Embedding K_5 and $K_{3,3}$ on orientable surfaces

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Abstract

The Kuratowski graphs K_5 and $K_{3,3}$ are fundamental non-planar graphs. We are interested in obtaining all their distinct 2-cell embeddings on orientable surfaces. The 2-cell embeddings of K_5 and $K_{3,3}$ on the torus are well-known. Using a constructive approach of expanding from minors, we obtain all 2-cell embeddings of these graphs on the double torus. As a consequence, several new polygonal representations of the double torus are described. Rotation systems for the one-face embeddings of K_5 on the triple torus are also found, using an exhaustive search approach.

Keywords : *non-planar graphs, topological surfaces,* 2*-cell embeddings, enumeration.*

1 Introduction

A graph G is embeddable on a surface if it can be drawn on the surface with no crossing edges. Such a drawing of G on the surface is called an embedding. An embedding is 2-cell (cellular) if its every face is homeomorphic to an open disk. The distinct (non-isomorphic) embeddings of the non-planar Kuratowski graphs K_5 and $K_{3,3}$ on the torus are well-known [4]. We want to find explicitly all their distinct 2-cell embeddings on the double torus, and all the distinct 2-cell embeddings of K_5 on the triple torus. This can serve as a first step in studying graphs embeddable on the double torus and orientable topological surfaces of higher genus.

The number of 2-cell embeddings of K_5 and $K_{3,3}$ on orientable surfaces was previously determined in [6] and [5] using Burnside's Lemma and automorphism groups of these graphs. We use a constructive approach to actually find the embeddings and to determine their orientability. By Euler's formula, the maximum orientable genus of $K_{3,3}$ and K_5 are two and three, and their orientable genus spectra are $\{1,2\}$ and $\{1,2,3\}$, respectively. 2-cell embeddings of $K_{3,3}$ and K_5 on the double torus must have respectively one and three faces, and a 2-cell embedding of K_5 on the triple torus must have only one face.

We consider some auxiliary graphs that can have parallel edges (i.e., be multi-graphs), but no loops. A 2-cell embedding of a graph G on an orientable surface is characterized by its *rotation system*. Given a labelling of the vertices and edges, a rotation system consists of a cyclic list of the incident edges for each vertex v, called the *rotation* at v. The rotation system uniquely determines the facial boundaries, and therefore the embedding of G on the surface. However, it does not determine a drawing on a polygonal representation of the surface. If τ is a rotation system for an embedding of G, we denote the embedding by G^τ. Two embeddings G^{τ_1} and G^{τ_2} are isomorphic if there is a permutation of the vertices $V(G)$ and of the edges $E(G)$ that transforms τ_1 into τ_2. See [4] for more details.

We denote by Θ_m a multi-graph consisting of two vertices $\{u,v\}$ and a set of m parallel edges between them ($m \geq 1$). We first construct all 2-cell embeddings of an auxiliary graph Θ_5 on the double torus, and then derive all possible embeddings of K_5 and $K_{3,3}$ on this surface by expanding Θ_5 and some other minors of K_5 and $K_{3,3}$ back to the original graphs. Based on these results, we provide different polygonal representations of the double torus. We also find all the rotation systems for distinct 2-cell embeddings of K_5 on the triple torus.

2 On the double torus

Three 2-cell embeddings of Θ_5 on the double torus are shown in Fig. 1. Here the double torus is represented by a standard octagon $a^+b^+a^-b^-c^+d^+c^-d^-$, traversed clockwise, with paired sides $\{a, b, c, d\}$. See [1, 2, 4] for more information on representations of the double torus. Each of these embeddings of Θ_5 has exactly one face homeomorphic to an open disc, with its facial boundary consisting of 10 edges. We prove that this list of 2-cell embeddings is complete.

Theorem 1 *There are exactly three distinct 2-cell embeddings of Θ_5 on the double torus.*

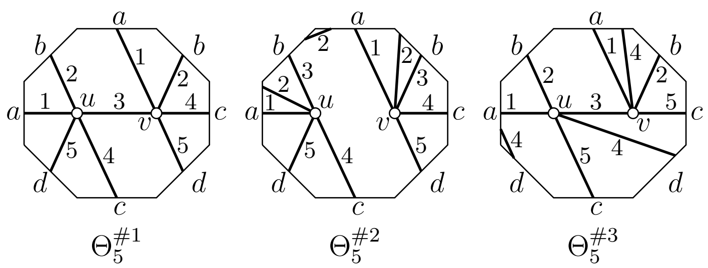


FIG. 1: The 2-cell embeddings of Θ_5 on the double torus (the edges are labelled).

The automorphisms of the embeddings of Fig. 1 are helpful in finding the embeddings of $K_{3,3}$. We need the rotations at vertices u and v for each embedding. An automorphism is a permutation of the vertices and edges that leaves the rotation system unchanged. The double torus is an orientable surface. If all the rotations of an embedding are reversed, an equivalent, but possibly non-isomorphic, embedding results. These embeddings will be considered equivalent. An embedding is *non-orientable* if the embedding obtained by reversing the rotations is isomorphic to the original embedding. Otherwise it is *orientable*. We can show that:

- Embeddings $\Theta_5^{\#1}$, $\Theta_5^{\#2}$, and $\Theta_5^{\#3}$ of Fig. 1 are non-orientable and have the automorphism groups of order two, ten, and five, respectively.
- All the edges in each of the embeddings $\Theta_5^{\#2}$ and $\Theta_5^{\#3}$ are equivalent.
- Vertices u and v are equivalent in each of the embeddings $\Theta_5^{\#1}$ and $\Theta_5^{\#2}$, but not in $\Theta_5^{\#3}$.

Denote now by A, B, C, D, E, F the vertices of $K_{3,3}$, and choose the edges AC, AE, BD, BF of $K_{3,3}$, as shown in Fig. 2. Contract these four edges to form a minor of $K_{3,3}$ isomorphic to Θ_5. If we start with a 2-cell embedding of $K_{3,3}$ on the double torus, and contract these edges, the resulting embedding of the minor will be a 2-cell embedding of Θ_5 on the double torus. Therefore, every 2-cell embedding of $K_{3,3}$ on the double torus can be contracted to a 2-cell embedding of Θ_5, and Θ_5 has only three 2-cell embeddings. To find the distinct 2-cell embeddings of $K_{3,3}$, we restore the contracted edges in all possible ways and compare the results for isomorphism.

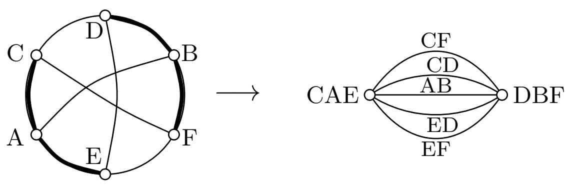


FIG. 2: $K_{3,3}$ and one of its minors isomorphic to Θ_5.

There are $5! = 120$ different ways to assign the labels AB, CD, CF, ED, EF to the five edges of the embeddings of Θ_5 shown in Fig. 1, before attempting to restore the contracted edges. However, many of them are equivalent. There are eight automorphisms of $K_{3,3}$ that map the bold subgraph of Fig. 2 to itself – they are generated by the permutations (CE), (DF), and $(AB)(CD)(EF)$. The only edge uv that doesn't cross the boundary of the octagon in the embeddings $\Theta_5^{\#1}$ and $\Theta_5^{\#3}$ of Fig. 1 is called the *central edge*. We prove that:

- There are four extensions of $\Theta_5^{\#1}$ to $K_{3,3}$ with central edge uv labelled CD.
- There are no extensions of $\Theta_5^{\#2}$ to $K_{3,3}$.
- Up to isomorphism, there are at most three extensions of $\Theta_5^{\#3}$ to $K_{3,3}$.

Using these results and checking the embeddings for isomorphism (using the graph isomorphism software [3]), we prove that the 2-cell embedding of $K_{3,3}$ on the double torus is actually unique (see Fig. 3). This embedding is non-orientable: the permutation (1)(4)(26)(35) of vertices of $K_{3,3}$ in Fig. 3, which is an automorphism of $K_{3,3}$, maps the rotations to their reversals.

Theorem 2 *Up to isomorphism, $K_{3,3}$ has a unique 2-cell embedding on the double torus.*

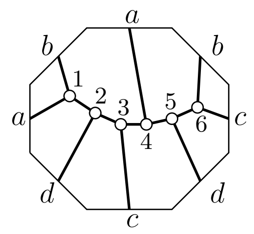


FIG. 3: The 2-cell embedding of $K_{3,3}$ on the double torus.

We find all distinct 2-cell embeddings of K_5 using Θ_5 and some intermediate graphs, which can be used as building blocks for 2-cell embeddings of various other graphs. First, we consider $T_{1,2,3}$, which is the graph of a triangle with edges of multiplicity 1, 2, and 3 (six edges in total). Then, we consider K_4^+, which is K_4 with one edge doubled, W_4 (the 4-*wheel*), which is K_5 without a 2-matching, and $K_5 - uv$, which is K_5 without an edge. A 2-cell embedding of $T_{1,2,3}$, K_4^+ or W_4 on the double torus must have only one face. We prove that, up to equivalence,

- $T_{1,2,3}$ has two orientable and four non-orientable 2-cell embeddings on the double torus;
- K_4^+ has two orientable and three non-orientable 2-cell embeddings on the double torus;
- W_4 has one orientable and three non-orientable 2-cell embeddings on the double torus.

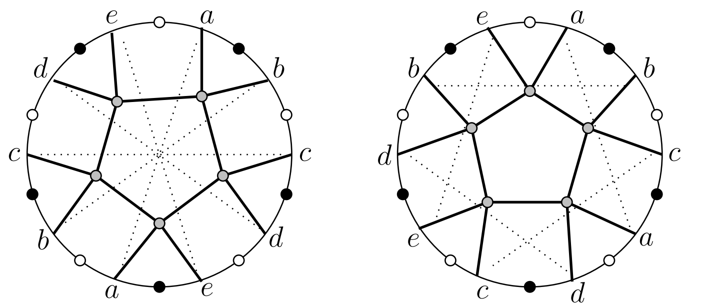
As a result, we prove the following theorem. The addition of edges to K_4^+, W_4, and $K_5 - uv$ in all possible ways in Theorem 3 was done by using a computer program. This provides a list of rotation systems for the 31 inequivalent 2-cell embeddings of K_5 on the double torus.

Theorem 3 *Up to equivalence, K_5 has* 14 *orientable and* 17 *non-orientable* 2*-cell embeddings on the double torus.*

3 Hyperbolic tilings and polygonal representations

The torus has symbolic representations $a^+b^+a^-b^-$ and $a^+b^+c^+a^-b^-c^-$, which also represent tilings of the Euclidean plane by rectangles and by regular hexagons, respectively. These polygonal representations correspond to one-face embeddings, one of which is given by Θ_3.

There are many more possibilities for such one-face embeddings on the double torus. The standard representation $a^+b^+a^-b^-c^+d^+c^-d^-$ of the double torus produces a tiling of the hyperbolic plane by regular octagons, in which eight octagons meet at each vertex. The three embeddings of Θ_5 produce tilings by regular 10-gons, i.e. polygons with 10 sides, having fundamental regions $a^+b^+c^+d^+e^+c^-d^-a^-b^-e^-$, $a^+b^+c^+d^+e^+a^-b^-c^-d^-e^-$, and $a^+b^+c^+a^-d^+c^-e^+d^-b^-e^-$. In each case, five hyperbolic 10-gons meet at each vertex. Two of the Θ_5-polygons are shown in Fig. 4, with 2-cell embeddings of K_5 drawn on them. The polygon boundaries are traversed in a clockwise direction, and for each pair of corresponding edges, the orientations of the two edges are opposite. Dotted lines are used to show the pairing of edges of the polygons. $K_{3,3}$ produces a tiling of the hyperbolic plane by regular 18-gons, i.e., polygons with 18 sides, with three 18-gons meeting at each vertex. Different polygonal representations of the double torus can be used, e.g., to have symmetric drawings of the embeddings (see Fig. 4).



$\Theta_5^{\#2}$ $\Theta_5^{\#3}$

FIG. 4: Embeddings of K_5 on two different 10-gon representations of the double torus.

4 On the triple torus

Using an exhaustive search of the rotation systems, we find all 13 inequivalent 2-cell embeddings of K_5 on the triple torus – 11 orientable and 2 non-orientable. Our approach using rotation systems for graphs on the double torus allows a drawing of the embedding to be constructed from its rotation system. However, in general, given a rotation system, it is a non-trivial task to find a drawing of the graph on a polygonal representation of the surface. An ad-hoc approach has been used to find drawings on the triple torus shown in Fig. 5.

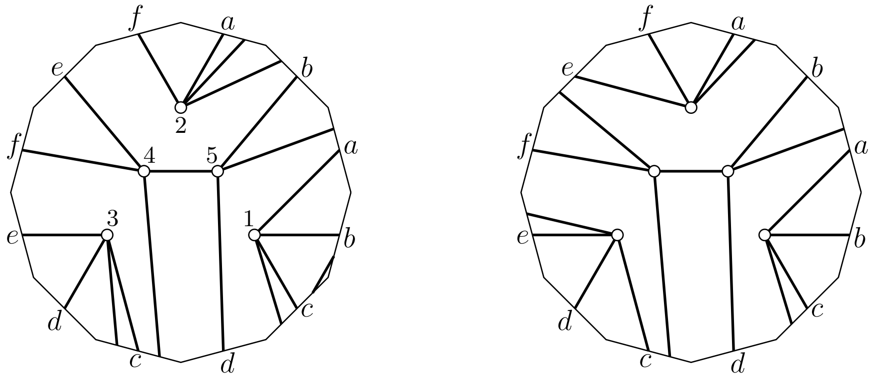


FIG. 5: Orientable and non-orientable embeddings of K_5 on the triple torus.

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