Binomial Polynomials Mimicking Riemann's Zeta Function

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ABSTRACT
The (generalised) Mellin transforms of Gegenbauer polynomials, have polynomial factors $p_n^\lambda(s)$, whose zeros all lie on the 'critical line' $\Re s = 1/2$ (called critical polynomials). The transforms are identified in terms of combinatorial sums related to H. W. Gould's S:4/3, S:4/2 and S:3/1 binomial coefficient forms. Their 'critical polynomial' factors are then identified in terms of ${}_3F_2(1)$ hypergeometric functions. Furthermore, we extend these results to a one-parameter family of critical polynomials that possess the functional equation $p_n(s;\beta) = \pm\, p_n\,(1-s;\beta)$.

Normalisation yields the rational function $q_n^\lambda(s)$ whose denominator has singularities on the negative real axis. Moreover as $s \to \infty$ along the positive real axis, $q_n^\lambda(s) \to 1$ from below.

For the Chebyshev polynomials we obtain the simpler S:2/1 binomial form, and with $\mathcal{C}_n$ the nth Catalan number, we deduce that $4\mathcal{C}_{n-1}p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield odd integers. The results touch on analytic number theory, special function theory, and combinatorics.

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critical polynomials, binomials coefficients, Gould combinatorial summations, Mellin transforms, hypergeometric functions.

1. Introduction

The motivation for this present work is to further understand the triangle of connections that exist between binomial coefficients, functions which only have critical zeros (those on the line $\Re s = 1/2$ or zeros on the real line, and henceforth referred to as *critical polynomials*), and prime numbers.

As stated by K. Dilcher and K. B. Stolarsky, [1]

> Two of the most ubiquitous objects in mathematics are the sequence of prime numbers and the binomial coefficients (and thus Pascal's triangle). A connection between the two is given by a well-known characterisation of the prime numbers: Consider the entries in the kth row of Pascal's triangle, without the initial and final entries. They are all divisible by k if and only if k is a prime".

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By considering a modified form of Pascal's triangle, whose kth row consists of the integers

$$a(k,j) := \frac{(2k-1)(2k+1)}{2j+3}\binom{k+j}{2j+1}, \qquad k \in \mathbb{N}, \qquad 0 \le j \le k-1, \tag{1.1}$$

Dilcher and Stolarsky obtained an analogous characterisation of pairs of twin prime numbers $(2k-1, 2k+1)$. This says that the entries in the kth row of the $a(k,s)$ number triangle are divisible by $2k-1$ with exactly one exception, and are divisible by $2k+1$ with exactly one exception, if and only if $(2k-1, 2k+1)$ are a pair of twin prime numbers.

The analogous sequence of polynomials $A_k(x)$ obtained from the kth row of the number triangle generated by the integers $a(k,j)$ is given by $A_k(x) = \sum_{j=0}^{k-1} a(k,j)x^j$. It was shown in [1] that this polynomial family satisfies the four-term recurrence relation

$$A_{k+4}(x) = (2x+4)\left(A_{k+3}(x) + A_{k+1}(x)\right) - (4x^2+4x+6)A_{k+2}(x) - A_k(x),$$

as opposed to a three-term recurrence relation required for orthogonality, and so they do not constitute an orthogonal polynomial system (e.g. [see 2, p.42-44]).

However it is also shown in [1] that the polynomials $A_k(x)$ are closely linked to the orthogonal system of Gegenbauer polynomials $C_n^\lambda(x)$ with $\lambda = 2$ by

$$A_k(x) = C_{k-1}^2\left((x+2)/2\right) + (x+6)C_{k-2}^2\left((x+2)/2\right) + C_{k-3}^2\left((x+2)/2\right).$$

The Genenbauer polynomials are defined for $\lambda > -1/2$, $\lambda \neq 0$ (e.g., [7]), by the hypergeometric series representation [see 3, p.773-802], and also in terms of binomial coefficients and powers of 2 such that

$$C_n^\lambda(x) = \frac{(2\lambda)_n}{n!}\, {}_2F_1\left(2\lambda+n, -n; \lambda+\frac{1}{2}; \frac{1-x}{2}\right) = \sum_{r=0}^{\lfloor n/2 \rfloor} (-1)^r \tbinom{n-r}{r}\tbinom{n-r-1+\lambda}{n-r}(2x)^{n-2r}. \tag{1.2}$$

The Legendre Polynomials $P_n(x)$ are the case $\lambda = 1/2$ of the Gegenbauer polynomials $C_n^{1/2}(x)$, and a close connection between these polynomials, the prime numbers and the absolute value of the Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, defined for $\Re(s) > 1$, was established in [4], where $|\zeta(s)|$ is expressed as an infinite sum over products of Legendre polynomials and functions derived from prime numbers.

The location of the zeros of the Riemann zeta function is famously known as the Riemann Hypothesis (1859), which states that all of the non-trivial zeros of $\zeta(s)$ (the trivial zeros lie at the negative even integers) lie on the critical line $\Re s = 1/2$. In 1901 von Koch reinforced the connection between $\zeta(s)$ and the prime numbers, demonstrating that the Riemann Hypothesis is equivalent to the statement that the error term for $\pi(x)$, the number of primes up to x, is of order of magnitude $O\left(\sqrt{x}\log(x)\right)$ [5]. Riemann had originally shown that

$$\pi(x) \sim \mathrm{Li}(x) + \sum_{n=2}^{\infty} \frac{\mu(n)}{n}\mathrm{Li}(x^{1/n}), \qquad \text{where} \qquad \mathrm{Li} = \int_2^x \frac{\mathrm{d}u}{\log u},$$

and with $\mu(n)$ the Möbius function, which returns 0 if n is divisible by a prime squared and $(-1)^k$ if n is the product of k distinct primes.

The Báez-Duarte equivalence to the Riemann Hypothesis [6] links the Riemann Hypothesis (and so the prime numbers) to binomial coefficients, via the infinite sequence of real numbers c_t, defined such that $c_t := \sum_{s=0}^{t}(-1)^s\binom{t}{s}\zeta(2s+2)^{-1}$, with the assertion that the Riemann hypothesis is true if and only if $c_t = O(t^{-3/4+\epsilon})$, for integers $t \geq 0$, and for all $\epsilon > 0$.

In relation to understanding the triangle of connections that exist between the three objects consisting of the prime numbers, the binomial coefficients, and functions which only have critical zeros, it is those between the binomial coefficients and the 'critical polynomials' that appears to be the least studied, thus motivating the results contained in this paper.

Before elaborating further, we mention some standard notation in which ${}_2F_1$ denotes the Gauss hypergeometric function, ${}_pF_q$ the generalized hypergeometric function, and

$$(a)_n = \Gamma(a+n)/\Gamma(a) = (-1)^n\Gamma(1-a)/\Gamma(1-a-n)$$

is the Pochhammer symbol, with Γ the gamma function [7,8]. We also set $\varepsilon = 0$ for n even and $\varepsilon = 1$ for n odd. Our starting point is the following definition:

Definition 1.1. For $\lambda > -1/2$, we define the generalised Mellin transform $M_n^\lambda(s)$, such that

$$M_n^\lambda(s) = \int_0^1 \frac{C_n^\lambda(x)x^{s-1}}{(1-x^2)^{3/4-\lambda/2}}dx = \int_0^{\pi/2} \cos^{s-1}\theta\; C_n^\lambda(\cos\theta)\sin^{\lambda-1/2}\theta\; d\theta, \tag{1.3}$$

wherein $x = \cos\theta$, and we assume that $\Re s > 0$ for n even and $\Re s > -1$ for n odd, denoting by $p_n^\lambda(s)$ the polynomial factor of $M_n^\lambda(s)$. Then for $\lambda = 1$ we have the generalised Mellin transform $M_n(s)$ of the Chebyshev functions [9] of the second kind

$$M_n(s) \equiv \int_0^1 x^{s-1}U_n(x)\frac{dx}{(1-x^2)^{1/4}}. \tag{1.4}$$

The integral transform (1.3) may be evaluated (see Theorem 2.1) using the formula below, which gives the more general class of integrals in terms of special functions such that [see 8, p.517 2.21.2(1)]

$$\int_0^a x^{\alpha-1}(a^2-x^2)^{\beta-1}C_{2n+\varepsilon}^\lambda(cx)\,\mathrm{d}x = \frac{(-1)^n(\lambda)_{n+\varepsilon}c^\varepsilon a^{\alpha+2\beta+\varepsilon-2}}{2^{1-\varepsilon}n!}B\left(\frac{\alpha+\varepsilon}{2},\beta\right)$$

$$\times {}_3F_2\left(-n, n+\lambda+\varepsilon, (\alpha+\varepsilon)/2; \varepsilon+(1/2), (\alpha+\varepsilon+2\beta)/2; a^2c^2\right), \tag{1.5}$$

where $\varepsilon \in \{0,1\}; a, \Re\beta > 0; \Re\alpha > -\varepsilon$, and $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, is the beta function.

In [10,11], Mellin transforms were used on $[0,\infty)$. Here we consider Mellin transformations for functions supported on $[0,1]$. For properties of the Mellin transform, we mention [12].

Our main results show that the polynomial factors $p_n^\lambda(s)$ of the Mellin transforms in (1.3) of the Gegenbauer (and so Chebyshev) functions $C_n^\lambda(x)$, yield families of 'critical polynomials' $p_n^\lambda(s)$, $n = 0, 1, 2, \ldots$, of degree $\lfloor n/2 \rfloor$, satisfying the functional equation $p_n^\lambda(s) = (-1)^{\lfloor n/2 \rfloor} p_n^\lambda(1-s)$. Additionally we find that (up to multiplication by a constant) these polynomials can be written explicitly as variants of Gould S:4/1 and S:3/2 binomial sums (see [13]), the latter form being

$$p_{2n+\varepsilon}^\lambda(s) = n!(2n+\varepsilon)!\binom{n+\lambda-1+\varepsilon}{n+\varepsilon}\binom{n+\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{r}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}}{\binom{n+r+\varepsilon}{r}\binom{\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}+r}{r}}. \tag{1.6}$$

In the case of the Chebyshev polynomials ($\lambda = 1$), this simplifies to the $S : 2/1$ form, due to cancellation of binomial factors, and with $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, s an integer, we show that polynomials $4\mathcal{C}_{n-1}p_{2n}(s)$ and $\mathcal{C}_n p_{2n+1}(s)$ yield integers with only odd prime factors.

The 'critical polynomials' under consideration here, in a sense motivate the Riemann hypothesis, and have many important applications to analytic number theory. For example, using the Mellin transforms of Hermite functions, Hermite polynomials multiplied by a Gaussian factor, Bump and Ng [10] were able to generalise Riemann's second proof of the functional equation of the zeta function $\zeta(s)$, and to obtain a new representation for it.

The polynomial factors of the Mellin transforms of Bump and Ng are realised as certain ${}_2F_1(2)$ Gauss hypergeometric functions [11]. In a different setting, the polynomials $p_n(x) = {}_2F_1(-n, -x; 1; 2) = (-1)^n\, {}_2F_1(-n, x+1; 1; 2)$ and $q_n(x) = i^n n! p_n(-1/2 - ix/2)$ were studied [14], and they directly correspond to the Bump and Ng polynomials with $s = -x$. Kirschenhofer, Pethö, and Tichy considered combinatorial properties of p_n, and developed Diophantine properties of them. Their analytic results for p_n include univariate and bivariate generating functions, and that its zeros are simple, lie on the line $x = -1/2 + it$, $t \in \mathbb{R}$, and that its zeros interlace with those of p_{n+1} on this line. These polynomials can be written as $p_n(x) = \binom{n+x}{n}\, {}_2F_1(-n, -x; -n-x; -1)$.

Example 1.2. The first few transformed polynomials $p_n^\lambda(s)$, are given by

$$\begin{aligned}
p_0^\lambda(s) =& 1/2, \quad p_1^\lambda(s) = \lambda, \quad p_2^\lambda(s) = \frac{1}{4}\lambda(2\lambda+1)(2s-1) = \frac{1}{2}\lambda(2\lambda+1)\left(s - \frac{1}{2}\right), \\
p_3^\lambda(s) =& \frac{1}{2}\lambda(\lambda+1)(2\lambda+1)(2s-1) = \lambda(\lambda+1)(2\lambda+1)\left(s - \frac{1}{2}\right), \\
p_4^\lambda(s) =& \frac{1}{8}\lambda(\lambda+1)(2\lambda+1)\left(8\lambda s^2 - 8\lambda s + 6\lambda + 12s^2 - 12s + 15\right), \\
=& \frac{1}{8}\lambda(\lambda+1)(2\lambda+1)\left(s - \left(\tfrac{1}{2} - \tfrac{i\sqrt{9+9\lambda+2\lambda^2}}{3+2\lambda}\right)\right)\left(s - \left(\tfrac{1}{2} + \tfrac{i\sqrt{9+9\lambda+2\lambda^2}}{3+2\lambda}\right)\right), \\
p_5^\lambda(s) =& \frac{1}{4}\lambda(\lambda+1)(\lambda+2)(2\lambda+1)\left(8\lambda s^2 - 8\lambda s + 14\lambda + 12s^2 - 12s + 51\right) \\
=& \frac{1}{4}\lambda(\lambda+1)(\lambda+2)(2\lambda+1)\left(s - \left(\tfrac{1}{2} - \tfrac{i\sqrt{3}\sqrt{2\lambda^2+11\lambda+12}}{3+2\lambda}\right)\right)\left(s + \left(\tfrac{1}{2} - \tfrac{i\sqrt{3}\sqrt{2\lambda^2+11\lambda+12}}{3+2\lambda}\right)\right)
\end{aligned}$$

Previous results obtained by the authors related to this area of research are discussed in [15,16], where in the former paper families of 'critical polynomials' are obtained from generalised Mellin transforms of classical orthogonal Legendre polynomials. In the latter paper sequences of 'critical polynomials' are considered which can also be obtained by generalised Mellin transforms of families of orthogonal polynomials whose coefficients are the weighted binomial coefficients defined by $B_k(x) = \sum_{j=0}^{k} \frac{2k+1}{2j+1} \binom{k+j}{2j} x^j$. There it was established that for $\Re s > -1/4$, the generalized Mellin transforms

$$M_n^B(s) = \int_{-4}^{0} \frac{B_n(x) x^{s-3/4}}{(4+x)^{3/4}} dx = (-1)^{s+5/4} 4^s 4^{-n-1} \Gamma(1/4) p_n(s) \frac{\Gamma\left(s+\frac{1}{4}\right)}{\Gamma\left(s+\frac{2n+1}{2}\right)},$$

yield critical polynomial factors $p_n(s)$, which obey the *perfect reflection* functional equation $p_n(s) = \pm p_n(1-s)$.

The 'perfect-reflection' functional equation $p_n^\lambda(s) = (-1)^{\lfloor n/2 \rfloor} p_n^\lambda(1-s)$, is similar to that for Riemann's xi function $\xi(s)$, defined by $\xi(s) = \frac{1}{2} s(s-1)\pi^{-s/2}\Gamma\left(\frac{1}{2}s\right)\zeta(s)$, and which satisfies $\xi(s) = \xi(1-s)$, so that for $t \in \mathbb{R}$, the zeros of $\xi(1/2+it)$ and $\zeta(1/2+it)$ are identical. Drawing upon this analogy, one interpretation is that the polynomials $p_n^\lambda(s)$ are normalised (from a functional equation perspective) polynomial forms of the rational functions $q_n^\lambda(s)$, defined for $n \in \mathbb{N}$ and $\varepsilon \in \{0,1\}$ by

$$q_{2n+\varepsilon}^\lambda(s) = \frac{2^\varepsilon\, p_{2n+\varepsilon}^\lambda(s)}{\lambda(n-1+\varepsilon)!(2n)!\binom{2n+2\lambda-1+\varepsilon}{2n-1+\varepsilon}\binom{n+\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}}{n}}, \tag{1.7}$$

where both numerator and denominator polynomials of $q_n^\lambda(s)$ are of degree $\lfloor n/2 \rfloor$.

For $\lambda > -1/2$, $\lambda \neq 0$, and $\Re s > 0$, the $\lfloor n/2 \rfloor$ linear factors of the denominator polynomials of $q_n^\lambda(s)$, are each non-zero, so that for these values of λ, we have $q_n^\lambda(s)$ has no singularities with $\Re s > 0$. Hence the rational function $q_n^\lambda(s)$ has the same 'critical zeros' as the polynomial $p_n^\lambda(s)$, and for $t \in \mathbb{R}$, the roots of $p_n^\lambda(1/2+it)$ and $q_n^\lambda(1/2+it)$ are identical. It obeys the binomial functional equation

$$q_n^\lambda(s) = (-1)^{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor + \frac{1-s+\lambda+\varepsilon}{2} - \frac{3}{4}}{\lfloor n/2 \rfloor} \binom{\lfloor n/2 \rfloor + \frac{s+\lambda+\varepsilon}{2} - \frac{3}{4}}{\lfloor n/2 \rfloor}^{-1} q_n^\lambda(1-s).$$

The $[n/2]$ poles of $q_n^\lambda(s)$ (so zeros of the denominator polynomial of $q_n^\lambda(s)$) occur on the negative real axis when $2s = 3 - \lambda - 4j$ or $2s = 1 - 2\lambda - 4j$ (depending on the parity of n) and for $s \in (1, \infty)$, we find that $q_n^\lambda(s)$ takes values on $(0,1)$, as detailed in Theorem 2.6 (b) from which it follows that on $\mathbb{R}_{>1}$, the behaviour of $q_n^\lambda(s)$ has similarities to that of $1/\zeta(s)$, albeit with a rate of convergence to the limit point 1, considerably slower than for $1/\zeta(s)$.

To give an overview, the present work is split into four sections, with the main results concerning the critical polynomials arising from Mellin transforms of Gegenbauer polynomials appearing after this introduction in the second section. In the third section we prove these results, utilising continuous Hahn polynomials to locate the 'critical zeros'. The fourth and concluding section then considers further possible extensions to these results.

2. Critical Polynomial Results

In the following statements we set

$$M_0^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s}{2}\right)}{2\Gamma\left(\frac{s+\lambda}{2}+\frac{1}{4}\right)}, \qquad M_1^\lambda(s) = 2\lambda M_0^\lambda(s+1), \quad \text{and} \quad N_n^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+n}{2}\right)}{2\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)}, \tag{2.1}$$

where as previously mentioned we take $\varepsilon \in \{0, 1\}$.

Theorem 2.1. *The Mellin transforms* (1.3) *may be written as* ${}_3F_2(1)$ *hypergeometric functions, such that*

$$M_{2n+\varepsilon}^\lambda(s) = (-1)^n M_0^\lambda(s+\varepsilon)(2n+2)^\varepsilon \binom{\lambda+n-1+\varepsilon}{n+\varepsilon}\, {}_3F_2\left(-n, \lambda+n+\varepsilon, \tfrac{s+\varepsilon}{2}; \tfrac{1}{2}+\varepsilon, \tfrac{\lambda+s+\varepsilon}{2}+\tfrac{1}{4}; 1\right), \tag{2.2}$$

or equivalently, and independently of ε *as*

$$M_n^\lambda(s) = N_n^\lambda(s)\binom{2\lambda+n-1}{n}\, {}_3F_2\left(\tfrac{\lambda}{2}+\tfrac{1}{4}, \tfrac{1-n}{2}, -\tfrac{n}{2}; \tfrac{1}{2}+\lambda, 1-\tfrac{(n+s)}{2}; 1\right). \tag{2.3}$$

Theorem 2.2. *The Mellin transforms* (1.3) *satisfy: (a) the recurrence relation*

$$nM_n^\lambda(s) = 2(\lambda+n-1)M_{n-1}^\lambda(s+1) - (2\lambda+n-2)M_{n-2}^\lambda(s),$$

(b) the generating function $G^\lambda(s,t) = \int_0^1 (1-x^2)^{\lambda/2-3/4}(1-2tx+t^2)^{-\lambda}x^{s-1}\,dx$

$$= \sum_{k=0}^\infty M_k^\lambda(s)t^k = \frac{\Gamma\left(\frac{1}{4}+\frac{\lambda}{2}\right)}{2(1+t^2)^\lambda}\left[\frac{\Gamma(\lambda)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}+\frac{1}{4}\right)}\, {}_3F_2\left(\frac{\lambda+1}{2}, \frac{\lambda}{2}, \frac{s}{2}; \frac{1}{2}, \frac{s+\lambda}{2}+\frac{1}{4}; \frac{4t^2}{(1+t^2)^2}\right)\right.$$

$$\left. + \frac{2t\Gamma(\lambda+1)}{(1+t^2)}\frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+\lambda}{2}+\frac{3}{4}\right)}\, {}_3F_2\left(\frac{\lambda+1}{2}, 1+\frac{\lambda}{2}, \frac{s+1}{2}; \frac{3}{2}, \frac{s+\lambda}{2}+\frac{3}{4}; \frac{4t^2}{(1+t^2)^2}\right)\right],$$

(c) the recurrence relation in s

$$-4(s-1)(s-2)M_n^\lambda(s-2) + [6 - 4(\lambda+2\lambda n+n^2) - 16s + 8s(s+1)]M_n^\lambda(s)$$

$$+[-9+4(n+\lambda)^2 + 16(s+2) - 4(s+2)(s+3)]M_n^\lambda(s+2) = 0,$$

with $M_0^\lambda(s)$ *and* $M_1^\lambda(s)$ *as defined in (2.1), and*
(d) the polynomial factors $p_n^\lambda(s)$ *of* $M_n^\lambda(s)$*, have zeros only on the critical line,*
(e) the polynomial factors satisfy the functional equation $p_n^\lambda(s) = (-1)^{\lfloor n/2\rfloor}p_n^\lambda(1-s)$.

Theorem 2.3. *The Mellin transforms* (1.3) *may be written as a constant multiplied by a variant on Gould's combinatorial S:4/2 and S:3/1 functions, such that*

$$M_{2n+\varepsilon}^\lambda(s) = M_0^\lambda(s+\varepsilon)\sum_{r=0}^n \frac{(-1)^{n-r}2^{2r+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{n+r+\varepsilon}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{s+\varepsilon-2}{2}+r}{r}\binom{n+\frac{s+\varepsilon+\lambda}{2}-\frac{3}{4}}{n-r}}{\binom{n}{r}\binom{n+\frac{s+\varepsilon+\lambda}{2}-\frac{3}{4}}{n}}$$

$$= M_0^\lambda(s+\varepsilon)\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{n+r+\varepsilon}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{s+\varepsilon-2}{2}+r}{r}}{\binom{\frac{s+\varepsilon+\lambda}{2}-\frac{3}{4}+r}{r}}.$$

Corollary 2.4. *When either $s = 2t$ or $s = 2t+1$, an even or odd positive integer then we respectively have*

$$M_0^\lambda(2t) = \frac{1}{4t}\binom{\frac{\lambda}{2}+\frac{1}{4}+t-1}{t}^{-1}, \qquad M_1^\lambda(2t+1) = 2\lambda M_0^\lambda(2t+2) = \frac{\lambda}{2t+2}\binom{\frac{\lambda}{2}+\frac{1}{4}+t}{t+1}^{-1}.$$

Hence, when s is a non-negative integer then the expressions for $M_n^\lambda(s)$ given in Theorem 2.3 can be written as variants of the Gould S:4/3 form.

Theorem 2.5. *Let $n \in \mathbb{N}_0$ and $\varepsilon = (1-(-1)^n)/2$. Then the Mellin transforms are of the form*

$$M_n^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+\varepsilon}{2}\right)}{\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)n!}\,p_n^\lambda(s) = L_n^\lambda(s)p_n^\lambda(s) \quad \textit{(say)},$$

where the polynomial factors $p_n^\lambda(s)$ can be written in terms of hypergeometric functions such that

$$p_n^\lambda(s) = \frac{2(n!)\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\binom{2\lambda+n-1}{n}\,{}_3F_2\left(\frac{\lambda}{2}+\frac{1}{4}, \frac{1-n}{2}, -\frac{n}{2}; \lambda+\frac{1}{2}, 1-\frac{(n+s)}{2}; 1\right), \tag{2.4}$$

satisfying the difference equation in s

$$[6-4(\lambda+2\lambda n+n^2)-16s+8s(s+1)]\left(\frac{s+\varepsilon}{2}-1\right)\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)p_n^\lambda(s)$$

$$+[-9+4(n+\lambda)^2-4(s-1)(s+2)]\left(\frac{s+\varepsilon}{2}\right)\left(\frac{s+\varepsilon}{2}-1\right)p_n^\lambda(s+2)$$

$$-4(s-1)(s-2)\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)\left(\frac{s+n+\lambda}{2}-\frac{3}{4}\right)p_n^\lambda(s-2) = 0. \tag{2.5}$$

Theorem 2.6 (Binomial 'critical polynomial' theorem)**.** *(a) The polynomials $p_n^\lambda(s)$, can be written (up to multiplication by a constant) in terms of binomial coefficients and powers of 2, as a variant of a Gould S:4/1 combinatorial function, such that*

$$p_{2n+\varepsilon}^\lambda(s) = n!(2n+\varepsilon)!\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{n+r+\varepsilon}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}\binom{n+\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}}{n-r}}{\binom{n}{r}},$$

or as a Gould S:3/2 combinatorial function variant, such that

$$p_{2n+\varepsilon}^\lambda(s) = n!(2n+\varepsilon)!\binom{n+\lambda-1+\varepsilon}{n+\varepsilon}\binom{n+\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{r}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}}{\binom{n+r+\varepsilon}{r}\binom{\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}+r}{r}},$$

thus establishing the binomial 'critical polynomial' relationship (1.6).
(b) Let $q_n^\lambda(s)$ denote the rational function in s derived from the S:3/2 form of the 'critical polynomial' $p_n^\lambda(s)$ such that

$$q_{2n+\varepsilon}^\lambda(s)=\frac{(2n+\varepsilon)\binom{n+\lambda-1+\varepsilon}{n+\varepsilon}}{\lambda\binom{2n+2\lambda-1+\varepsilon}{2n-1+\varepsilon}}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1+\varepsilon}\binom{n+r+\lambda-1+\varepsilon}{r}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}}{\binom{n+r+\varepsilon}{r}\binom{\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}+r}{r}},$$

Then we have the equivalent form for $q_{2n+\varepsilon}^\lambda(s)$ given in (1.7)

$$q_{2n+\varepsilon}^\lambda(s)=\frac{2^{2n+1}p_{2n+\varepsilon}^\lambda(s)}{(2\lambda)_{2n+\varepsilon}\prod_{j=1}^{n}(2(s+\varepsilon)+2\lambda+4j-3)},$$

where both numerator and denominator polynomials of $q_n^\lambda(s)$ are of degree $\lfloor n/2\rfloor$.
(c) For $\lambda>-1/2$, $\lambda\neq 0$, and $\Re s>0$, the rational function $q_n^\lambda(s)$ has no singularities, and has the same 'critical zeros' as the polynomial $p_n^\lambda(s)$, so that for $t\in\mathbb{R}$, the roots of $p_n^\lambda(1/2+it)$ and $q_n^\lambda(1/2+it)$ are identical.

When $s\in\mathbb{R}_{>1}$, $q_n^\lambda(s)$ takes values on $(0,1)$, with $\lim_{s\to\infty}q_n^\lambda(s)=1$ (from below), as does $1/\zeta(s)$, albeit with a rate of convergence considerably slower than than that for $1/\zeta(s)$. We also have that $q_n^\lambda(s)$ obeys the binomial ratio functional equation

$$q_n^\lambda(s)=(-1)^{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor+\frac{1-s+\lambda+\varepsilon}{2}-\frac{3}{4}}{\lfloor n/2\rfloor}\binom{\lfloor n/2\rfloor+\frac{s+\lambda+\varepsilon}{2}-\frac{3}{4}}{\lfloor n/2\rfloor}^{-1}q_n^\lambda(1-s). \tag{2.6}$$

Corollary 2.7. *The polynomial factors arising from the Mellin transform of the Chebyshev polynomials have the simpler form as a variant of a Gould S:2/1 combinatorial function, such that*

$$p_{2n+\varepsilon}(s)=n!(2n+\varepsilon)!\binom{n+\frac{s+\varepsilon}{2}-\frac{1}{4}}{n}\sum_{r=0}^{n}\frac{(-1)^{n-r}2^{2r-1+\varepsilon}\binom{n+r+\varepsilon}{2r+\varepsilon}\binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}}{\binom{\frac{s+\varepsilon}{2}-\frac{1}{4}+r}{r}},$$

For $s=t\in\mathbb{Z}$ an integer, and with $\mathcal{C}_n=\frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number, the polynomials $4\mathcal{C}_{n-1}p_{2n}(t)$ and $\mathcal{C}_np_{2n+1}(t)$ yield odd integers. Moreover the polynomials

$$(2^{2n+1}/(2n)!)p_{2n}(t),\quad \text{and}\quad (2^{2n+1}\mathcal{T}_{n+1})/((2n+2)!)p_{2n+1}(t), \tag{2.7}$$

with $\mathcal{T}_{n+1}$ the largest odd factor of $n+1$, yield odd integers with fewer prime factors.

Theorem 2.8 (Perfect reflection property theorem)**.** *We say that $f(s)$ has the 'perfect reflection property' to mean $f(\overline{s})=\overline{f(s)}$, $f(s)=\chi f(1-s)$, with $\chi=\pm1$, $f(s)=0$, only when $\Re s=1/2$.*

Let $n\in\mathbb{N}_0$ and $\varepsilon=(1-(-1)^n)/2$. Then the polynomials

$$p_n(s;\beta)=\frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)}\,{}_3F_2\left(1-\beta,\frac{1-n}{2},-\frac{n}{2};2(1-\beta),1-\frac{(n+s)}{2};1\right),$$

have the perfect reflection property *with $\chi(n)=(-1)^{\lfloor n/2\rfloor}$, wherein $\beta<1$, of degree $\lfloor n/2\rfloor$, satisfy the functional equation $p_n(s;\beta)=(-1)^{\lfloor n/2\rfloor}p_n(1-s;\beta)$. These poly-*

nomials have zeros only on the critical line, and all zeros $\neq 1/2$ occur in complex conjugate pairs.

Corollary 2.9. *(a) The properties of Theorem 2.8 are satisfied by the polynomials*

$$p_n(s;0) = \frac{2(n+s)}{(n+1)(n+2)} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)} \left[1 - \,_2F_1\left(-\frac{(n+1)}{2}, -\frac{n}{2} - 1; -\frac{(n+s)}{2}; 1\right)\right]$$

$$= \frac{2(n+s)}{(n+1)(n+2)} \frac{\Gamma\left(\frac{n+s}{2}\right)}{\Gamma\left(\frac{s+\varepsilon}{2}\right)} \left[1 - \frac{\Gamma\left(-\frac{(n+s)}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} \frac{\Gamma\left(\frac{n+3-s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\right].$$

(b) More generally, for β a negative integer, the properties of Theorem 2.8 are satisfied by the polynomials $p_n(s;-m)$, and these polynomials may be written in terms of elementary factors and the gamma function.

Table 1.: Table of values of $(2^{2n+1}/(2n)!)p_{2n}(t)$, for $0 \leq n \leq 5$, $-4 \leq t \leq 4$.

n \ t	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
0	1	1	1	1	1	1	1	1	1
1	-27	-21	-15	-9	-3	3	9	15	21
2	421	261	141	61	21	21	61	141	261
3	-7119	-3969	-1995	-861	-231	231	861	1995	3969
4	154665	80361	36729	13401	3465	3465	13401	36729	80361
5	-4029795	-1946637	-828135	-293073	-65835	65835	293073	828135	1946637

Table 2.: Table of values of $((2^{2n+1}\mathcal{T}_{n+1})/(2n+2)!)p_{2n+1}(t)$, for $0 \leq n \leq 5$, $-4 \leq t \leq 4$.

n \ t	**-4**	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**
0	1	1	1	1	1	1	1	1	1
1	-9	-7	-5	-3	-1	1	3	5	7
2	279	183	111	63	39	39	63	111	183
3	-1341	-819	-465	-231	-69	69	231	465	819
4	128637	72765	37581	17325	8157	8157	17325	37581	72765
5	-1809459	-959805	-465975	-197505	-52731	52731	197505	465975	959805

3. Proof of the Main Results

Proof of Theorem 2.1. Setting $a = 1$, $\alpha = s$, $\beta = \frac{\lambda}{2} + \frac{1}{4}$, and $c = 1$ in (1.5), we obtain (2.2). Taking $n = 0$ and $n = 1$ in (2.2) gives us (2.1).

To see the hypergeometric form (2.3) of $M_n^\lambda(s)$, we use the $C_n^\lambda(x)$ series representation [see 17, p.278 (6)], to obtain the expression

$$M_n^\lambda(s) = \sum_{k=0}^{[n/2]} \frac{(2\lambda)_n(-1)^k}{4^k k!(\lambda+1/2)_k(n-2k)!} \int_0^1 x^{s+n-2k-1}(1-x^2)^{k+\lambda/2-3/4}dx.$$

Applying the beta integral,

$$\int_0^1 x^{a-1}(1-x^2)^{b-1}dx = \frac{1}{2}B\left(\frac{a}{2},b\right), \qquad \Re a > 0, \quad \Re b > 0,$$

and replacing with gamma factors and rearranging we then obtain

$$M_n^\lambda(s) = \frac{\Gamma\left(\frac{\lambda}{2}+\frac{1}{4}\right)\Gamma\left(\frac{s+n}{2}\right)}{2\Gamma\left(\frac{s+n+\lambda}{2}+\frac{1}{4}\right)}\binom{2\lambda+n-1}{n}\,{}_3F_2\left(\tfrac{\lambda}{2}+\tfrac{1}{4},\tfrac{1-n}{2},-\tfrac{n}{2};\tfrac{1}{2}+\lambda,1-\tfrac{(n+s)}{2};1\right),$$

as required, where the above has used the identity $\frac{1}{\Gamma(n-2k+1)} = \frac{4^k}{n!}\left(-\frac{n}{2}\right)_k\left(\frac{1-n}{2}\right)_k$. $\square$

Proof of Theorem 2.2. (a) follows from ([see 7, p.303] or [see 8, p.1030])

$$(n+2)C_{n+2}^\lambda(x) = 2(\lambda+n+1)xC_{n+1}^\lambda(x) - (2\lambda+n)C_n^\lambda(x), \quad C_0^\lambda(x) = 1, \quad C_1^\lambda(x) = 2\lambda x.$$

(b) follows from $(1-2xt+t^2)^{-\lambda} = \sum_{n=0}^\infty C_n^\lambda(x)t^n$ (see [7, p,302], or [8, p.1029]).
(c) To obtain the difference equation for $M_n^\lambda(s)$, we apply the ordinary differential equation satisfied by Gegenbauer polynomials [see 8, p.1031],

$$(x^2-1)y''(x) + (2\lambda+1)xy'(x) - n(2\lambda+n)y(x) = 0.$$

If $f(x) \equiv C_n^\lambda(x)/(1-x^2)^{3/4-\lambda/2}$, we then substitute $C_n^\lambda(x) = (1-x^2)^{3/4-\lambda/2}f(x)$ into this differential equation. We then find that

$$\frac{1}{4}(1-x^2)^{-1/4-\lambda/2}\left[(6-4(\lambda+2\lambda n+n^2)+(-9+4(\lambda+n)^2)x^2)f(x)\right.$$

$$\left.+4(x^2-1)(-4xf'(x)+(1-x^2)f''(x))\right] = 0.$$

It follows that the quantity in square brackets is zero. We multiply it by x^{s-1} and integrate from $x=0$ to 1, integrating the f' term once by parts, and the f'' term twice by parts. We determine that the Mellin transforms satisfy the following difference equation:

$$-4(s-1)(s-2)M_n^\lambda(s-2) + [6-4(\lambda+2\lambda n+n^2) - 16s + 8s(s+1)]M_n^\lambda(s)$$

$$+[-9+4(n+\lambda)^2+16(s+2)-4(s+2)(s+3)]M_n^\lambda(s+2) = 0.$$

and hence the result.
(d) The case $\lambda = \frac{1}{2}$ was proven in [16] and the case $\lambda = \frac{3}{2}$ can be deduced similarly. To show that the resulting zeros of $p_n^\lambda(s)$ occur only on $\Re s = 1/2$ for general $\lambda > -\frac{1}{2}$ we apply a connection with continuous Hahn polynomials [see 7, p,331],

$$h_m(x;a,b,c,d) = i^m\frac{(a+c)_m(a+d)_m}{m!}\,{}_3F_2\left(-m,m+a+b+c+d-1,a+ix;a+c,a+d;1\right). \tag{3.1}$$

We use the transformation of a terminating ${}_3F_2(1)$ series

$$ {}_3F_2(-n,a,b;c,d;1) = \frac{(a)_n(c+d-a-b)_n}{(c)_n(d)_n}\, {}_3F_2(-n,c-a,d-a;1-a-n,c+d-a-b;1) $$

to obtain

$$ h_m(x;a,b,c,d) = \frac{i^m}{m!}(a+b+c+d+m-1)_m(1-b-m-ix)_m $$

$$ \times\, {}_3F_2(1-b-c-m,1-b-d-m,-m;2-a-b-c-d-2m,1-b-m-ix;1). $$

Then comparing with the ${}_3F_2(1)$ function for $M_n^\lambda(s)$ given in (2.3) we see when $n=2m$ is even, that setting $x=\frac{i}{2}\left(-s+\frac{1}{2}\right)$, $\quad a=c=\frac{1}{2}-\frac{\lambda}{2}-m, \quad b=d=\frac{1}{4}$, our polynomial factors $p_n^\lambda(s)$ are proportional to continuous Hahn polynomials such that

$$ p_n^\lambda(s) = (m!)^2 2^{2m-1}(-i)^m \binom{m+\lambda-1}{m} h_m\left(\frac{-i}{2}\left(s-\frac{1}{2}\right);\frac{1}{2}-\frac{\lambda}{2}-m,\frac{1}{4},\frac{1}{2}-\frac{\lambda}{2}-m,\frac{1}{4}\right). \tag{3.2} $$

Similarly setting $x=\frac{i}{2}\left(-s+\frac{1}{2}\right)$, $\quad a=c=-\frac{\lambda}{2}-m, \quad b=d=\frac{3}{4}$, when $n=2m+1$ is odd, our polynomial factors $p_n^\lambda(s)$ are proportional to continuous Hahn polynomials such that

$$ p_n^\lambda(s) = (m!)^2 2^{2m}(-i)^m \lambda \binom{m+\lambda}{m} h_m\left(\frac{-i}{2}\left(s-\frac{1}{2}\right);-\frac{\lambda}{2}-m,\frac{3}{4},-\frac{\lambda}{2}-m,\frac{3}{4}\right). \tag{3.3} $$

For fixed values of a,b,c,d, the continuous Hahn polynomials are an orthogonal system of polynomials which satisfy the recurrence relation [see 7, (6.10.11)]

$$ A_m\hat{h}_{m+1}(x) = ((a+ix)+A_m+C_m)\hat{h}_m(x) - C_m\hat{h}_{m-1}(x), $$

where

$$ \hat{h}_m(x) \;:= \hat{h}_m(x;a,\ b,\ c,\ d) = D_m h_m(x) = \frac{m!}{i^m(a+c)_m(a+d)_m} h_m(x;a,\ b,\ c,\ d), $$

so that $D_m = m!/(i^m(a+c)_m(a+d)_m)$, $\hat{h}_m(x)$ is the ${}_3F_2$ hypergeometric function given in (3.1), and

$$ A_m = -\tfrac{(m+a+b+c+d-1)(m+a+c)(m+a+d)}{(2m+a+b+c+d-1)(2m+a+b+c+d)}, \quad C_m = \tfrac{m(m+b+c-1)(m+b+d-1)}{(2m+a+b+c+d-2)(2m+a+b+c+d-1)}. $$

In the case that $a=\bar{c}$ and $b=\bar{d}$, we have $A_m\hat{h}_{m+1}(x) = ix\hat{h}_m(x) - C_m\hat{h}_{m-1}(x)$, and substituting for $\hat{h}_m$ and rearranging we have

$$ h_{m+1}(x) = \frac{ixD_m}{D_{m+1}A_m}h_m(x) - \frac{C_mD_{m-1}}{D_{m+1}A_m}h_{m-1}(x) = G_m\,x\,h_m(x) - H_m\,h_{m-1}(x), \tag{3.4} $$

say, where $G_m = \frac{iD_m}{D_{m+1}A_m}$, $\quad$ and $\quad H_m = \frac{C_mD_{m-1}}{D_{m+1}A_m}$.

As the a, b, c, d values must be constant for the conditions of the orthogonality theorems to be met, we set $m = u$, constant in the variable a, so that in (3.4) we take $a = c = \frac{1}{2} - \frac{\lambda}{2} - u, \quad b = d = \frac{1}{4}$ to obtain $h_{m+1}(x) = G_m\, x\, h_m(x) - H_m\, h_{m-1}(x)$, with

$$G_m = \frac{(-2\lambda + 4m - 4u + 1)(-2\lambda + 4m - 4u + 3)}{2(m+1)(-2\lambda + 2m - 4u + 1)}, \tag{3.5}$$

and

$$H_m = \frac{(2m-1)(2\lambda - 4m + 4u + 1)(-2\lambda + 4m - 4u + 3)(-\lambda + m - 2u)}{16(m+1)(-2\lambda + 2m - 4u + 1)}. \tag{3.6}$$

Similarly setting $a = c = -\frac{\lambda}{2} - u, \quad b = d = \frac{3}{4}$ in (3.4) gives us $h_{m+1}(x) = G_m\, x\, h_m(x) - H_m\, h_{m-1}(x)$, with G_m as in (3.5), but where H_m is now given by

$$H_m = \frac{(2m+1)(2\lambda - 4m + 4u + 1)(-2\lambda + 4m - 4u + 3)(-\lambda + m - 2u - 1)}{16(m+1)(-2\lambda + 2m - 4u + 1)}. \tag{3.7}$$

For the above coefficient G_m, and the two choices for the coefficient H_m, both of the resulting recurrence relations $h_{m+1}(x) = G_m\, x\, h_m(x) - H_m\, h_{m-1}(x)$ are of the form $P_{m+1}(x) = (A_m x + B_m)P_m(x) - C_m P_{m-1}(x)$, with $B_m = 0$, which is the form of the recurrence relation satisfied by a system of orthogonal polynomials that are not monic [see 18, p.19 (4.2)].

If $\tilde{P}_m(x)$ is the monic (scaled) polynomial corresponding to $P_m(x)$, so that $P_m(x) = k_m \tilde{P}_m$, where $k_{-1} = 1$, then in generality one finds that [see 18, p.19 (4.3)]

$$A_m = \frac{k_{m+1}}{k_m}, \quad , B_m = -c_{m+1}\frac{k_{m+1}}{k_m}, \quad C_m = \mu_n \frac{k_{m+1}}{k_{m-1}} = \mu_n A_m A_{m-1},$$

so that $c_m = 0$, $\mu_m = C_m/(A_m A_{m-1})$, and where the corresponding monic recurrence relation can be written as

$$\tilde{P}_m(x) = x\tilde{P}_{m-1}(x) - \mu_m \tilde{P}_{m-2}(x), \quad \tilde{P}_{-1}(x) = 0, \quad \tilde{P}_0(x) = 1, \quad m = 1, 2, 3, \ldots \tag{3.8}$$

For our two recurrence relations we find that $\mu_m = H_m/(G_m G_{m-1})$, so that in the cases $n = 2m$ is even, and $n = 2m + 1$ is odd, we respectively have

$$\mu_m = \frac{m(2m-1)(2\lambda-2m+4u+1)(-\lambda+m-2u)}{4(-2\lambda+4m-4u-3)(-2\lambda+4m-4u+1)}, \quad \mu_m = \frac{m(2m+1)(2\lambda-2m+4u+1)(-\lambda+m-2u-1)}{4(-2\lambda+4m-4u-3)(-2\lambda+4m-4u+1)}. \tag{3.9}$$

For fixed values of u and λ, and $m = 0, 1, 2, 3, \ldots$, with $\mu_m \neq 0$ and well defined, we obtain the pair of families of Hahn polynomials given by

$$h_m\left(x; \frac{1}{2} - \frac{\lambda}{2} - u, \frac{1}{4}, \frac{1}{2} - \frac{\lambda}{2} - u, \frac{1}{4}\right), \quad \text{and} \quad h_m\left(x; -\frac{\lambda}{2} - u, \frac{3}{4}, -\frac{\lambda}{2} - u, \frac{3}{4}\right). \tag{3.10}$$

In both expressions in (3.10) we find that $h_{-1}(s) = 0$ and $h_0(s) = 1$, and applying Favard's Theorem [see 18, p.21] concerning polynomial sequences satisfying the three-term recurrence relation given in (3.8), for $c_m = 0$, and noting that $\mu_m \neq 0$ it follows that the pair of expressions given in (3.10) form two families of orthogonal polynomial systems.

It is a well known fact (e.g. [18, p.27], or [2,9]) that systems of orthogonal polynomials have only real zeros which interlace. Hence the polynomial families corresponding to (3.10) have only real zeros which interlace on the real line. Setting $x = \left(\frac{-i}{2}\right)\left(s - \frac{1}{2}\right)$ in (3.10), the resulting polynomials will therefore have their zeros dilated by the factor of $\frac{1}{2}$ on the real line; rotated by $\frac{\pi}{2}$ clockwise onto the imaginary axis by the factor of $-i$, and then translated by $-\frac{1}{2}$ from the imaginary axis to the critical line $\Re s = \frac{1}{2}$. Henece setting $x = \left(\frac{-i}{2}\right)\left(s - \frac{1}{2}\right)$ in (3.10), yields families of polynomials which have zeros only on the critical line $\Re\, s = \frac{1}{2}$.

Restricting $u \in \mathbb{N}$, to be a positive integer, analysis of (3.10) shows that for $\lambda \neq \frac{1}{2}$ and $\lambda \neq \frac{3}{2}$, μ_m is well defined for $u \leq m$, and when $\lambda = 1$, μ_m is well defined for $u < 2m$. If λ is not an integer or half-integer then μ_m is well defined for all $u \in \mathbb{N}$.

Therefore, for each integer value $u \leq m$, and fixed $\lambda > -\frac{1}{2}$, $\lambda \neq \frac{1}{2}$, $\lambda \neq \frac{3}{2}$, we get two families of Hahn 'critical polynomials' in (3.10). For each particular family, there will exist one value of m, namely $m = u$, where the Hahn polynomial corresponds to that given for $p_n^\lambda(s)$ given in (3.2). This argument implies that all the zeros of our polynomial families $p_n^\lambda(s)$ lie on the critical line $\Re s = \frac{1}{2}$, as required. (e) The functional equation can be deduced directly from the fact that $B_m = 0$ in the above recurrence relations discussed in the proof of part (d).

□

Remark 3.1. If a family of polynomials with only critical zeros whose distribution of zeros is proportional to that of the Riemann zeta function is ever discovered, then it would be of great interest to apply the above arguments and see when the recurrence coefficients are well defined.

Proof of Theorem 2.3. The binomial forms of $M_n^\lambda(s)$ are obtained by rewriting the hypergeometric forms given in Theorem 2.1 in terms of Pochhammer symbols, and then rearranging them into binomial coefficients.

Corollary 2.4 then follows immediately by by replacing $M_0^\lambda(s)$ and $M_0^\lambda(s+1)$ with their equivalent binomial coefficients forms when s is respectively an even or an odd integer. □

Proof of Theorem 2.5. It follows from either part (c) of Theorem 2.2 or the hypergeometric form in Theorem 2.3, that the Mellin transforms are of the form given in (2.5).

The difference equation for $p_n^\lambda(s)$ given in (2.5) follows from part (c) or Theorem 2.2, where the Mellin transform expression (2.5) in terms of $p_n^\lambda(s)$ is substituted. Noting that the factor $\Gamma\left(\frac{\lambda}{2} + \frac{1}{4}\right)/(2n!)$ is independent of s, and repeatedly applying the functional equation $\Gamma(z+1) = z\Gamma(z)$, the result follows.

□

Proof of Theorem 2.6. (a) The S:4/1 type combinatorial expressions for the polynomial factors $p_n^\lambda(s)$ are obtained from the S:4/2 type expressions for $M_n^\lambda(s)$ in Theorem 2.3, by respectively multiplying through by the factors

$$\frac{n!(2n)!}{2M_0^\lambda(s)}\binom{n + \frac{s+\lambda}{2} - \frac{3}{4}}{n}, \qquad \text{or} \qquad \frac{n!(2n+1)!}{2M_0^\lambda(s+1)}\binom{n + \frac{s+1+\lambda}{2} - \frac{3}{4}}{n},$$

depending on whether n is odd or even.
(b) The two expressions for $q_n^\lambda(s)$ given can be verified by inserting the explicit expressions for p_n^λ given in part (a) into the latter expression for $q_n^\lambda(s)$ given in part (b)

and rearranging. The degree of both numerator and denominator polynomials of $q_n^\lambda(s)$ being $\lfloor n/2 \rfloor$, then follows from the degree of the polynomials $p_n^\lambda(s)$ given in Theorem 2.2, and the number of s-linear factors appearing in the denominator product of $q_n^\lambda(s)$.
(c) The zeros of the denominator polynomials (and so poles of q_n^λ), correspond to the zeros of the linear factors $2s+2\lambda+4j-3$, or $2s+2\lambda+4j-1$, with $1 \le j \le \lfloor n/2 \rfloor$. For $\lambda > -1/2$, $\lambda \neq 0$ and $\Re s > 0$, each linear factor is non-zero, ensuring that the rational function $q_n^\lambda(s)$ has no singularities. Hence the 'critical zeros' of the polynomials $p_n^\lambda(s)$, are the same as for $q_n^\lambda(s)$, and so for $t \in \mathbb{R}$, the roots of $p_n^\lambda(1/2+it)$ and $q_n^\lambda(1/2+it)$ are identical.

To see that the rational functions $q_n^\lambda(s)$ are normalised with limit 1 as $s \to \infty$, we consider the limit term by term as $s \to \infty$ in the S:3/2 sum

$$\lim_{s\to\infty} \sum_{r=0}^{n} \frac{(-1)^{n-r} 2^{2r-1+\varepsilon} \binom{n+r+\lambda-1+\varepsilon}{r} \binom{n+r+\varepsilon}{2r+\varepsilon} \binom{\frac{1}{2}(s+\varepsilon-2)+r}{r}}{\binom{n+r+\varepsilon}{r} \binom{\frac{1}{2}(s+\varepsilon+\lambda)-\frac{3}{4}+r}{r}} = \sum_{r=0}^{n} \frac{(-1)^{n-r} 2^{2r-1+\varepsilon} \binom{n+r+\lambda-1+\varepsilon}{r} \binom{n+r+\varepsilon}{2r+\varepsilon}}{\binom{n+r+\varepsilon}{r}},$$

Applying the combinatorial identity

$$\sum_{r=0}^{n} \frac{(-1)^{n-r} 2^{2r-1+\varepsilon} \binom{n+r+\lambda-1+\varepsilon}{r} \binom{n+r+\varepsilon}{2r+\varepsilon}}{\binom{n+r+\varepsilon}{r}} = \frac{n+\varepsilon}{2n+\varepsilon} \binom{2n+2\lambda-1+\varepsilon}{2n-1+\varepsilon} \binom{n+\lambda-1+\varepsilon}{n-1+\varepsilon}^{-1},$$

we then have the upper bounds for the combinatorial sums, so that $\lim_{s\to\infty} q_n^\lambda(s) = 1$ from below, as required. The functional equation follows from that for $p_n^\lambda(s)$, by considering the third and fourth displays in part (b) of the theorem.

To see Corollary 2.7, substituting $\lambda = 1$ in the S/3:2 forms for $p_n^\lambda(s)$, simplifies the sums to the Gould S/2:1 combinatorial functions stated. Term-by-term analysis of the $n+1$ terms in each sum then reveals that for s an integer, each term is an even integer apart form the $r=0$ term, given by $\frac{n!(2(n+\varepsilon))!}{2} \binom{n+\frac{s+\varepsilon}{2}-\frac{1}{4}}{n}$.

The binomial coefficients contribute the power of two 2^{-2n}, so that the power of 2 in the $r=0$ term is determined by $(2n)!/2^{2n+1}$ when n is even, and $(2n+2)!/2^{2n+1}$, when n is odd. Noting that the $n!$ terms cancel between numerator and denominator, we see that multiplying through by the reciprocal of these respective powers of 2 will produce odd integer values for the $r=0$ term, whilst leaving the others terms $r = 1, 2, \ldots, n$ even. Hence the summation results in odd integers, being the sum of n even numbers and one odd number.

Analysis of the nth Catalan number $\mathcal{C}_n = \frac{1}{n+1}\binom{2n}{n}$, shows that the power of 2 in the $\mathcal{C}_n$ is determined by $2^{2n+1}/(2n+2)!$ (A048881 in the OEIS) so that $4\mathcal{C}_{n-1}$ and $\mathcal{C}_n$ have the respective reciprocal powers of 2 to p_{2n} and p_{2n+1}. It follows that for $s \in \mathbb{Z}$ we have $4\mathcal{C}_{n-1}p_{2n}$ and $\mathcal{C}_n p_{2n+1}$ are odd integers. A slight modification of this argument also removes the odd factors arising in the $(2n)!$ and $(2n+1)!$ polynomial factors, where $\mathcal{T}_{n+1}$ is the largest odd factor of $n+1$, as required. □

Proof of Theorem 2.8 and Corollary 2.9. We note the integral representation

$$\int_0^1 (1-x)^{-\beta} x^{-\beta} \, {}_2F_1\left(\frac{1-n}{2}, -\frac{n}{2}; 1 - \frac{(n+s)}{2}; x\right) dx$$

$$= 2^{2\beta-1} \frac{\sqrt{\pi}\,\Gamma(1-\beta)}{\Gamma(3/2-\beta)} \, {}_3F_2\left(1-\beta, \frac{1-n}{2}, -\frac{n}{2}; 2(1-\beta), 1 - \frac{(n+s)}{2}; 1\right), \tag{2.4}$$

with $\beta < 1$, so that the ${}_2F_1(x)$ function is transformed to a ${}_2F_1(1-x)$ function and the result follows. The 'critical zeros' follow from Theorem 2.2 (d), setting $\lambda = 3/2 - 2\beta$.

To see Corollary 2.9 (a) The initial $\beta = 0$ reduction of Theorem 2.8 to ${}_2F_1$ form follows from the series definition of the ${}_3F_2$ function with a shift of summation index and the relations $(1)_j/(2)_j = 1/(j+1)$ and $(\kappa)_{j-1} = (\kappa-1)_j/(\kappa-1)$. The second reduction is a consequence of Gauss summation. (b) Similarly, with m a positive integer, $(m+1)_j/(2(m+1))_j$ may be reduced and partial fractions applied to this ratio. Then with shifts of summation index, the ${}_3F_2$ function may be reduced to a series of ${}_2F_1(1)$ functions. These in turn may be written in terms of ratios of gamma functions from Gauss summation.

□

4. Discussion

Given the Gould variant combinatorial expressions obtained for $p_n^\lambda(s)$ and $q_n^\lambda(s)$, our results invite several other research questions, such as: is there a combinatorial interpretation of $p_n^\lambda(s)$ or $q_n^\lambda(s)$, and more generally, of $p_n(s;\beta)$? Relatedly, is there a reciprocity relation for $p_n(s)$ and $p_n(s;\beta)$?

Two instances when the combinatorial sums produce "nice" combinatorial expressions are

$$q_{2n}^\lambda(1) = \sum_{r=0}^n \frac{(-1)^{n-r}2^{2r-1}\binom{n+r+\lambda-1}{r}\binom{n+r}{2r}\binom{r-\frac{1}{2}}{r}}{\binom{n+r}{r}\binom{\frac{\lambda}{2}-\frac{1}{4}+r}{r}} = \frac{1}{2}\binom{n+\frac{2\lambda-3}{4}}{n}\binom{n+\frac{2\lambda-1}{4}}{n}^{-1},$$

$$q_{2n+1}^\lambda(2) = \sum_{r=0}^n \frac{(-1)^{n-r}2^{2r}\binom{n+r+\lambda}{r}\binom{n+r+1}{2r+1}\binom{r+\frac{1}{2}}{r}}{\binom{n+r+1}{r}\binom{\frac{\lambda}{2}+\frac{3}{4}+r}{r}} = (n+1)\binom{n+\frac{2\lambda-3}{4}}{n}\binom{n+\frac{2\lambda+3}{4}}{n}^{-1}.$$

The Gegenbauer polynomials have the integral representation

$$C_n^\lambda(x) = \frac{1}{\sqrt{\pi}}\frac{(2\lambda)_n}{n!}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda)}\int_0^\pi (x+\sqrt{x^2-1}\cos\theta)^n \sin^{2\lambda-1}\theta\, d\theta. \tag{4.1}$$

Then binomial expansion of part of the integrand of $M_n^\lambda(s)$ is another way to obtain this Mellin transform explicitly. The representation (4.1) is also convenient for showing further special cases that reduce in terms of Chebyshev polynomials U_n or Legendre or associated Legendre polynomials P_n^m. We mention as examples

$$C_n^2(x) = \frac{1}{2(x^2-1)}[(n+1)xU_{n+1}(x) - (n+2)U_n(x)], \tag{4.2}$$

and

$$C_n^{3/2}(x) = \frac{(n+1)}{(x^2-1)}[xP_{n+1}(x) - P_n(x)] = -\frac{P_{n+1}^1(x)}{\sqrt{1-x^2}}.$$

The Gegenbauer polynomials are a special case of the two-parameter Jacobi polynomials $P_n^{\alpha,\beta}(x)$ [7] such that $C_n^\lambda(x) = (2\lambda)_n/\left(\lambda+\frac{1}{2}\right)_n P_n^{\lambda-1/2,\lambda-1/2}(x)$. The Jacobi polynomials are orthogonal on $[-1,1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$. Therefore, it is also of interest to consider Mellin transforms such as

$$M_n^{\alpha,\beta}(s) = \int_0^1 x^{s-1} P_n^{\alpha,\beta}(x)(1-x)^{\alpha/2-1/2}(1+x)^{\beta/2-1/2}dx,$$

especially as the Jacobi polynomials can be written in the binomial form

$$P_n^{\alpha,\beta}(x) = \sum_{j=0}^{n} \binom{n+\alpha}{k}\binom{n+\beta}{n-s}\left(\frac{x-1}{2}\right)^{n-s}\left(\frac{x+1}{2}\right)^{s}.$$

In fact this line of enquiry may provide a far more general approach to investigate 'critical polynomials' arising from combinatorial sums.

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