Approximate solution of nonlinear triad interactions of acoustic–gravity waves in cylindrical coordinates

Usama Kadri[1]†, Zhan Wang[2]

[1]School of Mathematics, Cardiff University, Cardiff, CF24 4AG, UK

[2]Key Laboratory for Mechanics in Fluid Solid Coupling Systems, Institute of Mechanics, Chinese Academy of Sciences, Beijing 100190, China

(Received xx; revised xx; accepted xx)

The three-dimensional radial propagation of wave disturbances over a slightly compressible fluid of constant depth is discussed. We focus on resonant triads comprising two gravity modes and one acoustic mode. The derivation of the evolution equations in a non-integral form is made possible by approximating the radial solution by cosine functions in two regions, inner and outer, that are matched at a location where all relevant derivatives are in agreement with the exact Bessel solution. When the interaction takes place in the inner region of all modes, the amplitude evolution equations are found to be similar to the two-dimensional case. However, focusing of one gravity mode and de-focusing of the other is observed when the interaction involves an inner region of the acoustic mode, and outer regions of the gravity modes.

1. Introduction

In classical water-wave theory, water is often treated as incompressible; an assumption that is valid for many applications in linear theory, where surface (gravity) waves and acoustic (compressible) waves are virtually decoupled. This assumption is appropriate when the wave disturbance propagates to the bottom at relatively short time compared to the period (Longuet-Higgins 1950), i.e. when $h/c \ll \mathcal{T}$, where h is the water depth, c is the speed of sound in water, and $\mathcal{T}$ is the wave period. Accounting for both compressibility and gravity effects reveals two types of wave motions: (1) surface waves with fluid motion that decays exponentially with depth; and (2) non-evanescent longitudinal compression waves. Among the first works who considered this setting were done by Pidduck (1910, 1912), who studied the propagation of an impulse applied on a water surface; Whipple & Lee (1935), who showed that the two types of waves exist when the wave period is a few seconds long; and Longuet-Higgins (1950) who focused on second-order pressure variations. For reviews of the general theory of wave interactions see Phillips (1981) and Craik (1985).

Longuet-Higgins (1950) demonstrated that quadratic nonlinear interactions of two trains of monochromatic gravity waves of identical frequencies and opposite directions can excite compression modes. Such special triad interactions comprise a standing compression mode at a cut-off frequency 2ω, and two subharmonic surface waves with frequency ω and opposite wavenumbers k. The resonances found by Longuet-Higgins (1950) are particular examples of resonant triads involving a propagating acoustic wave mode and two oppositely travelling subharmonic surface waves, as demonstrated numerically by

† Email address for correspondence: kadriu@cardiff.ac.uk

Kadri & Stiassnie (2013). The general two-dimensional (2D) asymptotic theory for resonant triad interactions of acoustic–gravity waves was developed by Kadri & Akylas (2016), and more recently a steady-state solution for resonant acoustic–gravity waves has been reported by Yang *et al.* (2018).

Here, we extend the general solution and develop an asymptotic theory for three-dimensional (3D) resonant triad interactions of acoustic–gravity wave disturbances in cylindrical coordinates. Similar to Kadri & Akylas (2016), such resonant interaction depends on a small parameter $\mu = gh/c^2$ that governs the effects of gravity relative to compressibility, where g is the gravitational acceleration. Thus, free-surface wave disturbances feature vastly different spatial and/or temporal scales from the acoustic mode.

A major challenge in the 3D analysis is the radial solution that is governed by Bessel functions, which makes collecting secular terms a cumbersome, if not an impossible, task. Alternatively, one could follow an integral form analysis, e.g. as in Miles (1984), though this would require a numerical evaluation of the integrals that is beyond the scope of the current study. Here, we provide a non-integral approximated closed form solution. To overcome the difficulty of collecting secular terms, we approximate the radial potential solution (and its derivatives) by cosine functions in two regions, inner and outer, matched at a location where all relevant derivatives are approximated to satisfactory agreement with the original Bessel functions, following Kadri (2019a). The amplitude evolution equations are then derived for three regions: (1) inner for all three modes; (2) mixed, i.e. inner for the acoustic and outer for the gravity; and (3) outer for all three modes. In the latter scenario, the waves disperse $\propto 1/\sqrt{r}$, thus the interaction is weak and effectively there is no energy share between the triad members. In the inner-inner region the solution is almost identical to that of the 2D case. In the mixed region the interaction is dictated by the vertical profile of the wave packets, and focusing of one gravity mode and de-focusing of the other is observed.

2. Preliminaries and formulation

Based on irrotationality, the surface gravity and acoustic–gravity wave problems are formulated in terms of the velocity potential $\varphi(r, z, t)$, where $\mathbf{u} = \nabla\varphi := (\varphi_r, \varphi_z)$ is the velocity field assuming radial symmetry ($\partial_\theta = 0$). Viscosity and surface tension are neglected, and we shall use dimensionless variables, employing μh as lengthscale and h/c as timescale. As in Longuet-Higgins (1950), the governing equation is the fully nonlinear wave equation

$$\varphi_{tt} - \frac{1}{\mu^2}\left(\varphi_{rr} + \frac{1}{r}\varphi_r + \varphi_{zz}\right) + \varphi_z + (|\nabla\varphi|^2)_t + \tfrac{1}{2}\mathbf{u}\cdot\nabla\left(|\nabla\varphi|^2\right) = 0. \tag{2.1}$$

The boundary condition on the rigid bottom at $z = -1/\mu$ reads

$$\varphi_z = 0 \qquad (z = -1/\mu). \tag{2.2}$$

Finally, on the free surface $z = \zeta(r, t)$, we consider the standard dynamic and kinematic boundary conditions

$$\left(\frac{\partial\varphi}{\partial t} + \frac{1}{2}|\nabla\varphi|^2 + z\right)_{z=\zeta} = 0\,; \qquad \Big(\zeta_t + \zeta_r\varphi_r - \varphi_z\Big)_{z=\zeta} = 0\,. \tag{2.3}$$

After expanding the two conditions about $z = 0$, correct up to cubic terms in the perturbation, and eliminating ζ yields the combined free-surface condition

$$\begin{aligned}&\varphi_{tt}+\varphi_z+(|\nabla\varphi|^2)_t+\tfrac{1}{2}\nabla\varphi\cdot\nabla\left(|\nabla\varphi|^2\right)-\left[\varphi_t(\varphi_{tt}+\varphi_z)\right]_z\\&-\left[\varphi_t(|\nabla\varphi|^2)_t\right]_z-\tfrac{1}{2}\left[(\varphi_{tt}+\varphi_z)\left(|\nabla\varphi|^2-\varphi_t^2\right)\right]_z=0\quad(z=0).\end{aligned}\tag{2.4}$$

2.1. *Linear solution*

By dropping the nonlinear terms in (2.1) and (2.4), we seek modes that radiate with frequency ω in the form

$$\varphi=\mathcal{R}(r)\mathcal{Z}(z)\exp\left(\frac{1}{2}\mu^2 z\right)\exp\left(-\mathrm{i}\omega t\right).\tag{2.5}$$

Upon substituting (2.5) into (2.1), a separation of variables produces a set of two ordinary differential equations, the first being in the vertical direction,

$$\mathcal{Z}_{zz}-\left(k^2-\mu^2\omega^2+\frac{1}{4}\mu^4\right)\mathcal{Z}=0,\tag{2.6}$$

and the second in the radial direction,

$$\mathcal{R}_{rr}+\frac{1}{r}\mathcal{R}_r+k^2\mathcal{R}=0\,,\tag{2.7}$$

where k stands for the eigenvalue of the problem. The detailed solution in the vertical direction is given by Kadri & Akylas (2016) who showed that for $\lambda = O(1)$ and $\mu \ll 1$ the solution decays exponentially into the fluid

$$\mathcal{Z}=\exp(|k|z)+O(\mu),\tag{2.8}$$

and ω satisfies the well known dispersion relation for surface gravity waves on deep water

$$\omega^2=|k|+O(\mu^4).\tag{2.9}$$

On the other hand, when $\lambda^2 < \mu^2\omega^2$ the vertical profile in (2.6) becomes oscillatory, with

$$\mathcal{Z}=\cos\left(\omega^2-\kappa^2\right)(Z+1)-\frac{\mu}{2\omega}\sin\left(\omega^2-\kappa^2\right)(Z+1)+O(\mu^2),\tag{2.10}$$

where $Z = \mu z$ and the eigenvalue k was rescaled to $\mu\kappa$ for $\kappa = O(1)$ to allow using h as a characteristic length, instead of μh, and ω satisfies, to leading order, the dispersion relation for acoustic modes

$$\omega^2=\omega_n^2+\kappa^2+\mu\frac{\omega_n^2-\kappa^2}{\omega_n^2+\kappa^2}+O(\mu^2)\qquad(n=0,1,2,...)\tag{2.11}$$

with $\omega_n = (n+1/2)\pi$. For the radial equation (2.7), we seek a propagating wave solution of the form

$$\mathcal{R}=C_1J_0(kr)+\mathrm{i}C_2J_0\left(kr-\frac{\pi}{2}\right),\tag{2.12}$$

where J_0 is the Bessel function of the first kind and C_1, C_2 are coefficients determined from the boundary-value problem. In the far field, J_0 oscillates and behaves like a damped cosine function, namely,

$$J_0(kr)\approx\sqrt{\frac{2}{\pi kr}}\cos\left(kr-\frac{\pi}{4}\right).\tag{2.13}$$

2.2. *Three-dimensional resonant triads*

The existence of resonant triads in the acoustic–gravity wave family has been presented numerically by Kadri & Stiassnie (2013), and various triad combinations were reported by Kadri (2015). However, the importance of cubic terms at the limit $\mu \ll 1$ was first demonstrated by Kadri & Akylas (2016). Here, we discuss acoustic–gravity wave triad resonance in 3D space. In particular, we consider the evolution equations of two gravity modes of complex amplitudes $S_{\pm}$ and frequencies $\omega_{\pm}$ interacting with an acoustic mode of amplitude A and frequency ω:

$$S_+ \mathrm{e}^{\mathrm{i}(k_+ r - \omega_+ t)}, \qquad S_- \mathrm{e}^{\mathrm{i}(k_- r - \omega_- t)}, \qquad A\, \mathrm{e}^{\mathrm{i}(\mu\kappa r - \omega t)}. \tag{2.14}$$

The gravity waves satisfy the dispersion relation (2.9), whereas the acoustic mode ($\mu\kappa$, ω) satisfies the dispersion relation (2.11). The triad resonance requires that

$$\omega_+ + \omega_- = \omega + \mu\beta, \qquad k_+ + k_- = \mu\kappa, \tag{2.15}$$

where $\beta = O(1)$ is a detuning parameter. It follows from the resonance condition (2.15) that two gravity modes are counter-propagating waves with nearly idential wavenumbers. We introduce a positive wavenumber k, such that

$$k_+ = k + \frac{1}{2}\mu\kappa\,, \quad k_- = -k + \frac{1}{2}\mu\kappa\,, \tag{2.16}$$

and therefore a direct calculation yields

$$\omega_{\pm} = k^{1/2}\left(1 \pm \frac{\mu\kappa}{4k}\right) + O\left(\mu^2\right). \tag{2.17}$$

It then follows that

$$\omega = 2k^{1/2} + O\left(\mu^2\right), \tag{2.18}$$

where

$$4k = \omega_n^2 + \kappa^2 + \mu\frac{\omega_n^2 - \kappa^2}{\omega_n^2 + \kappa^2}. \tag{2.19}$$

The velocity potential of gravity waves is $O(\epsilon)$, while the velocity potential of the excited acoustic wave grows to $\alpha = O(1)$ due to its small contributions to free-surface elevation. Based on the scaling argument carried out in Kadri & Akylas (2016), we have $\epsilon = \alpha\mu^{1/2}$. In the subsequent analysis, we derive the amplitude evolution equations subject to a resonant triad whose velocity potential is expanded as

$$\begin{aligned} \varphi = \epsilon \left\{ S_+(T)\mathcal{R}(k_+ r)\mathrm{e}^{|k_+|z}\mathrm{e}^{-\mathrm{i}\omega_+ t} + c.c. \right\} + \epsilon \left\{ S_-(T)\mathcal{R}(k_- r)\mathrm{e}^{|k_-|z}\mathrm{e}^{-\mathrm{i}\omega_- t} + c.c. \right\} \\ + \alpha \left\{ A(T)\mathcal{R}(\mu\kappa r)\cos\omega_n(Z+1)\mathrm{e}^{-\mathrm{i}\omega t} + \mathrm{c.c.} \right\} + \dots, \end{aligned} \tag{2.20}$$

where $T = \mu t$ is a slow time variable.

3. Approximate solution

The derivation of the evolution equations, which is presented in the following section, requires collecting secular terms. The collection process is rather straightforward when the motion is 2D as in Kadri & Akylas (2016), which uses complex exponents. However, here the collection involves quadratic and cubic terms of zeroth, first and second derivatives of J_0, which is a cumbersome exercise. To overcome this difficulty we partially implement a matching technique developed by Kadri (2019a) to approximate the Bessel function and its first and second derivatives about the origin. The matching comprises inner and outer solutions. The inner solution is approximated by a simple cosine function,

$$J_{0,in}(r) = \cos(\mathcal{M}r), \tag{3.1}$$

where $\mathcal{M} = 0.6967398$ is a matching parameter calculated originally by matching $J_{0,in}$ with J_0 at $r = \pi/\mathrm{e}$. We match the inner solution with an outer solution of the form

$$J_{0,out}(r) = \sqrt{\frac{2}{\pi b_1 r}} \cos\left(b_2 r - b_3\right), \tag{3.2}$$

where b_1, b_2 and b_3 are matching parameters. For the zeroth derivative we require that

$$-\cos(\mathcal{M}r) + \sqrt{\frac{2}{\pi b_1 r}} \cos\left(b_2 r - b_3\right) = 0, \tag{3.3}$$

and similarly for the first and second derivatives,

$$\mathcal{M}\sin(\mathcal{M}r) - b_2\sqrt{\frac{2}{\pi b_1 r}}\sin(b_2 r - b_3) - \frac{1}{2}\sqrt{\frac{2}{\pi b_1 r^3}}\cos(b_2 r - b_3) = 0, \tag{3.4}$$

$$\mathcal{M}^2\cos(\mathcal{M}r) - \left(\frac{b_2^2}{\sqrt{r}} - \frac{3}{4\sqrt{r^5}}\right)\sqrt{\frac{2}{\pi b_1}}\cos(b_2 r - b_3) + b_2\sqrt{\frac{2}{\pi b_1 r^3}}\sin(b_2 r - b_3) = 0. \tag{3.5}$$

Noting that the matching can be carried out at a range of r, where $r \leqslant \pi/e$, and we choose $r = 1$ for simplicity. With the values of b_1, b_2, and b_3 at hand, we are able to design a solution whose first and second derivatives are also matched. Note that the outer solution does not coincide with $J_0(r)$ as $r \to \infty$, since for the standard far-field solution to be met one should impose $b_1 = b_2 = 1$ and $b_3 = \pi/4$. To this end, a second matching with the far-field at multiple locations can be performed as detailed by Kadri (2019a). However, here we have no interest in the far-field since the waves disperse $\propto 1/\sqrt{r}$, and thus the interaction is weak. Moreover, a more accurate matching can be obtained by adopting a three-point matching at r_1, r_2, r_3, one for each derivative. However, this will require collecting secular terms at more regions, e.g. at $[0, r_1]$, $[r_1, r_2]$, $[r_2, r_3]$, and $[r_3, \infty)$, as opposed to two regions, $[0, 1]$ and $[1, \infty)$, when choosing a single matching point at $r = 1$, as demonstrated in the next section.

4. Amplitude evolution equations

Since the acoustic mode is much longer than the gravity mode, attention is focused on three scenarios of interest where the interaction could take place: (1) inner approximation for both acoustic and gravity modes; (2) inner approximation for the acoustic mode and outer for the gravity mode; and (3) outer approximation for both modes.

4.1. *Inner-inner region*

We employ the approximation in (3.1) for all modes so that the potential takes the form,

$$\begin{aligned}\varphi = \epsilon\left[S_+(T)\mathrm{e}^{|k_+|z}\mathrm{e}^{\mathrm{i}(\mathcal{M}k_+ r - \omega_+ t)} + \text{c.c.}\right] + \epsilon\left[S_-(T)\mathrm{e}^{|k_-|z}\mathrm{e}^{\mathrm{i}(\mathcal{M}k_- r - \omega_- t)} + \text{c.c.}\right] \\ +\alpha\left[A(T)\cos\omega_n(Z+1)\mathrm{e}^{\mathrm{i}(\mathcal{M}\mu\kappa r - \omega t)} + \text{c.c.}\right] + \cdots .\end{aligned} \tag{4.1}$$

Upon substituting (4.1) into (2.1), (2.2), and (2.4), collecting terms that result in secular behaviour, and imposing solvability conditions, the amplitude evolution equations for the acoustic and gravity modes are derived as detailed below.

4.1.1. *Acoustic mode*

For the acoustic mode, we collect terms $\propto \exp\left[\mathrm{i}(\mathcal{M}\mu\kappa r - \omega t)\right]$. The correction is posed as $\epsilon^2 \left\{F(Z,T)\exp\left[\mathrm{i}(\mathcal{M}\mu\kappa r - \omega t)\right] + \text{c.c.}\right\}$, where F satisfies the boundary-value problem

$$F_{ZZ} + \omega_n^2 F = R_1 \qquad (-1 < Z < 0), \tag{4.2}$$

$$-\omega^2 F = R_2 \qquad (Z = 0), \tag{4.3}$$

$$F_Z = 0 \qquad (Z = -1), \tag{4.4}$$

with

$$R_1 = -\frac{1}{\alpha}\left[2\mathrm{i}\omega\frac{\partial A}{\partial T}\cos\omega_n(Z+1) + \omega_n A\sin\omega_n(Z+1)\right] - 2\mathrm{i}\omega k^2 S_+ S_- \mathrm{e}^{2kz}\mathrm{e}^{\mathrm{i}\beta T}\left(1+\mathcal{M}^2\right), \tag{4.5}$$

and

$$R_2 = \frac{1}{\alpha}(-1)^n\omega_n A + 2\mathrm{i}\omega k^2 S_+ S_- \mathrm{e}^{\mathrm{i}\beta T}\left(1+\mathcal{M}^2\right). \tag{4.6}$$

Since $\cos\omega_n(Z+1)$ is a homogeneous solution, the forcing R_1 and R_2 must obey

$$\int_{-1}^{0} R_1\cos\omega_n(Z+1)\mathrm{d}Z = -(-1)^n\frac{\omega_n}{\omega^2}R_2. \tag{4.7}$$

Redefining $A \to A\mathrm{e}^{\mathrm{i}\beta T}$, in order to eliminate the exponential factor, it follows that

$$\frac{\partial A}{\partial T} = \mathrm{i}A\left(\frac{\kappa^2 - \omega_n^2}{2\omega^3} - 2\beta\right) + \frac{(-1)^n}{8}\alpha\omega_n\omega^2 S_+ S_-\left(1+\mathcal{M}^2\right). \tag{4.8}$$

Choosing $\mathcal{M} = 1$ results in a 2D solution which is in agreement with (Kadri & Akylas 2016). Note that the same solution has been confirmed using the software Maple. For the case of an infinitely long acoustic mode, or two gravity waves with the same wave numbers, $\kappa = 0$, $\omega_n = \omega$, the spatial dependence in the envelope dynamics can be invoked. Introducing the slow spatial variable $R = \mu^{3/2}r$, the evolution equation takes the form

$$\frac{\partial A}{\partial T} = -\frac{\mathrm{i}}{2\omega}\left(\frac{\partial^2 A}{\partial R^2} + \frac{1}{R}\frac{\partial A}{\partial R} + A\right) + \frac{(-1)^n}{8}\alpha\omega^3 S_+ S_-\left(1+\mathcal{M}^2\right). \tag{4.9}$$

The interested reader is referred to Kadri (2017) for more details about the derivation of (4.9). In the sequel we only derive the temporal evolution equations, as the spatial is identical to (4.9) in exact resonant cases and can be easily involved.

4.1.2. *Gravity modes*

On the other hand, for the gravity modes we collect terms $\propto \exp\left[\mathrm{i}(\mathcal{M}k_\pm r - \omega_\pm t)\right]$. Here, the correction is posed as $\epsilon^3\left\{G_\pm(Z)\exp\left[\mathrm{i}(\mathcal{M}k_\pm r - \omega_\pm t)\right] + \text{c.c.}\right\}$, where $G_\pm$ satisfies the boundary-value problem

$$(G_\pm)_{zz} - k_\pm^2 G_\pm = 0, \quad \text{for } -1/\mu < z < 0, \tag{4.10}$$

$$(G_\pm)_z - \omega_\pm^2 G_\pm = \frac{1}{\epsilon^3}\left[-2\mathrm{i}\omega_\pm\epsilon\mu\frac{\mathrm{d}S_\pm}{\mathrm{d}T} + R_\pm^{(3)} + R_\pm^{(4)}\right], \quad \text{at } z = 0, \tag{4.11}$$

$$(G_\pm)_z = 0, \quad \text{at } z = -1/\mu, \tag{4.12}$$

with

$$R_\pm^{(3)} = \frac{1}{4}\mathrm{i}\alpha\epsilon\mu(-1)^n\omega^3\omega_n A S_\mp^* \mathrm{e}^{-\mathrm{i}\beta T} - \frac{1}{4}\epsilon^3\omega^4 k^2(1+\mathcal{M}^2)|S_\mp|^2 S_\pm, \tag{4.13}$$

and

$$R_{\pm}^{(4)} = -\epsilon^3\Big[(-6-8\mathcal{M}^2-2\mathcal{M}^4)k^4|S_{\mp}|^2S_{\pm}+(-3+8\mathcal{M}^2-\mathcal{M}^4)k^4|S_{\pm}|^2S_{\pm}\Big]. \tag{4.14}$$

The solvability condition is thus

$$-2\mathrm{i}\omega_{\pm}\epsilon\mu\frac{\mathrm{d}S_{\pm}}{\mathrm{d}T}+R_{\pm}^{(3)}+R_{\pm}^{(4)}=0, \tag{4.15}$$

which finally yields

$$\begin{aligned}\frac{\mathrm{d}S_{\pm}}{\mathrm{d}T} = -\frac{(-1)^n}{4}\omega_n\omega^2\alpha A S_{\mp}^* - \frac{\mathrm{i}}{256}\omega^7\alpha^2\Big[&(-3+8\mathcal{M}^2-\mathcal{M}^4)|S_{\pm}|^2S_{\pm}^* \\ &-(2+4\mathcal{M}^2+2\mathcal{M}^4)|S_{\mp}|^2S_{\pm}\Big].\end{aligned} \tag{4.16}$$

Taking $\mathcal{M}=1$ reduces, in principle, to the 2D solution equation (5.9) by Kadri & Akylas (2016). However, there is a factor of two in the first term, and a factor of a half in the last third term of (4.16), that arise from higher order terms that were originally neglected by Kadri & Akylas (2016).

4.2. *Inner-outer region*

Since the wavelength of the acoustic mode far exceeds that of the gravity mode, when gravity modes generate an acoustic mode part of the interaction is expected to occur in the inner region of the acoustic mode, yet in the outer region of the gravity modes. Thus, the mixed potential takes the form

$$\begin{aligned}\varphi = \epsilon\left[S_+(T)\mathrm{e}^{|k_+|z}\sqrt{\frac{1}{r}}\mathrm{e}^{\mathrm{i}(b_2k_+r-\omega_+t)}+\text{c.c.}\right]+\epsilon\left[S_-(T)\mathrm{e}^{|k_-|z}\sqrt{\frac{1}{r}}\mathrm{e}^{\mathrm{i}(b_2k_-r-\omega_-t)}+\text{c.c.}\right] \\ +\alpha\left[A(T)\cos\omega_n(Z+1)\mathrm{e}^{\mathrm{i}(\mathcal{M}\mu\kappa r-\omega t)}+\text{c.c.}\right]+\cdots.\end{aligned} \tag{4.17}$$

By using a similar argument to those employed in the previous section, the solvability conditions lead to the governing equations for A and $S_{\pm}$,

$$\frac{\mathrm{d}A}{\mathrm{d}T} = \mathrm{i}A\left(\frac{\kappa^2-\omega_n^2}{2\omega^3}-2\beta\right)+(-1)^n\frac{2\alpha\omega_n}{\omega^2 r}\left[\frac{1}{4r^2}+(b_2^2+1)k^2\right]\mathrm{e}^{\mathrm{i}(b_2-\mathcal{M})\mu\kappa r}S_+S_-, \tag{4.18}$$

$$\frac{\mathrm{d}S_+}{\mathrm{d}T} = -\frac{(-1)^n}{4}\alpha\omega_n\omega^2AS_-^*\mathrm{e}^{\mathrm{i}(\mathcal{M}-b_2)\mu\kappa r}-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_1|S_+|^2S_+-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_2|S_-|^2S_+, \tag{4.19}$$

$$\frac{\mathrm{d}S_-}{\mathrm{d}T} = -\frac{(-1)^n}{4}\alpha\omega_n\omega^2AS_+^*\mathrm{e}^{\mathrm{i}(\mathcal{M}-b_2)\mu\kappa r}-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_1^*|S_-|^2S_--\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_2^*|S_+|^2S_-, \tag{4.20}$$

where

$$\begin{aligned}\mathcal{N}_1 =& 2\left(\frac{1}{4r^2}+b_2^2k^2\right)\left(\frac{3}{4r^2}-\frac{\mathrm{i}b_2k}{r}-b_2^2k^2+2k^2\right) \\ &+\left(\frac{1}{2r}-\mathrm{i}b_2k\right)^2\left(\frac{3}{4r^2}+\frac{\mathrm{i}b_2k}{r}-b_2^2k^2-4k^2\right)-3k^4,\end{aligned}$$

$$\begin{aligned}\mathcal{N}_2 =& 4\left(\frac{1}{4r^2}+b_2^2k^2\right)\left(\frac{3}{4r^2}-\frac{\mathrm{i}b_2k}{r}-b_2^2k^2\right) \\ &+2\left(\frac{1}{2r}-\mathrm{i}b_2k\right)^2\left(\frac{3}{4r^2}+\frac{\mathrm{i}b_2k}{r}-b_2^2k^2+2k^2\right)-2k^4.\end{aligned}$$

Note that r enters as a parameter in the evolution equations. Therefore, the spatially dependent case should have extra terms that are of the same form as the ones derived earlier in (4.9) where the gravity modes are essentially standing waves relative to the acoustic mode.

4.3. *Outer-outer region*

In the outer-outer region all modes obey (3.2), thus the velocity potential takes the form

$$\varphi = \epsilon\left[S_+(T)\mathrm{e}^{|k_+|z}\sqrt{\frac{1}{r}}\mathrm{e}^{\mathrm{i}(b_2k_+r-\omega_+t)}+\mathrm{c.c.}\right]+\epsilon\left[S_-(T)\mathrm{e}^{|k_-|z}\sqrt{\frac{1}{r}}\mathrm{e}^{\mathrm{i}(b_2k_-r-\omega_-t)}+\mathrm{c.c.}\right] \\ +\alpha\left[A(T)\cos\omega_n(Z+1)\sqrt{\frac{1}{r}}\mathrm{e}^{\mathrm{i}(b_2\mu\kappa r-\omega t)}+\mathrm{c.c.}\right]+\cdots, \tag{4.21}$$

where A was normalised by $\exp(-\mathrm{i}b_3)(2/\pi b_1\mu\kappa)^{1/2}$. Next, we substitute the ansatz into the governing equation (2.1) and the boundary condition (2.4), and derive the evolution equations. Repeating the same procedure as before, the governing equations for A and $S_\pm$ read

$$\frac{\mathrm{d}A}{\mathrm{d}T}=\mathrm{i}A\left(\frac{\kappa^2-\omega_n^2}{2\omega^3}-2\beta\right)+(-1)^n\frac{2\alpha\omega_n}{\omega^2\sqrt{r}}\left(\frac{1}{4r^2}+b_2^2k^2+k^2\right)S_+S_-, \tag{4.22}$$

$$\frac{\mathrm{d}S_+}{\mathrm{d}T}=\mathcal{L}AS_-^*-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_1|S_+|^2S_+-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_3|S_-|^2S_+, \tag{4.23}$$

$$\frac{\mathrm{d}S_-}{\mathrm{d}T}=\mathcal{L}^*AS_+^*-\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_1^*|S_-|^2S_--\frac{\mathrm{i}\alpha^2}{\omega r}\mathcal{N}_3^*|S_+|^2S_-, \tag{4.24}$$

where

$$\mathcal{L}=-\frac{\alpha\omega_n}{2\omega^2\sqrt{r}}(-1)^n\left[\frac{\omega^4}{2}-\frac{1}{r}\left(\frac{1}{2r}-\mathrm{i}b_2k\right)\right] \tag{4.25}$$

and

$$\begin{aligned}\mathcal{N}_3 =&4\left(\frac{1}{4r^2}+b_2^2k^2\right)\left(\frac{3}{4r^2}-\frac{\mathrm{i}b_2k}{r}-b_2^2k^2\right)\\ &-\frac{1}{r}\left(\frac{1}{2r}-\mathrm{i}b_2k\right)\left(\frac{1}{4r^2}+b_2^2k^2+k^2\right)\\ &+2\left(\frac{1}{2r}-\mathrm{i}b_2k\right)^2\left(\frac{3}{4r^2}+\frac{\mathrm{i}b_2k}{r}-b_2^2k^2+2k^2\right)-2k^4.\end{aligned} \tag{4.26}$$

Again, the spatial dependence in the exact resonant situation should be the same as derived earlier in (4.9).

5. Results and discussion

In the inner-inner region, the nature of interaction is almost identical to the 2D case. On the other hand, in the outer-outer region and far from the origin, the amplitudes tend to zero due to radial dispersion, and thus there is effectively no more resonant interactions. The more interesting scenario is the interaction involving the mixed regions, i.e. inner for the acoustic, and outer for two gravity modes.

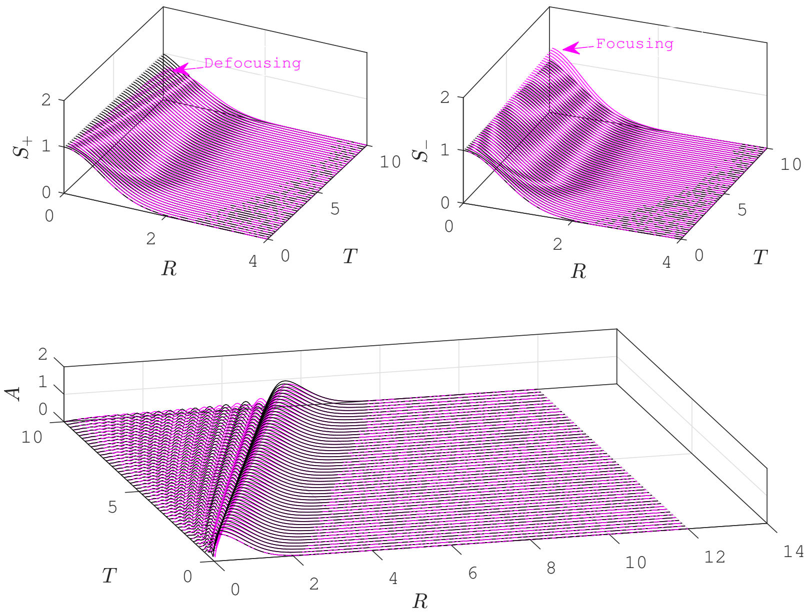

Figure 1: Amplitude spatiotemporal evolution of a triad comprising two gravity waves (top) and an acoustic mode (bottom), for the 3D mixed-field solution (magenta) and 2D solution by Kadri & Akylas (2016) (black). The initial conditions are $A(0) = S_{\pm}(0) = \exp(-r^2)$, $\alpha = 0.1$, $\gamma = 0$, $\kappa = 1$, grid size $\Delta r = 0.05$, and time step $\Delta t = 0.0025$.

To gain a quantitative understanding of the 3D resonant interaction we solved numerically the amplitude equations (4.18), (4.19), and (4.20) (including both temporal and spatial dependence) for two cases. In the first case, the computations focused on Gaussian initial amplitudes for all modes, $A(0) = \exp(-r^2)$, $S_{\pm}(0) = \exp(-r^2)$. In the second case, the acoustic mode was instead considered to be infinitely long ($\kappa = 0$), which acted as a vertically oscillating layer. In both cases an explicit Runge-Kutta method was implemented, and only the leading acoustic mode ($n = 0$) was considered.

In the first case, since the two gravity waves are effectively standing in space, they interact resonantly creating an acoustic 'shock' followed by acoustic ripples, as illustrated in figures 1 and 2. Here, the 3D solution (magenta) is similar to the 2D solution (black). However, a remarkable difference is that in the case of 3D waves, the gravity wave amplitudes either focus or defocus in time, whereas in the 2D case the gravity wave skewness changes. This is rather intuitive, as in the 2D case the gravity waves travel from one side to the other, whereas in the 3D case the gravity waves travel radially inwards or outwards. Since the two gravity waves exist simultaneously, the combined amplitude is comparable to the combined amplitude of the 2D case. The increase in amplitude of the acoustic wave reflects nonlinear interactions between three modes.

The second case is analogous to a surface gravity disturbance (Gaussian) of frequency ω over a fluid layer that is subject to vertical oscillation, e.g. due to underwater tremor, at double the frequency. The interaction excites subharmonic standing field of Faraday-type waves of frequency 2ω, as shown in figure 3. The analogy between the long-crested acoustic mode and an oscillating liquid is made possible since the interaction occurs

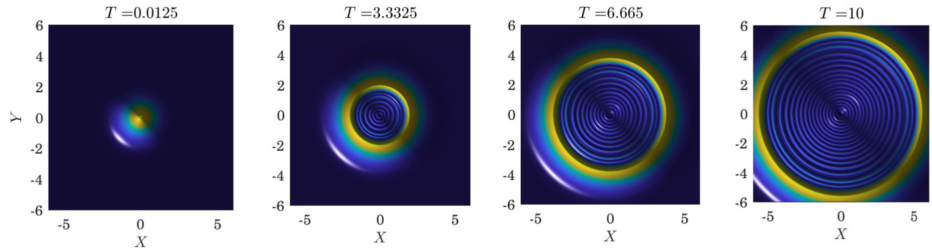

Figure 2: Evolution of waves at the surface from an acoustic mode $A(0) = \exp(-r^2)$, interacting with two gravity waves, $S_{\pm}(0) = \exp(-r^2)$, with $\alpha = 0.1$, $\gamma = 0$, $\kappa = 1$.

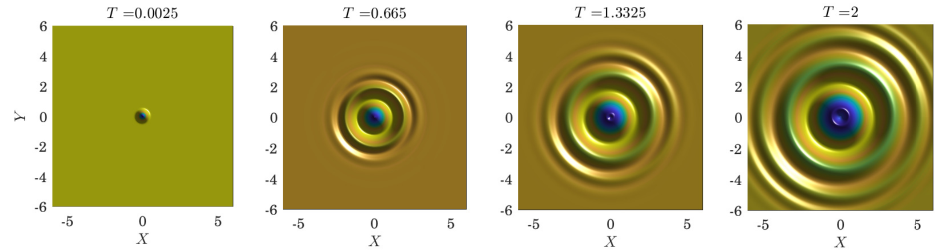

Figure 3: Evolution of Faraday-type waves from a gravity disturbance $S_{\pm}(0) = \exp(-r^2)$ $\alpha = 0.4$, $\gamma = 0.5$, interacting with a long-crested acoustic mode of $A(0) = 1$, and $\kappa = 0$.

at the surface, i.e. for the gravity waves the water layer is a half space. However, the applicability of the theory to laboratory scales, e.g. in an analogy to an oscillating bath, is by no means straightforward. The main difficulty is that under the current rigid bottom assumption, the frequency of the acoustic mode should be unrealistically high to exist. However, Kadri (2019b) showed recently that one could overcome this difficulty by considering an elastic bottom instead, which modifies the dispersion relations, though not the structure of the evolution equations:

$$\omega^2 = \omega_n^2 + \kappa^2 + \mu \frac{\omega_n^2 - \kappa^2 - (\omega_n^2 + \kappa^2)\mathcal{E}_1/\mu}{\omega_n^2 + \kappa^2 + \mu\mathcal{E}_2/2} + O(\mu^2), \tag{5.1}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ represent elastic terms that are found from the boundary conditions. Under these settings, the result in Figure 3, could be analogous to the generation of Faraday-type waves in a laboratory-size container.

As a final note, the motivation behind the proposed approximation is to allow collecting secular terms in a non-integral form, which otherwise becomes awkward when employing exact Bessel functions that involve quadratic and cubic products of zeroth, first and second derivatives of the Bessel functions. For example, the product of the first derivative of two modes can be of the form $J_{-1}(k_+r)J_{-1}(k_-r) - J_{-1}(k_+r)J_{+1}(k_-r) - J_{+1}(k_+r)J_{-1}(k_-r) + J_{+1}(k_+r)J_{+1}(k_-r)$, and it is not obvious how this term resonates with the third mode presented by $J_0(\mu\kappa r)$. On the other hand, the proposed approximation uses complex analysis which allows collecting secular terms in a straight forward way, e.g. $\exp(\mathrm{i}k_+r)\exp(\mathrm{i}k_-r) \equiv \exp(\mathrm{i}\mu\kappa r)$ clearly satisfies the resonance condition $k_+ + k_- = \mu\kappa$. To validate the results numerically, one could seek a full numerical solution of eqs.(2.1), (2.2) and (2.4), with (2.20). An alternative would be using exact Bessel solution with an attempt to numerically identify secular terms. Both options are beyond the scope of this study and are left for future work.

This work was supported in part by the National Natural Science Foundation of China (No.11772341), and the Key Research Program of Frontier Sciences, CAS (No. QYZDBSSW-SYS015).

REFERENCES

Craik, A. D. D. 1984. Wave Interactions and Fluid Flows, p. 322. Cambridge University Press.

Kadri, U. & Stiassnie, M. 2013 Generation of an acoustic–gravity wave by two gravity waves, and their subsequent mutual interaction. *J. Fluid Mech.* **735**, R6.

Kadri, U. 2015 Wave motion in a heavy compressible fluid: Revisited. *Eur. J. Mech. B Fluids* **49**, 50–57.

Kadri, U. & Akylas, T. R. 2016 On resonant triad interactions of acoustic–gravity waves. *J. Fluid Mech.* **788**, R1.

Kadri, U. 2017 Faraday waves by resonant triad interactions of surface-compression waves. axive.org/pdf/1701.00667.pdf

Kadri, U. 2019a Multiple-location matched approximation for Bessel function J_0 and its derivatives. *Commun. Nonlinear Sci.* **72**, 59–63.

Kadri, U. 2019b Time-reversal analogy by nonlinear acoustic–gravity wave triad resonance. *Fluids* **4**(2), 91.

Longuet-Higgins, M. S. 1950 A theory of the origin of microseisms. *Phil. Trans. R. Soc. Lond.* **243**, 1–35.

Miles, J. W. 1984. Internally resonant surface waves in a circular cylinder *J. Fluid Mech.* **149**, 1–14.

Phillips, O. M. 1981. Wave interactions – the evolution of an idea *J. Fluid Mech.* **106**, 215–227.

Pidduck, F. B. 1910 On the propagation of a disturbance in a fluid under gravity. *Proc. R. Soc. A* **83**, 347–356.

Pidduck, F. B. 1912 The wave-problem of Cauchy and Poisson for finite depth and slightly compressible fluid. *Proc. R. Soc. A* **86**, 396–405.

Whipple, F. J. W. & Lee, A. W. 1935 Notes on the theory of microseisms. *Mon. Not. R. Astron. Soc. Geophys. Suppl.* **3**, 287–297.

Yang, X., Dias, F. & Liao, S. 2018 On the steady-state resonant acoustic-gravity waves. *J. Fluid Mech.* **849**, 111–135.