(k, t). \quad (3.78)$$

Now solve the equation (3.78) for  $\mathcal{Z} P_d^u(k, t)$ , we obtain

$$\mathcal{Z} P_d^u(k, t) = A e^{-\lambda^{2d} \left( 1 - \frac{2u}{z} + \frac{1}{z^2} \right)^d t}.$$

Using initial conditions in (3.77), it follows that  $A = 1$ , and hence

$$\mathcal{Z} P_d^u(k, t) = e^{-\lambda^{2d} \left( 1 - \frac{2u}{z} + \frac{1}{z^2} \right)^d t}. \quad (3.79)$$

Further,

$$\begin{aligned} \mathcal{Z} P_d^u(k, t) &= \left[ 1 + \frac{(-\lambda)^{2dt}}{1!} \left\{ 1 + \frac{1 - 2uz}{z^2} \right\}^d + \dots + \frac{(-\lambda)^{2kdt^k}}{k!} \left\{ 1 + \frac{1 - 2uz}{z^2} \right\}^{kd} + \dots \right] \\ &= 1 + \frac{(-\lambda)^{2dt}}{1!} \left\{ \sum_{k=0}^{\infty} \binom{d}{k} \frac{\sum_{n=0}^k \binom{k}{n} (-2uz)^n}{z^{2k}} \right\} + \dots \\ &\quad + \frac{(-\lambda)^{2kdt^k}}{k!} \left\{ \sum_{k=0}^{\infty} \binom{kd}{k} \frac{\sum_{n=0}^k \binom{k}{n} (-2uz)^n}{z^{2k}} \right\} + \dots \end{aligned}$$

With the help of coefficient of  $z^{-k}$ , the inverse  $z$ -transform gives

$$P_d^u(k, t) = \sum_{r=0}^{\infty} \frac{(-\lambda)^{2rd} t^r}{r!} \left\{ \binom{rd}{p} \cdot \binom{p}{0} + (2u)^2 \binom{rd}{p+1} \cdot \binom{p+1}{2} + \dots \right. \\ \left. + (2u)^{2p-2} \binom{rd}{2p-1} \cdot \binom{2p-1}{2p-2} + 2^{2p} \binom{\alpha}{2p} \cdot \binom{2p}{2p} \right\}, \quad d \in (0, 1/2], \quad k = 2p.$$

We have the following proposition.

**Proposition 3.5.** *Solution of the initial value problem (3.75) is of the form*

$$P_d^u(k, t) = \sum_{r=0}^{\infty} \frac{(-\lambda)^{2rd} t^r}{r!} \left\{ \binom{rd}{p} \cdot \binom{p}{0} + (2u)^2 \binom{rd}{p+1} \cdot \binom{p+1}{2} + \dots \right. \\ \left. + (2u)^{2p-2} \binom{rd}{2p-1} \cdot \binom{2p-1}{2p-2} + 2^{2p} \binom{\alpha}{2p} \cdot \binom{2p}{2p} \right\}, \quad d \in (0, 1/2], \quad k = 2p. \quad (3.80)$$

**Remark 3.5.** *Taking  $\alpha = 2d, k = 2p$  in (3.36) and  $u = 1$  in (3.80), both the results coincides.*

Further, we also introduce the following Gegenbauer type space-time-fractional equation by replacing the integer order derivative in (3.75) by a fractional derivative of order  $\beta \in (0, 1]$ , as follows

$$\frac{d^\beta}{dt^\beta} Q_{d,\beta}^u(k, t) = -\lambda^d \nabla_u^d Q_{d,\beta}^u(k, t) \\ = (-\lambda)^{2d} (1 - 2uB + b^2 B)^d Q_{d,\beta}^u(k, t), \quad d \in (0, 1/2], \quad k \geq 1 \quad (3.81)$$

$$\frac{d^\beta}{dt^\beta} Q_{d,\beta}^u(0, t) = (-\lambda)^{2d} Q_{d,\beta}^u(0, t) \quad \text{with} \quad Q_{d,\beta}^{u,\beta}(k, 0) = \delta_{k,0}. \quad (3.82)$$

Using a similar approach, we have the result.

**Proposition 3.6.** *The solution for the Gegenbauer type space-time-fractional equation defined in (3.81) is*

$$Q_{d,\beta}^u(k, t) = \sum_{r=0}^{\infty} \frac{(-\lambda)^{2rd} t^{r\beta}}{\Gamma(1 + 2p\beta)} \left\{ \binom{rd}{p} \cdot \binom{p}{0} + (2u)^2 \binom{rd}{p+1} \cdot \binom{p+1}{2} + \dots \right. \\ \left. + (2u)^{2p-2} \binom{rd}{2p-1} \cdot \binom{2p-1}{2p-2} + 2^{2p} \binom{\alpha}{2p} \cdot \binom{2p}{2p} \right\}, \quad d \in (0, 1/2].$$

Note that if  $u \neq 1$  there is no stochastic process which marginals are the solutions of the equations (3.75). To see this, consider a process  $N_d^u(t)$  with marginal PMF  $P_d^u(k, t)$ , then

$$F(z, t) = \mathbb{E}[z^{-N_d^u(t)}] = \sum_{r=0}^{\infty} P_d^u(k, t) z^{-r}. \quad (3.83)$$

By normalization axiom of probability, it is necessary that  $F(1, t) = 1$  for all  $t > 0$ . From equation (3.79)

$$F(z, t) = e^{-\lambda^{2d}\left(1 - \frac{2u}{z} + \frac{1}{z^2}\right)^d t}.$$

With the condition of normalization

$$F(1, t) = e^{-2\lambda^{2d}(1-u)^d t}, \quad |u| \leq 1.$$

It is easy to see that  $F(1, t) < 1$ , for all  $u$  except the case when  $u = 1$ . So the solution  $P_d^u(k, t)$  will not satisfy the normalization condition. One can say that  $N_d^u(t)$  is a defective random variable which indicates that there is some positive mass concentrated at  $\infty$ .

One can also consider shift operators of the form,

$$\frac{d}{dt}P^*(k, t) = -\lambda[(1 - B)^{\alpha_1} + (1 - B)^{\alpha_2}]P^*(k, t), \quad (3.84)$$

with initial condition  $P^*(k, 0) = \delta_{k,0}$ . It is easy to show that

$$P^*(k, t) = (-1)^k \sum_{r=0}^{\infty} (-1)^r \frac{\lambda^r}{r!} \sum_{m=0}^r \binom{r}{m} \binom{\alpha_1 m + \alpha_2(r - m)}{k}, \quad k = 0, 1, \dots \quad (3.85)$$

Similar to the Gegenbauer shift operator case, the function  $P^*$  may not be a probability distribution.

## 4 Conclusion

In this article, we introduce and study tempered time-space-fractional Poisson processes, which may provide more flexibility in modeling of real life data. Further, we argue that  $z$ -transform is more useful than the PGF in solving the difference-differential equations since it is more general and hence may be used in the situations where the solution is not a probability distribution indeed. To support this, we work with the Gegenbauer type fractional shift operator. Our results generalize and complement the results available on time- and space-fractional Poisson processes.

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