Option pricing in illiquid markets: a fractional jump-diffusion approach

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Abstract

We study the pricing of European options when the underlying stock price is illiquid. Due to the lack of trades, the sample path followed by prices alternates between active and motionless periods that are replicable by a fractional jump-diffusion. This process is obtained by changing the time-scale of a jump-diffusion with the inverse of a Lévy subordinator. We prove that option prices are solutions of a forward partial differential equation in which the derivative with respect to time is replaced by a Dzerbayshan-Caputo (D-C) derivative. The form of the D-C derivative depends upon the chosen inverted Lévy subordinator. We detail this for inverted α stable and inverted Poisson subordinators. To conclude, we propose a numerical method to compute option prices for the two types of D-C derivatives.

1 Introduction

When looking to quotes of an illiquid asset e.g. in an emerging market, we often observe relatively long periods without any trade. As Brownian motions and Lévy processes are perpetually moving, they are not adapted for modelling periods with motionless stock returns. Notably, we observe similar behavior in physical systems exhibiting sub-diffusion. The periods without trades correspond to the trapping events in which the sub-diffusive particle gets immobilized, see e.g. Eliazar and Klafter (2004) or Metzler and Klafter (2004). Sub-diffusion is a well identified phenomenon in statistical physics and the density of a sub-diffusive process is described in terms of a Fractional Fokker-Planck (FFP) equation.

In the FFP equation, the derivative with respect to time is replaced by a fractional derivative. As explained in Acay et al. (2020 a), the classical derivative restricted by rate of change falls short to describe many phenomena that could not be constructed properly by integer order calculus encompassed by fractional calculus. Due to this fact, fractional derivatives are proposed for capturing the past history as in the classical integration. Hence, both fractional derivative and integral have past memory making them much more advantageous than classical counterparts. The beginning of the fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695 where the notation for differentiation of non-integer order 1/2 is discussed. Several famous mathematicians contributed to fractional calculus: Abel, Liouville, Rieman and more recently Caputo.

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The FFP equation for subdiffusions is e.g. studied in Barkai et al. (2000) and Metzler et al. (1999). Sub-diffusions are popular in econophysics (see Scalas, 2006, for a survey) and Magdziarz (2009 a) uses a geometric sub-diffusions to model illiquid asset prices. A sub-diffusion also admits a convenient representation as a time-changed Brownian motion, see e.g. Magdziarz (2009 b). Articles of Leonenko et al. (2013, a) Leonenko et al. (2013, b) go a step further and study fractional Pearson diffusions and their correlation. Whereas Hainaut (2020 a, 2020 b) explores fractional self-excited jump processes. Acay et al. (2020 b) develop economic models with various fractional dynamics.

Even if sub-diffusions appear as good candidates for modelling illiquidity, options pricing in this framework remains a challenging task. A first way consists to evaluate prices by simulations as in Magdziarz (2009 a). Solving the FFP offers an alternative solution that is nevertheless numerically unstable. Our article explores a new approach based on a fractional version of what is called the Dupire's equation. Dupire (1994) has established a forward partial differential equation for call options. In his model, the stock value is driven by a geometric diffusion with volatility, function of time and price. The Dupire's approach has enjoyed certain popularity with practitioners, at least partly because of its simplicity. It has subsequently been extended by many authors, notably to Markov process with jumps as in Andersen and Andreasen (2000) or Friz et al. (2014).

The contributions of this article are multiple. Firstly, the existing literature on option pricing in illiquid market mainly focuses on sub-diffusive dynamics that are continuous by nature. We consider instead a fractional jump-diffusion for the asset return. The sample path of such a process behaves like a Brownian motion with motionless periods but exhibits discontinuities caused by jumps. Secondly, we consider a wider family of fractional dynamics. In previously cited papers, Fractional processes are built by replacing the time scale by a random clock that is the inverse of an α stable Lévy process. Here, we establish a very general form of the fractional Dupire's equation valid for all invertible Lévy subordinators. As an illustration, we compare fractional dynamics based on inverted Poisson and α stable subordinators. Finally, we propose a numerical method to evaluate options in these two cases.

This work is organized as follows. Section 2 reviews well-known results about option pricing in a jump-diffusion setting. Section 3 presents the properties of inverted Lévy subordinators, used later as stochastic clock of our financial market. The Laplace exponent of these inverted Lévy processes may be used to define Dzerbayshan-Caputo derivatives as explained in Section 4. We retrieve such a derivative in the fractional version of the Dupire's equation for jump-diffusion developed in Section 5. The two last sections are devoted to the numerical pricing of options.

2 Non-fractional jump-diffusion model

This section review some well-known results about option valuation in a jump-diffusion framework. We will consider later a time-changed version of this model. For the moment, we consider a financial market in which is traded a risk-free bond and a stock. The risk-free bond, noted B_t, earns a constant interest rate r and satisfies the differential equation:

$$\frac{dB_t}{B_t} = rdt \quad B_0 = 1\,,\, t \geq 0\,.$$

The return of the stock is ruled by a Brownian motion $\left(W_t^P\right)_{t\geq 0}$ and by a compound Poisson process, $(J_t)_{t\geq 0}$. This compound Poisson process is defined as:

$$J_t \;=\; \sum_{k=1}^{N_t^J} Y_k\,,$$

where N_t^J is a Poisson process with parameter λ_J. Jumps are identically independent distributed random variables, $Y_k \sim Y$, on $[-1,\infty)$. We denote by $f_Y(.)$ the probability density function (pdf) of jumps and the jump expectation is denoted by $\xi = \mathbb{E}(Y)$. We assume that the stock price, A_t, is ruled by a following geometric jump-diffusion:

$$\frac{dA_t}{A_t} = \mu\, dt + \sigma_t\, dW_t^P + dJ_t\,, \tag{1}$$

where $\mu \in \mathbb{R}^+$ and $(\sigma_t)_{t\geq 0}$ is positive process. All processes are defined on a probability space Ω, endowed with their natural filtration $(\mathcal{F}_t)_{t\geq 0}$ and a probability measure, P. The volatility process is $\mathcal{F}_t-$adapted and square integrable, i.e. $\int_0^t \sigma_s^2(\omega) ds < \infty$ for all $\omega \in \Omega$ and $t >0$. By construction, the expected instantaneous return of the stock price is equal to $\mathbb{E}\left(dA_t|\mathcal{F}_t\right) = \left(\mu + \lambda_J \xi\right) A_t dt$. Applying the Itô's lemma to $d \ln A_t$ leads after integration to the following expression for the stock price:

$$A_t \;=\; A_0 \exp\left(\int_0^t \mu - \frac{\sigma_s^2}{2} ds + \int_0^t \sigma_s\, dW_s^P\right) \prod_{k=1}^{N_t^J} \left(1+Y_k\right). \tag{2}$$

This is the dynamics of the risky asset under the real measure P. Nevertheless, the pricing of financial derivatives is performed under an equivalent measure of probability, called "risk neutral" so as to exclude arbitrages. Under this measure, risky assets earn on average the risk free rate whatever their volatility. Equivalent probability measures are constructed as follows. Firstly, we define a function $\phi\left(.,\kappa\right) = \ln\left(\kappa \frac{f_Y^b(.)}{f_Y(.)}\right)$ where $\kappa \in \mathbb{R}^+$ and $f_Y^b(.)$ is a pdf on $[-1,\infty)$, eventually null on the same subdomain as $f_Y(.)$. Secondly, we introduce a $\mathcal{F}_t-$adapted and square integrable process, denoted by$(\theta_t)_{t\geq 0}$. Thirdly, we define a process $(Z_t)_{t\geq 0}$ as follows

$$Z_t \;=\; \exp\left(-\frac{1}{2}\int_0^t \theta_s^2 ds - \int_0^t \theta_s dW_s^P + \sum_{k=1}^{N_t^J} \phi\left(Y_k,\kappa\right) - (\kappa-1)\lambda_J t\right) \tag{3}$$

with $Z_0 = 1$. If we apply the Itô's lemma to Z_t, we immediately infer its infinitesimal dynamics:

$$dZ_t \;=\; Z_t\left(-\theta_t dW_t^P - (\kappa-1)\lambda_J dt + \left(\kappa \frac{f_Y^b(Y)}{f_Y(Y)} - 1\right) dN_t^J\right).$$

Given that $\mathbb{E}\left(\frac{f_Y^b(Y)}{f_Y(Y)}\right) = 1$, $\mathbb{E}\left(dZ_s|\mathcal{F}_t\right) = 0$ and $\mathbb{E}\left(Z_s|\mathcal{F}_t\right) = Z_t + \int_t^s \mathbb{E}\left(dZ_s|\mathcal{F}_t\right) = Z_t$. This proves that Z_t is a martingale. The process $(Z_t)_{t\geq 0}$ is a Radon-Nikodym derivative $Z_t = \left.\frac{dP^b}{dP}\right|_t$,that defines an equivalent probability measure, noted P^b. Under this measure, the process J_t still is a Poisson process but its frequency of jumps is equal to $\kappa\lambda_J$ whereas the jump pdf becomes $f_Y^b(.)$. Under P^b, the process $dW_t = dW_t^P + \theta_t dt$ is a Brownian motion. We refer the reader to e.g. Shreve (2004, chapter 11 section 6) for detailed explanations. The equivalent measure P^b is a risk neutral one, noted Q, if and only if discounted prices are martingales. If we denote expected jump size by $\xi^b = \mathbb{E}\left(Y^b\right)$, the following condition

$$r \;=\; \mu + \kappa\, \lambda_J\, \xi^b - \sigma_t \theta_t\,, \tag{4}$$

must be fulfilled to ensure that the equivalent measure is well a risk neutral one. As this condition is satisfied for an infinity of combinations of $(\theta_t)_{t\geq 0}$, κ and $f_Y^b(.)$, the risk neutral measure is not unique and the market is said incomplete. In practice, the risk neutral measure is chosen in order to replicate at best quoted prices of options.

In order to lighten notations in further developments, we assume without loss of generality that the distribution and frequency of jumps is identical under P and Q , ($\kappa = 1$ and $f_Y^b(.) = f_Y(.)$). In this case, the non-arbitrage condition imposes that $\theta_s = \frac{\mu+\lambda_J\xi-r}{\sigma_s}$. Under the risk neutral measure, the drift of the risky asset is equal to the risk free rate:

$$\frac{dA_t}{A_t} = (r - \lambda_J \xi)\, dt + \sigma_t\, dW_t + dJ_t. \tag{5}$$

We denote by $C(t,K)$ the value of a European call option paying the positive difference between the stock and strike prices at expiry (time t). According to the fundamental theorem of asset pricing, the value of this call option is the expected payoff under the risk neutral measure:

$$C(t,K) = \mathbb{E}^Q \left(e^{-rt} \left(A_t - K \right)_+ \right) .$$

Similarly, the European put option of maturity t and strike price K is equal to

$$D(t,K) = \mathbb{E}^Q \left(e^{-rt} \left(K - A_t \right)_+ \right) .$$

Call and Put prices are solution of a forward partial differential equation (PIDE) presented in the next proposition.

Proposition 2.1. *The call price is solution of the forward partial integro-differential equation (PIDE):*

$$\begin{aligned} &\frac{\partial C(t,K)}{\partial t} + (r-\lambda_J\xi)\, K \frac{\partial C(t,K)}{\partial K} - \frac{\mathbb{E}^Q \left(\sigma_t^2 \mid A_t = K \right) K^2}{2} \frac{\partial^2 C(t,K)}{\partial K^2} \\ &-\lambda_J \left(\mathbb{E}^Q \left((1+Y) C\left(t, \frac{K}{1+Y}\right) \right) - (1+\xi)\, C(t,K) \right) = 0. \end{aligned} \tag{6}$$

with the initial condition, $C(0,K) = (S_0 - K)_+$. The put price is solution of the same PIDE but with the initial condition $D(0,K) = (K - S_0)_+$.

The proof is recalled in Appendix and is similar to the one of Bergomi (2016) for a diffusion. Variants of this equation may be found in Andersen and Andreasen (2000),Carr et al. (2004), Cont and Voltchkova (2005) or Bentata and Cont (2010). In absence of jumps, Equation (6) is called the Dupire's equation (see Dupire, 1994). When the $\mathcal{F}_t$-adapted process σ_t is a function of time and of the asset price, $\sigma_t = \sigma(t, A_t)$, we say that the volatility is local. For example, in the constant elasticity volatility model (CEV), the volatility function is set to

$$\sigma(t, A_t)^2 = \sigma_0^2\, A_t^{2\gamma-2}\,, \tag{7}$$

where $\gamma \in \mathbb{R}$. The CEV process, introduced by Cox (1975), becames popular due to its ability to capture the implied volatility skew exhibited by options prices. A possible alternative local volatility with time dependence is:

$$\sigma(t, A_t)^2 = \beta_0 + \beta_1 \left(\frac{A_t}{F_t} \right)^{\gamma} + \beta_1 \left(\frac{A_t}{F_t} \right)^{2\gamma}\,, \tag{8}$$

where $F_t = A_0 e^{rt}$ is the forward stock price and where $\gamma \in \mathbb{R}^+$. This choice is motivated by the fact that in practice, implied volatilities reach their minimum when the strike is close or equal to the forward price ("at the money" options). Bergomi (2016) studies a similar local volatility function in a Brownian setting. In the remaining of this article, we explore the properties of a time-changed version of this model. The next section introduces the stochastic clocks that we use as time-changes.

3 Subordinators

A subordinator is a stochastic process noted $(U_t)_{t\geq 0}$, with positive, non-decreasing sample paths and taking value in $\mathbb{R}^+$. They are often used as stochastic clock for time-changed processes. We consider Lévy subordinators for which increments are independent and homogeneously distributed. We refer the reader to Applebaum (2004) for an introduction to Lévy processes. From the Lévy-Khintchine formula, we know that the Laplace transform of Lévy subordinators has the following form:

$$\mathbb{E}\left(e^{-\omega U_t}\right) = e^{-tf(\omega)} \tag{9}$$

where $f(\omega) = b\omega + \int_0^\infty \left(1 - e^{-\omega z}\right)\bar{\nu}(dz)$ and $b \in \mathbb{R}^+$. The function $\bar{\nu}(.)$ is a non-negative measure on $(0, \infty)$, referred to as the Lévy measure, satisfying the integrability condition

$$\int_0^\infty (z \wedge 1)\,\bar{\nu}(dz) < \infty\,.$$

The function $f(\omega)$ is also called a Bernstein function. For a detailed exposition of Bernstein functions, we refer to Schilling et al. (2010). The inverse of a subordinator is a process denoted by $(S_t)_{t\geq 0}$ that is defined as follows:

$$S_t = \inf\{\tau > 0 \,:\, U_\tau \geq t\}\,.$$

This is the time at which U_t hits the barrier t. This inverted Lévy subordinator is in general no more a Lévy process. However $(S_t)_{t\geq 0}$ is positive and non-decreasing and has all requisite properties to be used as stochastic clock. By construction, the inverted process may be constant. Therefore, any process subordinated by S_t exhibits motionless periods. This point is illustrated later in this section. The natural filtration of $(S_t)_{t\geq 0}$ is denoted by $(\mathcal{G}_t)_{t\geq 0}$. The probability density functions of U_t and S_t are respectively denoted by $p_U(t,u) = \frac{d}{d\tau}P(u \leq U_t \leq u + du)$ and $g(t,\tau) = \frac{d}{d\tau}P(\tau \leq S_t \leq \tau + d\tau)$. The survival function of S_t is $\bar{G}\,(t,s) = P\,(S_t > s)$ and by definition we have:

$$\bar{G}\,(t,s) \;=\; P\,(U_s < t)\,. \tag{10}$$

Toaldo (2015, lemma 3.1 and proposition 3.2) shows that the Laplace transform of $g(t,\tau)$ with respect to t, is linked to the Laplace exponent of U_t as follows:

$$\begin{aligned}\tilde{g}(\omega,\tau) &= \int_0^\infty e^{-\omega t} g(t,\tau)\,dt \\ &= \frac{f(\omega)}{\omega} e^{-\tau f(\omega)}\,.\end{aligned} \tag{11}$$

This relation plays an important role in later developments. Results in Section 4 and 5 are valid for all Lévy subordinators but we will grant more attention to inverse α stable and Poisson subordinators because they offer a certain level of analytical tractability.

Inverse α stable subordinators. In the α stable case, the process U_t is a $\frac{1}{\alpha}$ self-similar process, meaning that:

$$U_{at} \;\stackrel{d}{=}\; (at)^{\frac{1}{\alpha}}\, U_1\,.$$

This particular type of Lévy processes has has already been succesfully used in financial modelling e.g. in Carr and Wu (2003). It admits a simple moment generating function given by:

$$\mathbb{E}_0\left(e^{-uU_t}\right) \;=\; e^{-t\,u^\alpha}\,.$$

Since the Laplace exponent admits the following integral representation,

$$f(\omega) = \omega^\alpha = \frac{\alpha}{\Gamma(1-\alpha)}\int_0^\infty \frac{\left(1 - e^{-\omega x}\right)}{z^{1+\alpha}} dz\,,$$

the Lévy measure is $\bar{\nu}(dz) = \frac{\alpha}{\Gamma(1-\alpha)} \frac{dz}{z^{1+\alpha}}$. We infer that the Laplace transform of $g(t,\tau)$ with respect to t is given by:

$$\tilde{g}(\omega,\tau) = \omega^{\alpha-1} e^{-\tau\,\omega^{\alpha}} .$$

The Laplace's transform of S_t conditionally to the information available at time zero is given by :

$$\begin{aligned} \mathbb{E}_0\left[e^{-\omega S_t}\right] &= \int_0^{\infty} e^{-\omega\tau} g(t,\tau) d\tau \\ &= E_{\alpha}(-\omega t^{\alpha}) \end{aligned} \tag{12}$$

where E_α is the Mittag-Leffler function (for a proof see e.g. Piryatinska et al., 2005):

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+1)} , \tag{13}$$

where $\Gamma(.)$ is the gamma function. Equation (12) reveals that S_t is not a Lévy process since its Laplace's transform does not have an exponential form. The moments of S_t are obtained by deriving and cancelling its Laplace's transform:

$$\mathbb{E}_0\left(S_t^n\right) = \frac{n! t^{n\alpha}}{\Gamma\left(n\alpha+1\right)} .$$

A particular interesting case is when $\alpha = \frac{1}{2}$. Given that $\Gamma(\alpha) = \sqrt{\pi}$, the Lévy measure is equal to

$$\bar{\nu}(dz) = \frac{1}{2\sqrt{\pi}} \frac{dz}{z^{3/2}} .$$

The Laplace transform of the pdf of S_t has a simple expression

$$\tilde{g}(\omega,\tau) = \frac{e^{-\tau\,\sqrt{\omega}}}{\sqrt{\omega}} ,$$

that admits an analytical inverse. The probability density function of S_t for $\alpha = \frac{1}{2}$ is given by

$$g(t,\tau) = \frac{e^{-\frac{\tau^2}{4t}}}{\sqrt{\pi t}} \quad t \geq 0 ,$$

which is proportional to the density of a $N(0,\sqrt{2t})$ on the positive real line. If we denote the error function by $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, \mathrm{d}t$, the Mittag-Leffler exponential for $\alpha = \frac{1}{2}$ is equal to

$$E_{\frac{1}{2}}(z) = \exp(z^2)\mathrm{erfc}(-z) .$$

The first moment of S_t is :

$$\mathbb{E}_0\left(S_t\right) = \frac{t^{1/2}}{\Gamma\left(3/2\right)} = \frac{t^{1/2}}{\frac{1}{2}\sqrt{\pi}} .$$

The right plot of Figure 1 shows a simulated sample path of an α stable process and of its inverse for $\alpha = 0.90$. We clearly observe periods during which S_t is constant. They are caused by sharp increases of U_t.

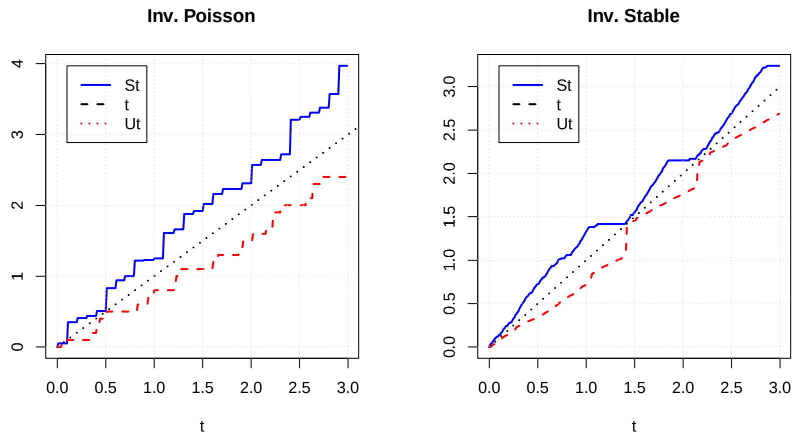

Figure 1: Left plot: sample path of an inverse Poisson subordinator ($\lambda = 10$). Right plot: sample path of an inverse α stable process ($\alpha = 0.90$).

Inverse Poisson subordinators. In this case, we invert a subordinator of the form:

$$U_t = \eta N_t \quad t, \mu \geq 0$$

where $(N_t)_{t\geq 0}$ is a Poisson process of parameter λ. We choose $\eta = \frac{1}{\lambda}$ in order to ensure that $\mathbb{E}\left(U_t\right) = t$. The variance of the subordinator is equal to ηt. The Laplace function of this subordinator is equal to:

$$\begin{aligned} \mathbb{E}\left(e^{-\omega U_t}\right) &= \mathbb{E}\left(e^{-\omega \eta N_t}\right) \\ &= e^{-tf(\omega\eta)}\,. \end{aligned}$$

where the Laplace exponent is a Bernstein function:

$$\begin{aligned} f(\omega\eta) &= \lambda(1 - e^{-\omega\eta}) \\ &= \int_0^\infty (1 - e^{-\omega z})\lambda \delta_{\{\eta\}}(z) dz\,. \end{aligned}$$

We denote by $\delta_{\{\eta\}}(z)$, the Dirac measure at point $z = \eta$. The Lévy measure is therefore $\bar{\nu}(dz) = \lambda\delta_{\{\eta\}}(z)dz$ and the Laplace's transform of the pdf of U_t is given by:

$$\tilde{g}(\omega, \tau) = \frac{\lambda(1 - e^{-\eta\,\omega})}{\omega} e^{-\tau\left(\lambda(1-e^{-\omega\eta})\right)}\,.$$

As N_t is a Poisson process, it takes its values in $\mathbb{N}$ and the survival function of S_t is equal to:

$$\begin{aligned} P(S_t \geq s) &= P(\eta N_s \leq t) \\ &= P(T_{\lfloor \frac{t}{\eta} \rfloor + 1} > s) \end{aligned}$$

where $T_{\lfloor \frac{t}{\eta} \rfloor + 1}$ is a gamma (or Erlang) random variable, $\Gamma\left(\left\lfloor \frac{t}{\eta} \right\rfloor + 1\,;\,\lambda\right)$. The pdf of S_t is therefore the following function:

$$g(t, \tau) = \frac{d}{d\tau} P(\tau \leq S_t \leq \tau + d\tau).$$

$$g(t, \tau) = P\left(\tau \leq S_t \leq \tau + d\tau\right) = \frac{\lambda^{\lfloor \frac{t}{\eta} \rfloor + 1} \tau^{\lfloor \frac{t}{\eta} \rfloor} \exp(-\lambda\tau)}{\left\lfloor \frac{t}{\eta} \right\rfloor !} \quad \tau \geq 0\,.$$

The Laplace transform is in this case

$$\begin{aligned}\mathbb{E}_0\left[e^{-\omega S_t}\right] &= \int_0^\infty e^{-\omega\tau} g(t,\tau)d\tau \\ &= \left(1+\frac{\omega}{\lambda}\right)^{-\left(\left\lfloor \frac{t}{\eta}\right\rfloor+1\right)}\end{aligned} \tag{14}$$

The variable $T_{\left\lfloor \frac{t}{\eta}\right\rfloor+1}$ is also the sum of $\left\lfloor \frac{t}{\eta}\right\rfloor+1$ exponential random variables, noted J_k, with parameter λ. The probability density function of J_k is equal to $f_J(x)=\lambda e^{-\lambda x}$. The process S_t^b defined as the sum of J_k:

$$S_t^b = \sum_{k=1}^{\left\lfloor \frac{t}{\eta}\right\rfloor+1} J_k\,, \tag{15}$$

has the same statistical distribution as S_t, for all time $t \geq 0$. Therefore, we can represent S_t by the sum (15) and we infer that the cumulative distribution function of $S_{t+\Delta}$, conditionally to the filtration $\mathcal{G}_t$ up to time t, is given by

$$P\left(S_{t+\Delta}\leq\tau\,|\,\mathcal{F}_t\right) = P\left(\sum_{k=1}^{\left\lfloor \frac{t+\Delta}{\eta}\right\rfloor+1} J_k \leq \tau\,|\, S_t=\sum_{k=1}^{\left\lfloor \frac{t}{\eta}\right\rfloor+1} J_k\right)$$

$$=\begin{cases}0 & S_t>\tau \\ P\left(T_{\left\lfloor \frac{t+\Delta}{\eta}\right\rfloor-\left\lfloor \frac{t}{\eta}\right\rfloor}\leq \tau-S_t\right) & \left\lfloor \frac{t+\Delta}{\eta}\right\rfloor > \left\lfloor \frac{\Delta}{\eta}\right\rfloor \; and\; S_t<\tau \\ 1 & \left\lfloor \frac{t+\Delta}{\eta}\right\rfloor = \left\lfloor \frac{t}{\eta}\right\rfloor \; and\; S_t<\tau\end{cases}$$

This also means that S_t is a stepwise function defined on a mesh with steps of size η: $S_t = S_{\left\lfloor \frac{t}{\eta}\right\rfloor\eta}\quad \forall t\in\mathbb{R}^+$. The process $(S_t)_{t\geq 0}$ is clearly not Markov since $P\left(S_{t+\Delta}\leq\tau\,|\,\mathcal{F}_t\right)$ depends upon the process value at time $\left\lfloor \frac{t}{\eta}\right\rfloor\eta\leq t$. The left plot of Figure 1 shows a simulated sample path of a Poisson process and of its inverse for $\lambda=10$. S_t is here an increasing stepwise function. Any process process time changed by an inverse Poisson subordinator has therefore piecewise constant functions.

4 The Dzerbayshan-Caputo derivatives

In order to introduce motionless phases in the dynamics of a jump-diffusion, we will consider a stochastic time scale ruled an inverse Lévy subordinator. We will see in Section 5 that option prices in this setting are solutions of a PIDE similar to Equation (6) in which the derivative with respect to time are replaced by a convolution-type derivative, called Dzerbayshan-Caputo (D-C) derivative. This section reviews the properties of this convolution-type derivative and its link with Lévy subordinators.

A Bernstein function is a function $f:(0,\infty)\to\mathbb{R}$ of class C^∞such that $f(x)\geq 0$ for all $x>0$ for which

$$(-1)^k f^{(k)}(x)\leq 0 \qquad \forall x>0\, k\in\mathbb{N}\,.$$

A Bernstein function also admits a similar representation to the Laplace exponent of a Lévy process:

$$f(x) = a+bx+\int_0^\infty\left(1-e^{-sz}\right)\bar{\nu}(dz) \tag{16}$$

where $a,b\geq 0$. $\bar{\nu}(.)$ is a positive Lévy measure on $(0,\infty)$. The triplet $(a,b,\bar{\nu})$ is the Lévy triplet of the Bernstein function. We denote by $\nu(s)$ the tail of the Lévy measure that is:

$$\nu(s)ds = \left(a+\int_s^\infty \bar{\nu}(dz)\right)ds\,.$$

Let us consider $f(.)$, a Bernstein function, and its tail Lévy measure $\nu(s)$ that is absolutely continuous on $(0,\infty)$. We also need a function $u(t)\in AC([0,\infty])$ that is the set of absolutely continuous function on $\mathbb{R}^{+}$. The generalized D-C derivative according to the Bernstein function $f(.)$ is defined as

$$ {}^{f}\mathcal{D}u(t) \quad = \quad b\,\frac{d}{dt}u(t)+\int_{0}^{t}\frac{\partial}{\partial t}u(t-s)\nu(s)ds \quad t\in[0,\infty)\,. \tag{17} $$

This integral is well defined if $|u(t)|\leq Me^{\omega_0 t}$ for some ω_0 and $M>0$. A direct calculation (see e.g. Lemma 2.2 in Toaldo (2015)) leads to the following Laplace transform of the D-C derivatives:

$$ \mathcal{L}\left[{}^{f}\mathcal{D}_t u(t)\right](\omega) \quad = \quad f(\omega)\,\tilde{u}(\omega)-\frac{f(\omega)}{\omega}\,u(0) \quad \mathcal{R}\omega\geq\omega_0 \tag{18} $$

where $\tilde{u}(\omega)$ is the Laplace transform of $u(t)$. We detail the form of ths D-C for α stable and Poisson subordinators.

α **stable subordinators**. Let us consider the α stable Lévy process, $(U_t)_{t\geq 0}$. The Lévy triplet is in this case $(0,0,\bar{\nu})$ where $\bar{\nu}(dz)=\frac{\alpha}{\Gamma(1-\alpha)}\frac{dz}{z^{1+\alpha}}$. The tail of the Lévy measure is given by:

$$ \begin{aligned} \nu(s)ds &= ds\int_{s}^{\infty}\frac{\alpha z^{-\alpha-1}}{\Gamma(1-\alpha)}dz \\ &= \frac{s^{-\alpha}}{\Gamma(1-\alpha)}ds\,. \end{aligned} $$

The D-C derivative becomes in this case the Caputo fractional derivative, that we denote by $\mathcal{D}_\alpha u(t)$. Indeed, if we perform the change of variable $s'=t-s$, we infer that

$$ \int_{0}^{t}\frac{\partial}{\partial t}u(t-s)\nu(s)ds=\int_{0}^{t}\frac{\partial}{\partial s'}u(s')\,\nu(t-s')\,ds' $$

and therefore

$$ {}^{f}\mathcal{D}u(t)=\mathcal{D}_\alpha u(t) \quad = \quad \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}\frac{u'(s)}{(t-s)^{\alpha}}ds\,. \tag{19} $$

Poisson subordinators. If $(U_t)_{t\geq 0}$ is a Poisson subordinator, the Lévy triplet is $(\mu,0,\bar{\nu})$ with $\bar{\nu}(dz)=\lambda\delta_{\{\eta\}}(z)dz$. The tail of the Lévy measure for $s\geq 0$ is:

$$ \begin{aligned} \nu(s)ds &= ds\int_{s}^{\infty}\lambda\delta_{\{\eta\}}(z)dz \\ &= \lambda\,\mathbb{I}_{\{s\leq\eta\}}\,ds \end{aligned} $$

where $\mathbb{I}_{\{s\leq\eta\}}$ is the indicator variable. The D-C derivative in this case is noted $\mathcal{D}_\lambda u(t)$ and given by

$$ {}^{f}\mathcal{D}u(t)=\mathcal{D}_\lambda u(t) \quad = \quad \lambda\int_{0}^{min(t,\eta)}\frac{\partial}{\partial t}u(t-s)\;ds \quad t\in[0,\infty)\,. \tag{20} $$

After a change of variable $s'=t-s$, we infer that

$$ \int_{0}^{min(t,\eta)}\frac{\partial}{\partial t}u(t-s)\;ds=u(t)-u(t-\min(t,\eta)) $$

and therefore

$$ \mathcal{D}_\lambda u(t) \quad = \quad \lambda\left(u(t)-u(t-\min(t,\eta))\right) \quad t\in[0,\infty)\,. \tag{21} $$

In the next section, we introduce a time-changed version of the financial market presented at the beginning of this article.

5 Fractional financial market

We recall that S_t is an inverse Lévy subordinator defined on $\left(\Omega, (\mathcal{G}_t)_{t\geq 0}, P\right)$ which is independent from the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of the asset price $(A_t)_{t\geq 0}$. In this section, We use $(S_t)_{t\geq 0}$ as stochastic clock and denote by $\mathcal{H}_t$ the augmented filtration $\mathcal{G}_t \vee \mathcal{F}_{S_t}$. This is the smallest filtration at the intersection of $\mathcal{G}_t$ and $\mathcal{F}_{S_t}$. The time-changed risk-free bond has a value at time t equal to:

$$B_{S_t} = e^{rS_t} \, , \, B_0 = 1. \tag{22}$$

Notice that in this framework, the bond return is now stochastic. The time-changed stock price is obtained by replacing the time by S_t in Equation (2):

$$A_{S_t} = A_0 \exp\left(\int_0^{S_t} \mu - \frac{1}{2}\sigma_s^2 ds + \int_0^{S_t} \sigma_s dW_s^P\right) \prod_{k=1}^{N_{S_t}^J} (1 + Y_k) \, . \tag{23}$$

The bond and stock values are also solutions of the time-changed stochastic differential equation:

$$\begin{aligned} \tfrac{dB_{S_t}}{B_{S_t}} &= r dS_t \, , \\ \tfrac{dA_{S_t}}{A_{S_t}} &= \mu \, dS_t + \sigma_{S_t} \, dW_{S_t}^P + dJ_{S_t} . \end{aligned}$$

Figures (2) and (3) show simulated sample paths of subordinated bond and stock prices when the time-change is an inverse α stable or Poisson subordinator. With an inverse α stable time-change, the sample path of the risky asset alternates between active and motionless phases. Whereas for an inverse Poisson time-change, the stock path has a piecewise constant sample path. These inverse subordinators describe two different kinds of illiquidity. Inverse Poisson processes are rather adapted to model recurring illiquidity at high or low frequency, depending upon the parameter λ. While the inverse α stable subordinator replicates temporary illiquidity periods for an asset that is most of the time actively traded.

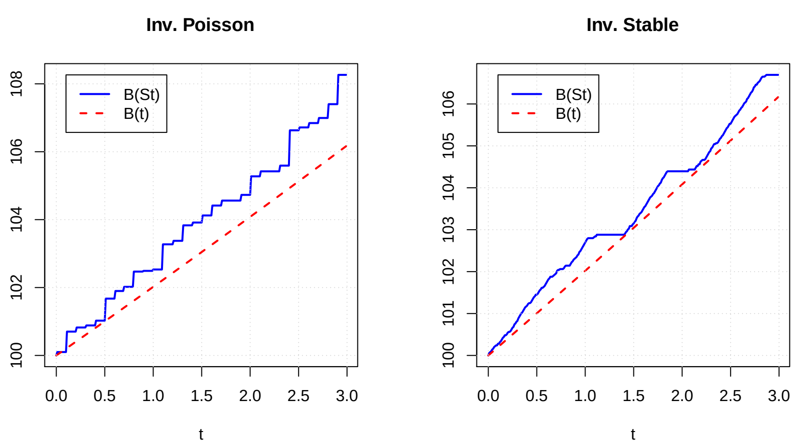

Figure 2: Left plot: sample path of B_{S_t} when S_t is an inverse Poisson subordinator ($\lambda = 10$). Right plot: sample path of B_{S_t} when S_t is an inverse α stable process ($\alpha = 0.90$). $r = 2\%$.

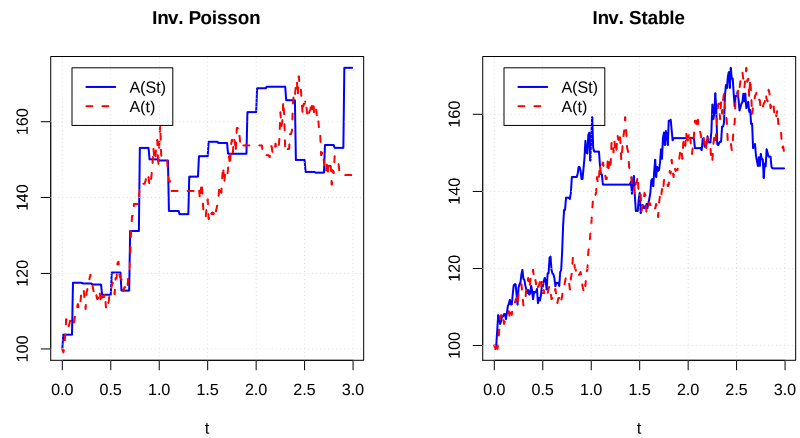

Figure 3: Left plot: sample path of A_{S_t} when S_t is an inverse Poisson subordinator ($\lambda = 10$). Right plot: sample path of A_{S_t} when S_t is an inverse α stable process ($\alpha = 0.90$). Here, $\mu = 5\%$, $\sigma_s = 15\%$ and no jump.

We have laid down the fractional dynamics under the real measure but need to determine a risk neutral measure in order to evaluate options. For this purpose, let us remember the definition of $\phi\left(.,\kappa\right) = \ln\left(\kappa \frac{f_Y^b(.)}{f_Y(.)}\right)$ where $\kappa \in \mathbb{R}^+$. For a $\mathcal{F}_t-$adapted and square integrable process, noted $(\theta_t)_{t\geq 0}$, we defined the time-changed process $(Z_{S_t})_{t\geq 0}$ as follows:

$$Z_{S_t} = \exp\left(-\frac{1}{2}\int_0^{S_t} \theta_s^2 ds - \int_0^{S_t} \theta_s dW_s^P + \sum_{k=1}^{N_{S_t}^J} \phi\left(Y_k,\kappa\right) - (\kappa-1)\lambda_J S_t\right), \tag{24}$$

which admits an equivalent infinitesimal representation:

$$dZ_{S_t} = -Z_{S_t}\theta_{S_t} dW_{S_t}^P - Z_{S_t}(\kappa-1)\lambda_J dS_t + Z_{S_t}\left(e^{\phi(Y,\kappa)}-1\right) dN_{S_t}^J .$$

We can check that $\mathbb{E}\left(dZ_t|\mathcal{H}_t \vee \mathcal{G}_T\right) = 0$ and therefore $\mathbb{E}\left(Z_T|\mathcal{H}_t \vee \mathcal{G}_T\right) = Z_t + \int_t^T \mathbb{E}\left(dZ_t|\mathcal{H}_t \vee \mathcal{G}_T\right) = Z_t$. Using nested expectations, the process Z_t is then a martingale with $Z_0 = 1$. Since Z_t is $\mathcal{H}_t$-adapted, it is a Radon-Nikodym derivative $Z_{S_t} = \left.\frac{dP^b}{dP}\right|_t$ defining a new measure P^b. The next proposition states the dynamics of Brownian motion and jump process under P^b.

Proposition 5.1. *Under the equivalent measure P_b defined by the Radon-Nykodym derivative (24),*

1. $dW_{S_t} = dW_{S_t}^P + \theta_{S_t} dS_t$ *is a time-changed Brownian motion.*

2. *the process J_{S_t} is a point process with an intensity equal to $\kappa\lambda_J\, dS_t$ and the pdf of jump is $f_Y^b(.)$.*

The proof of this result is reported in Appendix and mainly relies on the independence between filtrations of the time-change and of the Brownian and jump processes. We denote by $\xi_b = \mathbb{E}^{P_b}(Y)$ the expectation of a jump under the equivalent probability measure. From the previous proposition, we infer that the dynamics of the risky asset under P^b is given by

$$\frac{dA_{S_t}}{A_{S_t}} = \left(\mu - \theta_{S_t}\sigma_{S_t}\right) dS_t + \sigma_{S_t} dW_{S_t} + dJ_{S_t} .$$

Using again the independence between the clock and the filtration $(\mathcal{F}_t)_{t\geq 0}$, the expectation of the infinitesimal variation of A_t is such that

$$\mathbb{E}^{P_b}\left(\frac{dA_{S_t}}{A_{S_t}}|\mathcal{G}_t\right) = \left(\mu+\kappa\,\lambda_J\,\xi_b-\theta_{S_t}\sigma_{S_t}\right)\,dS_t\,.$$

Therefore, we obtain the conditions under κ, $f_Y(.)$ and $(\theta_{S_t})_{t\geq 0}$ defines a risk neutral measure.

Corollary 5.2. *The Radon-Nykodym derivative (24) defines an equivalent risk neutral measure Q, if ξ_b , κ and $(\theta_{S_t})_{t\geq 0}$ fulfill the following equality:*

$$\theta_{S_t} \quad = \quad \frac{\mu+\kappa\,\lambda_J\,\xi_b-r}{\sigma_{S_t}}\,. \tag{25}$$

To lighten notations in further developments, we assume without loss of generality that the frequency and size of jumps are identical under real and risk neutral measures ($\kappa=1$, $\xi_b=\xi$). The price of an European call option of maturity t, written on the time-changed asset is the discounted expected payoff under Q:

$$\begin{aligned} C_S(t,K) &= C(S_t,K) \\ &= \mathbb{E}^Q\left(e^{-rS_t}\left(A_{S_t}-K\right)_+|\mathcal{F}_0\right)\,. \end{aligned} \tag{26}$$

where S_t is a time-change. If we remember that the density of S_t is $g(t,\tau)d\tau = P(S_t\in[\tau,\tau+d\tau])$, we can rewrite the call option as an integral:

$$C_S(t,K) = \int_0^{\infty} C(\tau,K)\,g(t,\tau)d\tau\,. \tag{27}$$

If the non-fractional call and the density of S_t admits closed form expressions, we can evaluate the fractional call price by computing numerically the integral in the above equation. An alternative consists to calculate the price by Monte-Carlo simulations. These solutions are explored by Magdziarz (2009 a) in a Brownian setting. Unfortunately, call prices in a jump diffusion setting do not admit closed form expressions, excepted in the Merton model (1976). Another solution is provided by the next proposition.

Proposition 5.3. *The call option value in the fractional jump-diffusion setting is solution of a fractional PIDE equation:*

$$\begin{aligned} {}^f\mathcal{D}C_S(t,K) = &-\left(r-\lambda_J\xi\right)K\frac{\partial C_S(t,K)}{\partial K}+\frac{K^2}{2}\mathbb{E}^Q\left(\sigma^2_{S_t}\,|\,A_{S_t}=K\right) \\ &\times\frac{\partial^2 C_S(t,K)}{\partial K^2}+\lambda_J\left(\mathbb{E}^Q\left((1+Y)C_S\left(t,\frac{K}{1+Y}\right)\right)-(1+\xi)\,C_S(t,K)\right)\,. \end{aligned} \tag{28}$$

with the initial condition $C_S(0,K)=(S_0+K)_+$. The fractional put price, $D_S(t,K)=D(S_t,K)$, is solution of the same PIDE but with the initial condition $D(0,K)=(K-S_0)_+$.

The proof of this result is reported in Appendix. Notice that if $\sigma_t=\sigma(t,A_t)$ is a function of time and of the asset value, the conditional expectation in Equation (28) becomes

$$\begin{aligned} \mathbb{E}^Q\left(\sigma^2_{S_t}\,|\,A_{S_t}=K\right) &= \mathbb{E}^Q\left(\mathbb{E}^Q\left(\sigma^2_{S_t}\,|\,A_{S_t}=K\,,\,\mathcal{G}_t\right)\right) \\ &= \mathbb{E}^Q\left(\sigma^2(S_t,K)\right)\,. \end{aligned} \tag{29}$$

In the Black and Scholes (B&S) framework, the Brownian volatility is constant $\sigma^2(S_t, K) = \bar{\sigma}^2$ and there is no jumps. In this case, the fraction PIDE (28) can be rewritten as

$$ {}^f\mathcal{D}C_S(t,k) \quad = \quad -r\frac{\partial}{\partial k}C_S(t,k) + \frac{\bar{\sigma}^2}{2}\frac{\partial^2 C_S(t,k)}{\partial k^2}\,. \tag{30} $$

where $k = \ln(K)$. In case of an inverse α stable process, ${}^f\mathcal{D}C_S(t,k) = \mathcal{D}_\alpha C_S(t,k)$ and the fractional B&S call price is solution of

$$ \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\frac{\partial}{\partial s}C_S(s,K)}{(t-s)^\alpha}ds \quad = \quad -r\frac{\partial}{\partial k}C_S(t,k) + \frac{\bar{\sigma}^2}{2}\frac{\partial^2 C_S(t,k)}{\partial k^2}\,. \tag{31} $$

It the stochastic clock is an inverse Poisson subordinator, ${}^f\mathcal{D}C_S(t,k) = \mathcal{D}_\lambda C_S(t,k)$, the fractional B&S call price can be computed by iterating the following recursion:

$$ C_S(t,k) \quad = \quad C_S(\max(0;t-\eta),k) - \frac{r}{\lambda}\frac{\partial}{\partial k}C_S(t,k) + \frac{\bar{\sigma}^2}{2\lambda}\frac{\partial^2 C_S(t,k)}{\partial k^2} \quad t \geq 0\,. $$

In the constant elasticity volatility model (CEV), the expectation (29)is constant and equal to:

$$ \mathbb{E}^Q\left(\sigma(S_t,K)^2\right) \quad = \quad \sigma_0^2\,K^{2\gamma-2}\,. $$

If the Brownian volatility depends upon time as in Equation (8), the conditional expectation of the variance in Equation (28) is a function of the Laplace's transform of the time-change:

$$ \begin{aligned} \mathbb{E}\left(\sigma(S_t,K)^2\right) &= \beta_0 + \beta_1\mathbb{E}^Q\left(\left(\frac{K}{F_{S_t}}\right)^\gamma\right) + \beta_2\mathbb{E}^Q\left(\left(\frac{K}{F_{S_t}}\right)^{2\gamma}\right) \\ &= \beta_0 + \beta_1\left(\frac{K}{A_0}\right)^\gamma \mathbb{E}^Q\left(e^{-\gamma r S_t}\right) + \beta_2\left(\frac{K}{A_0}\right)^{2\gamma}\mathbb{E}^Q\left(e^{-2\gamma r S_t}\right)\,. \end{aligned} \tag{32} $$

If the clock is an inverse α stable process, this local volatility is a continuous function of time:

$$ \mathbb{E}^Q\left(\sigma(S_t,K)^2\right) \quad = \quad \beta_0 + \beta_1\left(\frac{K}{A_0}\right)^\gamma E_\alpha(-\gamma\,r\,t^\alpha) + \beta_2\left(\frac{K}{A_0}\right)^{2\gamma}E_\alpha(-2\,\gamma\,r\,t^\alpha)\,, $$

where $E_\alpha(.)$ is the Mittag-Leffler function, such as defined by Equation (13). For an inverse Poisson subordinator, the local volatility (8) is a stepwise function of time:

$$ \mathbb{E}^Q\left(\sigma(S_t,K)^2\right) \quad = \quad \beta_0 + \beta_1\left(\frac{K}{A_0}\right)^\gamma\left(1+\frac{\gamma\,r}{\lambda}\right)^{-\left(\left\lfloor\frac{t}{\eta}\right\rfloor+1\right)} + \beta_2\left(\frac{K}{A_0}\right)^{2\gamma}\left(1+\frac{2\,\gamma\,r}{\lambda}\right)^{-\left(\left\lfloor\frac{t}{\eta}\right\rfloor+1\right)}\,. $$

The next section proposes a numerical method for solving Equation (28).

6 Numerical framework

Andersen and Brotherton-Ratcliffe (1998) have demonstrated the reliability of a finite difference approach for solving the Dupire equation in a Brownian setting. We extend their framework to the fractional jump-diffusion and opt for an implicit method. We need to specify the distribution of jumps and the form of the local volatility. We do the common assumption that jumps are exclusively negative and then defined on $[-1,0]$. Furthermore, we consider a continuous pdf for Y. We also assume that the volatility is function of time and asset value: $\sigma_t = \sigma(t, A_t)$. Let us consider a domain $[0, t_{max}] \times [0, K_{max}]$ on which we wish to estimate fractional call prices. We choose two steps of discretization, noted Δ_t and Δ_K in order to define pairs (t_k, K_j) where

$$t_k = k\,\Delta_t \quad , \quad K_j = K_0 + j\,\Delta_K$$

for $k = 0, ..., n_t$ and $j = 0, ..., n_K$. The numbers n_t and n_K are integers equal to $n_t = \left\lfloor \frac{t_{max}}{\Delta_t} \right\rfloor$ and $n_K = \left\lfloor \frac{K_{max} - K_0}{\Delta_K} \right\rfloor$. To ligthen developments, we denote by $C_S(k, j)$ the approached value of $C_S(t_k, K_j)$. Under the assumption that $K_0 << A_0$, we have the following boundary conditions

$$\begin{aligned} C_S(0, j) &= (A_0 - K_j)_+ \, , \\ C_S(k, 0) &= \mathbb{E}^Q\left(e^{-rS_{t_k}} A_{S_{t_k}}\right) = A_0\,. \end{aligned}$$

We note $\sigma^2(k, j) = \mathbb{E}^Q\left(\sigma^2_{S_{t_k}} \,|\, A_{S_{t_k}} = K_j\right)$ while the $(n_t + 1) \times (n_K + 1)$ the matrix of variances is $\Sigma = (\sigma^2(k, j))_{k \in \{0,...,n_t\}, j \in \{0,...,n_K\}}$.

The first order derivative of $C_S(t, K)$ with respect to K in the right hand side of Equation (28) is approached by a central finite difference approximation:

$$\frac{\partial C_S(k, j)}{\partial K} \approx \frac{C_S(k, j+1) - C_S(k, j-1)}{2\Delta_K} \quad for\, 0 < j \leq n_K. \tag{33}$$

On the boundary, we set $\frac{\partial C_S(k,0)}{\partial K} = \frac{C_S(k,1) - C_S(k,0)}{\Delta_K}$ and $\frac{\partial C_S(k,n_K+1)}{\partial K} = \frac{C_S(k,n_K+1) - C_S(k,n_K)}{\Delta_K}$. The second order derivative is approached in the same way:

$$\frac{\partial^2 C_S(k, j)}{\partial K^2} \approx \frac{C_S(k, j+1) - 2C_S(k, j) + C_S(k, j-1)}{\Delta_K^2} \quad for\, 0 < j \leq n_K.$$

On lower and upper boundaries, we respectively set $\frac{\partial^2 C_S(k,0)}{\partial K^2} = \frac{\partial^2 C_S(k,1)}{\partial K^2}$ and $\frac{\partial^2 C_S(k,n_K+1)}{\partial K^2} = \frac{\partial^2 C_S(k,n_K)}{\partial K^2}$. In order to rewrite these partial derivatives in matrix form, we introduce $(n_K + 1) \times (n_K + 1)$ matrix R_1 and R_2 defined by:

$$R_1 = \frac{1}{\Delta_K} \begin{pmatrix} -1 & 1 & 0 & \dots & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} & \dots & 0 \\ 0 & -\frac{1}{2} & 0 & \frac{1}{2} & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & -1 & 1 \end{pmatrix} \quad R_2 = \frac{1}{\Delta_K^2} \begin{pmatrix} 1 & -2 & 1 & 0 & \dots & 0 \\ 1 & -2 & 1 & 0 & \dots & 0 \\ 0 & 1 & -2 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & \dots & 0 & 1 & -2 & 1 \end{pmatrix}.$$

The vector partial derivatives with respect to the strike at time t_k are then equal to following matrix products:

$$\frac{\partial C_S(k, .)}{\partial K} \approx \bar{K} R_1 C_S(k, .)^\top \quad , \quad \frac{\partial^2 C_S(k, .)}{\partial K^2} \approx R_2 C_S(k, .)^\top\,, \tag{34}$$

where $C_S(k, .)$ is the k^{th} line of the $(n_t + 1) \times (n_K + 1)$ matrix of call prices, denoted by C_S. The next step consists to discretize the continuous pdf of jumps. We denote by $y_m^{(j)} < 0$ the size of the jump for transiting from $K_m = \frac{K_j}{1 + y_m^{(j)}} > K_j$ to K_j . By definition, these $y_m^{(j)}$ are equal to

$$y_m^{(j)} \quad = \quad \frac{K_j}{K_m} - 1 = -\frac{(m - j)\,\Delta_k}{K_0 + m\Delta_k} \quad j = 0, ..., n_K\,,\; m = j, ..., n_k\,.$$

Notice that $y_m^{(j)}$ are ordered as follows: $y_{n_K}^{(j)} < ... < y_{m+1}^{(j)} < y_m^{(j)} < ... < y_j^{(j)} = 0$. The associated discrete probabilities of observing such jumps are:

$$\begin{aligned}
p_m^{(j)} &= P\left(Y \in \left[y_m^{(j)} - \frac{y_m^{(j)} - y_{m+1}^{(j)}}{2}\, ;\, y_m^{(j)} + \frac{y_{m-1}^{(j)} - y_m^{(j)}}{2}\right)\right) \quad j < m < n_K\,,\\
p_{n_K}^{(j)} &= P\left(Y \in \left(-\infty\, ;\, y_{n_K}^{(j)} + \frac{y_{n_K-1}^{(j)} - y_{n_K}^{(j)}}{2}\right)\right)\,,\\
p_j^{(j)} &= P\left(Y \in \left[\frac{y_{j+1}^{(j)}}{2}\, ;\, 0\right]\right)\,.
\end{aligned}$$

For a given $j \in \{0, ..., n_K\}$, we approximate the expectation related to the jump part in Equation (28) by the following sum:

$$\mathbb{E}^Q\left((1+Y)C_S\left(t_k, \frac{K_j}{1+Y}\right)\right) \approx \sum_{m=j}^{n_K} p_m^{(j)}\,(1 + y_m^{(j)})\,C_S\,(k, m)\,. \tag{35}$$

So as to rewrite this last expectation in matrix form, we denote by Y the $(n_K + 1) \times (n_K + 1)$ matrix of $y_m^{(j)}$:

$$Y = \begin{pmatrix} 0 & y_1^{(0)} & \dots & \dots & y_{n_K}^{(0)} \\ 0 & 0 & y_2^{(1)} & \dots & y_{n_K}^{(1)} \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & 0 & y_{n_K}^{(n_K-1)} \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix},$$

and by T, the $(n_K + 1) \times (n_K + 1)$ matrix of probabilities $T_{j,m} = p_m^{(j)}$ for $j \in \{0, ..., n_t\}$ and $m \in \{0, ..., n_K\}$:

$$T = \begin{pmatrix} p_0^{(0)} & p_1^{(0)} & \dots & \dots & p_{n_K}^{(0)} \\ 0 & p_1^{(1)} & p_2^{(1)} & \ddots & p_{n_K}^{(1)} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & 0 & p_{n_K-1}^{(n_K-1)} & p_{n_K}^{(n_K-1)} \\ 0 & \dots & \dots & 0 & p_{n_K}^{(n_K)} = 1 \end{pmatrix}.$$

The elementwise product (also called Hadamard product) of the matrix T and $(1 + Y)$ is denoted by $T \bullet (1 + Y)$. Using this notation allows us infer that

$$\left[\mathbb{E}^Q\left((1+Y)C_S\left(t_k, \frac{K_j}{1+Y}\right)\right)\right]_{k\in\{0,...,n_t\}, j\in\{0,...,n_K\}} = T \bullet (1+Y)\;\; C_S(k, .)^\top\,. \tag{36}$$

The Dzerbayshan-Caputo (D-C) derivative depends upon the chosen Bernstein function $f(.)$. Therefore there is no general procedure for approaching it numerically. In the two next subsections, we focus on D-C derivatives obtained with inverse α stable and Poisson subordinators.

Inverse α stable subordinators. In this case, the D-C derivative becomes the Caputo derivative and is numerically approached by the finite difference sum:

$$\begin{aligned}\mathcal{D}_\alpha C_S(k,j) &\approx \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{m=0}^{k-1}(k-m)^{-\alpha}\left(C_S(m+1,j)-C_S(m,j)\right) \qquad (37)\\ &= \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\left(C_S(k,j)-C_S(k-1,j)\right)+\\ &\quad \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\sum_{m=0}^{k-2}(k-m)^{-\alpha}\left(C_S(m+1,j)-C_S(m,j)\right).\end{aligned}$$

In order to rewrite this derivative under a matrix form, we define a matrix $D(k)$ of dimension $(k-1)\times k$ as follows:

$$D(k) = \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\begin{pmatrix} 0 & \dots & \dots & 0 & -(2)^{-\alpha} & (2)^{-\alpha} \\ \vdots & \iddots & 0 & -(3)^{-\alpha} & (3)^{-\alpha} & 0 \\ \vdots & \iddots & \iddots & \iddots & \iddots & \vdots \\ -(k)^{-\alpha} & (k)^{-\alpha} & \dots & \dots & \dots & 0 \end{pmatrix}. \qquad (38)$$

Therefore, The Caputo derivative at time t_k admits the following representation:

$$\mathcal{D}_\alpha C_S(k,.)^\top = \frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\left(C_S(k,.)-C_S(k-1,.)\right)^\top + \left(D(k)\,C_S(0:k-1\,,\,.)\right)^\top \mathbf{1}_{k-1}\,,$$

where $\mathbf{1}_{k-1}$ is a $(k-1)-$vector of ones. We denote by $\bar{K}=\mathrm{diag}\left(K_j\right)_{j\in\{0,1,\dots,n_K\}}$, the diagonal matrix of strikes and by $\bar{\Sigma}(k)=\mathrm{diag}\left(\Sigma(k,.)\right)$, the diagonal matrix of $\sigma(k,.)^2$. If we insert expressions (38) and (34) in the fractional Dupire's equation (28), we obtain its finite difference approximation:

$$\begin{aligned}&\frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}\left(C_S(k,.)-C_S(k-1,.)\right)^\top + \left(D(k)\,C_S(0:k-1\,,\,.)\right)^\top \mathbf{1}_{k-1}\\ &= \left(-(r-\lambda_J\xi)\,\bar{K}\,R_1 + \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2\right)C_S(k,.)^\top\\ &\quad +\lambda_J\left(T\bullet(1+Y)-(1+\xi)\,I_{n_K+1}\right)C_S(k,.)^\top\,.\end{aligned}$$

where I_{n_K+1} is the identity matrix. Finally, fractional call prices are computed iteratively from t_0 to t_{n_K} with the following recursion:

$$\begin{aligned}C_S(k,.)^\top &= \left[\frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}I_{n_K+1} + (r-\lambda_J\xi)\,\bar{K}\,R_1 - \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2\right. \qquad (39)\\ &\quad \left.-\lambda_J\left(T\bullet(1+Y)-(1+\xi)\,I_{n_K+1}\right)\right]^{-1}\times\\ &\quad \left[\frac{(\Delta_t)^{-\alpha}}{\Gamma(1-\alpha)}C_S(k-1,.)^\top - \left(D(k)\,C_S(0:k-1\,,\,.)\right)^\top\mathbf{1}_{k-1}\right].\end{aligned}$$

Inverse α stable subordinators. When the stochastic clock is the inverse Poisson subordinator and if η is a multiple of Δ_t (η/Δ_t in $\mathbb{N}$). The D-C derivative becomes in this case equal to

$$\mathcal{D}_\lambda C_S(k,.) = \begin{cases}\lambda\left(C_S(k,.)-C_S(0,.)\right) & k\,\Delta_t<\eta\\ \lambda\left(C_S(k,.)-C_S\left(k-\frac{\eta}{\Delta_t},.\right)\right) & k\,\Delta_t\geq\eta\end{cases}.$$

If η is not a multiple of Δ_t, $C_S\left(k-\frac{\eta}{\Delta_t},.\right)$ is computed by linear interpolation of nearest call prices. The finite difference version of fractional Dupire's equation (28) is in this case:

$$\lambda \left(C_S(k,.) - C_S\left(k - \frac{\eta}{\Delta_t},.\right)\right)^\top$$
$$= \left(-(r-\lambda_J \xi)\,\bar{K}\,R_1 + \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2\right) C_S(k,.)^\top$$
$$+\lambda_J \left(T \bullet (1+Y) - (1+\xi)\,I_{n_K+1}\right) C_S(k,.)^\top\,.$$

Fractional call prices are computed iteratively from t_0 to t_{n_K} with the following recursion for $k\,\Delta_t < \eta$:

$$\begin{aligned} C_S(k,.)^\top \;\; &= \;\; \Big[\lambda I_{n_K+1} + (r-\lambda_J\xi)\,\bar{K}\,R_1 - \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2 \\ &\quad -\lambda_J \left(T \bullet (1+Y) - (1+\xi)\,I_{n_K+1}\right)\Big]^{-1} \times \lambda\, C_S\,(0,.)\ , \end{aligned} \tag{40}$$

and for $k\,\Delta_t \geq \eta$:

$$\begin{aligned} C_S(k,.)^\top \;\; &= \;\; \Big[\lambda I_{n_K+1} + (r-\lambda_J\xi)\,\bar{K}\,R_1 - \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2 - \\ &\quad \lambda_J \left(T \bullet (1+Y) - (1+\xi)\,I_{n_K+1}\right)\Big]^{-1} \times \lambda\, C_S\left(k - \frac{\eta}{\Delta_t},.\right)\,. \end{aligned} \tag{41}$$

We conclude this section by presenting the recursion to estimate call prices in the non-fractional case. The approached solution of the Dupire Equation (6) may be obtained by the following recursion:

$$\begin{aligned} C_S(k,.)^\top \;\; &= \;\; \Big[I_{n_K+1} + (r-\lambda_J\xi)\,\Delta_t \bar{K}\,R_1 - \Delta_t \frac{\bar{K}^2}{2}\,\bar{\Sigma}(k)\,R_2 \\ &\quad -\lambda_J \Delta_t \left(T \bullet (1+Y) - (1+\xi)\,I_{n_K+1}\right)\Big]^{-1} \times C_S(k-1,.)^\top\,, \end{aligned}$$

for $k = 0, ..., n_t$ and $j = 0, ..., n_K$. In the next section, we test these numerical approximations and compare option prices in each of these cases.

7 Numerical illustration

We need to specify the statistical distribution of jumps. The size of a jump cannot exceed the current stock price otherwise the price would become negative. Therefore, we assume that $Y = e^Z - 1$ where Z is a negative exponential random variable of parameter $\rho > 0$, under the risk neutral measure. Clearly, $Y \in [0,1]$. The pdf of Z is $f_Z(z) = \rho\, e^{\rho\, z} I_{\{z \leq 0\}}$. A direct calculation yields that $P(Y \leq y) = (1+y)^\rho\, I_{\{y \in [0,1]\}}$ and its expectation is $\xi = \mathbb{E}(Y) = -\frac{1}{1+\rho}$. As the purpose of this section is to emphasize the impact of time-changing the dynamics of asset prices, we consider a constant volatility $\sigma_t = \bar{\sigma}$ but the numerical schemes proposed in Section 6 are well applicable with local volatility functions (7) and (8).

Figure 4 presents 1 year call prices obtained with inverse α stable and Poisson surbordinators for a range of strikes from $K = 50$ to 150. Market parameters are $r = 1\%$, $\lambda_J = 20$, $\bar{\sigma} = 25\%$, $\rho = 49$ and $A_0 = 100$. The number of steps of discretization are $n_t = 200$ and $n_K = 200$. The upper plot reveals that prices raises when α increases. The lower plot emphasizes that prices increases with λ.

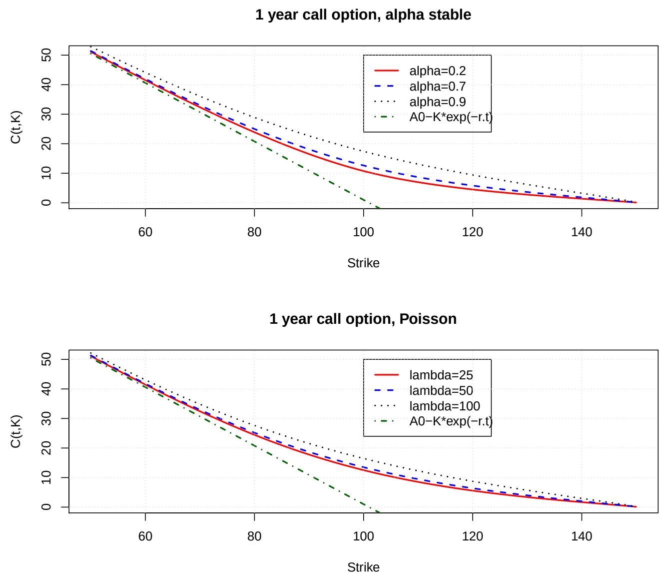

Figure 4: Prices of 1 year call option for varying strikes. Upper plot: inverse α stable subordinator. Lower plot: inverse Poisson subordinator.

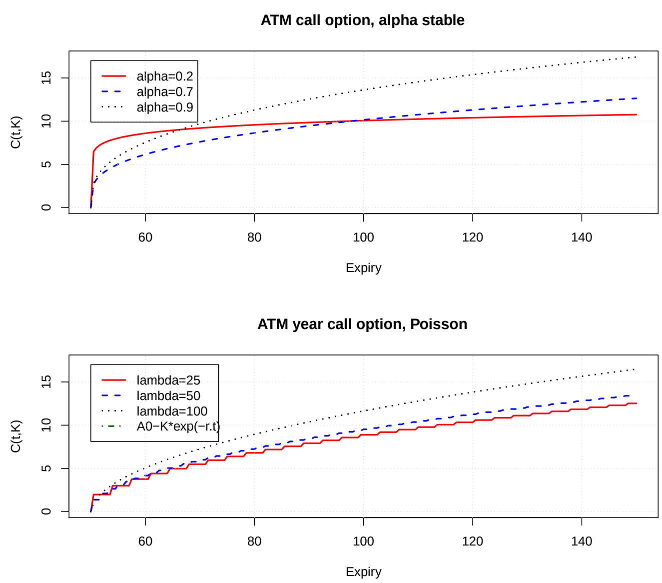

Figure 5: prices of “at the money” (ATM, $K = S_0$) call options for varying maturities. Upper plot: inverse α stable subordinator. Lower plot:inverse Poisson subordinator.

Figure 5 shows ATM call prices ($K = S_0$) for varying maturities. The upper plot reveals that increasing α reduces the concavity of the curve of prices with respect to time. The lower plot emphasizes firstly that call prices form a stepwise increasing function of expiry, for the inverse Poisson subordinator. The length of steps is inversely proportional to λ. Furthermore, increasing λ globally raises option prices.

To conclude this section, Figure 6 compares the implied volatility surfaces (obtained by inverting the Black & Scholes formula) in fractional and non-fractional cases. Here, we consider an inverse α stable subordinator with $\alpha = 0.8$. These plots reveals that time-changing the jump-diffusion leads to higher implicit volatilities than those obtained with a pure jump-diffusion. We also observe that implied volatilities are rather flat for ATM options whereas they decrease with the expiry in the fractional model.

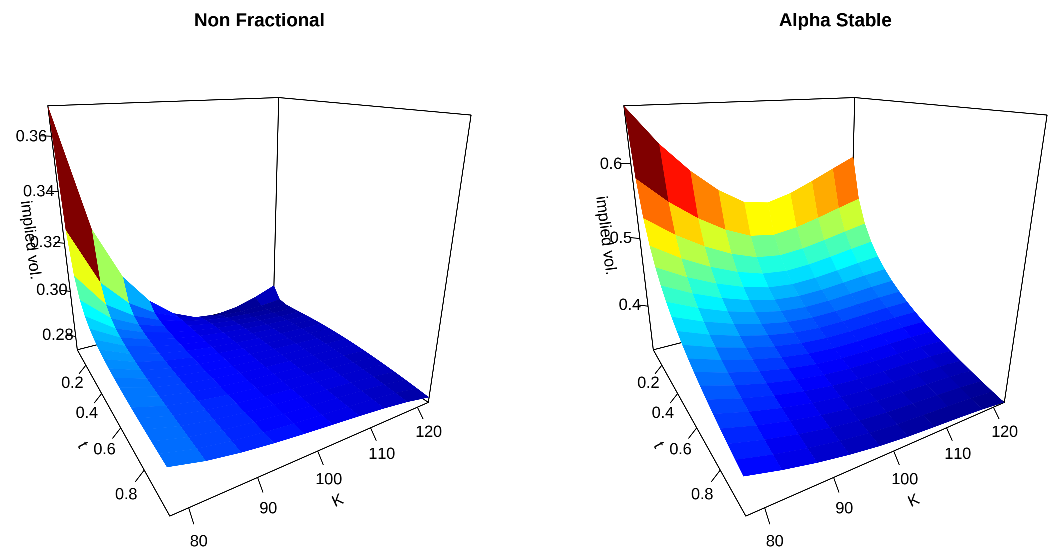

Figure 6: Left plot: Implied volatility surface for a jump-diffusion. Right plot: Implied volatility surface for the jump-diffusion subordinated by the inverse α stable subordinator.

8 Conclusions

Fractional processes are excellent candidates for modelling illiquidity in thestocks market. They are time-changed processes, ruled by an inverse Lévy subordinator. Within this approach, the sample path of prices alternates between active and motionless phases.

Nevertheless, the pricing of options in this framework remains a challenging task. This article explores a new approach based on a fractional version of the Dupire's equation for jump-diffusion. We establish a very general equation valid for all invertible Lévy subordinators.

As anillustration, we compare fractional dynamics based on inverted Poisson and α stable subordinators. For inverse Poisson subordinators, the stock path has piecewise constant sample paths. Inverse Poisson subordinators can model recurring illiquidity at high or low frequency. The inverse α stable subordinator has a different behavior and allows replicating temporary illiquidity periods for an asset that is most of the time actively traded.

Finally, we propose a numerical method to solve the fractional Dupire equation. The numerical illustration reveals that fractional and non-fractional version of the same jump-diffusion often leads to a very different structure of prices and implied volatilities.

Appendix

Proof of proposition 2.1. The Heaviside function, noted $\upsilon(x)$, is defined as:

$$\upsilon(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases},$$

and we denote the Dirac delta function by $\delta(.)$. Using the Itô-Tanaka's lemma for $e^{-rt}\left(A_t - K\right)_+$, we immediately obtain that:

$$\begin{aligned} d\left(e^{-rt}\left(A_t - K\right)_+\right) &= e^{-rt}\,\upsilon\left(A_t - K\right)\left(\left(r - \lambda_J \xi\right) A_t dt + \sigma_t A_t dW_t\right) \\ &\quad - re^{-rt}\left(A_t - K\right)_+ dt + \frac{1}{2} e^{-rt}\delta\left(A_t - K\right)\sigma_t^2 A_t^2 dt \\ &\quad + e^{-rt}\left(\left(A_t(1+Y) - K\right)_+ - \left(A_t - K\right)_+\right) dN_t^J\,. \end{aligned} \tag{42}$$

The next step consists to calculate the expectation of this infinitesimal variation. By definition of the call price, we have the useful relations:

$$\begin{aligned} \mathbb{E}^Q\left(e^{-rt}\upsilon\left(A_t - K\right)\right) &= -\frac{\partial C(t,K)}{\partial K}\,, \\ \mathbb{E}^Q\left(e^{-rt}\delta\left(A_t - K\right)\right) &= \frac{\partial^2 C(t,K)}{\partial K^2}\,. \end{aligned} \tag{43}$$

On the other hand, the call price can be rewritten with the Heaviside function as follows:

$$C(t,K) = \mathbb{E}^Q\left(e^{-rt}A_t\,\upsilon(A_t - K)\right) - K\,\mathbb{E}^Q\left(e^{-rt}\,\upsilon(A_t - K)\right)\,.$$

From this last equation, the discounted expectation of the product A_t and the Heaviside function is therefore equal to:

$$\mathbb{E}^Q\left(e^{-rt}A_t\,\upsilon(A_t - K)\right) = C(t,K) - K\frac{\partial C(t,K)}{\partial K}\,. \tag{44}$$

Given that jumps are independent from N_t^J, we have that

$$\begin{aligned} &\mathbb{E}^Q\left(e^{-rt}\left(\left(A_t(1+Y) - K\right)_+ - \left(A_t - K\right)_+\right) dN_t^J\right) \\ &= \lambda_J\left(\mathbb{E}^Q\left((1+Y)C\left(t, \frac{K}{1+Y}\right)\right) - C(t,K)\right) dt\,. \end{aligned} \tag{45}$$

Combining Equations (42), (43), (44) and (45) leads to the following forward equation:

$$\begin{aligned} \frac{\partial C(t,K)}{\partial t} &= -\left(r - \lambda_J \xi\right) K\frac{\partial C(t,K)}{\partial K} + \frac{K^2}{2} e^{-rt}\mathbb{E}^Q\left(\delta\left(A_t - K\right)\sigma_t^2\right) \\ &\quad + \lambda_J\left(\mathbb{E}^Q\left((1+Y)C\left(t, \frac{K}{1+Y}\right)\right) - (1+\xi)\,C(t,K)\right)\,. \end{aligned} \tag{46}$$

The expected variance of the Brownian term, conditioned by the asset value is equal to

$$\mathbb{E}^Q\left(\sigma_t^2 \,|\, A_t = K\right) = \frac{e^{-rt}\mathbb{E}^Q\left(\delta\left(A_t - K\right)\sigma_t^2\right)}{e^{-rt}\mathbb{E}^Q\left(\delta\left(A_t - K\right)\right)}\,.$$

Since $e^{-rt}\mathbb{E}^Q\left(\delta\left(A_t - K\right)\right) = \frac{\partial^2 C(t,K)}{\partial K^2}$, the PIDE (46) becomes Equation (6). The same reasoning holds for a put option.

$\blacksquare$

Proof of proposition 5.1. 1). Let us denote by $(\mathcal{V}_t)_{t\geq 0}$ the subfiltration of $(\mathcal{F}_t)_{t\geq 0}$ carrying exclusively information about the jump process. Using nested expectations, the moment generating function (mgf) of W_t under the equivalent measure is:

$$\mathbb{E}^{P^b}\left(e^{\omega W_{S_t}}|\mathcal{H}_0\right) = \mathbb{E}^{P^b}\left(e^{\omega W^P_{S_t}+\omega\int_0^{S_t}\theta_s ds}e^{\sum_{k=1}^{N^J_{S_t}}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J S_t}|\mathcal{H}_0\right)$$
$$= \mathbb{E}\left(e^{\sum_{k=1}^{N^J_{S_t}}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J S_t}\mathbb{E}\left(e^{\omega W^P_{S_t}+\omega\int_0^{S_t}\theta_s ds}|\mathcal{H}_0\vee\mathcal{V}_{S_t}\vee\mathcal{G}_t\right)|\mathcal{H}_0\right).$$

Since the Brownian motion is independent from the filtration of the jump process and time-change, we have that

$$\mathbb{E}\left(e^{\omega W^P_{S_t}+\omega\int_0^{S_t}\theta_s ds}|\mathcal{H}_0\vee\mathcal{V}_{S_t}\vee\mathcal{G}_t\right)$$
$$= e^{\frac{1}{2}\omega^2 S_t}\mathbb{E}\left(e^{-\frac{1}{2}\int_0^{S_t}(\theta_s-\omega)^2 ds-\int_0^{S_t}(\theta_s-\omega)dW^P_s}|\mathcal{H}_0\vee\mathcal{V}_{S_t}\vee\mathcal{G}_t\right)$$
$$= e^{\frac{1}{2}\omega^2 S_t}.$$

To pass from the second to last to the last line, we use the property that the Doleans-Dade exponential of a martingale is a martingale. We recognize the mgf of a Brownian motion, time-changed by S_t. Since, $e^{\sum_{k=1}^{N^J_t}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J t}$ is a martingale, nesting expectations leads to

$$\mathbb{E}^{P^b}\left(e^{\omega W_{S_t}}|\mathcal{H}_0\right) = \mathbb{E}\left(e^{\frac{1}{2}\omega^2 S_t}\mathbb{E}\left(e^{\sum_{k=1}^{N^J_{S_t}}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J S_t}|\mathcal{H}_0\vee\mathcal{G}_t\right)|\mathcal{H}_0\right)$$
$$= \mathbb{E}\left(e^{\frac{1}{2}\omega^2 S_t}|\mathcal{H}_0\right).$$

Therefore, we conclude that $W_{S_t} = W^P_{S_t} + \int_0^{S_t}\theta_s ds$ is a time-changed Brownian motion under P^b.

2) Using nested expectations, the mgf of the time-changed jump process under the measure P^b may be rewritten as follows:

$$\mathbb{E}^{P^b}\left(e^{\omega J_{S_t}}|\mathcal{H}_0\right) = \mathbb{E}\left(\mathbb{E}\left(e^{\omega J_{S_t}+\sum_{k=1}^{N^J_{S_t}}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J S_t}|\mathcal{H}_0\vee\mathcal{G}_t\right)|\mathcal{H}_0\right)$$

Given that jump sizes are independent from the number of jumps, we rewritte this last equation as:

$$\mathbb{E}\left(e^{\omega J_{S_t}+\sum_{k=1}^{N^J_{S_t}}\phi(Y_k,\kappa)-(\kappa-1)\lambda_J S_t}|\mathcal{H}_0\vee\mathcal{G}_t\right)$$
$$= e^{-(\kappa-1)\lambda_J S_t}\mathbb{E}\left(\left(\mathbb{E}\left(e^{\omega Y+\phi(Y,\kappa)}\right)\right)^{N^J_{S_t}}|\mathcal{H}_0\vee\mathcal{G}_t\right).$$

On one hand, we have that

$$\mathbb{E}\left(e^{\omega Y+\phi(Y,\kappa)}\right) = \int_{\mathbb{R}}\kappa e^{\omega y}f^b_Y(y)\,dy$$
$$= \kappa\mathbb{E}\left(e^{\omega Y}\right).$$

where Y has here the density $f^b_Y(.)$. On the other hand, we know that the moment generating function of a compound Poisson process is equal to:

$$\mathbb{E}\left(\exp\left(\omega J_t\right)\right) = \exp\left(\lambda_J t\left(\mathbb{E}\left(e^{\omega Y}\right)-1\right)\right).$$

The mgf of the jump process is therefore equal to:

$$\begin{aligned}
\mathbb{E}^{P^b}\left(e^{\omega J_t}|\mathcal{H}_0\right) &= \mathbb{E}\left(e^{-(\kappa-1)\lambda_J S_t}\mathbb{E}\left(\left(\kappa\mathbb{E}\left(e^{\omega Y}\right)\right)^{N_t^J}|\mathcal{H}_0\vee\mathcal{G}_t\right)|\mathcal{H}_0\right)\\
&= \mathbb{E}\left(e^{-(\kappa-1)\lambda_J\, S_t}\,\mathbb{E}\left(e^{N_t^J\,\ln\left(\kappa\mathbb{E}\left(e^{\omega Y}\right)\right)}|\mathcal{H}_0\vee\mathcal{G}_t\right)|\mathcal{H}_0\right)\\
&= \mathbb{E}\left(e^{-(\kappa-1)\lambda_J\, S_t}\,e^{\lambda_J\,t\left(\kappa\mathbb{E}\left(e^{\omega Y}\right)-1\right)}|\mathcal{H}_0\right)\\
&= \mathbb{E}\left(e^{\kappa\lambda_J\,S_t\,\left(\mathbb{E}\left(e^{\omega Y}\right)-1\right)}\,|\mathcal{H}_0\right)
\end{aligned}$$

and we recognize the mgf of a time-changed Poisson process with a intensity $\kappa\lambda_J$ and jump density $f_Y^b(.)$.

■

Proof of proposition 5.3. The Laplace's transform of $C_S(t,K)$ with respect to time t is equal to

$$\begin{aligned}
\tilde{C}_S(\omega,K) &= \int_0^\infty e^{-\omega t}\int_0^\infty C(\tau,K)\,g(t,\tau)d\tau\,dt \qquad (47)\\
&= \int_0^\infty C(\tau,K)\,\tilde{g}(\omega,\tau)\,d\tau\,,
\end{aligned}$$

where $\tilde{g}(\omega,\tau)$ is the Laplace transform, $\int_0^\infty e^{-\omega t}g(t,\tau)dt$, of the density of S_t. On the other hand, we have that:

$$\begin{aligned}
&\mathbb{E}\left((1+Y)\widetilde{C_S}\left(t,\frac{K}{1+Y}\right)\right)\\
&=\int_0^\infty e^{-\omega t}\int_{-\infty}^{+\infty}(1+y)C_S\left(t,\frac{K}{1+y}\right)f_Y(y)dy\,dt\\
&=\int_{-\infty}^{+\infty}(1+y)\int_0^\infty e^{-\omega t}\int_0^\infty C\left(\tau,\frac{K}{1+y}\right)\,g(t,\tau)\,d\tau\,dt\,f_Y(y)\,dy\\
&=\mathbb{E}^Q\left((1+Y)\tilde{C}_S\left(\omega,\frac{K}{1+Y}\right)\right)\,.
\end{aligned}$$

Let us adopt momentarily the following notations:

$$\begin{aligned}
h(t,K) &:= \mathbb{E}^Q\left(\sigma_t^2\,|\,A_t=K\right)\frac{\partial^2 C(t,K)}{\partial K^2}\,,\\
h_S(t,K) &:= \mathbb{E}^Q\left(\sigma_{S_t}^2\,|\,A_{S_t}=K\right)\frac{\partial^2 C(S_t,K)}{\partial K^2}\,.
\end{aligned}$$

The Laplace transform of $h(t,K)$ is equal to

$$\begin{aligned}
\tilde{h}_S(\omega,K) &= \int_0^\infty e^{-\omega t}\int_0^\infty h(\tau,K)\,g(t,\tau)d\tau\,dt\\
&= \int_0^\infty h(\tau,K)\,\tilde{g}(\omega,\tau)\,d\tau\,.
\end{aligned}$$

We have seen in Section 3 that the Laplace transform of $g(t,\tau)$ with respect to t, is related to the Laplace exponent $f(.)$ of U_t by Equation (11). Combining this expression with Equation (47) gives us

$$\begin{aligned}
\tilde{C}_S(\omega,K) &= \frac{f(\omega)}{\omega}\tilde{C}(f(\omega),K)\,,\\
\tilde{h}_S(\omega,K) &= \frac{f(\omega)}{\omega}h_S(f(\omega),K)\,,
\end{aligned}$$

and

$$\mathbb{E}^Q\left((1+Y)\widetilde{C_S\left(t, \frac{K}{1+Y}\right)}\right) = \frac{f(\omega)}{\omega}\mathbb{E}^Q\left((1+Y)\tilde{C}\left(f(\omega), \frac{K}{1+Y}\right)\right).$$

Hence, from the FPE (6) we deduce that $\tilde{C}(\omega, K)$ is solution of the following equation:

$$\begin{aligned}\omega\tilde{C}(\omega,K) - C(0,K) &= -\left(r-\lambda_J\xi\right)K\frac{\partial}{\partial K}\tilde{C}(\omega,K) + \frac{K^2}{2}\tilde{h}(\omega,K)\\ &+\lambda_J\left(\mathbb{E}^Q\left((1+Y)\tilde{C}\left(\omega, \frac{K}{1+Y}\right)\right) - (1+\xi)\ \tilde{C}(\omega,K)\right).\end{aligned}$$

As $\tilde{C}_S(\omega, K) = \frac{f(\omega)}{\omega}\tilde{C}(f(\omega), K)$, replacing ω by $f(\omega)$ leads to

$$\begin{aligned}&f(\omega)\tilde{C}(f(\omega),K) - C(0,K) =\\ &\quad -\left(r-\lambda_J\xi\right)K\frac{\partial}{\partial K}\tilde{C}(f(\omega),K) + \frac{K^2}{2}\tilde{h}(f(\omega),K)\\ &\quad +\lambda_J\left(\mathbb{E}^Q\left((1+Y)\tilde{C}\left(\omega, \frac{K}{1+Y}\right)\right) - (1+\xi)\ \tilde{C}(f(\omega),K)\right).\end{aligned}$$

Multiplying this last equation by $\frac{f(\omega)}{\omega}$ and since $C_S(0, K) = C(0, K)$, we infer that

$$\begin{aligned}&f(\omega)\tilde{C}_S(\omega,K) - \frac{f(\omega)}{\omega}C_S(0,K) =\\ &\quad -\left(r-\lambda_J\xi\right)K\frac{\partial}{\partial K}\tilde{C}_S(\omega,K) + \frac{K^2}{2}\tilde{h}_S(\omega,K)\\ &\quad +\lambda_J\left(\mathbb{E}^Q\left((1+Y)\tilde{C}\left(\omega, \frac{K}{1+Y}\right)\right) - (1+\xi)\ \tilde{C}_S(\omega,K)\right),\end{aligned}$$

which is well the Laplace transform of Equation (28).

■

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References

[1] Acay B., Bas E., Abdeljawad T., 2020 a. Fractional economic models based on market equilibrium in the frame of different type kernels. Chaos, Solitons & Fractals, 130, 109438.

[2] Acay B., Bas E., Abdeljawad T. 2020 b. Non-local fractional calculus from different viewpoint generated by truncated M-derivative. Journal of Computational and Applied Mathematics, 366, 112410.

[3] Andersen L., Andreasen J. 2000. Jump diffusion models: Volatility smile fitting and numerical methods for pricing, Review of Derivatives Research, 4, 231-262.

[4] Andersen L., Brotherton-Ratcliffe R., 1998. The equity option volatility smile: a finite difference approach. Journal of Computational Finance 1(2), 5–38.

[5] Appelbaum D. 2004. Lévy processes and stochastic calculus. Cambridge studies in advanced mathematics.

[6] Barkai E., Metzler R., Klafter J. 2000. From continuous time random walks to the fractional Fokker-Planck equation. Physical Review E, 61, 132–138.

[7] Bentata, A. and Cont, R. 2010, Forward equations for option prices in semimartingale models. Preprint, available at arxiv.org/abs/1001.

[8] Bergomi L., 2016. Stochastic volatility modeling. Chapman & Hall CRC financial mathematics series.

[9] Carr, P., Geman, H., Madan, D.P. and Yor, M., From local volatility to local Lévy models. Quant. Finance, 2004, 4, 581–588.

[10] Carr P., Wu L. 2003. The Finite Moment Log Stable Process and Option Pricing. The journal of Finance, 58 (2), 753-777.

[11] Cont, R. and Voltchkova, E., Integro-differential equations for option prices in exponential Lévy models. Finance Stoch., 2005, 9, 299– 325.

[12] Cox, J. C. 1975 (December 1996). Notes on option pricing I: constant elasticity of variance diffusions. Reprinted in The Journal of Portfolio Management, 23, 15–17.

[13] Dupire B., 1994. Pricing with a smile. Risk, 7, 18–20.

[14] Eliazar I., Klafter J. 2004. Spatial gliding, temporal trapping and anomalous transport. Physica D, 187, 30–50.

[15] Friz P.K., Gerhold S., Yor M., 2014. How to make Dupire's local volatility work with jumps. Quantitative finance 14(8) , 1327-1331.

[16] Hainaut D., 2020 a. Fractional Hawkes processes. Physica A: Statistical Mechanics and its Applications, 549, 124330.

[17] Hainaut D., 2020 b. Credit risk modelling with fractional self-excited processes. ISBA-UCLouvain discussion paper.

[18] Leonenko N., Meerschaert M., Sikorskii A. 2013 (a). Fractional Pearson diffusions. Journal of Mathematical Analysis and Applications, 403, 532-546.

[19] Leonenko N., Meerschaert M., Sikorskii A. 2013 (b). Correlation structure of fractional Pearson diffusions. Computers and Mathematics with Applications, 66, 737-745.

[20] Magdziarz M. 2009 (a). Black-Scholes formula in subdiffusive regime. Journal of Statistical Physics, 136, 553-564.

[21] Magdziarz M. 2009 (b). Stochastic representation of subdiffusion processes with time-dependent drift. Stochastic Processes and their Applications, 119, 3238-3252.

[22] Merton, R.C. 1976. Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics, 3, 125-144.

[23] Metzler R., Klafter J. 2004. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. Journal of Physics A: Mathematical and General 37 (31), R161.

[24] Schilling R.L., Song R., Vondracek Z. 2010. Bernstein functions. Theory and applications. Walter de Gruyter, Berlin.

[25] Schreve S.E., 2004. Stochastic calculus for finance II: continuous-time models. Springer Finance.

[26] Piryatinska A., Saichev A. I., Woyczynski W. A. 2005. Models of anomalous diffusion: the subdiffusive case. Physica A: Statistical Mechanics and its Applications, 349 (3), 375-420.

[27] Scalas E. 2006. Five years of continuous-time random walks in econophysics, in: A. Namatame, T. Kaizouji, Y. Aruka (Eds.), The Complex Networks of Economic Interactions, Springer, New York, pp. 3–16.

[28] Toaldo B. 2015. Convolution-type derivatives, hitting-times of subordinators and time-changed C_0-semigroups. Potential Analysis, 42, 115-140.