ON THE VOLUME OF HYPERPLANE SECTIONS OF A d-CUBE

ISKANDER ALIEV

Mathematics Institute, Cardiff University, Cardiff, Wales, UK

ABSTRACT. We obtain an optimal upper bound for the normalised volume of a hyperplane section of an origin-symmetric d-dimensional cube. This confirms a conjecture posed by Imre Bárány and Péter Frankl.

1. Statement of the results

Let $\mathcal{C}^d = [-1/2, 1/2]^d$ be the d-dimensional unit cube. Throughout this paper we assume that $d \geq 2$. For a nonzero vector $\boldsymbol{v} \in \mathbb{R}^d$ we will denote by $\boldsymbol{v}^\perp$ the hyperplane orthogonal to $\boldsymbol{v}$ and consider the section $\mathcal{C}^d \cap \boldsymbol{v}^\perp$ of the cube $\mathcal{C}^d$. Let further $\|\cdot\|_1$ and $\|\cdot\|_2$ denote ℓ_1 and ℓ_2 norms, respectively. In this paper we will be interested in the quantity

$$V_d = \max_{\boldsymbol{v}\in\mathbb{R}^d} \frac{\|\boldsymbol{v}\|_1}{\|\boldsymbol{v}\|_2} \cdot \mathrm{vol}_{d-1}(\mathcal{C}^d \cap \boldsymbol{v}^\perp)\,, \tag{1}$$

where $\mathrm{vol}_{d-1}(\cdot)$ stands for the $(d-1)$-volume. Imre Bárány and Péter Frankl [3] conjectured that the maximum in (1) is given by the vector $\boldsymbol{v} = \mathbf{1}_d := (1, \ldots, 1)$. Our main result confirms this conjecture.

Theorem 1. *We have*

$$V_d = \sqrt{d} \cdot \mathrm{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp)\,. \tag{2}$$

It is known that

$$\lim_{d\to\infty} \mathrm{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp) = \sqrt{\frac{6}{\pi}}$$

(see [9], [11] and e. g. [5]). The expression (2) also allows finding the exact value of V_d in the following way. Let $\boldsymbol{s} = (s_1, \ldots, s_d) \in \mathbb{S}^{d-1}$ be a unit vector. Then

$$\mathrm{vol}_{d-1}(\mathcal{C}^d \cap \boldsymbol{s}^\perp) = \frac{2}{\pi}\int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t}\,\mathrm{d}t \tag{3}$$

(see e. g. [2]). Consider the *sinc* integral [4]

$$\sigma_d = \frac{2}{\pi}\int_0^\infty \left(\frac{\sin t}{t}\right)^d \mathrm{d}t\ .$$

E-mail address: `alievi@cardiff.ac.uk`.
Date: June 4, 2020.
2010 *Mathematics Subject Classification.* 52A40, 52A38.

In view of (2) and (3) we have

$$V_d = \frac{2\sqrt{d}}{\pi}\int_0^\infty \left(\frac{\sin\frac{t}{\sqrt{d}}}{\frac{t}{\sqrt{d}}}\right)^d \mathrm{d}t = d\,\sigma_d\,. \tag{4}$$

Further

$$\sigma_d = \frac{d}{2^{d-1}} \sum_{0\le r<d/2,\, r\in\mathbb{Z}} \frac{(-1)^r(d-2r)^{d-1}}{r!(d-r)!}$$

(see e. g. [10]). The sequences of numerators and denominators of $\sigma_d/2$ can be found in [12].

Theorem 1 and (3) immediately imply the following lower bound for sinc integrals.

Corollary 2. *For any unit vector $\boldsymbol{s} = (s_1, \dots, s_d) \in \mathbb{S}^{d-1}$*

$$\frac{2||\boldsymbol{s}||_1}{\pi d}\int_0^\infty \prod_{i=1}^d \frac{\sin s_i t}{s_i t}\,\mathrm{d}t \le \sigma_d\,.$$

It is known that $0 < \sigma_{d+1}/\sigma_d < 1$ (see e. g. [1, Lemma 1]). Theorem 1 also implies the following lower bound for the ratio of consecutive sinc integrals.

Corollary 3. *We have*

$$\frac{d}{d+1} \le \frac{\sigma_{d+1}}{\sigma_d}\,.$$

2. Intersection body of $\mathcal{C}^d$

We can associate with each star body $\mathcal{L}$ the *distance function* $f_{\mathcal{L}}(\boldsymbol{x}) = \inf\{\lambda > 0 : \boldsymbol{x} \in \lambda\mathcal{L}\}$. The *intersection body* $I\mathcal{L}$ of a star body $\mathcal{L} \subset \mathbb{R}^d$ (recall that we assume $d \ge 2$) is defined as the $\mathbf{0}$–symmetric star body with distance function

$$f_{I\mathcal{L}}(\boldsymbol{x}) = \frac{\|\boldsymbol{x}\|_2}{\mathrm{vol}_{d-1}(\mathcal{L}\cap \boldsymbol{x}^\perp)}\,.$$

The Busemann theorem (see e. g. [6], Chapter 8) states that if $\mathcal{L}$ is $\mathbf{0}$–symmetric and convex, then $I\mathcal{L}$ is a convex set. For more details on intersection bodies we refer the reader to [7, 8].

For convenience, in what follows we will work with normalised cube

$$\mathcal{Q}^d = \frac{1}{\mathrm{vol}_{d-1}(\mathcal{C}^d \cap \mathbf{1}_d^\perp)^{1/(d-1)}}\cdot \mathcal{C}^d\,.$$

Then, in particular,

$$\mathrm{vol}_{d-1}(\mathcal{Q}^d \cap \mathbf{1}_d^\perp) = 1\,. \tag{5}$$

Lemma 4. *The affine hyperplane*

$$\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^d : x_1 + \dots + x_d = \sqrt{d}\}$$

is a supporting hyperplane of $I\mathcal{Q}^d$.

Proof. Let $f = f_{I\mathcal{Q}^d}$ denote the distance function of $I\mathcal{Q}^d$, so that $I\mathcal{Q}^d = \{\boldsymbol{x} \in \mathbb{R}^d : f(\boldsymbol{x}) \le 1\}$. By (5), for the point

$$\boldsymbol{h} := \frac{1}{\sqrt{d}}\mathbf{1}_d = \left(\frac{1}{\sqrt{d}}, \dots, \frac{1}{\sqrt{d}}\right) \tag{6}$$

we have $f(\boldsymbol{h}) = 1$. Therefore $\boldsymbol{h}$ is on the boundary of $I\mathcal{Q}^d$.

Suppose, to derive a contradiction, that $\mathcal{H}$ is not a supporting hyperplane of $I\mathcal{Q}^d$. Observe that $\boldsymbol{h} \in \mathcal{H} \cap I\mathcal{Q}^d$. Hence for any $\epsilon > 0$ there exists a point $\boldsymbol{p} = (p_1, \dots, p_d)$ in the interior of $I\mathcal{Q}^d$ with

$$\|\boldsymbol{h} - \boldsymbol{p}\|_2 < \epsilon \tag{7}$$

and $p_1 + \cdots + p_d > \sqrt{d}$.

By (7) we may assume that $\boldsymbol{p} \in \mathbb{R}^d_{>0}$. Further, as the point $\boldsymbol{p}$ is in the interior of $I\mathcal{Q}^d$ we may assume, for simplicity, that the entries of $\boldsymbol{p}$ are pairwise distinct: $p_i \neq p_j$ for $i \neq j$. Consider d points

$$\begin{array}{lcllll} \boldsymbol{p}_1 & = & (p_1, & \dots, & p_{d-1}, & p_d) \\ \boldsymbol{p}_2 & = & (p_2, & \dots, & p_d, & p_1) \\ \vdots & & & & & \\ \boldsymbol{p}_d & = & (p_d, & \dots, & p_{d-2}, & p_{d-1}) \,. \end{array}$$

For each i, the section $\mathcal{Q}^d \cap \boldsymbol{p}_i^{\perp}$ is the image of the section $\mathcal{Q}^d \cap \boldsymbol{p}_1^{\perp}$ under an orthogonal transformation defined by a permutation matrix. Therefore $\boldsymbol{p}_i \in I\mathcal{Q}^d$. Set

$$\boldsymbol{y} = \frac{1}{d}(\boldsymbol{p}_1 + \cdots + \boldsymbol{p}_d) = \frac{\sum_{i=1}^d p_i}{\sqrt{d}} \boldsymbol{h} \,.$$

By construction, $\boldsymbol{y}$ is a convex combination of the points $\boldsymbol{p}_1, \dots, \boldsymbol{p}_d$. Since $I\mathcal{Q}^d$ is convex, $\boldsymbol{y} = (y_1, \dots, y_d) \in I\mathcal{Q}^d$. Further

$$y_1 + \cdots + y_d = \sum_{i=1}^d p_i > \sqrt{d} \,.$$

Therefore the point $\boldsymbol{h}$ must be in the interior of $I\mathcal{Q}^d$. The derived contradiction completes the proof.

□

3. Proof of Theorem 1

It is sufficient to show that for any unit vector $\boldsymbol{v} \in \mathbb{S}^{d-1}$ the inequality

$$\|\boldsymbol{v}\|_1 \cdot \mathrm{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^{\perp}) \leq \sqrt{d} \cdot \mathrm{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{1}_d^{\perp}) = \sqrt{d} \tag{8}$$

holds.

In view of symmetry of $\mathcal{Q}^d$ we may assume without loss of generality that $\boldsymbol{v} \in \mathbb{R}^d_{\geq 0}$. Consider the plane $\mathcal{P}$ spanned by the vector $\boldsymbol{h}$, defined by (6), and the vector $\boldsymbol{v}$ and let α be the angle between these two vectors with $\cos(\alpha) = \boldsymbol{h} \cdot \boldsymbol{v}$ (Figure 1). It is not difficult to see that $\cos(\alpha) \geq 1/\sqrt{d}$ and, consequently, $\alpha < \pi/2$.

Notice that $\boldsymbol{h}$ is orthogonal to the line $\mathcal{H} \cap \mathcal{P}$. Let $\boldsymbol{u}$ denote the intersection point of the line spanned by $\boldsymbol{v}$ and $\mathcal{H} \cap \mathcal{P}$. Further, let $\boldsymbol{w}$ be the orthogonal projection of $\boldsymbol{v}$ onto the line spanned by $\boldsymbol{h}$.

Then we have

$$\cos(\alpha) = \|\boldsymbol{w}\|_2 = \frac{1}{\|\boldsymbol{u}\|_2} \,. \tag{9}$$

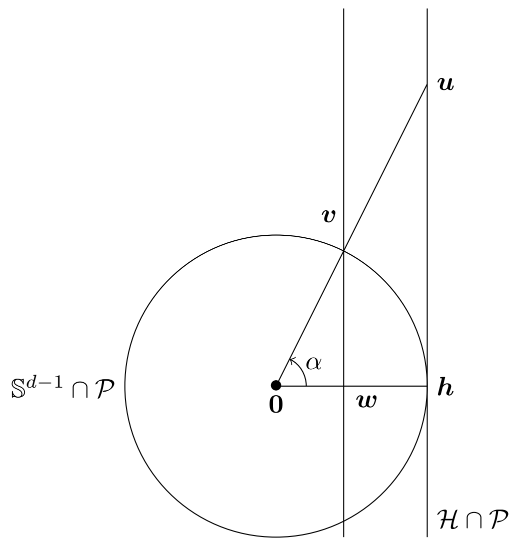

FIGURE 1. Geometric argument on the plane $\mathcal{P}$

Since $\boldsymbol{h} \in \mathcal{H}$, all points $\boldsymbol{x}$ on the line passing through the points $\boldsymbol{v}$ and $\boldsymbol{w}$ have $x_1 + \cdots + x_d = \sqrt{d}\,\|\boldsymbol{w}\|_2$. Therefore, we have $\|\boldsymbol{v}\|_1 = \sqrt{d}\,\|\boldsymbol{w}\|_2$. In was shown in Lemma 4 that $\mathcal{H}$ is a supporting hyperplane of $I\mathcal{Q}^d$. Hence we have

$$\mathrm{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^\perp) \le \|\boldsymbol{u}\|_2\,. \tag{10}$$

Finally, using (9) and (10), we have

$$\|\boldsymbol{v}\|_1 \cdot \mathrm{vol}_{d-1}(\mathcal{Q}^d \cap \boldsymbol{v}^\perp) \le \sqrt{d}\,\|\boldsymbol{w}\|_2 \|\boldsymbol{u}\|_2 = \sqrt{d}\,,$$

that confirms (8).

4. Proof of Corollary 3

It was observed in [3] that the sequence $\{V_d\}_{d=1}^{\infty}$ is increasing: $V_d \le V_{d+1}$ for all $d \ge 2$. It is now sufficient to note that, by Theorem 1 (see (4)), we can write $V_d = d\sigma_d$.

5. Acknowledgements

The author wishes to thank Gergely Ambrus, Imre Bárány and Martin Henk for valuable comments and suggestions.

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