Constructing Operating Theatre Schedules using Partitioned Graph Colouring Techniques

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ABSTRACT
In hospitals, scheduled operations can often be cancelled in large numbers due to the unavailability of beds for post-operation recovery. Operating theatre scheduling is known to be an $\mathcal{NP}$-hard optimisation problem. Previous studies have shown that the correct scheduling of surgical procedures can have a positive impact on the availability of beds in hospital wards, thereby allowing a reduction in number of elective operation cancellations. This study proposes an exact technique based on the partitioned graph colouring problem for constructing optimal master surgery schedules, with the goal of minimising the number of cancellations. The resultant schedules are then simulated in order to measure how well they cope with the stochastic nature of patient arrivals. Our results show that the utilisation of post-operative beds can be increased, whilst the number of cancellations can be decreased, which may ultimately lead to greater patient throughput and reduced waiting times. A scenario-based model has also been employed to integrate the stochastic-nature associated with the bed requirements into the optimisation process. The results indicate that the proposed model can lead to more robust solutions.

KEYWORDS
OR in Health Services; Optimisation; Scheduling; Graph Colouring; Integer Programming

1. Introduction

Operating theatres are very costly parts of a hospital system and often have a large impact on other departments (Beliën, Demeulemeester, & Cardoen, 2006; Macario, Vitez, Dunn, & McDonald, 1995). Banditori, Cappanera, and Visintin (2013) have shown that operating theatres are the source of almost 70% of hospital admissions. Operations in theatres are usually carried out in blocks of time that are allocated to a specific specialty. The schedule that determines these allocations is known as the Master Surgery Schedule (MSS), and typically it will specify a weekly timetable for each operating theatre, Monday to Friday (Rowse, Lewis, Harper, & Thompson, 2015). An example of a master surgery schedule is shown in Figure 1. Each surgery requires various members of hospital staff, including surgeons, anaesthetists, nurses and technicians, in addition to vital equipment and a variety of consumables. Therefore, careful planning is needed to ensure the smooth running of the hospital within tight resource

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	Theatre1	Theatre2	Theatre3	Theatre4	Theatre5	Theatre6	Theatre7	Theatre8	Theatre9	Theatre10	Theatre11
MON-AM	Trauma	OMFS	ENT	Renal	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
MON-PM	Trauma	OMFS	ENT	Renal	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
TUE-AM	Trauma	OMFS	Vascular	Thoracic	Urology	CEPOD	Colorectal	Cardiac	Cardiac	Neuro	Gynae
TUE-PM	Trauma	OMFS	Vascular	Thoracic	Urology	CEPOD	Colorectal	Cardiac	Cardiac	Neuro	Gynae
WED-AM	Trauma	Vascular	ENT	Gen Surg	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
WED-PM	Trauma	Vascular	ENT	Gen Surg	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
THU-AM	Trauma	Trauma	OMFS	Vascular	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
THU-PM	Trauma	Trauma	OMFS	Vascular	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
FRI-AM	Trauma	OMFS	Vascular	Renal	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae
FRI-PM	Trauma	OMFS	ENT	Renal	Urology	CEPOD	Colorectal	Thoracic	Cardiac	Neuro	Gynae

Figure 1. An example of a master surgery schedule

and budgetary constraints.

Surgical admissions into hospitals can be broadly classified into two categories: elective and non-elective admissions. Non-elective (emergency) patients often require an operation to be performed as soon as possible, whereas elective patients are scheduled in advance. Elective patients can also be grouped into inpatients and day cases. Inpatients require a ward bed for post-operative recovery, whereas day cases often leave the hospital on the same day as the operation. Generally, non-elective patients require treatment more urgently than elective patients. Indeed, their admission can often lead to the cancellation of elective operations, particularly if this results in no bed being available for an elective admission. Consequently, this work focuses on the scheduling of elective inpatients while taking into account the impact of non-elective admissions on hospital resources. Rowse (2015) reported that the cancellation of operations in hospitals due to the unavailability of beds post-surgery is a very common problem. Cancellations will often upset the flow of patients through the hospital system and will negatively affect the patient experience, including increased levels of anxiety and higher costs due to ongoing care issues. In this research, we aim to minimise the number of cancellations of elective surgery through careful scheduling of the operating theatres.

In this paper we discuss the development of a partitioned graph colouring based optimisation model for the construction of the MSS. The developed model captures resource constraints to ensure that the number of beds required does not exceed the number of beds available, and seeks to minimise the number of unused beds. This setting allows for the cancellations of elective operations arising due to the MSS to be minimised. The Critical Care Unit (CCU) is also accommodated in our model, which is a special ward in which patients who require the highest level of care are treated. The running costs of this ward are very high due to the fact that patients are often cared for on a one-to-one basis using specialist life-saving equipment. In practice, the beds in the CCU are prioritised to emergency patients over elective surgical patients. The model therefore needs to consider situations where patients are deemed well enough to leave the CCU and move onto another surgical ward for further post-operative recovery, or where they pass away while at the CCU. Our proposed model is then simulated to measure how well its solutions are able to cope with unexpected changes in patient demand.

Robustness can be defined as the ability to withstand or overcome changes in data, variables or assumptions. A robust schedule should be more impervious to uncertainty than a non-robust schedule. Solution methods are often designed to optimise a single problem instance, assuming that data is known precisely and accurately. If robustness is not taken into account this solution may actually be of very poor quality when used in real-life (Bertsimas & Sim, 2004). To improve the robustness of solutions, our proposed optimisation model is extended in this study to become a scenario-based optimisation model in which more scenarios of bed requirements are incorporated into a single optimisation model. The model allows a reduction in quality in return for a more robust solution when the data changes.

The paper is structured as follows. Section 2 overviews the master surgery scheduling problem. Sections 3 and 4 then describe the new partitioned graph colouring based optimisation model and introduce a deterministic model for the construction of the master surgery schedule that incorporates bed constraints. In Section 5, the results and the performance of this new model are presented. The deterministic model is also extended in this section into a scenario-based optimisation model to improve the robustness of solutions. Simulation of the resulting solutions is then performed in order to obtain a measure of their robustness. Section 6 provides conclusions and areas for further work.

2. Background

The field of automated timetabling is quite prominent in the wider area of operational research with applications in education, transport, advertising, in addition to healthcare. A number of studies have looked into the area of surgical scheduling over the past two decades, with Samudra et al. (2016) providing a good overview. The practical importance of the problem is due to the operating theatres having major impacts on other departments such as surgical wards and intensive care units.

Typically the problem involves assigning the correct patient to appropriate surgical teams at the right time, while maximising resource utilisation. However, van Oostrum, Bredenhoff, and Hans (2010) have noted that in many cases, the methodologies and processes used to create these timetables are not implemented. van Oostrum et al. (2008) argued that the main uncertainties related to operating theatre scheduling are the stochastic durations of surgical operations, personnel availability, no-shows of patients and the occurrence of emergency surgical procedures.

Despite the importance of the issue, there has also been very little previous research that considers the downstream effects of surgery schedules. Beliën et al. (2006) created a tool that visualises the effect of a schedule on resources, including beds, staff, equipment and so on; however, they did not create a model that closes the information feedback loop by generating a schedule that considers its effect on the demand for beds. Vanberkel and Blake (2007) simulated patients' waiting times in different scenarios of bed availability (but did not produce a schedule), and concluded that longer waiting times are more dependent on the availability of post-operative beds as opposed to the availability of operating theatres themselves. van Oostrum et al. (2010) and Santibáñez, Begen, and Atkins (2007) also discussed the potential benefits of a systematic approach to surgery scheduling, such as the use of a master surgical scheduling approach, including the increased utilisation of the resources, reduced cancellation of surgeries, increased efficiency and patient throughput and lower wait times. They also suggest that a systematic approach could increase transparency and fairness in surgeons' time allocation.

Several studies found that exact methods, such as mixed integer programming techniques, perform well on the construction of the surgery schedule (Cardoen, Demeulemeester, & Belin, 2010; Kumar, Costa, Fackrell, & Taylor, 2018; Marques, Captivo, & Barros, 2019). Blake and Carter (1997) discussed the scope of surgical process scheduling research and emphasised the importance of communication between the operating theatres and other hospital departments. Vissers, Adan, and Bekkers (2005) employed a mixed integer linear programming model for constructing a master surgery schedule for a thoracic surgery department with a cycle length of four weeks. Four resources were considered in their model including operating theatre time, medium care beds, intensive care beds and nursing staff. Kuo, Schroeder, Mahaffey, and Bollinger (2003) developed a linear programming technique to optimise the allocation of operating room time among a group of surgeons while simultaneously optimising the associated financial return. Their results indicated a potential 15% increase in revenue in their case studies. A column generation approach was developed by van Oostrum et al. (2008) to construct a master surgery schedule that maximises the utilisation of operation room and levels the subsequent hospital bed requirements in wards and critical care units. To deal with the stochastic nature of the duration of surgical procedures, planned slack is included in the construction of MSSs. Neyshabouri and Berg (2017) applied a two-stage robust optimisation to address the uncertainty in length of stay in the downstream unit and developed a column-and-constraint generation exact technique to solve the problem. Their model allows for the hospital manager to adjust the level of risk. M'Hallah and Visintin (2019) addressed the problem of constructing MSS using a stochastic model and solved it using a sample average approximation technique. Their research provides evidence that it is fundamental to consider the stochastic-nature of the problem.

Beliën and Demeulemeester (2007) have proposed a number of mixed integer programming models and a simulated annealing meta-heuristic method for constructing the master surgery schedule. They built a model that involves demand constraints for operating theatre blocks for each surgical group, and capacity constraints to limit the number of available operating theatre blocks on each day. Results revealed the success of the meta-heuristic approach but it was observed to need very long computation times. Their best performing exact method overall was achieved using quadratic programming models. A hybrid method that combines quadratic program technique and simulated annealing, yields satisfying results with regard to both quality of solutions and computation times. Beliën, Demeulemeester, and Cardoen (2009) subsequently developed a decision support system for the construction of a master surgery schedule in a medium-sized Belgian hospital. They used mixed integer programming techniques and a simulated annealing meta-heuristic approach and found, due to the multi-objective nature of the problem, that the different models may provide different solutions, and decision makers have the responsibility to then choose the *best* solution among these. Fügener, Hans, Kolisch, Kortbeek, and Vanberkel (2014) proposed a model for planning the MSS aiming to minimise downstream costs (fixed costs, overcapacity costs, staffing costs, and additional weekend staffing costs) by levelling bed demand and reducing weekend bed requests. Exact, incremental improvement heuristics, a 2-opt heuristic and simulated annealing algorithms were employed to minimise these costs. van Essen, Bosch, Hans, van Houdenhoven, and Hurink (2014) employed an ILP and a local search for optimising the operating room schedule with the goal of reducing the number of required beds. The empirical results show that the ILP with simplified objective function performs better than the local search heuristic method.

As an alternative to mathematical programming techniques and heuristic optimisation methods, Vanberkel et al. (2011) proposed a decision support tool based on a queueing theory approach to build the master surgery schedule. They argued that the

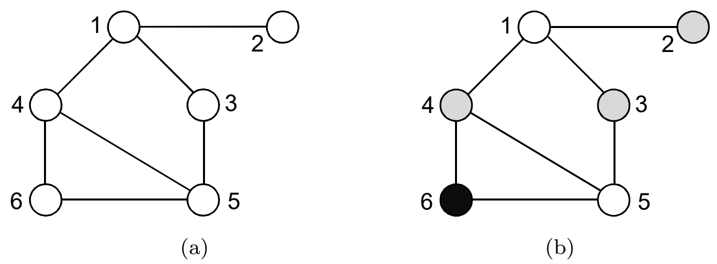

Figure 2. (a) A graph with $n = 6$ vertices, and (b) an optimal solution using 3 colours

department of surgery has the largest impact on other departments in the hospital, and in particular, the activities in the operating theatres. They also stated that more invasive surgeries require more care during a patient's post-surgery recovery. They suggest that giving consideration to this downstream effect of the operating theatre is essential for balancing the workload of the hospital. In their model, therefore, they described the workload for downstream departments as a function of the master surgery schedule.

3. Partitioned Graph Colouring Problem

We start this section with some basic definitions concerning graph colouring. For more details we refer to (Lewis, 2016). Let $G = (V, E)$ be a simple graph with n vertices and m edges. The standard graph colouring problem consists of assigning a colour to each vertex in V, such that no adjacent vertices are assigned to the same colour. The objective is to then minimise the number of colours used. Figure 2 shows a small graph coloured with three colours.

The partitioned graph colouring problem is a generalisation of the above. It is defined on an undirected graph $G = (V, E)$ as before, but the vertex set V has also been partitioned into u mutually exclusive nonempty subsets (partitions) $Q = (V_1, \ldots, V_u)$ with $V = V_1 \cup \cdots \cup V_u$ and $V_i \cap V_j = \emptyset \ \forall V_i, V_j \in V$ and $i \neq j$. The aim is to find a subset $V^* \subseteq V$ that contains exactly one vertex from each partition $V_k \in Q$ and a colouring for V^* so that, in the graph induced by V^*, no adjacent vertices have the same colour. As before, the objective is to minimise the number of colours used. Note that the partitioned graph colouring problem reduces to the classical graph colouring problem when all the subsets V_i have cardinality one. An example of the partitioned graph colouring problem is shown in Figure 3.

Li and Simha (2000) originally noted the partitioned graph colouring problem, and were motivated by wavelength routing and assignments in an optical network. The same authors proved that the problem is $\mathcal{NP}$-complete. In this context several approaches have since been proposed: a memetic algorithm (Pop, Hu, & Raidl, 2013), an ant colony optimisation algorithm (Fidanova & Pop, 2015), tabu search (Noronha & Ribeiro, 2006), branch-and-cut (Frota, Maculan, Noronha, & Ribeiro, 2010) and branch-and-price (Hoshino, Frota, & de Souza, 2011). Further information can be found at Campêlo, Corrêa, and Frota (2004).

Graph colouring has been applied to a wide area of operational research with applications in education and transport, in addition to healthcare. Our reason for choosing it was driven by three motivations: (i) Graph colouring algorithms are reasonably generic and are easy to implement and maintain. Thus we expect that our implementation

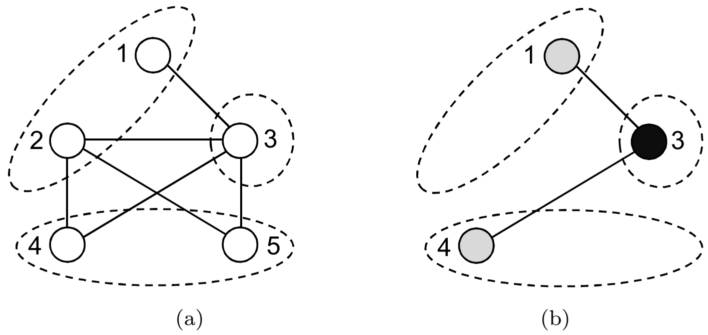

Figure 3. (a) Shows a partitioned graph colouring problem instance where $Q = \{\{1, 2\}, \{3\}, \{4, 5\}\}$, and (b) a solution using two colours

Table 1. Graph colouring in some selected timetabling and scheduling problems

Problem Domain	Reference
Round-robin sports scheduling	Lewis and Thompson (2011)
Exam timetabling	Hussin, Basari, Shibghatullah, Asmai, and Othman (2011)
Course timetabling	Burke, Marecek, Parkes, and Rudová (2010)
Job shop scheduling problem	Kouider, Haddadéne, Ourari, and Oulamara (2017)
Nurse scheduling	Anane (2014)

can be applicable to other variants of scheduling problems with minimal adaptation. (ii) The success of graph colouring in solving several $\mathcal{NP}$-hard optimisation problems generally and complex timetabling and scheduling problems specifically (Table 1). (iii) Partitioned graph colouring is appropriate for the construction of the MSS due to the $\mathcal{NP}$-hard nature of the problem, together with the benefit of being able to generate and limit the number of candidate schedules which can also help to reduce the size of the problem. To the best of the authors' knowledge, the use of graph colouring for surgery scheduling remains unexplored in the literature.

Before presenting a mathematical formulation for the partitioned graph colouring problem, it is worth noting that, as in Frota et al. (2010), a pre-processing phase is used that removes from the input graph all edges joining pairs of vertices belonging to the same set in the partition. Assuming that this pre-processing has been executed, a partitioned graph colouring problem can be formulated by the following binary integer programming problem. We define two matrices $X_{n\times q}$ and Y_q such that (n is the number of vertices and q is the maximum number of colours defined by the user):

$$x_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is coloured with colour } j \\ 0 & \text{otherwise, and} \end{cases} \tag{1}$$

$$y_j = \begin{cases} 1 & \text{if colour } j \text{ is used in the solution} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The objective is then to minimise the number of colours being used:

$$\text{min:} \quad \sum_{j=1}^{q} y_j \tag{3}$$

Figure 4. An example of the scheduling rules

subject to:

$$x_{ij} + x_{lj} \leq y_j, \quad \forall \{v_i, v_l\} \in E, \ \forall j \in \{1, \ldots, q\} \tag{4}$$

$$\sum_{i \in V_p} \sum_{j=1}^{q} x_{ij} = 1, \quad \forall V_p \in Q. \tag{5}$$

Here, Constraint (4) ensures that no pair of adjacent vertices have the same colour. Also, variable y_j is set to one if and only if colour j is being used. Constraint (5) then specifies that each *partition* has exactly one vertex that is assigned to a colour.

4. Optimisation of the Master Surgery Schedule

4.1. Overall Model

As noted, a partitioned graph colouring optimisation model is adopted here for the construction of the master surgery schedule, taking into account constraints on the operating theatres and post-surgery hospital ward bed requirements.

In our model, each surgical specialty represents a partition, and each partition has a number of vertices that represent *plans*. A plan for a specialty defines the operating theatre that the specialty has use of and on which sessions of the week. They reflect a specialty's preferences of theatres and days through the use of scheduling rules. The goal is to then select one plan (vertex) for each specialty (partition) that, when put together, forms the entire master surgery schedule. An example of the scheduling rules is shown in Figure 4. In this example, there are three operating theatres and three specialties S_1, S_2 and S_3. Operating theatre sessions coloured black denote that the specialty must not be allocated to these sessions; operating theatre sessions coloured grey denote that the specialty must be allocated to these sessions; operating theatre sessions coloured white denote that the specialty could be allocated to these sessions.

In this example, assume that S_1 requires 12 sessions, S_2 requires 1 session and S_3 requires 10 sessions. We now need to enumerate all possible plans (vertices) for each specialty. There are $\binom{10}{2} = 45$ possible plans for S_1, $\binom{10}{1} = 10$ possible plans for S_2 and only one possible plan for S_3. The first plans for S_1, S_2 and S_3 are shown in Figure 5. The construction of the master surgery schedule problem can now be considered as a partitioned graph colouring problem using the plans as the vertices, and adding edges

S1	Theatre1	Theatre2	Theatre3
MON-AM	S1	S1	
MON-PM	S1	S1	
TUE-AM	S1		
TUE-PM	S1		
WED-AM	S1		
WED-PM	S1		
THU-AM	S1		
THU-PM	S1		
FRI-AM	S1		
FRI-PM	S1		

S2	Theatre1	Theatre2	Theatre3
MON-AM		S2	
MON-PM			
TUE-AM			
TUE-PM			
WED-AM			
WED-PM			
THU-AM			
THU-PM			
FRI-AM			
FRI-PM			

S3	Theatre1	Theatre2	Theatre3
MON-AM			S3
MON-PM			S3
TUE-AM			S3
TUE-PM			S3
WED-AM			S3
WED-PM			S3
THU-AM			S3
THU-PM			S3
FRI-AM			S3
FRI-PM			S3

Figure 5. The first plans of specialty S_1, S_2 and S_3

between any pair of plans deemed to be conflicting. In this example, an edge between the first plan of S_1 and the first plan of S_2 would be added, because they require use of Theatre-2 at same time.

Note that in our case study and for some of the specialties, the enumeration algorithm can pair the allocated sessions so that, if a specialty has been allocated to an AM operating theatre session, then it will also be allocated to a PM operating theatre session if further sessions are required. In this way we reflect the hospital's policy that these specialties have whole day sessions rather than half day sessions. In our previous example, if S_1 is preferred to have whole day operating theatre sessions, then the number of possible plans reduces from 45 to 5.

Following the construction of the vertices (plans) in our partitioned graph colouring model, we now need to determine the number of patients who will require ward (including any CCUs) beds for each generated plan for each ward and for each day. We define bed requirements as the number of patients in beds for pre-operative and post-operative care. (The generation of the bed requirements will be discussed in Section 4.2.) This means that additional bed constraints must also be added to our model to ensure that the number of beds required does not exceed the number of beds available on each ward on each day. Figure 6 provides an example of bed requirements for two wards and one critical care unit, showing the number of surgical inpatients who require beds for the first plans of specialties S_1 and S_3 and the third plan of specialty S_2. In this example, the bed constraint is violated on Ward-1 on Wednesday, as shown in bold.

Mathematically, let:

- $a_{ik}^{(l)}$ be the number of beds required on ward k for plan i on day l,
- $b_{ik}^{(l)}$ be the number of beds required on CCU k for plan i on day l,
- $c_{k}^{(l)}$ be the number of beds available on ward k on day l, and
- $d_{k}^{(l)}$ be the number of beds available on CCU k on day l.

The bed constraints for each ward on each day can now be formulated as:

$$\sum_{i \in V} \sum_{j=1}^{q} a_{ik}^{(l)} x_{ij} \leq c_{k}^{(l)}, \quad \forall k, \ \forall l, \qquad (6)$$

S1	Theatre1	Theatre2	Theatre3
MON-AM	S1	S1	
MON-PM	S1	S1	
TUE-AM	S1		
TUE-PM	S1		
WED-AM	S1		
WED-PM	S1		
THU-AM	S1		
THU-PM	S1		
FRI-AM	S1		
FRI-PM	S1		

S2	Theatre1	Theatre2	Theatre3
MON-AM			
MON-PM			
TUE-AM		S2	
TUE-PM			
WED-AM			
WED-PM			
THU-AM			
THU-PM			
FRI-AM			
FRI-PM			

S3	Theatre1	Theatre2	Theatre3
MON-AM			S3
MON-PM			S3
TUE-AM			S3
TUE-PM			S3
WED-AM			S3
WED-PM			S3
THU-AM			S3
THU-PM			S3
FRI-AM			S3
FRI-PM			S3

	Ward1	Ward2	CCU1
MON	9	6	2
TUE	8	5	1
WED	9	4	2
THU	9	7	3
FRI	5	3	1
SAT	5	3	0
SUN	6	5	1

	Ward1	Ward2	CCU1
MON	0	3	0
TUE	0	3	0
WED	0	3	0
THU	0	2	0
FRI	0	0	0
SAT	0	0	0
SUN	0	1	0

	Ward1	Ward2	CCU1
MON	3	0	5
TUE	2	0	3
WED	4	0	2
THU	2	0	3
FRI	2	0	2
SAT	0	0	2
SUN	1	0	4

	Capacity
Ward1	12
Ward2	9
CCU1	10

	Ward1	Ward2	CCU1
MON	12	9	7
TUE	10	8	4
WED	13	7	4
THU	11	9	6
FRI	7	3	3
SAT	5	3	2
SUN	7	6	5

Figure 6. Illustrative diagram showing the number of beds required for three plans

and the bed constraints for each CCU on each day can be formulated as:

$$\sum_{i \in V} \sum_{j=1}^{q} b_{ik}^{(l)} x_{ij} \leq d_k^{(l)}, \quad \forall k, \ \forall l. \tag{7}$$

where $x_{ij} = 1$ if plan i is coloured with colour j, and 0 otherwise.

Our main objective in this problem is to utilise the beds on the wards and critical care units (which are expensive resources) as fully as possible. This is equivalent to minimising the number of unused beds on the wards and CCUs at any given time. This objective can be used as part of the hospitals' capacity planning strategy. If the optimal solution has spare capacity (i.e. empty beds on the wards) and there is enough operating theatre time for more operations, then there could be scope for increasing the number of patients brought in for surgery. The formulation of the partitioned graph colouring based model for the construction of the master surgery schedule is therefore as follows.

$$\text{min: } \alpha \sum_{j=1}^{q} y_j + \sum_{\forall k} \sum_{\forall l} (c_k^{(l)} - \sum_{x_i \in V} \sum_{j=1}^{q} a_{ik}^{(l)} x_{ij}) + \sum_{\forall k} \sum_{\forall l} (d_k^{(l)} - \sum_{x_i \in V} \sum_{j=1}^{q} b_{ik}^{(l)} x_{ij}) \quad (8)$$

subject to:

$$x_{ij} + x_{lj} \leq y_j, \quad \forall \{v_i, v_l\} \in E, \ \forall j \in \{1, \ldots, q\} \quad (9)$$

$$\sum_{i \in V_p} \sum_{j=1}^{q} x_{ij} = 1, \quad \forall V_p \in Q \quad (10)$$

$$\sum_{i \in V} \sum_{j=1}^{q} a_{ik}^{(l)} x_{ij} \leq c_k^{(l)}, \quad \forall k, \ \forall l \quad (11)$$

$$\sum_{i \in V} \sum_{j=1}^{q} b_{ik}^{(l)} x_{ij} \leq d_k^{(l)}, \quad \forall k, \ \forall l \quad (12)$$

The first term in Equation (8) gives the number of colours used in a solution multiplied by the weighting coefficient α. The value of α needs to be determined manually and is used to balance this objective against the sum of the remaining terms, which capture the level of bed utilisation in the solution.

4.2. Bed Requirements

Our method used to generate the bed requirements is presented in Algorithm 1. As in (Rowse et al., 2015), a probability table of which ward(s) or critical care unit(s) each specialty sends their patients to for post-operative recovery is used in the generation of bed requirements. In our model, patients admitted to the CCU may move onto another ward (i.e. another probability table is used) if they are deemed well enough to leave it or, with a given mortality rate, they die while in the CCU. Table 2 provides an example of these.

Table 2. An example of three specialties, two wards, and one CCU, showing (left) the probability table of which ward or CCU each specialty sends their patients to, and (right) which ward each specialty sends their patients to after leaving the CCU

First probability table pre-CCU				Second probability table post-CCU			
	W_1	W_2	CCU_1		W_1	W_2	CCU_1
S_1	0.80	0.17	0.03	S_1	0.95	0.05	0.00
S_2	0.00	1.00	0.00	S_2	0.00	0.00	0.00
S_3	0.98	0.00	0.02	S_3	1.00	0.00	0.00

Whenever a surgical specialty is scheduled to operate in a theatre, a bed is required for the total duration of the patients' pre-operative (pre-op) and post-operative (post-op) Length of Stay (LoS). The number of arrivals per session are user-specified parameters. Here, we employ the conditional probability of failure method (Kaplan & Meier, 1958) to estimate the post-operative LoS duration for each patient. Given the distribution of patients' post-operative LoSs for each specialty and CCU from activity data, the conditional probability of a given patient leaving the hospital or CCU on day l is $\frac{d(l)}{n(l)}$, where $d(l)$ is the number of patients leaving on day l, and $n(l)$ is the number of patients in the hospital at the start of day l. We may assign a specified proportion of patients into beds on wards few days before day of surgery for pre-operative LoS. For example, for the Trauma specialty 80% are admitted to hospital on the day of surgery, 10% has pre-operative LoSs of 1 day and 10% has pre-operative LoSs of 2 days. These are user-specified parameters. The number of operations per session controls how many arrivals enter the system and is calculated (for each specialty) by dividing the session duration into the average duration of surgical procedures. We appreciate that the duration of operations may vary depending on the medical needs of the patients. However, discussions with hospital staff have described that the number of sessions per specialty remain constant but often overrun, causing more time to be used in the session than was allocated.

5. Computational Experiments

We use a large teaching hospital in Cardiff, Wales as our case study. A recent study performed on the hospital revealed that over 25,000 surgical operations are performed annually, with approximately 18% of operations being cancelled, with non-clinical hospital reasons such as the lack of available beds post-surgery accounting for 54% of these. Additionally, around 30% of patients, after leaving the operation theatres, are assigned to an undesirable ward that does not necessarily have the equipment or specialist nurses due to unavailability of beds in the usual (preferred) ward. Such patients are referred to as *outliers* by the hospital.

In our case-study hospital there are 14 operating theatres ($OT_1, \ldots, OT_{14}$), eighteen specialties ($S_1, \ldots, S_{18}$) including Cardiac, Thoracic, Trauma, among others, ten wards ($W_1, \ldots, W_{10}$) including Vascular, Urology General/Liver, among others, and one critical care unit (CCU_1). One of the operating theatres is dedicated to non-elective surgical operations only (i.e., no elective patient can be admitted to this theatre). This theatre generates a high demand for beds, so it is deemed necessary to include it in our model. Elective operations are performed during one of two operating sessions per day (AM and PM), over five working days per week, Monday to Friday. The usage of beds at the hospital wards and critical care unit is modelled for seven days a week, Monday to Sunday.

In total, for our case study there were 1,449 vertices, 146,940 edges and 77 bed constraints. Table 3 provides the number of vertices for each specialty ($|V_i|$), the number of required sessions, the number of operations carried out per session and a list of operating theatres that the specialty can use. Table 4 provides the number of beds available at each hospital ward and critical care unit (CCU) and specifies the specialties that are assigned to each ward. These characteristics provide a rough idea about the size of the problem instance.

Table 5 shows a probability table indicating which ward or CCU each specialty sends their patients to. It also shows which ward each specialty sends their patients

Algorithm 1: Construction of bed requirements

```
foreach V_p ∈ Q do
    foreach vertex v_i ∈ V_p do
        foreach operating session j do
            if specialty p is scheduled in j then
                For each new arrival, decide on which ward or CCU they go to
                  given the probability table;
                if Some patients go to CCU then
                    Fill in beds with pre-op LoS before they go to CCU;
                    Send those patients to the CCU and update the bed
                      requirements with their CCU LoSs;
                    Calculate how many patients are discharged from the CCU on
                      each day of the week;
                    Put new arrivals straight from surgery into all other wards (not
                      CCU);
                    Fill in bed requirements with pre-op LoS for these patients;
                    Send these patients from surgery straight to other wards (not
                      the CCU) and update the bed requirements with their
                      post-op LoS;
                    Reduce the number of CCU discharges to other wards due to
                      some patients dying in CCU according to the mortality rate;
                else
                    Send patients from surgery straight to wards and update the
                      bed requirements with their pre-op and post-op LoSs;
                end
            end
        end
        foreach day l in the planning horizon do
            if anyone is discharged on day l then
                Distribute these patients between the other wards using a
                  probability table;
                foreach ward k do
                    if Any CCU discharges are sent to ward k then
                        Send these patients to ward k and update the bed
                          requirements with their post-op LoSs;
                    end
                end
            end
        end
    end
end
```

to after leaving the CCU.

Our experimentation was conducted on an Intel i7-6500U CPU at 2.50GHz and 2.60GHz with 8.00GB RAM, using the open-source optimisation software COIN Cbc 2.9.9 together with C++ under a Windows 10 operating system.

As mentioned earlier, our model allows more than one colour to be used in a solution,

Table 3. Specialties in our case-study problem

specialty	name	$\lvert V_i \rvert$	number of sessions	operations per session	accessible theatres
S_1	Cardiac	1	20	1	OT_{11}, OT_{12}
S_2	CEPOD	1	10	4	OT_6
S_3	Colorectal	5	8	1	OT_8
S_4	ENT	60	5	1	OT_3
S_5	General	210	8	3	OT_4, OT_9
S_6	Liver	5	2	2	OT_9
S_7	Neuro	1	20	2	OT_{13}, OT_{14}
S_8	Ophthalmology	5	2	1	OT_3
S_9	Oral	455	6	1	OT_3, OT_5, OT_{11}
S_{10}	Paeds ENT	10	1	3	OT_3
S_{11}	Paeds general	5	8	1	OT_5
S_{12}	Paeds ortho	5	2	2	OT_2
S_{13}	Renal	455	6	2	OT_4, OT_8, OT_9
S_{14}	Scoliosis	10	4	3	OT_2
S_{15}	Thoracic	5	8	1	OT_{11}
S_{16}	Trauma	5	12	5	OT_1, OT_2
S_{17}	Urology	1	10	2	OT_7
S_{18}	Vascular	210	8	2	OT_2, OT_4

though this is heavily penalised (in our case by setting $\alpha = 100,000$). When multiple colours are present in a solution, this implies that some of the solution's selected plans are clashing. We found this extra flexibility to be useful in practical circumstances in that it is often better for a user to be given a solution with some clashes, rather than no solution at all. Indeed, if a decision maker were to be given a solution featuring clashes, they might then choose to revisit the problem and loosen some constraints or add additional resources. On the other hand, we also found that limiting the number of colours was also useful in that it reduced the search space of the problem significantly. We therefore chose to set $q = 2$ in this study.

As part of our trials, a simulation tool was also developed to provide measures on how well our MSS schedules perform if they are to be used in a real-world setting. A snapshot of future bed requirements for each ward and each critical care unit for the optimal MSS schedule is produced using the same method described in Section 4. Each generated MSS is simulated 1000 times, each with a different set of bed constraints, and the number of times where the bed constraints are violated is reported.

Table 6 presents the objective function values obtained from five runs (each with a different seed value), and the percentage of simulations of the optimal MSS solutions in which at least one of the bed constraints is violated. A comparison of the optimal solution produced using our model and the current MSS used in the hospital is also

Table 4. Wards and critical care units in our case-study problem

ward	name	capacity	specialties using each ward
W_1	Paeds	28	S_2, S_{10}, S_{11}, S_{12}, S_{14}
W_2	ENT/Oral	19	S_2, S_4, S_8, S_9
W_3	Vascular	38	S_2, S_{18}
W_4	Trauma	83	S_2, S_{16}
W_5	Renal	20	S_{13}
W_6	General/Liver	76	S_2, S_5, S_6
W_7	Urology	19	S_{17}
W_8	Colorectal	20	S_3
W_9	Cardiothoracic	50	S_1, S_{15}
W_{10}	Neuro	53	S_2, S_7
CCU_1	Critical Care	27	S_2, S_5, S_7, S_{16}, S_{18}

Table 5. Probability tables

First probability table pre-CCU											
	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	CCU_1
S_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
S_2	0.18	0.11	0.08	0.05	0.00	0.27	0.00	0.00	0.00	0.14	0.17
S_3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
S_4	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_5	0.00	0.00	0.00	0.00	0.00	0.63	0.00	0.00	0.00	0.00	0.37
S_6	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
S_7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.93	0.07
S_8	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_9	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{10}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{11}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{12}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{13}	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{14}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{15}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
S_{16}	0.00	0.00	0.00	0.99	0.00	0.00	0.00	0.00	0.00	0.00	0.01
S_{17}	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
S_{18}	0.00	0.00	0.96	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04
Second probability table post-CCU											
	W_1	W_2	W_3	W_4	W_5	W_6	W_7	W_8	W_9	W_{10}	CCU_1
S_1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
S_2	0.22	0.13	0.09	0.06	0.00	0.33	0.00	0.00	0.00	0.17	0.00
S_3	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00
S_4	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_5	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
S_6	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00
S_7	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00
S_8	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_9	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{10}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{11}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{12}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{13}	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{14}	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{15}	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00
S_{16}	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
S_{17}	0.00	0.00	0.00	0.00	0.00	0.00	1.00	0.00	0.00	0.00	0.00
S_{18}	0.00	0.00	1.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

provided in Table 6. The table shows that in an average of 45% of the simulations, there were too many beds required on at least one of the wards i.e. this would have resulted in a violated bed constraint in our model and would also result in a cancellation in reality. In run 1 for instance, it is worse than for the current MSS used, though the utilisation of beds has been improved significantly.

5.1. Scenario-based Optimisation Model

Scenario-based optimisation was first proposed by Calafiore and Campi (2005) to help improve the robustness of solutions. Here we employ the same strategy and extend our model to become a scenario-based optimisation model in which more scenarios of bed requirements are incorporated. Algorithm 1 is used to generate ρ different scenarios which are then embedded into our model to create more bed constraints. This could potentially allow the produced schedule to cope with the stochastic-nature associated with the bed requirements. Constraints (11) and (12) are therefore replaced by Constraints (13) and (14).

Table 6. Summary of experimental results

Run	Unused beds	Violation	Execution time (s)
1	1110	57.0%	11
2	1066	50.9%	10
3	1069	43.6%	12
4	1174	35.9%	10
5	1075	38.8%	12
Average	1098.8	45.24%	11
Current MSS	1244	54.0%	

$$\sum_{i \in V} \sum_{j=1}^{q} a_{tik}^{(l)} x_{ij} \leq c_{tk}^{(l)}, \quad \forall k, \ \forall l, \ \forall t = 1, \ldots, \rho \tag{13}$$

$$\sum_{i \in V} \sum_{j=1}^{q} b_{tik}^{(l)} x_{ij} \leq d_{tk}^{(l)}, \quad \forall k, \ \forall l, \ \forall t = 1, \ldots, \rho \tag{14}$$

Here, the number of scenarios, ρ, is a parameter chosen by the decision maker. When $\rho = 1$, the model reduces to the model described in Section 4. Table 7 shows the average results from five runs on various ρ values.

Table 7. Summary of experimental results using scenario-based optimisation model

ρ	Feasibility	Unused beds	Violation	Execution time (s)
1	100%	1098.80	45.24%	11.0
2	80%	1110.50	45.18%	11.0
3	80%	1113.25	43.25%	12.0
4	80%	1113.50	44.20%	12.8
5	80%	1115.75	41.58%	13.8
6	80%	1124.00	39.58%	17.5
7	60%	1108.67	41.03%	19.0
8	40%	1109.50	39.50%	17.5
9	40%	1114.00	42.30%	20.5
10	40%	1115.50	38.55%	35.0

As expected, the results presented here indicate that, in general, increasing the number of scenarios ρ, worsens the average objective function values (i.e. the average number of unused beds) of feasible solutions; however, it also increases robustness of the resulting schedule with respect to the average percentage of simulations in which the optimal schedule would have resulted in at least one of the bed constraints being violated. Indeed, the results show that there is a trade-off between increasing the number of scenarios ρ, and the percentage of problem instances that result in no feasible solutions. This is due to the fact that the problem is now more constrained, because additional scenarios are included in the model, which seems to be the price to pay for more robust solutions. Table 7 also shows that the execution time increases rapidly as the value of ρ increases. However, even when ρ is assigned to 10 scenarios, the execution time remains reasonable, with an average execution time of just 35 seconds. Given that the construction of the MSS is a tactical level decision, made a few times per year, a run time of a few hours is reasonable. The main concern is that finding a feasible solution is extremely unlikely with a large number of scenarios (i.e. when $\rho > 10$ in this case).

6. Conclusions

In many hospitals there is an increasing number of operations being cancelled due to factors such as the unavailability of beds for post-operative recovery (Bowers, 2013). This study has investigated an optimisation model for tackling problems associated with theatre scheduling with the aim of reducing the number of cancellations of surgical procedures. To these ends a partitioned graph colouring formulation has been developed to solve the problem of constructing operating theatre schedules. The proposed model aims to choose an optimal plan from a set of possible plans for each surgical specialty, whilst minimising the number of unused bed days over the scheduling horizon, subject to constraints concerning the operating theatres and bed demands. The proposed model has then been validated to measure how well the produced schedules cope with unexpected changes in patient demand. Our model has the ability to efficiently manage the scarce hospital resources, and allow hospital managers to investigate various what-if scenarios such as the implications of increasing the number of beds in hospital wards. This study has shown that optimal and feasible master surgery schedules for our large case study hospital can usually be found in short amounts of computational time; however, the validation of the robustness of the produced solution has revealed that the percentage of simulations that violated bed constraints is reasonably high. In the context of elective operation scheduling, a timetable must be produced some time in advance of the actual dates of operations and will be based on several assumptions such as the current bed occupancy on different wards and staff availability. A robust schedule will mean that more operations will be able to take place, even if there are significant changes to these factors. The proposed partitioned graph colouring model is extended in this study to become a scenario-based optimisation model to ensure more robust master surgery schedules that safeguard against this uncertainty. A disadvantage to applying the scenario-based optimisation technique, however, is the massively increasing complexity of the solution space as the number of scenarios increases. In this situation, finding a feasible solution is highly unlikely. Given this conclusion, a sensible question to ask is whether it is better to use the scenario-based optimisation model to construct the master surgery schedule, or to continue with a more classical deterministic model? Our results suggest that the scenario-based optimisation model is better to use than the deterministic technique in terms of robustness of solutions. One may notice that the deterministic model is a special case of the scenario-based model with only one scenario, and hence will be at least as good as the deterministic model. As a future work, we would like to investigate the high number of violations. There is, therefore, additional scope to extend the scenario-based optimisation model to (for example) include bed transference between wards within each bed constraint.

Our method used to generate the bed requirements does not take into account that patients can be discharged at any point throughout the day. The estimation of the LoS duration for each patient has been calculated on a daily basis for each day after surgery. In reality, if a patient is discharged in the morning, a new patient can be admitted the same day that afternoon. It would be interesting to examine in future work whether this results in different optimal solutions being found. It would also be interesting to investigate the modelling of surgical patients based on their specific surgical procedure within each specialty, which may affect the optimal solutions that can be found from our proposed model. Decision makers in hospital may decide to move towards the use of whole day operating theatre sessions, as opposed to the current half-day sessions, and/or move to a seven-day, as opposed to the current five-day working week. It would be of interest to investigate these scenarios.

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