Categorical actions of generalised braids

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Summary

In [32], Khovanov and Thomas constructed a categorical action of the braid group Br_n on the derived category $D(T^* \mathrm{Fl}_n)$ of coherent sheaves on the cotangent bundle of the variety Fl_n of the complete flags in $\mathbb{C}^n$.

In this thesis, we define the generalised braid category $\mathcal{GB}r_n$, we define the notion of a skein-triangulated representation of $\mathcal{GB}r_3$, give a sufficient condition for the existence of a skein-triangulated representation of $\mathcal{GB}r_3$ and we construct a skein triangulated representation of $\mathcal{GB}r_3$ on $D(T^*(\mathrm{Fl}_3(\bar{i}))$ that generalises the Khovanov and Thomas categorical braid action on $D(T^* \mathrm{Fl}_3)$.

Declarations

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— Chapter 1 —

Introduction

The bounded derived category of coherent sheaves can be viewed as an ultimate cohomological invariant of an algebraic variety. The notion of the derived category, together with the notion of a triangulated category which axiomatises it, was invented by Grothendieck and Verdier in 1960s in their search for the natural context for the generalisations of Serre duality and for the existence of the right adjoint of a direct image functor [46].

In the past few decades it has become increasingly more relevant in many areas of algebraic geometry. In particular, in 1994 Kontsevich formulated the Homological Mirror Symmetry conjecture at the International Congress of Mathematicians in Zürich ([33]) which was a far reaching mathematical generalisation and interpretation of a certain duality between the families of 3-dimensional Calabi-Yau varieties observed several years earlier by string theorists. The duality was on the level of Hodge numbers: there were pairs of Calabi-Yau 3-folds X and X' for which:

$$\dim \mathrm{H}^{\mathrm{p}}(X, \Omega^q) = \dim \mathrm{H}^{\mathrm{n-p}}(X, \Omega^q).$$

Kontsevich's Homological Mirror Symmetry conjecture stated that if X and X' are two such dual Calabi-Yau manifolds, then the derived category of coherent sheaves $\mathrm{D^b}(\mathrm{Coh}(X))$ is equivalent to the derived Fukaya category of $D^b\mathrm{Fuk}(X')$ and vice versa: $\mathrm{D^b}(\mathrm{Coh}(X)) \simeq D^b\mathrm{Fuk}(X')$. The powerful intuition gained from this conjecture led to the discovery of many new mathematical structures, including spherical and $\mathbb{P}^n$-objects and their generalisations, which lie at the heart of this thesis.

Roughly, the main point of the notion of the derived category is that working with complexes is better than working with their (co)homologies. For example, there exist topological spaces X, Y such that their homologies are isomorphic $H_*(X) \cong H_*(Y)$, but X and Y are not homotopy equivalent; while the Whitehead theorem states that two simplicial complexes X and Y have homotopy equivalent geometric realizations $|X|$ and $|Y|$ if, and only if, there exists a simplicial complex Z and simplicial maps $f : Z \to X$ and $g : Z \to Y$ which are *quasi-isomorphisms*, that is they induce isomorphisms between the homology groups $H_i(X)$, $H_i(Z)$ and $H_i(Y)$. The notion of derived category is a realisation of the same principle: work with complexes of objects and formally invert the quasi-isomorphisms to identify any two complexes with naturally isomorphic cohomologies. We refer to [44] and [15] for an introduction to derived categories and to [34] and [27] for technical references.

Given an abelian category $\mathcal{A}$, define the category $C(\mathcal{A})$ to be the category of complexes of objects of $\mathcal{A}$. For technical purposes, we first invert all the homotopy equivalences: define the *homotopy category* $K(\mathcal{A})$ to be the additive category whose objects are complexes of objects in $\mathcal{A}$

$$\cdots \longrightarrow C^{n-1} \xrightarrow{d^{n-1}} C^n \xrightarrow{d^n} C^{n+1} \longrightarrow \cdots \qquad (n \in \mathbb{Z}, d^n \circ d^{n-1} = 0)$$

and whose morphisms spaces are the quotients of those in $C(\mathcal{A})$ by the subspaces of null-homotopic maps. The *derived category* $D(\mathcal{A})$ is the category whose objects are the same as those of $K(\mathcal{A})$, and whose morphisms $A^\bullet \to B^\bullet$ are certain equivalence classes fs^{-1} of pairs (s, f)

$$A^\bullet \xleftarrow{s} C^\bullet \xrightarrow{f} B^\bullet$$

of morphisms in $K(\mathcal{A})$ with s a quasi-isomorphim.

If X is a smooth quasi-projective variety we write $D^b(X)$ for the bounded derived category of the abelian category $Coh(X)$ of coherent sheaves on X, that is $D^b(X) := D^b(Coh(X))$. The structure of $D^b(X)$ can be studied by considering its autoequivalences and in this context categorical group actions play an important role. A categorical group action of a group G on $D^b(X)$ is an assignment of an autoequivalence F_g of $D^b(X)$ to every element $g \in G$ such that the group operation is compatible, up to isomorphism, with the composition of functors.

The classical result by Khovanov and Thomas in [32] states there exist n-1 autoequivalences T_i of the derived category of the total space of the complete flag variety in $\mathbb{C}^n$ which satisfy the braid relations:

$$\begin{aligned} T_iT_j &\cong T_jT_i && \text{for } |i-j| \geq 1. \\ T_iT_jT_i &\cong T_jT_iT_j && \text{for } |i-j| = 1. \end{aligned}$$

In other words, that the braid group Br_n acts categorically on $D^b(T^*Fl_n)$. Here braids are configurations of n disjoint pieces of string with n fixed endpoints, considered up to isotopies which keep the strands disjoint.

In the work of Khovanov and Thomas, configurations of n points represent the derived category of the cotangent space of complete flags in $\mathbb{C}^n$, and the cobordism between the two configurations represents an autoequivalence of this category. In this thesis, we generalise the Kohovanov and Thomas result in dimension 3 to a skein-triangulated action of the category $\mathcal{GB}r_3$ of generalised braids on the derived categories of the cotangent bundles of the varieties of complete and partial flags in $\mathbb{C}^3$. Generalised braids are the braids whose strands are allowed to touch in a certain way: they can join up (two at a time), continue as a multiple strand, and split apart. Moreover, we do not distinguish any permutations of individual strands within a multiple strand — only the multiplicity matters. Therefore, technically, we define generalised braids as a certain kind of trivalent coloured graphs with fixed univalent startpoints and endpoints and satisfying flow conditions.

Due to strands having multiplicity, instead of a single endpoint configuration consisting of n disjoint points, they have multiple endpoint configurations corresponding to the ordered partitions of n. This, together with the fact that such braids are no longer necessarily invertible, implies that generalised braids form a category rather than a group or a groupoid.

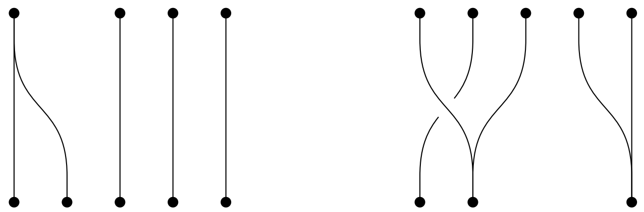


Figure 1.1: Some generalised braids

In Chapter 2 we describe complete and partial flag varieties as homogeneous spaces, describe their Picard group, and give a description of $T^*(\mathrm{Fl}_n)$ as Springer resolution of the nilpotent cone of $\mathfrak{sl}_n$.

In Chapter 3 we give an overview of autoequivalences of and braid group actions on the derived categories of coherent sheaves of smooth (quasi-)projective varieties.

In Chapter 4, we define the generalised braid category $\mathcal{GB}r_n$, we define the notion of a skein-triangulated representation of $\mathcal{GB}r_3$, give a sufficient condition for the existence of a skein-triangulated representation of $\mathcal{GB}r_3$ and we construct a skein triangulated representation of $\mathcal{GB}r_3$ on $D(T^*(\mathrm{Fl}_3(\bar{i}))$ that generalises the Khovanov and Thomas categorical braid action on $D(T^*\,\mathrm{Fl}_3)$.

— Chapter 2 —

Springer resolutions and flag varieties

Flag varieties are interesting geometrical objects, on one hand these Fano varieties are a natural generalisation of projective spaces and Grasmannians, on the other hand they are the model example of the notion of homogeneous spaces.

The aim of this chapter is to give a description of flag varieties as homogeneous spaces, understand their Picard group, give a convenient description of the total space of the cotangent bundle of complete and partial flag varieties. For the complete flag variety, we identify this total space with the Springer resolution of the nilpotent cone of $\mathfrak{sl}_n$.

In the first section, for a complex connected Lie group G, we define the nilpotent cone $\mathcal{N}_\mathfrak{g}$ of its Lie algebra $\mathfrak{g}$ and the Springer resolution $\tilde{\mathcal{N}}_\mathfrak{g}$ of $\mathcal{N}_\mathfrak{g}$. We then define the homogeneous space G/B, and identify the Springer resolution with the cotangent bundle T^*G/B. In the second section, we describe the Picard group of Grassmannians and flag varieties.

2.1 The nilpotent cone and its Springer resolution

In this section we describe the Springer resolution of the nilpotent cone of a semisimple Lie group following the Chapter 3 of [20].

Let G be a complex connected Lie group and let $\mathfrak{g}$ be its Lie algebra, viewed as the tangent space at the identity T_eG. Assume G to be semisimple, i.e. $\mathfrak{g}$ is semisimple. Let B_0 a Borel subgroup of G, let $T_0 \subset B_0$ be a maximal torus of G and let U_0 be the unipotent radical of B_0. Let moreover $\mathfrak{g}, \mathfrak{b}_0, \mathfrak{h}_0$ and $\mathfrak{n}_0$ be the respective Lie algebras of G, B, T and U.

Definition 2.1.1. Let G be a Lie group and $\mathfrak{g}$ be its Lie algebra. The *adjoint action* Ad_G of G on $\mathfrak{g}$ is the differential of the adjoint action of G on itself: $g \in G$ acts on $\mathfrak{g}$ by the differential $d(g(-)g^{-1}) : \mathfrak{g} \longrightarrow \mathfrak{g}$. The *coadjoint action* Ad^*_G of G on $\mathfrak{g}$ is the differential of the coadjoint action of G on itself: $g \mapsto d(g^{-1}(-)g)$.

Remark 2.1.2. *The differential of the adjoint action* $Ad_G : G \to GL(\mathfrak{g})$ *is the adjoint representation of Lie Algebras* $ad_\mathfrak{g} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$.

Example 2.1.3. Let $G = GL_n$ so that $\mathfrak{g} = \mathfrak{gl}(\mathbb{C}^n)$. The adjoint action Ad_G is the action by matrix conjugation.

Definition 2.1.4. Define $\mathcal{B}_{\mathfrak{g}}$ to be the set of all Borel subalgebras of $\mathfrak{g}$.

Proposition 2.1.5. *Let G be a Lie group and let $\mathfrak{g}$ be its Lie algebra. The adjoint action Ad_G of G on $\mathfrak{g}$ defines a transitive action of G on $\mathcal{B}_{\mathfrak{g}}$.*

Proof. Section 3.1 of [20] or [12]. □

Proposition 2.1.6. *The normaliser $N_B(G)$ of a Borel subgroup B of G is B.*

Proof. Lemma 3.12 of [20] or [12]. □

Remark 2.1.7. *We have $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0$ and we have $\mathfrak{n}_0 = [\mathfrak{b}_0, \mathfrak{b}_0]$. More generally, any $\mathfrak{b} \in \mathcal{B}_{\mathfrak{g}}$ contains the canonical subalgebra $\mathfrak{n}_{\mathfrak{b}} = [\mathfrak{b}, \mathfrak{b}]$ which consists of all ad-nilpotent elements of $\mathfrak{b}$. Under the adjoint action of G, if $\mathfrak{b}_0$ is sent to some $\mathfrak{b} \in \mathcal{B}_{\mathfrak{g}}$, then $\mathfrak{n}_0$ is sent to $\mathfrak{n}_{\mathfrak{b}}$, but $\mathfrak{h}_0$ can be sent to any of the Cartan subalgebras $\mathfrak{h} \subset \mathfrak{b}$.*

Definition 2.1.8. An element $x \in \mathfrak{g}$ is *ad-nilpotent* if $ad_x^n = 0$ for some $n \in \mathbb{N}$.

Remark 2.1.9. *If $\mathfrak{g} = \mathfrak{sl}_n$ then $x \in \mathfrak{g}$ is ad-nilpotent in the sense of Definition 2.1.8 if and only if its matrix is nilpotent.*

Definition 2.1.10. Define the *nilpotent cone* $\mathcal{N}_{\mathfrak{g}}$ of $\mathfrak{g}$ to be the set of all ad-nilpotent elements of $\mathfrak{g}$.

Proposition 2.1.11. *The set $\mathcal{N}_{\mathfrak{g}}$ has a natural structure of a quasi-projective variety: it is a closed subvariety of $\mathfrak{g}$ stable under Ad_G and $\mathbb{C}^*$ actions, i.e. a cone variety singular at the origin. The set $\mathcal{B}_{\mathfrak{g}}$ has the structure of a smooth projective variety: it is the closed subvariety of the Grassmannian $\mathrm{Gr}(\dim(\mathfrak{b}), \mathfrak{g})$ formed by all solvable Lie subalgebras of $\mathfrak{g}$.*

Proof. Section 3.1.6 of [20]. □

Example 2.1.12. Let $\mathfrak{g} = \mathfrak{sl}_2$ generated as Lie algebra by the elements

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.1}$$

subject to the relations

$$[x, y] = h \quad [h, x] = 2x \quad [h, y] = -2y.$$

Then the nilpotent cone is the space

$$\mathcal{N}_{\mathfrak{g}} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}_2 \;\middle|\; -a^2 - bc = 0 \right\}, \tag{2.2}$$

which is a quadratic cone in $\mathbb{C}^3$. Set

$$B_0 = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \quad \text{and } T_0 = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\},$$

their respective Lie algebras $\mathfrak{b}_0$ and $\mathfrak{h}_0$ consist of the traceless matrices which are upper triangular and diagonal, respectively. Hence, in the language of Example **??**:

$$\mathfrak{b}_0 = \langle x, h\rangle,\ \mathfrak{h}_0 = \langle h\rangle, \quad \text{and} \quad \mathfrak{n}_0 = \langle x\rangle,$$

and we see explicitly the decomposition $\mathfrak{b}_0 = \mathfrak{h}_0 \oplus \mathfrak{n}_0$ of Remark 2.1.7.

Denote by $\{e_1, e_2\}$ the standard basis of $\mathbb{C}^2$. We can describe $\mathfrak{b}_0$ as the traceless maps $\mathbb{C}^2 \to \mathbb{C}^2$ which preserve $\langle e_1\rangle$, $\mathfrak{h}_0$ as the traceless maps that preserve both $\langle e_1\rangle$ and $\langle e_2\rangle$, and $\mathfrak{n}_0$ as the maps which send the whole $\mathbb{C}^2$ to $\langle e_1\rangle$, and $\langle e_1\rangle$ to zero.

Since all $\mathfrak{b} \in \mathcal{B}_\mathfrak{g}$ are conjugate to $\mathfrak{b}_0$, each $\mathfrak{b} \in \mathcal{B}_\mathfrak{g}$ is the set of all traceless maps preserving a line in $\mathbb{C}^2$, whence $\mathcal{B}_\mathfrak{g}$ is isomorphic to $\mathbb{P}^1$.

Definition 2.1.13. For any Borel subgroup $B \subset G$, define G/B to be the set of all the left cosets $\{xB \mid x \in G\}$.

Remark 2.1.14. *By Propositions 2.1.5 and 2.1.6, the assignment*

$$g \mapsto g \cdot_{Ad_G} \mathfrak{b}_0$$

gives a bijection

$$G/B_0 \xrightarrow{\sim} \mathcal{B}_\mathfrak{g}. \tag{2.3}$$

It can be shown that the G/B_0 admits a natural structure of a smooth projective variety and that the bijection (2.3) is a G-equivariant isomorphism of varieties, see section 23.3 of [29] or [12].

Definition 2.1.15. Define

$$\tilde{\mathfrak{g}} = \{(x, \mathfrak{b}) \in \mathfrak{g} \times \mathcal{B}_\mathfrak{g} \mid x \in \mathfrak{b}\}, \tag{2.4}$$

$$\tilde{\mathcal{N}_\mathfrak{g}} \coloneqq \{(x, \mathfrak{b}) \in \mathcal{N}_\mathfrak{g} \times \mathcal{B}_\mathfrak{g} \mid x \in \mathfrak{b}\}. \tag{2.5}$$

and let the two maps

$$\begin{tikzcd} & \tilde{\mathcal{N}_\mathfrak{g}} \arrow[dl, "\mu"'] \arrow[dr, "\pi"] & \\ \mathcal{N}_\mathfrak{g} & & \mathcal{B}_\mathfrak{g}. \end{tikzcd} \tag{2.6}$$

be the corresponding natural projections.

Definition 2.1.16. The Springer morphism is the map

$$\mu : \tilde{\mathcal{N}_\mathfrak{g}} \longrightarrow \mathcal{N}_\mathfrak{g}. \tag{2.7}$$

Remark 2.1.17. *Consider the trivial vector bundle over $\mathcal{B}_\mathfrak{g}$ with fibers $\mathcal{N}_\mathfrak{g}$. We can view $\pi\colon \tilde{\mathcal{N}_\mathfrak{g}} \to \mathcal{B}_\mathfrak{g}$ as its subbundle whose fiber at any point $\mathfrak{b} \in \mathcal{B}_\mathfrak{g}$ is $\mathfrak{n}_\mathfrak{b}$.*

Remark 2.1.18. *$\tilde{\mathcal{N}_\mathfrak{g}}$ is a smooth variety since it is a vector bundle over $\mathcal{B}_\mathfrak{g}$.*

Example 2.1.19. If $\mathfrak{g} = \mathfrak{sl}_2$ as in example 2.1.12, then $\pi : \tilde{\mathcal{N}}_\mathfrak{g} \to \mathcal{B}_\mathfrak{g}$ is a line bundle over $\mathbb{P}^1$: a point $\mathfrak{b} \in \mathcal{B}_\mathfrak{g}$ corresponds to the choice of 1-dimensional subspace $l \subset \mathbb{C}^2$ and the corresponding fiber of π consists of the line $\mathfrak{n} \in \mathcal{N}_\mathfrak{g}$ of all nilpotent operators $\mathbb{C}^2 \to \mathbb{C}^2$ whose image is contained in l.

Definition 2.1.20. Define $\mathfrak{b}^\perp$ to be the *annihilator* of $\mathfrak{b}$, i.e. $\mathfrak{b}^\perp = \{x \in \mathfrak{g}^* \mid x(\mathfrak{b}) = 0\}$.

Definition 2.1.21. Define $G \times_{B_0} \mathfrak{b}_0^\perp$ to be the quotient of $G \times \mathfrak{b}_0^\perp$ by B_0 acting on G by right multiplication and on $\mathfrak{b}_0^\perp$ by the coadjoint action $Ad^*_{B_0}$.

Proposition 2.1.22. *There is an isomorphism*

$$T^*\mathcal{B}_\mathfrak{g} \simeq G \times_{B_0} \mathfrak{b}_0^\perp. \tag{2.8}$$

given by the map dual to the infinitesimal $\mathfrak{g}$*-action map* $G/B_0 \times \mathfrak{g} \to T(G/B_0)$*.*

Proof. Proposition 1.4.11 of [20]. □

Lemma 2.1.23. *For any* $\mathfrak{b} \in \mathcal{B}_\mathfrak{g}$*, the identification* $\mathfrak{g} \cong \mathfrak{g}^*$ *provided by the Killing form identifies* $\mathfrak{b}^\perp$ *with* $\mathfrak{n}_\mathfrak{b}$*.*

Proof. Section 8.1 of [28]. □

We can embed $G \times_{B_0} \mathfrak{b}_0^\perp$ into the trivial vector bundle $G/B_0 \times \mathfrak{g}^*$ over G/B_0:

$$\phi : ([g, \alpha]) \mapsto ([g], g \cdot_{AdG} \alpha).$$

The image of $G \times_B \mathfrak{b}_0^\perp$ under this embedding is the subbundle

$$\{(\mathfrak{b}, \alpha) \in G/B_0 \times \mathfrak{g}^* \mid \alpha \in \mathfrak{b}^\perp\}.$$

We thus have the following commutative diagram

$$\begin{tikzcd}
G \times_B \mathfrak{b}_0^\perp \arrow[d, "\sim"] \arrow[dr, hook, "\phi"] & \\
\{(\mathfrak{b}, \alpha) \mid \alpha \in \mathfrak{b}^\perp\} \arrow[r, hook] \arrow[d, "\sim", "\textit{Lemma } 2.1.23"'] & G/B \times \mathfrak{g}^* \arrow[d, "\sim", "\text{Killing form}"'] \\
\tilde{\mathcal{N}}_\mathfrak{g} \arrow[r, hook] & G/B \times \mathfrak{g}
\end{tikzcd}$$

Thus, we obtain

Proposition 2.1.24. *There exists a* G*-equivariant isomorphism of vector bundles over* G/B_0*:*

$$\tilde{\mathcal{N}}_\mathfrak{g} \cong G \times_B \mathfrak{b}^\perp. \tag{2.9}$$

Proof. Lemma 3.3.2 of [20] or [12]. □

Corollary 2.1.25. *There is a natural isomorphism $\tilde{\mathcal{N}}_{\mathfrak{g}} \cong T^*\mathcal{B}_{\mathfrak{g}}$.*

Remark 2.1.26. *By Corollary 2.1.25 we can think of the Springer resolution as a map $T^*\mathcal{B}_{\mathfrak{g}} \to \mathcal{N}_{\mathfrak{g}}$.*

2.2 Geometry of flag varieties

In this section we give a description of the Picard groups of flag varieties via Schubert calculus, following [14]. We begin with some preliminaries on Bruhat decompositions following [**?**]. Let G be a connected complex reductive algebraic group.

Definition 2.2.1. A *parabolic subgroup* P is a subgroup of G that contains a Borel subgroup B.

Proposition 2.2.2. *A subgroup $P \subset G$ is parabolic if and only if G/P is a projective variety.*

Proof. See [12] or [29]. □

Definition 2.2.3. Let $B \subset G$ a Borel subgroup and $T \subset B$ a maximal torus. The *Weyl group* of the Borel pair (T, B) is the group $W := N_G(T)/T$.

Definition 2.2.4. The *Bruhat decomposition* of G is the decomposition

$$G = \bigsqcup_{w \in W} BwB \tag{2.10}$$

as a disjoint union of double cosets of B parameterized by the elements of W.

More generally, any parabolic subgroup $B \subset P_J$ defines the *generalised Bruhat decomposition*

$$G = \bigsqcup_{w \in W/W_J} BwP_J \tag{2.11}$$

where $W_J = \{[v] \in W \mid v \in P_J\}$ and W/W_J is the set of right cosets of W_J in W.

See [12] and [29] for further details.

The double cosets of the Bruhat decomposition descend to the left B-cosets in the right coset quotients G/B and G/P_J. This gives the decomposition G/B and G/P_J into the B-orbits under the action of B by left multiplication. These are *Schubert cells*, and their closures are *Schubert varieties*:

Definition 2.2.5. For any $w \in W$, the corresponding *Schubert varieties* are

$$X_w = \overline{[BwB]} \subset G/B, \quad \text{and} \quad X_w^J = \overline{[BwP_J]} \subset G/P_J.$$

Definition 2.2.6. Define the *Bruhat order* on W by

$$w \leq w' \quad \Leftrightarrow \quad X_w \subseteq X_{w'}. \tag{2.12}$$

Define the *Bruhat order* on W_J by

$$w \leq w' \quad \Leftrightarrow \quad X_w^J \subseteq X_{w'}^J. \tag{2.13}$$

Example 2.2.7. Let $G = GL_3(\mathbb{C})$ and let T and B be the subgroups of the diagonal and the upper triangular matrices, respectively:

$$T = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \right\} \quad \text{and } B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

The normaliser $N_G(T)$ consists of all the matrices such that the induced change of basis keeps all the diagonal matrices diagonal. Up to scaling, any such change of basis is a permutation of basis vectors. It follows that

$$W := N_G(T)/T = S_3.$$

Moreover, we have the standard splitting $W \hookrightarrow N_G(T)$ given by the *permutation matrices*, which permute the standard basis e_1, e_2, e_3 of $\mathbb{C}^3$.

The upper triangular matrices are the matrices which preserve the standard flag

$$0 \subset E_1 \subset E_2 \subset E_3 = \mathbb{C}^3$$

where $E_i = \langle e_1, \ldots, e_i \rangle$. Hence any right coset of B consists of all the matrices which send the standard flag to a specific flag

$$0 \subset V_1 \subset V_2 \subset V_3 = \mathbb{C}^3.$$

This identifies the space G/B with the flag variety Fl_3 of complete flags in $\mathbb{C}^3$.

There are two parabolic subgroups of G containing B:

$$P_1 = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \qquad \text{and} \qquad P_2 = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \right\}.$$

Each P_i consists of all matrices which preserve the subspace E_i. Its right cosets consist of the matrices which send E_i to a specific i-dimensional subspace of $\mathbb{C}^3$. It follows that the homogeneous spaces G/P_1 and G/P_2 can be identified with the varieties $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$ of lines and planes in $\mathbb{C}^3$, respectively.

The Bruhat decomposition of a matrix $A \in GL_3$ corresponds to its reduction to column echelon form. Multiplication by an upper triangular matrix on the right is equivalent to performing a sequence of column operations where we only add to each column a linear combination of the columns lying to the left of it. Hence in the Bruhat decomposition

$$M = U_1 \sigma U_2$$

the (inverse of the) matrix U_2 encodes the column operations, the (inverse of the) matrix σ encodes the permutation of the columns, and the matrix U_1 is the resulting column echelon form of M.

Correspondingly, in G/B the Schubert cell corresponding to $\sigma \in W$ consists of all the points which can be represented by a matrix obtained by permuting the columns of an upper triangular matrix by σ. This translates naturally to the condition on the corresponding flag in $\mathbb{C}^3$. For example, the cell C_{123} which corresponds to the permutation (123) consists of the points representable by the matrix type

$$\begin{pmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{pmatrix}. \tag{2.14}$$

In terms of the corresponding flag $0 \subset V_1 \subset V_2 \subset \mathbb{C}^3$, the shape of the first column is equivalent to the condition

$$V_1 \not\subset E_2,$$

and then the shape of the second column is determined by the condition

$$E_1 \subset V_2.$$

The shape of the third column is determined by the first two.

By the first condition $V_1 \neq E_1$, thus we can replace the second by $V_2 = V_1 \oplus E_1$. Thus $C_{123} \simeq \mathbb{C}^2$ and can be identified with $\mathbb{P}^2$ of lines in $\mathbb{C}^3$ with the line $V_1 \subset E_2$ removed. The Schubert variety X_{123} is the closure of C_{123}. It is the subvariety $E_1 \subset V_2$ of Fl_3. Again, if $V_1 \neq E_1$, we have $V_2 = V_1 \oplus E_1$, but if $V_1 = E_1$ then we can take any V_2 containing E_1. Thus, X_{123} is the blowup of $\mathbb{P}^2$ at the point $V_1 = E_1$.

Similarly, we have:

$\sigma \in S_3$	Matrix type	Schubert cell C_σ	Schubert variety X_σ
Id	$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}$	$pt\colon V_1 = E_1, V_2 = E_2$	$pt\colon V_1 = E_1, V_2 = E_2$
(12)	$\begin{pmatrix} * & * & * \\ * & 0 & * \\ 0 & 0 & * \end{pmatrix}$	$\mathbb{C}^1\colon V_1 \neq E_1, V_2 = E_2$	$\mathbb{P}^1\colon V_2 = E_2$
(23)	$\begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & 0 \end{pmatrix}$	$\mathbb{C}^1\colon V_1 = E_1, V_2 \neq E_2$	$\mathbb{P}^1\colon V_1 = E_1$
(132)	$\begin{pmatrix} * & * & * \\ * & 0 & * \\ * & 0 & 0 \end{pmatrix}$	$\mathbb{C}^2\colon V_1 \not\subset E_2, V_2 = V_1 \oplus E_1$	$\mathbb{P}^2$ blown up at $V_1 = E_1$: $E_1 \subset V_2$
(123)	$\begin{pmatrix} * & * & * \\ * & * & 0 \\ 0 & * & 0 \end{pmatrix}$	$\mathbb{C}^2\colon V_1 = E_2 \cap V_2, E_1 \not\subset V_2$	$\mathbb{P}^{2\vee}$ blown up at $V_2 = E_2$: $V_1 \subset E_2$
(13)	$\begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}$	$\mathbb{C}^3\colon V_1 \not\subset E_2, E_1 \not\subset V_2$	Fl_3

Thus the Bruhat order on the Schubert subvarieties of Fl_3 is

$$\begin{tikzcd}
& & X_{13} = \mathrm{Fl}_3 & & \\
X_{132} \arrow[urr, hook] & & & & X_{123} \arrow[ull, hook'] \\
X_{12} \arrow[u, hook] \arrow[urrrr, hook] & & & & X_{23} \arrow[u, hook'] \arrow[ullll, hook'] \\
& & X_{\mathrm{Id}} = \bullet \arrow[ull, hook'] \arrow[urr, hook] \arrow[uuu, hook] & &
\end{tikzcd}$$

The parabolic subgroups P_1 and P_2 of G intersect W at subgroups $W_1 = \langle (12) \rangle$ and $W_2 = \langle (23) \rangle$. The generalised Bruhat decompositions of G for P_1 and P_2 merge the Bruhat cells corresponding to the elements of S_3 which get identified in W/W_1 and W/W_2, respectively. Thus:

$\sigma \in S_3/(23)$	Schubert cell C^1_σ	Schubert variety X^1_σ
Id, (23)	$pt\colon V_1 = E_1$	$pt\colon V_1 = E_1$
(12), (132)	$\mathbb{C}^1\colon V_1 \neq E_1, V_1 \subset E_2$	$\mathbb{P}^1\colon V_1 \subset E_2$
(123), (13)	$\mathbb{C}^2\colon V_1 \not\subset E_2$	$\mathbb{P}^2$

$\sigma \in S_3/(12)$	Schubert cell C^2_σ	Schubert variety X^2_σ
Id, (12)	$pt\colon V_2 = E_2$	$pt\colon V_2 = E_2$
(23), (132)	$\mathbb{C}^1\colon V_2 \neq E_2, E_1 \subset V_2$	$\mathbb{P}^1\colon E_1 \subset V_2$
(123), (13)	$\mathbb{C}^2\colon E_1 \not\subset V_2$	$\mathbb{P}^{2\vee}$

Grassmanians

For this section we follow [14]. Let $G = GL_n$, T and B be the standard torus and the standard Borel subgroup of GL_n consisting of diagonal and the upper triangular matrices, respectively. The latter are the matrices which preserve the standard flag

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n$$

obtained from the standard basis $\{e_1, \ldots, e_n\}$ of $\mathbb{C}^n$.

Definition 2.2.8. For any $0 < d < n$ define the subgroup $P_d \subset GL_n$ by

$$P_d \coloneqq \{A \in GL_n \mid A(E_d) \subseteq E_d\}\,.$$

The subgroups $P_1, \ldots, P_{n-1}$ are the maximal parabolic subgroups containing B. Any two matrices which send E_d to the same d-dimensional subspace $V_d \subset \mathbb{C}^n$ differ by an element of P_d, so the right cosets of P_d are the sets of all matrices which send E_d to the same V_d. Thus the homogeneous space G/P_d can be identified with:

Definition 2.2.9. The *Grassmanian* $\mathrm{Gr}(\mathrm{d},\mathrm{n})$ is the set of vector subspaces of $\mathbb{C}^n$ of dimension d:

$$\mathrm{Gr}(\mathrm{d},\mathrm{n}) \coloneqq \{V_d \subset \mathbb{C}^n \mid dim(V_d) = d\}\,.$$

The following gives $Gr(d,n)$ natural structure of a smooth projective variety:

Definition 2.2.10. The *Plücker embedding* is the map

$$\mathrm{Gr}(\mathrm{d},\mathrm{n}) \longrightarrow \mathbb{P}(\bigwedge^d \mathbb{C}^n)$$

defined by

$$\langle v_1, \ldots, v_n\rangle \mapsto [v_1 \wedge \cdots \wedge v_n].$$

The intersection of P_d with the Weyl group $W = S_n$ is the subgroup $S_d \times S_{n-d} \subset S_n$ which consists of all the permutations in S_n which preserve the subset $\{1, \ldots, d\}$. The right cosets of $S_d \times S_{n-d}$ in S_n consist therefore of all the permutations which send $\{1, \ldots, d\}$ to some fixed subset of $\{1, \ldots, n\}$. We therefore identify the elements of $W/(P_d \cap W)$, the set which indexes the generalised Bruhat cells of P_d, with size d subsets

$$I = \{i_1 < i_2 < \cdots < i_d\} \subset \{1, \ldots, n\}\,.$$

Any $\sigma \in S_n$ which lies in such I sends E_d to the subspace

$$E_I = \langle e_{j_1}, \ldots, e_{j_d}\rangle\,. \tag{2.15}$$

The corresponding Schubert cell C_I is the orbit of E_I under the left action of B, and hence the orbit of E_I under the action of its unipotent subgroup $U \simeq \mathbb{C}^{\frac{1}{2}n(n-1)}$. The stabiliser $\mathrm{Stab}_U(E_I)$ are the matrices with $u_{ij} = 0$ if $j \in I$ and $i \notin I$. This gives $\sum i_k - k$ conditions in total, whence $C_I \simeq U/\mathrm{Stab}_U(E_I) \simeq \mathbb{C}^{\sum i_k - k}$.

Any I as above is uniquely determined by $\dim(E_I \cap E_j)$ for all the subspaces E_j of the standard flag. Since the left action of B preserves the standard flag, we have

$$C_I = \{V_d \in Gr(d,n) \mid \dim(V_d \cap E_j) = a_j \text{ for all } 1 \leq j \leq n\},$$

where a_j is the number of i_k with $i_k \leq j$. The closure of $V_d \cap E_i = a_j$ in $Gr(d,n)$ is $V_d \cap E_i \geq a_j$. Since $a_{i_k} = k$, the Schubert variety X_I, the closure of C_I, is given by

$$X_I = \{V_d \in Gr(d,n) \mid \dim(V_d \cap E_{i_k}) = k \text{ for all } i_k \in I\}.$$

Since C_I is irreducible, X_I is an irreducible subvariety of $Gr(d,n)$.

Example 2.2.11. Since the condition $\dim(V_d \cap E_d) = d$ implies $E_d = V_d$, the Schubert variety $S_{1,\ldots,d}$ is the point $E_d \in Gr(d,n)$.

Example 2.2.12. Since by dimension considerations $\dim(V_d \cap E_j) \leq d + j - n$, the Schubert variety $S_{n-d+1,\ldots,n}$ is the whole of Grassmanian $Gr(d,n)$.

Example 2.2.13. The Schubert variety $X_{n-d,n-d+2,\ldots,n}$ consists of those V_d whose intersection with E_{n-d} is non-zero, i.e. whose projection onto $E_{n-d+1,\ldots,n}$ is non-invertible. Thus $X_{n-d,n-d+2,\ldots,n}$ is the hyperplane section $p_{n-d+1,\ldots,n} = 0$ of the Plücker embedding of $Gr(d,n)$. On the other hand, $\dim(V_d \cap E_{n-d+j}) \leq j$ so if $V_d \cap E_{n-d} = 0$, we must have $\dim(V_d \cap E_{n-d+j}) = j$, i.e. $V_d \in C_{n-d+1,\ldots,n}$. Thus

$$Gr(d,n) = C_{n-d+1,\ldots,n} \coprod X_{n-d,n-d+2,\ldots,n}. \tag{2.16}$$

Proposition 2.2.14. *The Picard group $Pic(Gr(d,n)$ of the Grasmannian $Gr(d,n)$ is freely generated by $\mathcal{O}(X_{n-d,n-d+2,\ldots,n})$.*

Proof. This follows from the decomposition (2.16) given that $C_{n-d+1,\ldots,n} \cong \mathbb{C}^{d(n-d)}$ and $X_{n-d,n-d+2,\ldots,n}$ is irreducible. □

More generally we have

Proposition 2.2.15. *The classes of the Schubert varieties X_I give an additive basis of the cohomology ring $H^*(Gr(d,n),\mathbb{Z})$.*

Flag varieties

As before, let $G = GL_n$, let T and B be the subgroups of diagonal and the upper triangular matrices, and let $E_\bullet$ be the standard flag

$$0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = \mathbb{C}^n$$

obtained from the standard basis $\{e_1,\ldots,e_n\}$ of $\mathbb{C}^n$.

Definition 2.2.16. Let $\lambda = (\lambda_1, \ldots, \lambda_m)$ be an ordered partition of n: $\sum \lambda_i = n$. Define $P_\lambda \subset G$ to be the subgroup of all the matrices which preserve the partial coordinate flag $E_\bullet^\lambda$:

$$0 \subset E_{\lambda_1} \subset E_{\lambda_1+\lambda_2} \subset \ldots \subset E_{\lambda_1+\cdots+\lambda_m} = \mathbb{C}^n. \tag{2.17}$$

It is the subgroup of block upper triangular matrices with diagonal blocks of sizes $\lambda_1, \ldots, \lambda_k$.

The groups P_λ are the standard parabolic groups of G. Any two matrices which send $E_\bullet^\lambda$ to the same partial flag $V_\bullet^\lambda$ differ by an element of P_λ, so the right cosets of P_λ are sets of all matrices which send $E_\bullet^\lambda$ to some fixed $V_\bullet^\lambda$. This identifies the homogeneous space G/P_λ with:

Definition 2.2.17. The *partial flag variety* $\mathrm{Fl}_n(\lambda)$ is the set of all partial flags

$$0 \subset V_{\lambda_1} \subset V_{\lambda_1+\lambda_2} \subset \ldots \subset V_{\lambda_1+\cdots+\lambda_m} = \mathbb{C}^n \tag{2.18}$$

with $\dim(V_i) = i$.

The following gives $\mathrm{Fl}_n(\lambda)$ the structure of a smooth projective variety of dimension $\sum_{i<j} \lambda_i \lambda_j$:

$$i_\lambda : \mathrm{Fl}_n(\lambda) \hookrightarrow \prod_{i=1}^{k} Gr(\lambda_1 + \cdots + \lambda_i, n). \tag{2.19}$$

Example 2.2.18. $\mathrm{Fl}_n(1, n-1)$ is the projective space $\mathbb{P}^{n-1}$ of lines in $\mathbb{C}^n$.

Example 2.2.19. $\mathrm{Fl}_n(n-1, 1)$ is the dual projective space $\mathbb{P}^{(n-1)\vee}$ of hyperplanes in $\mathbb{C}^n$.

Example 2.2.20. $\mathrm{Fl}(d, n-d)$ is the Grassmanian $Gr(d, n)$.

Example 2.2.21. $\mathrm{Fl}(1, n-2, 1)$ is the incidence variety of lines V_1 contained in the hyperplanes V_{n-1} of $\mathbb{C}^n$. It is a projective subvariety of $\mathbb{P}^{n-1} \times \mathbb{P}^{n-1\vee}$ carved out by the equation

$$x_1 y_1 + \cdots + x_n y_n = 0$$

where x_i and y_i are the standard i-th coordinate respectively of $\mathbb{C}^n$ and $\mathbb{C}^{n*}$.

Remark 2.2.22. *For every partition λ the quotient map $G/B \twoheadrightarrow G/P_\lambda$ is the forgetful morphism*

$$\pi_\lambda : \mathrm{Fl}_n \to \mathrm{Fl}_n(\lambda) \tag{2.20}$$

which sends each complete flag to its corresponding partial subflag. It is therefore a flat fibration whose fibers are isomorphic to the partial flag space of the flag type complementary to λ.

The intersection of P_λ with the Weyl group $W = S_n$ is the subgroup $\prod_i S_{\lambda_i}$ of all the permutations which respect the partition λ. Its right cosets can therefore be identified with partitions of the set $\{1, \ldots, n\}$ into the subsets of size λ_i. More precisely, define S_n^λ to be the subset of S_n consisting of permutations which are order preserving on each block of the partition λ. The permutations $\sigma \in S_n^\lambda$ uniquely represent all the cosets in $S_n / \prod_i S_{\lambda_i}$.

For each $\sigma \in S_n^\lambda$ the corresponding Schubert cell C_σ is the B-orbit of the partial flag $\sigma(E_\bullet^\lambda)$, where $\sigma(E_k) = \langle e_{\sigma(1)}, \ldots, e_{\sigma(k)} \rangle$. The Schubert variety X_σ is its Zariski closure.

Example 2.2.23. The Schubert variety X_{Id} is the single point $E_\bullet \in \mathrm{Fl}_n$.

Example 2.2.24. Let ρ be the order reversing permutation $(n\ n-1\ \ldots\ 1)$. The Schubert variety X_ρ is the whole of Fl_n.

Example 2.2.25. Let τ_i be the transposition $(i\ i{+}1)$. The Schubert variety $X_{\tau_i} \subset \mathrm{Fl}_n$ consists of the flags $V_\bullet$ with all $V_j = E_j$ except for V_i. Thus it can be identified with $\mathbb{P}^1$ of choices of i-dimensional space V_i with $E_{i-1} \subset W_i \subset E_{i+1}$.

Example 2.2.26. The Schubert variety $S_{\rho\tau_i}$ is a divisor in $\mathrm{Fl}_n(\lambda)$.

Example 2.2.27. Let λ be a partition of n and let $\sigma \in S_n^\lambda$. The inverse image of the Schubert variety $X_\sigma \subset \mathrm{Fl}_n(\lambda)$ under the map

$$\pi_\lambda : \mathrm{Fl}_n \to \mathrm{Fl}_n(\lambda)$$

is the Schubert variety $S_{\sigma\tilde{\sigma}} \subset \mathrm{Fl}_n$, where $\tilde{\sigma}$ is the Bruhat maximal element of $\prod_i S_{\lambda_i}$, i.e. the product of the order reversing permutations. This allows to reduce some questions about partial flag varieties to the study of complete flag varieties.

Proposition 2.2.28. *The Picard group Pic*(Fl) *of the complete flag variety* Fl *is freely generated by the line bundles of the Schubert divisors* $X_{\rho\tau_1}$, $X_{\rho\tau_2}$, $\ldots$, $X_{\rho\tau_n}$.

Proof. Proposition 1.4.1 of [14]. □

More generally:

Proposition 2.2.29. *Let λ be a partition of n. The classes of Schubert varieties X_σ give an additive basis of the cohomology ring $H^*(\mathrm{Fl}_n(\lambda), \mathbb{Z})$.*

Remark 2.2.30. *A nilpotent operator α preserving the flag* $0 \subset V_1 \subset \cdots \subset \mathbb{C}^n$ *has to satisfy the condition*

$$\alpha(V_i) \subset V_{i-1},$$

therefore Proposition 2.1.22 and Remark 2.1.23 give us the following description of the total space of $\mathrm{T}^* \mathrm{Fl}_n$*:*

$$\mathrm{T}^* \mathrm{Fl}_n \coloneqq \{(V_\bullet, \alpha) \mid \alpha : \mathbb{C}^n \to \mathbb{C}^n; \alpha(V_i) \subset V_{i-1}\}.$$

We use the following pictorial shorthand to denote such pairs $(V_\bullet, \alpha)$*:*

$$\begin{tikzcd} 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \ldots \arrow[ll, bend right, "\alpha"'] & \subset & V_{n-1} \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^n \arrow[ll, bend right, "\alpha"'] \end{tikzcd} .$$

Analogously (see section 1.2 of [20]), the total space of $\mathrm{T}^* \mathrm{Fl}_n(\lambda)$ *can be described as the space of pairs*

$$\mathrm{T}^* \mathrm{Fl}_n(\lambda) \cong \left\{ \begin{tikzcd} 0 & \subset & V_{\lambda_1} \arrow[ll, bend right, "\alpha"'] & \subset & \ldots \arrow[ll, bend right, "\alpha"'] & \subset & V_{\lambda_{k-1}} \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^n \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \middle| \dim(V_{\lambda_i}) = \sum_{j=1}^{i} \lambda_j \right\} .$$

— Chapter 3 —

Autoequivalences and braid group actions of derived categories

In this chapter, we focus on equivalences and autoequivalences of derived categories of coherent sheaves of reduced schemes of finite type and their relations with the braid groups.

Braid groups actions on derived categories occur in many different contexts.

In this chapter, we describe the braid group action arising from an $\mathcal{A}_n$ configuration of spherical objects on a smooth projective variety constructed by Seidel and Thomas in [40] and the braid group action on the cotangent bundle of complete flag variety constructed by Khovanov and Thomas in [32].

In this thesis we extend the latter to a generalised braid category action on the derived categories of coherent sheaves of the cotangent bundle of partial flag varieties; in Chapter 4 we will formulate this statement precisely and we prove it in its first non-trivial instance.

The first section of this chapter is a quick introduction to the language of DG-categories, twisted complexes and DG-enhancements.

In the second section, we introduce the notion of Fourier-Mukai transforms and give some results on standard kernels and the adjunction unit and counit maps between them.

In the third section, we introduce autoequivalences of bounded derived categories of coherent sheaves over smooth projective varieties, discuss some examples and give the results of Bondal and Orlov for the case of Fano and general type varieties in [11].

The fourth section is about the spherical objects and their twists: introduced by Seidel and Thomas they were the first example of genuinely derived autoequivalences of the derived category of coherent sheaves and they could be used to contract a braid group categorical action.

In section five we discuss a generalisation of the spherical object and their twists. Anno and Logvinenko in [6] introduced the notion of spherical functors. Given a scheme X, any $\mathcal{E} \in D^b(X)$ can be considered as the functor $(-) \otimes \mathcal{E} : D^b(pt) \to D^b(X)$, spherical functors are analogues of spherical objects where the point is replaced with a scheme Z. A spherical functor $F : D^b(Z) \to D^b(X)$ induces two autoequivalences: the twists T_F of $D^b(X)$ and the cotwist C_F of $D^b(X)$.

In section six, we discuss the other generalisation of spherical objects, the $\mathbb{P}^n$-objects due to Huybrechts and Thomas in [30].

Section seven covers the theory of $\mathbb{P}^n$-functors which unifies and generalises the notion of spherical functors and $\mathbb{P}^n$-objects due to Anno and Logvinenko in [7].

In section eight we focus on Mukai Flops in derived categories and on the example of the flop $\{T^*\mathbb{P}^{2\vee} \dashrightarrow T^*\mathbb{P}^2\}$ which will appear in the generalised braid action we construct in the case $n = 3$.

Section 9 contains some technical results on the excess bundle formula which computes the derived tensor product of two structure sheaves of two smooth subvarieties of a smooth variety.

In section ten, we define categorical group actions and we describe the Khovanov-Thomas braid group action on the total space of the cotangent bundle of a complete flag variety.

3.1 DG-categories and DG-enhancements

In this section we introduce DG-categories and DG-enhancements of triangulated categories, following [6].

We give the basic definitions and we will refer to [45] and [6] for further details.

Let for all this section R be a commutative ring.

Definition 3.1.1. A DG-category is a category $\mathcal{C}$ such that for every two objects $A, B \in Ob(\mathcal{C})$ the morphism space $\mathrm{Hom}^\bullet_{\mathcal{C}}(A,B)$ is a complex of R-modules and such that the composition map

$$\mathrm{Hom}^\bullet_{\mathcal{C}}(B,C) \otimes \mathrm{Hom}^\bullet_{\mathcal{C}}(A,B) \longrightarrow \mathrm{Hom}^\bullet_{\mathcal{C}}(A,C) \tag{3.1}$$

is map of complexes of R-modules.

Definition 3.1.2. Let $\mathcal{C}$ be a DG-category, the homotopy category $H^0(\mathcal{C})$ is the category whose objects are the objects of $\mathcal{C}$ and whose morphisms are given by $\mathrm{Hom}_{H^0(\mathcal{C})}(A,B) = H^0(\mathrm{Hom}^\bullet_{\mathcal{C}}(A,B))$.

Example 3.1.3. The DG-category $\mathbf{Mod}-R$ is the category of complexes of R-modules with morphisms complex $\mathrm{Hom}^\bullet_{\mathbf{Mod}-R}(M,N)$ defined by

$$\mathrm{Hom}^n_{\mathbf{Mod}-R}(M,N) := \bigoplus_{i+j=n} \mathrm{Hom}_R(M_i, N_j) \tag{3.2}$$

and the differential which sends $f \in \mathrm{Hom}^n_{\mathbf{Mod}-R}(M,N)$ to

$$df := d_N \circ f - (-1)^n f \circ d_M. \tag{3.3}$$

Definition 3.1.4. Let $\mathcal{C}$ be a DG-category. The opposite DG-category $\mathcal{C}^{opp}$ is DG-category whose objects are those of $\mathcal{C}$, Whose morphisms are

$$\mathrm{Hom}^{\bullet}_{\mathcal{C}^{opp}}(A,B) \coloneqq \mathrm{Hom}^{\bullet}_{\mathcal{C}}(B,A)$$

with the composition defined for every $\phi \in \mathrm{Hom}_{\mathcal{C}^{opp}}(A,B)$ and $\psi \in \mathrm{Hom}_{\mathcal{C}^{opp}}(B,C)$

$$\psi \circ_{\mathcal{C}^{opp}} \phi \coloneqq (-1)^{deg(\phi)deg(\psi)} \phi \circ_{\mathcal{C}} \psi.$$

Definition 3.1.5. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two DG-categories. A DG-functor F

$$F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2 \tag{3.4}$$

is a R-linear functor which commutes with the differentials of the morphism complexes and preserves the grading.

A natural transformation $t : F_1 \to F_2$ of degree n between two DG-functors from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a collection of morphisms

$$\left\{t(A) \in \mathrm{Hom}^n_{\mathcal{C}_2}(F_1(A), F_2(A))\right\}_{A \in Ob(\mathcal{C}_1)}$$

such that for every morphism $\phi \in \mathrm{Hom}^m_{\mathcal{C}_2}(A,B)$ the following equivalence hold

$$t(B) \circ F_1(\phi) = (-1)^{nm} F_2(\phi) \circ t(B).$$

Example 3.1.6. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two DG-categories. The DG-category $DG - Fun(\mathcal{C}_1, \mathcal{C}_2)$ is the category whose objects are the DG-functor from $\mathcal{C}_1$ to $\mathcal{C}_2$ and whose morphism complexes have as nth graded part all the natural transformations of degree n.

The grading is determined by the degree of the natural transformations, while the differentials and the composition are defined in $\mathcal{C}_2$ for each $A \in Ob(\mathcal{C}_1)$.

Example 3.1.7. Let $\mathcal{C}$ be a DG-category. The category $\mathbf{Mod} - \mathcal{C}$ is the DG-category $DG - Fun(\mathcal{C}^{opp}, \mathbf{Mod} - R)$: its objects are called *(right)* $\mathcal{C}-$*modules.*

Definition 3.1.8. Let $\mathcal{C}$ be a DG-category. A twisted complex $(A^{\bullet}, \phi_{\bullet,\bullet})$ over $\mathcal{C}$ is a collection

$$(A^{\bullet}, \phi_{\bullet,\bullet}) \coloneqq \{A_i, \phi_{i,j} : A_i \to A_j\}_{(i,j \in \mathbb{Z})} \tag{3.5}$$

where $A_i \in Ob(\mathcal{C})$, $A_i \neq 0$ for only a finite number of indexes and

$$\phi_{i,j} \in \mathrm{Hom}^{i-j+1}_{\mathcal{C}}(A_i, A_j)$$

satisfying

$$(-1)^j d\phi_{i,j} + \sum_k \phi_{k,j} \circ \phi_{i,k} = 0.$$

Remark 3.1.9. *The twisted complexes over a DG-category $\mathcal{C}$ have a natural structure of a DG-category with morphism complexes defined by*

$$\mathrm{Hom}^p((A^\bullet,\phi_{\bullet,\bullet}),(B^\bullet,\psi_{\bullet,\bullet})) := \amalg_{q+m-n=p}\mathrm{Hom}^q_{\mathcal{C}}(A_n,B_m) \tag{3.6}$$

with the differential which sends every $\alpha \in \mathrm{Hom}^q_{\mathcal{C}}(A_n,B_m)$ to

$$d\alpha = (-1)^m d_{\mathcal{C}}\alpha + \sum_{k\in\mathbb{Z}}(\psi_{m,k}\circ\alpha - (-1)^{q+m-n}\alpha\circ\phi_{i,j}. \tag{3.7}$$

Definition 3.1.10. Let $\mathcal{C}$ be a DG-category, let $(A^\bullet,\phi_{\bullet,\bullet})$ be a twisted complex over $\mathcal{C}$ and let $\oplus_i A_i[-i]$ be the $\mathcal{C}$-module where we implicitly use the Yoneda embedding to consider A_i as objects in $\mathbf{Mod}-\mathcal{C}$.

The *convolution* $\{A^\bullet,\phi_{\bullet,\bullet}\}$ of $(A^\bullet,\phi_{\bullet,\bullet})$ is the $\mathcal{C}$-module $\oplus_i A_i[-i]$ with the modified differential

$$d_{conv} := d_{old} + \sum_{n,m\in\mathbb{Z}}\phi_{n,m} \tag{3.8}$$

Definition 3.1.11. A DG-category $\mathcal{C}$ is *pretriangulated* if its image under the Yoneda embedding is a triangulated subcategory of $H^0(\mathbf{Mod}-\mathcal{C})$.

Definition 3.1.12. A functor $F:\mathcal{C}_1\longrightarrow\mathcal{C}_2$ between DG-categories is a *quasi-equivalence* if induces quasi-isomorphisms on morphism complexes and if the functor $H^0(F)$

$$H^0(F): H^0(\mathcal{C}_1)\longrightarrow H^0(\mathcal{C}_2)$$

is an equivalence.

Definition 3.1.13. An enhancement for a triangulated category $\mathcal{D}$ is a couple $(\mathcal{C},\theta)$, where $\mathcal{C}$ is a pretriangulated DG-category and

$$\theta: H^0(\mathcal{C}) \xrightarrow{\cong} \mathcal{T} \tag{3.9}$$

is an exact equivalence.

Two enhancements $(\mathcal{C}_1,\theta_1)$ and $(\mathcal{C}_2,\theta_2)$ are equivalent if there exists a quasi-equivalence

$$F:\mathcal{C}_1\longrightarrow\mathcal{C}_2.$$

Definition 3.1.14. Let $Ho(DG-Cat)$ be the localisation of the category $DG-Cat$ of small DG-categories by quasi-equivalences constructed using Tabuada's model structure on $DG-Cat$ (See [43] for more details). Let $\mathcal{C}_1$ and $\mathcal{C}_2$ two DG-categories, define the set $[\mathcal{C}_1,\mathcal{C}_2]$ of quasi-functors to be the set of morphism from $\mathcal{C}_1$ to $\mathcal{C}_2$ in $Ho(DG-Cat)$.

Remark 3.1.15. *The set $[\mathcal{C}_1,\mathcal{C}_2]$ is naturally bijective to the set of isomorphism classes in $D^{r-qr}(\mathcal{C}_1-\mathcal{C}_2)$, the derived category of right quasi-representable bimodules. The standard DG-enhancements of the latter can thus be described as a DG-category of quasi-functors $\mathcal{C}_1\to\mathcal{C}_2$, viewed as $R\mathcal{H}om(\mathcal{C}_1,\mathcal{C}_2)$.*

In our work we will always work in the context of Fourier-Mukai transforms. By a fundamental result of Toën ([45]), if $\mathcal{C}_1$ and $\mathcal{C}_2$ are Karoubi-complete enhancements of $D^b(X)$ and $D^b(Y)$, where X and Y are separated schemes of finite type over a field, then $H^0(R\mathcal{H}om(\mathcal{C}_1, \mathcal{C}_2))$ is isomorphic to the subcategory of $D^b(X \times Y)$ formed by Fourier-Mukai kernels of functors $D^b(X) \to D^b(Y)$. Thus we can work with Fourier-Mukai kernels as DG-enhancements of quasi-functors.

3.2 Fourier-Mukai kernels and their adjunction properties

In this section we introduce Fourier-Mukai transforms and their behaviour under adjunction; as explained at the end of section 3.1, Fourier-Mukai trasforms are the DG-enhanceable functors between the derived categories of coherent sheaves over separated schemes of finite type.

We refer to [27] and [7] for a more general treatment.

Through this section let k be an algebraically closed fild and let $D_{qcoh}(-)$ the unbounded derived category of quasi-coherent sheaves.

Definition 3.2.1. Let X and Y be two separated schemes of finite type over k and let $E \in D_{qcoh}(X \times Y)$. The *Fourier-Mukai* transform Φ_E with kernel E is the functor

$$\Phi_E : D_{qcoh}(X) \longrightarrow D_{qcoh}(Y)$$

defined as

$$\Phi_E(-) = \pi_{Y*}(E \otimes \pi_X^*(-))$$

where π_X and π_Y are the natural projections $X \times Y \to X, Y$.

Fourier-Mukai kernels admit the following composition operation:

Definition 3.2.2. Let X, Y, Z be separated schemes of finite type over k. Let $E_1 \in D_{qcoh}(X \times Y)$ and let $E_2 \in D_{qcoh}(Y \times Z)$.

The composition of Fourier-Mukai kernels $E_1 \star E_2$ is defined as

$$E_2 \star E_1 = \pi_{13*}(\pi_{12}^* E_1 \otimes \pi_{23}^* E_2) \in D^b(X \times Z) \tag{3.10}$$

where π_{12}, π_{23} and π_{13} are the natural projections

$$\begin{tikzcd} & X \times Y \times Z \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ X \times Y & X \times Z & Y \times Z \end{tikzcd}$$

The composition of Fourier-Mukai kernels induces the composition of the corresponding Fourier-Mukai transforms

Proposition 3.2.3. *Let X, Y, Z be separated schemes of finite type over k and let $E_1 \in D_{qcoh}(X \times Y)$ and $E_2 \in D_{qcoh}(Y \times Z)$. Then we have the following isomorphism*

$$\Phi_{E_2} \circ \Phi_{E_1} \cong \Phi_{E_2 \star E_1} \tag{3.11}$$

Proof. Section 5.1 of [27] or [35]. □

In this thesis we always work with certain standard Fourier-Mukai kernels for direct image, inverse image, twisted inverse image and tensor product functors. The following results present some convenient expressions for the kernels involved: although we prove some of them in Chapter 4 here we refer to section 2.6 of [7], since the discussion is more systematic and more general.

Lemma 3.2.4 (Standard kernels)**.** *Let X and Y be separated schemes of finite type over k and let $f : X \to Y$ be a map of separated schemes of finite type over k.*

1. *For any $E \in D^b(X)$, then the Fourier-Mukai kernel induces functor $E \otimes -$*

$$T_E = \pi_2^* E \otimes \Delta_* \mathcal{O}_X \qquad \in D^b(X \times X). \tag{3.12}$$

2. *The Fourier-Mukai kernel associated to the functor f_* is the object*

$$F_* = (Id_X \times f)_* \Delta_* \mathcal{O}_X \in D^b(X \times Y) \tag{3.13}$$

3. *The Fourier-Mukai kernel associated to the functor f^* is the object*

$$F^* = (Id_Y \times f)^* \Delta_* \mathcal{O}_Y \in D^b(Y \times X) \tag{3.14}$$

4. *If f is perfect, the Fourier-Mukai kernel associated to the functor $f^!$ is the object*

$$F^! = (Id_Y \times f)^! \Delta_* \mathcal{O}_Y \in D^b(Y \times X) \tag{3.15}$$

5. *If f is perfect, the Fourier-Mukai kernel associated to the functor $f_! = f_*(- \otimes f^!(\mathcal{O}_Y))$ is the object*

$$F_! = (Id_X \times f)_! \Delta_* \mathcal{O}_X \in D^b(X \times Y) \tag{3.16}$$

Proof. Lemma 2.18 in [7]. □

Lemma 3.2.5. *Let $f : X' \to X$ and $g : Y' \to Y$ be maps of separated schemes of finite type over k. Let $V \in D^b(X)$ and $W \in D^b(Y)$ and $K_1 \in D^b(X \times Y)$, $K_2 \in D^b(X' \times Y')$, then we have the following results:*

1. *The following isomorphism is functorial in K_2*

$$(f \times g)_* K_2 \cong G_* \star K_2 \star F^*. \tag{3.17}$$

2. *The following isomorphism is functorial in K_1*

$$(f \times g)^* K_1 \cong G^* \star K_1 \star F_*. \tag{3.18}$$

3. *If f and g are perfect maps, then the following isomorphism is functorial in K_1*

$$(f \times g)^! K_1 \cong G^! \star K_1 \star F_! \tag{3.19}$$

4. *If f and g are perfect maps, then the following isomorphism is functorial in K_2*

$$(f \times g)_! K_1 \cong G_! \star K_1 \star F^! \tag{3.20}$$

5. *The following isomorphism is functorial in K_1*

$$\pi_X^* V \otimes \pi_Y^* W \otimes K_1 \cong T_V \star K_1 \star T_W \tag{3.21}$$

Proof. Lemma 2.19 in [7]. □

In general the functor from Fourier-Mukai kernels to Fourier-Mukai transforms is neither full neither faithful, but for the Counit of the adjunction

$$\varepsilon : \Phi_{\mathcal{E}}^L \circ \Phi_{\mathcal{E}} \to \mathrm{Id} \tag{3.22}$$

it is possible to choose naturally a morphism

$$\epsilon : K \to \mathcal{O}_\Delta$$

such that it will be lifted to ε.

The following Lemma is a powerful tool for understanding adjunction unit and counit at level of Fourier-Mukai kernels.

Lemma 3.2.6. *Let $f \colon X \to Y$ be a map of separated schemes of finite type over a field.*

1. *There is an isomorphism*

$$F^* \star F_* \simeq (\mathrm{Id}_X \times f)^* (\mathrm{Id}, f)_* \mathcal{O}_X \tag{3.23}$$

which identifies the adjunction counit $F^* \star F_* \xrightarrow{\mu} \mathrm{Id}_X$ with the morphism

$$(\mathrm{Id}_X \times f)^* (\mathrm{Id}, f)_* \mathcal{O}_X \to \Delta_* \mathcal{O}_X \tag{3.24}$$

which is the base change map for the commutative square:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "\Delta"] & X \arrow[d, "(\mathrm{Id}, f)"] \\ X \times X \arrow[r, "\mathrm{Id}_X \times f"] & X \times Y. \end{tikzcd} \tag{3.25}$$

2. *There is an isomorphism*

$$F^* \star F_* \simeq (\mathrm{Id}_X \times f)^*(f \times \mathrm{Id}_Y)^* \Delta_* \mathcal{O}_Y \tag{3.26}$$

which identifies the adjunction counit $F^* \star F_* \xrightarrow{\mu} \mathrm{Id}_X$ *with the composition*

$$(\mathrm{Id}_X \times f)^*(f \times \mathrm{Id}_Y)^* \Delta_* \mathcal{O}_Y \xrightarrow{\sim} (\mathrm{Id}_X \times f)^*(\mathrm{Id}, f)_* \mathcal{O}_X \to i_* \mathcal{O}_{X \times_Y X} \to \Delta_* \mathcal{O}_X \tag{3.27}$$

where the first map is the base change isomorphism for the Tor*-independent fiber square at the bottom of the following commutative diagram:*

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times_Y X \arrow[r, "i"] \arrow[d, "\pi_1"] & X \times X \arrow[d, "\mathrm{Id}_X \times f"] \\ & X \arrow[r, "(\mathrm{Id}, f)"] \arrow[d, "f"] & X \times Y \arrow[d, "f \times \mathrm{Id}_Y"] \\ & Y \arrow[r, "\Delta"] & Y \times Y, \end{tikzcd} \tag{3.28}$$

the second map is the base change map for its top fiber square, and the third map is the natural restriction of sheaves which is the image under i_* *of the adjunction unit* $\mathcal{O}_{X \times_Y X} \to \Delta_* \Delta^* \mathcal{O}_{X \times_Y X}$.

3. *We have an isomorphism*

$$F_* \star F^* \simeq \Delta_* f_* \mathcal{O}_X \tag{3.29}$$

which identifies the adjunction unit $\mathrm{Id}_Y \xrightarrow{\epsilon} F_* \star F^*$ *with the morphism*

$$\Delta_* \mathcal{O}_Y \to \Delta_* f_* \mathcal{O}_X \tag{3.30}$$

which is the image under Δ_* *of the adjunction unit for* (f^*, f_*).

4. *If* f *is perfect and proper, we have an isomorphism*

$$F_* \star F^! \simeq \Delta_* f_* f^! \mathcal{O}_Y \tag{3.31}$$

which identifies the adjunction counit $F_* \star F^! \xrightarrow{\mu} \mathrm{Id}_Y$ *with the morphism*

$$\Delta_* f_* f^! \mathcal{O}_Y \to \Delta_* \mathcal{O}_Y \tag{3.32}$$

which is the image under Δ_* *of the adjunction counit for* $(f_*, f^!)$.

5. *If* f *is perfect and proper, we have an isomorphism*

$$F^! \star F_* \simeq (\mathrm{Id}_X \times f)^! (\mathrm{Id}, f)_* \mathcal{O}_X \tag{3.33}$$

which identifies the adjunction unit $\mathrm{Id}_X \xrightarrow{\epsilon} F^! \star F_*$ *with the morphism*

$$\Delta_* \mathcal{O}_X \to (\mathrm{Id}_X \times f)^! (\mathrm{Id}, f)_* \mathcal{O}_X \tag{3.34}$$

which is the twisted base change map $\Delta_* \mathrm{Id}^! \to (\mathrm{Id}_X \times f)^! (\mathrm{Id}, f)_*$ *for the commutative square* (3.25).

6. *If f is perfect and proper, we have an isomorphism*

$$F^! \star F_* \simeq (\mathrm{Id}_X \times f)^!(f \times \mathrm{Id}_Y)^* \Delta_* \mathcal{O}_Y \tag{3.35}$$

which identifies the adjunction unit $\mathrm{Id}_X \xrightarrow{\epsilon} F^! \star F_*$ *with the composition*

$$\Delta_* \mathcal{O}_X \to i_* \pi_1^! \mathcal{O}_X \to (\mathrm{Id}_X \times f)^!(\mathrm{Id}, f)_* \mathcal{O}_X \xrightarrow{\sim} (\mathrm{Id}_X \times f)^!(f \times \mathrm{Id}_Y)^* \Delta_* \mathcal{O}_Y \tag{3.36}$$

where the first map is the image under i_* *of the adjunction counit* $\Delta_* \Delta^! \pi_1^! \mathcal{O}_X \to \pi_1^! \mathcal{O}_X$*, the second map is the twisted base change map for the top fiber square in* (3.28)*, and the third map is the base change isomorphism for the* Tor*-independent bottom fiber square in* (3.28)*.*

Proof. Proposition 2.20 in [7]. □

3.3 Autoequivalences of derived categories

In order to study a mathematical object, one can study its decomposition into subobjects or study the transformations which preserve its structure. In the context of derived categories this means either studying semiorthogonal decompositions or studying derived autoequivalences.

We focus on the latter.

The group $\mathrm{Aut}(\mathrm{D^b(X)})$ of all isomorphism classes of autoequivalences of $D^b(X)$ can be used to investigate the structure of $D^b(X)$.

The following examples of autoequivalences are usually referred to as "standard" as they are either induced by autoequivalences of the abelian category Coh(X) or are the shift functors which every triangulated category is equipped with.

Example 3.3.1. Any automorphism $f : X \xrightarrow{\simeq} X$ induces the autoequivalence

$$f_* : D^b(X) \xrightarrow{\simeq} D^b(X)$$

and its inverse is given by $f^* : D^b(X) \xrightarrow{\simeq} D^b(X)$.

Example 3.3.2. For every line bundle $\mathcal{L}$ on X the functor

$$\mathcal{L} \otimes (-) : D^b(X) \longrightarrow D^b(X)$$

is an autoequivalence with inverse functor $\mathcal{L}^{-1} \otimes (-)$.

Example 3.3.3. The shift functor $[n]$ is an autoequivalence of $D^b(X)$ for every integer number; its inverse is the shift functor $[-n]$.

Do there exist any other autoequivalences of the bounded derived category of coherent sheaves on a smooth projective variety?

These would be genuinely derived in the sense of interacting non-trivially with the triangulated structure od $D^b(X)$.

The following Theorem gives an answer when X is a Fano variety or it is a variety of general type.

Proposition 3.3.4 (**Bondal, Orlov**). *Let X be a smooth projective variety with ample (anti)-canonical bundle. The group of autoequivalences $Aut(D^b(X))$ of $D^b(X)$ is generated by*

1. *Derived pushforwards f_* of automorphisms f of X.*
2. *Shift functors $[n](-)$, for $n \in \mathbb{Z}$.*
3. *Twists $\mathcal{L} \otimes -$ by line bundles, $\mathcal{L} \in Pic(X)$.*

Thus

$$Aut(D^b(X)) \cong \mathbb{Z} \times (Aut(X) \ltimes Pic(X)). \tag{3.37}$$

Proof. Section 4.2 of [27] or [11]. □

When the canonical bundle is neither ample or anti-ample, for example in the Calabi-Yau case, the group $Aut(D^b(X))$ has a richer structure.

3.4 Spherical twists

Consider a compact symplectic manifold (M, β) and a Lagrangian sphere S inside M. In [39] Seidel associated to such S a symplectic automorphism called the *generalised Dehn twist* along S.

In [40], Seidel and Thomas introduced the spherical twists, which are the analogues of the generalised Dehn twist under Homological Mirror Symmetry; these provided an early example of genuinely derived autoequivalences and were used to construct an example of braid group action.

For this section we refer to [40] and [27].

Let X be smooth projective variety and let $D^b(X)$ be the full subcategory of the derived category of $\mathcal{O}_X$-modules consisting of complexes with bounded and coherent cohomology.

Definition 3.4.1. An object $\mathcal{E}$ in $D^b(X)$ is called *spherical* if:

1. $Hom^r_{D(X)}(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C}, & \text{if } r = 0, dim(X), \\ 0, & \text{otherwise.} \end{cases}$
2. $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$, where ω_X is the canonical bundle.

Definition 3.4.2. Given two objects $\mathcal{F} \in D(X)$ and $\mathcal{G} \in D(Y)$ we define

$$\mathcal{F} \boxtimes \mathcal{G} = \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{G}. \tag{3.38}$$

Notice that the a bifunctor

$$\pi_1^*(-) \otimes \pi_2^*(-) : Coh(X) \times Coh(Y) \longrightarrow Coh(X \times Y)$$

is exact in each of its two arguments.

Definition 3.4.3. Let $\mathcal{E} \in D^b(X)$ over the scheme X and let $\Delta : X \to X \times X$ be the diagonal embedding. The *twist* functor of $T_{\mathcal{E}}$ is the Fourier-Mukai transform $\Phi_{\mathcal{P}}$ where

$$\mathcal{P} := Cone(\mathcal{E}^\vee \boxtimes \mathcal{E} \xrightarrow{\eta} \mathcal{O}_\Delta). \tag{3.39}$$

where η is the canonical pairing which sends

$$\bigoplus_{i-j=k} (\pi_1^* \mathcal{E}_{[j]})^\vee \otimes \pi_2^* \mathcal{E}_{[i]} \longrightarrow 0 \qquad \forall k \in \mathbb{Z},\ k \neq 0$$

and

$$\bigoplus_{i-j=0} (\pi_1^* \mathcal{E}_{[j]})^\vee \otimes \pi_2^* \mathcal{E}_{[i]} \longrightarrow \mathcal{O}_X$$

by the usual canonical pairing of sheaves.

Definition 3.4.4. An (A_m)-*configuration*, $m \geq 1$, in $D^b(X)$ is a collection of m spherical objects $\mathcal{E}_1, \dots, \mathcal{E}_m$ such that

$$\dim Hom^*_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j) = \begin{cases} 1 & |i-j| = 1, \\ 0 & |i-j| \geq 2. \end{cases}$$

Theorem 3.4.5 (**Seidel, Thomas**). *The twist $T_{\mathcal{E}}$ along any spherical object $\mathcal{E}$ is an autoequivalence of $D^b(X)$. Moreover, if $\mathcal{E}_1, \dots, \mathcal{E}_m$ is an (A_m)-configuration, the twists $T_{\mathcal{E}_i}$ satisfy the braid relations:*

$$\begin{aligned} &T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} \cong T_{\mathcal{E}_{i+1}} T_{\mathcal{E}_i} T_{\mathcal{E}_{i+1}} && \text{for } i = 1, \dots, m-1, \\ &T_{\mathcal{E}_i} T_{\mathcal{E}_j} \cong T_{\mathcal{E}_j} T_{\mathcal{E}_i} && \text{for } |i-j| \geq 2. \end{aligned}$$

Proof. See [40] or Propositions 8.6 and 8.22 of [27]. □

Remark 3.4.6. *The second part of Theorem 3.4.5 is an example of a braid group action on the derived category $D^b(X)$. See section 3.10 for details on categorical group actions.*

Example 3.4.7. Let C be a smooth projective curve and let $x \in C$ be a point on C. The skyscraper sheaf $\mathcal{O}_x$ is a spherical object and

$$T_{\mathcal{O}_x}(-) \cong \mathcal{O}_C(x) \otimes (-).$$

Example 3.4.8. If C is a smooth rational curve with $C^2 = -2$ in a smooth projective surface, then the structure sheaf $\mathcal{O}_C$ is as spherical object.

Example 3.4.9. If X be a Calabi-Yau variety, then any line bundle $\mathcal{L} \in Pic(X)$ is a spherical object.

Example 3.4.10. If C is a smooth projective curve contained in a Calabi-Yau threefold with normal bundle isomorphic to

$$\mathcal{N}_{C/X} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1), \tag{3.40}$$

Then $\mathcal{O}_C$ is a spherical object.

Example 3.4.11. If Y is a smooth projective subvariety of a Calaby-Yau variety X of dimension 2n+1, such that $Y \cong \mathbb{P}^n$ and the normal bundle of Y is isomorphic to

$$\mathcal{N}_{Y/X} \cong \mathcal{O}(-1)^{\oplus n+1}, \tag{3.41}$$

then $\mathcal{O}_Y$ is a spherical object.

Remark 3.4.12. *If X is Fano or of general type variety then ω_X is ample or anti-ample. The condition*

$$\mathcal{E} \otimes \omega_X \cong \mathcal{E} \tag{3.42}$$

forces the support of the spherical object to be 0*-dimensional.*

3.5 Spherical functors

Anno and Logvinenko in [6] introduced the notion of spherical functors.

Given a scheme X, any $\mathcal{E} \in D^b(X)$ can be considered as the functor

$$(-) \otimes \mathcal{E} : D^b(pt) \to D^b(X).$$

Spherical functors are analogues of spherical objects where the point is replaced with a scheme Z.

A spherical functor $F : D^b(Z) \to D^b(X)$ induces two autoequivalences: the twist T_F of $D^b(X)$ and the cotwist C_F of $D^b(X)$.

Definition 3.5.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be enhanced triangulated categories and let

$$F : D(\mathcal{C}_1) \to D(\mathcal{C}_2)$$

be an enhanceable functor with enhanceable left and right adjoints

$$L, R : D(\mathcal{C}_2) \to D(\mathcal{C}_1).$$

The *spherical twist* $T : D(\mathcal{C}_2) \to D(\mathcal{C}_2)$ of F is the enhanceable functor that fits in the natural exact triangle

$$FR \xrightarrow{tr} Id \to T \xrightarrow{t} FR[1]. \tag{3.43}$$

The *spherical cotwist* $C : D(\mathcal{C}_1) \to D(\mathcal{C}_1)$ of F is the enhanceable functor that fits in the natural exact triangle

$$C \to Id \xrightarrow{act} RF \xrightarrow{c} C[1]. \tag{3.44}$$

The functor F is *spherical* if the following conditions hold:

1. The twist T is an autoequivalence of $D(\mathcal{C}_2)$.

2. The cotwist C is an autoequivalence of $D(\mathcal{C}_1)$.

3. The following composition in an isomorphism:

$$LT \xrightarrow{Lt} LFR[1] \xrightarrow{trR} R[1]$$

4. The following composition is an isomorphism:

$$R \xrightarrow{Ract} RFL[1] \xrightarrow{cL} CL[1]$$

Theorem 3.5.2. *Any two of the conditions 1-4 in the definition above imply all four.*

Proof. Theorem 5.1 of [6]. □

Remark 3.5.3. *By the argument in Lemma 5.16 in [7] if condition (2) of definition 3.5 holds, it is enough to show*

$$R \simeq CL[1]$$

for condition (4) to hold.

Example 3.5.4. Let $Z = Spec(\mathbb{C})$, let X be a smooth projective variety, let E be an object in $D^b(X)$ and set F to be the functor

$$F : D(Z) \xrightarrow{-\otimes_{\mathbb{C}} E} D(X).$$

Then E is a spherical object if and only if F satisfies conditions 2 and 4 of Theorem 3.5.2, so if and only if F is a spherical functor. (See example 3.5 of [5].)

Example 3.5.5. If D is a divisor of an algebraic variety X with inclusion map

$$i : D \hookrightarrow X \tag{3.45}$$

Then $F \coloneqq i_*$ is a spherical functor.

Indeed, we have the adjunctions

$$L = i^* \dashv F \dashv R = i^! \tag{3.46}$$

The Fourier-Mukai kernels are:

- $i_* : \mathcal{O}_D \in D(D \times X)$;
- $i^* : \mathcal{O}_D \in D(X \times D)$;
- $i^! \simeq i^* \otimes \mathcal{O}(D)[-1] : \mathcal{O}_D(D)[-1] \in D(X \times D)$.

By Proposition 3.2.3 the composition RF has Fourier-Mukai kernel $\mathcal{O}_D \oplus \mathcal{O}_D(D)[-1]$.

Therefore the twist $T = Cone(\mathrm{Id} \to RF)$ has Fourier-Mukai kernel $\mathcal{O}_D(D)[-1] \in D(D \times D)$ and

$$CL \cong R.$$

Example 3.5.6. Let X be a variety of dimension n, let $i : D \hookrightarrow X$ be a divisorial inclusion and let $\pi : D \to Y$ be a projective bundle of rank k over the variety Y of dimension $n-1-k$.

$$\begin{tikzcd} D \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & X \\ Y & \end{tikzcd}$$

Define the functor F

$$F := i_* \pi^* : D(Y) \longrightarrow D(X). \tag{3.47}$$

The right adjoint R of F is the functor

$$R = \pi_* i^! : D(X) \longrightarrow D(Y) \tag{3.48}$$

while the left adjoint is the functor L

$$L = \pi_! i^* : D(X) \longrightarrow D(Y). \tag{3.49}$$

The functor F is spherical if and only if

$$\mathcal{N}_{D/X} \simeq \omega_{D/Y} \otimes \pi^* \mathcal{L}$$

for some $\mathcal{L} \in Pic(Y)$ (See [7]).

3.6 $\mathbb{P}^n$ twists

The notion of $\mathbb{P}^n$ twists was introduced by Huybrechts and Thomas in [30] as mirror Dehn twists of Lagrangian $\mathbb{CP}^n$s.

When $n = 2$ the notion of a $\mathbb{P}^n$ twist coincides with the notion of the square of a spherical twist, but in higher dimensions, the two notions are different.

Let $\mathcal{C}$ a triangulated category and let

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{3.50}$$

be a complex of objects in $\mathcal{C}$, so that $g \circ f = 0$.

Definition 3.6.1. A right Postnikov system of the complex 3.50 is a diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "j"'] & B \arrow[r, "g"] & C \arrow[dl, "h"] \\ & X \arrow[u, "i"] & \end{tikzcd} \tag{3.51}$$

where $B \to C \to X$ is a distinguished triangle and $f = i \circ j$.

Definition 3.6.2. The convolution of the right Postnikov system (3.51) is the cone $\mathrm{Cone}(A[1] \xrightarrow{j} X)$.

Definition 3.6.3. A left Postnikov system of the complex (3.50) is a diagram

$$\begin{tikzcd} A \arrow[r, "f"] & B \arrow[r, "g"] \arrow[d, "k"] & C \\ & Y \arrow[ul, "l"] \arrow[ur, "m"'] & \end{tikzcd} \tag{3.52}$$

where $A \to B \to Y$ is a distinguished triangle and $g = m \circ k$.

Definition 3.6.4. The convolution of the left Postnikov system (3.52) is the cone $\mathrm{Cone}(Y \xrightarrow{m} C)$.

Definition 3.6.5. An object $E \in \mathcal{C}$ is a convolution of the complex (3.50) if it is a convolution of a right or a left Postnikov system associated to it.

Definition 3.6.6. Let $\mathrm{D^b}(X)$ be the bounded derived category of coherent sheaves on a smooth projective variety.
An object $\mathcal{E} \in \mathrm{D^b}(X)$ is a $\mathbb{P}^n$-*object* if :

1. $Hom^r_{D(X)}(\mathcal{E},\mathcal{E}) = \begin{cases} \mathbb{C}, & \text{if } r = 2i, i \in \{0,\dots,n\} \\ 0, & \text{otherwise.} \end{cases}$

2. $\mathcal{E} \otimes \omega_X \cong \mathcal{E}$, where ω_X is the canonical bundle.

Remark 3.6.7. *Let $\mathcal{E}$ be a $\mathbb{P}^n$ object in $\mathrm{D^b}(X)$ with $\dim(X) = m$, then by Serre duality*

$$\mathrm{Ext}^i(\mathcal{E},\mathcal{E}) = \mathrm{Ext}^{m-i}(\mathcal{E},\mathcal{E}\otimes\omega_X)^* = \mathrm{Ext}^{m-i}(\mathcal{E},\mathcal{E})^*$$

that forces $m = n$.

Example 3.6.8. Let X be an hyperkähler manifold of dimension 2n , and take $\mathbb{P}^n \hookrightarrow X$.
In this case $\mathcal{N}_{\mathbb{P}^n/X} \cong \Omega_{\mathbb{P}^n}$, hence $\mathcal{E}xt(\mathcal{O}_{\mathbb{P}^n},\mathcal{O}_{\mathbb{P}^n}) = \Omega^q_{\mathbb{P}^n}$.

Thus the local to global spectral sequence

$$E_2^{p,q} = H^p(X, \mathcal{E}xt(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})) \Rightarrow \mathrm{Ext}_X^{p+q}(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n})$$

gives the following isomorphism of rings

$$\mathrm{Ext}_X^*(\mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}) \cong H^*(\mathbb{P}^n, \Omega^*_{\mathbb{P}^n}) \cong H^*(\mathbb{P}^n, \mathbb{C}).$$

Thus $\mathcal{O}_{\mathbb{P}^n}$ is a $\mathbb{P}^n$-object.

Example 3.6.9. Let X be a K3 surface an let C be a curve on it; we have that $C \cong \mathbb{P}^1 \subset X$; so by example 3.6.8 $\mathcal{O}_C \in \mathrm{D^b}(X)$ is a $\mathbb{P}^1$ object.
Moreover, by Example 3.4.7 $\mathcal{O}_C$ is also a spherical object, which agrees with $S^2 \cong \mathbb{P}^1$.

Remark 3.6.10. *Suppose that $\mathcal{E}$ is a $\mathbb{P}^n$ object, then $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E})$ is one dimensional vector space let*

$$\phi : \mathcal{E}[-2] \longrightarrow \mathcal{E}$$

We have $\mathrm{Ext}^2(\mathcal{E}, \mathcal{E}) \cong \mathrm{Ext}^2(\mathcal{E}^\vee, \mathcal{E}^\vee)$, so just define $\phi^\vee$ to be the image of ϕ under the group isomorphisms of the Exts; $\phi^\vee$ is a generator for $\mathrm{Ext}^2(\mathcal{E}^\vee, \mathcal{E}^\vee)$ and can be represented by a morphism

$$\phi : \mathcal{E}^\vee[-2] \longrightarrow \mathcal{E}^\vee. \tag{3.53}$$

Let $\tilde{\phi} = \phi^\vee \boxtimes \mathrm{Id} - \mathrm{Id} \boxtimes \phi$,

$$\tilde{\phi} : \mathcal{E}^\vee \boxtimes \mathcal{E}[-2] \longrightarrow \mathcal{E}^\vee \boxtimes \mathcal{E}. \tag{3.54}$$

The trace map $tr : \mathcal{E}^\vee \boxtimes \mathcal{E} \longrightarrow \mathcal{O}_\Delta$ factorise throughout the cone $Cone(\tilde{\phi})$ of $\tilde{\phi}$

$$\begin{tikzcd}
\mathcal{E}^\vee \boxtimes \mathcal{E} \arrow[r] \arrow[dr, "tr"'] & Cone(\tilde{\phi}) \arrow[d, "\psi"] \\
& \mathcal{O}_\Delta
\end{tikzcd}$$

Definition 3.6.11. Let $\mathcal{E} \in D^b(X)$ be a $\mathbb{P}^n$-object and let $P_\mathcal{E} = Cone(\psi)$. The $\mathbb{P}^n$ twist $T^{\mathbb{P}}_{\mathcal{P}}$ associated to $\mathcal{E}$ is the Fourier-Mukai transform $\Phi_{P_\mathcal{E}}$ with kernel $P_\mathcal{E}$

$$T^{\mathbb{P}}_{\mathcal{P}} := \Phi_{P_\mathcal{E}} : D^b(X) \longrightarrow D^b(X) \tag{3.55}$$

Theorem 3.6.12. *If $\mathcal{P} \in D^b(X)$ is a $\mathbb{P}^n$ object, then the $\mathbb{P}^n$ twist $T^{\mathbb{P}}_{\mathcal{P}}$ is an autoequivalence of $D^b(X)$.*

Proof. See Proposition 2.6 of [30] or Proposition 8.19 of [27]. □

3.7 $\mathbb{P}^n$ functors

The notion of $\mathbb{P}^n$-functor unifies and generalises those of spherical functors and of $\mathbb{P}^n$-objects. In its general form, it is due to Anno and Logvinenko, we refer the reader to [7] for all technical details.

Definition 3.7.1. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be enhanced triangulated categories and let

$$F : \mathcal{C}_1 \to \mathcal{C}_2$$

be an enhanceable functor with enhanceable left and right adjoints

$$L, R : \mathcal{C}_2 \to \mathcal{C}_1.$$

F is a *$\mathbb{P}^n$-functor* if it can be equipped with a triple (H, Q_n, γ):

- H is an enhanced autoequivalence of $\mathcal{C}_1$ such that $H(Ker(F)) = Ker(F)$.
- Q_n is cyclic degree n coextension of Id by H of the form:

$$\begin{tikzcd}[column sep=small]
\mathrm{Id} \arrow[rr, "\iota_1", "\star"'] & & Q_1 \arrow[dl, "\mu_1"] \arrow[rr, "\iota_2", "\star"'] & & Q_2 \arrow[dl, "\mu_2"] \arrow[r] & \cdots \arrow[r] & Q_{n-2} \arrow[dl] \arrow[rr, "\iota_{n-1}", "\star"'] & & Q_{n-1} \arrow[dl, "\mu_{n-1}"] \arrow[rr, "\iota_n", "\star"'] & & Q_n \arrow[dl, "\mu_n"] \\
& H \arrow[ul, dashed] & & H^2 \arrow[ul, dashed] \arrow[ll, dashed] & & \cdots \arrow[ul, dashed] \arrow[ll, dashed] & & H^{n-1} \arrow[ul, dashed] \arrow[ll, dashed] & & H^n \arrow[ul, dashed] \arrow[ll, dashed]
\end{tikzcd}$$

  Here all starred triangles are exact and all the remaining triangles are commutative. Let $\iota = \iota_n \circ \cdots \circ \iota_1$.

- The map γ is an isomorphism

  $$\gamma : Q_n \xrightarrow{\sim} RF$$

  that interweaves the adjunction unit $\mathrm{Id} \xrightarrow{\varepsilon} RF$ with the map $\mathrm{Id} \xrightarrow{\iota} Q_n$;

  Note that as $F \xrightarrow{F\varepsilon} FRF$ is a retract, so is $F\iota$. Hence the exact triangle $FR \xrightarrow{F\iota_1 R} FQ_1R \to FHR$ is also split. Choose any splitting $FHR \hookrightarrow FQ_1R$ and denote by ϕ the composition

  $$FHR \hookrightarrow FQ_1R \xrightarrow{\iota_n \circ \ldots \circ \iota_2} FQ_nR \xrightarrow{F\gamma R} FRFR. \tag{3.56}$$

  Define the map $FHR \xrightarrow{\psi} FR$ to be the composition

  $$FHR \xrightarrow{\phi} FRFR \xrightarrow{FR\,\mathrm{tr} - \mathrm{tr}\,FR} FR.$$

  Note that any choice of the splitting $FHR \hookrightarrow FQ_1R$ in the definition of ϕ will produce the same map ψ, since the composition $(FR\,\mathrm{tr} - \mathrm{tr}\,FR) \circ F\epsilon R$ is zero.

satisfying the following conditions:

1. *The monad condition.* The map

$$v : FHQ_{n-1} \xrightarrow{FH\iota_n} FHRF \xrightarrow{\psi F} FRF \xrightarrow{F\kappa} FC[1] \tag{3.57}$$

is an isomorphism, where C is the spherical cotwist of F defined by the exact triangle $\mathrm{Id} \xrightarrow{\varepsilon} RF \xrightarrow{\kappa} C[1]$.

2. *The adjoints condition.* The map

$$w : FR \xrightarrow{FR\varepsilon} FRFL \xrightarrow{F\mu_n L} FH^n L \tag{3.58}$$

is an isomorphism.

3. *The highest degree term condition.* There exists an isomorphism

$$u : FH^n L \xrightarrow{\sim} FHH^n H'L$$

that makes the following diagram commutate:

$$\begin{tikzcd} FHQ_{n-1}L \arrow[r, "FH\iota_n L"] \arrow[d, "\mathrm{Id}"'] & FHRFL \arrow[r, "\psi FL"] & FRFL \arrow[r, "F\mu_n L"] & FH^n L \arrow[d, "\sim"] \\ FHQ_{n-1}L \arrow[r, "FH\iota_n L"] & FHRFL \arrow[r, "FHR\psi'"] & RHRFH'L \arrow[r, "FH\mu_n H'L"] & FHH^n H'L \end{tikzcd} . \tag{3.59}$$

where H' is the inverse of H the map $\psi' \colon FL- > FH'L$ is the left dual of $\psi \colon FHR \to FR$.

Theorem 3.7.2. *Let $F \colon C_1 \to C_2$ equipped with the triple (H, Q_n, γ), as above, be a $\mathbb{P}^n$-functor.*

Let the $\mathbb{P}$-twist *P_F of F to be the unique convolution of the two-step complex*

$$FHR \xrightarrow{\psi} FR \xrightarrow{tr} Id, \tag{3.60}$$

see [4] for the uniqueness of the convolution.

Then P_F is an autoequivalence of the category $\mathcal{C}_2$.

Proof. See Theorem 4.1 in [7]. □

Remark 3.7.3. *Definition 3.7.1 is the more general version of the definition of a* split $\mathbb{P}^n$-functor, *an enhanced functor F equipped with an isomorphism*

$$RF \simeq \mathrm{Id} \oplus H \dots H^n$$

for which the following conditions hold:

- Strong monad condition *: The matrix A_l of the left multiplication by H in RF has the form

$$\begin{pmatrix} * & * & \dots & * & * \\ 1 & * & \dots & * & * \\ 0 & 1 & \dots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & * \end{pmatrix}, \tag{3.61}$$

i.e., $a_{kj} = 0$ for $k > j + 1$ and $a_{j+1,j}$ are identities for $0 \leq j < n$.

- Weak adjoints condition*: There exists* some isomorphism $R \simeq H^n L$.

Example 3.7.4. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be enhanced triangulated categories and let

$$F : D(\mathcal{C}_1) \to D(\mathcal{C}_2)$$

be a enhanceable functor with enhanceable left and right adjoints

$$L, R : D(\mathcal{C}_2) \to D(\mathcal{C}_1).$$

F is a spherical functor with twists T and cotwist C if and only if is a $\mathbb{P}^1$-functor with

$$H \cong C[1]$$

and the degree 1 coextension of Id by H structure on RF defined by the exact triangle

$$C \to Id \xrightarrow{act} RF \xrightarrow{c} C[1]. \tag{3.62}$$

(See Proposition 7.1 [7]).

Example 3.7.5. Let X be a smooth projective variety and let $\mathcal{P} \in D^b(X)$ be a $\mathbb{P}^n$ object, let $Z = Spec(\mathbb{C})$ and consider the functor F

$$F : D^b(Z) \xrightarrow{-\otimes_{\mathbb{C}}\mathcal{P}} D^b(X) \tag{3.63}$$

with $\mathcal{P} \in D^b(X) \cong D^b(Z \times X)$.

We have

$$RF \cong \mathrm{Hom}(\mathcal{P}, \mathcal{P}) \cong \mathbb{C} \oplus \mathbb{C}[-2] \oplus \cdots \oplus \mathbb{C}[-2n]$$

which decomposes RF as

$$RF \cong \mathrm{Id} \oplus H \oplus \cdots \oplus H^n \tag{3.64}$$

where $H = [-2]$ and (3.64) gives F a structure of split $\mathbb{P}^n$-functor.

Moreover, from the weak adjoint condition we have that

$$R \simeq H^n L.$$

(See [1] for details.)

Example 3.7.6. Let X be the Hilbert scheme of n points on a projective K3 surface Z so that

$$X = Z^{[n]}$$

and define F to be the functor

$$F : D^b(Z) \longrightarrow D^b(X)$$

realised by the Fourier-Mukai transform with Fourier-Mukai kernel the universal ideal sheaf $\mathcal{I} \in D^b(Z \times X)$.

There exists an isomorphism

$$RF \cong \mathrm{Id} \oplus H \oplus H^2 \oplus \cdots \oplus H^{n-1} \tag{3.65}$$

where $H = \mathrm{Id}[-2]$ and (3.65) gives F the structure of a split $\mathbb{P}^{n-1}$ functor.

(See Theorem 2 of [1] for details).

Example 3.7.7. Let $\mathcal{V}$ be a vector bundle on a smooth projective variety Z, with $\mathbb{P}^n$ fibration

$$\pi : \mathbb{P}\mathcal{V} \longrightarrow Z$$

and embedding in a smooth projective variety X

$$i : \mathbb{P}\mathcal{V} \hookrightarrow X$$

with normal bundle of rank n isomorphic to

$$\mathcal{N}_{\mathbb{P}\mathcal{V}/X} \cong \Omega^1_{\mathbb{P}\mathcal{V}/X}.$$

Define the functors f_k to be

$$f_k := i_* \circ (\mathcal{O}_{\mathbb{P}\mathcal{V}}(k) \otimes \pi^*(-)) : D^b(Z) \longrightarrow D^b(X). \tag{3.66}$$

let r_k be their right adjoints and define

$$h := \mathrm{Id}[-2] : D^b(Z) \to D^b(Z). \tag{3.67}$$

Let F_k, R_k,L_K and H be their standard enhancements.

There exist a structure of cyclyc extension of degree n of Id by H on the adjunction monad R_kF_k that makes F_k a $\mathbb{P}^n$-functor (see Theorem 7.2 in [7]) .

3.8 Mukai flops in derived categories

For the first part of this section we refer to section 11.4 of [27], for the second part we refer to section 5 of [3].

Let X be a smooth projective variety of even dimension and let P be a smooth subvariety of X with dimension half the dimension of X.

Let moreover P be isomorphic to $\mathbb{P}^n$, with

$$\mathcal{N}_{P/X} \cong \Omega_P. \tag{3.68}$$

Let $\tilde{X} = Bl_P(X)$ be the blow up of X along P with projective morphism

$$q : \tilde{X} \longrightarrow X$$

and exceptional divisor $E \cong \mathbb{P}(\mathcal{N}_{P/X}) \cong \mathbb{P}(\Omega_P)$.

If we consider $P \cong \mathbb{P}(V)$, with V an $n+1$ dimensional complex vector space, then from the Euler sequence

$$0 \to \Omega_P \to V^* \otimes \mathcal{O}(-1) \to \mathcal{O}_P \to 0 \tag{3.69}$$

we have that

$$\mathbb{P}(\Omega_P) \hookrightarrow \mathbb{P}(V^* \otimes \mathcal{O}(-1)) \cong \mathbb{P}(V^*) \times \mathbb{P}(V)$$

The exceptional divisor $E \cong \mathbb{P}$ inside $P \times P^\vee \cong \mathbb{P}(V^*) \times \mathbb{P}(V)$ is the incidence variety of pairs (l, H) of lines $l \subset V$ and planes $H \subset V$ such that $l \subset H$.

Indeed, the fiber $\Omega_P(l)$ over a line $[l] \subset P$ is naturally isomorphic to the space of linear functions $\alpha : V \to \mathbb{C}$ which vanish on l.

Therefore, because $E \in |\mathcal{O}(-1,1)|$, from the adjuction formula we deduce that the canonical bundle of the exceptional divisor

$$\omega_E \cong \mathcal{O}(-n,-n)|_E \tag{3.70}$$

Moreover, using the adjuction formula for $E \subset \tilde{X}$, we have that the same canonical bundle is isomorhic to

$$\omega_E \cong (\omega_{\tilde{X}} \otimes \mathcal{O}(E))|_E \cong (q^*\omega_X \otimes \mathcal{O}((n-1)E))|_E \otimes \mathcal{O}_E(E) \cong \pi^*(\omega_X|_P) \otimes \mathcal{O}_E(nE)$$

where π is the natural map $\pi : E \to P$.

Since by assumption $\omega_X|P$ is the trivial bundle, we have that $\omega_E \cong \mathcal{O}(-1,-1)$ and conclude that

$$\mathcal{O}_E(E) \cong \mathcal{O}(-1,-1) \tag{3.71}$$

The previous isomorphism, ensure that exists a birational morphism

$$p : \tilde{X} \to X'$$

which restricted to E is the second projection $E \subset P \times P^\vee \to P^\vee$ and away from E is an isomorphism (by the Fujiki-Nakano criterion).

Moreover,

$$\mathcal{N}_{P^\vee/X'} \cong \Omega_{P^\vee} \text{ and } \omega_{X'}|_{P^\vee} \cong \mathcal{O}_{P^\vee}$$

$$\begin{tikzcd}
 & & E \arrow[r, hook] \arrow[d, hook] \arrow[ddll, "\pi"'] \arrow[ddrr, "\pi'"] & P \times P^\vee & \\
 & & \tilde{X} \arrow[dl, "q"'] \arrow[dr, "p"] & & \\
P \arrow[r, hook] & X & & X' & P^\vee \arrow[l, hook']
\end{tikzcd}$$

Proposition 3.8.1. *The Fourier-Mukai transform induced by the compositon of the functors*

$$p_* \circ q^* : D^b(X) \longrightarrow D^b(X') \tag{3.72}$$

is not fully faithful.

Proof. Proposition 11.28 of [27]. □

Theorem 3.8.2. *Given the reduced subvariety $\tilde{X} \cup (P \times P^\vee)$ inside $X \times X'$, and the object $\mathcal{O}_{\tilde{X}\cup(P\times P^\vee)} \in D^b(X \times X')$, the Fourier-Mukai transform with kernel $\mathcal{O}_{\tilde{X}\cup(P\times P^\vee)}$*

$$\Phi_{\mathcal{O}_{\tilde{X}\cup(P\times P^\vee)}} : D^b(X) \longrightarrow D^b(X') \tag{3.73}$$

is an equivalence.

Proof. Proposition 11.28 of [27] or [31] and [36]. □

Consider the three dimensional complex vector space $\mathbb{C}^3$ and the three dimensional flag variety Fl_3, with the natural projection maps to $\mathbb{P}^{2*}$ and $\mathbb{P}^2$

$$\begin{tikzcd}
 & Fl_3 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
\mathbb{P}^2 & & \mathbb{P}^{2*}.
\end{tikzcd}$$

Let C be the total space of the cotangent bundle of Fl_3

$$C = T^*Fl_3.$$

Let A and E be the quasi-projective varieties defined as the total space of the cotangent bundle of $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$:

$$A = T^*\mathbb{P}^2 \qquad E = T^*\mathbb{P}^{2\vee}.$$

Let B and D be the quasi-projective varieties defined as the total space of the pullback via p_1 and p_2 on Fl_3 of the contangent bundles of $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$:

$$B = p_1^*T^*\mathbb{P}^2 \qquad E = p_2^*T^*\mathbb{P}^{2\vee}.$$

Then we have the following diagram

$$\begin{tikzcd} B \arrow[r, hook, "i_B"'] \arrow[d, two heads, "\pi_A"'] & C & D \arrow[l, hook', "i_D"'] \arrow[d, two heads, "\pi_E"'] \\ A & & E \end{tikzcd} \tag{3.74}$$

where i_B, i_D are divisorial inclusions and π_A, π_E are $\mathbb{P}^1$-bundles respectively on A and E.

From remark 2.2.30, we have the following descriptions of the quasi-projective varieties

$$A = \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} ;\ \dim(V_1) = 1 \right\},$$

and

$$E = \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_2 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} ;\ \dim(V_2) = 2 \right\}.$$

Moreover the four dimensional subvariety of C defined as the (transverse) intersection of B and D can be described as

$$B \cap D = \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] & V_2 \arrow[ll, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[ll, bend left, "\alpha"] \end{tikzcd} \right\}.$$

equipped with the two natural forgetful maps

$$\begin{tikzcd} & B \cap D \arrow[dl, "q_1"'] \arrow[dr, "q_2"] & \\ A & & E \end{tikzcd}$$

where q_i is the map that forget the choice of the $n-i$-th space.

Both q_1 and q_2 are isomorphic to the blow up of the zero section carved out by $\{\alpha = 0\}$ in respectively A and E.

Both the bow-ups have the same exceptional divisor Fl_3 carved out by $\{\alpha = 0\}$ and the resulting birational transformations

$$q_2 \circ q_1^{-1} : A \dashrightarrow E$$

$$q_1 \circ q_2^{-1} : E \dashrightarrow A$$

are a local model of a four dimensional Mukai flop.

Therefore, from Theorem 3.8.2 we have the following

Corollary 3.8.3. *Given the reduced subvariety $(B \cap D) \cup (\mathbb{P}^{2*} \times \mathbb{P}^2)$ inside $E \times A$, and the object $\mathcal{O}_{(B \cap D) \cup (\mathbb{P}^{2*} \times \mathbb{P}^2)} \in D^b(E \times A)$, the Fourier-Mukai transform with kernel $\mathcal{O}_{(B \cap D) \cup (\mathbb{P}^{2*} \times \mathbb{P}^2)}$*

$$\Phi_{\mathcal{O}_{(B \cap D) \cup (\mathbb{P}^{2*} \times \mathbb{P}^2)}} : D^b(E) \longrightarrow D^b(A) \tag{3.75}$$

is an equivalence.

3.9 The excess bundle formula

This section is on the excess bundle formula which computes the derived tensor product of two structure sheaves of two smooth subvarieties of a smooth variety: we refer to [38] and [7] for more details.

Let Z_1, Z_2 be two locally complete intersection subvarieties of a smooth algebraic variety Z with their intersection $W = Z_1 \cap Z_2$ being a locally complete subvariety of Z_1 and Z_2.

Let i_1, i_2, j_1, j_2 be the inclusion of the following fiber square

$$\begin{tikzcd} Z_1 \cap Z_2 = & W \arrow[r, hook, "j_1"] \arrow[d, hook, "j_2"'] & Z_1 \arrow[d, hook, "i_1"] \\ & Z_2 \arrow[r, hook, "i_2"'] & Z \end{tikzcd}$$

Definition 3.9.1. The *excess bundle* $\mathcal{E}_W$ of the insersection $W = Z_1 \cap Z_2$, is the locally free sheaf which fits in the short exact sequence of sheaves on W

$$0 \longrightarrow \mathcal{N}_{W/Z} \longrightarrow j_1^* \mathcal{N}_{Z_1/Z} \oplus j_2^* \mathcal{N}_{Z_2/Z} \longrightarrow \mathcal{E}_W \longrightarrow 0 \tag{3.76}$$

Theorem 3.9.2. *Under the previous hypothesis, the cohomology sheaves of*

$$i_2^* i_{1*} \mathcal{O}_{Z_1} \in D^b(Z_2)$$

are

$$H^{-q}(i_2^* i_{1*} \mathcal{O}_{Z_1}) = j_{1*}(\bigwedge^q \mathcal{E}_W^\vee) \tag{3.77}$$

Proof. See [38]. □

Definition 3.9.3. The intersection $W = Z_1 \cap Z_2$ of two locally complete intersection subvarieties Z_1, Z_2 of a smooth algebraic variety Z is called *transverse* if both the following conditions are satisfied:

1. W is a smooth subvariety of Z.

2. $codim_Z(W) = codim_Z(Z_1) + codim_Z(Z_2)$.

Corollary 3.9.4. *If the intersection $W = Z_1 \cap Z_2$ is transverse then we have the following isomorphism*

$$\mathcal{O}_{Z_1} \otimes \mathcal{O}_{Z_2} \cong \mathcal{O}_W. \tag{3.78}$$

3.10 A braid group action on $D^b(T^*Fl_n)$

In this section, we present the Khovanov and Thomas braid group action on the derived category of coherent sheaves of the cotangent bundle of complete flag varieties $D^b(T^*Fl_n)$: we refer to [32] for more details.

Definition 3.10.1. An *action* of a group G on a category $\mathcal{C}$ is an assignment of an invertible functor

$$F_g : \mathcal{C} \longrightarrow \mathcal{C}$$

to each $g \in G$ such that $F_g \circ F_h \cong F_{g \cdot h}$. Moreover, the following diagram has to be commutative

$$\begin{tikzcd} F_f \circ F_g \circ F_h \arrow[r, "\cong"] \arrow[d, "\cong"] & F_f \circ F_{g\cdot h} \arrow[d, "\cong"] \\ F_{f\cdot g} \circ F_h \arrow[r, "\cong"] & F_{f\cdot g\cdot h} \end{tikzcd} .$$

Remark 3.10.2. *The n-braid group* Br_n *is generated by the elements* $\{t_1, \ldots, t_n\}$

subjects to the relations

$$\begin{cases} t_i t_j = t_j t_j & \text{if } \mid i-j \mid > 1 \quad \text{"}\textit{commutation}\text{"} \\ t_i t_j t_i = t_j t_i t_j & \text{if } \mid i-j \mid = 1 \quad \text{"}\textit{braiding}\text{"}. \end{cases}$$

Recall that the the complete flag variety is defined

$$\mathrm{Fl}_n = \left\{ 0 \subset V_1 \subset \ldots \subset V_{n-1} \subset V \;\middle|\; \dim(V_i) = i \right\}$$

and the total space of its cotangent bundle can be described as the space

$$\mathrm{T}^*\,\mathrm{Fl}_n \cong \left\{ \begin{tikzcd}[column sep=small] 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \dots \arrow[ll, bend right, "\alpha"'] & \subset & V_{n-1} \arrow[ll, bend right, "\alpha"'] & \subset & V \arrow[ll, bend right, "\alpha"'] \end{tikzcd} ;\ \dim(V_i) = i \right\}.$$

Denote by $C = T^*Fl_n(1,\dots,1)$ the total space of the cotangent bundle on the complete flag and by $A_i = T^*Fl_n(\hat{i})$ the total space of the cotangent bundle on the flag where the choice of V_i is skipped.

Let the map p_i

$$p_i : Fl_n \to Fl_n(\hat{i})$$

be the projection which just skip the choice of V_i containing a V_{i-1} and contained in V_{i+1}, therefore a $\mathbb{P}^1$ bundle over $Fl_n(\hat{i})$.

Denote $B_i = p_i^* T^*Fl_n(\hat{i})$ the total space of the pullback via the morphism p_i on Fl_n of the cotangent bundle of $Fl_n(\hat{i})$.

As before, B_i can be described as the space

$$\left\{ \begin{tikzcd}[column sep=small] 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \dots \arrow[ll, bend right, "\alpha"'] & \subset & V_{i-1} \arrow[ll, bend right, "\alpha"'] & \subset V_i \subset & V_{i+1} \arrow[ll, bend right, "\alpha"'] & \subset & \dots \arrow[ll, bend right, "\alpha"'] & \subset & V \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \;\middle|\; \dim(V_i) = i \right\}.$$

For every integer $i \in \{1,\dots,n\}$ we have a couple of natural maps j_i and π_i

$$\begin{tikzcd} B_i \arrow[r, hook, "j_i"] \arrow[d, two heads, "\pi_i"'] & C \\ A_i & \end{tikzcd} \tag{3.79}$$

which are respectively the divisorial inclusion of B_i in C and the canonical projection of B_i onto A_i.

Lemma 3.10.3. *Let $\mathcal{V}_i$ be the pullback on Fl_n of the tautological bundle of $\mathrm{Gr}(i,n)$. The normal bundle $\mathcal{O}_{B_i}$ of the divisor B_i inside C is isomorphic to*

$$\mathcal{O}(B_i) \cong (\Lambda^i \mathcal{V}_i^*)^{-2} \otimes \Lambda^{i-1}\mathcal{V}_{i-1}^* \otimes \Lambda^{i+1}\mathcal{V}_{i+1}^*.$$

Proof. Section 4.1 of [32]. □

For every $i \in \{1,\dots,n-1\}$, define the functors

$$F_i = j_{i*} \circ p_i^* : D^b(A_i) \to D^b(C)$$

and denote their right adjoints by

$$R_i = p_{i*} \circ j_i^! : D^b(C) \to D^b(A_i)$$

Since we are in the hypothesis of example 3.5.6 F_i are spherical functors and their spherical twists are autoequivalences.

Moreover, since B_i is a divisor inside C the right adjoint of the functor $j_i^!$ is the functor $j_{i*}(-) \otimes \mathcal{O}(-B_i)$, so by Lemma 3.10.3 we have the isomorphism

$$j_{i*}(p_i^*(-) \otimes \omega_{p_i})) \otimes \mathcal{O}(-B_i)[2] \cong j_{i*}p_i^*[2]$$

thus, the following adjuctions hold

$$F_i \dashv R_i \dashv F_i[2] \tag{3.80}$$

with the respective counit and unit maps

$$\varepsilon_i : F_iR_i \to Id \qquad \epsilon_i : Id[-2] \to F_iR_i.$$

and, therefore, the cotwist around F_i is [2].

Define the functors

$$T_i = \text{Cone}(F_iR_i \xrightarrow{\varepsilon_i} Id) \qquad T_i' = \text{Cone}(Id \xrightarrow{\epsilon_i} F_iR_i). \tag{3.81}$$

Theorem 3.10.4. *The functors T_i defined in 3.81 are autoquivalences of* $\mathrm{D^b(T^*Fl_n)}$ *with respective inverses T_i', i.e.*

$$T_i' \circ T_i \cong Id \cong T_i \circ T_i'$$

Moreover the $n-1$ autoequivalences T_i satisfy the braid relations:

$$\begin{aligned} T_iT_j &\cong T_jT_i && \textit{for } |i-j| \geq 1. \\ T_iT_jT_i &\cong T_jT_iT_j && \textit{for } |i-j| = 1. \end{aligned}$$

Thus, there is a categorical action of the braid group Br_n *on* $\mathrm{D^b(T^*Fl_n)}$.

Proof. Theorem 4.1 in [32]. □

To conclude this section we prove two results on the canonical bundle of the quasi-projective varieties C, B_i and A_i.

Proposition 3.10.5. *Let X be a smooth projective variety and let $p : E \to X$ be the cotangent vector bundle Ω_X Let $Y = Spec(Sym(E^*))$ be the total space of E, then the canonical bundle of ω_Y is trivial.*

Proof. Consider the projection π

$$\pi : Y \longrightarrow X,$$

then the relative tangent bundle $T_{Y/X}$ of Y over X is isomorphic to

$$T_{Y/X} \cong \pi^* E$$

thus the relative canonical bundle ω_{Y_X} is isomorphic to

$$\omega_{Y_X} \cong \pi^* det(E).$$

Therefore the canonical bundle ω_Y of the total space of E is isomorphic to

$$\omega_Y \cong \pi^*\omega_X \otimes \omega_{Y/X} \cong \pi^*(\omega_X \otimes det(T_X^*)) \cong \pi^*\mathcal{O}_X \cong \mathcal{O}_Y$$

since the dual commutes with the determinant. □

As a corollary of Proposition 3.10.5, the canonical bundle of C and A_i is trivial for every i.

Proposition 3.10.6. *The canonical bundle of B_i is*

$$\omega_{B_i} \cong \mathcal{O}_{B_i}(-B_i) \cong (\Lambda^i \mathcal{V}_i^*)^2 \otimes (\Lambda^{i-1}\mathcal{V}_{i-1}^*)^{-1} \otimes (\Lambda^{i+1}\mathcal{V}_{i+1}^*)^{-1} \tag{3.82}$$

Proof. Using adjunction formula and Proposition 3.10.5 we have that

$$\omega_{B_i} \cong \mathcal{O}_{B_i}(-B_i)$$

and from Lemma 3.10.3 follows that

$$\omega_{B_i} \cong (\Lambda^i \mathcal{V}_i^*)^2 \otimes (\Lambda^{i-1}\mathcal{V}_{i-1}^*)^{-1} \otimes (\Lambda^{i+1}\mathcal{V}_{i+1}^*)^{-1}.$$

□

— Chapter 4 —

Categorical action of generalised braids

Braids are topological configurations of n disjoint pieces of string with n fixed endpoints, considered up to isotopies which keep the strands disjoint. In [32] Khovanov and Thomas constructed a categorical action of the braid group Br_n on the derived category $D(T^* \mathrm{Fl}_n)$ of coherent sheaves on the cotangent bundle of the variety Fl_n of the complete flags in $\mathbb{C}^n$. The configuration of n distinct fixed endpoints is represented by $D(T^* \mathrm{Fl}_n)$ and a braid starting and ending at such configuration is represented by an auto-equivalence of this category.

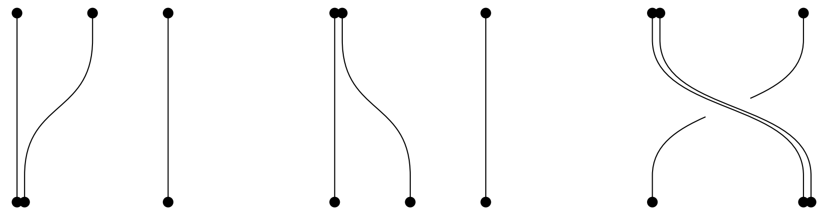
In this chapter we describe a more general structure: $\mathcal{GB}r_n$, the *generalised braid category* on n-strands. Our goal is to study its categorical representations. The definition of $\mathcal{GB}r_n$ resembles that of the category $\mathcal{W}eb_n$ of $\mathfrak{sl}_n$-webs ([19], [37]), but unlike the latter category, $\mathcal{GB}r_n$ is not additive. Like in $\mathcal{W}eb_n$, the objects of $\mathcal{GB}r_n$ are ordered partitions $\bar{i} = (i_1, \dots, i_k)$ of n and the morphisms are generated by certain elementary diagrams modulo relations. The key difference is that $\mathcal{W}eb_n$ is enriched over $\mathbb{Z}[q, q^{-1}]$, and some relations are additive. The category $\mathcal{GB}r_n$ is topological in nature, and its relations stem from isotopies. Our main interest, however, lies in *skein triangulated representations* of $\mathcal{GB}r_n$ where we impose triangulated relations on functors that conjecturally categorify those of $\mathcal{W}eb_n$.

In particular, it is expected that there is a certain skein-triangulated action of $\mathcal{GB}r_n$ on the derived categories of coherent sheaves of the cotangent bundles of $\mathrm{Fl}_n(\bar{i})$, the varieties of complete and partial flags in $\mathbb{C}^n$. This action consists of a network of functors between these derived categories, some of which are well-known in geometric representation theory: the Khovanov-Thomas braid group action [32] comprises a limited subset of the endofunctors of the full flag variety, a single node in the network. The Cautis-Kamnitzer-Licata categorical $\mathfrak{sl}_2(\mathbb{C})$-action [18] comprises a limited subset of our functors between the Grassmanians, a few of the nodes in the network. To obtain the Cautis-Kamnitzer tangle calculus [17], on the other hand, we restrict a part of our network to a small slice of each flag variety, the resolution of (n, n)-Slodowy slice. On these slices, the generalised braid relations simplify and become the tangle calculus.

In Section 1, we define the generalised braid category $\mathcal{GB}r_n$. In Section 2, we define the notion of a skein-triangulated representation of $\mathcal{GB}r_3$ and give the conditions on fork generator functors which allow us to construct such representation out of them. In Section 3, we introduce the setup for a generalised braid action on $D^b(\mathrm{T}^* \mathrm{Fl}_3)$, we define the merge and the fork functors and compute their compositions at level of Fourier-Mukai kernels. In Section 4, we describe a conjectural program which allows us to inductively compute the multiple crossing functors and verify it for $\mathcal{GB}r_3$. In Section 5, we construct the conjectured skein triangulated representation of $\mathcal{GB}r_3$ on $D(T^*(\mathrm{Fl}_3(\bar{i})))$.

4.1 Generalised braid category

Intuitively, generalised braids should be thought of as braids where we remove the restriction that the strands are not allowed to touch each other. They can now come together, continue as a strand with multiplicity, and then possibly split apart:



Any two strands with multiplicities p and q can join up and continue as a strand with multiplicity $p + q$. Any strand with multiplicity $p + q$ can split up into two strands with multiplicities p and q. Instead of a single configuration of n disjoint endpoints, we have multiple configurations indexed by the ordered partitions of n. Finally, we want to consider the generalised braids up to isotopies which preserve the intervals on which strands come together. An isotopy can make such interval shorter or longer, but can't make it vanish completely or join two such intervals into one.

This intuitive idea needs to be coarsened: we do not want to distinguish individual strands within a strand with multiplicity. One approach would be to take the definition above, and factorise by the action of the permutation group S_p on each multiplicity p strand which permutes the individual strands within it.

We take another approach, that of embedded trivalent graphs:

Definition 4.1.1. The *generalised braid category* $\mathcal{GB}r_n$ is the category with:

- Objects: ordered partitions of n:

$$\left\{ \bar{i} = (i_1 \dots i_k) \;\middle|\; \sum_{s=1}^{k} \bar{i}_s = n \right\}.$$

- Morphisms: The morphisms

$$\bar{i} = (i_1 \dots i_k) \quad \to \quad \bar{j} = (j_1 \dots j_l)$$

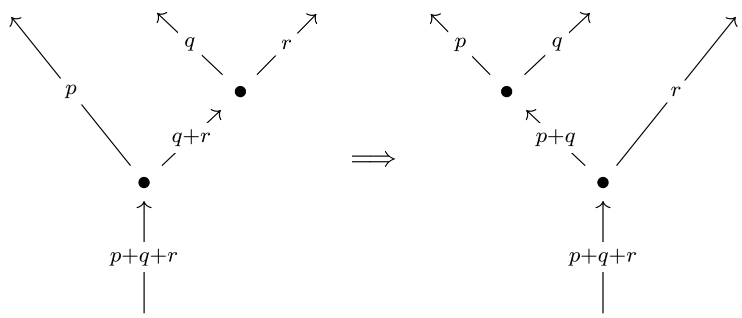
are the *generalised braids* with startpoint/endpoint configurations $\bar{i}$ and $\bar{j}$. Such braid is an oriented graph with edges colored by integers from 1 to n. We refer to the colors as the multiplicities. The graph must have:

- k ordered 1-valent startpoint vertices with the emerging edges of multiplicities $i_1, , \ldots, i_k$,
- l ordered 1-valent endpoint vertices with the terminating edges of multiplicities $j_1, \ldots, j_l$,
- The remaining vertices are trivalent with the flow condition respected: the total multiplicity of the terminating edges equals that of the emerging edges.

We consider this oriented graph together with an embedding into $\mathbb{R}^2 \times [0, 1]$, which satisfies:

- The startpoint vertices are $(1, 0, 0), (2, 0, 0), \ldots, (k, 0, 0)$.
- The endpoint vertices are $(1, 0, 1), (2, 0, 1), \ldots, (l, 0, 1)$.
- The orientation at any point must project positively onto $[0, 1]$.

We considered these generalised braids up to equivalence generated by two types of relations. One is the isotopy of embedded trivalent graphs. The other is the multifork and multimerge relations, where we identify the graphs which can be obtained one from another by the modification:



(4.1)

and its obvious analogue for merge vertices.

- <u>The composition</u>: The composition is given by concatenation of graphs.

- <u>Identity morphisms</u>: The identity morphism from $\bar{i} = (i_1, \ldots, i_k)$ to itself is the graph which consists of k vertical edges: from the j-th startpoint vertex to the j-th endpoint vertex for all $l \leq j \leq k$.

4.2 Skein-triangulated representations of $\mathcal{GB}r_3$

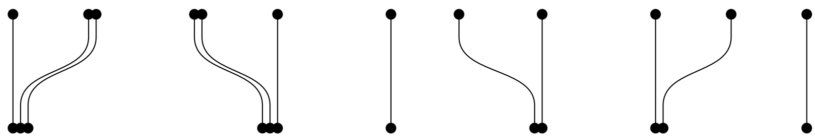
General notation

Let $\bar{i} = (i_1, \dots, i_k)$ denote a partition of n. The general principle is that the source partition and the target partition are denoted by the subscript and the superscript, respectively. Specifically, if $\bar{i}$ and $\bar{j}$ allow a fork between them, let $f_{\bar{i}}^{\bar{j}}$ denote this unique fork. Similarly, let $g_{\bar{i}}^{\bar{j}}$ denote the unique merge and when $\bar{i} \neq \bar{j}$, let $t_{\bar{i}}^{\bar{j}}$ and $d_{\bar{i}}^{\bar{j}}$ denote the unique positive and the negative crossings. When $\bar{i} = \bar{j}$, there is an ambiguity, so we overscore the subscript indices which correspond to the strands being crossed, e.g. $t_{\bar{1}\bar{1}1}^{111}$ or $t_{1\bar{1}\bar{1}}^{111}$.

The generators of $\mathcal{GB}r_3$

The generalised braid category $\mathcal{GB}r_3$ has the following generators:

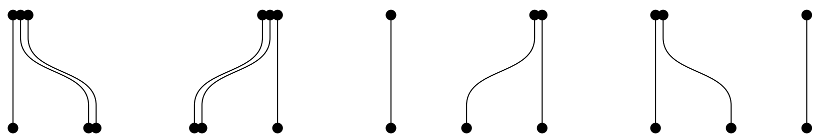
1. Four forks:



f_3^{12} f_3^{21} f_{12}^{111} f_{21}^{111}

Figure 4.1: Forks

2. Four merges:

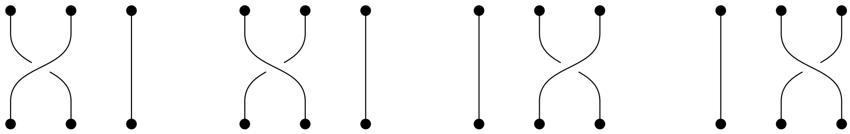


g_{12}^3 g_{21}^3 g_{111}^{12} g_{111}^{21}

Figure 4.2: Merges

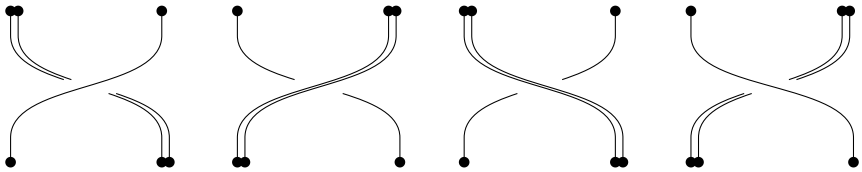
3. Two positive and two negative $(1,1)$-crossings:

4. Two positive and two negative $(2,1)$- and $(1,2)$-crossings:



$t^{111}_{\bar{1}\bar{1}1}$ $d^{111}_{1\bar{1}\bar{1}}$ $t^{111}_{1\bar{1}\bar{1}}$ $d^{111}_{\bar{1}\bar{1}1}$

Figure 4.3: $(1,1)$-crossings



t^{21}_{12} t^{12}_{21} d^{21}_{12} d^{12}_{21}

Figure 4.4: $(1,2)$- and $(2,1)$-crossings

Insipired by the results in [8], we formulate the following conjecture.

Conjecture 4.2.1. *All the relations between these generators in $\mathcal{GB}\mathrm{r}_3$ can be obtained from the following four basic relations via three operations:* vertical reflection *(swap the source and the target partitions, change all forks into merges and vice versa, reverse the parity of the crossings);* horizontal reflection *(swap the partitions* 12 *and* 21*, reverse the parity of the crossings); and* blackboard reflection *(reverse the parity of the crossings).*

1. *Multifork relation:* $f^{111}_{21} f^{21}_{3} = f^{111}_{12} f^{12}_{3}$.

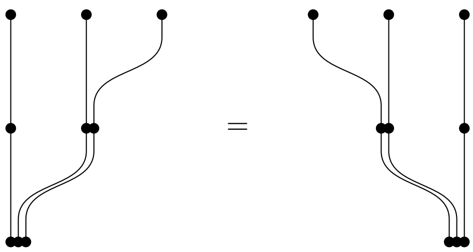


Figure 4.5: The Multifork relation

2. *The braid relation:* $t^{111}_{\bar{1}\bar{1}1} t^{111}_{1\bar{1}\bar{1}} t^{111}_{\bar{1}\bar{1}1} = t^{111}_{1\bar{1}\bar{1}} t^{111}_{\bar{1}\bar{1}1} t^{111}_{1\bar{1}\bar{1}}$.

3. *Inverses relations:* $t^{12}_{21} d^{21}_{12} = id_{12}, d^{21}_{12} t^{12}_{21} = id_{21}$, $t^{111}_{\bar{1}\bar{1}1} d^{111}_{\bar{1}\bar{1}1} = t^{111}_{1\bar{1}\bar{1}} d^{111}_{1\bar{1}\bar{1}} = id_{111}$.

4. *The pitchfork relation:* $f^{111}_{12} t^{12}_{21} = t^{111}_{1\bar{1}\bar{1}} t^{111}_{\bar{1}\bar{1}1} f^{111}_{21}$.

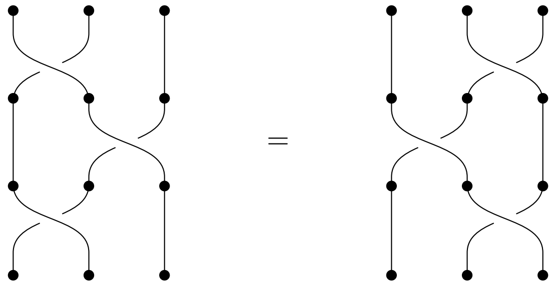


Figure 4.6: The braid relation

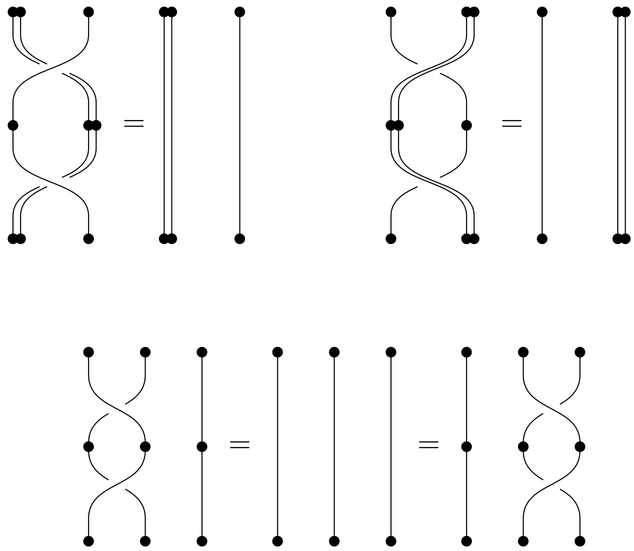


Figure 4.7: Inverses relations

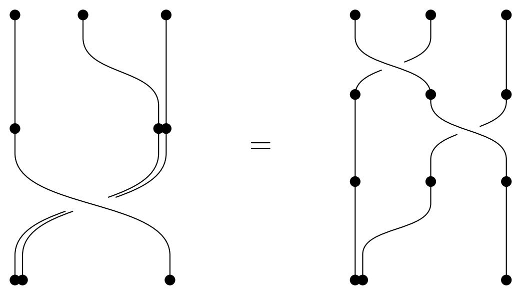


Figure 4.8: The pitchfork relation

Remark 4.2.2. *The category $\mathcal{GB}\mathrm{r}_2$ has a simpler structure; indeed, the generators of $\mathcal{GB}\mathrm{r}_2$ are f_2^{11}, g_{11}^2, $t_{\overline{11}}^{11}$, $d_{\overline{11}}^{11}$ and the only relation between them is*

$$t_{\overline{11}}^{11} d_{\overline{11}}^{11} = \mathrm{Id} = d_{\overline{11}}^{11} t_{\overline{11}}^{11}. \tag{4.2}$$

f_2^{11} g_{11}^2 $t_{\bar{1}\bar{1}}^{11}$ $d_{\bar{1}\bar{1}}^{11}$

Figure 4.9: Generators of $\mathcal{GB}r_2$

Skein-triangulated representations for $n = 3$

In this section, we describe a special type of categorical representations of $\mathcal{GB}r_3$ which we call *skein-triangulated*. In these representations, $\mathcal{GB}r_3$ acts on enhanced triangulated categories and certain additional relations are satisfied which make use of the triangulated structure of the target categories.

Let $\mathcal{C}_3$, $\mathcal{C}_{12}$, $\mathcal{C}_{21}$, and $\mathcal{C}_{111}$ be enhanced triangulated categories. Let

$$F_{21}^{111}\colon D(\mathcal{C}_{21}) \to D(\mathcal{C}_{111}), \tag{4.3}$$

$$F_{12}^{111}\colon D(\mathcal{C}_{12}) \to D(\mathcal{C}_{111}), \tag{4.4}$$

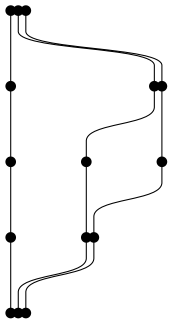
$$F_{3}^{21}\colon D(\mathcal{C}_{3}) \to D(\mathcal{C}_{21}), \tag{4.5}$$

$$F_{3}^{12}\colon D(\mathcal{C}_{3}) \to D(\mathcal{C}_{12}), \tag{4.6}$$

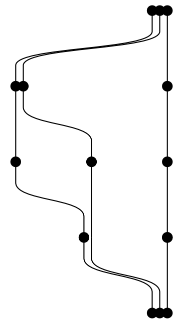
be enhanced functors with enhanced 2-categorical left and right adjoints. Denote the left and right adjoints of each $F_{\bar{i}}^{\bar{j}}$ by $L_{\bar{j}}^{\bar{i}}$ and $R_{\bar{j}}^{\bar{i}}$.

Assume that Conjecture 4.2.1 holds and assume moreover that

1. $(1,1)$-forks F_{12}^{111} and F_{21}^{111} are split spherical functors with cotwist $[-2]$.
2. $(1,2)$-fork F_3^{12} and $(2,1)$-fork F_3^{21} are split $\mathbb{P}^2$-functors with $H = [-2]$.
3. There exists a multifork isomorphism
$$\alpha\colon F_{21}^{111}F_3^{21} \xrightarrow{\sim} F_{12}^{111}F_3^{12}. \tag{4.7}$$
4. There exist isomorphisms
$$R_{12}^3R_{111}^{12}F_{12}^{111}F_3^{12} \simeq \mathrm{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6] \simeq R_{21}^3R_{111}^{21}F_{21}^{111}F_3^{21}$$
which together with $\mathbb{P}^2$-functor structures on F_3^{12} and F_3^{21} identify the maps
$$R_{12}^3F_3^{12} \xrightarrow{R_{12}^3\,\mathrm{act}\,F_3^{12}} R_{12}^3R_{111}^{12}F_{12}^{111}F_3^{12}, \tag{4.8}$$
$$R_{21}^3F_3^{21} \xrightarrow{R_{21}^3\,\mathrm{act}\,F_3^{21}} R_{21}^3R_{111}^{21}F_{21}^{111}F_3^{21}, \tag{4.9}$$
with the maps
$$\mathrm{Id}_3 \oplus [-2] \oplus [-4] \to \mathrm{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6]$$
which are the direct summand embeddings whose images are $\mathrm{Id}_3 \oplus$ the first $[-2] \oplus$ the first$[-4]$, and $\mathrm{Id}_3 \oplus$ the second $[-2] \oplus$ the second $[-4]$, respectively.

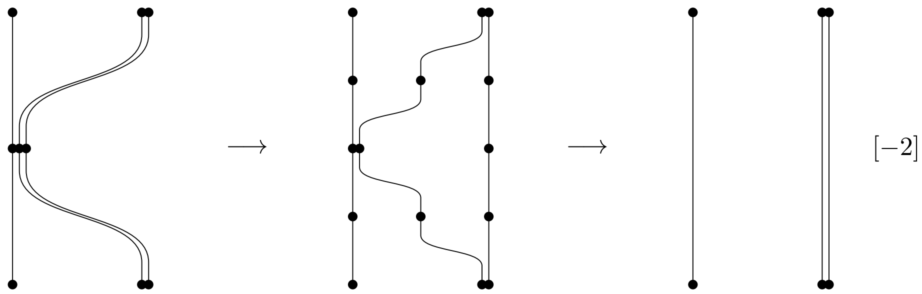


$$\simeq \mathrm{Id}_3 \oplus [-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6] \simeq$$



5. The following diagram can be completed to an exact triangle in $D(\mathcal{C}_{12}\text{-}\mathcal{C}_{12})$:

$$F_3^{12} R_{12}^{3} \to R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \to \mathrm{Id}_{12}[-2]. \tag{4.10}$$



Here the first map is the composition

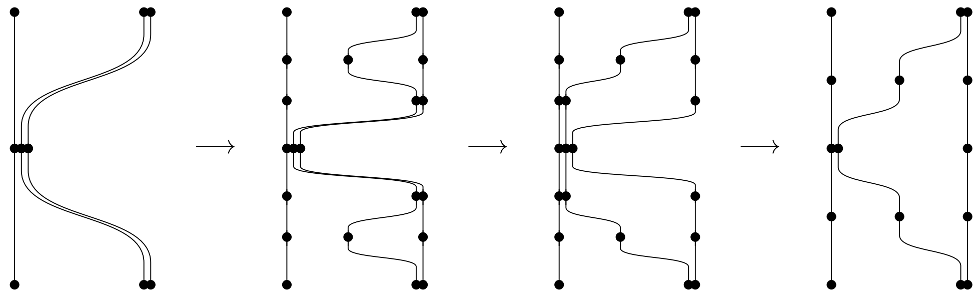
$$\begin{array}{c}
F_3^{12} R_{12}^{3} \\
\Big\downarrow \scriptstyle \mathrm{act}\, F_3^{12} R_{12}^{3}\, \mathrm{act} \\
R_{111}^{12} F_{12}^{111} F_3^{12} R_{12}^{3} R_{111}^{12} F_{12}^{111} \\
\sim \Big\downarrow \scriptstyle \mathrm{multifork} \\
R_{111}^{12} F_{21}^{111} F_3^{21} R_{21}^{3} R_{111}^{21} F_{12}^{111} \\
\Big\downarrow \scriptstyle R_{111}^{12} F_{21}^{111}\, \mathrm{tr}\, R_{111}^{21} F_{12}^{111} \\
R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111},
\end{array} \tag{4.11}$$

and the second map is the composition

$$R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} \simeq L_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111}[-2] \xrightarrow{\mathrm{tr}[-2]} \mathrm{Id}_{12}[-2]. \tag{4.12}$$

6. The following diagram can be completed to an exact triangle in $D(\mathcal{C}_{21}\text{-}\mathcal{C}_{21})$:

$$F_3^{21} R_{21}^{3} \to R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \to \mathrm{Id}_{21}[-2]. \tag{4.13}$$



Its two maps are defined analogously to (4.11) and (4.12):

$$
\begin{array}{c}
F_3^{21} R_{21}^3 \\
\Big\downarrow {\scriptstyle \mathrm{act}\, F_3^{21} R_{21}^3\, \mathrm{act}} \\
R_{111}^{21} F_{21}^{111} F_3^{21} R_{21}^3 R_{111}^{21} F_{21}^{111} \\
\sim \Big\downarrow {\scriptstyle \mathrm{multifork}} \\
R_{111}^{21} F_{12}^{111} F_3^{12} R_{12}^3 R_{111}^{12} F_{21}^{111} \\
\Big\downarrow {\scriptstyle R_{111}^{21} F_{12}^{111}\, \mathrm{tr}\, R_{111}^{12} F_{21}^{111}} \\
R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111},
\end{array}
\tag{4.14}
$$

and

$$R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} \simeq L_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111}[-2] \xrightarrow{\mathrm{tr}[-2]} \mathrm{Id}_{21}[-2]. \tag{4.15}$$

Theorem 4.2.3. *Under the assumptions above, the following assignments define a categorical action of $\mathcal{GB}\mathrm{r}_3$:*

1. *Each partition $\bar{i}$ of 3 is represented by the enhanced triangulated category $\mathcal{C}_{\bar{i}}$.*

2. *Each fork $f_{\bar{i}}^{\bar{j}}$ is represented by the functor $F_{\bar{i}}^{\bar{j}}$.*

3. *$(1,1)$-merges $g_{\bar{i}}^{\bar{j}}$ are represented by the functors $G_{\bar{i}}^{\bar{j}}$ defined by*

$$L_{\bar{i}}^{\bar{j}}[-1] \simeq G_{\bar{i}}^{\bar{j}} \simeq R_{\bar{i}}^{\bar{j}}[1]. \tag{4.16}$$

4. *$(1,2)$- and $(2,1)$-merges $g_{\bar{i}}^{\bar{j}}$ are represented by the functors $G_{\bar{i}}^{\bar{j}}$ defined by*

$$L_{\bar{i}}^{\bar{j}}[-2] \simeq G_{\bar{i}}^{\bar{j}} \simeq R_{\bar{i}}^{\bar{j}}[2]. \tag{4.17}$$

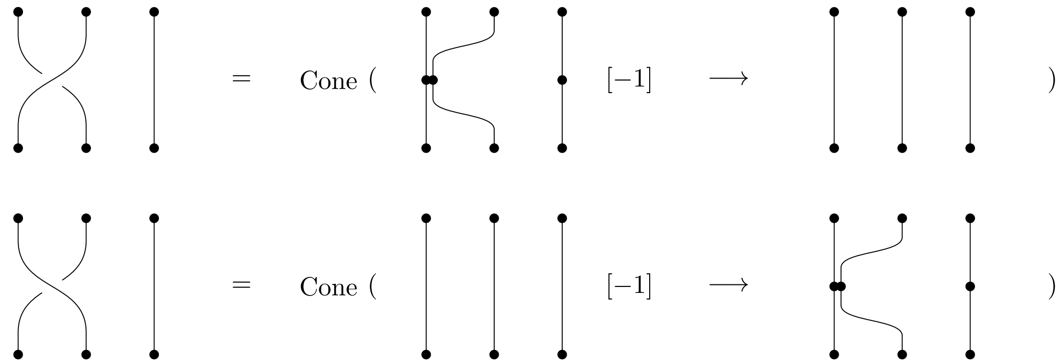
5. $(1,1)$*-crossings are represented by the spherical twists of* $(1,1)$*-forks:*

$$T^{111}_{\bar{1}\bar{1}1} = \mathrm{Cone}\left(F^{111}_{21}G^{21}_{111}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111}\right), \tag{4.18}$$

$$T^{111}_{1\bar{1}\bar{1}} = \mathrm{Cone}\left(F^{111}_{12}G^{12}_{111}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111}\right), \tag{4.19}$$

$$D^{111}_{\bar{1}\bar{1}1} = \mathrm{Cone}\left(\mathrm{Id}_{111}[-1] \xrightarrow{\mathrm{act}} F^{111}_{21}G^{21}_{111}\right), \tag{4.20}$$

$$D^{111}_{1\bar{1}\bar{1}} = \mathrm{Cone}\left(\mathrm{Id}_{111}[-1] \xrightarrow{\mathrm{act}} F^{111}_{12}G^{12}_{111}\right), \tag{4.21}$$



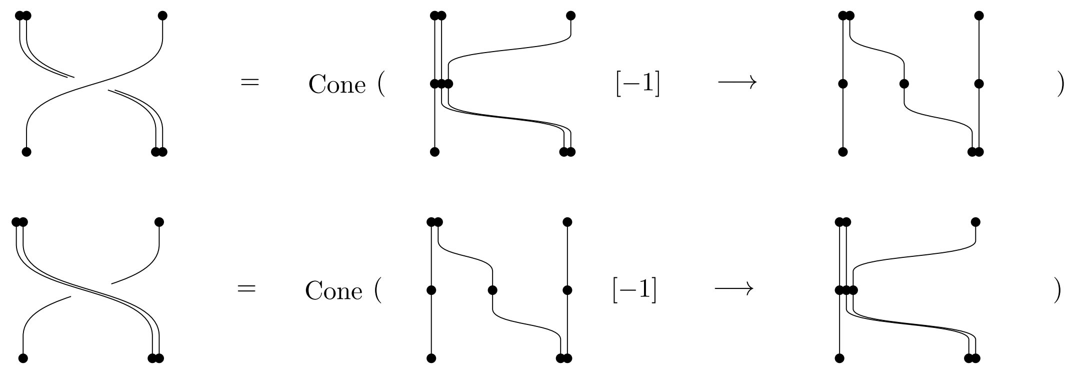
6. $(1,2)$*- and* $(2,1)$*-crossings are represented by the cones:*

$$T^{21}_{12} = \mathrm{Cone}\left(F^{21}_{3}G^{3}_{12}[-1] \xrightarrow{\lambda} G^{21}_{111}F^{111}_{12}\right), \tag{4.22}$$

$$T^{12}_{21} = \mathrm{Cone}\left(F^{12}_{3}G^{3}_{21}[-1] \xrightarrow{\mu} G^{12}_{111}F^{111}_{21}\right), \tag{4.23}$$

$$D^{21}_{12} = \mathrm{Cone}\left(G^{21}_{111}F^{111}_{12}[-1] \xrightarrow{\lambda'} F^{21}_{3}G^{3}_{12}\right), \tag{4.24}$$

$$D^{12}_{21} = \mathrm{Cone}\left(G^{12}_{111}F^{111}_{21}[-1] \xrightarrow{\mu'} F^{12}_{3}G^{3}_{21}\right), \tag{4.25}$$



where λ is defined by either of the two equal compositions

$$\begin{tikzcd}
F_3^{21} R_{12}^3 \arrow[d, "F_3^{21} R_{12}^3 \,\mathrm{act}"] & F_3^{21} R_{12}^3 \arrow[d, "\mathrm{act}\, F_3^{21} R_{12}^3"] \\
F_3^{21} R_{12}^3 R_{111}^{12} F_{12}^{111} \arrow[d, "\sim"', "\mathit{multifork}"] & R_{111}^{21} F_{21}^{111} F_3^{21} R_{12}^3 \arrow[d, "\sim"', "\mathit{multifork}"] \\
F_3^{21} R_{21}^3 R_{111}^{21} F_{12}^{111} \arrow[d, "\mathrm{tr}\, R_{111}^{21} F_{12}^{111}"] & R_{111}^{21} F_{12}^{111} F_3^{12} R_{12}^3 \arrow[d, "R_{111}^{21} F_{12}^{111}\, \mathrm{tr}"] \\
R_{111}^{21} F_{12}^{111} & R_{111}^{21} F_{12}^{111}
\end{tikzcd} , \tag{4.26}$$

λ' *is defined by either of the two equal compositions*

$$\begin{tikzcd}
L_{111}^{21} F_{12}^{111} \arrow[d, "L_{111}^{21} F_{12}^{111} \,\mathrm{act}"] & L_{111}^{21} F_{12}^{111} \arrow[d, "\mathrm{act}\, L_{111}^{21} F_{12}^{111}"] \\
L_{111}^{21} F_{12}^{111} F_3^{12} L_{12}^3 \arrow[d, "\sim"', "\mathit{multifork}"] & F_3^{21} L_{21}^3 L_{111}^{21} F_{12}^{111} \arrow[d, "\sim"', "\mathit{multifork}"] \\
L_{111}^{21} F_{21}^{111} F_3^{21} L_{12}^3 \arrow[d, "\mathrm{tr}\, F_3^{21} L_{12}^3"] & F_3^{21} L_{12}^3 L_{111}^{12} F_{12}^{111} \arrow[d, "F_3^{21} L_{12}^3\, \mathrm{tr}"] \\
F_3^{21} L_{12}^3 & F_3^{21} L_{12}^3
\end{tikzcd} , \tag{4.27}$$

and μ and μ' are defined similarly.

Moreover, this categorical action also satisfies the following condition:

- *Flop + Flop = Twist:*
  1. *$T_{21}^{12} T_{12}^{21}$ is isomorphic in $D(\mathcal{C}_{12}\text{-}\mathcal{C}_{12})$ to the $\mathbb{P}$-twist of F_3^{12},*
  2. *$T_{12}^{21} T_{21}^{12}$ is isomorphic in $D(\mathcal{C}_{21}\text{-}\mathcal{C}_{21})$ to the $\mathbb{P}$-twist of F_3^{21}.*

Motivated by the previous result, we give the following definition.

Definition 4.2.4. A *skein-triangulated representation* of $\mathcal{GB}r_3$ is a system of categories and functors satisfying the assignments properties of Theorem 4.2.3.

Proof of Theorem 4.2.3. It suffices to prove that the four basic relations between the generators of $\mathcal{GB}r_3$ listed in (4.2.1) hold under the assumptions of the theorem. The proofs for the relations obtained from these four by vertical, horizontal and blackboard reflections are identical.

$(T_{\bar{1}\bar{1}1}^{111}, D_{\bar{1}\bar{1}1}^{111})$ **and** $(T_{1\bar{1}\bar{1}}^{111}, D_{1\bar{1}\bar{1}}^{111})$ **are pairs of mutually inverse equivalences.**

This follows from F_{21}^{111} and F_{12}^{111} being spherical functors, see [6], Theorem 5.1.

The pitchfork relation

We need to show the existence of an isomorphism

$$F_{12}^{111}T_{21}^{12} \simeq T_{\bar{1}\bar{1}1}^{111}T_{1\bar{1}\bar{1}}^{111}F_{21}^{111}.$$

By definition we have

$$\begin{aligned} T_{\bar{1}\bar{1}1}^{111} &= \operatorname{Cone}\left(F_{21}^{111}G_{111}^{21}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111}\right), \\ T_{1\bar{1}\bar{1}}^{111} &= \operatorname{Cone}\left(F_{12}^{111}G_{111}^{12}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111}\right), \end{aligned}$$

therefore $T_{\bar{1}\bar{1}1}^{111}T_{1\bar{1}\bar{1}}^{111}F_{21}^{111}$ is isomorphic in $D(\mathcal{C}_{111} - \mathcal{C}_{111})$ to the convolution of the twisted complex

$$F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111} \xrightarrow{\begin{pmatrix} trF_{12}^{111}R_{111}^{12}F_{21}^{111} \\ -F_{21}^{111}R_{111}^{21}trF_{21}^{111} \end{pmatrix}} F_{12}^{111}R_{111}^{12}F_{21}^{111} \oplus F_{21}^{111}R_{111}^{21}F_{21}^{111} \xrightarrow{\begin{pmatrix} trF_{21} & trF_{21}^{111} \end{pmatrix}} \underset{\text{deg.0}}{F_{21}^{111}}$$

By Lemma 5.10 of [6], we have a homotopy equivalence of twisted complexes between $F_{21}^{111}\underset{\text{deg.0}}{R_{111}^{21}}F_{21}^{111} \xrightarrow{trF_{21}^{111}} F_{21}^{111}$ and $F_{21}^{111} \xrightarrow{F_{21}^{111}act} F_{21}^{111}\underset{\text{deg.0}}{R_{111}^{21}}F_{21}^{111}$.

It follows by the Replacement Lemma that $T_{\bar{1}\bar{1}1}^{111}T_{1\bar{1}\bar{1}}^{111}F_{21}^{111}$ is further isomorphic to the convolution of

$$F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111} \oplus F_{21}^{111} \xrightarrow{\begin{pmatrix} trF_{12}^{111}R_{111}^{12}F_{21}^{111} & 0 \\ -F_{21}^{111}R_{111}^{21}trF_{21}^{111} & F_{21}^{111}act \end{pmatrix}} F_{12}^{111}R_{111}^{12}F_{21}^{111} \underset{\text{deg.-1}}{\oplus} F_{21}^{111}R_{111}^{21}F_{21}^{111}.$$

Moreover, there exists an homotopy equivalence from $F_{21}^{111}F_3^{21}R_{21}^3$ to

$$F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111} \oplus F_{21}^{111} \xrightarrow{\begin{pmatrix} -F_{21}^{111}R_{111}^{21}trF_{21}^{111} & F_{21}^{111}act \end{pmatrix}} F_{21}^{111}\underset{\text{deg.0}}{R_{111}^{21}}F_{21}^{111}$$

whose $F_{21}^{111}F_3^{21}R_{21}^3 \to F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}$ component is the map $F_{21}^{111}(4.11)$.

It follows by the Replacement Lemma, that $T_{\bar{1}\bar{1}1}^{111}T_{1\bar{1}\bar{1}}^{111}F_{21}^{111}$ is therefore isomorphic to the convolution of

$$F_{21}^{111}F_3^{21}R_{21}^3 \xrightarrow{trF_{12}^{111}R_{111}F_{21}^{111} \circ F_{21}^{111}(4.11)} F_{12}^{111}\underset{\text{deg.-1}}{R_{111}^{12}}F_{21}^{111}$$

and hence by the multifork isomorphism to the convolution of

$$F_{12}^{111}F_3^{12}R_{21}^3 \xrightarrow{trF_{12}^{111}R_{111}F_{21}^{111} \circ F_{21}^{111}(4.11) \circ multifork} F_{12}^{111}\underset{\text{deg.-1}}{R_{111}^{12}}F_{21}^{111}.$$

On the other hand, by the definition of T_{21}^{12} the object $F_{12}^{111}T_{21}^{12}$ is isomorphic in $D(\mathcal{C}_{111} - \mathcal{C}_{111})$ to the convolution if the twisted complex

$$F_{12}^{111} F_3^{12} R_{21}^{3} \xrightarrow{F_{12}^{111}\mu} F_{12}^{111} \underset{\text{deg.-1}}{R_{111}^{12}} F_{21}^{111}.$$

And since

$$tr F_{12}^{111} R_{111} F_{21}^{111} \circ F_{21}^{111}(4.11) \circ multifork = F_{12}^{111}\mu$$

we have

$$F_{12}^{111} T_{21}^{12} \simeq T_{\bar{1}\bar{1}1}^{111} T_{1\bar{1}\bar{1}}^{111} F_{21}^{111}.$$

The braid relation

Consider the following twisted complexes of enhanced functors:

$$F_{21}^{111} R_{111}^{21} F_{12}^{111} \underset{\text{deg.0}}{R_{111}^{12} F_{21}^{111}} R_{111}^{21} \oplus F_{21}^{111} R_{111}^{21} \xrightarrow{F_{21}^{111} R_{111}^{21}\, \mathrm{tr}\, F_{21}^{111} R_{111}^{21} \oplus F_{21}^{111}\, \mathrm{act}\, R_{111}^{21}} F_{21}^{111} R_{111}^{21} F_{21}^{111} R_{111}^{21}, \tag{4.28}$$

$$F_{12}^{111} R_{111}^{12} F_{21}^{111} \underset{\text{deg.0}}{R_{111}^{21} F_{12}^{111}} R_{111}^{12} \oplus F_{12}^{111} R_{111}^{12} \xrightarrow{F_{12}^{111} R_{111}^{12}\, \mathrm{tr}\, F_{12}^{111} R_{111}^{12} \oplus F_{12}^{111}\, \mathrm{act}\, R_{111}^{12}} F_{12}^{111} R_{111}^{12} F_{12}^{111} R_{111}^{12}. \tag{4.29}$$

There are natural maps from both to $F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} \oplus F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21}$ induced by the maps

$$\begin{array}{c} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21} \\ \Big\downarrow {\scriptstyle \mathrm{tr}\, F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21} \oplus F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12}\, \mathrm{tr}} \\ F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} \oplus F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21} \end{array} \tag{4.30}$$

and

$$\begin{array}{c} F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} \\ \Big\downarrow {\scriptstyle F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21}\, \mathrm{tr} \oplus \mathrm{tr}\, F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12}} \\ F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} \oplus F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21}. \end{array} \tag{4.31}$$

By [6], Theorem 6.2, if there exists a $D(\mathcal{C}_{111}\text{-}\mathcal{C}_{111})$ isomorphism between the convolutions of the twisted complexes (4.28) and (4.29) which intertwines the maps induced by (4.30) and (4.31), then the braid relation holds for $T_{\bar{1}\bar{1}1}^{111}$ and $T_{1\bar{1}\bar{1}}^{111}$.

Claim: There exist homotopy equivalences from the objects $F_{21}^{111} F_3^{21} R_{21}^{3} R_{111}^{21}$ and $F_{12}^{111} F_3^{12} R_{12}^{3} R_{111}^{12}$ to the twisted complexes (4.28) and (4.29) whose

$$F_{21}^{111} F_3^{21} R_{21}^{3} R_{111}^{21} \to F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21},$$

$$F_{12}^{111} F_3^{12} R_{12}^{3} R_{111}^{12} \to F_{12}^{111} R_{111}^{12} F_{21}^{111} R_{111}^{21} F_{12}^{111} R_{111}^{12},$$

components are the maps $F_{21}^{111}(4.14)R_{111}^{21}$ and $F_{12}^{111}(4.11)R_{111}^{12}$.

Indeed, since (4.5) fits into an exact triangle in $D(\mathcal{C}_{21}\text{-}\mathcal{C}_{21})$, there exist a homotopy equivalence of twisted complexes of the form

$$\begin{tikzcd} \underset{\text{deg.0}}{F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21}} \arrow[d, "F_{21}^{111}(4.14)R_{111}^{21}"] \arrow[drr, "?"] & & \\ \underset{\text{deg.0}}{F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}} \arrow[rr, "F_{21}^{111}(4.15)R_{111}^{21}"] & & F_{21}^{111}R_{111}^{21}[-2]. \end{tikzcd} \tag{4.32}$$

Since F_{21}^{111} is split-spherical with cotwist [-3], there is a split exact triangle

$$F_{21}^{111}R_{111}^{21} \xrightarrow{F_{21}^{111}\text{ act }R_{111}^{21}} F_{21}^{111}R_{111}^{21}F_{21}^{111}R_{111}^{21} \xrightarrow{\pi} F_{21}^{111}R_{111}^{21}[-2].$$

Therefore, the object $F_{21}^{111}R_{111}^{21}[-2]$ is homotopy equivalent to the twisted complex $F_{21}^{111}R_{111}^{21} \xrightarrow{F_{21}^{111}\text{ act }R_{111}^{21}} \underset{\text{deg.0}}{F_{21}^{111}R_{111}^{21}F_{21}^{111}R_{111}^{21}}$. It follows by the Replacement Lemma ([7], Lemma 2.1) that there exists a homotopy equivalence of form

$$\begin{tikzcd} \underset{\text{deg.0}}{F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}} \arrow[rr, "F_{21}^{111}(4.15)R_{111}^{21}"] \arrow[d, "\mathrm{Id}\oplus 0"] & & F_{21}^{111}R_{111}^{21}[-2] \arrow[d, "?"] \arrow[dll, "0\oplus ?"'] \\ \underset{\text{deg.0}}{F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}} \oplus F_{21}^{111}R_{111}^{21} \arrow[rr, "F_{21}^{111}R_{111}^{21}\text{ tr }F_{21}^{111}R_{111}^{21}\oplus F_{21}^{111}\text{ act }R_{111}^{21}"'] & & F_{21}^{111}R_{111}^{21}F_{21}^{111}R_{111}^{21} \end{tikzcd} \tag{4.33}$$

The composition of (4.32) and (4.33) is the desired homotopy equivalence from $F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21}$ to the twisted complex (4.28).

Therefore, we have proven the claim for the twisted complex (4.28); the proof for (4.29) are similar.

Now, recall that we have the multifork isomorphism $\alpha\colon F_{21}^{111}F_3^{21} \xrightarrow{\sim} F_{12}^{111}F_3^{12}$. Let β denote its inverse $F_{12}^{111}F_3^{12} \xrightarrow{\sim} F_{21}^{111}F_3^{21}$, and let $\beta^R\colon R_{21}^3R_{111}^{21} \xrightarrow{\sim} R_{12}^3R_{111}^{12}$.

We now claim that the $D(\mathcal{C}_{111}\text{-}\mathcal{C}_{111})$ isomorphism between the convolutions of (4.28) and (4.29) which is induced via the isomorphisms provided by the Claim from the isomorphisms

$$F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21} \xrightarrow{\alpha\beta^R} F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12}$$

is the requisite isomorphism intertwining the maps (4.30) and (4.31).

The target of the maps is the direct sum

$$F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12} \oplus F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}.$$

We prove that the isomorphism intertwines the components of (4.30) and (4.31) which go into the second direct summand. The proof that it intertwines the first direct summand components is similar.

It suffices to show that the following diagram commutes in $D(\mathcal{C}_{111}\text{-}\mathcal{C}_{111})$:

$$\begin{tikzcd}
F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21} \arrow[rr, "\alpha\beta^R"] \arrow[d, "F_{21}^{111}(4.14)R_{111}^{21}"'] & & F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12} \arrow[d, "F_{12}^{111}(4.11)R_{111}^{12}"] \\
F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21} \arrow[d, "\mathrm{tr}\,F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}"'] & & F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12} \arrow[d, "F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}\,\mathrm{tr}"] \\
F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21} \arrow[rr, equal] & & F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}.
\end{tikzcd}$$

This can be simplified to:

$$\begin{tikzcd}
F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21} \arrow[rr, "\alpha\beta^R"] \arrow[d, "F_{21}^{111}F_3^{21}R_{21}^3\,\mathrm{act}\,R_{111}^{21}"'] & & F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12} \arrow[d, "F_{12}^{111}\,\mathrm{act}\,F_3^{12}R_{12}^3R_{111}^{12}"] \\
F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21}F_{21}^{111}R_{111}^{21} \arrow[d, "\alpha\beta^R F_{21}^{111}R_{111}^{21}"'] & & F_{12}^{111}R_{111}^{12}F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12} \arrow[d, "F_{12}^{111}R_{111}^{12}\beta\alpha^R"] \\
F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12}F_{21}^{111}R_{111}^{21} \arrow[d, "F_{12}^{111}\,\mathrm{tr}\,R_{111}^{12}F_{21}^{111}R_{111}^{21}"'] & & F_{21}^{111}R_{111}^{21}F_{21}^{111}F_3^{21}R_{12}^3R_{111}^{12} \arrow[d, "F_{21}^{111}R_{111}^{21}F_{21}^{111}\,\mathrm{tr}\,R_{111}^{12}"] \\
F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21} \arrow[rr, equal] & & F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}.
\end{tikzcd}$$

The commutativity of this diagram reduces to the commutativity of the diagram

$$\begin{tikzcd}
 & F_{12}^{111}F_3^{12}R_{21}^3R_{111}^{21} \arrow[dl, "F_{12}^{111}F_3^{12}R_{21}^3\,\mathrm{act}\,R_{111}^{21}"'] \arrow[dr, "F_{12}^{111}\,\mathrm{act}\,F_3^{12}R_{21}^3R_{111}^{21}"] & \\
F_{12}^{111}F_3^{12}R_{21}^3R_{111}^{21}F_{21}^{111}R_{111}^{21} \arrow[d, "F_{12}^{111}F_3^{12}\beta^R F_{21}^{111}R_{111}^{21}"'] & & F_{12}^{111}R_{111}^{12}F_{12}^{111}F_3^{12}R_{21}^3R_{111}^{21} \arrow[d, "F_{21}^{111}R_{111}^{21}\beta R_{21}^3R_{111}^{21}"] \\
F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12}F_{21}^{111}R_{111}^{21} \arrow[dr, "F_{12}^{111}\,\mathrm{tr}\,R_{111}^{12}F_{21}^{111}R_{111}^{21}"'] & & F_{12}^{111}R_{111}^{12}F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21} \arrow[dl, "F_{12}^{111}R_{111}^{12}F_{21}^{111}\,\mathrm{tr}\,R_{111}^{21}"] \\
 & F_{12}^{111}R_{111}^{12}F_{21}^{111}R_{111}^{21}. &
\end{tikzcd}$$

and then further to the commutativity of

$$\begin{tikzcd}
 & F_{12}^{111}F_3^{12}R_{21}^3R_{111}^{21} \arrow[dl, "F_{12}^{111}F_3^{12}\beta^R"'] \arrow[dr, "\beta R_{21}^3R_{111}^{21}"] & \\
F_{12}^{111}F_3^{12}R_{12}^3R_{111}^{12} \arrow[dr, "\mathrm{tr}"'] & & F_{21}^{111}F_3^{21}R_{21}^3R_{111}^{21} \arrow[dl, "\mathrm{tr}"] \\
 & \mathrm{Id}_{111}, &
\end{tikzcd}$$

which commutes since β^R is the right dual of β.

Flop+flop=twist

We only prove the first of the "flop-flop = twist" relations, the second is proved identically. We need to show that

$$\{F_3^{12}R_{21}^3 \xrightarrow{\mu} \underset{\text{deg.0}}{R_{111}^{12}F_{21}^{111}}\}\{F_3^{21}R_{12}^3 \xrightarrow{\lambda} \underset{\text{deg.0}}{R_{111}^{21}F_{12}^{111}}\}[2] \tag{4.34}$$

is isomorphic to

$$F_3^{12}R_{12}^3[-2] \xrightarrow{\psi} F_3^{12}R_{12}^3 \xrightarrow{\text{tr}} \mathrm{Id}_{\mathcal{C}_{12}}\,.$$

The tensor of product of convolutions in (4.34) is isomorphic to the convolution of the twisted complex

$$\left(\begin{array}{c} F_3^{12}R_{21}^3F_3^{21}R_{12}^3 \\ \downarrow\begin{pmatrix} \mu F_3^{21}R_{12}^3 \\ -F_3^{12}R_{21}^3\lambda \end{pmatrix} \\ R_{111}^{12}F_{21}^{111}F_3^{21}R_{12}^3 \oplus F_3^{12}R_{21}^3R_{111}^{21}F_{12}^{111} \\ \downarrow\begin{pmatrix} R_{111}^{12}F_{21}^{111}\lambda & \mu R_{111}^{21}F_{12}^{111} \end{pmatrix} \\ \underset{\text{deg.0}}{R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}}. \end{array}\right)[2] \tag{4.35}$$

Since F_{21}^{111} and F_{12}^{111} are split spherical with cotwist $[-3]$, we have

$$R_{111}^{12}F_{21}^{111}F_3^{21}R_{12}^3 \simeq R_{111}^{12}F_{12}^{111}F_3^{12}R_{12}^3 \simeq F_3^{12}R_{12}^3 \oplus F_3^{12}R_{12}^3[-2],$$

$$F_3^{12}R_{21}^3R_{111}^{21}F_{12}^{111} \simeq F_3^{12}R_{12}^3R_{111}^{12}F_{12}^{111} \simeq F_3^{12}R_{12}^3 \oplus F_3^{12}R_{12}^3[-2],$$

Since F_3^{21} is split $\mathbb{P}^2$-functor with $H = [-2]$ we also have

$$F_3^{12}R_{21}^3F_3^{21}R_{12}^3 \simeq F_3^{12}R_{12}^3 \oplus F_3^{12}R_{12}^3[-2] \oplus F_3^{12}R_{12}^3[-4].$$

Thus three out of four objects in the twisted complex (4.35) are isomorphic to direct sums of shifted copies of F_2R_2. Under these identifications (4.35) becomes:

$$\begin{array}{c} F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3 \oplus F_3^{12}R_{12}^3[-2] \\ \Bigg\downarrow\begin{pmatrix} \mathrm{Id} & 0 & 0 \\ 0 & \mathrm{Id} & \psi_1 \\ -\mathrm{Id} & 0 & 0 \\ 0 & -\mathrm{Id} & -\psi_2 \end{pmatrix} \\ \left(F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3\right) \oplus \left(F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3\right) \\ \Big\downarrow\begin{pmatrix} (4.11) & \phi_1 & (4.11) & \phi_2 \end{pmatrix} \\ \underset{\text{deg.0}}{R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2]} \end{array}, \tag{4.36}$$

where ψ_1, ψ_2 are the compositions

$$F_3^{12}R_{12}^3[-2] \hookrightarrow F_3^{12}R_{12}^3F_3^{12}R_{12}^3 \xrightarrow{\text{tr}\,F_3^{12}R_{12}^3,\quad F_3^{12}R_{12}^3\,\text{tr}} F_3^{12}R_{12}^3,$$

and ϕ_1 and ϕ_2 are the compositions

$$\begin{array}{ccc}
\begin{array}{c}
F_3^{12}R_{12}^3 \\
\big\downarrow \\
R_{111}^{12}F_{12}^{111}F_3^{12}R_{12}^3[2] \\
\big\downarrow {\scriptstyle R_{111}^{12}(\mathrm{multifork})R_{12}^3} \\
R_{111}^{12}F_{21}^{111}F_3^{21}R_{12}^3[2] \\
\big\downarrow {\scriptstyle R_{111}^{12}F_{21}^{111}\ \mathrm{act}\ F_3^{21}R_{12}^3} \\
R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{21}^{111}F_3^{21}R_{12}^3[2] \\
\big\downarrow {\scriptstyle R_{111}^{12}F_{21}^{111}R_{111}^{21}(\mathrm{multifork})R_{12}^3} \\
R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}F_3^{12}R_{12}^3[2] \\
\big\downarrow {\scriptstyle R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}\ \mathrm{tr}} \\
R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2]
\end{array}
& \text{and} &
\begin{array}{c}
F_3^{12}R_{12}^3 \\
\big\downarrow \\
F_3^{12}R_{12}^3R_{111}^{12}F_{12}^{111}[2] \\
\big\downarrow {\scriptstyle R_{111}^{12}(\mathrm{multifork})R_{12}^3} \\
F_3^{12}R_{21}^3R_{111}^{21}F_{12}^{111}[2] \\
\big\downarrow {\scriptstyle F_3^{12}R_{21}^3\ \mathrm{act}\ R_{111}^{21}F_{12}^{111}} \\
F_3^{12}R_{21}^3R_{111}^{21}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2] \\
\big\downarrow {\scriptstyle F_3^{12}(\mathrm{multifork})F_{21}^{111}R_{111}^{21}F_{12}^{111}} \\
F_3^{12}R_{12}^3R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2] \\
\big\downarrow {\scriptstyle \mathrm{tr}\ R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}} \\
R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2]
\end{array}
\end{array}$$

Note that since the map (4.12) is the composition

$$R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111} \xrightarrow{R_{111}^{12}\,\mathrm{tr}\,F_{12}^{111}} R_{111}^{12}F_{12}^{111} \twoheadrightarrow \mathrm{Id}_{111}[-2],$$

we have the following equality of maps $F_3^{12}R_{12}^3 \to \mathrm{Id}_3$:

$$\phi_1 \circ (4.12) = \mathrm{tr} = \phi_2 \circ (4.12).$$

The map ψ in the definition of the $\mathbb{P}$-twist of F_3^{12} is $\psi_1 - \psi_2$. Changing the basis of the middle term of (4.36) to the diagonal and the antidiagonal we obtain:

$$\begin{array}{c}
F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3 \oplus F_3^{12}R_{12}^3[-2] \\
\Bigg\downarrow {\scriptstyle \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \psi \\ 2\,\mathrm{Id} & 0 & 0 \\ 0 & 2\,\mathrm{Id} & \psi_1+\psi_2 \end{pmatrix}} \\
\big(F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3\big) \oplus \big(F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3\big) \\
\big\downarrow {\scriptstyle \begin{pmatrix} 2(4.11) & \phi_1+\phi_2 & 0 & \phi_1-\phi_2 \end{pmatrix}} \\
\underset{\mathrm{deg.0}}{R_{111}^{12}F_{21}^{111}}R_{111}^{21}F_{12}^{111}[2].
\end{array} \tag{4.37}$$

We can remove the following acyclic subcomplex of (4.37)

$$F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3 \xrightarrow{\begin{pmatrix} 2\,\mathrm{Id} & 0 \\ 0 & 2\,\mathrm{Id} \end{pmatrix}} \underset{\mathrm{deg.-1}}{F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3}$$

using the Replacement Lemma. Since the subcomplex has no external arrows emerging from its degree -2 part, no other differentials are affected. We obtain:

$$F_3^{12}R_{12}^3[-2] \xrightarrow{\begin{pmatrix}0\\ \psi\end{pmatrix}} F_3^{12}R_{12}^3[2] \oplus F_3^{12}R_{12}^3 \xrightarrow{\begin{pmatrix}2(4.11) & \phi_1+\phi_2\end{pmatrix}} \underset{\text{deg.0}}{R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2]}. \tag{4.38}$$

Since (4.5) fits into an exact triangle, we have a homotopy equivalence of form

$$\begin{tikzcd} F_3^{12}R_{12}^3[2] \arrow[r, "2(4.11)"] \arrow[dr, "*"'] & R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}[2] \arrow[d, "\frac{1}{2}(4.12)"] \\ & \underset{\text{deg.0}}{\mathrm{Id}_{12}} \end{tikzcd} \tag{4.39}$$

By the Replacement Lemma, we finally obtain:

$$F_3^{12}R_{12}^3[-2] \xrightarrow{\psi} F_3^{12}R_{12}^3[2] \xrightarrow{\frac{1}{2}(4.12)\circ(\phi_1+\phi_2)} \underset{\text{deg.0}}{\mathrm{Id}_{12}}. \tag{4.40}$$

Since $(4.12) \circ \phi_i = \mathrm{tr}$, this is homotopy equivalent to the $\mathbb{P}$-twist of F_3^{12}:

$$F_3^{12}R_{12}^3[-2] \xrightarrow{\psi} F_3^{12}R_{12}^3[2] \xrightarrow{\mathrm{tr}} \underset{\text{deg.0}}{\mathrm{Id}_{12}}. \tag{4.41}$$

$(T_{21}^{12}, D_{12}^{21})$ and $(T_{21}^{12}, D_{12}^{21})$ are pairs of mutually inverse equivalences.

This follows from the "twist-twist=flop" relations. Indeed, $T_{21}^{12}T_{12}^{21}$ and $T_{12}^{21}T_{21}^{12}$ are isomorphic to $\mathbb{P}$-twists of F_3^{12} and F_3^{21} and hence are both autoequivalences. Therefore T_{12}^{21} and T_{21}^{12} are also autoequivalences. On the other hand, the maps λ' and μ' which define the functors D_{12}^{21} and D_{21}^{12} are the left duals of the maps μ and λ which define the functors T_{21}^{12} and T_{12}^{21}. Hence D_{12}^{21} and D_{21}^{12} are the left adjoints of T_{21}^{12} and T_{12}^{21}. The claim now follows. □

4.3 A skein triangulated action of $\mathcal{GB}r_3$ on $T^*\,\mathrm{Fl}_3(\bar{\imath})$

The aim of the rest of this chapter is to define a network of categories and functors that satisfy the assumption of Theorem 4.2.3.

In particular, we construct such a network on $T^*\,\mathrm{Fl}_3(\bar{\imath})$ and we prove that all the hypothesis for a skein triangulated action of GBr_3 are verified.

In the following, let $D^b(X)$ be the derived category of coherent sheaves on a smooth quasi-projective variety X and assume all the functors derived, i.e. we omit $\mathbf{R}$ and $\mathbf{L}$.

We omit the pushforward i_* applied to structure sheaves when i is an embedding and the pullbacks applied to line bundles; we also write, when D_1 and D_2 are divisors respectively in X and Y, $\mathcal{O}_{X\times_Z Y}(D_1, D_2)$ for the line bundle $\mathcal{O}_X(D_1)\boxtimes\mathcal{O}_Y(D_2)$ on the fiber product $X\times_Z Y$.

We write the total space $T^*\,\mathrm{Fl}_n(\bar{\imath})$ of the contangent bundle of the flag $\mathrm{Fl}_n(\bar{\imath})$ with the little abuse of notation, as in remark 2.2.30

$$T^*\,\mathrm{Fl}_n(\bar{\imath}) = \left\{ \begin{tikzcd}[column sep=small] 0 & \subset & V_{i_1} \arrow[ll, bend right, "\alpha"'] & \subset & \cdots \arrow[ll, bend right, "\alpha"'] & \subset & V_{i_{k-1}} \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^n \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \right\},$$

and we also write

$$\left\{ \begin{tikzcd}[column sep=small] 0 & \subset & V_{\lambda_1} \arrow[ll, bend right, "\alpha"'] & \subset & \cdots \arrow[ll, bend right, "\alpha"'] & \subset & V_{\lambda_{k-1}} \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^n \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd}[column sep=small] 0 & \subset & V_{j_1} \arrow[ll, bend right, "\alpha"'] & \subset & \cdots \arrow[ll, bend right, "\alpha"'] & \subset & V_{j_{h-1}} \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^n \arrow[ll, bend right, "\alpha"'] \end{tikzcd} \right\}$$

for the subspace of $T^*\,\mathrm{Fl}_n(\bar{\imath})\times T^*\,\mathrm{Fl}_n(\bar{\jmath})$ where the maps αs need to satisfy $\alpha(V_{i_k})\subset V_{i_{k-1}}$ and $\alpha(V_{j_k})\subset V_{j_{k-1}}$.

We write $\mathcal{V}_i$ for the pullback on Fl_n of the tautological bundle of $\mathrm{Gr}(i,n)$.

Recall that if X and Y are quasi-projective subvarieties of Z such that their intersection $X\cap Y$ is a Cartier divisor in Y than we have the short exact sequence of coherent sheaves of Z

$$0\to\mathcal{O}_Y(-X\cap Y)\to\mathcal{O}_{X\cup Y}\to\mathcal{O}_X\to 0. \tag{4.42}$$

Recall moreover that if $f: X\to Y$ is proper, then we have the following adjunction of derived functors

$$f^*\dashv f_*\dashv f^! \tag{4.43}$$

where $f^! = f^*(-)\otimes\omega_{X/Y}[dimY - dimX]$.

If f is a divisorial embedding we write $\mathcal{O}_{X\times_X Y}\in D^b(X\times Y)$ and $\mathcal{O}_{Y\times_X X}\in D^b(Y\times X)(X,0)[-1]$ for the Fourier-Mukai kernels representing respectively f_* and $f^!$.

If f is a fibration we write $\mathcal{O}_{X\times_X Y} \in D^b(X \times Y)$ and $\mathcal{O}_{Y\times_X X} \in D^b(Y \times X)$ for the Fourier-Mukai kernels representing respectively f_* and f^*.

We write moreover $\mathcal{O}_{X\times_X X} \in D^b(X \times X)$ for the Fourier-Mukai kernel representing the identity on X.

As in section 3.8, let C be our ambient variety the total space of the contangent bundle of Fl_3

$$C = T^*Fl_3.$$

Let A and E be the quasi-projective varieties defined as the total space of the contangent bundle of $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$:

$$A = T^*\mathbb{P}^2 \qquad E = T^*\mathbb{P}^{2\vee}.$$

Let B and D be the quasi-projective varieties defined as the total space of the pullback via p_1 and p_2 on Fl_3 of the contangent bundle of $\mathbb{P}^2$ and $\mathbb{P}^{2\vee}$:

$$B = p_1^*T^*\mathbb{P}^2 \qquad D = p_2^*T^*\mathbb{P}^{2\vee}.$$

We have therefore the following diagram

$$\begin{tikzcd}
B \arrow[r, hook, "i_B"'] \arrow[d, two heads, "\pi_A"'] & C & D \arrow[l, hook', "i_D"] \arrow[d, two heads, "\pi_E"'] \\
A & & E \\
\mathbb{P}^2 \arrow[u, hook, "i_{\mathbb{P}^2}"'] \arrow[r, two heads, "\pi_{pt2}"'] & pt & \mathbb{P}^{2\vee} \arrow[u, hook, "i_{\mathbb{P}^{2\vee}}"'] \arrow[l, two heads, "\pi_{pt1}"]
\end{tikzcd} \tag{4.44}$$

where i_B, i_D,$i_{\mathbb{P}^{2\vee}}$ and $i_{\mathbb{P}^2}$ are divisorial inclusions and $B \xrightarrow{\pi_A} A$ and $D \xrightarrow{\pi_E} E$ are $\mathbb{P}^1$ bundles.

Definition 4.3.1. We define the categories $\mathcal{C}_{(3)} = D^b(pt)$, $\mathcal{C}_{(1,2)} = D^b(A)$, $\mathcal{C}_{(2,1)} = D^b(E)$, and $\mathcal{C}_{(1,1,1)} = D^b(C)$.

4.4 A skein triangulated action of $\mathcal{GB}r_3$: generators

In this section we define the generators of skein triangulated action of GBr_3, the forks and the merges.

The functors F_{12}^{111} F_{21}^{111} and their respective right adjoints R_{111}^{12} R_{111}^{21} are the spherical functors F_i and their right adjoints R_i of section 3.10.

After giving a description of them as Fourier-Mukai transforms, we will compute the first compositions of them.

In the following sections, we use $D^b(pt)$, $D^b(A)$, $D^b(E)$, $D^b(C)$ instead of the notation of Definition 4.3.1.

Definition 4.4.1. We define the first fork functor

$$F_{21}^{111} = i_{D*} \circ \pi_E^* : D^b(E) \to D^b(C) \tag{4.45}$$

and define the second fork functor

$$F_{12}^{111} = i_{B*} \circ \pi_A^* : D^b(A) \to D^b(C). \tag{4.46}$$

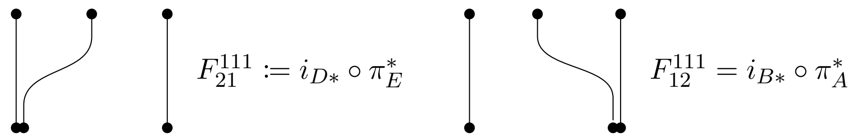


Figure 4.10: First and second forks

Proposition 4.4.2. *The Fourier-Mukai kernel of F_{21}^{111} and F_{12}^{111} are respectively the sheaves $\mathcal{O}_{E\times_E D} \in D^b(E \times C)$ and $\mathcal{O}_{A\times_A B} \in D^b(A \times C)$.*

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{tikzcd} & E \times D \times C \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ E \times D & D \times C & D \times E \end{tikzcd} . \tag{4.47}$$

By Proposition 3.9.4, the composition F_{21}^{111} of $i_{D*} \circ \pi_E^*$ is represented by the convolution of their Fourier-Mukai kernels, thus

$$\pi_{13*}(\pi_{12}^*\mathcal{O}_{E\times_E E} \otimes \pi_{23}^*\mathcal{O}_{D\times_D D}) = \pi_{13*}(\mathcal{O}_{E\times_E E\times C} \otimes \mathcal{O}_{E\times D\times_D D})$$

By Corollary 3.9.4 the latter derived tensor product of the structure sheaves is isomorphic to the structure sheaf of the intersection

$$\mathcal{O}_{E\times_E E\times C} \otimes \mathcal{O}_{E\times D\times_D D} \simeq \mathcal{O}_{E\times_E E\times_E D}$$

Indeed the subvariety $E \times_E D \times C$ is of codimension 4 inside $E \times D \times C$ while the condimension of the subvariety $E \times D \times_D D$ is 6 inside $E \times D \times C$.

Their intersection $(E \times_E D \times C) \cap (E \times D \times_D D)$ is smooth and of codimension 9, therefore they intersect transversally.

Notice that $E \times_E E \times_E D$ is isomorphic to a copy of D in the third component, therefore $\pi_{13} : E \times_E E \times_E D \to E \times C$ is an embedding, so the derived pushforward of $\mathcal{O}_{E\times_E E\times_E D}$ is just the structure sheaf of the image of the support.

So in conclusion the Fourier-Mukai kernel of F^{111}_{21} is isomorphic to

$$\mathcal{O}_{E\times_E D}.$$

The same argument applies to $i_{B*} \circ \pi^*_A$ for showing that the kernel of F^{111}_{12} is isomorphic to

$$\mathcal{O}_{A\times_A B}.$$

□

Remark 4.4.3. *The first merge*

$$R^{21}_{111} = \pi_{E*} \circ i^!_D : D^b(C) \to D^b(E) \tag{4.48}$$

and the second merge

$$R^{12}_{111} = \pi_{A*} \circ i^!_B : D^b(C) \to D^b(A) \tag{4.49}$$

are respectively the right adjoints of F^{111}_{21} and F^{111}_{12}.

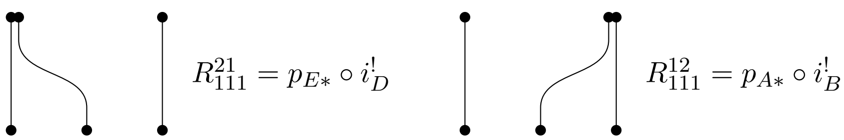


Figure 4.11: First and second merges

Proposition 4.4.4. *The Fourier-Mukai kernel to R^{21}_{111} and R^{12}_{111} are respectively the objects $\mathcal{O}_{D\times_E E}(D,0)[-1] \in D^b(C \times E)$ and $\mathcal{O}_{B\times_A A}(B,0)[-1] \in D^b(C \times A)$.*

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections as in diagram (4.59)

By Proposition 3.2.3, the composition R^{21}_{111} of $p_{E*} \circ i^!_D$ is represented by the convolution of their Fourier-Mukai kernels, thus

$$\pi_{13*}(\pi^*_{23}\mathcal{O}_{D\times_D D}(D,0) \otimes \pi^*_{12}\mathcal{O}_{E\times_E E}) \simeq \pi_{13*}(\mathcal{O}_{E\times D\times_D D}(D,0,0) \otimes \mathcal{O}_{E\times_E E\times C})$$

As in the proof of Proposition (4.59), by transversality of the intersection of $E \times D \times_D D$ with $E \times_E E \times C$, we can apply Corollary 3.9.4 and obtaining the isomorphism

$$\mathcal{O}_{E\times D\times_D D}(D,0,0) \otimes \mathcal{O}_{E\times_E E\times C}) \simeq \mathcal{O}_{E\times_E E\times_E D}(D,0,0).$$

Notice that $E \times_E E \times_E D$ is isomorphic to a copy of D in the third component, therefore $\pi_{13} : E \times_E E \times_E D \to E \times C$ is an embedding, so the derived pushforward of $\mathcal{O}_{E\times_E E\times_E D}$ is just the structure sheaf of the image of the support.

Since π_{13} is an embedding restricted to $E \times_E E \times_E D$, we conclude that the Fourier-Mukai kernel associated to R^{21}_{111} is isomorphic to

$$\mathcal{O}_{D\times_E E}(D,0)[-1].$$

The same argument applies to $i_{B*} \circ \pi_A^*$ for showing that the Fourier-Mukai associated to R^{12}_{111} is isomorphic to

$$\mathcal{O}_{B\times_A A}(B,0)[-1].$$

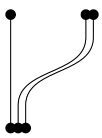
$\square$

Definition 4.4.5. We define the third and fourth forks

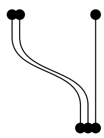
$$F_3^{12} \coloneqq i_{\mathbb{P}^2*} \circ \pi^*_{pt1} : D^b(pt) \to D^b(A), \quad F_3^{21} \coloneqq i_{\mathbb{P}^{2\vee}*} \circ \pi^*_{pt2} : D^b(pt) \to D^b(E)$$

and the third and fourth merges

$$R^3_{12} = \pi_{pt1*} \circ i^!_{\mathbb{P}^2} : D^b(A) \to D^b(pt), \quad R^3_{21} = \pi_{pt2*} \circ i^!_{\mathbb{P}^{2\vee}} : D^b(E) \to D^b(pt).$$

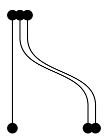


$F_3^{12} \coloneqq i_{\mathbb{P}^2*} \circ \pi^*_{pt1}$

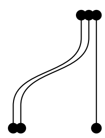


$F_3^{21} \coloneqq i_{\mathbb{P}^{2\vee}*} \circ \pi^*_{pt2}$

Figure 4.12: Third and fourth forks



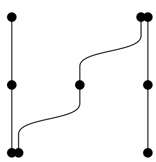
$R^3_{12} = \pi_{pt1*} \circ i^!_{\mathbb{P}^2}$



$R^3_{21} = \pi_{pt2*} \circ i^!_{\mathbb{P}^{2\vee}}$

Figure 4.13: Third and fourth merges

Remark 4.4.6. *The functor F_3^{12} (and similarly for the functor F_3^{21}) can be also described as the functor that maps the 1-dimensional vector space to $\mathcal{O}_{\mathbb{P}^2}$ and its right adjoint R^3_{12} is the functor* $\mathrm{RHom}(\mathcal{O}_{\mathbb{P}^2}, -)$.



R^{12}_{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{B\times_A A}(B,0)[-1]$.

F^{111}_{21} is represented by the Fourier-Mukai kernel $\mathcal{O}_{E\times_E D}$.

Proposition 4.4.7. *The Fourier-Mukai kernel associated to* $R^{12}_{111}F^{111}_{21}$ *is* $\mathcal{O}_{Z_1}\otimes\mathcal{V}_1^*\otimes(\Lambda^2\mathcal{V}_2)^2[-1]$, *where*

$$Z_1 := \left\{\begin{tikzcd} 0 & \subset & V_2 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}\times\left\{\begin{tikzcd} 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}$$

is a line bundle over Fl_3 *and it is also the blow up of* A *or* E *along their zero sections.*

Proof. Let $\pi_{12},\pi_{23},\pi_{13}$ be the natural projections

$$\begin{tikzcd} & E\times C\times A \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ E\times C & E\times A & C\times A \end{tikzcd}.$$

By Proposition 3.2.3, the composition is represented by the convolution of their Fourier-Mukai kernels, thus the kernel of $R^{12}_{111}F^{111}_{21}$ is isomorphic to

$$\pi_{13*}(\pi_{12}^*\mathcal{O}_{E\times_E D}\otimes\pi_{23}^*\mathcal{O}_{B\times_A A}(B,0)[-1]) = \pi_{13*}(\mathcal{O}_{E\times_E D\times A}\otimes\mathcal{O}_{E\times B\times_A A}(0,B,0))[-1]$$

The subvariety $E\times_E D\times A$ is of codimension 5 inside $E\times C\times A$; the same codimension is the one of $E\times B\times_A A$ in $E\times C\times A$.

Their intersection $(E\times_E D\times A)\cap(E\times B\times_A A) = E\times_E(D\cap B)\times_A A$ is smooth and of codimension 10, therefore $E\times_E D\times A$ and $E\times B\times_A A$ intersect transversally in $E\times C\times A$.

By Corollary 3.9.4,

$$\mathcal{O}_{E\times_E D\times A}\otimes\mathcal{O}_{E\times B\times_A A}\simeq\mathcal{O}_{E\times_E(D\cap B)\times_A A}$$

The subvariety $E\times_E(D\cap B)\times_A A$ can be described as the space

$$\left\{\begin{tikzcd} 0 & \subset & V_2 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}\times\left\{\begin{tikzcd} 0 & \subset V_1 \subset & V_2 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[lll, bend left, "\alpha"] \end{tikzcd}\right\}\times\left\{\begin{tikzcd} 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}.$$

It is the clear that $\pi_{13}: E\times_E(D\cap B)\times_A A\to E\times A$ is an embedding.

The image Z_1 of $E\times_E(D\cap B)\times_A A$ under π_{13} is of the form

$$Z_1 = \left\{\begin{tikzcd} 0 & \subset & V_2 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}\times\left\{\begin{tikzcd} 0 & \subset & V_1 \arrow[ll, bend right, "\alpha"'] & \subset & \mathbb{C}^3 \arrow[ll, bend right, "\alpha"'] \end{tikzcd}\right\}.$$

Notice that by 3.10.3 $\mathcal{O}_{Z_1}(0, B, 0) \simeq \mathcal{V}_1^* \otimes (\Lambda^2 \mathcal{V}_2)^2$, so the Fourier-Mukai kernel of $R^{12}_{111} F^{111}_{21}$ is isomorphic to

$$\mathcal{O}_{Z_1} \otimes \mathcal{V}_1^* \otimes (\Lambda^2 \mathcal{V}_2)^2[-1].$$

□

The single crossings are the autoequivalences T_i of section 3.10 that induces the Khovanov-Thomas braid group action on $D^b(C)$ of Theorem 3.10.4.

Remark 4.4.8. *The simple crossing functors $T^{111}_{\bar{1}\bar{1}1}$ $T^{111}_{1\bar{1}\bar{1}}$ are*

$$T^{111}_{\bar{1}\bar{1}1} = \mathrm{Cone}\left(F^{111}_{21} R^{21}_{111}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111} \right),$$

$$T^{111}_{1\bar{1}\bar{1}} = \mathrm{Cone}\left(F^{111}_{12} R^{12}_{111}[-1] \xrightarrow{\mathrm{tr}} \mathrm{Id}_{111} \right),$$

where *tr* is the counit of the adjunction.

Proposition 4.4.9. *The Furier-Mukai kernel associated to $T^{111}_{\bar{1}\bar{1}1}$ is the sheaf*

$$\mathcal{O}_{(C \times_C C) \cup (D \times_E D)}(D, 0).$$

The Furier-Mukai kernel associated to $T^{111}_{1\bar{1}\bar{1}}$ is the sheaf $\mathcal{O}_{(C \times_C C) \cup (B \times_A B)}(B, 0)$.

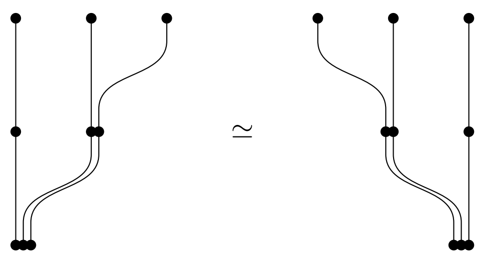
Proof. Proposition 4.4 in [32]. □

4.5 A skein triangulated action of $\mathcal{GB}r_3$: main theorem

In this section, we prove the main theorem of the thesis verify assumptions for a skein triangulated action of $\mathcal{GB}r_3$ on $D^b(T^* \mathrm{Fl}_3(\bar{\imath}))$.

Lemma 4.5.1. *The following isomomorphism holds*

$$F^{111}_{21} F^{21}_{3} \simeq F^{111}_{12} F^{12}_{3}$$



Proof. The Fourier-Mukai kernel representing $F_{21}^{111}F_3^{21}$ is isomorphic to

$$\begin{aligned} &\pi_{13*}(\pi_{12}^*(\mathcal{O}_{pt\times_{pt}A})\otimes\pi_{23}^*(\mathcal{O}_{A\times_A B}))\\ &\simeq\pi_{13*}(\mathcal{O}_{pt\times_{pt}\mathbb{P}^{2\vee}\times C}\otimes\mathcal{O}_{pt\times A\times_A B})\\ &\simeq\pi_{13*}(\mathcal{O}_{pt\times_{pt}\mathbb{P}^{2\vee}\times_{\mathbb{P}^2}Fl_3})\simeq\mathcal{O}_{pt\times Fl_3}\end{aligned}\tag{4.50}$$

Similarly the Fourier-Mukai kernel associated to $F_{12}^{111}F_3^{12}$ is isomorphic to

$$\mathcal{O}_{pt\times Fl_3}.$$

∎

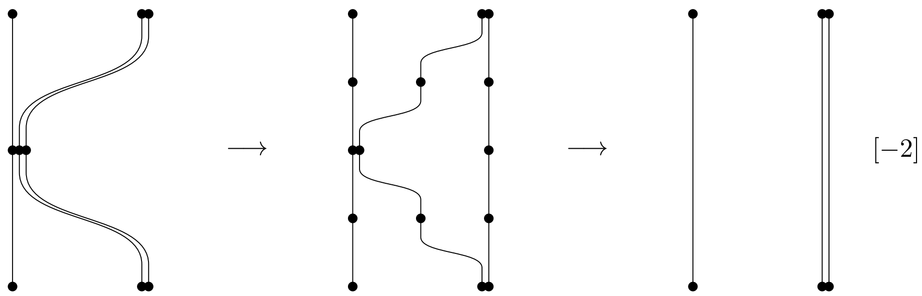
Remark 4.5.2. *From Remark 4.4.6, Lemma 4.5.1 can be also proved showing that functors $F_{21}^{111}F_3^{21}$ and $F_{12}^{111}F_3^{12}$ both map the 1-dimensional vector space to $\mathcal{O}_{\mathrm{Fl}_3}$.*

Lemma 4.5.3. *The mapping cone of the morphism*

$$R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}\xrightarrow{tr[-2]}\mathrm{Id}[-2]$$

is isomorphic to

$$Cone(tr[-2])\simeq F_3^{12}R_{12}^3.\tag{4.51}$$



Proof. The Fourier-Mukai kernel representing $F_3^{12}R_{12}^3$ is isomorphic to

$$\pi_{13*}(\pi_{12}^*(\mathcal{O}_{A\times_{pt}pt})\otimes\pi_{23}^*(\mathcal{O}_{pt\times_{pt}A}(0,\mathbb{P}^2))[-1]\simeq$$

$$\simeq\pi_{12}^*(\mathcal{O}_{\mathbb{P}^2\times pt\times A})\otimes\mathcal{O}_{A\times pt\times\mathbb{P}^2}(0,0,\mathbb{P}^2))[-1]\simeq$$

$$\pi_{13*}(\mathcal{O}_{\mathbb{P}^2\times pt\times\mathbb{P}^2}(0,0,\mathbb{P}^2)[-1]\simeq\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(0,\mathbb{P}^2)[-1].$$

The Fourier-Mukai kernel representing $R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}$ is isomorphic, by the proof of Lemma 4.4.7, to

$$\pi_{15*}(\mathcal{O}_{A\times_A(B\cap D)\times_E E\times C\times A}(0,D,0,0,0)\otimes\mathcal{O}_{A\times C\times E\times_E(B\cap D)\times_A A}(0,0,0,B,0))[-2].$$

The subvarieties $A\times_A(B\cap D)\times_E E\times C\times A$ and $A\times C\times E\times_E(B\cap D)\times_A A$ are both of codimension 10 in $A\times C\times E\times C\times A$. Their intersection $A\times_A(B\cap D)\times_E$

$E \times (B \cap D) \times_A A$ is of codimesion 20 in $A \times C \times E \times C \times A$ therefore by Corollary 3.9.4 the Fourier-Mukai kernel representing $R^{12}_{111} F^{111}_{21} R^{21}_{111} F^{111}_{12}$ is isomorphic to

$$\pi_{15*}(\mathcal{O}_{A \times_A (B \cap D) \times_E E \times_E (B \cap D) \times_A A}(0, D, 0, B, 0))[-2]$$

whose support is isomorphic to $(B \cap D) \times_E (B \cap D)$.

The variety $(B \cap D) \times_E (B \cap D)$ has two irreducible components, one of them X_1 is isomorphic to Z_1 of Proposition 4.4.7, and the other one X_2 is the blow up of $\mathbb{P}^2 \times \mathbb{P}^2$ along the diagonal.

The intersection $X_1 \cap X_2$ of the two components is isomorphic to Fl_3 which is the exceptional divisor inside X_2.

Thus, by 4.42 we have the following short exact sequence

$$0 \to \mathcal{O}_{X_2}(-(X_1 \cap X_2)) \to \mathcal{O}_{X_1 \cup X_2} \to \mathcal{O}_{X_1} \to 0$$

hence we have the isomorphism

$$\mathcal{O}_{X_1 \cup X_2} \simeq Cone(\mathcal{O}_{X_1}[1] \to \mathcal{O}_{X_2}(-(X_1 \cap X_1)))$$

and therefore in $D^b(A \times C \times E \times C \times A)$ $\mathcal{O}_{A \times_A (B \cap D) \times_E E \times_E (B \cap D) \times_A A}(0, D, 0, B, 0)$ is isomorphic to

$$Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_1))) \otimes \mathcal{O}(0, D, 0, B, 0).$$

When we project via π_* to $A \times A$ the first component X_1, it surjects onto the diagonal, while the second component X_2 surjects onto $\mathbb{P}^2 \times \mathbb{P}^2$. Since both maps are blow-downs, the projection is an isomorphism except over the diagonal in $\mathbb{P}^2 \times \mathbb{P}^2$ where it is a $\mathbb{P}^1$-bundle.

Thus, taking the derived pushforward π_{15*} of $Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_1))) \otimes \mathcal{O}(0, D, 0, B, 0)$ and applying Corollary 4.5 of [3] to the map

$$R^{12}_3 F^3_{12} \to R^{12}_{111} F^{111}_{21} R^{21}_{111} F^{111}_{12}$$

we have that

$$F^{12}_3 R^3_{12} \to R^{12}_{111} F^{111}_{21} R^{21}_{111} F^{111}_{12} \to \mathrm{Id}_A[-2]$$

is a distinguished triangle. □

The following Lemma holds in a more general context.

Lemma 4.5.4. *Let X be smooth projective variety over k, $\pi : X \to \operatorname{Spec} k$ be the structure morphism, and $\iota \colon X \hookrightarrow T*(X)$ be the embedding of the zero section:*

$$\begin{array}{ccc} X & \overset{\iota}{\hookrightarrow} & T^*(X) \\ \downarrow{\scriptstyle \pi} & & \\ pt. & & \end{array} \qquad (4.52)$$

Let P^* and P_* be the standard Fourier-Mukai kernels of the exact functors

$$\pi^* \colon D(pt) \to D(X),$$

$$\pi_* \colon D(X) \to D(pt),$$

and let I_* and $I^!$ be the standard Fourier-Mukai kernels of the exact functors

$$\iota_* \colon D(X) \to D(T^*X),$$

$$\iota^! \colon D(T^*X) \to D(X).$$

See [7], Section 2.6.2 for the details on the standard kernels.

Then we have an isomorphism in $D(pt)$:

$$P_* I^! I_* P^* \simeq \Delta_* H^*(X, k).$$

Proof. By Lemma 2.19 of [7], we have

$$P_* I^! I_* P^* \simeq (\pi \times \pi)_* I^! I_*. \tag{4.53}$$

By Proposition 7.8 of [7], the object $I^! I_* \in D(X \times X)$ has the cohomology sheaves:

$$H^i(I^! I_*) \simeq \Delta_* \wedge^i \mathcal{N}_{X/T^*X},$$

in degrees $0 \leq i \leq n$ and 0 elsewhere.

Moreover, by Theorem 1.8(6) of [9], the object $I^! I_*$ is formal, and hence

$$I^! I_* \simeq \bigoplus_{i=0}^{n} \Delta_* \wedge^i \mathcal{N}_{X/T^*X}.$$

Since $\mathcal{N}_{X/T^*X} \simeq \Omega^1_{X/k}$, we conclude that

$$I^! I_* \simeq \Delta_* \left(\bigoplus_{i=0}^{n} \Omega^i_{X/k} \right).$$

Thus we have

$$P_* I^! I_* P^* \simeq (\pi \times \pi)_* I^! I_* \simeq (\pi \times \pi)_* \Delta_* \left(\bigoplus_{i=0}^{n} \Omega^i_{X/k} \right) \simeq \Delta_* \pi_* \left(\bigoplus_{i=0}^{n} \Omega^i_{X/k} \right).$$

Since π_* is isomorphic to the derived global section functor $\mathrm{R}\,\Gamma(-)$, the assertion of the lemma follows by the degeneration of the Hodge-de-Rham spectral sequence. $\square$

The main theorem of the thesis is the following:

Theorem 4.5.5. *Assume that Conjecture 4.2.1 holds; then the assignment of:*

1. *the partition* (111) *to the category* $D^b(\mathrm{T}^* \mathrm{Fl}_3)$*;*

2. *the partition* (12) *to the category* $D^b(\mathrm{T}^*\,\mathbb{P}^2)$*;*
3. *the partition* (21) *to the category* $D^b(\mathrm{T}^*\,\mathbb{P}^{2\vee})$*;*
4. *the partition* (3) *to the category* $D^b(pt)$*;*

and the assignment of:

1. *the fork* f^{111}_{21} *to the functor* F^{111}_{21}*;*
2. *the fork* f^{111}_{12} *to the functor* F^{111}_{12}*;*
3. *the fork* f^{21}_{3} *to the functor* F^{21}_{3}*;*
4. *the fork* f^{12}_{3} *to the functor* F^{12}_{3}*;*

define a skein triangulated representation of $\mathcal{GB}\mathrm{r}_3$ *on* $\mathrm{T}^*\,\mathrm{Fl}_3(\bar{\imath})$.

Proof. By Examples 3.5.6 and 3.7.5, F^{111}_{21} and F^{111}_{12} are split spherical functors with cotwist $[-2]$, while F^{21}_3 and F^{12}_3 are split $\mathbb{P}^2$ functors with $H = [-2]$.

As a consequence of Lemma 4.5.1, there exists a multifork isomorphism.

By Lemma 4.5.4, there exists an isomorphism

$$R^3_{12}R^{12}_{111}F^{111}_{12}F^{12}_3 \simeq \mathrm{Id}_3 \oplus[-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6] \simeq R^3_{21}R^{21}_{111}F^{111}_{21}F^{21}_3$$

and moreover, from Theorem 7.2 of [7] , it identifies together with the $\mathbb{P}^2$ functor structure of F^{21}_3 and F^{12}_3 the maps 4.8, 4.9 with

$$\mathrm{Id}_3 \oplus[-2] \oplus [-4] \to \mathrm{Id}_3 \oplus[-2] \oplus [-2] \oplus [-4] \oplus [-4] \oplus [-6].$$

Finally, by Lemma 4.5.3, the following two diagrams can be completed to two distinguished triangles

$$F^{12}_3R^3_{12} \to R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12} \to \mathrm{Id}_{12}[-2],$$

$$F^{21}_3R^3_{21} \to R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21} \to \mathrm{Id}_{21}[-2].$$

Thus, by Theorem 4.2.3 such assignment define a categorical action of $\mathcal{GB}r_3$ on $\mathrm{T}^*\,\mathrm{Fl}_3(i)$. □

Remark 4.5.6. *From Remark 4.2.2, a categorical action of* $\mathcal{GB}\mathrm{r}_2$ *needs only to satisfy relation 4.2.*

Therefore, it is a case already covered by Theorem 3.10.4 of Section 3.10.

4.6 Further developments

As a continuation of the results of the previous section, the long term research plan is to generalise our theory to the $\mathcal{GB}r_n$ case.

In this section we present some computations that we expect to be helpful to understand how to generalise Theorem 4.5.5 in arbitrary dimensions.

In particular, for computing higher dimensional multiple crossings we use the invariance of the generalised braids under isotopies and an induction on the construction of figure 4.14.

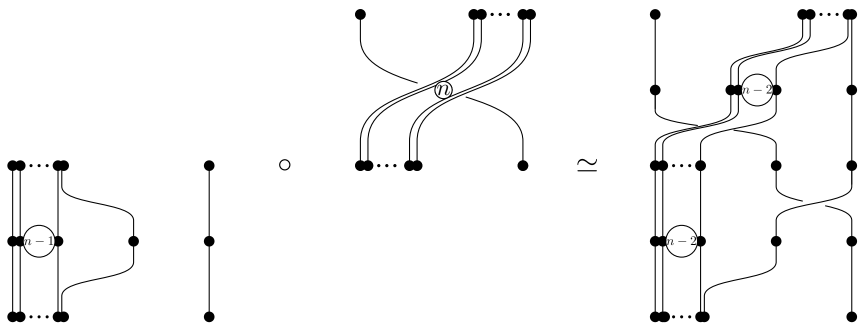


Figure 4.14: Higher dimensional induction

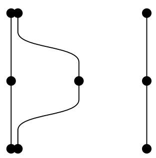
The idea behind this computation is a conjectural program which allows us to inductively compute the multiple crossing functors in higher dimensions and obtain equivalences of type $(pq) - (qp)$.

To motivate the general argument, in this thesis we provide the case $n = 3$.

Therefore, we construct the functor T^{12}_{21} as the difference

$$T^{12}_{21} \circ R^{12}_{111} F^{111}_{12} \simeq R^{21}_{111} T^{111}_{1\bar{1}\bar{1}} T^{111}_{\bar{1}\bar{1}1} F^{111}_{12}. \tag{4.54}$$

We first compute the loop functor $R^{21}_{111} F^{111}_{21}$



R^{21}_{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{D \times_D E}(D, 0)[-1]$.

F^{21}_{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{E \times_E D}$.

Lemma 4.6.1. *The Fourier-Mukai kernel associated to $R^{21}_{111} F^{111}_{21}$ is isomorphic to $\pi_{13*}(\mathcal{E})$ where the cohomologies of $\mathcal{E}$ are*

$$H^r(\mathcal{E}) = \begin{cases} \mathcal{O}_{E \times_E D \times_E E}(0, D, 0), & \textit{if } r = 0; \\ \mathcal{O}_{E \times_E D \times_E E}, & \textit{if } r = -1; \\ 0, & \textit{otherwise.} \end{cases}$$

and where π_{13} is the natural projection $\pi_{13} : A \times C \times A \to A \times A$.

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{tikzcd} & A \times C \times A \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ A \times C & A \times A & C \times A \end{tikzcd} . \tag{4.55}$$

By the standard technique of Proposition 3.2.3, the Fourier-Mukai kernel of the composition $R^{21}_{111}F^{111}_{21}$ is isomorphic to

$$\pi_{13*}(\pi_{12}^*\mathcal{O}_{E\times_E D} \otimes \pi_{23}^*\mathcal{O}_{D\times_E D}(D,0)[-1]) \simeq$$

$$\simeq \pi_{13*}(\mathcal{O}_{E\times_E D\times E} \otimes \mathcal{O}_{E\times D\times_E E}(0,D,0))[-1].$$

We observe that the the supports $E \times_E D \times E$ and $E \times D \times_E E$ of the sheaves involved in the derived tensor product are both of codimesion 5 inside $E \times C \times E$.

The intersection $E \times_E D \times_E E = (E \times_E D \times A) \cap (E \times D \times_E D)$ is smooth and codimension 9 inside $E \times C \times E$, therefore with a rank 1 excess bundle $\mathcal{E}$.

The excess line bundle $\mathcal{E}$ is equal to $\mathcal{O}(0,-D,0)$; this follows from the adjuction formula and by fact that $\mathcal{E}$ is of rank 1. By Proposition 3.9.2 the cohomologies of $\mathcal{O}_{E\times_E D\times E} \otimes \mathcal{O}_{E\times D\times_E E}(0,D,0)$ are therefore

$$H^r(\mathcal{O}_{E\times_E D\times E} \otimes \mathcal{O}_{E\times D\times_E E}(0,D,0)) = \begin{cases} \mathcal{O}_{E\times_E D\times_E E}(0,D,0), & \text{if } r = 0, \\ \mathcal{O}_{E\times_E D\times_E E}, & \text{if } r = -1; \\ 0, & \text{otherwise.} \end{cases}$$

□

The following Lemma holds for general varieties and vector bundles.

Lemma 4.6.2. *Let Y be a variety and let $E \to Y$ be a vector bundle. If $X = \mathbb{P}(E)$ is the projectivisation of E and is the projection $\pi : X = \mathbb{P}(E) \to Y$ then $\pi_*(\Omega^k_{X/Y}) = \mathcal{O}_Y[k]$.*

Proof. Consider the short exact sequence

$$0 \to \Omega_{X/Y} \to \pi^*(E^\vee)(-1) \to \mathcal{O}_X \to 0 \tag{4.56}$$

which is the dual of the Euler sequence.

If we apply the functor π_* to (4.56) we obtain the short exact sequence

$$0 \to \pi_*(\Omega_{X/Y}) \to 0 \to \mathcal{O}_Y \to 0 \tag{4.57}$$

indeed, since X is a projective bundle, $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $\pi_*\mathcal{F}(-1) = 0$.

Thus, from the long exact sequence associated to (4.57) we get

$$\pi_*(\Omega^k_{X/Y}) \simeq \mathcal{O}_Y[k].$$

□

Lemma 4.6.3. $\pi_{13*}(\mathcal{O}_{E\times_E D\times E}\otimes\mathcal{O}_{E\times D\times_E E}(0,D,0))[-1]$ *is formal in* $E\times E$.

Proof. From the adjunction (4.43) we have the distinguished triangle

$$id \to R^{21}_{111}F^{111}_{21} \to id[-2]$$

which at level of Fourier-Mukai kernels correspond to the distinguished triangle

$$\Delta_*\mathcal{O}_E \to X \to \Delta_*\mathcal{O}_E[-2] \tag{4.58}$$

where $\Delta : E \to E\times E$ is diagonal embedding and X is the Fourier-Mukai kernel of $R^{21}_{111}F^{111}_{21}$. he abelian group of equivalence classes of distinguished triangles of the form

$$\mathcal{O}_{E\times_E E} \to Y \to \mathcal{O}_{E\times_E E}[-2]$$

with $Y\in D^b(E\times E)$ is isomorphic to the group

$$Ext^1(\mathcal{O}_{E\times_E E}[-2],\mathcal{O}_{E\times_E E}) \simeq Ext^3(\mathcal{O}_{E\times_E E},\mathcal{O}_{E\times_E E}).$$

Therefore, in order to prove that $\pi_{13*}(\mathcal{O}_{E\times_E D\times E}\otimes\mathcal{O}_{E\times D\times_E E}(0,D,0))[-1]$ is formal in $E\times E$, it is sufficient to prove that the distinguished triangle (4.58) correspond to the zero class in $Ext^3(\mathcal{O}_{E\times_E E},\mathcal{O}_{E\times_E E})$.

The functor $F^{111}_{21}R^{21}_{111}F^{111}_{21}$ is a retract, so it correspond to the zero class of

$$Ext^1(F^{111}_{21}[-2],F^{111}_{21})$$

Let $\pi_{12},\pi_{23},\pi_{13}$ be the natural projections

$$\begin{tikzcd} & E\times E\times C \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ E\times E & E\times C & E\times C \end{tikzcd} . \tag{4.59}$$

The Fourier-Mukai kernel of $F^{111}_{21}R^{21}_{111}F^{111}_{21}$ is then by Proposition 3.2.3 by base change around the commutative square

$$\begin{tikzcd} E\times D \arrow[r, "{(\mathrm{id},\pi,\mathrm{i})}"] \arrow[d, "\pi_2"'] & E\times E\times C \arrow[d, "\pi_{23}"] \\ D \arrow[r, "{(\pi,\mathrm{i})}"] & E\times C \end{tikzcd} \tag{4.60}$$

isomorphic to

$$\pi_{13*}(\pi_{12}^*X \otimes \pi_{23}^*(\pi,i)_*\mathcal{O}_D) \simeq \pi_{13*}(\pi_{12}^*X \otimes \pi_{23}^*(id,\pi,i)_*\mathcal{O}_{E\times D})$$

by projection formula we have the isomophism

$$\pi_{13*}(\pi_{12}^*X \otimes \pi_{23}^*(id,\pi,i)_*\mathcal{O}_{E\times D}) \simeq \pi_{13}(id,\pi,i)^*(id,\pi,i)_*\pi_{12}^*X.$$

Moreover, since

$$\pi_{13*}(id,\pi,i)_* = (\pi_{13}\circ(id,\pi,i))_* = (\pi,i)_*$$

and

$$(id,\pi,i)^*\pi_{12}^* = (\pi_{12}\circ(id,\pi,i))^* = (id,\pi)^*$$

we have the isomorphism

$$\pi_{13}(id,\pi,i)^*(id,\pi,i)_*\pi_{12}^*X \simeq (\pi,i)_*(id,\pi)^*X.$$

Consider now the following commutative diagram

$$\begin{tikzcd} Ext^3_{A\times B}((\pi,id)_*\mathcal{O}_D,(\pi,id)_*\mathcal{O}_D) \arrow[d, "g"'] \arrow[dr, "i_*\circ\pi_{2*}"] & \\ Ext^3_{A\times C}((\pi,i)_*\mathcal{O}_D,(\pi,i)_*\mathcal{O}_D) \arrow[r, "h"] & Ext^3_C(i_*\mathcal{O}_D,i_*\mathcal{O}_D) \end{tikzcd} \tag{4.61}$$

The zero element $0_{A\times C}\in Ext^3_{A\times C}((\pi,i)_*\mathcal{O}_D,(\pi,i)_*\mathcal{O}_D)$ is sent by h to the zero element of $Ext^3_C(i_*\mathcal{O}_D,i_*\mathcal{O}_D)$; showing that the homomorphism $(\pi,i)_*$ is injective proves therefore that g is injective and that $g^{-1}(0_{A\times C}) = 0_{A\times B}$.

Indeed, since $\pi_2^*i^*\pi_{2*}(\pi,id)_* = \pi_2^*i^*i_*$, we have the isomorphism,

$$\pi_2^*i^*i_*\pi_{2*}(\pi,id)_*\mathcal{O}_D \simeq \pi_2^*i^*i_*\mathcal{O}_D$$

but $i : D\hookrightarrow C$ is a divisorial inclusion, hence $i^*i_*\mathcal{O}_D = \mathcal{O}_D\oplus\mathcal{O}_D(D)[1]$ and

$$\pi_2^*i^*i_*\mathcal{O}_D \simeq \pi_2^*(\mathcal{O}_D\oplus\mathcal{O}_D(D)[1])$$

which correspond to the zero element as it splits and therefore $i_*\pi_{2*}$ is injective.

Since $(\pi,i)_*\mathcal{O}_{E\times D}\simeq\mathcal{O}_{E\times_E E}$, because $D\xrightarrow{\pi}E$ is $\mathbb{P}^1$-bundle, and $(id,\pi)^*$ is fully faithful, we have that under the isomorphism

$$f : Ext^3(\Delta_*\mathcal{O}_E,\Delta_*\mathcal{O}_E) \xrightarrow{\simeq} Ext^3((\pi,id)_*\mathcal{O}_D,(\pi,id)_*\mathcal{O}_D)$$

the Fourier-Mukai the element $0_{A\times C}\in Ext^3_{A\times C}((\pi,i)_*\mathcal{O}_D,(\pi,i)_*\mathcal{O}_D)$ that correspond to the Fourier-Mukai kernel representing $F^{111}_{21}R^{21}_{111}F^{111}_{21}$ has preimage under $f\circ g$ the zero element of $Ext^3(\Delta_*\mathcal{O}_E,\Delta_*\mathcal{O}_E)$ and therefore X is formal.

□

Proposition 4.6.4. *The Fourier-Mukai kernel associated to* $R^{21}_{111}F^{111}_{21}$ *is* $\mathcal{O}_{E\times_E E} \oplus \mathcal{O}_{E\times_E E}[-2]$

Proof. Consider the projections $\pi_{12}, \pi_{23}, \pi_{13}$ as in (4.66).

By Lemma 4.6.1 the non-zero cohomologies of $\mathcal{O}_{E\times_E D\times E} \otimes \mathcal{O}_{E\times D\times_E E}(0, D, 0)$ are $\mathcal{O}_{E\times_E D\times_E E}(0, D, 0)$ in degree -1 and $\mathcal{O}_{E\times_E D\times_E E}$ in degree zero.

We have $\pi_{13*}\mathcal{O}_{E\times_E D\times_E E} \simeq \mathcal{O}_{E\times_E E}$ since $E\times_E D\times_E E \xrightarrow{\pi_{13}} E\times_E E$ is a $\mathbb{P}^1$ bundle, while by Lemma 4.6 we have that $\pi_{13*}\mathcal{O}_{E\times_E D\times_E E}(0, D, 0)[-1] \simeq \mathcal{O}_{E\times_E E}[-2]$.

Finally by Lemma 4.6.3 we conclude that $F_1R_1 \simeq \mathcal{O}_{E\times_E E} \oplus \mathcal{O}_{E\times_E E}[-2]$. □

By Proposition 4.6.4 we know that the Fourier Mukai kernel associated to the functor $R^{12}_{111}F^{111}_{12}$ is $\mathcal{O}_{E\times_E E} \oplus \mathcal{O}_{E\times_E E}[-2]$. While $R^{21}_{111}T^{111}_{1\bar{1}\bar{1}}T^{111}_{\bar{1}\bar{1}1}F^{111}_{12}$ is by definition of $T^{111}_{1\bar{1}\bar{1}}$ and $T^{111}_{\bar{1}\bar{1}1}$ isomorphic to

$$R^{21}_{111}T^{111}_{1\bar{1}\bar{1}}T^{111}_{\bar{1}\bar{1}1}F^{111}_{12} \simeq R^{21}_{111}Cone(F^{111}_{12}R^{12}_{111} \xrightarrow{tr_2} id)Cone(F^{111}_{21}R^{21}_{111} \xrightarrow{tr_1} id)F^{111}_{12} \tag{4.62}$$

where $T^{111}_{1\bar{1}\bar{1}} = Cone(F^{111}_{12}R^{12}_{111} \xrightarrow{tr} Id)$ and tr is the counit of the adjunction $F^{111}_{12} \dashv R^{12}_{111}$.

Therefore the functor $R^{21}_{111}T^{111}_{1\bar{1}\bar{1}}T^{111}_{\bar{1}\bar{1}1}F^{111}_{12}$ is the convolution of the diagram

$$\begin{tikzcd}
 & R^{12}_{111}F^{111}_{21}R^{21}_{111}\mathrm{Id}F^{111}_{21} \arrow[dr, "tr_1"] & \\
R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21} \arrow[ur, "tr_2"] \arrow[dr, "tr_1"'] & \bigoplus & R^{12}_{111}F^{111}_{21} \\
 & R^{12}_{111}\mathrm{Id}F^{111}_{12}R^{12}_{111}F^{111}_{21} \arrow[ur, "tr_2"'] &
\end{tikzcd} \tag{4.63}$$

Notice that by Lemma 4.6.3 we have the following the isomorphisms

$$R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{21} \simeq R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2]$$

and similarly for $R^{12}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21}$

$$R^{12}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21} \simeq R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2].$$

Thus, the convolution of diagram (4.63) is isomorphic to the convolution of diagram

$$\begin{tikzcd}
 & R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2] \arrow[dr, "tr_1"] & \\
R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{12}R^{12}_{111}F^{111}_{21} \arrow[ur, "tr_2"] \arrow[dr, "tr_1"'] & \bigoplus & R^{12}_{111}F^{111}_{21} \\
 & R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2] \arrow[ur, "tr_2"'] &
\end{tikzcd} \tag{4.64}$$

Thus, to compute the double crossing functor T_{21}^{12} we have the following recipe:

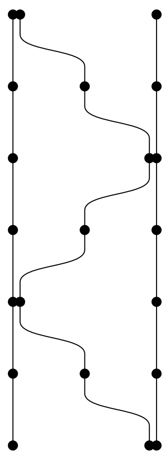
1. Compute the Fourier-Mukai transform associated to
$$R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}.$$

2. Determine the Fourier-Mukai kernel of $R_{111}^{21}T_2T_{\bar{1}\bar{1}1}^{111}F_{12}^{111}$ as a convolution of diagram (4.64).

3. Define the double crossing functor T_{21}^{12} using the isomorphism (4.54).

STEP 1: the Fourier Mukai kernel of $R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}$

The first step of the recipe for the computation of the double crossing T_{21}^{12} is to compute the Fourier-Mukai transform associated to $R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}$.



F_{111}^{21} is represented by the Fourier-Mukai kernel $\mathcal{O}_{E\times_E D}$.

R_{12}^{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{B\times_A A}(B,0)[-1]$.

F_{111}^{12} is represented by the Fourier-Mukai kernel $\mathcal{O}_{A\times_A B}$.

R_{21}^{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{D\times_D E}(D,0)[-1]$.

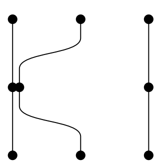
F_{111}^{21} is represented by the Fourier-Mukai kernel $\mathcal{O}_{E\times_E D}$.

R_{12}^{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{B\times_A A}(B,0)[-1]$.

Figure 4.15: The functor $R_{111}^{12}F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}F_{21}^{111}$

We begin by spitting the big composition in smaller ones in order make it more accessible.

Lemma 4.6.5. *The Fourier-Mukai kernel associated to $R_{111}^{21}F_{21}^{111}$ and $R_{111}^{12}F_{21}^{111}$ are respectively $\mathcal{O}_{D\times_E D}(D,0)[-1]$ and $\mathcal{O}_{B\times_A B}(B,0)[-1]$.*



R_{21}^{111} is represented by the Fourier-Mukai kernel $\mathcal{O}_{D\times_D E}(D,0)[-1]$.

F_{111}^{21} is represented by the Fourier-Mukai kernel $\mathcal{O}_{E\times_E D}$.

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{tikzcd} & C \times E \times C \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ C \times E & E \times E & E \times C \end{tikzcd} . \tag{4.65}$$

The Fourier-Mukai kernel associated to $R^{21}_{111} F^{111}_{21}$ is by Proposition 3.2.3

$$\pi_{13*}(\pi_{12}^* \mathcal{O}_{D \times_E E}(D,0)[-1] \otimes \pi_{23}^* \mathcal{O}_{E \times_E D}) = \pi_{13*}(\mathcal{O}_{D \times_E E \times C}(D,0) \otimes \mathcal{O}_{C \times E \times_E D})[-1]$$

Let us compare the codimensions of the supports of the sheaves involved in the derived tensor product: it easy to see that $D \times_E E \times C$ and $C \times E \times_E D$ are both of codimension 5 inside $C \times E \times C$.

Their intersection $(C \times E \times_E D) \cap (D \times_E E \times C) = D \times_E E \times_E D$ is smooth and of codimension 10, therefore their intersection is tranverse and we can use Lemma 3.9.4 and obtain the isomorphism

$$\mathcal{O}_{D \times_E E \times C}(D,0) \otimes \mathcal{O}_{C \times E \times_E D} \simeq \mathcal{O}_{D \times_E E \times_E D}(D,0,0).$$

Notice that the morphism $\pi_{13} : D \times_E E \times_E D \to C \times C$ is an embedding. Indeed is is easy to see that the space $D \times_E E \times_E D$

$$\left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] & V_2 \arrow[r, phantom, "\subset"] \arrow[ll, bend right, "\alpha"'] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_2 \arrow[r, phantom, "\subset"] \arrow[l, bend right, "\alpha"'] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1' \arrow[r, phantom, "\subset"] & V_2 \arrow[r, phantom, "\subset"] \arrow[ll, bend right, "\alpha"'] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}$$

is naturally embedded into $C \times C$

$$C \times C = \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1' \arrow[r, phantom, "\subset"] \arrow[l, bend right, "\alpha"'] & V_2' \arrow[r, phantom, "\subset"] \arrow[l, bend right, "\alpha"'] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] \arrow[l, bend right, "\beta"'] & V_2 \arrow[r, phantom, "\subset"] \arrow[l, bend right, "\beta"'] & \mathbb{C}^3 \arrow[l, bend right, "\beta"'] \end{tikzcd} \right\}$$

and the image $\pi_{13}(D \times_E E \times D) = D \times_E D$.

In conclusion, the Fourier-Mukai kernel associated to $R^{21}_{111} F^{111}_{21}$ is isomorphic to

$$\pi_{13*}(\mathcal{O}_{D \times_E E \times_E D}(D,0))[-1] \simeq \mathcal{O}_{D \times_E D}(D,0)[-1].$$

The same argument shows that the kernel of $R^{12}_{111} F^{111}_{12}$ is isomorphic to

$$\mathcal{O}_{B \times_A B}(B,0)[-1].$$

$\square$

Lemma 4.6.6. *The Fourier-Mukai kernel associated to $F_{21}^{111}R_{111}^{21}F_{12}^{111}R_{111}^{12}$ is isomorphic to $\mathcal{O}_{Z_3}(D,0)[-2]$ where*

$$Z_3 = \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & V_2' \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1' \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & V_2 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}.$$

Proof. Let $\pi_{12}, \pi_{23}, \pi_{13}$ be the natural projections

$$\begin{tikzcd} & C \times C \times C \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{13}"] \arrow[dr, "\pi_{23}"] & \\ C \times C & C \times C & C \times C \end{tikzcd}. \tag{4.66}$$

Denote by $F_1R_1F_2R_2$ the Fourier-Mukai kernel of $F_{21}^{111}R_{111}^{21}F_{21}^{111}R_{111}^{12}$.

By Proposition 3.2.3 and by the previous Lemma 4.6.5 $F_1R_1F_2R_2$ is isomorphic in $D^b(C \times C)$ to

$$F_1R_1F_2R_2 \simeq \pi_{13*}(\pi_{12}^*\mathcal{O}_{B\times_A B}(B,0)[-1] \otimes \pi_{23}^*\mathcal{O}_{D\times_E D}(D,0)[-1]) =$$

$$= \pi_{13*}(\mathcal{O}_{B\times_A B\times C}(B,0,0) \otimes \mathcal{O}_{C\times D\times_E D}(0,D,0))[-2]$$

The supports of $\mathcal{O}_{B\times_A B\times C}(B,0,0)$ and $\mathcal{O}_{C\times D\times_E D}(0,D,0)$ are both of codimension 5 inside $C \times C \times C$, their intersection $D \times_E (B \cap D) \times_A B = (B \times_A B \times C) \cap (C \times D \times_E D)$ is smooth and of codimension 10.

$$\mathcal{O}_{B\times_A B\times C}(B,0,0) \otimes \mathcal{O}_{C\times D\times_E D}(0,D,0)) \simeq \mathcal{O}_{D\times_E(B\cap D)\times_A B}(B,D,0)$$

The projection $\pi_{13} : D \times_E (B \cap D) \times_A B \to C \times C$ is an embedding.

Indeed, $B \times_A (B \cap D) \times_E D$ is isomorphic to

$$\left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & V_2' \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \end{tikzcd} \right\} \times \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] & V_2 \arrow[ll, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[ll, bend left, "\alpha"] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1' \arrow[r, phantom, "\subset"] & V_2 \arrow[ll, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}$$

and therefore id a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle over the $D \cap B$ copy in the central component.

Its image under π_{13} is the subvariety Z_3 of $C \times C$

$$Z_3 = \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & V_2' \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd}[column sep=small] 0 \arrow[r, phantom, "\subset"] & V_1' \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & V_2 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}$$

which is the same $\mathbb{P}^1 \times \mathbb{P}^1$ over the $B \cap D$ viewed as a line bundle over the flag $\{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\}$

$$B \cap D = \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[r, phantom, "\subset"] & V_2 \arrow[ll, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[ll, bend left, "\alpha"] \end{tikzcd} \right\}.$$

Thus, in conclusion we have that the Fourier-Mukai kernel of $F^{111}_{21} R^{21}_{111} F^{111}_{21} R^{12}_{111}$ is

$$\pi_{13*}(\mathcal{O}_{D \times_E (D \cap B) \times_A B}(B, D, 0))[-2] = \mathcal{O}_{Z_3}(D, 0)[-2].$$

□

Before merging all the Fourier-Mukai kernel together, we prove the following technical Lemma which holds in a more general context.

Lemma 4.6.7. *Let W_1, W_2, W_3 be smooth subvarieties of the variety Z. and consider the following commutative diagram of inclusion maps*

$$\begin{tikzcd} & W_1 \cap W_2 \arrow[ddl, "p"'] \arrow[dddr, "h"] \arrow[dr, "q"] & & W_1 \cap W_2 \cap W_3 \arrow[ll, "r"] \arrow[dddl, "t"'] \arrow[ddr, "s"] & \\ & & W_1 \arrow[dd, "i"] & & \\ W_2 \arrow[drr, "j"] & & & & W_3 \arrow[dll, "w"] \\ & & Z & & \end{tikzcd} \quad . \tag{4.67}$$

The relative canonical bundle $\omega_{W_1 \cap W_2 \cap W_3}/Z$ is isomorphic to the sheaf

$$\omega_{W_1 \cap W_2 \cap W_3}/Z \simeq r^* h^! \mathcal{O}_Z \otimes s^* w^! \mathcal{O}_Z. \tag{4.68}$$

Proof. By base change, the relative canonical bundle $\omega_{W_1 \cap W_2/Z}$ is isomorphic to

$$\begin{aligned} \omega_{W_1 \cap W_2/Z} &\simeq q^* i^! \mathcal{O}_Z \otimes q^! \mathcal{O}_{W1} \\ &\simeq q^* i^! \mathcal{O}_Z \otimes q^! i^* \mathcal{O}_Z \\ &\simeq q^* i^! \mathcal{O}_Z \otimes p^* j^! \mathcal{O}_Z. \end{aligned} \tag{4.69}$$

Similarly we can iterate this argument for the relative canonical bundle on $W_1 \cap W_2 \cap W_3/Z$

$$\begin{aligned}\omega_{W_1\cap W_2\cap W_3/Z} &\simeq r^*h^!\mathcal{O}_Z\otimes h^!\mathcal{O}_{W_1\cap W_2}\\ &\simeq r^*h^!\mathcal{O}_Z\otimes r^!h^*\mathcal{O}_Z\\ &\simeq r^*h^!\mathcal{O}_Z\otimes s^*w^!\mathcal{O}_Z.\end{aligned}\tag{4.70}$$

□

Proposition 4.6.8. *The Fourier-Mukai kernel representing*

$$R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{21}R^{12}_{111}F^{111}_{21}$$

is isomorphic to $\pi_{15*}(\mathcal{F})$*, where* $E\times C\times C\times C\times A\xrightarrow{\pi_{15}}E\times A$ *and*

$$\mathcal{F}\in D^b(E\times C\times C\times C\times A)$$

is the derived tensor product of the Fourier-Mukai kernels involved in the composition with cohomologies

$$H^r(\mathcal{F})=\begin{cases}\mathcal{O}_{Y_1\cup Y_2}(0,(1,-2),(-2,1),(1,-2),0), & \textit{if } r=0,\\ \mathcal{O}_{Y_1}(0,(3,-3),(-3,0),(0,0),0), & \textit{if } r=-1;\\ 0, & \textit{otherwise.}\end{cases}$$

The subvarieties Y_1 *and* Y_2 *are the total spaces respectively of the zero section and the complement of the zero section of* $E\times_E(B\cap D)\times_A(B\cap D)\times_E(B\cap D)\times_A A$.

Moreover, the cohomologies of $\mathcal{F}$ *split under* π_{15*}.

Proof. Let $\pi_{12},\pi_{234},\pi_{45}$ and π_{15} the natural projections

$$\begin{tikzcd} & E\times C\times C\times C\times A \arrow[r, "\pi_{15}"] \arrow[dl, "\pi_{12}"'] \arrow[d, "\pi_{234}"] \arrow[dr, "\pi_{45}"] & E\times A \\ E\times C & C\times C\times C & C\times A \end{tikzcd}.$$

By Proposition 3.2.3 and the previous Lemmas 4.6.5 and 4.6.6 the Fourier-Mukai kernel $R_2F_1R_1F_2R_2F_1$ associated to $R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{21}R^{12}_{111}F^{111}_{21}$ is isomorphic to

$$\pi_{15*}(\pi_{12}^*\mathcal{O}_{E\times_E D}\otimes\pi_{234}^*\mathcal{O}_{B\times_A(D\cap B)\times_E D}(B,D,0))[-2]\otimes\pi_{45}^*\mathcal{O}_{B\times_A A}(B,0)[-1])$$

Let's deal first with the transverse part of the intersection, since the derived tensor product is commutative.

As usual we want to use Corollary 3.9.4 in order to obtain the isomorphism

$$\pi_{12}^*\mathcal{O}_{E\times_E D}\otimes\pi_{45}^*\mathcal{O}_{B\times_A A}(B,0)[-1]\simeq$$

$$\simeq \mathcal{O}_{E\times_E D\times C\times C\times A} \otimes \mathcal{O}_{E\times C\times C\times B\times_A A}(0,0,0,B,0)[-1]$$

The subvariety $E\times_E D\times C\times B\times_A A = (E\times_E D\times C\times C\times A)\cap(E\times C\times C\times B\times_A A)$ is smooth and of codimension 10 in $E\times C\times C\times C\times A$, while the codimensions of $E\times_E D\times C\times C\times A$ and of $E\times C\times C\times B\times_A A$ of $E\times_E D\times C\times C\times A$ are both of codimension 5.

The Fourier-Mukai kernel $R_2F_1R_1F_2R_2F_1$ is isomorphic, since the intersection is transverse, to

$$\pi_{15*}(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0)[-1] \otimes \mathcal{O}_{E\times B\times_A (D\cap B)\times_E D\times A}(0,B,D,0,0))[-2])$$

The subvariety $E\times_E D\times C\times B\times_A A$ is of dimension 16 so is codimension 10 in $E\times C\times C\times C\times A$, the subvariety $E\times B\times_A (D\cap B)\times_E D\times A$ is of dimension 14 so is codimension 12 in $E\times C\times C\times C\times A$. Their intersection $(E\times_E D\times C\times B\times_A A)\cap(E\times B\times_A (D\cap B)\times_E D\times A) = E\times_E (B\cap D)\times_A (B\cap D)\times_E (B\cap D)\times_A A$ is a reducible variety; indeed

$$E\times_E (B\cap D)\times_A (B\cap D)\times_E (B\cap D)\times_A A = Y_1\cup Y_2$$

where Y_1 is the zero section $\{\alpha=0\}$

$$Y_1 = \{0\subset V_2'\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2'\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1'\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1'\subset \mathbb{C}^3\}$$

which is a $\mathbb{P}^1\times\mathbb{P}^1$-bundle over Fl_3 and of codimension 21 in $E\times C\times C\times C\times C\times A$, so with one dimensional excess bundle $\mathcal{E}_1$;

Y_2 is the component outside of the zero section

$$Y_2 = \{0\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset \mathbb{C}^3\}$$

which is isomorphic to $B\cap D$, so smooth and of codimension 22: therefore the intersection is transverse on this component.

The intersection between the two components of $E\times_E (B\cap D)\times_A (B\cap D)\times_E (B\cap D)\times_A A$ is

$$Y_1\cap Y_2 = \{0\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset V_2\subset \mathbb{C}^3\}\times\{0\subset V_1\subset \mathbb{C}^3\}$$

so isomorphic to Fl_3, hence a Cartier divisor in $D\cap B\simeq Y_2$.

Thus, by 4.42 we have the following short exact sequence

$$0\to \mathcal{O}_{Y_2}(-(Y_1\cap Y_1))\to \mathcal{O}_{Y_1\cup Y_2}\to \mathcal{O}_{Y_1}\to 0$$

hence we have the isomorphism

$$\mathcal{O}_{Y_1\cup Y_2}\simeq Cone(\mathcal{O}_{Y_1}[1]\to \mathcal{O}_{Y_2}(-(Y_1\cap Y_1)))$$

therefore in $D^b(E\times C\times C\times C\times A)$ $\mathcal{O}_{E\times_E(B\cap D)\times_A(B\cap D)\times_E(B\cap D)\times_A A}(0,B,D,B,0)$ is isomorphic to

$$Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1\cap Y_1)))\otimes \mathcal{O}(0,B,D,B,0).$$

For computing the excess conormal bundle

$$\mathcal{E}_1 = (\mathcal{N}_{E\times_E D\times C\times B\times_A A} \oplus \mathcal{N}_{E\times B\times_A(D\cap B)\times_E D\times A})/\mathcal{N}_{Y_1}$$

we will use the fact that it is a line bundle, hence $det(\mathcal{E}_1)=\mathcal{E}_1$ and the short exact sequence

$$0\to \mathcal{N}_{Y_1} \to \mathcal{N}_{E\times_E D\times C\times B\times_A A} \oplus \mathcal{N}_{E\times B\times_A(D\cap B)\times_E D\times A} \to \mathcal{E}_1 \to 0 \tag{4.71}$$

thus,

$$\mathcal{E}_1 = det(\mathcal{N}_{E\times_E D\times C\times B\times_A A})\otimes det(\mathcal{N}_{E\times B\times_A(D\cap B)\times_E D\times A})\otimes det(\mathcal{N}_{Y_1})^{-1}$$

So the excess bundle will be

$\mathcal{E}_1 = \mathcal{O}(0,-D,0,-B,0)\otimes\mathcal{O}(0,-B,-B-D,-D,0)\otimes\mathcal{O}(0,\omega_{Fl_3}\otimes\omega_{\mathbb{P}^2}^{-1},\omega_{Fl_3},\omega_{Fl_3}\otimes\omega_{\mathbb{P}^{2\vee}}^{-1},0)$. Recall that the tautological bundles can be written as $\mathcal{O}(-1,0)=\mathcal{V}_1$ and $\mathcal{O}(0,-1)=\Lambda^2\mathcal{V}_2$.

The canonical bundles of $Fl_3, \mathbb{P}^2, \mathbb{P}^{2\vee}$ are respectively

$$\omega_{Fl_3} = \mathcal{O}(-3,-3)\otimes\mathcal{O}(1,1) = \mathcal{O}(-2,-2), \quad \omega_{\mathbb{P}^2} = \mathcal{O}(-3,0), \quad \omega_{\mathbb{P}^{2\vee}} = \mathcal{O}(0,-3)$$

The normal bundles of B and D inside C could be written respectively as

$$\mathcal{O}(B) = (\Lambda_2\mathcal{V}_2^*)^{-2}\otimes\mathcal{V}_1^* = \mathcal{O}(1,-2), \quad \mathcal{O}(D) = \Lambda_2\mathcal{V}_2^*\otimes(\mathcal{V}_1^*)^{-2} = \mathcal{O}(-2,1)$$

and we have that $\omega_{Fl_3}\otimes\omega_{\mathbb{P}^2}^{-1} = \mathcal{O}(1,-2) = \mathcal{O}(B)$, $\omega_{Fl_3}\otimes\omega_{\mathbb{P}^{2\vee}}^{-1} = \mathcal{O}(-2,1) = \mathcal{O}(D)$
Then the excess conormal bundle is $\mathcal{E}_1 = \mathcal{O}(0,(2,-1),(-1,-1),(-1,2),0)$
and if $\mathcal{L} = \mathcal{O}(0,B,D,B,0) = \mathcal{O}(0,(1,-2),(-2,1),(1,-2),0)$ then

$$\mathcal{E}_1\otimes\mathcal{L} = \mathcal{O}(0,(3,-3),(-3,0),(0,0),0)$$

So the cohomologies of the derived tensor product are

$$H^r(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0)\otimes\mathcal{O}_{E\times B\times_A(D\cap B)\times_E D\times A}(0,B,D,0,0))) =$$
$$= \begin{cases} \mathcal{O}_{Y_1\cup Y_2}(0,(1,-2),(-2,1),(1,-2),0), & \text{if } r=0,\\ \mathcal{O}_{Y_1}(0,(3,-3),(-3,0),(0,0),0), & \text{if } r=-1;\\ 0, & \text{otherwise.} \end{cases}$$

Consider the map

$$\pi_{13}: Y_1 \longrightarrow E\times A$$

As Y_1 lies inside the zero section of $E\times C\times C\times C\times A$, its image under π_{13} is $\mathbb{P}^2\times\mathbb{P}^{2\vee}$. Moreover:

1. If $(V_1', V_2') \notin Fl_3$ then V_1 is forced to be different from V_1', thus V_2' is determined: so Y_1 is a $\mathbb{P}^1$ bundle over the open $\mathbb{P}^2 \times \mathbb{P}^{2\vee} \setminus Fl_3$.

2. If $(V_1', V_2') \in Fl_3$, then $\begin{cases} V_1 \neq V_1' \text{ implies } V_2 = V_2'; \\ V_2 \neq V_2' \text{ implies } V_1 = V_1'; \end{cases}$

   This implies that over $Fl_3 \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$, Y_1 is a $\mathbb{P}^1 \cup \mathbb{P}^1$-bundle.

The morphism π_{13} is flat on Y_1.

Since flatness is a local open condition, on the open $\mathbb{P}^2 \times \mathbb{P}^{2\vee} \setminus Fl_3$ is flat because it is a $\mathbb{P}^1$-bundle.

For the closed subvariety $(V_1', V_2') \in Fl_3$, by definition, the morphism π_{13} is flat if and only if $\mathcal{O}_{Y_1,p}$ is a flat $\mathcal{O}_{E\times A,\pi_{13}(p)}$-module.

Choose x, y and α, β to be local coordinates respectively for $\mathbb{P}^2$ and for $\mathbb{P}^{2\vee}$, so that

$$\mathcal{O}_{E\times A,\pi_{13}(p)} = \mathbb{C}[\alpha, \beta, x, y]_{(\alpha,\beta,x,y)}$$

Denote moreover δ, γ and w, z local coordinates for V_1 and V_2 respectively. In $\mathbb{P}^{2\vee} \times \mathbb{P}^2$ we have the following relations that we want to be satisfied

$$V_1 \subset V_2, \qquad V_1' \subset V_2, \qquad V_1 \subset V_2'.$$

Let's take the local chart for $V_2' = (1 : \alpha : \beta)$, the argument for the other two charts will be completely specular.

Moreover, since all the linear spaces V_1, V_1', V_2, V_2' lie in the same ambient space $\mathbb{C}^3$, the incidence relations are independent by the action of $GL(\mathbb{C}, 3)$; we can then assume that the point (α, β) correspond to the origin and

$$V_2' = (1 : 0 : 0)^{\perp}$$

If $V_1 \neq V_1'$ then $V_2 = V_2'$, so the local coordinates of V_1 and V_1' can be :

- $V_1' = (0 : 1 : \beta)$, $V_1 = (0 : 1 : \delta)$, with β forced to be different from δ; thus, the local ring is $\mathcal{O}_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\alpha,\delta,x,y)}$

  By symmetry, this case cover also the chart $V_1' = (0 : \alpha : 1)$, $V_1 = (0 : \gamma : 1)$

- $V_1' = (0 : 1 : \beta)$, $V_1 = (0 : \gamma : 1)$ with β and γ are both forced to be different from 1. Then, the local ring is $\mathcal{O}_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\gamma,\beta,x,y)}$

  By symmetry, this case cover also the chart $V_1' = (0 : \alpha : 1)$, $V_1 = (0 : 1 : \delta)$

If $V_1 \neq V_2'$ then $V_1 = V_1'$: if we choose as local coordinates of $V_1 = V_1' = (0 : 1 : \beta)$, then V_2 is forced to have the coordinates $V_2 = (0 : 1 : \frac{-1}{\beta})^{\perp}$ with $\beta \neq 0$ ($V_2 = (0 : 1 : 0)^{\perp}$ otherwise) or $V_2 = (0 : -\alpha : 1)^{\perp}$.
Therefore, the local ring $\mathcal{O}_{Y_1,p} \simeq \mathbb{C}[\alpha, \beta, \gamma, \delta, x, y, w, z]_{(\frac{-1}{\beta},\beta,x,y)}$.
The argument is symmetric in the case we take coordinates $V_1 = V_1' = (0 : \alpha : 1)$.

Finally, if $V_2' = V_2$ and $V_1 = V_1'$ then the two charts for $V_1' = V_1'$ are $(0:1:\beta)$ and $(0:\alpha:1)$.
So, $\mathcal{O}_{Y_{1,p}} \simeq \mathbb{C}[\alpha,\beta,\gamma,\delta,x,y,w,z]_{(\beta,x,y)}$ and $\mathcal{O}_{Y_{1,p}} \simeq \mathbb{C}[\alpha,\beta,\gamma,\delta,x,y,w,z]_{(\alpha,x,y)}$

Since π_{13} is flat, using base change around the commutative square

$$\begin{tikzcd} Y_{1,p} \arrow[r, "u"] \arrow[d, "v"'] & Y_1 \arrow[d, "\pi_{13}"] \\ p \arrow[r, "w"] & \mathbb{P}^{2\vee} \times \mathbb{P}^{2\vee} \end{tikzcd} \tag{4.72}$$

The map π_{13} is flat and the map w is proper, so the commutative square (4.72) is Tor-independent.
By flat base change theorem $w^* \circ \pi_{13*} \simeq v_* \circ u^*$.
So $w^*(\pi_{13*}\mathcal{O}_{Y_1}) \simeq v_*(u^*\mathcal{O}_{Y_1})$.

Moreover, we have that

$$u^*\mathcal{O}_{Y_1} = \mathcal{O}_{Y_{1,p}},$$

$$v_*\mathcal{O}_{Y_{1,p}} = \Gamma(Y_{1,p}, \mathcal{O}_{Y_{1,p}}) = \mathbb{C}.$$

Thus, $w^*\pi_{13*}\mathcal{O}_{Y_1} = \mathbb{C}$, therefore the fibers of $\pi_{13*}\mathcal{O}_{Y_1}$ are one dimensional, so $\pi_{13*}\mathcal{O}_D \in Pic(\mathbb{P}^{2\vee} \times \mathbb{P}^2)$.

In particular the the higher dimensional cohomologies of the derived pushforward $\pi_{13*}\mathcal{O}_{Y_1}$ vanish and

$$\pi_{13*}\mathcal{O}_{Y_1} \simeq \pi_{13*}\mathcal{O}_{Y_1}.$$

Since for every point q of $E \times A$, the fiber $\pi_{13}^{-1}(q)$ is compact, then the only regular functions of $\pi_{13*}\mathcal{O}_{Y_1}$ on an open set U containing q are the constant. Therefore we have the local isomorphisms $(\pi_{13*}\mathcal{O}_{Y_1})_q \simeq \mathbb{C} \simeq (\mathcal{O}_{\mathbb{P}^{2\vee}\times\mathbb{P}^2})_q$ induced by $\pi_{13}^{\#}$.

Thus, we deduce that $\pi_{13*}\mathcal{O}_{Y_1} = \mathcal{O}_{\mathbb{P}^{2\vee}\times\mathbb{P}^2}$.

Since $\pi_{12} : Y_2 \to E \times A$ is an embedding, we conclude that

$$\begin{aligned} \pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2}) &\simeq \pi_{13*}Cone(\mathcal{O}_{Y_1}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_2))) \\ &\simeq Cone(\pi_{13*}\mathcal{O}_{Y_1}[1] \to \pi_{13*}\mathcal{O}_{Y_2}(-(Y_1 \cap Y_2))) \\ &\simeq Cone(\mathcal{O}_{\mathbb{P}^{2\vee}\times\mathbb{P}^2}[1] \to \mathcal{O}_{Z_4}(-(Fl_3))) \end{aligned} \tag{4.73}$$

where Z_4 is the subvariety of $E \times A$

$$Z_4 = \left\{ 0 \subset V_2 \subset \mathbb{C}^3 \right\} \times \left\{ 0 \subset V_1 \subset \mathbb{C}^3 \right\}.$$

Since Fl_3 is a Cartier divisor inside Z_1 we have the isomorphism

$$\mathcal{O}_{Z_1 \cup \mathbb{P}^2 \times \mathbb{P}^{2\vee}} \simeq Cone(\mathcal{O}_{\mathbb{P}^{2\vee} \times \mathbb{P}^2}[-1] \to \mathcal{O}_{Z_1}(-(Fl_3))),$$

we conclude that

$$\pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2}) = \mathcal{O}_{Z_4 \cup \mathbb{P}^2 \times \mathbb{P}^{2\vee}}$$

Denote $\mathcal{L}$ the line bundle $\mathcal{O}(0, (1, -2), (-2, 1), (1, -2), 0))$

$$\begin{aligned} \pi_{13*}(\mathcal{O}_{Y_1 \cup Y_2} \otimes \mathcal{L}) &\simeq \pi_{13*} Cone(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \to \mathcal{O}_{Y_2}(-(Y_1 \cap Y_2)) \otimes \mathcal{L}) \\ &\simeq Cone(\pi_{13*}\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \to \pi_{13*}\mathcal{O}_{Y_2}(-(Y_1 \cap Y_2)) \otimes \mathcal{L}) \\ &\simeq Cone(0[1] \to \mathcal{O}_{Z_4}(-(Fl_3) - D)) \end{aligned} \tag{4.74}$$

Indeed $\pi_{13*}\mathcal{O}_{Y_1} \otimes \mathcal{L}[1] \simeq 0$: by flat base change theorem around the commutative square (4.72), the following functors are isomorphic $w^* \circ \pi_{13*} \simeq v_* \circ u^*$.
So $w^*(\pi_{13*}(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1])) \simeq v_*(u^*(\mathcal{O}_{Y_1} \otimes \mathcal{L}[1]))$.

Consider the following commutative square

$$\begin{tikzcd} Y_1 \arrow[r, "i"] \arrow[dr, "p"'] & \mathbb{P}^{2\vee} \times \mathbb{P}^2 \times \mathbb{P}^{2\vee} \times \mathbb{P}^2 \arrow[d, "\pi_{14}"] \\ & \mathbb{P}^{2\vee} \times \mathbb{P}^2 \end{tikzcd} \tag{4.75}$$

Then by Lemma 4.6.7 , the relative canonical bundle $\omega_{Y_1/\mathbb{P}^{2\vee} \times \mathbb{P}^2} \simeq \pi_{13}^! \mathcal{O}_{\mathbb{P}^{2\vee}\mathbb{P}^2}$ is isomorphic to

$$\begin{aligned} \pi_{13}^! \mathcal{O}_{\mathbb{P}^{2\vee}\mathbb{P}^2} &\simeq i^* p^! \mathcal{O}_{\mathbb{P}^{2\vee} \times P^2} \otimes i^! \mathcal{O}_{\mathbb{P}^{2\vee} \times \mathbb{P}^2 \times \mathbb{P}^{2\vee} \times \mathbb{P}^2} \\ &\simeq i^* \mathcal{O}(0, -3, -3, 0) \otimes i^* \mathcal{O}(1, 2, 2, 1) \\ &\simeq i^* \mathcal{O}(1, -1, -1, 1) \end{aligned} \tag{4.76}$$

Recall that

$$\mathcal{L} = \mathcal{O}(-2, -1, -1, 1)$$

So

$$\mathcal{L} = \pi_{13}^! \mathcal{O}_{\mathbb{P}^{2\vee} \times \mathbb{P}^2} \otimes \pi_{13}^* \mathcal{O}(-3, 0)$$

and therefore

$$\begin{aligned} \pi_{13*}\mathcal{L} &\simeq \pi_{13*}(\pi_{13}^! \mathcal{O}_{\mathbb{P}^{2\vee} \times P^2} \otimes \pi_{13}^* \mathcal{O}(-3, 0)) \\ &\simeq \pi_{13*} \mathcal{H}om(\mathcal{O}_{Y_1}, \pi_{13}^! \mathcal{O}_{\mathbb{P}^{2\vee} \times P^2}) \otimes \mathcal{O}(-3, 0) \\ &\simeq \pi_{13*} \mathcal{H}om(\pi_{13*}\mathcal{O}_{Y_1}, \mathcal{O}_{\mathbb{P}^{2\vee} \times P^2})[-1] \otimes \mathcal{O}(-3, 0) \\ &\simeq \mathcal{O}(-3, 0)[-1] \end{aligned} \tag{4.77}$$

Therefore we have the isomorphisms

$$\pi_{13*}(\mathcal{O}_{Y_1} \otimes \mathcal{L}) \simeq \mathcal{O}_{\mathbb{P}^{2\vee} \times \mathbb{P}^2}(-3,0)[-1].$$

and

$$\pi_{13*}(\mathcal{O}_{Y_1}(0,(3,-3),(-3,0),(0,0),0)[1]) = \mathcal{O}_{\mathbb{P}^{2\vee}\times\mathbb{P}^2}(-3,0)[1] \simeq \mathcal{O}_{\mathbb{P}^{2\vee}\times\mathbb{P}^2}(2B+D)[1]$$

since on the fibers the line bundle is trivial.

The last statement of the proposition follows therefore from relative Bott vanishing.

$\square$

STEP 2: The Fourier Mukai kernel of $R^{12}_{111}T^{111}_{\bar{1}\bar{1}1}T^{111}_{1\bar{1}\bar{1}}F^{111}_{21}$

The second step of the recipe for constructing the double crossing T^{12}_{21} is computing the Fourier-Mukai kernel of the functor

$$R^{21}_{111}T^{111}_{1\bar{1}\bar{1}}T^{111}_{\bar{1}\bar{1}1}F^{111}_{21} = R^{21}_{111}(F^{111}_{21}R^{12}_{111} \to id)(F^{111}_{21}R^{21}_{111} \to id)F^{111}_{21} \tag{4.78}$$

as the convolution of diagram

$$\begin{tikzcd}
 & R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2] \arrow[dr, "tr_1"] & \\
R^{12}_{111}F^{111}_{21}R^{21}_{111}F^{111}_{21}R^{12}_{111}F^{111}_{21} \arrow[ur, "tr_2"] \arrow[dr, "tr_1"'] & \bigoplus & R^{12}_{111}F^{111}_{21} \\
 & R^{12}_{111}F^{111}_{21} \oplus R^{12}_{111}F^{111}_{21}[-2] \arrow[ur, "tr_2"'] &
\end{tikzcd} \tag{4.79}$$

at level of Foirier-Mukai transforms.

Proposition 4.6.9. *The cohomologies of the derived tensor product of Fourier-Mukai kernels involved in the composition of $R^{12}_{111}F^{111}_{21}R^{21}_{111}\,\mathrm{Id}\,F^{111}_{21}$ before taking the derived pushforward π_{15*} are*

$$H^r(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0)\otimes\mathcal{O}_{E\times D\times_D D\times_E D\times A}(0,0,D,0,0)) =$$
$$= \begin{cases} \mathcal{O}_{Y_3}(0,0,D,B,0), & \textit{if } r=0, \\ \mathcal{O}_{Y_3}(0,0,0,B,0), & \textit{if } r=-1; \\ 0, & \textit{otherwise.} \end{cases}$$

where

$$Y_3 \coloneqq (E \times_E D \times_D D \times_E (B \cap D) \times_A A)$$

and

$$\pi_{15} : E \times C \times C \times C \times A \to E \times A.$$

Proof. Consider the following diagram

$$\begin{tikzcd}
E \times C & E \times C \times C \times C \times A \arrow[l, "\pi_{12}"'] \arrow[r, "\pi_{15}"] \arrow[dl, "\pi_{23}"'] \arrow[d, "\pi_{34}"'] \arrow[dr, "\pi_{45}"] & E \times A \\
C \times C & C \times C & C \times A
\end{tikzcd}.$$

By change base theorem and lemma 4.6.5 the Fourier-Mukai kernel K_5 associated to $R^{12}_{111} F^{111}_{21} R^{21}_{111} \operatorname{Id} F^{111}_{21}$ is

$$\begin{aligned}
K_5 &\simeq \pi_{15*}(\pi^*_{12}\mathcal{O}_{E\times_E D} \otimes \pi^*_{23}\mathcal{O}_{C\times_C C} \otimes \pi^*_{34}\mathcal{O}_{D\times_E D}(D,0)[-1] \otimes \pi^*_{45}\mathcal{O}_{B\times_A A}(B,0)[-1]) \\
&\simeq \pi_{15*}(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0)[-1] \otimes \mathcal{O}_{E\times D\times_D D\times_E D\times A}(0,0,D,0,0))[-1])
\end{aligned} \tag{4.80}$$

The subvariety $E \times_E D \times C \times B \times_A A$ is of dimension 16 so is codimension 10 in $E \times C \times C \times C \times A$ while the subavariety $E \times D \times_D D \times_E D \times A$ is of dimension 14 so is codimension 12 in $E \times C \times C \times C \times A$

$Y_4 = (E \times_E D \times C \times B \times_A A) \cap (E \times D \times_D D \times_E D \times A) \simeq D \times_E (D \cap B)$ which is is a $\mathbb{P}^1$-bundle over $B \cap D$ and at the same time the blow up $Bl_{\{\alpha=0\}}(D)$ over the zero section of D, so it is 5 dimensional and its codimension in $E \times C \times C \times C \times A$ is 21, so with one dimensional excess bundle $\mathcal{E}_3$.

For computing $\mathcal{E}_3$ we will use the short exact sequence 4.71 to get the isomorphism

$$\begin{aligned}
\mathcal{E}_3 &\simeq det(\mathcal{N}_{E\times_E D\times C\times B\times_A A}) \otimes det(\mathcal{N}_{E\times D\times_D D\times_E D\times A}) \otimes det(\mathcal{N}_{Y_4})^{-1} \\
&\simeq \mathcal{E}_1 = \mathcal{O}(0,-D,0,-B,0) \otimes \mathcal{O}(0,-D,+\omega_D - D,-D,0) \otimes \mathcal{O}(0,D,-\omega_D + D,+B+D,0) \\
&\simeq \mathcal{O}(0,-D,0,0,0)
\end{aligned}$$

So by Proposition 3.9.2 the cohomologies of the derived tensor product are

$$H^r(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0) \otimes \mathcal{O}_{E\times D\times_D D\times_E D\times A}(0,0,D,0,0)) = \\
= \begin{cases} \mathcal{O}_{Y_3}(0,0,D,B,0), & \text{if } r = 0, \\ \mathcal{O}_{Y_3}(0,0,0,B,0), & \text{if } r = -1; \\ 0, & \text{otherwise.} \end{cases}$$

□

Analogously a symmetric proof proves the following

Proposition 4.6.10. *The cohomologies of the derived tensor product of Fourier-Mukai kernels involved in the composition of $R^{12}_{111}\,\mathrm{Id}\,F^{111}_{21}R^{12}_{111}F^{111}_{21}$ before taking the derived pushforward π_{15*} are*

$$H^r(\mathcal{O}_{E\times_E D\times C\times B\times_A A}(0,0,0,B,0)\otimes\mathcal{O}_{E\times B\times_A B\times_B B\times A}(0,B,0,0,0)) =$$
$$= \begin{cases} \mathcal{O}_{Y'_3}(0,B,0,B,0), & \text{if } r=0,\\ \mathcal{O}_{Y'_3}(0,0,0,B,0), & \text{if } r=-1;\\ 0, & \text{otherwise.}\end{cases}$$

where

$$Y'_3 := E\times_E (B\cap D)\times_A B\times_A B\times_A A.$$

and

$$\pi_{15} : E\times C\times C\times C\times A \to E\times A.$$

The following Lemma holds in the context of triangulated categories.

Lemma 4.6.11. *Let the following be a distinguished triangle in a triangulated category $\mathcal{C}$*

$$V[-1]\xrightarrow{f} W \xrightarrow{g} U \xrightarrow{h} V \tag{4.81}$$

and let $\gamma\in Hom^0(A[-1],A')$ for $A'\in\mathcal{T}$. Let $X[1]$ be the cone of $\gamma\circ h : C\to A'[1]$ so that

$$V'[-1]\to W \longrightarrow X \xrightarrow{\gamma\circ h} V'$$

(4.82)

is a distinguished triangle. Then $X\simeq Cone(V[-1]\xrightarrow{f\oplus\gamma} W\oplus V')$

Proof. Consider the following commutative square

$$\begin{array}{ccc} V[-1] & \xrightarrow{f\oplus\gamma} & W\oplus V' \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle(\mathrm{Id},0)} \\ W & \xrightarrow{\mathrm{Id}} & W \end{array} \tag{4.83}$$

and complete it to the following commutative diagram

$$\begin{tikzcd} V[-1] \arrow[r, "f\oplus\gamma"] \arrow[d, "\alpha"'] & W\oplus V' \arrow[r] \arrow[d, "(\mathrm{Id},0)"] & Q \arrow[d, dashed] \\ W \arrow[r, "\mathrm{Id}"] \arrow[d] & W \arrow[r, "0"] \arrow[d] & 0 \arrow[d, dashed] \\ U \arrow[r, dashed, "\gamma\circ h"] \arrow[d, "\gamma"'] & V' \arrow[r, dashed] \arrow[d, "(0,\mathrm{Id})"] & Q[1] \arrow[d, "\mathrm{Id}"] \\ V \arrow[r, "f\oplus\gamma"] & W[1]\oplus V' \arrow[r] & Q[1] \end{tikzcd} \tag{4.84}$$

Since all the rows and all the colums are distinguished triangle we can aplly the octaedral axiom of triangualate categories in order to obtain the isomorphism $X \simeq Q$. $\square$

Proposition 4.6.12. *The Fourier-Mukai kernel representing the functor*

$$R^{12}_{111} T^{111}_{\bar{1}\bar{1}1} T^{111}_{1\bar{1}\bar{1}} F^{111}_{21}$$

is the object $\mathcal{O}_{Z_1 \cup (\mathbb{P}^{2\vee} \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^{2\vee} \times \mathbb{P}^2)}[-2]$ *where*

$$Z_1 := \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_2 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}.$$

Proof. By Proposition 4.6.8, Lemmas 4.6.9, 4.6.10 and Proposition 4.4.7, the convolution of diagram (4.64) and therefore the Fourier-Mukai kernel of $R^{12}_{111} T^{111}_{\bar{1}\bar{1}1} T^{111}_{1\bar{1}\bar{1}} F^{111}_{21}$ is the derived pushforward of the projection on $E \times A$ of the diagram of the cohomologies

$$\begin{tikzcd}[column sep=small]
 & \mathcal{O}_{Y_3}(0,0,0,B,0)[-2] \oplus \mathcal{O}_{Y_3}(0,0,D,B,0)[-1] & \\
\mathcal{O}_{Y_1}(0,(3,-3),(-3,0),(0,0),0)[-3] \oplus \mathcal{O}_{Y_1 \cup Y_2}(0,B,D,D,0)[-2] \arrow[ur, bend left, "f"] \arrow[ur, bend right, "g_1"] \arrow[dr, bend left, "g_2"'] & & \mathcal{O}_{E \times_E (D \cap B) \times_E E}(0,D,0)[-1] \\
 & \mathcal{O}_{Y_3'}(0,B,0,B,0)[-2] \oplus \mathcal{O}_{Y_3'}(0,0,0,B,0)[-2] \arrow[ur, bend left, "h"] &
\end{tikzcd} \tag{4.85}$$

We want to compute the cone of the subdiagram

$$\begin{tikzcd}
 & \mathcal{O}_{Y_3}(0,0,D,B) \\
\mathcal{O}_{Y_1 \cup Y_2}(0,B,D,B,0) \arrow[ur, "g_2"] \arrow[dr, "g_1"] & \bigoplus \\
 & \mathcal{O}_{Y_3'}(0,B,0,B,0)
\end{tikzcd} \tag{4.86}$$

using Lemma 4.6.11.

The intersection of Y_3 with $Y_1 \cup Y_2$ is a divisor in Y_3; indeed

$$Y_3 \cap (Y_1 \cup Y_2) = (Y_3 \cap Y_1) \cup (Y_3 \cap Y_2)$$

where

$$Y_3 \cap Y_2 = Y_2$$

and Y_2 is a divisor in Y_3, and

$$Y_3 \cap Y_1 = Y_{3|\alpha=0}$$

so the restriction of Y_3 to the zero section, which is a divisor in Y_3 and will be denoted by $\tilde{Y_3}$.

So from 4.42 we have the short exact sequence

$$0 \to \mathcal{O}_{Y_3}(-\tilde{Y_3} - Y_2) \to \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3} \to \mathcal{O}_{Y_1 \cup Y_2} \to 0 \tag{4.87}$$

From Proposition 4.6.8 we have that the zero cohomology of $R_2 F_1 R_1 F_2 R_2 F_1$ is

$$Y_1 \cup Y_2 = \mathcal{O}_{E \times_E B \cap D \times_A B \cap D \times_E B \cap D \times_A A}(0, B, D, B, 0)$$

meanwhile from Lemma 4.6.10 we know that the zero cohomology of $R_2 F_1 R_1 \operatorname{Id} F_1$ is

$$Y_3 = \mathcal{O}_{E \times_E D \times_D D \times_E B \cap D \times_A A}(0, 0, D, B, 0)$$

Their intersection is the reducible variety

$$Y_3 \cap (Y_1 \cup Y_2) = \mathcal{O}_{E \times_E D \cap B \times_{D \cap B} D \cap B \times_E B \cap D \times_A A}(0, 0, D, B, 0)$$

which is a divisor in Y with conormal bundle $\mathcal{N}_1 = \mathcal{O}(0, -B, 0, 0, 0)$.

So from 4.42, they fit in the short exact sequence

$$0 \to \mathcal{O}_{E \times_E D \times_D D \times_E B \cap D \times_A A}(0, 0, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3}(0, B, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2}(0, B, D, B, 0) \to 0$$

Similarly the zero cohomology of $R_2 \operatorname{Id} F_2 R_2 F_1$

$$Y_3' = \mathcal{O}_{E \times_E B \cap D \times_A B \times_B B \times_A A}(0, B, 0, B, 0)$$

the intersection of Y_3' with $Y_1 \cup Y_2$ is

$$Y_3' \cap (Y_1 \cup Y_2) = \mathcal{O}_{E \times_E D \cap B \times_D D \cap B \times_{D \cap B} B \cap D \times_A A}(0, B, 0, B, 0)$$

which is a Cartier divisor in Y_3' with conormal bundle $\mathcal{N}_2 = \mathcal{O}(0, 0, -D, 0, 0)$.

As before, the zero cohomologies of the functors fits in the short exact sequence

$$0 \to \mathcal{O}_{E \times_E B \cap D \times_E B \times_B B \times_A A}(0, B, 0, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2 \cup Y_3'}(0, B, D, B, 0) \to \mathcal{O}_{Y_1 \cup Y_2}(0, B, D, B, 0) \to 0 \tag{4.88}$$

By Lemma 4.6.11 we have that

$$Cone(g_1 \oplus g_2) \simeq \mathcal{O}_{Y_1 \cup Y_3 \cup Y_3'}(0, B, D, B, 0). \tag{4.89}$$

Similarly f is the Ext^1 map coming from the short exact sequence (4.42) which glues Y_1 on top of Y_3, while h became the identity after taking the derived pushforward since both the varieties Y_3' and $E \times_E (D \cap B) \times_E E$ are mapped to Z_1 under the projections to $E \times A$. Using Lemma , the derived pushforward of the cone of (4.85) and therefore the Fourier-Mukai kernel representing the functor $R^{12}_{111} T^{111}_{\bar{1}\bar{1}1} T^{111}_{1\bar{1}\bar{1}} F^{111}_{21}$ is isomorphic to

$$\mathcal{O}_{Z_1 \cup (\mathbb{P}^{2\vee} \times \mathbb{P}^2)} \oplus \mathcal{O}_{Z_1 \cup (\mathbb{P}^{2\vee} \times \mathbb{P}^2)}[-2].$$

□

STEP 3: The double crossing functor T^{12}_{21}

The last step for the construction of the double crossing functor T^{12}_{21} is to defined it as the difference of Figure 4.16.

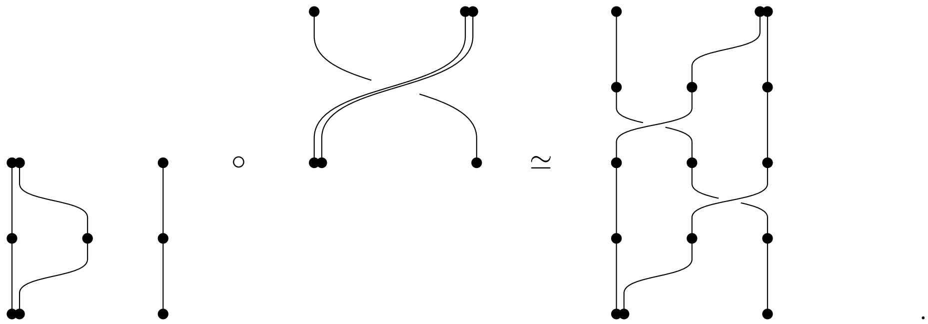


Figure 4.16: The double crossing functor T^{12}_{21}

Theorem 4.6.13. *The Fourier-Mukai kernel representing the double crossing functor T^{12}_{21} is $\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}$ where*

$$Z_1 := \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_2 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\} \times \left\{ \begin{tikzcd} 0 \arrow[r, phantom, "\subset"] & V_1 \arrow[l, bend right, "\alpha"'] \arrow[r, phantom, "\subset"] & \mathbb{C}^3 \arrow[l, bend right, "\alpha"'] \end{tikzcd} \right\}.$$

Proof. By Proposition 4.6.12 the Fourier-Mukai kernel representing the functor

$$R^{12}_{111}T^{111}_{\bar{1}\bar{1}1}T^{111}_{1\bar{1}\bar{1}}F^{111}_{21}$$

is the object $\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}\oplus\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}[-2]$, while by Proposition 4.6.4 the functor $R^{12}_{111}F^{111}_{21}$ is represented by $\mathcal{O}_{E\times_E E}\oplus\mathcal{O}_{E\times_E E}[-2]$.

Therefore

$$X\star(\mathcal{O}_{E\times_E E}\oplus\mathcal{O}_{E\times_E E}[-2])\simeq\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}\oplus\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}[-2]. \tag{4.90}$$

Indeed 4.90 leads to the isomorphism

$$X\simeq\mathcal{O}_{Z_1\cup(\mathbb{P}^{2\vee}\times\mathbb{P}^2)}. \tag{4.91}$$

□

Remark 4.6.14. *The Fourier-Mukai kernel of Theorem 4.90 is the Kawamata-Namikawa Fourier-Mukai kernel of Theorem 3.8.2.*

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