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# ON THE VOLUME OF HYPERPLANE SECTIONS OF A d-CUBE 

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#### Abstract

We obtain an optimal upper bound for the normalised volume of a hyperplane section of an origin-symmetric $d$-dimensional cube. This confirms a conjecture posed by Imre Bárány and Péter Frankl.


## 1. Statement of the results

Let $\mathcal{C}^{d}=[-1 / 2,1 / 2]^{d}$ be the $d$-dimensional unit cube. Throughout this paper we assume that $d \geq 2$. For a nonzero vector $\boldsymbol{v} \in \mathbb{R}^{d}$ we will denote by $\boldsymbol{v}^{\perp}$ the hyperplane orthogonal to $\boldsymbol{v}$ and consider the section $\mathcal{C}^{d} \cap \boldsymbol{v}^{\perp}$ of the cube $\mathcal{C}^{d}$. Let further $\|\cdot\|_{1}$ and $\|\cdot\|_{2}$ denote $\ell_{1}$ and $\ell_{2}$ norms, respectively. In this paper we will be interested in the quantity

$$
\begin{equation*}
V_{d}=\max _{\boldsymbol{v} \in \mathbb{R}^{d}} \frac{\|\boldsymbol{v}\|_{1}}{\|\boldsymbol{v}\|_{2}} \cdot \operatorname{vol}_{d-1}\left(\mathcal{C}^{d} \cap \boldsymbol{v}^{\perp}\right) \tag{1}
\end{equation*}
$$

where $\operatorname{vol}_{d-1}(\cdot)$ stands for the $(d-1)$-volume. Imre Bárány and Péter Frankl [3] conjectured that the maximum in (1) is given by the vector $\boldsymbol{v}=\mathbf{1}_{d}:=(1, \ldots, 1)$. Our main result confirms this conjecture.
Theorem 1. We have

$$
\begin{equation*}
V_{d}=\sqrt{d} \cdot \operatorname{vol}_{d-1}\left(\mathcal{C}^{d} \cap \mathbf{1}_{d}^{\perp}\right) \tag{2}
\end{equation*}
$$

It is known that

$$
\lim _{d \rightarrow \infty} \operatorname{vol}_{d-1}\left(\mathcal{C}^{d} \cap \mathbf{1}_{d}^{\perp}\right)=\sqrt{\frac{6}{\pi}}
$$

(see [9], [11] and e. g. [5]). The expression (2) also allows finding the exact value of $V_{d}$ in the following way. Let $\boldsymbol{s}=\left(s_{1}, \ldots, s_{d}\right) \in \mathbb{S}^{d-1}$ be a unit vector. Then

$$
\begin{equation*}
\operatorname{vol}_{d-1}\left(\mathcal{C}^{d} \cap s^{\perp}\right)=\frac{2}{\pi} \int_{0}^{\infty} \prod_{i=1}^{d} \frac{\sin s_{i} t}{s_{i} t} \mathrm{~d} t \tag{3}
\end{equation*}
$$

(see e. g. [2]). Consider the sinc integral [4]

$$
\sigma_{d}=\frac{2}{\pi} \int_{0}^{\infty}\left(\frac{\sin t}{t}\right)^{d} \mathrm{~d} t
$$

[^0]In view of (2) and (3) we have

$$
\begin{equation*}
V_{d}=\frac{2 \sqrt{d}}{\pi} \int_{0}^{\infty}\left(\frac{\sin \frac{t}{\sqrt{d}}}{\frac{t}{\sqrt{d}}}\right)^{d} \mathrm{~d} t=d \sigma_{d} \tag{4}
\end{equation*}
$$

Further

$$
\sigma_{d}=\frac{d}{2^{d-1}} \sum_{0 \leq r<d / 2, r \in \mathbb{Z}} \frac{(-1)^{r}(d-2 r)^{d-1}}{r!(d-r)!}
$$

(see e. g. [10]). The sequences of numerators and denominators of $\sigma_{d} / 2$ can be found in [12].
Theorem 1 and (3) immediately imply the following lower bound for sinc integrals.
Corollary 2. For any unit vector $\boldsymbol{s}=\left(s_{1}, \ldots, s_{d}\right) \in \mathbb{S}^{d-1}$

$$
\frac{2\|\boldsymbol{s}\|_{1}}{\pi d} \int_{0}^{\infty} \prod_{i=1}^{d} \frac{\sin s_{i} t}{s_{i} t} \mathrm{~d} t \leq \sigma_{d}
$$

It is known that $0<\sigma_{d+1} / \sigma_{d}<1$ (see e. g. [1, Lemma 1]). Theorem 1 also implies the following lower bound for the ratio of consecutive sinc integrals.

## Corollary 3. We have

$$
\frac{d}{d+1} \leq \frac{\sigma_{d+1}}{\sigma_{d}}
$$

## 2. Intersection body of $\mathcal{C}^{d}$

We can associate with each star body $\mathcal{L}$ the distance function $f_{\mathcal{L}}(\boldsymbol{x})=\inf \{\lambda>0: \boldsymbol{x} \in$ $\lambda \mathcal{L}\}$. The intersection body $I \mathcal{L}$ of a star body $\mathcal{L} \subset \mathbb{R}^{d}$ (recall that we assume $d \geq 2$ ) is defined as the $\mathbf{0}$-symmetric star body with distance function

$$
f_{I \mathcal{L}}(\boldsymbol{x})=\frac{\|\boldsymbol{x}\|_{2}}{\operatorname{vol}_{d-1}\left(\mathcal{L} \cap \boldsymbol{x}^{\perp}\right)}
$$

The Busemann theorem (see e. g. [6], Chapter 8 ) states that if $\mathcal{L}$ is $\mathbf{0}$-symmetric and convex, then $I \mathcal{L}$ is a convex set. For more details on intersection bodies we refer the reader to $[7,8]$.

For convenience, in what follows we will work with normalised cube

$$
\mathcal{Q}^{d}=\frac{1}{\operatorname{vol}_{d-1}\left(\mathcal{C}^{d} \cap \mathbf{1}_{d}^{\perp}\right)^{1 /(d-1)}} \cdot \mathcal{C}^{d}
$$

Then, in particular,

$$
\begin{equation*}
\operatorname{vol}_{d-1}\left(\mathcal{Q}^{d} \cap \mathbf{1}_{d}^{\perp}\right)=1 \tag{5}
\end{equation*}
$$

Lemma 4. The affine hyperplane

$$
\mathcal{H}=\left\{\boldsymbol{x} \in \mathbb{R}^{d}: x_{1}+\cdots+x_{d}=\sqrt{d}\right\}
$$

is a supporting hyperplane of $I \mathcal{Q}^{d}$.
Proof. Let $f=f_{I \mathcal{Q}^{d}}$ denote the distance function of $I \mathcal{Q}^{d}$, so that $I \mathcal{Q}^{d}=\left\{\boldsymbol{x} \in \mathbb{R}^{d}: f(\boldsymbol{x}) \leq\right.$ $1\}$. By (5), for the point

$$
\begin{equation*}
\boldsymbol{h}:=\frac{1}{\sqrt{d}} \mathbf{1}_{d}=\left(\frac{1}{\sqrt{d}}, \ldots, \frac{1}{\sqrt{d}}\right) \tag{6}
\end{equation*}
$$

we have $f(\boldsymbol{h})=1$. Therefore $\boldsymbol{h}$ is on the boundary of $I \mathcal{Q}^{d}$.
Suppose, to derive a contradiction, that $\mathcal{H}$ is not a supporting hyperplane of $I \mathcal{Q}^{d}$. Observe that $\boldsymbol{h} \in \mathcal{H} \cap I \mathcal{Q}^{d}$. Hence for any $\epsilon>0$ there exists a point $\boldsymbol{p}=\left(p_{1}, \ldots, p_{d}\right)$ in the interior of $I \mathcal{Q}^{d}$ with

$$
\begin{equation*}
\|\boldsymbol{h}-\boldsymbol{p}\|_{2}<\epsilon \tag{7}
\end{equation*}
$$

and $p_{1}+\cdots+p_{d}>\sqrt{d}$.
By (7) we may assume that $\boldsymbol{p} \in \mathbb{R}_{>0}^{d}$. Further, as the point $\boldsymbol{p}$ is in the interior of $I \mathcal{Q}^{d}$ we may assume, for simplicity, that the entries of $\boldsymbol{p}$ are pairwise distinct: $p_{i} \neq p_{j}$ for $i \neq j$. Consider $d$ points

$$
\begin{aligned}
& \boldsymbol{p}_{1}=\left(\begin{array}{llll}
p_{1}, & \ldots, & p_{d-1}, & p_{d}
\end{array}\right) \\
& \boldsymbol{p}_{2}=\left(\begin{array}{llll}
p_{2}, & \ldots, & p_{d}, & p_{1}
\end{array}\right) \\
& \vdots \\
& \boldsymbol{p}_{d}=\left(\begin{array}{llll}
p_{d}, & \ldots, & p_{d-2}, & p_{d-1}
\end{array}\right) .
\end{aligned}
$$

For each $i$, the section $\mathcal{Q}^{d} \cap \boldsymbol{p}_{i}^{\perp}$ is the image of the section $\mathcal{Q}^{d} \cap \boldsymbol{p}_{1}^{\perp}$ under an orthogonal transformation defined by a permutation matrix. Therefore $\boldsymbol{p}_{i} \in I \mathcal{Q}^{d}$. Set

$$
\boldsymbol{y}=\frac{1}{d}\left(\boldsymbol{p}_{1}+\cdots+\boldsymbol{p}_{d}\right)=\frac{\sum_{i=1}^{d} p_{i}}{\sqrt{d}} \boldsymbol{h}
$$

By construction, $\boldsymbol{y}$ is a convex combination of the points $\boldsymbol{p}_{1}, \ldots, \boldsymbol{p}_{d}$. Since $I \mathcal{Q}^{d}$ is convex, $\boldsymbol{y}=\left(y_{1}, \ldots, y_{d}\right) \in I \mathcal{Q}^{d}$. Further

$$
y_{1}+\cdots+y_{d}=\sum_{i=1}^{d} p_{i}>\sqrt{d}
$$

Therefore the point $\boldsymbol{h}$ must be in the interior of $I \mathcal{Q}^{d}$. The derived contradiction completes the proof.

## 3. Proof of Theorem 1

It is sufficient to show that for any unit vector $\boldsymbol{v} \in \mathbb{S}^{d-1}$ the inequality

$$
\begin{equation*}
\|\boldsymbol{v}\|_{1} \cdot \operatorname{vol}_{d-1}\left(\mathcal{Q}^{d} \cap \boldsymbol{v}^{\perp}\right) \leq \sqrt{d} \cdot \operatorname{vol}_{d-1}\left(\mathcal{Q}^{d} \cap \mathbf{1}_{d}^{\perp}\right)=\sqrt{d} \tag{8}
\end{equation*}
$$

holds.
In view of symmetry of $\mathcal{Q}^{d}$ we may assume without loss of generality that $\boldsymbol{v} \in \mathbb{R}_{>0}^{d}$. Consider the plane $\mathcal{P}$ spanned by the vector $\boldsymbol{h}$, defined by (6), and the vector $\boldsymbol{v}$ and let $\alpha$ be the angle between these two vectors with $\cos (\alpha)=\boldsymbol{h} \cdot \boldsymbol{v}$ (Figure 1). It is not difficult to see that $\cos (\alpha) \geq 1 / \sqrt{d}$ and, consequently, $\alpha<\pi / 2$.

Notice that $\boldsymbol{h}$ is orthogonal to the line $\mathcal{H} \cap \mathcal{P}$. Let $\boldsymbol{u}$ denote the intersection point of the line spanned by $\boldsymbol{v}$ and $\mathcal{H} \cap \mathcal{P}$. Further, let $\boldsymbol{w}$ be the orthogonal projection of $\boldsymbol{v}$ onto the line spanned by $\boldsymbol{h}$.

Then we have

$$
\begin{equation*}
\cos (\alpha)=\|\boldsymbol{w}\|_{2}=\frac{1}{\|\boldsymbol{u}\|_{2}} \tag{9}
\end{equation*}
$$



Figure 1. Geometric argument on the plane $\mathcal{P}$

Since $\boldsymbol{h} \in \mathcal{H}$, all points $\boldsymbol{x}$ on the line passing through the points $\boldsymbol{v}$ and $\boldsymbol{w}$ have $x_{1}+\cdots+x_{d}=$ $\sqrt{d}\|\boldsymbol{w}\|_{2}$. Therefore, we have $\|\boldsymbol{v}\|_{1}=\sqrt{d}\|\boldsymbol{w}\|_{2}$. In was shown in Lemma 4 that $\mathcal{H}$ is a supporting hyperplane of $I \mathcal{Q}^{d}$. Hence we have

$$
\begin{equation*}
\operatorname{vol}_{d-1}\left(\mathcal{Q}^{d} \cap \boldsymbol{v}^{\perp}\right) \leq\|\boldsymbol{u}\|_{2} . \tag{10}
\end{equation*}
$$

Finally, using (9) and (10), we have

$$
\|\boldsymbol{v}\|_{1} \cdot \operatorname{vol}_{d-1}\left(\mathcal{Q}^{d} \cap \boldsymbol{v}^{\perp}\right) \leq \sqrt{d}\|\boldsymbol{w}\|_{2}\|\boldsymbol{u}\|_{2}=\sqrt{d}
$$

that confirms (8).

## 4. Proof of Corollary 3

It was observed in [3] that the sequence $\left\{V_{d}\right\}_{d=1}^{\infty}$ is increasing: $V_{d} \leq V_{d+1}$ for all $d \geq 2$. It is now sufficient to note that, by Theorem 1 (see (4)), we can write $V_{d}=d \sigma_{d}$.

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