ON STOCHASTIC POROUS-MEDIUM EQUATIONS WITH CRITICAL-GROWTH CONSERVATIVE MULTIPLICATIVE NOISE

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ABSTRACT. First, we prove existence, nonnegativity, and pathwise uniqueness of martingale solutions to stochastic porous-medium equations driven by conservative multiplicative power-law noise in the Ito-sense. We rely on an energy approach based on finite-element discretization in space, homogeneity arguments and stochastic compactness. Secondly, we use Monte-Carlo simulations to investigate the impact noise has on waiting times and on free-boundary propagation. We find strong evidence that noise on average significantly accelerates propagation and reduces the size of waiting times – changing in particular scaling laws for the size of waiting times.

1. INTRODUCTION

In this paper, we present results on wellposedness (existence, nonnegativity, pathwise uniqueness) and qualitative behaviour of solutions to the stochastic porous-medium equation[1]

$$du = (|u|^m u)_{xx}\, dt + (|u|^{\frac{m+2}{2}}\, dW)_x, \qquad m > 0. \tag{1.1}$$

Our motivation is twofold. Observing that in previous studies [14, 15, 11, 10], see also [3, 19], the authors confined themselves to multiplicative noise terms of at most linear growth in u – inside convective terms or as a source term - , the existence of solutions with multiplicative noise which grows according to a power law in u, is still open. Interestingly, in case of Ito-noise, formal computations based on Ito's-formula and results on deterministic porous-medium equations by Djie [13] indicate that for conservative noise terms, the exponent $\frac{m+2}{2}$ in (1.1) is distinguished in two aspects.

Apparently, it is the only one for which a priori-estimates can be derived in a form such that the $L^\infty(0,T;L^1(\mathcal{O}))$- and the $L^2(0,T;H^1(\mathcal{O}))$-norms of appropriate powers of solutions u are controlled almost surely.

The second feature which suggests to consider this exponent is the following. The deterministic porous-medium is well-known to allow for solutions which exhibit finite speed of propagation – see the monograph [32] by Vazquez and the references therein. For sufficiently smooth initial data, even the occurrence of waiting time phenomena, i.e. a locally delayed onset of spreading, has been proven.

Following [4, 16, 20], solutions to stochastic porous-medium equations exhibit finite speed of propagation, too, provided the noise appears inside a source term and has at most linear growth in u.

In [16], sufficient conditions for the occurrence of a waiting fine phenomenon have been identified as well.

It seems natural to pose the question in which way noise influences propagation speeds or the size of waiting times.

In [23], the case of multiplicative linear noise inside a source term has been studied. Monte-Carlo simulations based on convergent numerical schemes show that on average the effect of noise on the size of waiting times is rather negligible.

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[1] Here, W is a Q-Wiener process, and under periodic boundary conditions the total mass of solutions is constant in time. In this sense, we call multiplicative noise inside a convective term "conservative noise".

In the light of Djie's work [13] on the size of waiting times associated with nonnegative radial symmetric solutions to deterministic porous medium equations with source term

$$\partial_t w - \Delta(w^{m+1}) + \nu w^\beta = 0,$$

these findings are not surprising that much:

If $1 < \beta < m+1$ and initial data satisfy the critical bound $w_0 \geq A(R-\|x\|)_+^{\frac{2}{m}}$, the upper bounds on waiting times are independent of ν. If $\beta = 1$, this is still true for A sufficiently large and $\nu > 0$.

In the convective case, however, the power $\beta_0 = \frac{m+2}{2}$ turns out to be critical. For $\beta > \beta_0$, the equation

$$\partial_t w - (w^m w_x)_x + \nu(w^\beta)_x = 0$$

behaves like for $\nu = 0$.

For $1 < \beta \leq \frac{m+2}{2}$, however, pronounced effects in the scaling behaviour have been identified. Already for $\beta = \frac{m+2}{2}$, bounds on waiting times become $|\nu|$-dependent.

In Section 8, we present Monte-Carlo simulations which indicate that on average conservative power-law noise decreases the size of waiting times and changes corresponding scaling laws. In the same spirit, noise increases the propagation speed. However, we do not observe any change in the average spreading rate. These simulations are based on a modification to conservative noise of the schemes which have been proven in [23] to converge against a martingale solution of the stochastic porous-medium equation with multiplicative linear source-term noise.

To provide a theoretical fundament for these numerical experiments, we are for most of the paper concerned with rigorous existence results for nonnegative and pathwise unique solutions of (1.1). Technically, our approach is based on a stochastic Faedo-Galerkin method using – as in [17] – conforming linear finite elements as ansatz spaces. The formal Ito-estimate mentioned before carries over to the discrete setting, and we succeed to prove existence of martingale solutions in an $(H^1_{0,per}(\mathcal{O}))'$-setting[2], provided the noise amplitude is sufficiently small.

In addition, nonnegativity almost surely and pathwise uniqueness of the martingale solutions can be established. Let us emphasize that our results differ from previous results for degenerate equations with conservative multiplicative noise. In a series of papers, Fehrmann and Gess [14, 15], Dareiotis, Gerencscer, and Gess [10], Dareiotis and Gess [11] show existence results using Stratonovich noise. They assume sufficiently regular nonlinear noise terms having at most linear growth, and they base their argument on rough path theory.

It is not the least the Ito-Stratonovich-correction term appearing due to the authors' special choice of Stratonovich noise which turns out to be advantageous. This term has a regularizing effect, and it helps to control critical growth contributions in Ito's formula. We expect that in our setting existence results could be established for different growth exponents than $\frac{m+2}{2}$, too, if the Ito integral were replaced by the Stratonovich integral.

Beyond the distinction Ito vs Stratonovich noise discussed above, other types of noise like Lévy noise have been studied in the context of this nonlinear equation, see e.g. [8, 6]. In general, the stochastic porous medium equation has attracted a lot of attention in the scientific community. An exhaustive survey would be beyond the scope of this paper. For further information we refer to the monographs [30] and [2].

Our paper is organized as follows.

In Section 2, we introduce the discretization approach and we specify the setting, in particular the assumptions on the Q-Wiener processes driving the SPDE. In Section 3, the solution concept and our results on existence, nonnegativity, and pathwise uniqueness are presented. Section 4 is devoted to results on compactness in space and in time for discrete solutions. Note that the existence of strong solutions of the discretized equation is straightforward, as it essentially reduces to become a system of ordinary stochastic differential equations. Then, the strategy is to consider a discretized version of the energy

$$F(u) := \frac{1}{m+2}\int_{\mathcal{O}} |u|^{m+2}\, dx \tag{1.2}$$

and to apply Ito's formula to it. In order to obtain a Gronwall structure, we need to absorb terms in the dissipation which arise due to Ito's formula and which are different from those in the continuous

[2] Here, $H^1_{0,per}(\mathcal{O})$ denotes the closure of smooth $\mathcal{O}$-periodic functions in $H^1(\mathcal{O})$.

setting due to index shifts inherent in finite-element approaches. Lacking any h-independent control of the expected value of the oscillation of discrete solutions, we take advantage of the power-law structure of the nonlinearities to conclude by homogeneity arguments. Finally, compactness in space follows by a Burkholder-Davis-Gundy argument.

For compactness in time, we establish estimates uniform in h for discrete stochastic integrals in the space $L^\sigma(\Omega; C^\gamma([0,T_{max}];(H^1_{0,per}(\mathcal{O}))'))$ which – combined with inverse estimates for finite-element functions – allow to bound discrete solutions almost surely in Sobolev-Slobodecky spaces mapping the time interval onto $(H^1_{0,per}(\mathcal{O}))'$.

In Sections 5 and 6, we follow the strategy of [27] (see also [7] and [26]) to extract weakly convergent subsequences – using in particular a generalization of the Skorokhod theorem due to Jakubowski, [28] – and to establish existence of martingale solutions in the sense of Definition 3.1, provided the noise amplitude is sufficiently small. For these solutions, nonnegativity for appropriate initial data is established in Subsection 7.1. Formally, the idea is based on estimates for appropriate powers of the negative part of the solution. The rigorous argument is based on an Ito-formula which is justified by combining convolution arguments with boundedness results for maximal functions – modifying ideas of [16]. Subsection 7.2 is about pathwise uniqueness for the martingale solutions constructed in this paper – again relying on an Ito-argument.

Finally, in Section 8, a space-time discrete numerical scheme is introduced and applied to initial data with compact support. It turns out that in expectation, the size of waiting times significantly decreases with increasing noise amplitude, while a change of the sufficient condition for the occurrence of a waiting time phenomenon has not been observed. Moreover, we find a change in the scaling law for the size of waiting times: Taking initial data satisfying a critical growth condition $u_0(x) = S^{1/m}(x)_+^{2/m}$, deterministically[3] we expect a scaling $T^* \sim S^{-1}$. In our experiments, however, we observe – depending on the noise amplitude – exponents between -0.998 and -0.793, i.e. a change of about 20 percent (see Table 3).
In the same spirit, the expected propagation of the free boundary is accelerated. Concerning rates, however, our studies do not suggest a change in expectation.

Notation. Throughout the paper, we use standard notation for Lebesgue, Sobolev, and Hölder spaces and from stochastic analysis. Besides this, $H^1_{0,per}(\mathcal{O})$ denotes the closure of smooth $\mathcal{O}$-periodic functions in $H^1(\mathcal{O})$. For a Banach space E, the Nikolskii-space $N^{\alpha,p}(I;E)$ is defined for $\alpha \in (0,1)$, $1 \le p \le \infty$ by

$$N^{\alpha,p}(I;E) := \left\{ f \in L^p(I;E) \ : \ \sup_{h>0} \|f(\cdot+h) - f(\cdot)\|_{L^p((0,T-h);E)} \le h^\alpha \right\}.$$

The spatial domain $\mathcal{O}$ is given by the interval $(0,L)$, and we use the abbreviation $(v)_{\mathcal{O}}$ for the mean value of a function v over a domain $\mathcal{O}$. The notation $a \wedge b$ stands for the minimum of a and b, and $L_2(X,Y)$ denotes the set of Hilbert-Schmidt operators from X to Y. For a stopping time T, we write χ_T to denote the (ω-dependent) characteristic function of the time interval $[0,T]$.
Further notation related to the discretization will be introduced in Section 2.

2. Preliminaries on the Discretisation

In this section, we will introduce a semi-discrete scheme which will serve to obtain spatially discrete approximate solutions to the stochastic porous-medium equation. Existence of those approximate solutions will be established in Lemma 4.1.

- For $h = Ln^{-1}$ for a real number $L > 0$, and a natural number n, let X_h denote the space of periodic linear finite elements, i.e. the space of periodic continuous functions on $[0,L]$ that are affine polynomials on each of the intervals $[0,h]$, $[h,2h]$, $\ldots$, $[L-h,L]$.
- Let $C_{per}([0,L])$ be the space of periodic continuous functions on $[0,L]$. By $\mathcal{I}_h : C_{per}([0,L]) \to X_h$, we denote the nodal interpolation operator uniquely defined by $(\mathcal{I}_h\psi)(ih) := \psi(ih)$ for all $i \in \{1,\cdots,L_h\}$ where $L_h := Lh^{-1}$ is the dimension of X_h.
- On the Hilbert space X_h, we introduce the scalar product

$$(\phi^h,\psi^h)_h := \sum_{i=1}^{L_h} h\phi^h(ih)\psi^h(ih)$$

[3]see, e.g. [22] for lower bounds and [13] for upper bounds, respectively.

and the corresponding norm

$$||\psi^h||_h := \left(\sum_{i=1}^{L_h} h|\psi^h(ih)|^2 \right)^{1/2} .$$

Note that the norm $||\cdot||_h$ is equivalent to the $L^2(\mathcal{O})$-norm on X_h, uniformly in h. With a slight misuse of notation, we will frequently abbreviate $(\mathcal{I}_h\phi, \psi_h)_h$ for functions $\phi \in C_{per}([0,L])$ and $\psi_h \in X_h$ by $(\phi, \psi_h)_h$.

- By ∂_x^{+h} and ∂_x^{-h}, we denote the forward and backward difference quotient, respectively, i.e. $\partial_x^{+h} f(x) := h^{-1}(f(x+h) - f(x))$ (with f extended outside of $[0,L]$ by periodicity). The discrete Laplacian $\Delta_h u$ is defined by $\partial_x^{+h}(\partial_x^{-h} u)$.

Formally, equation (1.1) satisfies an integral estimate which allows to control the energy (1.2). Analytically, we rely on discrete counterparts of this estimate. For this, we introduce

$$F_h[v] := \tfrac{1}{m+2} \int_{\mathcal{O}} \mathcal{I}_h(|v|^{m+2})\, dx. \tag{2.1}$$

Now we are in the position to formulate the general assumptions on the data.

(H1) Let Λ be a probability measure on $L^{m+2}(\mathcal{O})$ equipped with the Borel σ-algebra which is supported on the subset of absolutely continuous nonnegative functions[4] such that there is a positive constant C with the property that

$$\operatorname{esssup}_{z \in \operatorname{supp} \Lambda} \left\{ F_h[\mathcal{I}_h z] + \left(\int_{\mathcal{O}} z\, dx \right) + \left(\int_{\mathcal{O}} z\, dx \right)^{-1} \right\} \leq C$$

for any $h > 0$ with $F_h[\cdot]$ as defined in (2.1).

(H2) Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration such that
- W is a Q-Wiener process on Ω adapted to $(\mathcal{F}_t)_{t\geq 0}$ which admits a decomposition of the form $W = \sum_{\ell=1}^{\infty} \mu_\ell g_\ell \beta_\ell$ for a sequence of independent standard Brownian motions β_ℓ and nonnegative real numbers $(\mu_\ell)_{\ell \in \mathbb{N}}$ (Here, the functions $(g_\ell)_{\ell\in\mathbb{N}}$ form a complete set of orthonormal functions in $L^2(\mathcal{O})$ which are in addition L-periodic.),
- the noise W is colored in the sense that $\sum_{\ell=1}^{\infty} \ell^2 \mu_\ell^2$ is bounded and sufficiently small,
- there exists a $\mathcal{F}_0$-measurable random variable u_0 such that $\Lambda = \mathbb{P} \circ u_0^{-1}$.

Let us define our scheme for approximation. On a stochastic basis satisfying (H2), given a positive time T_{max} and introducing $F_{max,h} := \frac{1}{m+2} h^{-\Theta}$ for $\Theta > 0$, we consider solutions

$$u^h \in L^2(\Omega; C([0, T_{max}]; X_h)),$$

to the system of stochastic evolution equations

$$\begin{aligned} (u^h(T), \phi^h)_h =& (u_0, \phi^h)_h \\ &- \int_0^{T \wedge T_h} \int_{\mathcal{O}} M_1^h(u^h) u_x^h \phi_x^h \, dx dt \\ &- \sum_{k=1}^{N_h} \int_0^{T\wedge T_h} \int_{\mathcal{O}} \mu_k M_2^h(u^h) g_k \phi_x^h \, dx d\beta_k \qquad \forall \phi^h \in X_h. \end{aligned} \tag{2.2}$$

where T_h is the stopping time defined by $T_h := T_{max} \wedge \inf\{t \geq 0 : F_h[u^h(t)] \geq F_{max,h}\}$. Note that $u^h(\cdot, t)$ is constant for $t \in [T_h, T_{max}]$. Here, $N_h \in \mathbb{N}$ is a cut-off for the noise for the purpose of discretization, subject only to the condition $N_h \to \infty$ for $h \to 0$. Discrete initial data are computed by the formula

$$u_0^h(\omega) := \mathcal{I}_h u_0(\omega). \tag{2.3}$$

The degenerate diffusion coefficient $M_1^h(u^h)$ is defined by

$$M_1^h(u)(\tau)|_{[x_{i-1}, x_i]} := (m+1) \fint_{u((i-1)h,\tau)}^{u(ih,\tau)} |s|^m ds,$$

while $M_2^h(u)$ is defined by

$$M_2^h(u) := \mathcal{I}_h(|u|^{\frac{m+2}{2}}).$$

[4]Note that this implies the estimate $\left| \int_{\mathcal{O}} u(x) - \mathcal{I}_h(u)(x)\, dx \right| \leq Ch \int_{\mathcal{O}} |\partial_x u(x)|\, dx$ guaranteeing the h-convergence of discrete masses.

Note in particular the formula

$$M_1^h(u^h)\partial_x u^h = \left(\mathcal{I}_h\left(|u^h|^m u^h\right)\right)_x. \tag{2.4}$$

3. Main Results

Definition 3.1. *Let Λ be a probability measure satisfying (H1). A triple $((\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde u, \tilde W)$ is called a weak martingale solution to the stochastic porous-medium equation* (1.1) *with initial data Λ on the time interval $[0,T]$ provided*

i) *$(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ is a stochastic basis with a complete, right-continuous filtration,*
ii) *$\tilde W$ satisfies (H2) with respect to $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$,*
iii) *$\tilde u$ is contained in*

$$L^{\bar p}(\Omega; L^\infty([0,T_{max}]; L^{m+2}(\mathcal{O}))) \cap L^2(\Omega; L^{2(m+1)}((0,T_{max}); C^\sigma(\mathcal{O})))$$

for any $\sigma \in (0, \frac{1}{2(m+1)})$,
iv) *the mean-value deviation $\tilde u - (\tilde u_0)_{\mathcal{O}}$ is contained in $L^2\left(\Omega; C\left([0,T_{max}]; (H^1_{0,per})'\right)\right)$, and*

$$|\tilde u|^m\, \tilde u \in L^2(\Omega; L^2\left(0, T_{max}; H^1(\mathcal{O})\right),$$

v) *there exists an $\tilde{\mathcal{F}}_0$-measurable $L^{m+2}(\mathcal{O})$-valued random variable $\tilde u_0$ such that $\Lambda = \tilde{\mathbb{P}} \circ \tilde u_0^{-1}$, and the equation*

$$\begin{aligned}\int_{\mathcal{O}} \tilde u(t)\phi\, dx = \int_{\mathcal{O}} \tilde u_0 \phi\, dx - \int_0^t \int_{\mathcal{O}} (m+1)|\tilde u|^m \tilde u_x \phi_x\, dxds \\ - \sum_{k=1}^\infty \mu_k \int_0^t \int_{\mathcal{O}} |\tilde u|^{\frac{m+2}{2}} g_k \phi_x\, dxd\beta_k\end{aligned} \tag{3.1}$$

holds $\tilde{\mathbb{P}}$-almost surely for all $t\in[0,T]$ and all $\phi \in H^1_{per}(\mathcal{O})$.

We are going to establish the existence of a weak martingale solution via approximation by solutions to the semi-discrete scheme (2.2).

Theorem 3.2. *Let the assumptions (H1)-(H2) be satisfied and let $T_{max} > 0$ be given. Assume $(u^h)_{h\to 0}$ to be a sequence of solutions to the approximation scheme* (2.2) *for the stochastic convective porous-medium equation* (1.1) *with $F_{max,h} = \frac{1}{m+2} h^{-\Theta}$. Let $\bar\gamma \in \{0, 1/4\}$ be given. If $\sum_{\ell=1}^\infty \ell^2 \mu_\ell^2$ is sufficiently small, then there exist a stochastic basis $(\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ as well as processes $(\tilde u^h)_{h\to 0}$ and $\tilde u \in L^2(\tilde\Omega; L^\infty(0,T; L^{m+2}(\mathcal{O}))) \cap L^2(\Omega; W^{\bar\gamma,p}(0,T_{max}; (H^1_{0,per}(\mathcal{O}))'))$ such that the following holds:*
The processes $\tilde u^h$ have the same law as the processes u^h, and for a subsequence $\tilde{\mathbb{P}}$-almost surely the following convergence results hold:

- *$\tilde u^h \rightharpoonup \tilde u$ weakly$*$ in $L^\infty(0,T_{max}; L^{m+2}(\mathcal{O}))$ $\tilde{\mathbb{P}}$-a.s.,*
- *$\tilde u^h - (\mathcal{I}_h u_0)_{\mathcal{O}} \to \tilde u - (\tilde u_0)_{\mathcal{O}}$ in $C([0,T_{max}]; (H^1_{0,per}(\mathcal{O}))')$ $\tilde{\mathbb{P}}$-a.s.,*
- *$\tilde u^h \rightharpoonup \tilde u$ weakly$*$ in $W^{\bar\gamma,p}(0,T_{max}; (H^1_{0,per}(\mathcal{O}))')$ $\tilde{\mathbb{P}}$-a.s.,*
- *$\chi_{\tilde T_h} \mathcal{I}_h(|\tilde u^h|^m \tilde u^h) \rightharpoonup |\tilde u|^m \tilde u$ in $L^2(0,T_{max}; H^1_{per}(\mathcal{O}))$ $\tilde{\mathbb{P}}$-a.s.,*
- *$\mathcal{I}_h \tilde u_0 \to \tilde u_0$ in $L^{m+2}(\mathcal{O})$ $\tilde{\mathbb{P}}$-a.s..*

Moreover, there exists a Q-Wiener process $\tilde W$ such that $((\tilde\Omega, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}), \tilde u, \tilde W)$ is a martingale solution in the sense of Def. 3.1, and for any $\bar p \geq 1$, we have the estimate

$$\mathbb{E}\left[\sup_{t\in[0,T_{max}]} F[\tilde u]^{\bar p}\right] + \mathbb{E}\left[\int_0^{T_{max}} \int_{\mathcal{O}} \left(|\tilde u|^m \tilde u\right)_x^2\, dxds\right] \leq C(u_0, \bar p, T_{max}). \tag{3.2}$$

Under a smallness assumption on the noise amplitude, we may prove nonnegativity and pathwise uniqueness.

Theorem 3.3. *If $\sum_{\ell\in\mathbb{N}} \ell^2 \mu_\ell^2$ is sufficiently small, then the following holds true: For any $m > 0$, a martingale solution u as constructed in Theorem 3.2 satisfies*

$$\tilde{\mathbb{P}}\left(\left\{\int_0^{T_{\max}} \int_{\mathcal{O}} (u)_-(x,s)\, dxds > 0\right\}\right) = 0,$$

where $(u)_- := \max\{-u, 0\}$ denotes the negative part of u.

Theorem 3.4. *Let u and v be two solutions in the sense of Defintion 3.1 on the same stochastic basis $(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}})$ sharing the same initial data u_0. If $\sum_{\ell=1}^{\infty} \mu_\ell^2$ is sufficiently small, then $u \equiv v$.*

These theorems are proven in Sections 6, 7.1, and 7.2, respectively.

4. A Priori Estimates

4.1. Existence of Solutions for the Semidiscrete Scheme.

Let us state an existence result for solutions to the Faedo-Galerkin scheme (2.2).

Lemma 4.1. *Let T_{max} be a positive real number and $F_{max,h} = \frac{1}{m+2} h^{-\Theta}$ with $\theta \in \mathbb{R}^+$ being a cut-off parameter. Then there exist stopping times T_h and stochastic processes $u^h \in L^2(\Omega; C([0, T_{max}]; X_h))$ with the following properties:*

- *Almost surely, we have $T_h = T_{\max} \wedge \inf\{t \in [0, \infty) : F_h[u^h(\cdot, t)] \geq F_{max,h}\}$.*
- *Almost surely, the process u^h solves (2.2) on $[0, T_{max}]$. In particular, it is constant for $t \in [T_h, T_{max}]$.*

The proof is based on combining standard existence results for SDEs with a smooth cut-off involving $F_h[u^h(\cdot, t)]$ and $F_{max,h}$ – for more details on this technique, see [17].

4.2. Compactness in Space – the Energy Estimate.

We now demonstrate that our spatial semi-discretization satisfies an estimate for moments of the discrete energy (2.1) as long as the energy remains below a critical threshold. As before, we choose $F_{max,h} = \frac{1}{m+2} h^{-\Theta}$ to be the threshold energy. In particular, it becomes infinite in the limit $h \to 0$.

Writing $u^h(x, t) = \sum_{i=1}^{L_h} a_i(t) e_i(x)$, we first note that (2.2) may be rewritten as

$$da_i = \frac{1}{h} L_i(s) ds + \sum_{\ell=1}^{N_h} Z_i(\mu_\ell g_\ell) d\beta_\ell, \tag{4.1}$$

where we have introduced

$$L_i(s) := -\chi_{T_h}(s) \int_{\mathcal{O}} M_1^h(u^h(s)) u_x^h(s) (e_i^h)_x \, dx \tag{4.2}$$

and $Z_i : L^2(\mathcal{O}) \to \mathbb{R}$ defined by

$$Z_i(w) := \chi_{T_h} \frac{1}{h} \sum_{\ell=1}^{\infty} \int_{\mathcal{O}} (g_\ell, w)_{L^2(\mathcal{O})} \left(M_2^h(u^h) g_\ell\right)_x e_i^h \, dx. \tag{4.3}$$

For a given parameter $\kappa > 0$ we consider the integral quantity

$$R(\kappa, s) := \kappa + F_h[u^h(s)]. \tag{4.4}$$

Note that we often abbreviate $R(s) := R(\kappa, s)$. Using Ito's formula, we derive the following integral estimates.

Proposition 4.2. *Let $\bar{p} \geq 1$ be arbitrary and let u^h be a solution to (2.2) for a parameter $0 < h < 1$. Then, for sufficiently small $\sum_{\ell \in \mathbb{N}} \mu_\ell^2 \ell^2$ there exist positive constants C_1, γ and $\overline{\gamma}$ independent of h and initial data such that for all $t \in [0, T_{max}]$ the following inequality holds:*

$$\begin{aligned}
&\mathbb{E}[R(t \wedge T_h)^{\bar{p}}] \\
&+ C_1 \mathbb{E}\left[\int_0^{t \wedge T_h} R(s)^{\bar{p}-1} \int_{\mathcal{O}} \left|\left(\mathcal{I}_h(|u^h(s)|^m u^h(s))\right)_x\right|^2 dx ds\right] \qquad (4.5)\\
&\leq \gamma(\kappa, \mu, \bar{p}) \left\{\mathbb{E}[R(0)^{\bar{p}}] + t^{1/\bar{p}} \left\{\mathbb{E}\left[\int_0^{t \wedge T_h} R(s)^{\bar{p}} ds\right] + \mathbb{E}\left[|(u_0)_{\mathcal{O}}|^{\bar{p}(2m+2)}\right]\right\}\right\} \\
&\leq \gamma(\kappa, \mu, \bar{p}) \mathbb{E}\left[R(0)^{\bar{p}} + |(u_0)_{\mathcal{O}}|^{\bar{p}(2m+2)}\right] \exp(\bar{\gamma} t^{\frac{\bar{p}+1}{\bar{p}}}). \qquad (4.6)
\end{aligned}$$

Moreover, for $T_{max} > 0$ arbitrary but fixed and sufficiently large κ , there exists a positive constant C depending only on $\bar{p}$, $(\mu_\ell)_{\ell \in \mathbb{N}}$, T_{max}, and initial data such that

$$\mathbb{E}\Big[\sup_{t \in [0, T_{max}]} R(t \wedge T_h)^{\overline{p}}\Big] \leq C(T_{max}, \bar{p}, (\mu_\ell)_{\ell \in \mathbb{N}}). \tag{4.7}$$

Proof. Using the notation

$$\varphi(h,s) := \frac{1}{h}\sum_{i=1}^{L_h} L_i(s)e_i \tag{4.8}$$

and

$$\Phi(h,s)(w) := \sum_{i=1}^{L_h} Z_i(w)e_i, \tag{4.9}$$

we may rewrite (2.2) as

$$du^h = \varphi(h,s)ds + \Phi(h,s)(dW_Q^h) \tag{4.10}$$

with

$$W_Q^h := \sum_{\ell=1}^{N_h} \mu_\ell g_\ell \beta_\ell. \tag{4.11}$$

By Ito's formula, we deduce

$$\begin{aligned}
R(t\wedge T_h)^{\bar p} &= R(0)^{\bar p} - \bar p\int_0^{t\wedge T_h} R(s)^{\bar p-1}\int_{\mathcal{O}} M_1^h[u^h(s)]u_x^h(s)\Big(\mathcal{I}_h\big(|u^h(s)|^m u^h(s)\big)\Big)_x\,dxds\\
&\quad + \bar p\sum_{\ell=1}^{N_h}\int_0^{t\wedge T_h} R(s)^{\bar p-1}\left(u^h(s)|u^h(s)|^m, \sum_{i=1}^{L_h} Z_i(\mu_\ell g_\ell)e_i\right)_h d\beta_\ell\\
&\quad + \frac{\bar p}{2}\sum_{\ell=1}^{N_h}\int_0^{t\wedge T_h} R(s)^{\bar p-1}\Big((m+1)|u^h(s)|^m, \Big[\sum_{i=1}^{L_h} Z_i(\mu_\ell g_\ell)e_i\Big]^2\Big)_h ds\\
&\quad + \frac{\bar p(\bar p-1)}{2}\sum_{\ell=1}^{N_h}\int_0^{t\wedge T_h} R(s)^{\bar p-2}\left(u^h(s)|u^h(s)|^m, \sum_{i=1}^{L_h} Z_i(\mu_\ell g_\ell)e_i\right)_h^2 ds\\
&=: R^{\bar p}(0) + I_1 + \ldots + I_4.
\end{aligned} \tag{4.12}$$

Using (2.4), I_1 becomes

$$I_1 = -\bar p\int_0^{t\wedge T_h} R(s)^{\bar p-1}\int_{\mathcal{O}} \left|\big(\mathcal{I}_h(|u^h(s)|^m u^h(s)\big)_x\right|^2 dxds. \tag{4.13}$$

Note that

$$Z_i(\mu_\ell g_\ell) = \frac{\mu_\ell}{h^2}\int_{(i-1)h}^{ih}\big(M_2^h[u^h](\cdot+h) - M_2^h[u^h]\big)\,g_\ell\,dx + \frac{\mu_\ell}{h^2}\int_{(i-1)h}^{ih} M_2^h[u^h](\cdot+h)\,(g_\ell(\cdot+h)-g_\ell)\,dx, \tag{4.14}$$

where the dependence of u^h on t is suppressed.

Using the splitting (4.14) in I_3 of (4.12), we get

$$I_3 = \frac{\bar p(m+1)}{2}\int_0^{t\wedge T_h} R(s)^{\bar p-1}\sum_{\ell=1,i=1}^{N_h,L_h} h(|U_i|^m)I_3(i,\ell)ds$$

with

$$I_3(i,\ell) := \left[\frac{\mu_\ell}{h^2}\int_{(i-1)h}^{ih}\big(M_2^h[u^h](\cdot+h) - M_2^h[u^h]\big)\,g_\ell\,dx + \frac{\mu_\ell}{h^2}\int_{(i-1)h}^{ih} M_2^h[u^h](\cdot+h)\,(g_\ell(\cdot+h)-g_\ell)\,dx.\right]^2$$

Note that we may assume

$$C_g := \sup_\ell\big(\|g_\ell\|_\infty + \|\ell^{-1}(g_\ell)_x\|_\infty\big) < \infty. \tag{4.15}$$

Obviously,

$$I_3(i,\ell) \le 2C_g^2\mu_\ell^2\left[\int_{(i-1)h}^{ih}\left|\frac{M_2^h[u^h(\cdot+h)] - M_2^h[u^h]}{h}\right|^2 dx + \ell^2\int_{(i-1)h}^{ih}|M_2^h[u^h(\cdot+h)]|^2\,dx\right].$$

In order to simplify notation, we will use U_i for the value of u^h at node i. Using convexity, we have

$$\begin{aligned}\int\limits_{(i-1)h}^{ih} \left|\frac{M_2^h[u^h](\cdot+h)-M_2^h[u^h]}{h}\right|^2 dx \\ \leq \frac{h}{2}\left(\left|\frac{|U_i|^{\frac{m+2}{2}}-|U_{i-1}|^{\frac{m+2}{2}}}{h}\right|^2+\left|\frac{|U_{i+1}|^{\frac{m+2}{2}}-|U_i|^{\frac{m+2}{2}}}{h}\right|^2\right),\end{aligned} \tag{4.16}$$

so $I_3 \leq C(I_{3,1}+I_{3,2})$ with $C=C_g$ as above,

$$I_{3,1} \leq \frac{\bar{p}(m+1)}{2}\int_0^{t\wedge T_h} R(s)^{\bar{p}-1}\left(\sum_{\ell=1}^{N_h}\mu_\ell^2\right)\sum_{i=1}^{L_h} L_1(i)ds,$$

$$L_1(i)=\frac{h}{2}|U_i|^m\left(\left|\frac{|U_i|^{\frac{m+2}{2}}-|U_{i-1}|^{\frac{m+2}{2}}}{h}\right|^2+\left|\frac{|U_{i+1}|^{\frac{m+2}{2}}-|U_i|^{\frac{m+2}{2}}}{h}\right|^2\right)$$

and

$$I_{3,2} \leq \frac{\bar{p}(m+1)}{2}\int_0^{t\wedge T_h} R(s)^{\bar{p}-1}\left(\sum_{\ell=1}^{N_h}\ell^2\mu_\ell^2\right)\sum_{i=1}^{L_h} L_2(i)ds$$

with

$$L_2(i)=|U_i|^m\int_{ih}^{(i+1)h}|M_2^h[u^h]|^2\,dx.$$

Let us turn to $I_{3,1}$. After an index shift we can estimate

$$\sum_{i=1}^{L_h} L_1(i) \leq \sum_{i=1}^{L_h}\frac{h}{2}\left(|U_i|^m+|U_{i-1}|^m\right)\left|\frac{|U_i|^{\frac{m+2}{2}}-|U_{i-1}|^{\frac{m+2}{2}}}{h}\right|^2.$$

The following lemma allows us to absorb $I_{3,1}$ in I_1 as in (4.13), provided the noise amplitude $\sum_{\ell=1}^\infty(1+\ell^2)\mu_\ell^2$ is sufficiently small.

Lemma 4.3. *There exists a constant $C>0$ such that for all $(U_i,U_{i-1})\in\mathbb{R}^2$*

$$\left(|U_i|^m+|U_{i-1}|^m\right)\left||U_i|^{\frac{m+2}{2}}-|U_{i-1}|^{\frac{m+2}{2}}\right|^2 \leq C\left(|U_i|^mU_i-|U_{i-1}|^mU_{i-1}\right)^2.$$

Proof: By the Minkowski inequality, it is sufficient to show the corresponding inequality with right- hand side $C\left(|U_i|^{m+1}-|U_{i-1}|^{m+1}\right)^2$. Hence, we can assume that both U_i and U_{i-1} are positive. Now, the result follows in a standard way combining a scaling argument with L'Hopital's rule. □

Now consider $I_{3,2}$.

By convexity we get that

$$L_2(i)=|U_i|^m\int\limits_{ih}^{(i+1)h}\left|M_2^h[u^h(\cdot)]\right|^2\,dx \leq \left(\frac{h}{2}|U_i|^m\left(|U_i|^{m+2}+|U_{i+1}|^{m+2}\right)\right). \tag{4.17}$$

As by Young's inequality

$$|U_i|^m|U_{i+1}|^{m+2} \leq \frac{m}{2m+2}|U_i|^{2m+2}+\frac{m+2}{2m+2}|U_{i+1}|^{2m+2} \leq (|U_i|^{2m+2}+|U_{i+1}|^{2m+2}),$$

we see that

$$L_2(i) \leq 2\int_{(i-1)h}^{ih}\mathcal{I}_h[|u^h|^{2m+2}]\,dx. \tag{4.18}$$

Moreover we have for $a,b\in\mathbb{R}$ by $||a|^ma|b|^mb| \leq \frac{1}{2}((|a|^{m+1})^2+(|b|^{m+1})^2)$ that

$$\begin{aligned}&\int_0^1(\lambda|a|^ma+(1-\lambda)|b|^mb)^2d\lambda=\frac{1}{3}\left(|a|^{2m+2}+|b|^{2m+2}+|a|^ma|b|^mb\right)\\ &\geq \frac{1}{6}\left(|a|^{2m+2}+|b|^{2m+2}\right)=\frac{1}{3}\int_0^1(|a|^{2m+2}\lambda+|b|^{2m+2}(1-\lambda))d\lambda.\end{aligned}$$

Hence,

$$\int_{(i-1)h}^{ih} \mathcal{I}_h[|u^h|^{2m+2}]\,dx \le 3\int_{(i-1)h}^{ih} (\mathcal{I}_h[|u^h|^m u^h])^2\,dx$$

and $L_2(i) \le 6\int_{(i-1)h}^{ih}(\mathcal{I}_h[|u^h|^m u^h])^2\,dx$ and in conclusion for some C depending only on $m, \overline{p}$ and C_g,

$$I_{3,2} \le C\int_0^{t\wedge T_h} R(s)^{\bar p-1}\left(\int_{\mathcal{O}} \left(\mathcal{I}_h[|u^h|^m u^h]\right)^2\,dx\right)ds. \tag{4.19}$$

We use that the linear interpolation yields a Sobolev function, hence we can apply Poincaré-inequality and the Gagliardo-Nirenberg inequality for Sobolev functions, without re-proving them at a discrete level.

Case 1: $\min_{1\le i\le L_h} U_i \le 0$. For sufficiently small noise amplitudes we can use the one-dimensional Poincaré inequality to absorb $I_{3,2}$ in I_1.

Case 2: $\min_{1\le i\le L_h} U_i \ge 0$. If, on the other hand, the values at the nodes are all nonnegative, then we identify $\int_{\mathcal{O}} u^h\,dx = \|u^h\|_{L^1}$, and we use an appropriate version of Gagliardo-Nirenberg's inequality. More precisely, a straightforward computation shows the existence of a positive constant C independent of h such that

$$\int_{\mathcal{O}} |\mathcal{I}_h((u^h)^{m+1})|^2\,dx \le C\int_{\mathcal{O}} (u^h)^{2m+2}\,dx.$$

Gagliardo-Nirenberg's inequality in the version of [21] gives for any $\varepsilon > 0$ the existence of a constant $C_\varepsilon > 0$ such that

$$\int_{\mathcal{O}} |\mathcal{I}_h((u^h)^{m+1})|^2\,dx \le C\int_{\mathcal{O}} \left((u^h)^{m+1}\right)^2\,dx \le \varepsilon\int_{\mathcal{O}} \left|((u^h)^{m+1})_x\right|^2\,dx + C_\varepsilon\left(\int_{\mathcal{O}} u^h\,dx\right)^{2m+2}. \tag{4.20}$$

By a straightforward computation, we find a positive constant independent of h and u^h such that

$$\int_{\mathcal{O}} \left|((u^h)^{m+1})_x\right|^2\,dx \le C\int_{\mathcal{O}} \left|(\mathcal{I}_h(u^h)^{m+1})_x\right|^2\,dx.$$

Finally, $\int_{\mathcal{O}} u^h = \left(u^h, 1\right)_h$ is controlled by the mass of initial data.

Combining both estimates we obtain for sufficiently small noise amplitude $\sum_{\ell\in\mathbb{N}}(1+\ell^2)\mu_\ell^2$ that I_3 is absorbed in I_1 up to a constant depending only on the initial condition, i.e. there exists a constant $\delta > 0$ such that

$$I_1 + I_3 \le \delta I_1 + C(u_0)\int_0^{t\wedge T_h} R(s)^{p-1}ds. \tag{4.21}$$

We turn to estimate I_4. By (4.14) we get

$$\frac{2}{\bar p(\bar p-1)}I_4 \le 2\int_0^{t\wedge T_h} R(s)^{\bar p-2}(T_1+T_2)ds, \tag{4.22}$$

where

$$T_1 = \sum_{\ell=1}^{N_h}\mu_\ell^2\left(h\sum_{i=1}^{L_h}|U_i|^{m+1}\frac{1}{h^2}\int\limits_{(i-1)h}^{ih}\left|M_2^h[u^h(\cdot+h)-M_2^h[u^h]\right||g_\ell|\,dx\right)^2 \tag{4.23}$$

and

$$T_2 = \sum_{\ell=1}^{N_h}\mu_\ell^2\left(h\sum_{i=1}^{L_h}|U_i|^{m+1}\frac{1}{h^2}\int\limits_{(i-1)h}^{ih}\left|M_2^h[u^h(\cdot)]\right||g_\ell(\cdot+h)-g_\ell|\,dx\right)^2. \tag{4.24}$$

Then for again $C_g := \sup_\ell\left(\|g_\ell\|_\infty + \|\ell^{-1}(g_\ell)_x\|_\infty\right)$, we have $T_2 \le C_g\left(\sum_{\ell=1}^{N_h}\ell^2\mu_\ell^2\right)\hat T_2$ with

$$\hat T_2 = \left(\sum_{i=1}^{L_h}|U_i|^{m+1}\int\limits_{ih}^{(i+1)h}\left|M_2^h[u^h(\cdot)]\right|\,dx\right)^2. \tag{4.25}$$

By splitting $|U_i|^{m+1} = \sqrt{h}|U_i|^{1+\frac{m}{2}} \frac{1}{\sqrt{h}}|U_i|^{\frac{m}{2}}$ we can estimate with Cauchy-Schwarz

$$\hat{T}_2 \leq \left[h \sum_{i=1}^{L_h} \frac{1}{2} \left(|U_i|^{m+2} + |U_{i-1}|^{m+2} \right) \right] \left[\sum_{i=1}^{L_h} |U_i|^m \int_{ih}^{(i+1)h} \left| M_2^h[u^h(\cdot)] \right|^2 \, dx \right]. \tag{4.26}$$

Combining (4.17) with Young's inequality yields

$$\hat{T}_2 \leq \hat{C} F_h[u^h] \sum_{i=1}^{L_h} \int_{(i-1)h}^{ih} \mathcal{I}_h[|u^h|^{2m+2}] \, dx \tag{4.27}$$

for some constant $\hat{C}$ depending only on m and $\sup_\ell(\|g_\ell\|_{W^{1,\infty}})$. Concluding, we have

$$\hat{T}_2 \leq C F_h[u^h] \int_{\mathcal{O}} \left(\mathcal{I}_h[|u^h|^m u^h] \right)^2 \, dx. \tag{4.28}$$

Therefore we can use again the Poincaré inequality in the case of a sign change or the Gagliardo-Nirenberg inequality otherwise in order to estimate

$$\hat{T}_2 \leq C \left(F_h[u^h] \int_{\mathcal{O}} \left| (\mathcal{I}_h(|u^h|^m u^h))_x \right|^2 \, dx + F_h[u^h] \left| \int_{\mathcal{O}} u_0 \, dx \right|^{2m+2} \right). \tag{4.29}$$

So for sufficiently small noise amplitude, this can be absorbed in I_1 or leads to a Gronwall structure.

Let us now consider T_1. As before, when estimating T_2, split $|U_i|^{m+1} = |U_i|^{1+\frac{m}{2}} |U_i|^{\frac{m}{2}}$, then use Cauchy-Schwarz for the sum and the integral to get

$$\begin{aligned} T_1 &\leq C_g F_h(u^h) \hat{T}_1 \\ \hat{T}_1 &= \sum_{i=1}^{L_h} \int_{(i-1)h}^{ih} |U_i|^m \left| \frac{M_2^h[u^h](\cdot + h) - M_2^h[u^h](\cdot)}{h} \right|^2 \, dx. \end{aligned}$$

Using again (4.16), we get

$$\hat{T}_1 \leq \sum_{i=1}^{L_h} \frac{h}{2} \left(|U_{i+1}|^m + |U_i|^m \right) \left| \frac{|U_{i+1}|^{\frac{m+2}{2}} - |U_i|^{\frac{m+2}{2}}}{h} \right|^2. \tag{4.30}$$

This implies that for a sufficiently small noise amplitude T_1 can be absorbed in the dissipative term, provided there exists $C > 0$ such that for all $a, b \geq 0$

$$(|a|^m + |b|^m)(|a|^{\frac{m+2}{2}} - |b|^{\frac{m+2}{2}}) \leq C \left(|a|^m a - |b|^m b \right)^2,$$

which is confirmed by Lemma 4.3. Collecting the estimates for I_3 and I_4 gives (4.5) which together with Gronwall's lemma entails (4.6).

In order to estimate the *supremum* $\mathbb{E}(\sup_{t \geq 0} R(t \wedge T_h)^{\bar{p}})$ it remains to estimate the martingale term I_2 of (4.12) by the Burkholder-Davis Gundy inequality, which requires an estimate of the time integral over the quadratic variation.

Let us denote the martingale term by $\mathcal{M}(t)$, then we have to estimate $\mathbb{E}\left[(\langle \mathcal{M}(t) \rangle)^{\frac{1}{2}} \right]$. As the discrete energy solves an ordinary stochastic differential equation driven by the independent Brownian motions β_ℓ, we find that the quadratic variation of the martingale in I_2 of (4.12) is given by

$$\langle \mathcal{M}(t) \rangle = \bar{p}^2 \sum_{\ell=1}^{N_h} \int_0^{t \wedge T_h} (R(s)^{\bar{p}-1})^2 \left(u^h(s)|u^h(s)|^m, \sum_{i=1}^{L_h} Z_i(\mu_\ell g_\ell) \right)_h^2 ds.$$

Note that this is, up to a different power of $R(s)$ and a constant factor, exactly the term in I_4, which is already estimated against the dissipative term and a Gronwall term. Therefore there exists a constant

$C > 0$ such that

$$
\begin{aligned}
\mathbb{E}\left(\langle \mathcal{M}(t)\rangle^{\frac{1}{2}}\right) &\le C\mathbb{E}\left[\left(\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\int_0^{t\wedge T_h}(R(s))^{2\bar{p}-2}\int_{\mathcal{O}}(\mathcal{I}_h(|u^h|^m u^h))_x^2\,dxds\right)^{\frac{1}{2}}\right]\\
&\quad + C\mathbb{E}\Bigg[\Bigg(\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\int_0^{t\wedge T_h}(R(s))^{2\bar{p}-2}\Big\{F_h[u^h(s)]\int_{\mathcal{O}}(\mathcal{I}_h(|u^h|^m u^h))_x^2\,dx \\
&\qquad\qquad + F_h[u^h(s)]\,(u_0)_{\mathcal{O}}^{2m+2}\Big\}\,ds\Bigg)^{\frac{1}{2}}\Bigg]\\
&=: \mathcal{M}_1 + \mathcal{M}_2.
\end{aligned}
\tag{4.31}
$$

To estimate $\mathcal{M}_1$, we use

$$
\begin{aligned}
\mathcal{M}_1 &\le C\mathbb{E}\left[\left(\sup_{[0,t\wedge T_h]} R(s)\right)^{\frac{\bar{p}}{2}}\left(\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\int_0^{t\wedge T_h}(R(s))^{\bar{p}-2}\int_{\mathcal{O}}|(\mathcal{I}_h[(u^h(s))^m]u^h)_x|^2\,dxds\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\left(\sup_{[0,t\wedge T_h]} R(s)\right)^{\bar{p}}\right]\\
&\qquad + \frac{C}{2}\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\mathbb{E}\left[\left(\int_0^{t\wedge T_h}(R(s))^{\bar{p}-2}\int_{\mathcal{O}}|(\mathcal{I}_h[(u^h(s))^m u^h])_x|^2\,dxds\right)\right]\\
&\le \frac{1}{2}\mathbb{E}\left[\left(\sup_{[0,t\wedge T_h]} R(s)\right)^{\bar{p}}\right]\\
&\qquad + \frac{C(\kappa)}{2}\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\mathbb{E}\left[\left(\int_0^{t\wedge T_h}(R(s))^{\bar{p}-1}\int_{\mathcal{O}}|(\mathcal{I}_h[(u^h(s))^m u^h])_x|^2\,dxds\right)\right]
\end{aligned}
\tag{4.32}
$$

for a suitable choice of $\kappa > 0$ in the definition of R, which bounds $R(s)$ uniformly away from zero. The first part can be absorbed in the left-hand side and the second term can be absorbed in the dissipative term provided $\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2$ is sufficiently small.

For $\mathcal{M}_2$, we get

$$
\begin{aligned}
\mathcal{M}_2 &\le \mathbb{E}\left[\left(\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\int_0^{t\wedge T_h}(R(s))^{2\bar{p}-1}\left\{(u_0)_{\mathcal{O}}^{2m+2} + \int_{\mathcal{O}}\left|(\mathcal{I}_h(|u^h|^m u^h))_x^2\right|\,dx\right\}ds\right)^{\frac{1}{2}}\right]\\
&\le \frac{1}{4}\mathbb{E}\left[\left(\sup_{[0,t\wedge T_h]} R(s)\right)^{\bar{p}}\right]\\
&\quad + C\sum_{\ell=1}^{\infty}(1+\ell^2)\mu_\ell^2\mathbb{E}\left[t + \int_0^{t\wedge T_h}R(s)^{\bar{p}}ds + \int_0^{t\wedge T_h}R(s)^{\bar{p}-1}\int_{\mathcal{O}}(\mathcal{I}_h(|u^h|^m u^h))_x^2\,dxds\right].
\end{aligned}
\tag{4.33}
$$

Note that the third term on the right-hand side is bounded due to (4.6) and the remaining terms on the right-hand side can be controlled analogously to those in (4.32).

□

4.3. **Compactness in Time.**

Lemma 4.4. *Let $(u^h)_{h\to 0}$ be a family of discrete solutions as constructed in Lemma 4.1, satisfying in particular inequality 4.7 for arbitrary $\bar{p} \ge 1$. Then there is a constant $C_{\bar{p}}$ – independent of $h > 0$ such that the mappings*

$$
\begin{aligned}
Z_h : [0,T_{\max}] \times L^2(\mathcal{O}) &\to (H^1_{0,per}(\mathcal{O}))'\\
(s,w) &\mapsto \chi_{T_h}(s)\sum_{i=1}^{L_h}\sum_{j=1}^{N_h}h^{-1}\int_{\mathcal{O}}\left(\mathcal{I}_h(M_2\left(u^h(s)\right))\mu_j(g_j,w)_{L^2}g_j\right)_x e_i\,dx\,e_i(y)
\end{aligned}
$$

are contained in $L^{2\bar p}(\Omega\times[0,T_{\max}];L_2(L^2(\mathcal{O});(H^1_{0,per}(\mathcal{O}))'))$ *satisfying* $\|Z_h\|_{L^{2\bar p}(\Omega\times[0,T_{\max}];L_2(L^2(\mathcal{O});(H^1_{0,per}(\mathcal{O}))'))}\le C_{\bar p}$ *uniformly in* $h\searrow 0$.

Proof of Lemma 4.4. First, we rewrite $Z_h(s)(g_j)$ for any basis function $g_j\in L^2(\mathcal{O})$ – using the abbreviation $f_j(u^h):=\mu_j\mathcal{I}_h(M_2^h(u^h(s)))g_j(x)$ – as

$$\begin{aligned}
Z_h(s)[g_j](y) =&\chi_{T_h}\sum_{i=1}^{L_h}\mu_j h^{-1}\int_{\mathcal{O}}\left(\mathcal{I}_h(M_2^h(u^h(s)))g_j\right)_x e_i(x)\,dx e_i(y)\\
=&-\chi_{T_h}\sum_{i=1}^{L_h}h^{-2}\left\{\int_{(i-1)h}^{ih}f_j(u^h(s))\,dx-\int_{ih}^{(i+1)h}f_j(u^h(s))\,dx\right\}e_i(y)\\
=&\chi_{T_h}\sum_{i=1}^{L_h}\mu_j h^{-2}\int_{ih}^{(i+1)h}\mathcal{I}_h(M_2^h(u^h(s)))g_j\,dx\,\{e_i(y)-e_{i+1}(y)\}
\end{aligned}$$

Observe that $\operatorname{supp}(e_i-e_{i+1})=[(i-1)h,(i+2)h]$ and $\int_{\mathcal{O}}(e_i-e_{i+1})\,dx\equiv 0$. Hence, a primitive is given by

$$\tilde q_i(y):=\begin{cases}\frac12(x-(i-1)h)^2 & \text{on } [(i-1)h,ih),\\ -(x-ih)^2+h(x-ih)+\frac12 h^2 & \text{on } [ih,(i+1)h),\\ \frac12\left((i+2)h-x\right)^2 & \text{on } [(i+1)h,(i+2)h],\\ 0 & \text{else.}\end{cases}\tag{4.34}$$

Note that

$$\max|\tilde q_i|\le Ch^2 \qquad\text{and}\qquad \operatorname{supp}\tilde q_i=[(i-1)h,(i+2)h].\tag{4.35}$$

Hence, $\left|\fint_{\mathcal{O}}\tilde q_i(y)dy\right|\le Ch^3$. Defining $-\Delta^{-1}(e_i-e_{i-1})$ to be the periodic function with mean-value zero satisfying $\int_{\mathcal{O}}(\Delta^{-1}(e_i-e_{i+1}))_x\phi_x=\int_{\mathcal{O}}(e_i-e_{i+1})\phi$ for all $\phi\in H^1_{0,\mathrm{per}}(\mathcal{O})$, we find $q_i:=-(\Delta^{-1}(e_i-e_{i+1}))_x$ to have mean-value zero, hence $q_i(y)=\tilde q_i(y)+\kappa h^3$ with a parameter κ which is bounded independently of h and i.
Abbreviating $\rho_{ij}:=\int_{ih}^{(i+1)h}\mu_j\mathcal{I}_h(M_2^h(u^h(s)))g_j\,dx$, we estimate the Hilbert-Schmidt norm of $Z_h(s)$ by

$$\begin{aligned}
\|Z_h(s)\|^2_{L_2(L^2(\mathcal{O});(H^1_{0,per}(\mathcal{O}))')} =&\chi_{T_h}\sum_{j=1}^{N_h}\left\|\sum_{i=1}^{L_h}h^{-2}\int_{ih}^{(i+1)h}\mu_j\mathcal{I}_h(M_2^h(u^h(s)))g_j\,dx q_i(y)\right\|^2_{L^2(\mathcal{O})}\\
=&\chi_{T_h}\sum_{j=1}^{N_h}\left\|\sum_{i=1}^{L_h}h^{-2}\rho_{ij}q_i\right\|^2_{L^2(\mathcal{O})}\\
\le&\chi_{T_h}\sum_{j=1}^{\infty}h^{-4}\sum_{i,k=1}^{L_h}\left\{\int_{\mathcal{O}}\rho_{ij}^2q_i^2dx+\int_{\mathcal{O}}\rho_{kj}^2q_k^2dx\right\}=(*_1)
\end{aligned}$$

By (4.35), we have $\int_{\mathcal{O}}q_k^2dx\le Ch^5$ with a constant independent of h and k. Hence,

$$(*_1)\le\chi_{T_h}\sum_{j=1}^{\infty}h\sum_{i,k=1}^{L_h}\left(\rho_{ij}^2+\rho_{kj}^2\right)\le C\chi_{T_h}\sum_{j=1}^{\infty}\mu_j^2\int_{\mathcal{O}}\left(\mathcal{I}_h(M_2^h(u^h(s)))\right)^2dx,$$

where we used Cauchy-Schwarz inequality combined with the L^2-orthonormality of $(g_j)_{j\in\mathbb{N}}$. In addition, by convexity

$$\int_{\mathcal{O}}\left(\mathcal{I}_hM_2^h(u^h)\right)^2\,dx\le\int_{\mathcal{O}}\mathcal{I}_h\left(\left|u^h\right|^{m+2}\right)dx.\tag{4.36}$$

Now, use the boundedness of $\sum_{j=1}^\infty\mu_j^2$ together with the uniform bound on $\mathbb{E}\left(\sup_t\left(\int_{\mathcal{O}}\mathcal{I}_h\left(\left|u^h\right|^{m+2}\right)dx\right)^{\bar p}\right)$ which is valid for arbitrary $\bar p\ge 1$. □

Combining Lemma 4.4 with Lemma 2.1 in [18] we obtain

Corollary 4.5. *Let* $h\in(0,1],T_{\max}>0,u^h$ *be a discrete solution constructed in Lemma 4.1. Then*

$$I_h(t):=\sum_{i=1}^{L_h}\sum_{j=1}^{N_h}h^{-1}\int_0^{t\wedge T_h}\int_{\mathcal{O}}\left(\mathcal{I}_h(M_2(u^h))\mu_j\,g_j\right)_x e_i\,dx\,d\beta_j(s)e_i$$

is contained in $L^\sigma(\Omega; C^\gamma([0,T_{\max}];(H^1_{0,per}(\mathcal{O}))'))$ *for any* $\sigma \geq 1$ *and* $\gamma \in (0,\frac{1}{2})$.
In particular, there is a positive constant $C_{\sigma,\gamma}$ *independent of* $h \in (0,1]$ *such that*

$$\|I_h\|_{L^\sigma(\Omega;C^\gamma([0,T_{\max}];(H^1_{0,per}(\mathcal{O}))'))} \leq C_{\sigma,\gamma}. \tag{4.37}$$

Lemma 4.6. *Under the assumptions of Corollary 4.5 solutions* u^h *constructed in Lemma 4.1 are contained and uniformly bounded in* $L^2(\Omega; C^{\frac{1}{4}}([0,T_{\max}];(H^1_{0,per}(\mathcal{O}))'))$. *In particular, we have the bound*

$$\mathbb{E}\left[\left(\sup_{t_1,t_2\in[0,T_{max}]} \frac{\|u^h(t_2)-u^h(t_1)\|_{(H^1_{0,per}(\mathcal{O}))'}}{(t_2-t_1)^{\frac{1}{4}}}\right)^2\right] \leq C_2 \tag{4.38}$$

with a constant C_2 independent of h.

Proof. Let $\varphi \in H^1_{0,per}(\mathcal{O})$ be arbitrary. Note that the orthogonal projection $P_h : L^2(\mathcal{O}) \to (X_h)_{per}$ maps $H^1_{0,per}(\mathcal{O})$ onto $(X_h)_{per} \cap H^1_{0,per}(\mathcal{O})$. Then, we estimate – using (9.1) and the weak formulation

$$\begin{aligned}
&\left|\langle u^h(T_2)-u^h(T_1),\varphi\rangle_{(H^1_{0,per}(\mathcal{O}))'\times H^1_{0,per}}\right| = \left|(u^h(T_2)-u^h(T_1),P_h\varphi)_{L^2}\right| \\
&\leq \left|(u^h(T_2)-u^h(T_1),P_h\varphi)_h\right| + \frac{h}{3}\left\|u^h(T_2)-u^h(T_1)\right\|_h \|\nabla P_h\varphi\|_{L^2} \\
&\leq \left|\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}} M_1(u^h)\nabla u^h \nabla P_h\varphi dx\right| + |(I_h(T_2)-I_h(T_1),P_h\varphi)| \\
&\quad + \frac{h}{3}\left\|u^h(T_2)-u^h(T_1)\right\|_h \|\nabla P_h\varphi\|_{L^2} + \frac{h}{3}\|I_h(T_2)-I_h(T_1)\|_h \|\nabla P_h\varphi\| = R_1+\cdots+R_4.
\end{aligned} \tag{4.39}$$

ad (R_1):

$$R_1 \leq C\|\nabla\varphi\|_{L^2}\sqrt{T_2-T_1}\left(\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}}\left|\nabla\mathcal{I}_h(\left|u^h\right|^m u^h)\right|^2 dx\,dt\right)^{\frac{1}{2}}. \tag{4.40}$$

ad (R_2):

$$R_2 \leq C\|I_h(T_2)-I_h(T_1)\|_{(H^1_{0,per}(\mathcal{O}))'}\|\nabla\varphi\|_{L^2} \leq C(\omega)(T_2-T_1)^\gamma\cdot\|\nabla\varphi\|_{L^2} \tag{4.41}$$

ad (R_3): Following the exposition in [17], we obtain the estimate

$$\begin{aligned}
\frac{\left\|u^h(T_2)-u^h(T_1)\right\|_h^2}{\|u^h(T_2)-u^h(T_1)\|_{L^2}} &\leq \frac{1}{\|u^h(T_2)-u^h(T_1)\|_{L^2}} \\
&\left|\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}} M_1(u^h)\nabla u^h\nabla(u^h(T_2)-u^h(T_1))dx\,dt\right| \\
&+\|I_h(T_2)-I_h(T_1)\|_{L^2}.
\end{aligned} \tag{4.42}$$

Similar to 4.49 in [17], we get

$$\begin{aligned}
\left\|u^h(T_2)-u^h(T_1)\right\|_h^2 &\leq C\left|\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}} M_1(u^h)\nabla u^h\nabla(u^h(T_2)-u^h(T_1))dx\,dt\right| \\
&+\|I_h(T_2)-I_h(T_1)\|_{L^2}^2.
\end{aligned} \tag{4.43}$$

Multiplying by h^2, we obtain – using (9.3), (9.4) and Corollary 4.5

$$\begin{aligned}
h^2\left\|u^h(T_2)-u^h(T_1)\right\|_h^2 &\leq Ch^2\sqrt{T_2-T_1}\left(\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}}\left|M_1(u^h)\nabla u^h\right|^2 dx\,dt\right)^{\frac{1}{2}} \\
&\quad\cdot\left\|\nabla(u^h(T_2)-u^h(T_1))\right\|_{L^2} + Ch^2\|I_h(T_2)-I_h(T_1)\|_{L^2}^2 \\
&\leq C\cdot h\sqrt{T_2-T_1}\left(\int_{T_1\wedge T_h}^{T_2\wedge T_h}\int_{\mathcal{O}}\left|M_1(u^h)\nabla u^h\right| dx\,dt\right)^{\frac{1}{2}}\cdot\sup_t\left\|u^h(t)\right\|_h + C(T_2-T_1)^{2\gamma}.
\end{aligned} \tag{4.44}$$

Hence,

$$
\begin{aligned}
&|R_3| \\
&\le C\,\|\nabla\varphi\| \left\{ h^{\frac{1}{2}}(T_2-T_1)^{\frac{1}{4}} \left(\int_{T_1\wedge T_h}^{T_2\wedge T_h} \int_{\mathcal{O}} \left|M_1(u^h)\nabla u^h\right| dx\,dt \right)^{\frac{1}{4}} \sup_t \left\|u^h(t)\right\|_h^{\frac{1}{2}} + (T_2-T_1)^{\gamma} \right\}.
\end{aligned}
\tag{4.45}
$$

For R_4, we get – using (9.4) and the equivalence of $\|\cdot\|_h$ and $\|\cdot\|_{L^2}$ on X_h –

$$
\begin{aligned}
&\frac{h}{3}\left\|I_h(T_2)-I_h(T_1)\right\|_h \left\|\nabla P_h\varphi\right\| \\
&\qquad\le C\left\|\nabla\varphi\right\|_{L_2} h \left\|I_h(T_2)-I_h(T_1)\right\|_{L^2} \\
&\qquad\le C\left\|I_h(T_2)-I_h(T_1)\right\|_{((H^1_{0,per}(\mathcal{O}))'} \left\|\nabla\varphi\right\|_{L^2} \le C(T_2-T_1)^{\gamma}\cdot\left\|\nabla\varphi\right\|_{L^2}.
\end{aligned}
\tag{4.46}
$$

Hence, due to (4.40), (4.41), (4.45), and (4.46),

$$
\begin{aligned}
\left\|u^h(T_2)-u^h(T_1)\right\|_{(H^1_{0,per}(\mathcal{O}))'} &= \sup_{\varphi} \frac{\left|\langle u^h(T_2)-u^h(T_1),\varphi\rangle\right|}{\left\|\nabla\varphi\right\|_{L^2}} \\
&\le C\sqrt{T_2-T_1}\left(\int_{T_1\wedge T_h}^{T_2\wedge T_h} \int_{\sigma} \left|M_1(u^h)\nabla u^h\right|^2 dx\,dt \right)^{\frac{1}{2}} \\
&\qquad + C_\sigma(\omega)(T_2-T_1)^{\gamma} \\
&\quad + Ch^{\frac{1}{2}}(T_2-T_1)^{\frac{1}{4}} \left(\int_{T_1\wedge T_h}^{T_2\wedge T_h} \int_{\mathcal{O}} \left|M_1(u^h)\nabla u^h\right|^2 dx\,dt \right)^{\frac{1}{4}} \sup_t \left\|u^h(t)\right\|_h^{\frac{1}{2}}.
\end{aligned}
\tag{4.47}
$$

Hence, the result. ☐

Using the standard embedding

$$
C^s(I;E) \hookrightarrow N^{s,p}(I;E) \hookrightarrow W^{r,p}(I;E) \qquad \forall r<s,\ \forall p\in[1,\infty],
$$

we have the following corollary to Lemma 4.6.

Corollary 4.7. *For any $\bar\gamma\in(0,\frac{1}{4})$ and any $p\in[1,\infty]$, there is a positive constant $C_{\bar\gamma,p}$ such that*

$$
\left\|u^h\right\|_{L^2\left(\Omega;W^{\bar\gamma,p}\left((0,T_{\max});(H^1_{0,per}(\mathcal{O}))'\right)\right)} \le C_{\bar\gamma,p}
$$

independently of $h>0$.

5. Existence of Solutions in the Continuous Setting

5.1. **Path Spaces and Compactness Results.** Following the strategy in [26, 5], we will apply the Jakubowski-Skorokhod Theorem 9.2 to find a stochastic basis such that a subsequence of finite-element solutions can be constructed which almost surely converges in topologies consistent with the nonlinearities of the equation. Finally, the limit process will be shown to be a martingale solution in the sense of Definition 3.1.

Let us collect the estimates obtained so far.

i) $\mathcal{I}_h\left(\chi_{T_h}\left|u^h\right|^m u^h\right)$ is uniformly bounded in

$$
L^2\left(\Omega;L^2\left((0,T_{\max});H^1(\mathcal{O})\right)\right)
\tag{5.1}
$$

as a consequence of the energy estimate and the estimate

$$
\mathbb{E}\left[\left\|\chi_{T_h}\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|^2_{L^2(\mathcal{O}_T)}\right] \le C\mathbb{E}\left[\sup_{t\in[0,T]} F_h\left(u^h\right)^{\frac{2m+2}{m+2}}\right].
$$

ii) For any $p\in[1,\infty)$

$$
\mathcal{I}_h\left(\left|u^h\right|^{m+2}\right) \in L^p\left(\Omega;L^\infty\left((0,T_{\max});L^1(\mathcal{O})\right)\right).
\tag{5.2}
$$

ii)′ By convexity, $\int_{\mathcal{O}}\left|u^h\right|^{m+2}dx \le \int_{\mathcal{O}}\mathcal{I}_h\left(\left|u^h\right|^{m+2}\right)dx$ and hence u^h is uniformly bounded in

$$
L^p\left(\Omega;L^\infty\left((0,T_{\max});L^{m+2}(\mathcal{O})\right)\right).
\tag{5.3}
$$

iii)

$$\int_{\mathcal{O}} (u^h)_{\mathcal{O}} = \int_{\mathcal{O}} \mathcal{I}_h u_0 \, dx \text{ is uniformly bounded in } L^{\bar{p}}(\Omega;\mathbb{R}) \tag{5.4}$$

for any $\bar{p} \in [1,\infty)$.

iv) u^h is uniformly bounded in

$$L^2\left(\Omega; W^{\bar{\gamma},p}\left((0,T_{\max})\,; \left(H^1_{\text{per}}(\mathcal{O})\right)'\right)\right) \tag{5.5}$$

for any $\bar{\gamma} \in (0,\frac{1}{4})$ and any $p \in [1,\infty]$.

In our setting, we consider the path spaces

$$\begin{aligned}
\chi_{u_1} :=& L^\infty\left((0,T)\,; L^{m+2}(\mathcal{O})\right)_{\text{weak}*}, && (5.6)\\
\chi_{u_2} :=& W^{\bar{\gamma},p}\left((0,T_{\max})\,; (H^1_{0,per}(\mathcal{O}))'\right)_{\text{weak}} && (5.7)\\
& \text{for } \bar{\gamma} \in (0,\tfrac{1}{4}) \text{ and } p \in [1,\infty) \text{ sufficiently large,} &&\\
\chi_{u_0} :=& L^{m+2}(\mathcal{O})_{\text{weak}}, && (5.8)\\
\chi_p :=& L^2\left((0,T_{\max})\,; H^1_{\text{per}}(\mathcal{O})\right)_{\text{weak}} && (5.9)
\end{aligned}$$

associated with the solutions u^h of our discrete scheme, initial data u_0, where we introduced

$$p_h := \chi_{T_h} \mathcal{I}_h \left(\left|u^h\right|^m u^h\right). \tag{5.10}$$

To cope with the deviation of u^h from its mean value $(u^h)_{\mathcal{O}}$, we introduce the path space

$$\chi_{\mathfrak{u}} := C\left([0,T]\,; (H^1_{0,per}(\mathcal{O}))'\right). \tag{5.11}$$

In particular, the solution of the problem being contained in that space allows the application of Ito's formula as it gives the only result on continuity with respect to time. As a sixth path space, we introduce

$$\chi_W := C\left([0,T]\,; L^2(\mathcal{O})\right). \tag{5.12}$$

Due to (5.1) and (5.5), we expect that the approximate solutions converge strongly in appropriate Lebesgue-Hölder spaces. However, to avoid technical difficulties due to the fact that H^1-estimates for $\mathcal{I}_h(\left|u^h\right|^m u^h)$ in (5.1) are valid only on the interval $[0,T_h(\omega)]$, we apply the corresponding Aubin-Lions-type interpolation argument not before Jakubowski's theorem.
Here is our first result on tightness.

Lemma 5.1. *Let $\mu_{\mathfrak{u}^h}$ denote the law of $\mathfrak{u}^h = u^h - (u^h)_{\mathcal{O}}$ in $C([0,T];(H^1_{0,per}(\mathcal{O}))')$. Then, for $h \in (0,1]$, the family of laws $(\mu_{\mathfrak{u}^h})_{h\searrow 0}$ is tight.*

Proof. We apply Theorem 9.3 to the spaces $X = L^{m+2}_0(\mathcal{O}) := \left\{u \in L^{m+2}(\mathcal{O}) \;:\; ⨍_{\mathcal{O}} u = 0\right\}$ and $B = Y = (H^1_{0,per}(\mathcal{O}))'$. Note that $L^{m+2}_0(\mathcal{O}) \subset\subset (H^1_{0,per}(\mathcal{O}))'$ as a topological injection. Observe that bounded families A in $L^\infty\left((0,T)\,; L^{m+2}(\mathcal{O})\right) \cap C^{\frac{1}{4}}\left([0,T]\,; (H^1_{0,per}(\mathcal{O}))'\right)$ satisfy in particular that

$$\sup_{f\in A} \|f(\cdot+\sigma) - f(\cdot)\|_{L^\infty\left(0,T-\sigma;(H^1_{0,per}(\mathcal{O}))'\right)} \to 0 \qquad \text{for } \sigma \to 0.$$

Hence, the ball $\overline{B_R} \subset L^\infty\left((0,T)\,; L^{m+2}(\mathcal{O})\right) \cap C^{\frac{1}{4}}\left([0,T]; (H^1_{0,per}(\mathcal{O}))'\right)$ is a compact subset of the space $C\left([0,T]\,; (H^1_{0,per}(\mathcal{O}))'\right)$. Furthermore, we have

$$\begin{aligned}
&\mu_{\mathfrak{u}^h}\left(C\left([0,T]\,; (H^1_{0,per}(\mathcal{O}))'\right) \setminus \overline{B_R}\right)\\
&\quad= \mathbb{P}\left(\left\|u^h - (u^h)_{\mathcal{O}}\right\|_{L^\infty(0,T;L^{m+2}(\mathcal{O}))} + \left\|u^h - (u^h)_{\mathcal{O}}\right\|_{C^{\frac{1}{4}}\left([0,T];(H^1_{0,per}(\mathcal{O}))'\right)} > R\right)\\
&\quad\le C\frac{\mathbb{E}\left[\left\|u^h - (u^h)_{\mathcal{O}}\right\|^2_{L^\infty(0,T;L^{m+2}(\mathcal{O}))} + \left\|u^h - (u^h)_{\mathcal{O}}\right\|_{C^{\frac{1}{4}}\left([0,T];(H^1_{0,per}(\mathcal{O}))'\right)}\right]}{R^2} \le \frac{C}{R^2}
\end{aligned}$$

due to (5.5) and the energy estimate combined with conservation of mass. This gives the result. ☐

Denoting the laws corresponding to the path spaces χ_{u_0}, χ_{u_1}, χ_{u_2}, χ_p, and χ_W by μ_{u_0}, μ_{u_1}, μ_{u_2}, μ_p, and μ_W, respectively, we obtain the following result on tightness.

Lemma 5.2. *Let $T_{\max} > 0$ be arbitrary, but fixed. Let (u^h, p^h) be a sequence of discrete solutions as constructed in Lemma 4.1. Then, for $h \in (0,1]$, the families of laws $(\mu_{u_1})_h$, $(\mu_{u_2})_h$, and $(\mu_p)_h$ are tight.*

Proof. Let us begin with $(\mu_{u_1})_h$. Using Markov's inequality together with the a priori bound (5.2), we have

$$\mathbb{P}\left(\left\|u^h\right\|_{L^\infty((0,T_{\max});L^{m+2}(\mathcal{O}))} > R\right) \le \frac{\mathbb{E}\left(\left\|u^h\right\|^p_{L^\infty(0,T_{\max};L^{m+2}(\mathcal{O}))}\right)}{R^p}.$$

Observing that closed balls in X' are weakly$*$ sequentially compact, we deduce tightness of $(\mu_{u_1})_h$ on χ_{u_1} subjected to the weak$*$-topology. For the tightness of $(\mu_{u_2})_h$ we use the bounds (5.7) combined with the fact that closed balls in χ_{u_2} are compact in the weak topology. A similar reasoning is applied for $(\mu_p)_h$ where we combine the bound

$$\mathbb{E}\left[\left\|\chi_{T_h}\nabla\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)^2\right\|_{L^2(\mathcal{O}_T)}\right] \le \text{const.}$$

with the estimate

$$\mathbb{E}\left[\left\|\chi_{T_h}\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|^2_{L^2(\mathcal{O}_T)}\right] \le C\mathbb{E}\left[\sup_{t\in[0,T_{\max}]} F_h\left(u^h\right)^{\frac{2m+2}{m+2}}\right] \le \text{const.} \tag{5.13}$$

(note that we used (5.19) together with the energy estimate to obtain (5.13)) . This proves the lemma. □

Observing that χ_{u_0} and χ_W are both Polish spaces and hence μ_{u_0} and μ_W are Radon measures, and recalling that marginal tightness implies tightness, we get the following result.

Proposition 5.3. *Let $\bar{\gamma} \in (0, \frac{1}{4})$ and $p \in [1,\infty)$ be given and assume u^h to be a sequence of solutions to our semi-discrete scheme ((2.2)) in the sense of Lemma 4.1, defined on the same stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ with respect to the Wiener process W. Then there exists a subsequence (not relabeled), a stochastic basis $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \tilde{\mathbb{P}})$, sequences of random variables*

$$\begin{aligned}
\tilde{\mathfrak{u}}^h : \widetilde{\Omega} \to & C\left([0,T_{\max}]; (H^1_{0,per}(\mathcal{O}))'\right),\\
\tilde{u}^h : \widetilde{\Omega} \to & L^\infty\left((0,T_{\max}); L^{m+2}(\mathcal{O})\right)_{weak*} \cap W^{\bar{\gamma},p}\left((0,T_{\max}); (H^1_{0,per}(\mathcal{O}))'\right)_{weak},\\
\widetilde{p}_h : \widetilde{\Omega} \to & L^2\left((0,T_{\max}); H^1_{per}(\mathcal{O})\right)_{weak},\\
\tilde{u}^h_0 : \widetilde{\Omega} \to & L^{m+2}(\mathcal{O}),
\end{aligned}$$

a sequence of $L^2(\mathcal{O})$-valued continuous processes $\widetilde{W}_h$ on $\widetilde{\Omega}$, and random variables

$$\begin{aligned}
\tilde{\mathfrak{u}} \in & C\left([0,T_{\max}]; (H^1_{0,per}(\mathcal{O}))'\right),\\
\tilde{u} \in & L^\infty\left((0,T_{\max}); L^{m+2}(\mathcal{O})\right)_{weak*} \cap W^{\bar{\gamma},p}\left((0,T_{\max}); (H^1_{0,per}(\mathcal{O}))'\right)_{\text{weak}},\\
\tilde{p} \in & L^2\left((0,T_{\max}); H^1_{per}(\mathcal{O})\right),\\
\tilde{u}_0 \in & L^1(\widetilde{\Omega}; L^{m+2}(\mathcal{O}))
\end{aligned}$$

as well as an $L^2(\mathcal{O})$-valued process $\widetilde{W}$ on $\widetilde{\Omega}$ such that

i) *the law of $\left(\tilde{\mathfrak{u}}^h, \tilde{u}^h, \widetilde{p}_h, \widetilde{W}_h, \tilde{u}^h_0\right)$ on $\chi := \chi_{\mathfrak{u}} \times \chi_{u_1} \times \chi_{u_2} \times \chi_p \times \chi_W \times \chi_{u_0}$ under $\tilde{\mathbb{P}}$ coincides for any h with the law of $\left(\mathfrak{u}^h, u^h, p_h, W, u_0\right)$ under $\mathbb{P}$,*

ii) *the sequence $\left(\tilde{\mathfrak{u}}^h, \tilde{u}^h, \widetilde{p}_h, \widetilde{W}_h, \tilde{u}^h_0\right)$ converges $\tilde{\mathbb{P}}$-almost surely to $\left(\tilde{\mathfrak{u}}, \tilde{u}, \tilde{p}, \widetilde{W}, \tilde{u}^h_0\right)$ in the topology of χ.*

We introduce the random times

$$\widetilde{T}_h := T_{\max} \wedge \inf\left\{t \ge 0 \;:\; F_h\left[\tilde{u}^h(t)\right] \ge F_{\max,h}\right\}$$

with $F_{\max,h} := \frac{1}{m+2}h^{-\Theta}$.

Lemma 5.4. *Along a subsequence, the convergence $\lim_{h\to 0}\widetilde{T}_h = T_{\max}$ holds $\tilde{\mathbb{P}}$-almost surely.*

Proof. Observing that F_h is pathwise continuous, as u^h solves a stochastic ODE, the result follows along the lines of Lemma 5.5 in [17]. □

As a consequence of coincidence of laws and the energy estimate, we have the following lemma.

Lemma 5.5. *Under the assumptions of Proposition 5.3, we have $\widetilde{P}_h = \mathcal{I}_h\left(\chi_{\tilde{T}_h}\left|\tilde{u}^h\right|^m \tilde{u}^h\right)$.*

We note that $\tilde{W}$ and $\tilde{W}^h$ are Q-Wiener processes adapted to filtrations $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t^h)_{t\geq 0}$, defined by follows:

We take $(\tilde{\mathcal{F}}_t)_{t\geq 0}$ to be the $\tilde{\mathbb{P}}$-augmented canonical filtration associated with $(\tilde{u}, \tilde{W}, \tilde{u}_0)$, i.e.

$$\tilde{\mathcal{F}}_t := \sigma(\sigma(r_t\tilde{u}, r_t\tilde{W}) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}_0)). \tag{5.14}$$

Here, r_t is the restriction of a function defined on $[0, T_{max}]$ to the interval $[0, t]$, $t \in [0, T_{max}]$.

Note that we do not need an explicit dependence of the filtration on $r_t\tilde{\mathfrak{u}}$ and $r_t\tilde{p}$, as the quantities $\tilde{\mathfrak{u}}^h$ and $\tilde{p}^h$ depend in a measurable way on $\tilde{u}^h$ and – later on – we will identify $\tilde{\mathfrak{u}} = \lim_{h\to 0}\tilde{\mathfrak{u}}^h = \tilde{u} - \overline{(\tilde{u}_0)}$ and $\tilde{p} = \lim_{h\to 0}\tilde{p}^h = |\tilde{u}|^m\,\tilde{u}$.

Analogously, we introduce the filtrations $(\tilde{\mathcal{F}}_t^h)_{t\geq 0}$ as the $\tilde{\mathbb{P}}$-augmented canonical filtration associated with $(\tilde{u}^h, \tilde{W}^h, \tilde{u}_0^h)$

$$\tilde{\mathcal{F}}_t^h := \sigma(\sigma(r_t\tilde{u}^h, r_t\tilde{W}^h) \cup \{N \in \tilde{\mathcal{F}} : \tilde{\mathbb{P}}(N) = 0\} \cup \sigma(\tilde{u}_0^h)). \tag{5.15}$$

Similarly as in [17], we obtain

Lemma 5.6. *The processes $\tilde{W}^h$ and $\tilde{W}$ are Q-Wiener processes adapted to the filtrations $(\tilde{\mathcal{F}}_t^h)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, respectively. They can be written as*

$$\tilde{W}^h(t) = \sum_{\ell\in\mathbb{N}} \lambda_\ell \tilde{\beta}_\ell^h(t) g_\ell \tag{5.16}$$

and

$$\tilde{W}(t) = \sum_{\ell\in\mathbb{N}} \lambda_\ell \tilde{\beta}_\ell(t) g_\ell, \tag{5.17}$$

respectively. Here, $(\tilde{\beta}_\ell^h)_{\ell\in\mathbb{N}}$ and $(\tilde{\beta}_\ell)_{\ell\in\mathbb{N}}$ are families of i. i. d. Brownian motions with respect to $(\tilde{\mathcal{F}}_t^h)_{t\geq 0}$ and $(\tilde{\mathcal{F}}_t)_{t\geq 0}$, respectively.

5.1.1. *Additional Convergence Results for Discrete Solutions $(\tilde{u}^h)_{h\searrow 0}$.* In this section, we show the sequence $(\tilde{u}^h)_{h\searrow 0}$ obtained in Proposition 5.3 to converge strongly to $\tilde{u}$ in $L^{2(m+1)}((0,T); C^\gamma(\mathcal{O}))$ for any $\gamma \in \left(0, \frac{1}{2(m+1)}\right)$ and for any $T \in [0, T_{\max})$ $\tilde{\mathbb{P}}$-almost surely. We begin with a preliminary result.

Lemma 5.7. *Let $T \in [0, T_{\max})$ and let $(\tilde{u}^h)_{h\searrow 0}$ be a sequence of periodic finite-element functions in $C^0\left([0,T]; X_{\mathrm{per}}^h\right)$ satisfying*

- (A1) $\left(\nabla\mathcal{I}_h\left(\left|\tilde{u}^h\right|^m \tilde{u}^h\right)\right)_{h\searrow 0}$ *weakly convergent in $L^2\left((0,T); L^2(\mathcal{O})\right)$.*
- (A1)$'$ $(\tilde{u}^h)_{h\searrow 0}$ *weakly$*$-convergent in $L^\infty\left((0,T); L^{m+2}(\mathcal{O})\right)$ $\tilde{\mathbb{P}}$-almost surely.*
- (A2) $(\tilde{u}^h)_{h\searrow 0}$ *weakly convergent in $W^{\bar{\gamma},p}\left((0,T_{\max}); (H^1_{0,per}(\mathcal{O}))'\right)$ for $\bar{\gamma} \in (0, \frac{1}{4})$ and $p > \frac{1}{\bar{\gamma}}$.*

Then, for each $\sigma \in (0, \frac{1}{2(m+1)})$, there is a convergent subsequence in $L^{2(m+1)}\left((0,T); C^\sigma(\overline{\mathcal{O}})\right)$.

Proof. First, we show that

$$(u^h)_{h\searrow 0} \text{ uniformly bounded in } L^{2(m+1)}((0,T); C^{\gamma_0}(\mathcal{O})) \tag{5.18}$$

with $\gamma_0 = \frac{1}{2(m+1)}$.

Indeed, by the mean-value Poincaré-inequality, we have

$$\left\|\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|_{L^2(\mathcal{O})}^2 \leq C\left\{\left\|\nabla\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|_{L^2(\mathcal{O})}^2 + \left(\int_{\mathcal{O}} \mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\,dx\right)^2\right\}.$$

By Hölder's inequality,

$$\left(\int_{\mathcal{O}} \mathcal{I}_h\left(\left|u^h\right|^m u^h\right)dx\right)^2 \leq C\mathcal{E}\left(u^h\right)^{\frac{2m+2}{m+2}}. \tag{5.19}$$

Using (A1) and the Sobolev embedding theorem, we have

$$\left(\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right)_{h\searrow 0} \text{ uniformly bounded in } L^2\left((0,T); C^{\frac{1}{2}}(\mathcal{O})\right). \tag{5.20}$$

Using $\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)$ to be continuous and piecewise linear, (5.20) implies

$$\left(\left|u^h\right|^m u^h\right)_{h\searrow 0} \text{ uniformly bounded in } L^2\left((0,T);C^{\frac{1}{2}}(\mathcal{O})\right). \tag{5.21}$$

Now, consider $H(W) := \operatorname{sign} W \sqrt[m+1]{W\cdot \operatorname{sign} W}$ which is the inverse of $G(u) := |u|^m u$. One easily shows that $H \in C^{\frac{1}{m+1}}(\mathbb{R}^+)$. Now, standard estimates for compositions of Hölder-continuous functions imply

$$\frac{\left|H\left(\left|u^h(x_2)\right|^m u^h(x_2)\right) - H\left(\left|u^h(x_1)\right|^m u^h(x_1)\right)\right|}{|x_2-x_1|^{\frac{1}{2(m+1)}}} \le \text{Höl}_{\frac{1}{m+1}}(H)\cdot \text{Höl}_{\frac{1}{2}}\left(\left|u^h\right|^m u^h\right)^{\frac{1}{m+1}}. \tag{5.22}$$

Using

$$\left\|\left|u^h\right|^m u^h\right\|_{C^{\frac{1}{2}}(\mathcal{O})} \le F_h\left(u^h\right)^{\frac{m+1}{m+2}} + \left\|\nabla \mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|_{L^2(\mathcal{O})},$$

we observe (using the $L^\infty(0,T)$-bound on $F_h(u^h)$ and the $L^2(0,T)$-bound on $\left\|\nabla\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|_{L^2(\mathcal{O})}$) that

$$\left\|u^h\right\|_{L^{2(m+1)}(0,T;C^{\frac{1}{2(m+1)}}(\mathcal{O}))} \le C\left\{\sup_{t\in[0,T]} F_h\left(u^h\right)^{\frac{1}{m+2}} + \left\|\nabla\mathcal{I}_h\left(\left|u^h\right|^m u^h\right)\right\|^{\frac{1}{m+1}}_{L^2(0,T;L^2(\mathcal{O}))}\right\}. \tag{5.23}$$

Secondly, we establish Nikolsk'ii-regularity with respect to time.
From (A2), we infer

$$\sup_{\substack{t\in[0,T-\sigma]\\ h\searrow 0}} \left\|u^h(t+\sigma) - u^h(t)\right\|_{(H^1_{0,per}(\mathcal{O}))'} \le C\sigma^\delta$$

for $\delta = \bar{\gamma} - \frac{1}{p} > 0$. Hence, $\left(u^h\right)_{h\searrow 0}$ is uniformly bounded in $N^{\delta,\infty}\left(0,T;(H^1_{0,per}(\mathcal{O}))'\right)$. Observing that $C^\gamma(\bar{\mathcal{O}}) \subset\subset L^1(\mathcal{O}) \hookrightarrow (H^1_{0,per}(\mathcal{O}))'$ for any $\gamma\in(0,1)$, we may apply Theorem 9.3 with $X = C^{\gamma_0}(\mathcal{O})$, $B = C^\gamma(\mathcal{O})$, $Y = (H^1_{0,per}(\mathcal{O}))'$, $p = 2(m+1)$ to obtain the result of the lemma. □

Now, the main result reads as follows.

Proposition 5.8. *Let $(h_n)_{n\in\mathbb{N}}$ converge monotonically to zero and let $\bar{T}\in[0,T_{\max})$ be arbitrary, but fixed. Then, the sequence $(\tilde{u}^{h_n})_{n\to\infty}$ converges to $\tilde{u}$ in $L^{2(m+1)}\left((0,\bar{T});C^\sigma(\mathcal{O})\right)$ for any $\sigma\in\left(0,\frac{1}{2(m+1)}\right)$ $\tilde{\mathbb{P}}$-almost surely.*

Proof. From Proposition 5.3, we infer that $(\tilde{u}^{h_n})_{n\in\mathbb{N}}$ is weakly∗-convergent in $L^\infty\left((0,T);L^{m+2}(\mathcal{O})\right)$ $\tilde{\mathbb{P}}$-a.s.. Hence – using $\tilde{u}^h(t)\in X^h_{\text{weak}}$ – $\mathcal{I}_h\left(\left|\tilde{u}^{h_n}\right|^{m+2}\right)$ is uniformly bounded in $L^\infty\left((0,T_{\max});L^1(\mathcal{O})\right)$. The estimate

$$F_{h_n}\left[\tilde{u}^{h_n}\left(\bar{T}_{h_n}\right)\right] = \tfrac{h_n}{m+2}\sum_{i=1}^{L_{h_n}}\left|\tilde{u}^{h_n}_i\right|^{m+2} \le \frac{1}{m+2}h_n^{-\Theta},$$

entails

$$\max_{i=1}^{L_{h_n}}\left|\tilde{u}^{h_n}_{(i)}\right| \le h_n^{-\frac{\Theta}{m+2}}.$$

As a consequence, there is a constant $\widetilde{C}$ depending only on h_n such that

$$\left|\nabla\mathcal{I}_h\left(\left|\tilde{u}^{h_n}\right|\tilde{u}^{h_n}\right)\right|\Big|_{[\widetilde{T}_{h_n},\bar{T}]} \le \widetilde{C}(h_n).$$

Using $\widetilde{P}_h = \mathcal{I}_h\left(\chi_{\widetilde{T}_{h_n}}\left|\tilde{u}^{h_n}\right|^m\tilde{u}^{h_n}\right)$ to be uniformly bounded in $L^2\left((0,T_{\max});(H^1_{0,per}(\mathcal{O}))'\right)$ and the identity $\lim_{n\to\infty}\widetilde{T}_{h_n} = T_{\max}$ $\tilde{\mathbb{P}}$-a.s., we find the sequence $\left(\mathcal{I}_h\left(\left|\tilde{u}^{h_n}\right|^m\tilde{u}^{h_n}\right)\right)_{n\in\mathbb{N}}$ to be uniformly bounded in $L^2\left((0,\bar{T});(H^1_{0,per}(\mathcal{O}))'\right)$. Using in addition the $W^{\bar{\gamma},p}\left((0,T_{\max});(H^1_{0,per}(\mathcal{O}))'\right)$-weak convergence of $\left(\tilde{u}^{h_n}\right)_{n\in\mathbb{N}}$, we find a subsequence to satisfy the assumptions of Lemma 5.7. Hence, the desired convergence is established for this subsequence. On the other hand, weak limits $\tilde{u}$ are prescribed – hence the whole sequence converges to $\tilde{u}$. □

5.1.2. *Convergence of the Deterministic Terms.* In this section, we relate $\tilde{p}$ and $\tilde{\mathfrak{u}}$ to $\tilde{u}$.

Lemma 5.9. *For the limits $\tilde{u} = \lim_{h\to 0} \tilde{u}^h$ and $\tilde{p} = w - \lim_{h\to 0} \tilde{p}^h$, we have $\tilde{p} = |\tilde{u}|^m \tilde{u}$ pointwise almost everywhere and $\tilde{\mathbb{P}}$-almost surely. Moreover,*

$$\lim_{h\to 0} \tilde{\mathfrak{u}}^h = \tilde{\mathfrak{u}} = \tilde{u} - (\tilde{u}_0)_{\mathcal{O}} \tag{5.24}$$

in $L^{2(m+1)}(0,\bar{T};(H^1_{0,per}(\mathcal{O}))')$ for all $\bar{T}\in[0,T_{max})$ and $\tilde{\mathbb{P}}$-a.s..

Proof. The starting point is the identity

$$\lim_{h\to 0}\int_0^{T_{max}}\int_{\mathcal{O}} \tilde{p}^h\phi\, dxds = \int_0^{T_{max}}\int_{\mathcal{O}} \tilde{p}\phi\, dxds \qquad \forall \phi\in C(\mathcal{O}_T)$$

which holds $\tilde{\mathbb{P}}$-almost surely according to Proposition 5.3. Since $\tilde{u}^h \to \tilde{u}$ strongly in $L^{2(m+1)}\left((0,T);C^\delta(\mathcal{O})\right)$ $\tilde{\mathbb{P}}$-almost surely as proven in Lemma 5.7, we have for a subsequence (not relabeled)

$$\left|\tilde{u}^h\right|^m \tilde{u}^h(t) \to |\tilde{u}|^m \tilde{u}(t) \tag{5.25}$$

in $C^\delta(\mathcal{O})$ for almost all $t\in[0,T_{max}]$. Observing that we have in addition the convergence

$$\left|\mathcal{I}_h\left(\left|\tilde{u}^h\right|^m \tilde{u}^h\right) - \left|\tilde{u}^h\right|^m \tilde{u}^h\right| \to 0$$

for $h\to 0$, we identify $\tilde{p} = |\tilde{u}|^m\tilde{u}$ for almost all $t\in[0,T_{max}]$. Now using $\mathcal{I}_h(\left|\tilde{u}^h\right|^m\tilde{u}^h)$ to be uniformly bounded in $L^2(\mathcal{O}_{T_{max}})$, we find

$$\lim_{h\to 0}\int_0^{T_{max}}\int_{\mathcal{O}} \tilde{p}^h\phi\, dxdt = \int_0^{T_{max}}\int_{\mathcal{O}} |\tilde{u}|^m\tilde{u}\phi\, dxdt$$

by means of Vitali's theorem. Finally, we combine the identity

$$\mathbb{E}\left[\left|\int_0^{T_{max}}\int_{\mathcal{O}} \tilde{p}^h\phi dxdt - \int_0^{T_{max}}\int_{\mathcal{O}} \mathcal{I}_h\left(\left|\tilde{u}^h\right|^m\tilde{u}^h\right)\phi dxdt\right|\right] = 0$$

with Fatou's lemma.
Let us identify $\tilde{\mathfrak{u}}$ with $\tilde{u} - (\tilde{u}_0)_{\mathcal{O}}$. From Proposition 5.3, Proposition 5.8, and (H1) together with (2.3), we infer

$$\tilde{u}^h - (\tilde{u}^h)_{\mathcal{O}} \to \tilde{\mathfrak{u}} \quad \text{in } C\left([0,T_{\max}];(H^1_{0,per}(\mathcal{O}))'\right), \tag{5.26}$$

$$\tilde{u}^h \to \tilde{u} \quad \text{in } L^{2(m+1)}\left([0,\bar{T}];C^\sigma(\mathcal{O})\right), \tag{5.27}$$

$$(\tilde{u}_0^h)_{\mathcal{O}} \to (\tilde{u}_0)_{\mathcal{O}} \quad \text{in } \mathbb{R} \tag{5.28}$$

$\tilde{\mathbb{P}}$-a.s. for any $\bar{T}\in(0,T_{max})$. Combining (5.27) and (5.28) gives

$$\tilde{u}^h - (\tilde{u}_0^h)_{\mathcal{O}} \to \tilde{u} - (\tilde{u}_0)_{\mathcal{O}} \qquad \text{in } L^{2(m+1)}\left(\left[0,\tilde{T}\right];(H^1_{0,per}(\mathcal{O}))'\right) \tag{5.29}$$

and in $L^{2(m+1)}\left([0,\bar{T}];C^\sigma(\mathcal{O})\right)$, as $\|u\|_{(H^1_{0,per}(\mathcal{O}))'} \le C\|u\|_{C^\sigma(\overline{\mathcal{O}})}$. On the other hand, (5.26) entails

$$\tilde{u}^h - (\tilde{u}^h)_{\mathcal{O}} \to \tilde{\mathfrak{u}} \quad \text{in } L^{2(m+1)}\left([0,\bar{T}];(H^1_{0,per}(\mathcal{O}))'\right), \tag{5.30}$$

which together with (5.29) gives (5.24). □

6. Convergence of the Stochastic Integral

Consider for $v\in H^2_{\text{per}}(\mathcal{O})$ arbitrary, but fixed, the operator $\mathcal{M}_{h,v}:\Omega\times[0,T_{\max}]\to\mathbb{R}$ defined by

$$\begin{aligned}\mathcal{M}_{h,v}(t) :&= \left(u^h(t)-u_0^h, P_h v\right)_h + \int_0^{t\wedge T_h}\int_{\mathcal{O}} M_1^h\left(u^h\right)\nabla u^h\nabla\left(P_h v\right)\, dxds \\ &= -\int_{l=1}^{N_h}\int_0^{t\wedge T_h}\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(u^h\right)\mu_l g_l\right)P_h v\, dxd\beta_l.\end{aligned} \tag{6.1}$$

Here, $P_h: H^1_{\text{per}}(\mathcal{O})\to X_h$ is a projection operator satisfying

$$\lim_{h\to 0}\|P_h v - v\|_{H^1_{\text{per}}(\mathcal{O})} = 0 \qquad \forall v\in H^1_{\text{per}}(\mathcal{O}). \tag{6.2}$$

Observe that by the optional stopping theorem, $\mathcal{M}_{h,v}$ is a real valued martingale; that is, denoting by r_s the restriction of a function on $[0,T_{\max}]$ onto $[0,s]$, we have

$$\mathbb{E}\left(\left[\mathcal{M}_{h,v}(t)-\mathcal{M}_{h,v}(s)\right]\Psi\left(r_s u^h, r_s W\right)\right)=0 \tag{6.3}$$

for all $0\le s\le t\le T_{\max}$ and for all $[0,1]$-valued continuous functions Ψ defined on $L^{2(m+1)}\left((0,s)\,;C^\gamma(\mathcal{O})\right)\times C\left([0,s]\,;L^2(\mathcal{O})\right)$.

Following the lines of proof of Lemma 5.10 in [17], and using Jensen's inequality, we find the quadratic variation of $\mathcal{M}_{h,v}$ to satisfy

$$\begin{aligned}\langle\langle\mathcal{M}_{h,v}\rangle\rangle_t &= \int_0^{t\wedge T_h}\sum_{l=1}^{N_h}\mu_l^2\left(\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(u^h\right)g_l\right)P_h v\;dx\right)^2 ds\\ &\le C\,\|v\|^2_{H^1_{\mathrm{per}}(\mathcal{O})}\int_0^{t\wedge T_h}\left\|\mathcal{I}_h M_2^h\left(u^h\right)(s)\right\|^2_{L^2(\mathcal{O})}ds\\ &\le C\,\|v\|^2_{H^1_{\mathrm{per}}(\mathcal{O})}\int_0^{t\wedge T_h}\int_{\mathcal{O}}\mathcal{I}_h\left|u^h\right|^{m+2}(s)\;dxds.\end{aligned} \tag{6.4}$$

Note that (6.4) combined with the energy estimates guarantees $\mathcal{M}_{h,v}$ to be a square-integrable martingale. For the identification of the stochastic integral in the limit $h\to 0$, we will study processes

$$\beta_l(t)=\int_{\mathcal{O}}\int_0^t\frac{1}{\mu_l}g_l\;dW dx$$

and their cross variations with $\mathcal{M}_{h,v}$. Following the lines of proof of Lemma 5.12 in [17], we readily compute

$$\langle\langle\mathcal{M}_{h,v},\beta_l\rangle\rangle_t=\begin{cases}\mu_l\int_0^{t\wedge T_h}\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(u^h\right)g_l\right)_x P_h v\;dxds, & l\le N_h,\\ 0, & l>N_h.\end{cases} \tag{6.5}$$

By equality of laws, we deduce that

$$\widetilde{\mathcal{M}}_{h,v}(t):=\left(\tilde{u}^h(t)-\tilde{u}^h(0),P_h v\right)_h+\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}M_1^h\left(\tilde{u}^h\right)\nabla\tilde{u}^h\nabla\left(P_h v\right)dxds, \tag{6.6}$$

$$\widetilde{\mathcal{M}}^2_{h,v}(t)-\int_0^{t\wedge\widetilde{T}_h}\sum_{l=1}^{N_h}\mu_l^2\left(\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(u^h\right)g_l\right)P_h v\;dx\right)^2 ds,$$

$$\widetilde{\mathcal{M}}_{h,v}(t)\tilde{\beta}_l^h(t)-\mu_l\int_0^{t\wedge\widetilde{T}_h}\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(u^h\right)g_l\right)_x P_h v\;dxds$$

for $l\le N_h$, and

$$\widetilde{\mathcal{M}}_{h,v}(t)\tilde{\beta}_l^h(t)$$

for $l>N_h$ are $(\tilde{\mathcal{F}}_t)$-martingales, where

$$\tilde{\beta}_l^h(t):=\int_{\mathcal{O}}\int_0^t\mu_l^{-1}g_l\;d\widetilde{W}^h dx.$$

Starting point for the passage to the limit $h\to 0$ are the identities

$$\mathbb{E}\left[\left(\widetilde{\mathcal{M}}_{h,v}(t)-\widetilde{\mathcal{M}}_{h,v}(s)\right)\Psi\left(r_s\tilde{u}^h,r_s\widetilde{W}_h\right)\right]=0, \tag{6.7}$$

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(\widetilde{\mathcal{M}}^2_{h,v}(t)-\widetilde{\mathcal{M}}^2_{h,v}(s)-\int_{s\wedge\widetilde{T}_h}^{t\wedge\widetilde{T}_h}\sum_{l=1}^{N_h}\mu_l^2\left(\int_{\mathcal{O}}\mathcal{I}_h M_2^h\left(\tilde{u}^h\right)g_l\nabla\left(P_h v\right)dx\right)^2 d\tau\Bigg)\cdot\\ \cdot\Psi\left(r_s\tilde{u}^h,r_s\widetilde{W}_h\right)\Bigg]=0,\end{aligned} \tag{6.8}$$

$$\begin{aligned}\mathbb{E}\Bigg[\Bigg(&\widetilde{\mathcal{M}}_{h,v}(t)\tilde{\beta}_l^h(t)-\widetilde{\mathcal{M}}_{h,v}(s)\tilde{\beta}_l^h(s)\\ &-\int_{s\wedge\widetilde{T}_h}^{t\wedge\widetilde{T}_h}\mu_l\int_{\mathcal{O}}\nabla\left(\mathcal{I}_h M_2^h\left(\tilde{u}^h\right)g_l\right)P_h v\;dxd\tau\Bigg)\Psi\left(r_s\tilde{u}^h,r_s\widetilde{W}_h\right)\Bigg]=0,\end{aligned} \tag{6.9}$$

for $l \le N_h$, which hold for all $s \le t \in [0, T_{\max}]$ and for all $[0,1]$-valued continuous functions defined on

$$L^{2(m+1)}\left((0,s)\,; C^\gamma(\mathcal{O})\right) \times C\left([0,s]\,; L^2(\mathcal{O})\right).$$

Let us pass to the limit in (6.7).

Lemma 6.1. *For all* $[0,1]$*-valued continuous functions on* $L^{2(m+1)}\left((0,s)\,; C^\gamma(\mathcal{O})\right) \times C\left([0,s]\,; L^2(\mathcal{O})\right)$*, we have*

$$\mathbb{E}\left[\left(\int_{\mathcal{O}} \left(\tilde{u}(t) - \tilde{u}(s)\right) v \; dx + \int_s^t \int_{\mathcal{O}} |\tilde{u}|^m \, \tilde{u}_x v_x \; dx d\tau\right) \Psi\left(r_s \tilde{u}, r_s \widetilde{W}\right)\right] = 0 \tag{6.10}$$

for all $0 \le s \le t < T_{\max}$.

Proof. Starting from (6.7), for the first term in (6.10), we use the strong convergence of $\tilde{u}^h - (\tilde{u}^h)_{\mathcal{O}}$ in $C\left([0,T]\,; (H^1_{0,per}(\mathcal{O}))'\right)$ to $\tilde{u} - (\tilde{u})_{\mathcal{O}}$. Let $0 \le s \le t < T_{\max}$ be arbitrary. Then, using conservation of mass,

$$\begin{aligned}
&\left|\left(\tilde{u}(t) - \tilde{u}^h(s), P_h v\right)_h - \left(\tilde{u}(t) - \tilde{u}(s), v\right)\right| \\
&\quad \le \left|\left(\tilde{u}^h(t) - \tilde{u}^h(s), P_h v\right)_h - \left(\tilde{u}^h(t) - \tilde{u}^h(s), P_h v\right)\right| \\
&\qquad\quad + \left|\left(\tilde{u}^h(t) - \tilde{u}^h(s), P_h v\right) - \left(\tilde{u}(t) - \tilde{u}(s), v\right)\right| \\
&\quad =: R_1 + R_2.
\end{aligned}$$

For R_1, we have according to

$$R_1 \le \frac{2h}{3} \left\|\tilde{u}^h(t) - \tilde{u}^h(s)\right\|_h \|\nabla v\|_{L^2(\mathcal{O})} \to 0 \qquad \text{for } h \to 0 \tag{6.11}$$

due to the energy estimate. Using conservation of mass for $\tilde{u}^h$ as well as for $\tilde{u}$ (which is a consequence of the $L^{2(m+1)}\left(0,T; C^\gamma(\mathcal{O})\right)$ convergence, adapting the mean, if necessary, on a set of $\mathcal{L}^1$-measure zero), we get for R_2

$$\begin{aligned}
R_2 &= \left|\left(\tilde{u}^h(t) - \tilde{u}^h(s), P_h v - (P_h v)_{\mathcal{O}}\right) - \left(\tilde{u}(t) - \tilde{u}(s), v - (v)_{\mathcal{O}}\right)\right| \\
&\le \left|\left(\tilde{u}^h(t) - \tilde{u}(t), P_h v - (P_h v)_{\mathcal{O}}\right)\right| + \left|\left(\tilde{u}^h(s) - \tilde{u}(s), P_h v - (P_h v)_{\mathcal{O}}\right)\right| \\
&\quad + \left|\left(\tilde{u}(s), (P_h v - (P_h v)_{\mathcal{O}}) - (v - (v)_{\mathcal{O}})\right)\right| + \left|\left(\tilde{u}(t), (P_h v - (P_h v)_{\mathcal{O}}) - (v - (v)_{\mathcal{O}})\right)\right| \\
&\le 2C \sup_{t \in [0,T]} \left\|\tilde{u}^h - \tilde{u}\right\|_{(H^1_{0,per}(\mathcal{O}))'} \cdot \|\nabla v\|_{L^2(\mathcal{O})} \\
&\quad + 2 \sup_{t \in [0,T]} \|\tilde{u}\|_{(H^1_{0,per}(\mathcal{O}))'} \cdot \|\nabla \left(P_h v - v\right)\|_{L^2(\mathcal{O})} \qquad \to 0 \quad \text{for } h \to 0.
\end{aligned} \tag{6.12}$$

On the other hand, $\Psi\left(r_s \tilde{u}^h, r_s \widetilde{W}_h\right)$ converges $\tilde{\mathbb{P}}$-almost surely to $\Psi\left(r_s \tilde{u}, r_s \widetilde{W}\right)$ in $\mathbb{R}$ by continuity of Ψ, the $L^{2(m+1)}\left((0,T_{\max})\,; C^\gamma(\mathcal{O})\right)$- and the $C\left([0,T_{\max}]\,; (H^1_{0,per}(\mathcal{O}))'\right)$-convergence of $\tilde{u}^h$ to $\tilde{u}$, and the $C\left([0,T_{\max}]\,; L^2(\mathcal{O})\right)$-convergence of $\widetilde{W}_h$. From Proposition 5.3 and Lemma 6.1, we infer

$$\int_s^t \int_{\mathcal{O}} M_1^h\left(\tilde{u}^h\right) \nabla \tilde{u}^h \cdot \nabla P_h(v) \; dx ds = \int_s^t \int_{\mathcal{O}} \nabla \mathcal{I}_h\left(\left|\tilde{u}^h\right|^m \tilde{u}^h\right) \cdot \nabla P_h(v) \; dx ds$$

to converge to $\int_s^t \int_{\mathcal{O}} \nabla\left(|\tilde{u}|^m \tilde{u}\right) \cdot \nabla v$ $\tilde{\mathbb{P}}$-almost surely for any $v \in H^2_{\text{per}}(\mathcal{O})$. Here, we used in particular (6.2). Since $|\Psi| \le 1$, we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_s^t \int_{\mathcal{O}} \nabla \mathcal{I}_h\left(\left|\tilde{u}^h\right|^m \tilde{u}^h\right) \cdot \nabla P_h(v) \; dx d\tau \cdot \Psi\left(r_s \tilde{u}^h, r_s \tilde{u}^h\right)\right)^2\right] \\
&\qquad \le C \cdot \|\nabla v\|^2_{L^2(\mathcal{O})} \, \mathbb{E}\left[\int_s^t \int_{\mathcal{O}} \left|\nabla \mathcal{I}_h\left(\left|\tilde{u}^h\right|^m \tilde{u}^h\right)\right|^2 dx d\tau\right] \le \text{const.}
\end{aligned}$$

due to the energy estimate. Similarly $\mathbb{E}\left[\left(\int_{\mathcal{O}} \left(\tilde{u}^h(t) - \tilde{u}^h(s)\right) P_h v \; dx\right)^2 \cdot \Psi\left(r_s \tilde{u}^h, r_s \widetilde{W}_h\right)^2\right]$ is bounded by $C \|\nabla v\|^2_{L^2(\mathcal{O})} \cdot \mathbb{E}\left[\sup_{t \in [0,T]} \int_{\mathcal{O}} \left|\tilde{u}^h(t)\right|^2\right]$, which, again, is controlled by the energy estimate. Hence, we may apply Vitali's theorem to obtain (6.10). ☐

Let us discuss the convergence behaviour of (6.8).

Lemma 6.2. *For all $[0,1]$-valued continuous functions Ψ defined on the space $L^{2(m+1)}\left((0,s);C^\gamma(\mathcal{O})\right)\times C\left([0,s];(H^1_{0,per}(\mathcal{O}))'\right)\times C\left([0,s];L^2(\mathcal{O})\right)$, we have*

$$\begin{aligned}\mathbb{E}\Bigg[\left(\widetilde{\mathcal{M}_v}^2(t)-\widetilde{\mathcal{M}_v}^2(s)-\int_s^t\sum_{l=1}^\infty\mu_l^2\left(\int_{\mathcal{O}}|\tilde u|^{\frac{m+2}{2}}g_l\nabla v\,dx\right)^2d\tau\right)\cdot\\ \cdot\Psi\left(r_s\tilde u,r_s\left(\tilde u-(\tilde u)_{\mathcal{O}}\right),r_s\widetilde{W}\right)\Bigg]=0\end{aligned}\tag{6.13}$$

for all $0\le s\le t\le T$ with

$$\widetilde{\mathcal{M}_v}(t):=\int_{\mathcal{O}}\left(\tilde u(t)-\tilde u(0)\right)v\,dx+(m+1)\int_0^t\int_{\mathcal{O}}|\tilde u|^m\nabla\tilde u\cdot\nabla v\,dxd\tau.$$

Proof. From the proof of Lemma 6.1, we infer the convergence of $\widetilde{\mathcal{M}}^2_{h,v}(t)$ to $\widetilde{\mathcal{M}_v}^2(t)$ $\tilde{\mathbb{P}}$-almost surely. To prove a corresponding result in expectation, we need higher integrability for $\widetilde{\mathcal{M}}^2_{h,v}(t)$. Combining the martingale moment inequality ([29], Proposition 3.26)

$$\mathbb{E}\left(\left|\widetilde{\mathcal{M}}_{h,v}(t)\right|^{2q}\right)\le C_q\mathbb{E}\left(\left\langle\left\langle\widetilde{\mathcal{M}}_{h,v}\right\rangle\right\rangle_t^q\right),\qquad q>0,$$

with the identity (6.1) and with the analogue of (6.4) in terms of $\widetilde{\mathcal{M}}_{h,v}$, we have

$$\begin{aligned}\mathbb{E}\left(\left|\widetilde{\mathcal{M}}_{h,v}(t)\right|^{2q}\right)&\le C\|v\|^{2q}_{H^1_{\text{per}}(\mathcal{O})}\mathbb{E}\left[\left(\int_0^t\int_{\mathcal{O}}\mathcal{I}_h\left|\tilde u^h\right|^{m+2}dxds\right)^q\right]\\&\le C\|v\|^{2q}_{H^1_{\text{per}}(\mathcal{O})}T^q\mathbb{E}\left[\left(\sup_{t\in[0,T]}\int_{\mathcal{O}}\mathcal{I}_h\left|\tilde u^h\right|^{m+2}\right)^q\right]\le\text{const.}.\end{aligned}\tag{6.14}$$

Using Vitali's theorem and the boundedness of Ψ, we get

$$\lim_{h\to0}\mathbb{E}\left(\widetilde{\mathcal{M}}^2_{h,v}(t)\cdot\Psi\left[r_s\tilde u^h,r_s\left(\tilde u^h-(\tilde u^h)_{\mathcal{O}}\right),r_s\widetilde{W}_h\right]\right)\tag{6.15}$$

$$=\mathbb{E}\left(\widetilde{\mathcal{M}_v}^2(t)\cdot\Psi\left[r_s\tilde u,r_s\left(\tilde u-(\tilde u)_{\mathcal{O}}\right),r_s\widetilde{W}\right]\right).\tag{6.16}$$

It remains to show

$$\begin{aligned}&\lim_{h\to0}\mathbb{E}\left[\left\langle\left\langle\widetilde{\mathcal{M}}_{h,v}\right\rangle\right\rangle_t\Psi\left(r_s\tilde u^h,r_s\left(\tilde u^h-(\tilde u^h)_{\mathcal{O}}\right),r_s\widetilde{W}_h\right)\right]\\&\quad=\mathbb{E}\left[\int_0^t\sum_{l=1}^\infty\mu_l^2\left(\int_{\mathcal{O}}|\tilde u|^{\frac{m+2}{2}}g_l\nabla v\,dx\right)^2d\tau\cdot\Psi\left(r_s\tilde u,r_s\left(\tilde u-(\tilde u)_{\mathcal{O}}\right),r_s\widetilde{W}\right)\right].\end{aligned}\tag{6.17}$$

From Proposition 5.3, we infer $\tilde u^h$ to converge strongly in $L^{2(m+1)}\left((0,T);L^\infty(\mathcal{O})\right)$ to $\tilde u$ $\tilde{\mathbb{P}}$-almost surely. Using in addition (6.2), the uniform L^∞-bound of g_l, $l\in\mathbb{N}$, we find $\left\langle\left\langle\widetilde{\mathcal{M}}_{h,v}\right\rangle\right\rangle_t$ to converge to $\int_0^t\sum_{l=1}^\infty\mu_l^2\left(\int_{\mathcal{O}}|\tilde u|^{\frac{m+2}{2}}g_l\nabla v\,dx\right)^2d\tau$ $\tilde{\mathbb{P}}$-almost surely for any $t\in[0,T_{\max}]$. To conclude and to establish (6.17), we use the boundedness of Ψ and Vitali's theorem combined with (6.14) and the energy estimate. □

In a similar fashion, we get

Lemma 6.3. *For all $[0,1]$-valued continuous functions Ψ defined on the space $L^{2(m+1)}\left((0,s);C^\gamma(\mathcal{O})\right)\times C\left([0,s];(H^1_{0,per}(\mathcal{O}))'\right)\times C\left([0,s];L^2(\mathcal{O})\right)$, we have*

$$\begin{aligned}\mathbb{E}\Bigg[\left(\widetilde{\mathcal{M}_v}(t)\tilde\beta_l(t)-\widetilde{\mathcal{M}_v}(s)\tilde\beta_l(s)-\int_s^t\mu_l\int_{\mathcal{O}}\nabla\left(|\tilde u|^{\frac{m+2}{2}}g_l\right)v\,dxd\tau\right)\cdot\\ \cdot\Psi\left(r_s\tilde u,r_s\left(\tilde u-(\tilde u)_{\mathcal{O}}\right),r_s\widetilde{W}\right)\Bigg]=0\end{aligned}$$

for all $l\in\mathbb{N}$ and all $0\le s\le t\le T_{\max}$.

Along the lines of proof of Lemma 5.16 in [17], we have

Lemma 6.4. *We have*

$$\widetilde{\mathcal{M}_v}(t) = \sum_{l=1}^{\infty} \int_0^t \mu_l \int_{\mathcal{O}} \nabla \left(|\tilde{u}|^{\frac{m+2}{2}} g_l \right) v \; dx d\tilde{\beta}_l.$$

Finally, we prove Theorem 3.2.

Proof of Theorem 3.2. From Proposition 5.3, Lemma 5.6, and Lemma 5.9, we infer the existence of a Wiener process $\tilde{W}(t) = \sum_{\ell=1}^{\infty} \mu_\ell g_\ell \tilde{\beta}_\ell(t)$ and of a random variable $\tilde{u}$ in $L^{\tilde{p}}\left(\tilde{\Omega}; L^\infty\left(0, T_{max}; L^{m+2}(\mathcal{O})\right)\right) \cap L^2\left(\tilde{\Omega}; W^{\bar{\gamma},p}\left(0, T_{max}; (H^1_{0,per}(\mathcal{O}))'\right)\right)$ such that

$$\begin{aligned}
&\tilde{p} = |\tilde{u}|^m \tilde{u} \in L^2\left(\tilde{\Omega}; L^2\left(0, T_{max}; H^1_{per}(\mathcal{O})\right)\right),\\
&\tilde{\mathfrak{u}} = \tilde{u} - (\tilde{u})_{\mathcal{O}} \in L^2\left(\tilde{\Omega}; C\left([0, T_{max}]; (H^1_{0,per}(\mathcal{O}))'\right)\right),\\
&\Lambda = \tilde{\mathbb{P}} \circ \tilde{u}_0^{-1}.
\end{aligned}$$

Lemma 6.1 implies

$$\tilde{\mathcal{M}}_v(t) := \int_{\mathcal{O}} (\tilde{u}(t) - \tilde{u}(s))\, v dx + \int_s^t \int_{\mathcal{O}} |\tilde{u}|^m \tilde{u}_x v_x dx d\tau$$

to be an $\left(\tilde{\mathcal{F}}_t\right)_{t\geq 0}$-martingale. Lemma 6.4 gives the identity (3.1). Combining Fatou's lemma and Proposition 4.2, the energy estimate (3.2) follows. □

7. Miscellaneous Results

7.1. **Nonnegativity.** In this subsection, we prove Theorem 3.3.

Proof of Theorem 3.3: Formally, our proof of nonnegativity is based on an Ito-argument. To obtain a rigorous statement, some approximations are necessary. The idea is to justify an appropriate version of Ito's formula by a convolution argument. For this, choose a smooth, symmetric kernel $K : \mathbb{R} \to [0, \infty)$ such that its support is contained in the open interval $(-1, 1)$ and $\int_{(-1,1)} K(x)dx = 1$. Fix $\varepsilon > 0$ and define for a periodic, measurable function $v : \mathcal{O} \to \mathbb{R}$ the smooth, periodic function

$$(K^\varepsilon * v)(x) := \int_{\mathbb{R}} \varepsilon^{-1} K(\varepsilon^{-1} y) v(x - y) dy,$$

where v is extended to $\mathbb{R}$ periodically. For a solution u as constructed in Theorem 3.2, we will abbreviate $u^\varepsilon := K^\varepsilon * u$ where we have extended the convolution with K^ε to $(H^1_{0,per})'$ via

$$(K^\varepsilon * u)(x, t) := \langle u(\cdot, t), \varepsilon^{-1} K(\varepsilon^{-1}(x - \cdot) - 1\rangle + \int_{\mathcal{O}} u_0(x) dx.$$

Here, we use that the spatial mean of u is preserved. In this subsection, we will always use this extended definition of convolution.

For $0 < \varepsilon < 1$, we consider the functional

$$F_\varepsilon : \; (H^1_{0,per})' \to \mathbb{R} : v \mapsto \frac{1}{m+2} \int_{\mathcal{O}} [(v^\varepsilon)_-]^{m+2} \, dx$$

which is well-defined and twice continuously differentiable for $m > 0$. Let $C_m := \frac{m+1}{2}$. Obviously,

$$\langle DF_\varepsilon(v), \zeta\rangle = \int_{\mathcal{O}} [(v^\varepsilon)_-]^{m+1} (K^\varepsilon * \zeta) dx,$$

$$\langle D^2 F_\varepsilon(v)\zeta, \psi\rangle = 2C_m \int_{\mathcal{O}} [(v^\varepsilon)_-]^m (K^\varepsilon * \zeta)(K^\varepsilon * \psi) dx$$

for $\zeta, \psi \in H^1_{0,per}(\mathcal{O})$. The Ito-Formula in [9] combined with (1.1) and Jensen's inequality gives

$$
\begin{aligned}
\mathbb{E}\left(F_\varepsilon(u(t))-F_\varepsilon(u_0)\right) &= \mathbb{E}\Bigg[\int_0^t\int_{\mathcal{O}} -\left[(u^\varepsilon)_-^{m+1}\right]_x\left[K^\varepsilon * (|u|^m u)\right]_x dxds \\
&\qquad + C_m\sum_\ell \mu_\ell^2\int_0^t\int_{\mathcal{O}}[(u^\varepsilon)_-]^m\left[K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell + |u|^{\frac{m+2}{2}}(g_\ell)_x\right)\right]^2 dxds\Bigg] \\
&\le \mathbb{E}\Bigg[\int_0^t\int_{\mathcal{O}} -\left[(u^\varepsilon)_-^{m+1}\right]_x\left[K^\varepsilon * (|u|^m u)\right]_x dxds \\
&\qquad + 2C_m\sum_\ell \mu_\ell^2\int_0^t\int_{\mathcal{O}}[(u^\varepsilon)_-]^m\left(\left[K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell\right)\right]^2 + C_g\ell^2 K^\varepsilon * \left(|u|^{m+2}\right)\right) dxds\Bigg] \\
&\le \mathbb{E}\Bigg[\int_0^t\int_{\mathcal{O}} -\left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|^2 dxds \\
&\qquad + 2C_m\sum_\ell \mu_\ell^2\int_0^t\int_{\mathcal{O}}\left(\left[(u^\varepsilon)_-^m\right]\left(\left[(u^\varepsilon)^{\frac{m+2}{2}}\right]_x g_\ell\right)^2 + C_g\ell^2\left[(u_-^\varepsilon)^{m+1}\right]^2\right) dxds\Bigg] + R_\varepsilon \\
&\le \mathbb{E}\Bigg[\int_0^t\int_{\mathcal{O}} -\left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|^2 dxds \\
&\qquad + 2C_m\sum_\ell \mu_\ell^2\int_0^t\int_{\mathcal{O}}\left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|^2 + C_g\ell^2\left[(u_-^\varepsilon)^{m+1}\right]^2 dxds\Bigg] + R_\varepsilon
\end{aligned}
\tag{7.1}
$$

where C_g is the constant from (4.15) and R_ε is defined below.

Note that for sufficiently small noise amplitudes the first integral in the second term on the right-hand side can be absorbed in the dissipative term, and for the second integral this is possible after using Poincaré's inequality as well. By Fubini, $\int_{\mathcal{O}}(K^\varepsilon * u)dx = \int_{\mathcal{O}} u$, where the latter is initially strictly positive by (H1) and conserved. So $(K^\varepsilon * u)$ cannot be negative everywhere.

So it remains to show that the remainder actually converges to zero. We decompose

$$
\begin{aligned}
R_\varepsilon &:= R_\varepsilon^1 + 2C_m R_\varepsilon^2 + 2C_m R_\varepsilon^3, \\
R_\varepsilon^1 &:= \mathbb{E}\left[\int_0^t\int_{\mathcal{O}}\left(\left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|^2 - \left[(u^\varepsilon)_-^{m+1}\right]_x\left[K^\varepsilon * \left(u^{m+1}\right)\right]_x\right) dxds\right] \\
R_\varepsilon^2 &:= \sum_\ell \mu_\ell^2\mathbb{E}\left[\int_0^t\int_{\mathcal{O}}[(u^\varepsilon)_-]^m\left[\left[K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell\right)\right]^2 - \left(\left[|u^\varepsilon|^{\frac{m+2}{2}}\right]_x g_\ell\right)^2\right] dxds\right] \\
R_\varepsilon^3 &:= \sum_\ell C_g\ell^2\mu_\ell^2\mathbb{E}\left[\int_0^t\int_{\mathcal{O}}\left([(u^\varepsilon)_-]^m K^\varepsilon * \left(|u|^{m+2}\right) - \left[(u^\varepsilon)_-^{m+1}\right]^2\right) dxds\right]
\end{aligned}
$$

For R_ε^1 note that almost surely

$$
\left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|^2 = \left|\left[(u^\varepsilon)_-^{m+1}\right]_x\right|\left|\left[(u^\varepsilon)^{m+1}\right]_x\right|,
$$

so $R_\varepsilon^1 = \mathbb{E}\int_0^t\int_{\mathcal{O}} r_\varepsilon^1 dxdt$ with

$$
r_\varepsilon^1 = [(u^\varepsilon)_-^{m+1}]_x\Big(\left([(u^\varepsilon)^{m+1}]_x - [u^{m+1}]_x\right) + \left([u^{m+1}]_x - [K^\varepsilon * (u^{m+1})]_x\right)\Big).
$$

Here, the convergence to zero follows by Cauchy-Schwarz' inequality, Lemma 4.1 in [16], and the standard convolution convergence for the L^2-function $(|u|^m u)_x$. The argument for R_ε^3 is analogous.

This leaves us with R_ε^2. Note that by Lemma 4.1 in [16], it is sufficient to consider

$$
\hat{R}(\ell,\varepsilon) := \mu_\ell^2\mathbb{E}\left[\int_0^t\int_{\mathcal{O}}\left([(u^\varepsilon)_-]^m\left[K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell\right)\right]^2 - (u_-)^m\left(\left[u^{\frac{m+2}{2}}\right]_x g_\ell\right)^2\right) dxds\right] \tag{7.2}
$$

for $\ell \in \mathbb{N}$, $\varepsilon \in (0,1)$.

First, let us show the pointwise convergence almost everywhere of the integrand in (7.2) on $\Omega\times\mathcal{O}\times[0,T]$: By (3.2), we have $L^2(\Omega, L^2([0,T]\times\mathcal{O}))$-regularity of $(|u|^m u)_x$ as well as bounds on higher moments of the supremum of the energy over time. By standard embedding, $|u|^m u$ is contained in $C^{1/2}(\mathcal{O})$ for almost all t and ω. In particular, $u(\cdot,t,\omega)$ itself is continuous for those (t,ω). Let us now fix such a pair (t,ω) and distinguish the following cases:

(1) Assume x_0 to be an interior point of $S_0(t,\omega) := \{x \in \mathcal{O} : \ u(x,t,\omega) \geq 0\}$. By continuity, the integrand of $\hat{R}(\ell,\varepsilon)$ is zero for sufficiently small $\varepsilon > 0$ if evaluated in (x_0,t,ω).
(2) $u(x_0,t) < 0$: As $|u|^m u_x \in L^2(\mathcal{O})$ and u is continuous, all functions under the integral are at least L^1 in a small neighbourhood of x_0. Therefore for ε sufficiently small, all integrands converge pointwise by the standard convolution estimate.
(3) x_0 is a boundary point of $S_0(t,\omega)$. By continuity, this means $u(x,t) = 0$, and there are only countably many such points for fixed $(t,\omega) \in [0,T] \times \Omega$. Hence, for fixed (t,ω), the spatial Lebesgue measure of these points is zero and their contribution to the integral can be neglected.

To show the integral in (7.2) to vanish in the limit $\varepsilon \to 0$, it will be sufficient to find an integrable dominating function for

$$\left(|u^\varepsilon|^{\frac{m}{2}} K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell\right)\right)^2.$$

We sketch a proof which is inspired by Lemma 4.1 in [16] and which uses the maximal function

$$\mathcal{M}(f)(x) := \sup_{r>0} |B_r(x)|^{-1} \int_{B_r(x)} |f| dy.$$

Fix $x_0 \in \mathcal{O}$ and consider for $\varepsilon > 0$

$$\beta_\varepsilon(x_0) := 2^{-\frac{1}{m+1}} \fint_{B_\varepsilon(x_0)} |u(x)| \ dx, \tag{7.3}$$

where $B_\varepsilon(x_0)$ is the open ball of radius ε around x_0 on the L-torus $(0,L]$. In the sequel, we will simply write β instead of $\beta_\varepsilon(x_0)$ – just for the ease of presentation. Introduce for $\beta \geq 0$ the piecewise linear, monotone increasing function F_β defined by

$$F_\beta(\alpha) := \begin{cases} \alpha, & \alpha > \beta \\ 2\alpha - \beta, & \frac{\beta}{2} \leq \alpha \leq \beta \\ 0, & 0 \leq \alpha < \frac{\beta}{2}. \end{cases}$$

Obviously, $F_\beta \in W^{1,\infty}_{loc}(\mathbb{R})$ and

$$F'_\beta(\alpha) \leq 2\chi_{[\frac{\beta}{2},\beta]}(\alpha) + \chi_{(\beta,\infty)}(\alpha) \tag{7.4}$$

almost everywhere. By appropriate scaling arguments, we have for $\gamma_1, \gamma_2 \in \mathbb{R}_0^+$, $\alpha \in \mathbb{R}_0^+$, the inequalities

$$\beta^{\gamma_1} \left(F_\beta(\alpha)^{\gamma_2}\right)_\alpha \leq 2^{\gamma_1+1} \alpha^{\gamma_1+\gamma_2-1}, \tag{7.5}$$

$$0 \leq \alpha^{\gamma_1} - (F_\beta(\alpha))^{\gamma_1} \leq C(\gamma_1)\beta^{\gamma_1}\chi_{[0,\beta]}(\alpha). \tag{7.6}$$

By construction, there is a positive constant C_K depending only on K, such that

$$|u_\varepsilon(x_0)| \leq C_K \fint_{B_\varepsilon(x_0)} |u(x)| \ dx \leq C_K 2^{-\frac{1}{m+1}} \beta_\varepsilon(x_0), \tag{7.7}$$

using (7.3), too. To construct the desired dominating function, it is sufficient to control

$$Q^\varepsilon(x_0) := |u^\varepsilon|^{\frac{m}{2}} K^\varepsilon * \left(\left(|u|^{\frac{m+2}{2}}\right)_x g_\ell\right)(x_0). \tag{7.8}$$

We decompose

$$Q^\varepsilon(x_0) = T_1(x_0) + T_2(x_0) + T_3(x_0) \tag{7.9}$$

with

$$T_1(x_0) := \left[|u^\varepsilon|^{\frac{m}{2}} K^\varepsilon * \left(\left[F_\beta(|u|)\right]^{\frac{m+2}{2}}\right)_x\right](x_0) g_\ell(x_0), \tag{7.10}$$

$$T_2(x_0) := \left[|u^\varepsilon|^{\frac{m}{2}} (K^\varepsilon)_x * \left(|u|^{\frac{m+2}{2}} - \left[F_\beta(|u|)\right]^{\frac{m+2}{2}}\right)\right](x_0) g_\ell(x_0) \tag{7.11}$$

$$T_3(x_0) := \left[|u^\varepsilon|^{\frac{m}{2}} K^\varepsilon * \left(|u|^{\frac{m+2}{2}} - \left[F_\beta(u)\right]^{\frac{m+2}{2}}\right)\right](x_0) \left(g_\ell(x_0)\right)_x. \tag{7.12}$$

Let us discuss T_1. By (7.5) and (7.7) for the choices $\gamma_1 = \frac{m}{2}$, $\gamma_2 = \frac{m+2}{2}$, we have

$$\begin{aligned}
|T_1|(x_0) &\le C_K \left(2^{\frac{1}{m+1}}\beta\right)^{\frac{m}{2}} \left[K^\varepsilon * \left(\left|\left(F_\beta(|u|)^{\frac{m+2}{2}}\right)_x\right| |g_\ell|\right)\right](x_0) \\
&\le \tilde{C}\left[K^\varepsilon * (|u|^m |u_x|)\right](x_0) \\
&\le \tilde{C} \fint_{B_\varepsilon(x_0)} \left|\left(|u|^{m+1}(x)\right)_x\right| \, dx.
\end{aligned} \tag{7.13}$$

For T_2, we get – using the standard estimate $(K^\varepsilon)_x \le C\varepsilon^{-2}$ as well as (7.6) and (7.7) with $\gamma_1 = \frac{m}{2}$ – that

$$\begin{aligned}
|T_2(x_0)| &\le \tilde{C}\beta^{\frac{m}{2}}\varepsilon^{-2} \int_{B_\varepsilon(x_0)} \left||u|^{\frac{m+2}{2}} - F_\beta(|u|)^{\frac{m+2}{2}}\right| |g_\ell| \, dy \\
&\le \tilde{C}\beta^{\frac{m}{2}}\varepsilon^{-2}\beta^{\frac{m+2}{2}} \mathcal{L}\left(\{x \in T_L \mid |x - x_0| \le \varepsilon, \ |u(x)| < \beta\}\right),
\end{aligned} \tag{7.14}$$

where $\mathcal{L}$ denotes the Lebesgue measure on the torus T_L. Similarly, using for T_3 the estimate $K^\varepsilon \le C\varepsilon^{-1}$, we get

$$|T_3(x_0)| \le C\varepsilon^{-1}\ell\beta^{m+1}\mathcal{L}\left(\{x \in T_L \mid |x - x_0| \le \varepsilon, \ |u(x)| < \beta\}\right). \tag{7.15}$$

Using Jensen's inequality and the mean-value Poincaré-inequality, we estimate

$$\begin{aligned}
&\mathcal{L}\left(\{x \in T_L \mid |x - x_0| \le \varepsilon, \ |u(x)| < \beta\}\right) \\
&\quad\le \frac{1}{\beta^{m+1}} \left(2\beta^{m+1} - |v|^{m+1}\right)_+ \, dx \\
&\quad\le \frac{1}{\beta^{m+1}} \int_{B_\varepsilon(x_0)} \left(\fint_{B_\varepsilon(x_0)} |u|^{m+1}(y) \, dy - |u|^{m+1}(x)\right)_+ \, dx \\
&\quad\le \frac{1}{\beta^{m+1}} \int_{B_\varepsilon(x_0)} \left|\fint_{B_\varepsilon(x_0)} |u|^{m+1}(y) \, dy - |u|^{m+1}(x)\right| \, dx \\
&\quad\le C\frac{\varepsilon^2}{\beta^{m+1}} \fint_{B_\varepsilon(x_0)} \left|\left(|u|^{m+1}(x)\right)_x\right| \, dx.
\end{aligned} \tag{7.16}$$

Combining (7.13), (7.14), (7.15), (7.16) with (7.9), we find

$$|Q_\varepsilon(x_0)| \le C(1 + \varepsilon\ell)\mathcal{M}\left([|u|^m u]_x\right)(x_0).$$

As $[|u|^m u]_x$ is in $L^2(\Omega \times \mathcal{O} \times [0,T])$, the integrability of this dominating function follows by the standard properties of the maximal function. Note that the factor ℓ^2 does not cause any problems with respect to summation over all the terms $\hat{R}(\ell,\varepsilon)$ arising in (7.2) as it is weighted by a factor μ_ℓ^2 and $\sum_{\ell\in\mathbb{N}} \ell^2\mu_\ell^2$ is bounded by assumption. Using Fatou's lemma on the left-hand side, the result follows. $\square$

7.2. **Uniqueness.** In this subsection, we prove Theorem 3.4 by combining Ito's formula with an homogeneity argument.

Proof of Theorem 3.4: Let u and v be two solutions to the same initial data on the same probability space. Starting from the process

$$d(u - v) = (m+1)\left(\partial_x(|u|^m u_x) - \partial_x(|v|^m v_x)\right) dt + \left(\left(|u|^{\frac{m+2}{2}} - |v|^{\frac{m+2}{2}}\right) dW\right)_x, \tag{7.17}$$

we will apply the following Theorem (see [30], Theorem 4.2.5) with $\alpha = m + 2$.

Theorem. *Let $\alpha \in (1,\infty)$, $X_0 \in L^2(\Omega; \mathcal{F}_0, \mathbb{P}; H)$ and*

$$\begin{aligned}
Y &\in L^{\frac{\alpha}{\alpha-1}}\left([0,T] \times \Omega, dt \otimes \mathbb{P}; V'\right) \\
Z &\in L^2\left([0,T] \times \Omega, dt \otimes \mathbb{P}; L_2(U,H)\right),
\end{aligned}$$

both be progressively measurable. Define the continuous V'-valued process

$$X(t) := X_0 + \int_0^t Y(s)ds + \int_0^t Z(s)dW(s), \qquad t \in [0,T].$$

If for its $dt\otimes\mathbb{P}$-equivalence class $\hat{X}$ we have $\hat{X}\in L^\alpha\left([0,T]\times\Omega, dt\otimes\mathbb{P}; V\right)$ and if $\mathbb{E}(\|X(t)\|_H^2)<\infty$ for dt-a.e. $t\in[0,T]$ (which is always true for $\alpha\geq 2$), then X is a continuous H-valued $(\mathcal{F}_t)$-adapted process, and the following Ito-formula

$$\mathbb{E}\left(\|X(t)\|_H^2\right)=\mathbb{E}\left(\|X_0\|_H^2\right)+\mathbb{E}\left(\int_0^t 2\langle Y(s),\bar{X}(s)\rangle_{V'\times V}+\|Z(s)\|^2_{L_2(U,H)}\,ds\right) \tag{7.18}$$

holds for all $t\in[0,T]$ and for any V-valued progressively measurable $dt\otimes\mathbb{P}$-version $\bar{X}$ of X.

Introducing

$$X(s):=u(\cdot,s)-v(\cdot,s),$$
$$Y(s):=\partial_x\left(|u|^m u_x-|v|^m v_x\right)(\cdot,s),$$

and $Z(s)$ as a mapping from $L^2(\mathcal{O})$ to $(H^1_{0,\mathrm{per}}(\mathcal{O}))'$ by

$$Z(s)[w]:=\sum_{k\in\mathbb{N}}\mu_k\left[\left(|u|^{\frac{m+2}{2}}-|v|^{\frac{m+2}{2}}\right)(w,g_k)_{L^2(\mathcal{O})}\,g_k\right]_x,$$

we may rewrite (7.17) in the form

$$X(t)=\int_0^t Y(s)ds+\int_0^t Z(s)[dW(s)],\qquad t\in[0,T]. \tag{7.19}$$

Choosing $V:=L^{m+2}(\mathcal{O})$ and $H:=(H^1_{0,\mathrm{per}}(\mathcal{O}))'$, we find V to be densely embedded in H. Using the Riesz-isomorphism $(-\Delta)^{-1}:(H^1_{0,\mathrm{per}}(\mathcal{O}))'\to H^1_{0,\mathrm{per}}(\mathcal{O})$ where $(-\Delta)^{-1}f\in H^1_{0,\mathrm{per}}(\mathcal{O})$ is given as the unique solution of the variational equation

$$\int_{\mathcal{O}}\partial_x\left((-\Delta)^{-1}f\right)\cdot\partial_x\phi=\langle f,\phi\rangle_{(H^1_{0,\mathrm{per}}(\mathcal{O}))'\times H^1_{0,\mathrm{per}}(\mathcal{O})}\qquad\forall\phi\in H^1_{0,\mathrm{per}}(\mathcal{O}),$$

we identify H with H' to obtain

$$V\subset H\subset V' \tag{7.20}$$

continuously and densely with $V'=L^{\frac{m+2}{m+1}}(\mathcal{O})$. In particular,

$$\langle z,v\rangle_{V'\times V}=(z,v)_H \tag{7.21}$$

for all $z\in H$, $v\in V$. Inferring from Theorem 3.2 the deviation of $|u|^m u$ (and of $|v|^m v$, respectively) from their spatial mean values to be contained in $L^2\left(\Omega;L^2\left([0,T];H^1_{0,\mathrm{per}}(\mathcal{O})\right)\right)$, we get

$$\partial_x\left(|u|^m u_x-|v|^m v_x\right)\in L^2\left(\Omega;L^2\left([0,T];(H^1_{0,\mathrm{per}}(\mathcal{O}))'\right)\right).$$

Hence, by the embedding $H\subset V'$,

$$Y\in L^2\left(\Omega;L^2\left([0,T];V'\right)\right)\subset L^{\frac{m+2}{m+1}}\left(\Omega;L^{\frac{m+2}{m+1}}\left([0,T],V'\right)\right). \tag{7.22}$$

By a standard computation, we find

$$\|Z(s)\|^2_{L_2(L^2(\mathcal{O});(H^1_{0,\mathrm{per}}(\mathcal{O}))')}=\sum_{k\in\mathbb{N}}\mu_k^2\int_{\mathcal{O}}\left(|u|^{\frac{m+2}{2}}-|v|^{\frac{m+2}{2}}\right)^2(\cdot,s)g_k^2(\cdot)dx. \tag{7.23}$$

By Theorem 3.2, we have u and v to be element of $L^p(\Omega;L^\infty(0,T;L^{m+2}(\mathcal{O})))$ for arbitrary $1\leq p<\infty$. Together with (7.20) and (7.23), we deduce

$$Z(\cdot)\in L^2\left(\Omega;L^2\left([0,T];L_2(L^2(\mathcal{O});(H^1_{0,\mathrm{per}}(\mathcal{O}))')\right)\right).$$

In particular, we may assume Y,Z to be progressively measurable. Indeed, let us replace the solutions u and v by convolutions with a smooth kernel K^ε in the spirit of the proof of Theorem 3.3. As the mean-value deviations of our solutions are by construction continuous with values in $(H^1_{0,\mathrm{per}}(\mathcal{O}))'$, the resulting v^ε and u^ε will be continuous in both space and time and converge a.s. and in those L^p-spaces for which a-priori estimates on u and v exist. The corresponding Y^ε and Z^ε are continuous in time and so in particular progressively measurable. Letting $\varepsilon\to 0$, we obtain measurability of the limit. (Note that convergence in L^p implies a.s. convergence for a subsequence which implies measurability of the limit if the σ-algebra is complete, i.e. if it satisfies the usual conditions.)

Theorem 3.2 furthermore implies that

$$u-v\in L^2\left(\Omega;C^0\left([0,T];(H^1_{0,\mathrm{per}}(\mathcal{O}))'\right)\right)\cap L^p\left(\Omega;L^\infty\left([0,T];L^{m+2}(\mathcal{O})\right)\right)$$

for any $p > 1$. By the same convolution argument as before, its $dt \otimes \mathbb{P}$-equivalence class $\hat{X}$ is contained in $L^{m+2}(\Omega \times [0,T]; V)$. Hence, we may apply the theorem above to get

$$\begin{aligned}\mathbb{E}\left(\|u(t)-v(t)\|^2_{(H^1_{0,\mathrm{per}}(\mathcal{O}))'}\right) =& 2(m+1)\mathbb{E}\left[\int_0^t \left(u-v, \partial_x\left[|u|^m u_x - |v|^m v_x\right]\right)_{(H^1_{0,\mathrm{per}}(\mathcal{O}))'} ds\right] \\ &+ \mathbb{E}\left[\int_0^t \sum_{k=1}^\infty \mu_k^2 \int_{\mathcal{O}} \left(|u|^{\frac{m+2}{2}} - |v|^{\frac{m+2}{2}}\right)^2 (\cdot, s) g_k^2 dx ds\right] =: R_1 + R_2.\end{aligned} \tag{7.24}$$

Here, we used the identity

$$\langle Y(s), \bar{X}(s)\rangle_{V'\times V} = \big(Y(s), \bar{X}(s)\big)_H$$

which holds true due to $\bar{X}(s) \in H$ and (7.21). R_1 can be rewritten as

$$R_1 = -2\mathbb{E}\left[\int_0^t \left((u-v), |u|^m u - |v|^m v\right)_{L^2(\mathcal{O})} ds\right]$$

which has a sign due to the convexity of $|u|^{m+2}$. R_2 is readily estimated by

$$R_2 \le \sum_{\ell=1}^\infty \|g_\ell\|_{L^\infty(\mathcal{O})} \mu_\ell^2 \mathbb{E}\left[\int_0^t \int_{\mathcal{O}} \left(|u|^{\frac{m+2}{2}} - |v|^{\frac{m+2}{2}}\right)^2 dx ds\right].$$

Now, for sufficiently small $\sum_{\ell=1}^\infty \mu_\ell^2$, this term can be absorbed in R_1. Indeed, both terms have the same homogeneity, we have

$$\Big(|u| - |v|, |u|^{m+1} - |v|^{m+1}\Big)_{L^2(\mathcal{O})} \le \left(u-v, |u|^m u - |v|^m v\right)_{L^2(\mathcal{O})}$$

and $(1-\alpha)\cdot(1-\alpha^{m+1})$ and $(1-\alpha^{\frac{m+2}{2}})^2$ both have a second order root in $\alpha = 1$ as their only roots. This gives the result. $\square$

8. Monte-Carlo Simulations

We present numerical experiments to investigate which quantitative impact conservative multiplicative noise has on the size of waiting times and on the speed of propagation. The simulations are based on the convergent schemes for stochastic porous-medium equations with linear multiplicative source-term noise presented in [23], combined with the upwind discretization for stochastic thin-film equations with nonlinear multiplicative conservative noise proposed in [24]. Our Monte-Carlo simulations on the propagation of the solution's support and on the size of waiting times indicate that in expectation noise

- decreases the size of waiting times,
- changes scaling laws for the size of waiting times, and
- increases the propagation speed while keeping spreading rates fixed in expectation.

For completeness, here is a definition for local waiting times.

Definition 8.1. *Let $u : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ with $\operatorname{supp} u(\cdot, 0) =: [a, x_0] \subset\subset \mathcal{O} := (-L, L)$. We say that u has positive waiting time T^* in x_0, if*

$$T^* := \inf\{T \ge 0 \ : \ \operatorname{supp} u(\cdot, T) \cap (x_0, L) \neq \emptyset\} > 0.$$

As we expect the solution's support to increase continuously, we consider homogeneous boundary conditions and compactly supported initial data in what follows. We emphasize that we choose the computational domain in relation to the time interval under consideration such large that practically it is excluded that the support of solutions reaches the boundary of the domain.

In this sense, we are free to replace periodic boundary conditions by homogeneous Dirichlet boundary conditions.

8.1. **A Numerical Scheme for the Stochastic Porous-Medium Equation.** In this subsection, we propose a fully practical space-time discrete numerical scheme for stochastic porous-medium equations with multiplicative noise inside a convective term.

We consider the domain $\mathcal{O} := (-L, L)$. We follow a finite-element approach using continuous, piecewise linear ansatz functions with homogeneous boundary conditions, based on uniform triangulations $\mathcal{T}_h$ of $\mathcal{O}$. Here, h denotes the gridsize, $L_{0,h} := \frac{|\mathcal{O}|}{h} - 1 \in \mathbb{N}$ is the number of degrees of freedom, and $\mathcal{T}_h = \{[(i-1)h, ih] \; : \; i = 1, \ldots, L_{0,h}+1\}$. Let $X_{0,h}$ denote the corresponding ansatz space. We define a basis $\{\phi_i\}_{i=1}^{L_{0,h}}$ of $X_{0,h}$ by

$$\phi_i(jh) := \delta_{ij} \qquad \forall i, j = 1, \ldots, L_{0,h}.$$

$X_{0,h}$ is equipped with the lumped scalar product $(\cdot,\cdot)_h$ (see Section 2). The advantages are evident: First, $\{\phi_i\}_{i=1}^{L_{0,h}}$ is an orthogonal basis of $(X_{0,h}, (\cdot,\cdot)_h)$. Moreover, in contrast to the standard $L^2(\mathcal{O})$-scalar product, the lumped scalar product prevents unphysical oscillations at the free boundary of discrete solutions.

As an approximation for the degenerate diffusion coefficient $s \mapsto (m+1)\,|s|^m$, we use the following elementwise constant function $M_{1,\sigma}$ suggested for thin-film equations in [25]. Let $\sigma > 0$ be a fixed regularization parameter, take $m_\sigma(s) := m \max\{\sigma, |s|\}^{m-1}$. Let $\psi \in X_{0,h}$ and $\psi_i := (ih)$, $i = 0, 1, \ldots, L_{0,h}$ and define the discrete mobility $M_{1,\sigma}$

$$M_{1,\sigma}(\psi)\big|_{((i-1)h, ih)} := \begin{cases} m_\sigma(\psi_i) & \text{if } \psi_{i-1} = \psi_i \\ \left(\fint_{\psi_{i-1}}^{\psi_i} \frac{1}{m_\sigma(s)} ds\right)^{-1} & \text{if } \psi_{i-1} \neq \psi_i \end{cases}$$

for $i \in \{1, \cdots, L_{0,h}+1\}$. Taking a finite number $N_h \in \mathbb{N}$ of modes into account, our finite-element formulation reads as follows: We search for $u_h \in C([0,T]; X_{0,h})$ such that

$$\begin{aligned}(u_h(t), \phi)_h = \left(u_h^0, \phi\right)_h - \int_0^t \left(M_{1,\sigma}(u_h)(u_h)_x, \phi_x\right) ds \\ + \sum_{k=1}^{N_h} \int_0^t \mu_k \left((M_2(u_h) g_k \; d\beta_k)_x, \phi\right) \qquad \forall \phi \in X_{0,h},\end{aligned}$$

where $u_h^0 \in X_{0,h}$ is an approximation of u_0. We use a semi-implicit Euler scheme for an equidistant time discretization with stepsize $\tau = \tau(h)$ and stochastic increments $b_{h,k}^n$, which are $N(0,1)$-distributed independent random numbers. This leads to the following fully-discrete scheme: We search for $u_h^n \in X_{0,h}$, $n = 1, \ldots, \frac{T}{\tau}$, such that

$$\begin{aligned}(u_h^n, \phi)_h = \left(u_h^{n-1}, \phi\right)_h - \tau \left(M_{1,\sigma}(u_h^n)(u_h^n)_x, \phi_x\right)_{L^2(\mathcal{O})} \\ \sum_{k=1}^{N_h} \mu_k \sqrt{\tau} \left(\left(M_2(u_h^{n-1}) g_k b_{h,k}^n\right)_x, \phi\right)_{L^2(\mathcal{O})} \qquad \forall \phi \in X_{0,h}.\end{aligned} \tag{8.1}$$

For the convective term, we suggest a stochastic upwind-scheme in the spirit of [24]. To this aim, we introduce the elementwise integral-mean of the discrete noise

$$N_{h,i-\frac{1}{2}}^n := \frac{\sqrt{\tau}}{h} \int_{x_{i-1}}^{x_i} \sum_{k=1}^{N_h} \mu_k b_{h,k}^n \mathcal{I}_h g_k(x) \; dx, \qquad i = 1, \cdots, L_{0,h}+1$$

and the following family of discrimination parameters

$$B_{h,i-\frac{1}{2}}^n := \begin{cases} M_2\left(u_h^n\left((i-1)h\right)\right) & \text{if } N_{h,i-\frac{1}{2}}^n \geq 0 \\ M_2\left(u_h^n\left(ih\right)\right) & \text{if } N_{h,i-\frac{1}{2}}^n < 0 \end{cases}, \qquad i = 1, \cdots, L_{0,h}+1.$$

Summing up, we define the nodal coefficient vector $\overline{s_h^n}$ of our upwind discretization

$$(\overline{s_h^n})_i := N_{h,i+\frac{1}{2}}^n B_{h,i+\frac{1}{2}}^n - N_{h,i-\frac{1}{2}}^n B_{h,i-\frac{1}{2}}^n, \qquad i = 1, \cdots, L_{0,h}. \tag{8.2}$$

Let us rewrite (8.1) in combination with (8.2) in matrix formulation. For this, we denote the coefficient vector of an element $v_h \in X_{0,h}$ with respect to the nodal basis $\{\phi_i\}_{i=1}^{L_{0,h}}$ by $\overline{v_h}$. Furthermore, let $L_h(\overline{v_h})$ denote the weighted stiffness matrix with respect to the elliptic term $(M_{1,\sigma}(v_h)(\cdot)_x, \phi_x)_{L^2(\mathcal{O})}$. Taking

	$\nu = 0$	$\nu =$ 0.0125	$\nu =$ 0.025	$\nu = 0.05$	$\nu = 0.1$	$\nu = 0.2$
$\bar{S} = 1$	63.8	39.0	25.7	15.2	8.36	4.43
$\bar{S} = 2$	31.1	20.0	13.6	8.31	4.67	2.54
$\bar{S} = 4$	15.0	10.1	7.22	4.51	2.61	1.43
$\bar{S} = 8$	7.14	5.05	3.74	2.44	1.44	0.838
$\bar{S} = 16$	3.39	2.45	1.86	1.28	0.791	0.491

TABLE 1. Average waiting times $\cdot 10^3$.

into account that the mass matrix for a lumped scalar product is given by h times identity matrix, we end up with the nonlinear system

$$\left(\mathrm{I} + \frac{1}{h} L_h(\overline{u_h^n})\right) \overline{u_h^n} = \overline{u_h^{n-1}} - \overline{s_h^{n-1}}.$$

An analogous scheme without upwinding has been proposed and applied in [23] for the stochastic porous-medium equation with linear multiplicative noise inside a source term. Therein, stability and convergence of the scheme are proven for $m \in (1,2)$.

8.2. **Experiments on Waiting Times.** The scaling of waiting times for the deterministic porous-medium equation

$$\begin{aligned} \partial_t w - \Delta w^{m+1} &= 0, \\ w(\cdot, 0) &= w_0 \end{aligned}$$

is well understood. If $\operatorname{supp} w_0 = [0, x_0]$, $x_0 > 0$, and

$$|w_0(x)| \leq S\, |x - x_0|^{\frac{2}{m}} \tag{8.3}$$

as well as

$$\lim_{x \nearrow x_0} \frac{w_0(x)}{|x - x_0|^{\frac{2}{m}}} = S, \tag{8.4}$$

then the waiting time T^* in x_0 is proportional to S^{-m}, see [1]. Let us consider the following test case. We choose initial data

$$w_0(x) := \bar{S}^{\frac{1}{m}}\, (1 - |x|)_+^{\frac{2}{m}}$$

which satisfy (8.3) and (8.4) in $x_0 = 1$. Therefore, the waiting time in x_0 is proportional to $\frac{1}{\bar{S}}$. We perform the computations on the domain $\mathcal{O} = [-1.1, 1.1]$ for $m = 0.5$. We set $T = 100$ to ensure that our algorithm terminates in finite time. We choose $h := 2.1 \cdot 10^{-3}$, $\tau = 1.9 \cdot 10^{-5}$, and $\sigma = 10^{-15}$. This choice of discretization parameters is in accordance with numerical experiments conducted in [23] for the case of linear multiplicative noise inside a source term. For the Wiener noise, we consider $N_h = 1025$ modes given by

$$g_k := \begin{cases} \sqrt{\frac{1}{L}} \cos\left(\pi k \frac{x}{L}\right) & \text{for } k = -1, \dots, -512 \\ \sqrt{\frac{1}{L}} \sin\left(\pi k \frac{x}{L}\right) & \text{for } \ k = 1, \dots, 512 \\ \frac{1}{2L} & \text{for } k = 0 \end{cases} . \tag{8.5}$$

As we wish to investigate the influence of the size of the noise amplitude, we write

$$\mu_k := \nu \hat{\mu}_k \tag{8.6}$$

where $\nu \in \mathbb{R}$ is a chosen constant and

$$\hat{\mu}_k := \begin{cases} \frac{1}{|k|} & \text{if } k \neq 0 \\ 1 & \text{if } k = 0 \end{cases} . \tag{8.7}$$

The average waiting times for $m = 0.5$ and different values of $\bar{S}$ and ν can be seen in Table 1. We have considered 100 sample paths for each constellation of $\nu, \bar{S}$. The corresponding approximate variances of the waiting times are gathered in Table 2.

	$\nu = 0.0125$	$\nu = 0.025$	$\nu = 0.05$	$\nu = 0.1$	$\nu = 0.2$
$\bar{S} = 1$	34.6	60.2	45.6	39.2	17.2
$\bar{S} = 2$	19.6	20.0	23.8	15.3	9.14
$\bar{S} = 4$	8.0	11.2	9.68	6.74	4.62
$\bar{S} = 8$	2.73	5.43	5.05	4.68	2.95
$\bar{S} = 16$	1.47	1.91	1.65	1.93	1.38

TABLE 2. Estimated variances $\cdot 10^8$.

	$\nu = 0$	$\nu = 0.0125$	$\nu = 0.025$	$\nu = 0.05$	$\nu = 0.1$	$\nu = 0.2$
p_ν	1.06	0.998	0.947	0.892	0.85	0.793

TABLE 3. Average scaling of waiting times w.r.t. $\bar{S}^{-1}$.

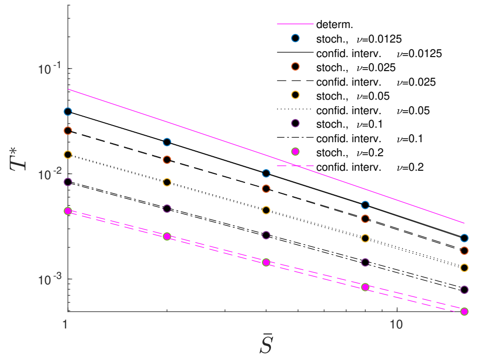

FIGURE 1. Log-log plot of the average size of waiting times in terms of $\bar{S}$ for different noise amplitudes ($m = 0.5$).

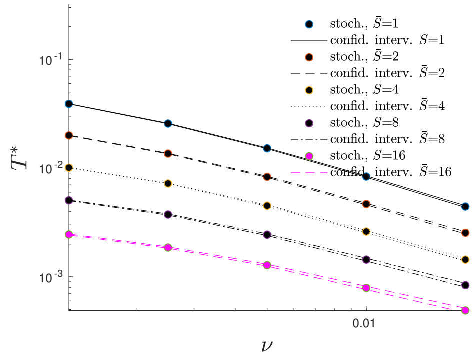

FIGURE 2. Log-log plot of the average size of waiting times T^* in dependence of the noise amplitude ν ($m = 0.5$).

Let us also consider the average scaling p_ν of the waiting times $T^* := T^*(\nu, \bar{S})$, see Table 3, which are given by

$$p_\nu := \frac{1}{4} \sum_{j=1}^{4} \log_2 \left(\frac{T^*(\nu, \frac{\bar{S}}{2})}{T^*(\nu, \bar{S})} \right) \Big|_{\bar{S}=2^j}.$$

Regarding the results presented in Table 1 and Table 2, we observe that an increase of the noise amplitude causes the waiting time to decrease. The log-log-plot in Figure 1 indicates that on average the dependence of the expected size of waiting times in terms of $\bar{S}$ still follows a power-law. We observe, however, a deviation from the scaling parameter (-1) in $T^* \sim \bar{S}^{-1}$ by up to 20 percent with increasing noise amplitude, see also Table 3.

Let us investigate the dependence of waiting times $T^*(\nu, \bar{S})$ on the noise amplitude ν. First, plotting the data from Table 1 in a logarithmic ν-T^*-plot, we see an initially concave line which becomes almost linear for larger values of ν (see Figure 2). This behaviour correlates with the dependency ratio

$$T^* \lesssim \frac{1}{1+\nu}$$

which is consistent with bounds by Djie [13] on waiting times for deterministic doubly nonlinear parabolic equations. More precisely, Djie considers the deterministic convection-diffusion equations

$$\partial_t w - \left(|w|^m w\right)_{xx} + \lambda \left(w^\beta\right)_x = 0, \qquad m > 1,\ \lambda \in \mathbb{R},$$

with initial data satisfying $w(x,0) \geq A^{\frac{1}{m+1}} |x - x_0|^{\frac{2}{m}}$ in a neighborhood of the free boundary point x_0. For this deterministic setting, it is proven that the corresponding waiting time $T^*_w(\lambda, A)$ is bounded by

$$T^*_w(\lambda, A) \leq \frac{C(m)}{A^{\frac{m}{2m+2}} \lambda}, \qquad \text{if } \lambda > 0 \tag{8.8}$$

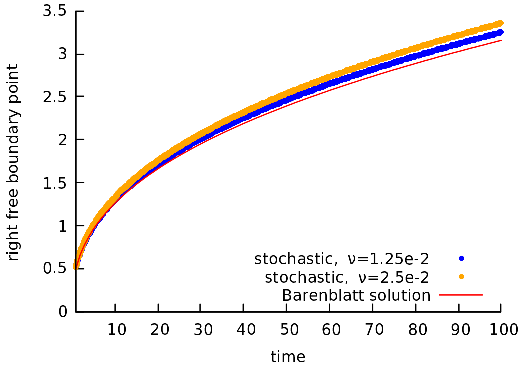

FIGURE 3. Average value of free-boundary location for $m = 0.5$ over the time intervall [0,100].

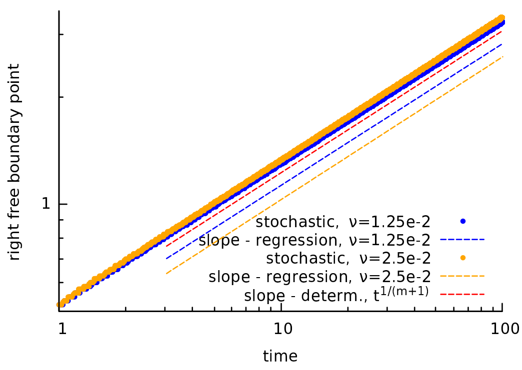

FIGURE 4. Log-log plot of the average free-boundary location in terms of time ($m = 0.5$).

and

$$T_w^*(\lambda, A) \leq \frac{C(m)}{A^{\frac{m}{m+1}} + A^{\frac{m}{2m+2}}\,|\lambda|}, \qquad \text{if } \lambda < 0, \tag{8.9}$$

see Theorem 2.3.1 in [13]. We expect an interplay of both effects (8.8) and (8.9) for the waiting times of conservative stochastic porous-medium equations like (1.1), as the probabilistic term has no fixed sign. In particular, such a behaviour would be consistent with Figure 2.

8.3. **Experiments on Free-Boundary Propagation.** In order to get a better understanding of the average propagation of the free boundary subjected to noise, we perform simulations over a large time interval. We choose the final time in such a way that we may practically exclude the free boundary to reach the boundary of the spatial domain. In particular, we will choose the profile of a self-similar solution of

$$\partial_t w - \Delta w^{m+1} = 0 \qquad \text{in } \mathbb{R} \times [0, T]$$

as initial data. Thus, we can compare our simulations with a deterministic, analytical solution. Such self-similar solutions – the famous Barenblatt solutions – are given by

$$w(x,t) := \frac{1}{t^{\frac{1}{m+2}}} \left(b - \frac{m}{(2m+2)(m+2)} \frac{x^2}{t^{\frac{2}{m+2}}} \right)_+^{\frac{1}{m}}, \qquad (x,t) \in \mathbb{R} \times (0, \infty)$$

(cf. [32]). Choosing $b = \frac{1}{8}\frac{m}{(m+1)(m+2)}$, $u_0(x) = w(x,1)$, $N_h = 1025$, and $\{g_k\}_{k=1}^{N_h}$, $\{\mu_k\}_{k=1}^{N_h}$ as in (8.5), (8.7) with ν to be specified below, we consider the following two cases for which we compute the empirical average of the location of the right boundary of the solution's support. We use 500 sample paths each together with the following specific data

- $m = 0.5$,
- $\mathcal{O} \times [0, T) = [-3.5, 3.5] \times [0, 100)$,
- $h = 1.37 \cdot 10^{-2}$, $\tau = 1.22 \cdot 10^{-4}$,
- different noise amplitudes (cf. (8.6)), given by $\nu = 1.25 \cdot 10^{-2},\ 2.5 \cdot 10^{-2}$.

The results are depicted in Figures 3, 4. We refrain from plotting any confidence intervals for the expected values of the localizations of the free boundary as they can hardly be distinguished from the average propagation itself. In fact, the estimated standard deviations $\hat{\sigma}_{1.25}(t)$ and $\hat{\sigma}_{2.5}(t)$ for the free-boundary point at time t with $\nu = 1.25 \cdot 10^{-2}$ and $2.5 \cdot 10^{-2}$, respectively, are uniformly bounded by

$$\|\hat{\sigma}_{1.25}\|_{L^\infty} \leq 7.3 \cdot 10^{-3}, \qquad \|\hat{\sigma}_{2.5}\|_{L^\infty} \leq 1.2 \cdot 10^{-2}.$$

Concerning the propagation of the free boundary, we see that the presence of conservative noise increases the propagation speed of the free boundary. This effect seems to be correlated to the strength of the noise, as a reduced noise amplitude entails a reduced empirical average propagation speed in our experiments.

Furthermore, the scaling of the average propagation speed is independent of the strength of the noise. Indeed, a log-log-plot of the expected location of the free-boundary point on the right-hand side over time reveals essentially the same slopes for different noise amplitudes, see Figure 4. Again, due to the small standard deviation, confidence intervals are not depicted, either. A change in the spreading of solutions towards faster propagation − including a change in propagation rates − has been numerically observed by Davidovitch, Moro, and Stone [12] for the thin-film equation with multiplicative noise inside a convective term. These authors considered a specific degenerate mobility which corresponds physically to a *no-slip paradoxon*. This means that the spreading has been triggered by assuming initial data to be strictly positive. Therefore, it is not excluded that the propagation depends on the size of this artificial precursor layer.

Summing up, changes in the expected values of the size of waiting times and the propagation speeds might be a typical feature of solutions to degenerate parabolic equations with multiplicative noise inside convective terms which merits further studies, numerically as well as theoretically.

9. Appendix

Lemma 9.1. *For elements $u, v \in (X_h)_{per}$, the following estimates hold true.*

$$|(u,v)-(u,v)_h| \le \frac{1}{3}h\,\|u\|_h\,\|\nabla v\|_{L^2}\,, \tag{9.1}$$

$$h^{\frac{1}{2}}\,\|u\|_{L^2} \le \sqrt{2}\,\|u\|_h^{\frac{1}{2}}\,\|u\|_{(H^1_{0,per})'}^{\frac{1}{2}}\,, \tag{9.2}$$

$$h\,\|\nabla u\|_{L^2} \le 2\,\|u\|_h\,, \tag{9.3}$$

$$h\,\|u\|_{L^2} \le 4\,\|u\|_{(H^1_{0,per})'}\,. \tag{9.4}$$

Theorem 9.2 (Jakubowski [28]). *Let $(\mathcal{X},\tau)$ be a topological space and assume that there exists a countable family $\{f_i : \mathcal{X} \to [-1,1]\}_{i\in\mathcal{I}}$ of τ-continuous functions which separate points of X.*

Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of $\mathcal{X}$-valued random variables. Suppose for each $\epsilon > 0$ there exists a compact subset $K_\epsilon \subset \mathcal{X}$ such that

$$\mathbb{P}\{X_n \in K_\epsilon\} > 1-\epsilon, \qquad \textit{for all } n \in \mathbb{N}. \tag{9.5}$$

Then, there exist a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$, a sequence $(X_{n_k})_{k\in\mathbb{N}}$, and a sequence $(Y_k)_{k\in\mathbb{N}}$ of $\mathcal{X}$-valued random variables on $\tilde{\Omega}$ with the following properties:
The law of X_{n_k} on $\mathcal{X}$ coincides with the law of Y_k for all $k \in \mathbb{N}$. Furthermore, there exists a random variable $Y_\infty : \tilde{\Omega} \to \mathcal{X}$ such that for almost every $\omega \in \tilde{\Omega}$ the convergence $Y_k(\omega) \to Y_\infty(\omega)$ holds in the topology of $\mathcal{X}$.

Theorem 9.3 (cf. Theorem 5 in [31]). *Let $X \subset\subset B \hookrightarrow Y$ be Banach spaces and assume*

- *F bounded subset in $L^p(0,T;X)$,*
- *$\|f(\cdot+\sigma) - f(\cdot)\|_{L^p(0,T-\sigma;Y)} \overset{\sigma\to 0}{\rightrightarrows} 0$ uniformly for $f \in F$.*

Then, F is relatively compact in $L^p(0,T;B)$ (and in $C(0,T;B)$ if $p=\infty$).

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