DISTANCE-SPARSITY TRANSFERENCE FOR VERTICES OF CORNER POLYHEDRA

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Abstract. We obtain a transference bound for vertices of corner polyhedra that connects two well-established areas of research: proximity and sparsity of solutions to integer programs. In the knapsack scenario, it implies that for any vertex $\boldsymbol{x}^*$ of an integer feasible knapsack polytope $P(\boldsymbol{a}, b) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \boldsymbol{a}^\top \boldsymbol{x} = b\}$, $\boldsymbol{a} \in \mathbb{Z}^n_{>0}$, there exists an integer point $\boldsymbol{z}^* \in P(\boldsymbol{a}, b)$ such that, denoting by s the size of the support of $\boldsymbol{z}^*$ and assuming $s > 0$,

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^{s-1}}{s} < \|\boldsymbol{a}\|_\infty \,,$$

where $\|\cdot\|_\infty$ stands for the ℓ_∞-norm. The bound gives an exponential in s improvement on previously known proximity estimates. In addition, for general integer linear programs we obtain a resembling result that connects the minimum absolute nonzero entry of an optimal solution with the size of its support.

1. Introduction and statement of results

The main contribution of this paper shows a surprising relation that holds between two well-established areas of research, proximity and sparsity of solutions to integer programs, in the case of Gomory's corner polyhedra.

The proximity-type results provide estimates for the distance between optimal vertex solutions of linear programming relaxations and feasible integer points, with seminal works by Cook et al. [11] and, more recently, by Eisenbrand and Weismantel [13]. The sparsity-type results, in their turn, provide bounds for the size of support of feasible integer points and solutions to integer programs. Bounds of this type are dated back to the classical integer Carathéodory theorems of Cook, Fonlupt and Schriver [10] and Sebő [17]. More recent contributions include results of Eisenbrand and Shmonin [12] and Aliev et al. [1, 2, 3]. Further, in a very recent work Lee, Paat, Stallknecht and Xu [15] apply new sparsity-type bounds to refine the bounds for proximity.

To state the main results of this paper, we will need the following notation. Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, and let $\tau = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$ with $i_1 < i_2 < \cdots < i_k$. We will use the notation A_τ for the $m \times k$ submatrix of A with columns indexed by τ. In the same manner, given $\boldsymbol{x} \in \mathbb{R}^n$, we will denote by $\boldsymbol{x}_\tau$ the vector $(x_{i_1}, \ldots, x_{i_k})^\top$. The complement of τ in $\{1, \ldots, n\}$ will be denoted as $\bar{\tau}$. We will say that τ is a *basis* of A if $|\tau| = m$ and the submatrix A_τ is nonsingular. By $\Sigma(A)$ we will denote the maximum absolute $m \times m$ subdeterminant of A:

$$\Sigma(A) = \max\{|\det(A_\tau)| : \tau \subset \{1, \ldots, n\} \text{ with } |\tau| = m\} \,.$$

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When $\Sigma(A)$ is positive, $\gcd(A)$ will denote the greatest common divisor of all $m \times m$ subdeterminants of A.

For $\boldsymbol{x} = (x_1, \dots, x_n)^\top \in \mathbb{R}^n$, the ℓ_∞-norm of $\boldsymbol{x}$ will be denoted as $\|\boldsymbol{x}\|_\infty$. We will denote by $\operatorname{supp}(\boldsymbol{x}) := \{i \in \{1, \dots, n\} : x_i \neq 0\}$ the *support* of $\boldsymbol{x}$. Further, $\|\boldsymbol{x}\|_0 := |\operatorname{supp}(\boldsymbol{x})|$ will denote the 0-"norm", widely used in the theory of compressed sensing [6, 8], which counts the cardinality of the support of $\boldsymbol{x}$. The notation $\log(\cdot)$ will be used for logarithm with base two.

Let $A \in \mathbb{Z}^{m\times n}$ with $m < n$ and $\boldsymbol{b} \in \mathbb{Z}^m$. Without loss of generality, we will assume that A has rank m. Consider the polyhedron

$$P(A, \boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : A\boldsymbol{x} = \boldsymbol{b}\}$$

and, assuming $P(A, \boldsymbol{b})$ is not empty, take any vertex $\boldsymbol{x}^*$ of $P(A, \boldsymbol{b})$. There is a basis γ of A such that

$$\boldsymbol{x}^*_\gamma = A_\gamma^{-1}\boldsymbol{b} \text{ and } \boldsymbol{x}^*_{\bar{\gamma}} = \boldsymbol{0}\,. \tag{1}$$

In general, for a given vertex there can be many choices for the basis γ in (1). However, if $\boldsymbol{x}^*$ is *nondegenerate*; that is, if the size of the support of $\boldsymbol{x}^*$ is m, then there is a unique choice for γ, namely $\gamma = \operatorname{supp}(\boldsymbol{x})$.

For a set $S \subset \mathbb{R}^n$ we will denote by $\operatorname{conv}(S)$ the *convex hull* of S. Let τ be a subset of $\{1, \dots, n\}$. Following Gomory [14] and Thomas [18, §2] we define the *corner polyhedron* $C_\tau(A, \boldsymbol{b})$ associated with τ as

$$C_\tau(A, \boldsymbol{b}) = \operatorname{conv}(\{\boldsymbol{x} \in \mathbb{Z}^n : A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x}_{\bar{\tau}} \geq \boldsymbol{0}\}).$$

Theorem 1. *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m and $\boldsymbol{b} \in \mathbb{Z}^m$. Let $\boldsymbol{x}^*$ be a vertex of the polyhedron $P(A, \boldsymbol{b})$ given by a basis γ as in* (1) *and let $C_\gamma(A, \boldsymbol{b}) \neq \emptyset$. Let $\boldsymbol{z}^*$ be a vertex of $C_\gamma(A, \boldsymbol{b})$ and let $r = \|\boldsymbol{z}^*_{\bar{\gamma}}\|_0$. Then*

$$\boldsymbol{x}^* = \boldsymbol{z}^* \ \textit{if } r = 0\,,$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \leq \frac{\Sigma(A)}{\gcd(A)} - 1 \ \textit{if } r = 1 \textit{ and} \tag{2}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^r}{r} \leq \frac{\Sigma(A)}{\gcd(A)} \ \textit{if } r \geq 2\,. \tag{3}$$

The bound (2) is optimal, as it is attained already in the knapsack scenario (with the choice of parameters as in (16)). The following example shows that the bound (3) is optimal when $r = 2$. Given the data

$$A = \begin{pmatrix} 2 & 0 & 5 & 5 \\ 0 & 4 & 2 & -1 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 20 \\ 3 \end{pmatrix},$$

the point $\boldsymbol{x}^* = (10, 3/4, 0, 0)^\top$ is a vertex of $P(A, \boldsymbol{b})$. For this choice of parameters, $\Sigma(A) = 20$, $\gcd(A) = 1$ and the corner polyhedron $C_{\{1,2\}}(A, \boldsymbol{b})$ has the unique vertex $\boldsymbol{z}^* = (0, 1, 1, 3)^\top$. We remark that this example was obtained by analyzing the tight cases of inequality (20) in Lemma 9, which take on a special form when $r = 2$.

Theorem 1 shows that for the corner polyhedron associated with a vertex $\boldsymbol{x}^*$ of $P(A, \boldsymbol{b})$ a strong *proximity-sparsity transference* holds: the distance from $\boldsymbol{x}^*$ to any vertex $\boldsymbol{z}^*$ of the corner polyhedron *exponentially* drops with the size of support of $\boldsymbol{z}^*$ and, vice versa, the size of support of $\boldsymbol{z}^*$ reduces with the growth of its distance to $\boldsymbol{x}^*$.

Suppose next that the polyhedron $P(A, \boldsymbol{b})$ is integer feasible and consider its *integer hull* $P_I = \operatorname{conv}(P(A, \boldsymbol{b}) \cap \mathbb{Z}^n)$. A natural direction for further research would be to derive a distance-sparsity transference bound for the vertices of P_I. Notice that the set $P(A, \boldsymbol{b}) \cap \mathbb{Z}^n$ is obtained from $C_\gamma(A, \boldsymbol{b}) \cap \mathbb{Z}^n$ by enforcing back the nonnegativity constraints $\boldsymbol{x}_\gamma \geq 0$ and this may potentially result in cutting off all vertices of the corner polyhedron. In Section 1.1 we show that in the knapsack scenario at least one vertex of $C_\gamma(A, \boldsymbol{b})$ avoids the cut and Theorem 1 implies an optimal distance-sparsity transference bound for lattice points in the knapsack polytope. We expect that a certain transference bound holds for vertices of P_I in the general setting.

The next result of this paper, Theorem 2 below, provides additional information in the case when $\boldsymbol{x}^*$ is degenerate; that is, $\tau := \operatorname{supp}(\boldsymbol{x}^*)$ has size strictly less than m. In particular, this result applies to a tighter relaxation of the integer program, where we enforce back the constraints $\boldsymbol{x}_\gamma \geq 0$ that are tight at $\boldsymbol{x}^*$, at the cost of a slightly weaker bound. Theorems 1 and 2 coincide, however, when $\boldsymbol{x}^*$ is nondegenerate.

When $\boldsymbol{x}^*$ is degenerate, the choice of basis γ in (1) is typically not unique. However, what we show is that there exists at least one basis γ for which the conclusions of Theorem 1 remain valid for this polyhedron, up to a factor which depends on the number of zero coordinates of $\boldsymbol{x}^*_\tau$.

Theorem 2. *Let $A \in \mathbb{Z}^{m \times n}$, $m < n$, be a matrix of rank m and let $\boldsymbol{b} \in \mathbb{Z}^m$. Let $\boldsymbol{x}^*$ be a vertex of the polyhedron $P(A, \boldsymbol{b})$ with $\tau = \operatorname{supp}(\boldsymbol{x}^*)$. Let $C_\tau(A, \boldsymbol{b}) \neq \emptyset$ and let $\boldsymbol{z}^*$ be a vertex of $C_\tau(A, \boldsymbol{b})$. Then there exists a basis γ of A with $\tau \subset \gamma$ such that, letting $r = \|\boldsymbol{z}^*_{\bar{\gamma}}\|_0$ and $d = m - |\tau|$, we have*

$$\boldsymbol{x}^* = \boldsymbol{z}^* \text{ if } r = 0\,,$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \leq \frac{\Sigma(A)}{\gcd(A)} - 1 \text{ if } r = 1 \text{ and} \tag{4}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^r}{r^{d+1}} \leq \frac{\Sigma(A)}{\gcd(A)} \text{ if } r \geq 2\,.$$

Although the statement of Theorem 2 is quite similar to the statement of Theorem 1, the proofs are quite different. The proof of Theorem 2 is carried out in Section 4 using convex-geometric arguments, in addition to the lattice-based arguments in the proof of Theorem 1 found in Section 3. This is due to the fact that the affine cone $A\boldsymbol{x} = \boldsymbol{b}$, $\boldsymbol{x}_{\bar{\tau}} \geq \boldsymbol{0}$ can have considerably more complicated geometry when τ has cardinality strictly smaller than m. In particular, this cone need not be a simplicial cone, whose orthogonal projection onto the $\bar{\gamma}$ coordinates of $\mathbb{R}^n$ is simply the nonnegative orthant.

1.1. **Distance-sparsity transference for knapsacks.** We will now separately consider the case $A \in \mathbb{Z}^{1 \times n}_{>0}$, known as *knapsack scenario.* We will follow a traditional vector notation and replace A and $\boldsymbol{b}$ with a positive integer vector $\boldsymbol{a} = (a_1, \ldots, a_n)^\top \in \mathbb{Z}^n_{>0}$ and integer $b \in \mathbb{Z}$. In this setting, $P(A, \boldsymbol{b})$ is referred to as the *knapsack polytope*

$$P(\boldsymbol{a}, b) = \{\boldsymbol{x} \in \mathbb{R}^n_{\geq 0} : \boldsymbol{a}^\top \boldsymbol{x} = b\}\,.$$

Note that the polytope $P(\boldsymbol{a}, b)$ is an $(n-1)$-dimensional simplex in $\mathbb{R}^n$ with vertices $(b/a_1)\boldsymbol{e}_1, \ldots, (b/a_n)\boldsymbol{e}_n$, where $\boldsymbol{e}_i$ denotes the i-th standard basis vector. Given a vertex $\boldsymbol{x}^*$ of $P(\boldsymbol{a}, b)$ with $\gamma = \operatorname{supp}(\boldsymbol{x}^*)$, the corner polyhedron associated with $\boldsymbol{x}^*$ can be written as $C_\gamma(\boldsymbol{a}, b) = \operatorname{conv}(\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{a}^\top \boldsymbol{x} = b, \boldsymbol{x}_{\bar{\gamma}} \geq 0\})$.

In what follows, we will exclude the trivial case $n = 1$ and assume that $n \geq 2$. We also assume that the polytope $P(\boldsymbol{a}, b)$ contains integer points. Equivalently, b belongs to the *semigroup*

$$Sg(\boldsymbol{a}) = \{\boldsymbol{a}^\top \boldsymbol{z} : \boldsymbol{z} \in \mathbb{Z}^n_{\geq 0}\}$$

generated by the entries of the vector $\boldsymbol{a}$. Note that any element of the semigroup $Sg(\boldsymbol{a})$ must be divisible by the greatest common divisor $\gcd(a_1, \ldots, a_n)$ of $a_1, \ldots, a_n$. Hence, we may assume without loss of generality that $\boldsymbol{a}$ satisfies the following conditions:

$$\boldsymbol{a} = (a_1, \ldots, a_n)^\top \in \mathbb{Z}^n_{>0}\,, n \geq 2\,, \text{ and } \gcd(a_1, \ldots, a_n) = 1\,. \tag{5}$$

Aliev et al [4, Theorem 1] proved that for any vertex $\boldsymbol{x}^*$ of the polytope $P(\boldsymbol{a}, b)$ there exists an integer point $\boldsymbol{z} \in P(\boldsymbol{a}, b)$ such that

$$\|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty \leq \|\boldsymbol{a}\|_\infty - 1 \tag{6}$$

and that the bound (6) is sharp in the following sense. For any positive integer k and any dimension n there exist $\boldsymbol{a}$ satisfying (5) with $\|\boldsymbol{a}\|_\infty = k$ and $b \in \mathbb{Z}$ such that the knapsack polytope $P(\boldsymbol{a}, b)$ contains exactly one integer point $\boldsymbol{z}$ and $\|\boldsymbol{x}^* - \boldsymbol{z}\|_\infty = \|\boldsymbol{a}\|_\infty - 1$.

The best known sparsity-type estimate in the knapsack scenario (5), obtained in [1, Theorem 6], guarantees existence of an integer point $\boldsymbol{z} \in P(\boldsymbol{a}, b)$ that satisfies the bound

$$\|\boldsymbol{z}\|_0 \leq 1 + \log(\min\{a_1, \ldots, a_n\})\,. \tag{7}$$

The next result will combine and refine the bounds (6) and (7) as follows.

Theorem 3. *Let $\boldsymbol{a}$ satisfy (5), $b \in Sg(\boldsymbol{a})$ and let $\boldsymbol{x}^*$ be a vertex of $P(\boldsymbol{a}, b)$ with basis $\gamma = \mathrm{supp}(\boldsymbol{x}^*)$. Then $P(\boldsymbol{a}, b)$ contains a vertex $\boldsymbol{z}^*$ of $C_\gamma(\boldsymbol{a}, b)$ such that, letting $r = \|\boldsymbol{z}^*_{\bar{\gamma}}\|_0$,*

$$\boldsymbol{x}^* = \boldsymbol{z}^* \text{ if } r = 0\,, \tag{8}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \leq \|\boldsymbol{a}\|_\infty - 1 \text{ if } r = 1 \text{ and} \tag{9}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^r}{r} < \|\boldsymbol{a}\|_\infty \text{ if } r \geq 2\,. \tag{10}$$

It should be pointed out that Theorem 3 guarantees that the vertex $\boldsymbol{z}^*$ of $C_\gamma(\boldsymbol{a}, b)$ belongs to the knapsack polytope $P(\boldsymbol{a}, b)$, while in Theorems 1 and 2 the corresponding vertex of the corner polyhedron may be infeasible. Theorem 3 can be viewed as a transference result that allows strengthening the distance bound (6) if integer points in the knapsack polytope are not sparse and, vice versa, strengthening the sparsity bound (7) if feasible integer points are sufficiently far from a vertex of the knapsack polytope.

Given a cost vector $\boldsymbol{c} \in \mathbb{Z}^n$, we will now consider the *integer knapsack problem*

$$\min\{\boldsymbol{c}^\top \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{a}, b) \cap \mathbb{Z}^n\}\,. \tag{11}$$

Note that (11) is feasible since $b \in Sg(\boldsymbol{a})$.

Let $IP(\boldsymbol{c}, \boldsymbol{a}, b)$ and $LP(\boldsymbol{c}, \boldsymbol{a}, b)$ denote the optimal values of (11) and its linear programming relaxation

$$\min\{\boldsymbol{c}^\top \boldsymbol{x} : \boldsymbol{x} \in P(\boldsymbol{a}, b)\}\,, \tag{12}$$

respectively. The *integrality gap* $IG(\boldsymbol{c}, \boldsymbol{a}, b)$ of (11) is defined as

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) = IP(\boldsymbol{c}, \boldsymbol{a}, b) - LP(\boldsymbol{c}, \boldsymbol{a}, b)\,.$$

As a corollary of Theorem 3, we obtain the following bound for the integrality gap.

Corollary 4. *Let $\boldsymbol{a}$ satisfy (5), $b \in Sg(\boldsymbol{a})$ and $\boldsymbol{c} \in \mathbb{Z}^n$. Let $\boldsymbol{x}^*$ be an optimal vertex solution to* (12) *with basis γ. Let further $\boldsymbol{z}^*$ be any vertex of $C_\gamma(\boldsymbol{a}, b)$ such that $\boldsymbol{z}^* \in P(\boldsymbol{a}, b)$. Then, letting $r = \|\boldsymbol{z}^*_{\bar{\gamma}}\|_0$, we have*

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) = 0 \text{ if } r = 0\,, \tag{13}$$

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) \leq 2(\|\boldsymbol{a}\|_\infty - 1)\|\boldsymbol{c}\|_\infty \text{ if } r = 1 \text{ and} \tag{14}$$

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) < \frac{r(r+1)}{2^r}\|\boldsymbol{a}\|_\infty \|\boldsymbol{c}\|_\infty \text{ if } r \geq 2\,. \tag{15}$$

It follows from the proof of Theorem 1 (ii) in [4] that the bound (9) (and hence (2) and (4) for $m = 1$) corresponding to the case $r = 1$ is optimal. For completeness, we recall that it is sufficient to choose a positive integer k and set

$$\boldsymbol{a} = (k, \ldots, k, 1)^\top, b = k - 1 \text{ and } \boldsymbol{x}^* = \frac{k-1}{k} \cdot \boldsymbol{e}_1\,. \tag{16}$$

Then the knapsack polytope $P(\boldsymbol{a}, b)$ contains precisely one integer point, $\boldsymbol{z}^* = (k-1) \cdot \boldsymbol{e}_n$ and we obtain $\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty = k - 1 = \|\boldsymbol{a}\|_\infty - 1$. The next result of this paper shows that the bounds in Theorems 1 - 3 are optimal in the knapsack scenario for $r \geq 2$.

Theorem 5. *Fix integer $n \geq 3$. For any $\epsilon > 0$ there exist an integer vector $\boldsymbol{a} \in \mathbb{Z}^n$ satisfying (5) and $b \in Sg(\boldsymbol{a})$ such that for a vertex $\boldsymbol{x}^*$ of the knapsack polytope $P(\boldsymbol{a}, b)$ with $\gamma = \operatorname{supp}(\boldsymbol{x}^*)$ and a vertex $\boldsymbol{z}^*$ of $C_\gamma(\boldsymbol{a}, b)$ with $\|\boldsymbol{z}^*_{\bar{\gamma}}\|_0 = n - 1$*

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^{n-1}}{n-1} > (1 - \epsilon)\|\boldsymbol{a}\|_\infty\,.$$

1.2. **A refined sparsity-type bound for solutions to integer programs.** The next result of this paper aims to refine the general sparsity-type bound obtained in [2, Theorem 1]. Let

$$\rho(\boldsymbol{x}) = \min\{|x_i| : i \in \operatorname{supp}(\boldsymbol{x})\}$$

denote the minimum absolute nonzero entry of $\boldsymbol{x}$. Let $\boldsymbol{c} \in \mathbb{Z}^n$. We will consider the general integer linear problem in standard form

$$\max\left\{\boldsymbol{c}^\top \boldsymbol{x} : \boldsymbol{x} \in P(A, \boldsymbol{b}) \cap \mathbb{Z}^n\right\}\,. \tag{17}$$

We assume that $P(A, \boldsymbol{b})$ contains integer points so that (17) is feasible.

It was shown in [2, Theorem 1] that there exists an optimal solution $\boldsymbol{z}^*$ for (17) satisfying the bound

$$\|\boldsymbol{z}^*\|_0 \leq m + \log\left(\frac{\sqrt{\det(AA^\top)}}{\gcd(A)}\right)\,. \tag{18}$$

Note that any vertex solution for (17) has the size of support $\leq m$. Any non-vertex solution $\boldsymbol{z}^*$, in its turn, belongs to the interior of the face $\mathcal{F} = P(A, \boldsymbol{b}) \cap \{\boldsymbol{x} \in \mathbb{R}^n : x_i = 0 \text{ for } i \notin \operatorname{supp}(\boldsymbol{z}^*)\}$ of the polyhedron $P(A, \boldsymbol{b})$. Then the minimum absolute nonzero entry $\rho(\boldsymbol{z}^*)$ is the ℓ_∞-distance from $\boldsymbol{z}^*$ to the boundary of $\mathcal{F}$. To obtain a refinement of the bound (18) we will link the minimum absolute nonzero entry and the size of support of solutions to (17).

Theorem 6. *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m, $\boldsymbol{b} \in \mathbb{Z}^m$, $\boldsymbol{c} \in \mathbb{Z}^n$ and suppose that* (17) *is feasible. Then there is an optimal solution $\boldsymbol{z}^*$ to* (17) *such that, letting $s = \|\boldsymbol{z}^*\|_0$,*

$$(\rho(\boldsymbol{z}^*) + 1)^{s-m} \le \frac{\sqrt{\det(AA^\top)}}{\gcd(A)}\,. \tag{19}$$

2. Lattices and corner polyhedra

For linearly independent $\boldsymbol{b}_1, \dots, \boldsymbol{b}_l$ in $\mathbb{R}^d$, the set $\Lambda = \{\sum_{i=1}^{l} x_i\boldsymbol{b}_i,\ x_i \in \mathbb{Z}\}$ is an l-dimensional *lattice* with *basis* $\boldsymbol{b}_1, \dots, \boldsymbol{b}_l$ and *determinant* $\det(\Lambda) = (\det(\boldsymbol{b}_i \cdot \boldsymbol{b}_j)_{1\le i,j\le l})^{1/2}$, where $\boldsymbol{b}_i \cdot \boldsymbol{b}_j$ is the standard inner product of the basis vectors $\boldsymbol{b}_i$ and $\boldsymbol{b}_j$. Recall that the *Minkowski sum* $X + Y$ of the sets $X, Y \subset \mathbb{R}^d$ consists of all points $\boldsymbol{x} + \boldsymbol{y}$ with $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in Y$. The *difference set* $X - X$ is the Minkowski sum of X and $-X$. For a lattice $\Lambda \subset \mathbb{R}^d$ and $\boldsymbol{y} \in \mathbb{R}^d$, the set $\boldsymbol{y} + \Lambda$ is an *affine lattice* with determinant $\det(\Lambda)$.

Let $\Lambda \subset \mathbb{Z}^d$ be a d-dimensional integer lattice. The point $\boldsymbol{x} \in \mathbb{Z}^d_{\ge 0}$ is called *irreducible (with respect to* Λ*)* if for any two points $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{Z}^d_{\ge 0}$ with $0 \le y_i \le x_i$, $0 \le y'_i \le x_i$, $i \in \{1, \dots, d\}$ the inclusion $\boldsymbol{y} - \boldsymbol{y}' \in \Lambda$ implies $\boldsymbol{y} = \boldsymbol{y}'$.

Lemma 7 (Theorem 1 in [14])**.** *If $\boldsymbol{x} \in \mathbb{Z}^d_{\ge 0}$ is irreducible with respect to the lattice Λ then*

$$\prod_{i=1}^{d}(x_i + 1) \le \det(\Lambda)\,.$$

Proof. The lattice Λ can be viewed as a subgroup of the additive group $\mathbb{Z}^d$. The number of points $\boldsymbol{y} \in \mathbb{Z}^d_{\ge 0}$ with $0 \le y_i \le x_i$, $i \in \{1, \dots, d\}$ is equal to $\prod_{i=1}^{d}(x_i + 1)$. Since $\boldsymbol{x}$ is irreducible, each such $\boldsymbol{y}$ corresponds to a unique coset (affine lattice) $\boldsymbol{y} + \Lambda$ of Λ. Finally notice that there are only $\det(\Lambda)$ different cosets. □

Let $\boldsymbol{r} \in \mathbb{Z}^d$ and consider the affine lattice $\Gamma = \boldsymbol{r} + \Lambda$. We will call the set $E(\Gamma) = \mathrm{conv}(\Gamma \cap \mathbb{R}^d_{\ge 0})$ the *sail* associated with Γ.

Lemma 8. *Every vertex of the sail $E(\Gamma)$ is irreducible.*

Proof. Let $\boldsymbol{x}$ be a vertex of $E(\Gamma)$. Suppose, to derive a contradiction, that $\boldsymbol{x}$ is reducible. Then there are distinct points $\boldsymbol{y}, \boldsymbol{y}' \in \mathbb{Z}^d_{\ge 0}$ with $0 \le y_i \le x_i$, $0 \le y'_i \le x_i$, $i \in \{1, \dots, d\}$ such that $\boldsymbol{y} - \boldsymbol{y}' \in \Lambda$.

Since $\boldsymbol{x} - \boldsymbol{y} \in \mathbb{Z}^d_{\ge 0}$ and $\boldsymbol{x} - \boldsymbol{y}' \in \mathbb{Z}^d_{\ge 0}$, the vectors $\boldsymbol{v}_1 = \boldsymbol{x} - \boldsymbol{y} + \boldsymbol{y}'$ and $\boldsymbol{v}_2 = \boldsymbol{x} - \boldsymbol{y}' + \boldsymbol{y}$ have nonnegative integer entries. Further, $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \Gamma$ and $\boldsymbol{x} = (\boldsymbol{v}_1 + \boldsymbol{v}_2)/2$. Therefore $\boldsymbol{x}$ is not a vertex of $E(\Gamma)$. □

Lemma 9. *For $r \ge 2$ and $x_1, \dots, x_r \ge 1$ the inequality*

$$x_1 + \cdots + x_r \le \frac{r(x_1 + 1)\cdots(x_r + 1)}{2^r} \tag{20}$$

holds.

Proof. Suppose that (20) is satisfied for $x_1 = y_1, \dots, x_r = y_r$. We will first show that for any $\epsilon > 0$ and any $i \in \{1, \dots, r\}$ the inequality (20) is satisfied for $x_1 = y_1, \dots, x_{i-1} = y_{i-1}, x_i = y_i + \epsilon, x_{i+1} = y_{i+1}, \dots, x_r = y_r$. After possible renumbering, it is sufficient to consider the case $i = 1$. We have

$$\begin{aligned}(y_1+\epsilon)+y_2+\cdots+y_r &\le \frac{r(y_1+1)\cdots(y_r+1)}{2^r}+\epsilon\\ &\le \frac{r(y_1+1)\cdots(y_r+1)}{2^r}+\epsilon\frac{r(y_2+1)\cdots(y_r+1)}{2^r}\\ &= \frac{r(y_1+1+\epsilon)\cdots(y_r+1)}{2^r}\,.\end{aligned}$$

To complete the proof it is sufficient to observe that (20) holds for $y_1 = \cdots = y_r = 1$. □

Given $A \in \mathbb{Z}^{m\times n}$ and $\boldsymbol{b} \in \mathbb{Z}^m$, we will denote by $\Gamma(A, \boldsymbol{b})$ the set of integer points in the affine subspace

$$H(A,\boldsymbol{b}) = \{\boldsymbol{x} \in \mathbb{R}^n : A\boldsymbol{x} = \boldsymbol{b}\}\,,$$

that is

$$\Gamma(A,\boldsymbol{b}) = H(A,\boldsymbol{b}) \cap \mathbb{Z}^n\,.$$

The set $\Gamma(A, \boldsymbol{b})$ is either empty or an affine lattice of the form $\Gamma(A, \boldsymbol{b}) = \boldsymbol{r} + \Gamma(A)$, where $\boldsymbol{r}$ is any integer vector with $A\boldsymbol{r} = \boldsymbol{b}$ and $\Gamma(A) = \Gamma(A, \boldsymbol{0})$ is the lattice formed by all integer points in the kernel of the matrix A.

Fix a basis γ of A and let π_γ denote the projection map from $\mathbb{R}^n$ to $\mathbb{R}^{n-m}$ that forgets the coordinates indexed by γ, that is $\pi_\gamma : \boldsymbol{u} \mapsto \boldsymbol{u}_{\bar\gamma}$. Recall that A_γ is nonsingular. It follows that the restricted map $\pi_\gamma|_{H(A,\boldsymbol{b})} : H(A, \boldsymbol{b}) \to \mathbb{R}^{n-m}$ is bijective. Specifically, any $\boldsymbol{u}_{\bar\gamma} \in \mathbb{R}^{n-m}$ is mapped by $\pi_\gamma^{-1}|_{H(A,\boldsymbol{b})}$ to a point $\boldsymbol{u} \in H(A, \boldsymbol{b})$ with

$$\boldsymbol{u}_\gamma = A_\gamma^{-1}(\boldsymbol{b} - A_{\bar\gamma}\boldsymbol{u}_{\bar\gamma})\,. \tag{21}$$

For technical reasons, it is convenient to consider the projected affine lattice $\Lambda_\gamma(A, \boldsymbol{b}) = \pi_\gamma(\Gamma(A, \boldsymbol{b}))$ and the projected lattice $\Lambda_\gamma(A) = \pi_\gamma(\Gamma(A))$.

Lemma 10. *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix with a basis γ. Then*

$$\det(\Lambda_\gamma(A)) = \frac{|\det(A_\gamma)|}{\gcd(A)}\,. \tag{22}$$

Proof. Without loss of generality, we may assume that $\gamma = \{1, \dots, m\}$. Let $\boldsymbol{g}_1, \dots, \boldsymbol{g}_{n-m}$ be a basis of $\Gamma(A)$. Since the map $\pi_\gamma|_{H(A,\boldsymbol{0})}$ is bijective, the vectors $\boldsymbol{b}_1 = \pi_\gamma(\boldsymbol{g}_1), \dots, \boldsymbol{b}_{n-m} = \pi_\gamma(\boldsymbol{g}_{n-m})$ form a basis of the lattice $\Lambda_\gamma(A)$. Let $G \in \mathbb{Z}^{n\times(n-m)}$ be the matrix with columns $\boldsymbol{g}_1, \dots, \boldsymbol{g}_{n-m}$. We will denote by F the $(n-m)\times(n-m)$-submatrix of G consisting of the last $n-m$ rows; hence, the columns of F are $\boldsymbol{b}_1, \dots, \boldsymbol{b}_{n-m}$. Then $\det(\Lambda_\gamma(A)) = |\det(F)|$. The rows of the matrix A span the m-dimensional rational subspace of $\mathbb{R}^n$ orthogonal to the $(n-m)$-dimensional rational subspace spanned by the columns of G. Therefore, by Lemma 5G and Corollary 5I in [16], we have $|\det(F)| = |\det(A_\gamma)|/\gcd(A)$ and, consequently, (22) holds. □

Theorem 11. *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix with a basis γ and $\boldsymbol{b} \in \mathbb{Z}^m$. For any vertex $\boldsymbol{z}^*$ of the corner polyhedron $C_\gamma(A, \boldsymbol{b})$ the bound*

$$\prod_{j\in\bar\gamma}(z_j^*+1) \le \frac{|\det(A_\gamma)|}{\gcd(A)}$$

holds.

Proof. Since $\pi_\gamma|_{H(A,\boldsymbol{b})}$ is a bijection, the point $\boldsymbol{y}^* = \pi_\gamma(\boldsymbol{z}^*)$ is a vertex of the sail $E(\Lambda_\gamma(A,\boldsymbol{b}))$. The result now follows by Lemma 8, Lemma 7 and (22). □

3. Proof of Theorem 1

Theorem 1 is an immediate consequence of Theorem 11 and the following lemma:

Lemma 12. *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m and $\boldsymbol{b} \in \mathbb{Z}^m$. Let $\boldsymbol{x}^*$ be a vertex of $P(A,\boldsymbol{b})$, and let γ be any basis of A containing* supp$(\boldsymbol{x}^*)$. *Let $\boldsymbol{z}^*$ be an integral vector satisfying $A\boldsymbol{z}^* = \boldsymbol{b}$, with $\boldsymbol{z}^*_{\bar\gamma} \geq \boldsymbol{0}$, and let $r = \|\boldsymbol{z}^*_{\bar\gamma}\|_0$. Then*

$$\boldsymbol{x}^* = \boldsymbol{z}^* \text{ if } r = 0\,, \tag{23}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \leq \frac{\Sigma(A)}{|\det(A_\gamma)|}\prod_{j\in\bar\gamma}(z^*_j+1) - 1 \text{ if } r = 1 \text{ and} \tag{24}$$

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^r}{r} \leq \frac{\Sigma(A)}{|\det(A_\gamma)|}\prod_{j\in\bar\gamma}(z^*_j+1) \text{ if } r \geq 2\,. \tag{25}$$

Proof. If $r = \|\boldsymbol{z}^*_{\bar\gamma}\|_0 = 0$ the vector $\boldsymbol{z}^*_\gamma$ is the unique solution to the system $A_\gamma \boldsymbol{x}_\gamma = \boldsymbol{b}$. Therefore (23) holds.

In the rest of the proof we assume that $r \geq 1$. Without loss of generality, we may also assume in this proof that $\gamma = \{1,\dots,m\}$. We will set $\delta = \|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty$ and consider the following two cases. First suppose that there exists an index $j \in \bar\gamma$ such that $\delta = |x^*_j - z^*_j| = z^*_j$. Observe that r of the numbers $z^*_{m+1},\dots,z^*_n$ are nonzero. Hence,

$$(\delta+1)2^{r-1} \leq \prod_{j\in\bar\gamma}(z^*_j+1)$$

and so

$$\delta\frac{2^r}{r} \leq \frac{2}{r}\prod_{j\in\bar\gamma}(z^*_j+1) - \frac{2^r}{r}\,. \tag{26}$$

Since $\Sigma(A) \geq |\det(A_\gamma)|$, inequality (26) justifies both (24) and (25).

Now suppose that $\delta = x^*_j - z^*_j$ for $j \in \gamma$. We can write

$$A_\gamma \boldsymbol{z}^*_\gamma + A_{\bar\gamma}\boldsymbol{z}^*_{\bar\gamma} = \boldsymbol{b} \text{ and } A_\gamma \boldsymbol{x}^*_\gamma = \boldsymbol{b}\,.$$

Therefore

$$A_\gamma(\boldsymbol{x}^*_\gamma - \boldsymbol{z}^*_\gamma) = A_{\bar\gamma}\boldsymbol{z}^*_{\bar\gamma}\,. \tag{27}$$

Given a vector $\boldsymbol{v} \in \mathbb{R}^m$, we will denote by $A^j_\gamma(\boldsymbol{v})$ the matrix obtained from A_γ by replacing its j-th column with $\boldsymbol{v}$. Let $A_1,\dots,A_n$ be the columns of the matrix A. Solving (27) by Cramer's rule, we have

$$\begin{aligned}\delta &= x^*_j - z^*_j \\ &= \frac{\det(A^j_\gamma(A_{\bar\gamma}\boldsymbol{z}^*_{\bar\gamma}))}{\det(A_\gamma)} \\ &= \frac{1}{\det(A_\gamma)}(z^*_{m+1}\det(A^j_\gamma(A_{m+1})) + \cdots + z^*_n \det(A^j_\gamma(A_n)))\,.\end{aligned} \tag{28}$$

If $r = 1$, then for some $i \in \bar{\gamma}$ we can write

$$\delta = \frac{z_i^* \det(A_\gamma^j(A_i))}{\det(A_\gamma)} = (z_i^* + 1)\frac{\det(A_\gamma^j(A_i))}{\det(A_\gamma)} - \frac{\det(A_\gamma^j(A_i))}{\det(A_\gamma)}\,. \tag{29}$$

Equation (29) implies (24).

To settle the case $r \geq 2$, observe that (28) implies

$$\delta \leq (z_{m+1}^* + \cdots + z_n^*)\frac{\Sigma(A)}{|\det(A_\gamma)|}\,. \tag{30}$$

Without loss of generality, assume that $z_i^* \neq 0$ for $i \in \{m+1, \ldots, m+r\}$ and $z_i^* = 0$ for $m + r < i \leq n$. Then, by (30) and Lemma 9, we have

$$\delta \leq \frac{r(z_{m+1}^* + 1)\cdots(z_{m+r}^* + 1)\Sigma(A)}{2^r|\det(A_\gamma)|}\,.$$

This establishes inequality (25). □

4. Proof of Theorem 2

Theorem 2 is an immediate consequence of Lemma 12 and the generalization of Theorem 11 given below. Recall that τ denotes the support of $\boldsymbol{x}^*$, and $C_\tau(A, \boldsymbol{b})$ denotes the polyhedron

$$C_\tau(A, \boldsymbol{b}) = \operatorname{conv}\left(\{\boldsymbol{x} \in \mathbb{Z}^n : A\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x}_{\bar{\tau}} \geq \boldsymbol{0}\}\right).$$

Let $\boldsymbol{z}^*$ be a vertex of $C_\tau(A, \boldsymbol{b})$.

Theorem 13. *There exists a basis γ of A containing τ such that*

$$\prod_{j \in \overline{\gamma}} (\boldsymbol{z}_j^* + 1) \leq r^d \frac{|\det(A_\gamma)|}{\gcd(A)},$$

where $r = \left\|\boldsymbol{z}_{\bar{\gamma}}^*\right\|_0$ and $d = m - |\tau|$.

Theorem 13 is proved over the remainder of this section, by constructing a convex set P such that

$$\prod_{j \in \overline{\gamma}} (\boldsymbol{z}_j^* + 1) \leq \operatorname{vol}_{n-m}(P) \leq r^d \frac{|\det(A_\gamma)|}{\gcd(A)}. \tag{31}$$

Here $\operatorname{vol}_k(S)$ denotes the k-dimensional volume, or Lebesgue measure, of S. Subsection 4.1 collects some facts from convex geometry that are used in this proof. Subsection 4.2 establishes the inequalities (31), and hence Theorem 13, in the special case when τ and $\operatorname{supp}(\boldsymbol{z}^*)$ together cover $[n]$. Finally, Subsection 4.3 uses this special case to establish the general case of Theorem 13.

4.1. **Convex geometry lemmas.**

Lemma 14 (Blichfeldt's lemma [9, Chapter III, Theorem I])**.** *Let $K \subseteq \mathbb{R}^d$ be bounded, nonempty, Lebesgue measurable and let Λ be a full-dimensional lattice in $\mathbb{R}^d$. Suppose that the difference set $K - K$ contains no nonzero lattice points from Λ. Then $\operatorname{vol}_d(K) \leq \det(\Lambda)$.*

Theorem 15 (Brunn's concavity principle [5, Theorem 1.2.1])**.** *Let K be a convex body, and let F be a k-dimensional subspace of $\mathbb{R}^d$. Then the function $g : F^\perp \to \mathbb{R}$ defined by*

$$g(x) = \operatorname{vol}_k(K \cap (F + x))^{1/k}$$

is concave on its support.

By a *slab* in $\mathbb{R}^d$ we mean the closed region bounded by two distinct parallel hyperplanes. Let $\boldsymbol{q} \in \mathbb{R}^d$ be nonzero. The *width* of a set $K \subseteq \mathbb{R}^d$ along $\boldsymbol{q}$ is defined to be

$$w_{\boldsymbol{q}}(K) := \left(\sup_{\boldsymbol{x}\in K} \boldsymbol{q}^\top \boldsymbol{x}\right) - \left(\inf_{\boldsymbol{x}\in K} \boldsymbol{q}^\top \boldsymbol{x}\right).$$

Proposition 16. *Let K be a centrally symmetric convex body with centre $\boldsymbol{c}$. Let S be a slab containing $\boldsymbol{c}$, such that $\boldsymbol{c}$ is equidistant from the two facets defining S with respect to Euclidean distance. Let $\boldsymbol{q}$ be a normal vector to either of the hyperplanes bounding S. If S does not contain K, then*

$$\mathrm{vol}_d(K \cap S) \geq \frac{w_{\boldsymbol{q}}(S)}{w_{\boldsymbol{q}}(K)} \cdot \mathrm{vol}_d(K).$$

Proof. Without loss of generality, we may assume $\boldsymbol{c}$ is the origin. For $\lambda \in [-1,1]$, define the affine hyperplane

$$L_\lambda := \left\{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{q}^\top \boldsymbol{x} = \lambda \cdot w_{\boldsymbol{q}}(K)/2\right\}.$$

Let $K_\lambda := K \cap L_\lambda$, and define the cross-sectional volume

$$f(\lambda) := \mathrm{vol}_{d-1}\left(K_\lambda\right).$$

By symmetry, we have $K_\lambda = -K_{-\lambda}$. Hence, f is an even function on $[-1,1]$, which means that $g(\lambda) := (f(\lambda))^{1/(d-1)}$ is an even function as well. Since g is concave on $[-1,1]$ by Brunn's concavity principle, we have that g, and therefore f, is a decreasing function on $[0,1]$.

Now let $\delta := w_{\boldsymbol{q}}(S)/w_{\boldsymbol{q}}(K)$. By Fubini's theorem, symmetry, and monotonicity on $[0,1]$, we conclude

$$\mathrm{vol}_d(K \cap S) = \int_{-\delta}^{\delta} f(\lambda)d\lambda = 2\int_0^{\delta} f(\lambda)d\lambda \geq 2\delta \int_0^1 f(\lambda)d\lambda = \frac{w_{\boldsymbol{q}}(S)}{w_{\boldsymbol{q}}(K)} \cdot \mathrm{vol}_d(K). \qquad \square$$

The notion of irreducibility from Lemma 8 can be mildly generalized as follows. Let C be a pointed cone. Let $\Lambda \subset \mathbb{Z}^d$ be a d-dimensional integer lattice. The point $\boldsymbol{x} \in C \cap \mathbb{Z}^d$ is called *irreducible (with respect to Λ and C)* if

$$(-\boldsymbol{x} + C) \cap (\boldsymbol{x} - C) \cap \Lambda = \{\boldsymbol{0}\}.$$

Let $\boldsymbol{r} \in \mathbb{Z}^d$ and consider the affine lattice $\Gamma = \boldsymbol{r} + \Lambda$. We will call the set $E(\Gamma, C) = \mathrm{conv}(\Gamma \cap C)$ the *sail* associated with Γ and C.

Lemma 17. *Every vertex of the sail $E(\Gamma, C)$ is irreducible.*

Proof. Let $\boldsymbol{x}$ be a vertex of $E(\Gamma, C)$. Suppose, to derive a contradiction, that $\boldsymbol{x}$ is reducible. Then there exists nonzero $\boldsymbol{\lambda} \in \Lambda$ and vectors $\boldsymbol{y}, \boldsymbol{y}' \in C$ such that $\boldsymbol{\lambda} = -\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{x} - \boldsymbol{y}'$. The fact that $\boldsymbol{x}$ is a vertex of $E(\Gamma, C)$ implies $\boldsymbol{x} \in \Gamma$, and therefore both $\boldsymbol{y} = \boldsymbol{\lambda} + \boldsymbol{x}$ and $\boldsymbol{y}' = -\boldsymbol{\lambda} + \boldsymbol{x}$ are contained in $\Gamma \cap C$, and hence in $E(\Gamma, C)$. Since $\boldsymbol{\lambda}$ is nonzero, we conclude that $\boldsymbol{x} = (\boldsymbol{y} + \boldsymbol{y}')/2$ is not a vertex of $E(\Gamma, C)$. $\square$

4.2. **A special case of Theorem 13.** We next choose the basis γ of Theorem 13, define the convex set P in terms of this γ, and establish the lower and upper bounds of (31) in the special case when τ and $\mathrm{supp}(\boldsymbol{z}^*)$ together cover $[n]$.

Proposition 18. *Assume* $\bar{\tau} \subseteq \operatorname{supp}(\boldsymbol{z}^*)$. *Then there exists a basis* γ *of* A *containing* τ *satisfying, for each* $i \in \gamma \backslash \tau$,

$$z_i^* + 1 \geq \frac{1}{r} \sum_{j \in \bar{\gamma}} \left| (A_\gamma^{-1} A_{\bar{\gamma}})_{i,j} (z_j^* + 1) \right|, \tag{32}$$

where $r = \|\boldsymbol{z}_{\bar{\gamma}}^*\|_0$.

Proof. Among all bases of A containing τ, choose a basis γ so that the quantity $|\det A_\gamma| \cdot \prod_{i \in \gamma} (z_i^* + 1)$ is as large as possible. If $i \in \gamma$ and $j \in \bar{\gamma}$, then by Cramer's rule we have

$$(A_\gamma^{-1} A_{\bar{\gamma}})_{i,j} = \frac{\det(A_\gamma^i(A_j))}{\det(A_\gamma)},$$

where $A_\gamma^i(A_j)$ denotes the matrix obtained by replacing column i of A_γ with column j of A. The choice of γ implies that if $i \in \gamma \backslash \tau$ and $j \in \bar{\gamma}$, then

$$z_i^* + 1 \geq \left| \frac{\det(A_\gamma^i(A_j))}{\det(A_\gamma)} (z_j^* + 1) \right| = \left| (A_\gamma^{-1} A_{\bar{\gamma}})_{i,j} (z_j^* + 1) \right|.$$

The condition $\bar{\tau} \subseteq \operatorname{supp}(\boldsymbol{z}^*)$ implies $r = |\bar{\gamma}|$. Hence, for all $i \in \gamma \backslash \tau$, we have

$$z_i^* + 1 \geq \frac{1}{r} \sum_{j \in \bar{\gamma}} \left| (A_\gamma^{-1} A_{\bar{\gamma}})_{i,j} (z_j^* + 1) \right|. \qquad \square$$

We now fix a basis γ of A satisfying the hypotheses of Proposition 18, so that in particular $\bar{\tau} \subseteq \operatorname{supp}(\boldsymbol{z}^*)$. We also let $r = \|\boldsymbol{z}_{\bar{\gamma}}^*\|_0$. Without loss of generality, we may assume $\gamma = \{1, 2, \ldots, m\}$ and we further assume $\gamma \backslash \tau = \{1, 2, \ldots, d\}$. We denote the rows of the matrix $-A_\gamma^{-1} A_{\bar{\gamma}}$ by $\boldsymbol{q}_1^\top, \boldsymbol{q}_2^\top, \ldots, \boldsymbol{q}_m^\top$. Note that the equality $A\boldsymbol{z}^* = \boldsymbol{b}$ implies by (21) that $\boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* = z_i^* - x_i^*$ for all $i \in \gamma$.

Let $\mathbf{1}_{n-m} \in \mathbb{R}^{n-m}$ be the vector of all ones, and define, for each $i \in \gamma \backslash \tau$,

$$S_i := \left\{ \boldsymbol{x} \in \mathbb{R}^{n-m} : -\tfrac{1}{2} < \boldsymbol{q}_i^\top \boldsymbol{x} < \boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* + \tfrac{1}{2} \right\}.$$

Also define

$$B := \left\{ \boldsymbol{x} \in \mathbb{R}^{n-m} : -\tfrac{1}{2}\mathbf{1}_{n-m} < \boldsymbol{x} < \boldsymbol{z}_{\bar{\gamma}}^* + \tfrac{1}{2}\mathbf{1}_{n-m} \right\},$$

and for each $i \in \gamma \backslash \tau$, let $P_i := P_{i-1} \cap S_i$ with $P_0 = B$. Let $P := P_d$.

Lemma 19. *We have*

$$\operatorname{vol}_{n-m}(P) \geq \frac{1}{r^d} \prod_{j \in \bar{\gamma}} (z_j^* + 1).$$

Proof. Suppose $i \in \gamma \backslash \tau$. If S_i contains P_{i-1} then $P_{i-1} = P_i$, and hence

$$\operatorname{vol}_{n-m}(P_i) = \operatorname{vol}_{n-m}(P_{i-1}).$$

Otherwise, define

$$\lambda_i := \frac{w_{\boldsymbol{q}_i}(S_i)}{w_{\boldsymbol{q}_i}(P_{i-1})}.$$

The fact that $\bar{\tau} \subseteq \operatorname{supp}(\boldsymbol{z}^*)$ implies $z_i^* \geq 1$ for all $i \in \gamma \backslash \tau$. Applying Proposition 18, we get

$$\lambda_i \geq \frac{w_{\boldsymbol{q}_i}(S_i)}{w_{\boldsymbol{q}_i}(B)} = \frac{\boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* + 1}{\sum_{j \in \bar{\gamma}} \left| q_{i,j}(z_j^* + 1) \right|} = \frac{z_i^* + 1}{\sum_{j \in \bar{\gamma}} \left| q_{i,j}(z_j^* + 1) \right|} \geq \frac{z_i^* + 1}{r(z_i^* + 1)} = \frac{1}{r}.$$

Since S_i does not contain P_{i-1}, Proposition 16 applies, and so

$$\mathrm{vol}_{n-m}(P_i) \geq \frac{w_{\boldsymbol{q}_i}(S_i)}{w_{\boldsymbol{q}_i}(P_{i-1})}\mathrm{vol}_{n-m}(P_{i-1}) = \lambda_i \mathrm{vol}_{n-m}(P_{i-1}) \geq \frac{1}{r}\mathrm{vol}_{n-m}(P_{i-1}).$$

Applying induction to the sequence of polytopes $P = P_d, \ldots, P_1, P_0 = B$, we get

$$\mathrm{vol}_{n-m}(P) \geq \frac{1}{r^d}\mathrm{vol}_{n-m}(B) = \frac{1}{r^d}\prod_{j\in\bar{\gamma}}(z_j^* + 1). \qquad \square$$

Lemma 20. *We have*

$$\mathrm{vol}_{n-m}(P) \leq \frac{|\det(A_\gamma)|}{\gcd(A)}.$$

Proof. Recall we defined the lattice $\Lambda_\gamma(A) = \pi_\gamma(\ker(A) \cap \mathbb{Z}^n)$, whose determinant is given by $|\det(A_\gamma)| \,/\, \gcd(A)$ by (22). We show that $(P - P) \cap \Lambda_\gamma(A) = \{\boldsymbol{0}\}$. The conclusion then follows from Lemma 14.

Suppose that $\boldsymbol{u}, \boldsymbol{v} \in P$ and $\boldsymbol{u} - \boldsymbol{v} \in \Lambda_\gamma(A)$. Since P is symmetric, $P - P$ is the origin-symmetric translate of $2P$, and therefore

$$\begin{gathered} -\boldsymbol{z}_{\bar{\gamma}}^* - \boldsymbol{1}_{n-m} < \boldsymbol{u} - \boldsymbol{v} < \boldsymbol{z}_{\bar{\gamma}}^* + \boldsymbol{1}_{n-m} \\ -\boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* - 1 < \boldsymbol{q}_i^\top(\boldsymbol{u} - \boldsymbol{v}) < \boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* + 1 \text{ for all } i \in \gamma\backslash\tau. \end{gathered} \tag{33}$$

The lattice $\Lambda_\gamma(A)$ can be characterized as the set of points $\boldsymbol{x} \in \mathbb{Z}^{n-m}$ such that $\boldsymbol{q}_i^\top \boldsymbol{x} \in \mathbb{Z}$ for each $i \in \gamma$. Hence, the inequalities from (33) imply

$$\begin{gathered} -\boldsymbol{z}_{\bar{\gamma}}^* \leq \boldsymbol{u} - \boldsymbol{v} \leq \boldsymbol{z}_{\bar{\gamma}}^* \\ -\boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* \leq \boldsymbol{q}_i^\top(\boldsymbol{u} - \boldsymbol{v}) \leq \boldsymbol{q}_i^\top \boldsymbol{z}_{\bar{\gamma}}^* \text{ for all } i \in \gamma\backslash\tau. \end{gathered}$$

In particular, $\boldsymbol{u} - \boldsymbol{v}$ lies in the polyhedron $(-\boldsymbol{z}_{\bar{\gamma}}^* + C) \cap (\boldsymbol{z}_{\bar{\gamma}}^* - C)$, where

$$C := \left\{\boldsymbol{x} \in \mathbb{R}^{n-m} : \boldsymbol{x} \geq \boldsymbol{0},\ \boldsymbol{q}_i^\top \boldsymbol{x} \geq 0 \text{ for all } i \in \gamma\backslash\tau\right\}.$$

By assumption, $\boldsymbol{z}_{\bar{\gamma}}^*$ is a vertex of the sail $E(\Lambda_\gamma(A, \boldsymbol{b}), C)$. Hence, $\boldsymbol{z}_{\bar{\gamma}}^*$ is irreducible by Lemma 17, and therefore $\boldsymbol{u} = \boldsymbol{v}$. $\square$

To summarize this subsection, we have proven the following special case of Theorem 13:

Corollary 21. *Suppose τ and $\mathrm{supp}(\boldsymbol{z}^*)$ together cover $[n]$. Then there exists a basis γ of A containing τ such that*

$$\prod_{j\in\bar{\gamma}}(\boldsymbol{z}_j^* + 1) \leq r^d\,\frac{|\det(A_\gamma)|}{\gcd(A)}.$$

4.3. **Proof of Theorem 13.** To complete the proof of Theorem 13, it remains to deal with the case when $\bar{\tau}$ is not necessarily contained in $\mathrm{supp}(\boldsymbol{z}^*)$. Fix a vertex $\boldsymbol{z}^*$ of $C_\tau(A, \boldsymbol{b})$, and let $\mu = \tau \cup \mathrm{supp}(\boldsymbol{z}^*)$ which we may assume without loss of generality is given by $\mu = \{1, 2, \ldots, |\mu|\}$. Let $\bar{A}_\mu$ be any full row rank integer matrix with the same rowspace as A_μ. We have that $\boldsymbol{x}_\mu^*$ is a basic feasible solution of the system

$$\bar{A}_\mu \boldsymbol{x}_\mu = \bar{\boldsymbol{b}},\ \boldsymbol{x}_\mu \geq \boldsymbol{0}, \tag{34}$$

where $\bar{\boldsymbol{b}} := \bar{A}_\mu \boldsymbol{x}_\mu^*$. Moreover, letting

$$C_\tau(\bar{A}_\mu, \bar{\boldsymbol{b}}) := \mathrm{conv}(\{\boldsymbol{x}_\mu \in \mathbb{Z}^{|\mu|} : \bar{A}_\mu \boldsymbol{x}_\mu = \bar{\boldsymbol{b}},\ \boldsymbol{x}_{\mu\backslash\tau} \geq \boldsymbol{0}\}),$$

we have that $C_\tau(\bar{A}_\mu, \bar{\boldsymbol{b}}) \times \{\boldsymbol{0}\}^{|\bar{\mu}|}$ is the face of $C_\tau(A, \boldsymbol{b})$ for which the constraints $\boldsymbol{x}_{\bar{\mu}} \geq \boldsymbol{0}$ are tight, and this face contains $\boldsymbol{z}^* = (\boldsymbol{z}_\mu^*, \boldsymbol{0})$. Hence, $\boldsymbol{z}_\mu^*$ is a vertex of $C_\tau(\bar{A}_\mu, \bar{\boldsymbol{b}})$ such that

$\mu\backslash\tau \subseteq \operatorname{supp}(\boldsymbol{z}^*_\mu)$. We may therefore apply Corollary 21 to $\boldsymbol{x}^*_\mu, \boldsymbol{z}^*_\mu$, and the system (34), to obtain a basis $\sigma \subseteq \mu$ of $\bar{A}_\mu$ containing τ satisfying

$$\prod_{j\in\mu\backslash\sigma} (\boldsymbol{z}^*_j + 1) \le r^{|\sigma|-|\tau|} \frac{\left|\det(\bar{A}_\sigma)\right|}{\gcd(\bar{A}_\mu)}.$$

Now let γ be a basis of A containing σ. Then μ and $\gamma\backslash\sigma$ partition $\mu \cup \gamma$. Up to invertible row operations, we can write

$$A_{\mu\cup\gamma} = \begin{pmatrix} \bar{A}_\mu & \bar{A}_{\gamma\backslash\sigma} \\ \mathbf{0} & \bar{\bar{A}}_{\gamma\backslash\sigma} \end{pmatrix} = \begin{pmatrix} \bar{A}_{\mu\backslash\sigma} & \bar{A}_\sigma & \bar{A}_{\gamma\backslash\sigma} \\ \mathbf{0} & \mathbf{0} & \bar{\bar{A}}_{\gamma\backslash\sigma} \end{pmatrix},$$

where both $\bar{A}_\sigma$ and $\bar{\bar{A}}_{\gamma\backslash\sigma}$ are both invertible. Now, every nonzero maximal subdeterminant of $A_{\mu\cup\gamma}$ is the product of $\det(\bar{\bar{A}}_{\gamma\backslash\sigma})$ with a maximal subdeterminant of $\bar{A}_\mu$. It follows that

$$\gcd(A_{\mu\cup\gamma}) = \left|\det(\bar{\bar{A}}_{\gamma\backslash\sigma})\right| \cdot \gcd(\bar{A}_\mu),$$

and hence

$$\frac{\left|\det(\bar{A}_\sigma)\right|}{\gcd(\bar{A}_\mu)} = \frac{\left|\det(\bar{A}_\sigma)\right| \cdot \left|\det(\bar{\bar{A}}_{\gamma\backslash\sigma})\right|}{\gcd(A_{\mu\cup\gamma})} = \frac{\left|\det(A_\gamma)\right|}{\gcd(A_{\mu\cup\gamma})}.$$

We conclude

$$\prod_{j\in\bar{\gamma}}(\boldsymbol{z}^*_j + 1) \le \prod_{j\in\mu\backslash\sigma} (\boldsymbol{z}^*_j + 1) \le r^{|\sigma|-|\tau|} \frac{\left|\det(A_\gamma)\right|}{\gcd(A_{\mu\cup\gamma})} \le r^d \frac{\left|\det(A_\gamma)\right|}{\gcd(A)}.$$

5. Proof of Theorem 3

Without loss of generality, we assume in this proof that $\gamma = \{1\}$. Hence, we assume that the vertex $\boldsymbol{x}^*$ has the form

$$\boldsymbol{x}^* = \frac{b}{a_1}\boldsymbol{e}_1 \,.$$

The corner polyhedron associated with the vertex $\boldsymbol{x}^*$ can be written as

$$C_\gamma(\boldsymbol{a}, b) = \operatorname{conv}(\{\boldsymbol{x} \in \mathbb{Z}^n : \boldsymbol{a}^\top \boldsymbol{x} = b, x_2 \ge 0, \dots, x_n \ge 0\})\,.$$

First we will show that the knapsack polytope $P(\boldsymbol{a}, b)$ contains a vertex of the corner polyhedron $C_\gamma(\boldsymbol{a}, b)$. Let $\boldsymbol{z}^*$ be a vertex of $C_\gamma(\boldsymbol{a}, b)$ that gives an optimal solution to the linear program

$$\max\{x_1 : \boldsymbol{x} = (x_1, \dots, x_n)^\top \in C_\gamma(\boldsymbol{a}, b)\}\,.$$

By definition of $C_\gamma(\boldsymbol{a}, b)$ the vertex $\boldsymbol{z}^*$ is in $P(\boldsymbol{a}, b)$ if and only if $z^*_1 \ge 0$. Since $P(\boldsymbol{a}, b) \subset C_\gamma(\boldsymbol{a}, b)$, it is now sufficient to choose any integer point $\boldsymbol{z} = (z_1, \dots, z_n)^\top \in P(\boldsymbol{a}, b)$ and observe that $z^*_1 \ge z_1 \ge 0$.

Applying Theorem 1 with the vertex $\boldsymbol{z}^* \in P(\boldsymbol{a}, b)$ we immediately obtain (8) and (9). Further, the bound (3) implies for $r \ge 2$ the non-strict inequality

$$\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \frac{2^r}{r} \le \|\boldsymbol{a}\|_\infty \,. \tag{35}$$

To show that (35) is strict (and hence that (10) holds), it is sufficient to prove that the bound (30) in the proof of Theorem 1 is strict in the knapsack scenario. Specifically, we need to prove that for the vertex $\boldsymbol{z}^*$

$$\delta = |x_1^* - z_1^*| < \frac{(z_2^* + \cdots + z_n^*)\|\boldsymbol{a}\|_\infty}{a_1}\,.$$

Set $A = (a_1, \ldots, a_n) \in \mathbb{Z}^{1\times n}$ and consider the affine lattice $\Lambda(\boldsymbol{a}, b) := \Lambda_\gamma(A, b)$. We can write

$$\Lambda(\boldsymbol{a}, b) = \{(\lambda_2, \ldots, \lambda_n)^\top \in \mathbb{Z}^{n-1} : \lambda_2 a_2 + \cdots + \lambda_n a_n \equiv b\,(\bmod a_1)\}\,. \tag{36}$$

Following (21), the map $\pi_\gamma|_{H(A,b)}$ is a bijection. It follows that the point $\boldsymbol{y}^* = \pi_\gamma(\boldsymbol{z}^*)$ is a vertex of the sail $E(\Lambda(\boldsymbol{a}, b))$.

Suppose, to derive a contradiction, that the equality

$$\delta = \frac{(z_2^* + \cdots + z_n^*)\|\boldsymbol{a}\|_\infty}{a_1} \tag{37}$$

holds. By (28) we have

$$\delta = \frac{z_2^* a_2 + \cdots + z_n^* a_n}{a_1}$$

and, consequently, (37) implies $a_2 = \cdots = a_n = \|\boldsymbol{a}\|_\infty$. Therefore, using (36), the affine lattice $\Lambda(\boldsymbol{a}, b)$ contains the points

$$(z_2^* + \cdots + z_n^*)\boldsymbol{e}_j\,,\ j \in \{1, \ldots, n-1\}\,. \tag{38}$$

The point $\boldsymbol{y} = (z_2^*, \ldots, z_n^*)^\top$, in its turn, belongs to the simplex with vertices (38) and has $\|\boldsymbol{y}\|_0 = r \geq 2$. Therefore $\boldsymbol{y}$ cannot be a vertex of the sail $E(\Lambda(\boldsymbol{a}, b))$. The derived contradiction completes the proof of Theorem 3.

6. Proof of Corollary 4

By Theorem 3 the knapsack polytope $P(\boldsymbol{a}, b)$ contains a vertex $\boldsymbol{z}^*$ of $C_\gamma(\boldsymbol{a}, b)$. Therefore

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) \leq \|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \sum_{i\in \mathrm{supp}(\boldsymbol{x}^* - \boldsymbol{z}^*)} |c_i|\,. \tag{39}$$

If $r = 0$ we have $\boldsymbol{x}^* = \boldsymbol{z}^*$ that justifies (13). Further, (39) implies the bound

$$IG(\boldsymbol{c}, \boldsymbol{a}, b) \leq (r+1)\|\boldsymbol{x}^* - \boldsymbol{z}^*\|_\infty \|\boldsymbol{c}\|_\infty$$

that immediately gives (14) and (15).

7. Proof of Theorem 5

For $n \geq 2$ set $\boldsymbol{a}^{(n)} = (2^{n-1}, 2^{n-2}, \ldots, 1)^\top$ and $b^{(n)} = \mathbf{1}_n^\top \boldsymbol{a}^{(n)} = 2^n - 1$. Let $P_I(\boldsymbol{a}^{(n)}, b^{(n)}) = \mathrm{conv}(P(\boldsymbol{a}^{(n)}, b^{(n)}) \cap \mathbb{Z}^s)$ be the integer hull of the knapsack polytope $P(\boldsymbol{a}^{(n)}, b^{(n)})$.

We will need the following observations.

Lemma 22. *The point $\mathbf{1}_n$ is a vertex of the polytope $P_I(\boldsymbol{a}^{(n)}, b^{(n)})$.*

Proof. We will use induction on n. The basis step $n = 2$ holds as there are only two integer points $\mathbf{1}_2$ and $(0, 3)^\top$ in the polytope $P(\boldsymbol{a}^{(2)}, b^{(2)})$. To verify the inductive step, suppose that the result does not hold for some $n \geq 3$. Observe that any integer point $\boldsymbol{z} = (z_1, \ldots, z_n)^\top \in P(\boldsymbol{a}^{(n)}, b^{(n)})$ has $z_1 \leq 1$. Consequently, $\mathbf{1}_n$ belongs to the face $P_I(\boldsymbol{a}^{(n)}, b^{(n)}) \cap \{\boldsymbol{x} \in \mathbb{R}^s : x_1 = 1\}$ of the polyhedron $P_I(\boldsymbol{a}^{(n)}, b^{(n)})$. Hence $\mathbf{1}_n$ is a convex combination of some integer points in $P(\boldsymbol{a}^{(n)}, b^{(n)})$ that have the first entry 1. Therefore, removing the first entry we obtain a convex combination of integer points from $P(\boldsymbol{a}^{(n-1)}, \boldsymbol{b}^{(n-1)})$ equal to $\mathbf{1}_{n-1}$. The obtained contradiction completes the proof. □

For the rest of the proof we assume $n \geq 3$.

Lemma 23. *The point $\mathbf{1}_{n-1}$ is a vertex of the sail $E(\Lambda(\boldsymbol{a}^{(n)}, \boldsymbol{b}^{(n)}))$.*

Proof. Using (36), the affine lattice $\Lambda(\boldsymbol{a}^{(n)}, b^{(n)})$ can be written as

$$\Lambda(\boldsymbol{a}^{(n)}, b^{(n)}) = \{\boldsymbol{x} \in \mathbb{Z}^{n-1} : 2^{n-2}x_2 + \cdots + x_n \equiv -1 \pmod{2^{n-1}}\}.$$

Therefore

$$\mathcal{H} = \{\boldsymbol{x} \in \mathbb{R}^{n-1} : 2^{n-2}x_2 + \cdots + x_n = 2^{n-1} - 1\}$$

is a supporting hyperplane of $E(\Lambda(\boldsymbol{a}^{(n)}, b^{(n)}))$. Consequently,

$$P_I(\boldsymbol{a}^{(n-1)}, b^{(n-1)}) = \mathcal{H} \cap E(\Lambda(\boldsymbol{a}^{(n)}, b^{(n)}))$$

is a face of $E(\Lambda(\boldsymbol{a}^{(n)}, b^{(n)}))$. The result now follows by Lemma 22.

□

For a positive integer t set

$$\boldsymbol{a}^{(n)}(t) = (a_1^{(n)}(t), \ldots, a_n^{(n)}(t))^\top = (2^{n-1}, 2^{n-2} + t2^{n-1}, \ldots, 1 + t2^{n-1})^\top$$

and $b^{(n)}(t) = \mathbf{1}_n^\top \boldsymbol{a}^{(n)}(t) = 2^n + (n-1)t2^{n-1} - 1$. Consider the vertex $\boldsymbol{v}^{(n)}(t) = (b^{(n)}(t)/a_1^{(n)}(t))\boldsymbol{e}_1$ of the knapsack polytope $P(\boldsymbol{a}^{(n)}(t), b^{(n)}(t))$.

In view of (36), we have $\Lambda(\boldsymbol{a}^{(n)}, b^{(n)}) = \Lambda(\boldsymbol{a}^{(n)}(t), b^{(n)}(t))$. Therefore, by Lemma 23 the point $\mathbf{1}_{n-1}$ is a vertex of the sail $E(\Lambda(\boldsymbol{a}^{(n)}(t), b^{(n)}(t)))$. Observe that the sail $E(\Lambda(\boldsymbol{a}^{(n)}(t), b^{(n)}(t)))$ is the image of the corner polyhedron $C_{\{1\}}(\boldsymbol{a}^{(n)}, b^{(n)})$ under the bijective linear map $\pi_{\{1\}}|_{H(\boldsymbol{a}^{(n)}(t), b^{(n)}(t))}(\cdot)$. Using (21), the point

$$\mathbf{1}_n = \pi^{-1}_{\{1\}}|_{H(\boldsymbol{a}^{(n)}(t), b^{(n)}(t))}(\mathbf{1}_{n-1})$$

is a feasible vertex of $C_{\{1\}}(\boldsymbol{a}^{(n)}, b^{(n)})$. Note also that $\mathbf{1}_n \in P(\boldsymbol{a}^{(n)}(t), b^{(n)}(t))$.

It is now sufficient to show that for any $\epsilon > 0$

$$\|\boldsymbol{v}^{(n)}(t) - \mathbf{1}_n\|_\infty \frac{2^{n-1}}{n-1} > (1-\epsilon)\|\boldsymbol{a}^{(n)}(t)\|_\infty \tag{40}$$

for sufficiently large t. We have

$$\|\boldsymbol{v}^{(n)}(t) - \mathbf{1}_n\|_\infty = \frac{b^{(n)}(t)}{a_1^{(n)}(t)} - 1 = (n-1)t + 1 - \frac{1}{2^{n-1}}.$$

Finally,

$$\frac{\|\boldsymbol{v}^{(n)}(t) - \mathbf{1}_n\|_\infty}{\|\boldsymbol{a}^{(n)}(t)\|_\infty} = \frac{(n-1)t + 1 - 2^{-(n-1)}}{2^{n-2} + t2^{n-1}} \longrightarrow \frac{n-1}{2^{n-1}}$$

as $t \to \infty$, that implies (40).

8. Proof of Theorem 6

The result is a consequence of the following theorem by Bombieri and Vaaler [7].

Theorem 24 ([7, Theorem 2])**.** *Let $A \in \mathbb{Z}^{m\times n}$, $m < n$, be a matrix of rank m. There exist $n-m$ linearly independent integral vectors $\boldsymbol{y}_1, \dots, \boldsymbol{y}_{n-m} \in \Gamma(A)$ satisfying*

$$\prod_{i=1}^{n-m} \|\boldsymbol{y}_i\|_\infty \le \frac{\sqrt{\det(AA^T)}}{\gcd(A)} .$$

Let $\boldsymbol{z}^*$ be a vertex of the integer hull $P_I(A, \boldsymbol{b})$ that gives an optimal solution to (17). We will show that $\boldsymbol{z}^*$ satisfies (19). First, we argue that it suffices to consider the case $\|\boldsymbol{z}^*\|_0 = n$. Suppose that $\|\boldsymbol{z}^*\|_0 < n$. For $\tau = \mathrm{supp}(\boldsymbol{z}^*)$ set $\bar{A} = A_\tau$, $\bar{\boldsymbol{b}} = \boldsymbol{b}$, $\bar{\boldsymbol{c}} = \boldsymbol{c}_\tau$, and $\bar{\boldsymbol{z}}^* = \boldsymbol{z}^*_\tau$. By removing linearly dependent rows, we may assume that $\bar{A}$ has full row rank. Let $\bar{m} = \mathrm{rank}(\bar{A}) \le m$. Observe that $\bar{\boldsymbol{z}}^*$ is an optimal solution for the corresponding problem (17) with minimal support. Furthermore, note that $\bar{\boldsymbol{z}}^*$ has full support. Now, if (19) holds true for $\bar{\boldsymbol{z}}^*$, then

$$(\rho(\boldsymbol{z}^*) + 1)^{s-m} \le (\rho(\bar{\boldsymbol{z}}^*) + 1)^{s-\bar{m}} \le \frac{\sqrt{\det(\bar{A}\bar{A}^\top)}}{\gcd(\bar{A})} . \tag{41}$$

Further, using [3, Lemma 2.3] we have

$$\frac{\sqrt{\det(\bar{A}\bar{A}^\top)}}{\gcd(\bar{A})} \le \frac{\sqrt{\det(AA^\top)}}{\gcd(A)} . \tag{42}$$

Combining (41) and (42), we obtain (19).

From now on, we may assume that $\|\boldsymbol{z}^*\|_0 = n$. Suppose, to derive a contradiction, that (19) does not holds, that is

$$(\rho(\boldsymbol{z}^*) + 1)^{n-m} > \frac{\sqrt{\det(AA^\top)}}{\gcd(A)} .$$

By Theorem 24, there exists a vector $\boldsymbol{y} \in \mathbb{Z}^n \setminus \{\boldsymbol{0}\}$ such that

$$A\boldsymbol{y} = \boldsymbol{0} \quad \text{and} \quad \|\boldsymbol{y}\|_\infty \le \left(\frac{\sqrt{\det(AA^\top)}}{\gcd(A)}\right)^{\frac{1}{n-m}} < \rho(\boldsymbol{z}^*) + 1.$$

It follows that both $\boldsymbol{z}^* + \boldsymbol{y}$ and $\boldsymbol{z}^* - \boldsymbol{y}$ are in the knapsack polytope $P(A, \boldsymbol{b})$. Therefore $\boldsymbol{z}^*$ is not a vertex of $P_I(A, \boldsymbol{b})$. The obtained contradiction completes the proof.

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References

[1] I. Aliev, G. Averkov, J. A. De Loera, and T. Oertel. Optimizing sparsity over lattices and semigroups. *IPCO, Lecture Notes in Comput. Sci.*, 12125:40–51, 2020.

[2] I. Aliev, J. A. De Loera, F. Eisenbrand, T. Oertel, and R. Weismantel. The support of integer optimal solutions. *SIAM J. Optim.*, 28(3):2152–2157, 2018.

[3] I. Aliev, J. A. De Loera, T. Oertel, and C. O'Neill. Sparse solutions of linear Diophantine equations. *SIAM J. Appl. Algebra Geom.*, 1(1):239–253, 2017.

[4] I. Aliev, M. Henk, and T. Oertel. Distances to lattice points in knapsack polyhedra. *Math. Program.*, 182(1-2, Ser. A):175–198, 2020.

[5] S. Artstein-Avidan, A. Giannopoulos, and V.D. Milman. *Asymptotic Geometric Analysis, Part I*. Mathematical Surveys and Monographs. American Mathematical Society, 2015.

[6] H. Boche, R. Calderbank, G. Kutyniok, and J. Vybíral. A survey of compressed sensing. In *Compressed sensing and its applications*, Appl. Numer. Harmon. Anal., pages 1–39. Birkhäuser/Springer, Cham, 2015.

[7] E. Bombieri and J. Vaaler. On Siegel's lemma. *Invent. Math.*, 73(1):11–32, 1983.

[8] E. J. Candes and T. Tao. Decoding by linear programming. *IEEE Trans. Inform. Theory*, 51(12):4203–4215, 2005.

[9] J. W. S. Cassels. *An Introduction to the Geometry of Numbers*. Classics in Mathematics. Springer Berlin Heidelberg, 1996.

[10] W. Cook, J. Fonlupt, and A. Schrijver. An integer analogue of Carathéodory's theorem. *J. Combin. Theory Ser. B*, 40(1):63–70, 1986.

[11] W. Cook, A. M. H. Gerards, A. Schrijver, and É. Tardos. Sensitivity theorems in integer linear programming. *Math. Program.*, 34(3):251–264, 1986.

[12] F. Eisenbrand and G. Shmonin. Carathéodory bounds for integer cones. *Oper. Res. Lett.*, 34(5):564–568, 2006.

[13] F. Eisenbrand and R. Weismantel. Proximity results and faster algorithms for integer programming using the Steinitz lemma. In *Proceedings of the Twenty-Ninth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 808–816. SIAM, Philadelphia, PA, 2018.

[14] R. E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra Appl.*, 2:451–558, 1969.

[15] J. Lee, J. Paat, I. Stallknecht, and L. Xu. Improving proximity bounds using sparsity. In: Baïou M., Gendron B., Günlük O., Mahjoub A. (eds) Combinatorial Optimization. ISCO 2020. *Lecture Notes in Computer Science*, vol 12176. Springer, Cham. *arXiv:2001.04659*.

[16] W. M. Schmidt. *Diophantine approximations and Diophantine equations*. Lecture Notes in Mathematics. Springer-Verlag, 1991.

[17] A. Sebő. Hilbert bases, Carathéodory's theorem and combinatorial optimization. In *Proceedings of the 1st Integer Programming and Combinatorial Optimization Conference*, pages 431–455. University of Waterloo Press, 1990.

[18] R. R. Thomas. The structure of group relaxations. *Handbooks in Operations Research and Management Science*, 12:123–170, 2005.

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