ON UNIQUENESS OF $\mathbb{P}$-TWISTS

RINA ANNO AND TIMOTHY LOGVINENKO

Abstract. We prove that for any $\mathbb{P}^n$-functor all the convolutions (double cones) of the three-term complex $FHR \xrightarrow{\psi} FR \xrightarrow{\epsilon} \mathrm{Id}$ defining its $\mathbb{P}$-twist are isomorphic. We also introduce a new notion of a non-split $\mathbb{P}^n$-functor.

1. Introduction

A $\mathbb{P}^n$-*object* E in the derived category $D(X)$ of a smooth projective variety X has $\mathrm{Ext}^*_X(E,E) \simeq H^*(\mathbb{P}^n)$ as graded rings and $E \otimes \omega_X \simeq E$. These were introduced by Huybrechts and Thomas in [HT06] as mirror symmetric analogues of Lagrangian $\mathbb{CP}^n$s in a Calabi Yau manifold. Moreover, there is an analogue of the Dehn twist: the $\mathbb{P}$-*twist* P_E about E is the Fourier-Mukai transform defined by a certain convolution (double cone) of the three term complex

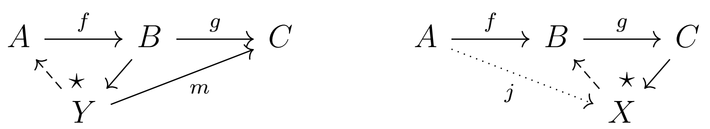
$$E^\vee \boxtimes E[-2] \xrightarrow{h^\vee \otimes \mathrm{Id} - \mathrm{Id} \otimes h} E^\vee \boxtimes E \xrightarrow{\epsilon} \mathcal{O}_\Delta \tag{1.1}$$

where h is the degree 2 generator of $\mathrm{Ext}^*_X(E,E)$. It was shown in [HT06] to be an auto-equivalence of $D(X)$.

A *convolution* of a three term complex in a triangulated category $\mathcal{D}$

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1.2}$$

is any object obtained via one of the following two constructions. A *left Postnikov system* is where we first take the cone Y of f, then lift g to a morphism $m\colon Y \to C$, and take the cone of m. A *right Postnikov system* is where we first take cone X of g, then lift f to a morphism $j\colon A[1] \to X$, and take a cone of j.



Apriori, convolutions are not unique. For example, the convolutions of $A[-2] \to 0 \to C$ are the extensions of A by C in $\mathcal{D}$. If $\mathcal{D}$ admits a DG-enhancement $\mathcal{C}$, the convolutions of a complex in $\mathcal{D}$ up to isomorphism are in bijection with the *twisted complex* structures on it in $\mathcal{C}$ up to homotopy equivalence, cf. §2.2.

In [HT06, Lemma 2.1] it was shown that the complex (1.1) has a unique left Postnikov system and defined the $\mathbb{P}$-twist to be its convolution. Later Addington noted in [Add16] that

$$\operatorname{Hom}^{-1}_{D(X\times X)}(E^\vee \boxtimes E[-2], \mathcal{O}_\Delta) \simeq \operatorname{Hom}^1_X(E,E) = 0 \tag{1.3}$$

which by a simple homological argument implies that the complex (1.1) has a unique convolution. At the heart of this note is Lemma 2.2 where we establish an abstract criterion for the unicity of the convolution of an arbitrary three-term complex (1.2). This criterion is trivially satisfied whenever (1.3) holds, see Remark 2.3(1) after Lemma 2.2.

In [Add16] and [Cau12] Addington and Cautis introduced the notion of a *(split)* $\mathbb{P}^n$*-functor* to generalise $\mathbb{P}^n$-objects in a similar way to spherical functors [AL17] generalising spherical objects [ST01]. It was a brilliant idea and numerous applications followed [Kru15][Kru14] [ADM16][ADM19].

For Z and X smooth projective varieties a *split* $\mathbb{P}^n$*-functor* is a Fourier-Mukai functor $F\colon D(Z) \to D(X)$ with left and right Fourier-Mukai adjoints L, R such that for some autoequivalence H of $D(Z)$ we have an isomorphism

$$RF \simeq H^n \oplus H^{n-1} \oplus \cdots \oplus H \oplus \operatorname{Id} \tag{1.4}$$

satisfying the *monad condition* and the *adjoints condition* which generalise the $\mathbb{P}$-object requirements of $\operatorname{Ext}^*_X(E,E) \simeq H^*(\mathbb{P}^n)$ respecting the graded ring structure and of $E \simeq E \otimes \omega_X$. The $\mathbb{P}$*-twist* about F is then the convolution of a certain canonical right Postnikov system of the three-term complex

$$FHR \xrightarrow{\psi} FR \xrightarrow{\epsilon} \operatorname{Id} \tag{1.5}$$

where ϵ is the adjunction co-unit and ψ the corresponding component of the map $FRFR \xrightarrow{FR\epsilon - \epsilon FR} FR$ after the identification (1.4).

Addington noted in [Add16, §4.3] that Postnikov systems for (1.5) are not necessarily unique. This caused technical difficulties in applications. They were aggravated by the fact that it was sometimes simpler to calculate left Postnikov systems associated to (1.5). In a word, it was often easy to compute some convolution of (1.5) but difficult to prove that it was indeed the convolution defined in [Add16] as the $\mathbb{P}$-twist for split $\mathbb{P}^n$-functors.

The main result of this paper is that contrary to the expectations of specialists, including the authors of this paper, the three term complex (1.1) has a unique convolution. Thus we can compute the $\mathbb{P}$-twist via any Postnikov system, taking cones in any order and using any lifts. To prove this we prove a more general fact:

Theorem (see Theorem 3.1)**.** *Let Z, X be separated schemes of finite type over a field. Let $F\colon D(Z) \to D(X)$ be an exact functor with a right adjoint R. Let $\epsilon\colon FR \to \operatorname{Id}_X$ be the adjunction co-unit. Let*

$G\colon D(X) \to D(Z)$ be any exact functor and $f\colon FG \to FR$ any natural transformation with $\epsilon \circ f = 0$. Finally, assume these are all Fourier-Mukai functors and natural transformations thereof.

Then all convolutions of the following three-term complex are isomorphic:

$$FG \xrightarrow{f} FR \xrightarrow{\epsilon} \mathrm{Id}_X\,. \tag{1.6}$$

Our proof shows that the complex (1.6) has a unique right Postnikov system. We then prove in Lemma 2.1 that in an arbitrary triangulated category for any left Postnikov system there exists a right Postnikov system with the same convolution, and vice versa.

In a DG-enhanced setting we can work more generally and give a more direct proof. In Prop. 3.1 we construct a homotopy equivalence between any two twisted complex structures on (1.6). We thus obtain:

Theorem (see Theorem 3.2)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced triangulated categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor with a right adjoint R. Let $\epsilon\colon FR \to \mathrm{Id}_{\mathcal{B}}$ be the adjunction counit. Let $G\colon \mathcal{B} \to \mathcal{A}$ be any exact functor and $f\colon FG \to FR$ any natural transformation with $\epsilon \circ f = 0$. Finally, assume that all these are also enhanceable.*

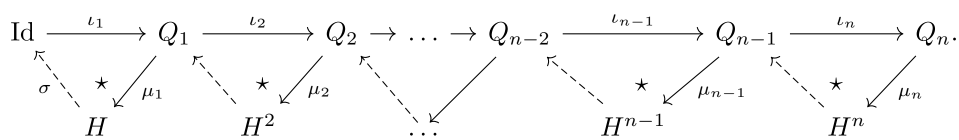
Then all convolutions of the following three-term complex are isomorphic:

$$FG \xrightarrow{f} FR \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{B}}\,. \tag{1.7}$$

The uniqueness of $\mathbb{P}$-twists as established by these two theorems removes a significant roadblock in the way of research into $\mathbb{P}^n$-functors. Our results were immediately applied in a number of papers including [HK17], [KM17], [MR19].

Finally, Addington and Cautis referred to the notion which they introduced as $\mathbb{P}^n$-functors. We propose to use the name *split* $\mathbb{P}^n$*-functors* instead. This is because in their definition the monad RF splits into a direct sum of Id and powers of an autoequivalence H. On the other hand, in the definition of a spherical functor the monad RF can be a non-trivial extension of Id by an autoequivalence, and this is the case in many interesting examples. Indeed, it was later noted by Addington, Donovan, and Meachan in [ADM19, Remark 1.7] that it would be nice to allow RF to have a filtration with quotients $\mathrm{Id}, H, \dots, H^n$, however it would then be difficult to formulate the monad condition and to construct the $\mathbb{P}^n$-twist as a convolution of a three-term complex.

In §2.3 we propose a general notion of a (non-split) $\mathbb{P}^n$*-functor* which deals with all of these issues. These are the functors F for which RF is isomorphic to a repeated extension of Id by $H, \dots, H^n$ of the form



This has to satisfy three conditions: the monad condition, the adjoints condition, and the highest degree term condition, see §2.3. The definition in [Add16] only asks for two conditions. However, in the non-split situation, the analogue of the monad condition in [Add16] is complicated to state on the level of triangulated categories. We weaken it to the point where it can be easily stated on the triangulated level, but at the price of introducing the highest degree term condition. However, as explained in §2.3, if the non-split analogues of the two conditions in [Add16] hold, they do imply our three conditions. Thus our definition is strictly more general. Indeed, in [AL19, §7] we give examples of four families of non-split $\mathbb{P}^n$-functors, while in [AL19, Appendix] our study of the case of the derived category of a point demonstrates the existence of split $\mathbb{P}^n$-functors which do not satisfy the strong monad condition of [Add16], yet do satisfy our weaker monad condition.

We define the $\mathbb{P}$-*twist* about such F to be the unique convolution of the three-term complex

$$FHR \xrightarrow{\psi} FR \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{B}} \tag{1.8}$$

where ψ is again the corresponding component of $FR\epsilon - \epsilon FR$ after appropriate identifications. The uniqueness follows by Theorem 3.2 of this paper. In [AL19] we show that this $\mathbb{P}$-twist is indeed an autoequivalence and give examples of non-split $\mathbb{P}^n$-functors.

On the structure of this paper. In §2.1 and §2.2 we give preliminaries on Postnikov systems and on twisted complexes, respectively. In §2.3 we give the definition of a (non-split) $\mathbb{P}^n$-functor. Then in §3.1 and §3.2 we prove our main results via triangulated and DG-categorical techniques, respectively. Those only interested in the triangulated approach should read §2.1, §3.1, and, possibly, §2.3.

Acknowledgements: We would like to thank Alexei Bondal and Mikhail Kapranov for introducing the notions of a DG-enhancement and a twisted complex in [BK90].

2. Preliminaries

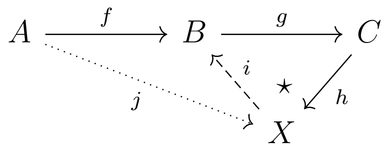
2.1. **Postnikov systems and convolutions.**

Let $\mathcal{D}$ be a triangulated category and let

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.1}$$

be a complex of objects of $\mathcal{D}$, that is $g \circ f = 0$.

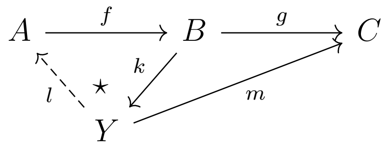
A *right Postnikov system* associated to the complex (2.1) is a diagram



(2.2)

where the starred triangle is exact and the other triangle is commutative. The dashed and dotted arrows denote maps of degree 1 and -1 respectively. The *convolution* of (2.2) is the cone of the map $A[1] \xrightarrow{j} X$.

Similarly, a *left Postnikov system* associated to the complex (2.1) is a diagram



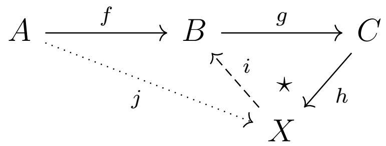
(2.3)

Its *convolution* is the cone of the map $Y \xrightarrow{m} C$.

We say that an object $E \in \mathcal{D}$ is a *convolution* of the complex (2.1) if it is a convolution of some right or left Postnikov system associated to it. The following is a direct proof of the three-term complex case of the more general fact about arbitrary Postnikov systems whose proof is sketched out in [GM03, §IV.2, Exercise 1]:

Lemma 2.1. *For every right (resp. left) Postnikov system associated to the complex* (2.1) *there is a left (resp. right) Postnikov system with an isomorphic convolution.*

Proof. Let



(2.4)

be any right Postnikov system associated to (2.1). Then we have a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle j[-1]}\downarrow & & \| \\ X[-1] & \xrightarrow[i]{} & B. \end{array} \tag{2.5}$$

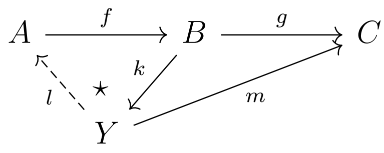
Let

$$A \xrightarrow{f} B \xrightarrow{k} Y \xrightarrow{l} A[1]$$

be any exact triangle incorporating the map f. By [May01, Lemma 2.6] it follows from the octahedral axiom that (2.5) can be completed to the following 3×3 diagram with exact rows and columns

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{k} & Y \\ {\scriptstyle j[-1]}\downarrow & & \| & & \downarrow \\ X[-1] & \xrightarrow[i]{} & B & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Cone}(j)[-1] & \longrightarrow & 0 & \longrightarrow & Z. \end{array} \tag{2.6}$$

Let m be the map $Y \to C$ in the right column of (2.6). Since the top right square in (2.6) commutes



(2.7)

is a left Postnikov system associated to (2.1). Since the bottom row is exact the object Z is isomorphic to $\mathrm{Cone}(j)$, i.e. the convolution of the right Postnikov system (2.4). On the other hand, since the right column is exact, Z is isomorphic to $\mathrm{Cone}(m)$, i.e. the convolution of the left Postnikov system (2.7). Thus (2.7) is a left Postnikov system whose convolution is isomorphic to that of (2.4), as desired.

The proof that given a left Postnikov system associated to (2.1) we can construct a right Postnikov system with an isomorphic convolution is analogous. □

Lemma 2.2. *If the natural map*

$$\mathrm{Hom}^{-1}(A,B) \xrightarrow{g\circ(-)} \mathrm{Hom}^{-1}(A,C) \tag{2.8}$$

is surjective then the convolutions of all right Postnikov systems associated to (2.1) are isomorphic.

Similarly, if the natural map

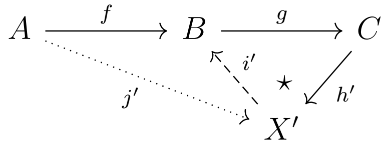
$$\mathrm{Hom}^{-1}(B,C) \xrightarrow{(-)\circ f} \mathrm{Hom}^{-1}(A,C) \tag{2.9}$$

is surjective then the convolutions of all left Postnikov systems associated to (2.1) are isomorphic.

Proof. We only prove the first assertion as the second assertion is proved similarly. Take any exact triangle incorporating the map g

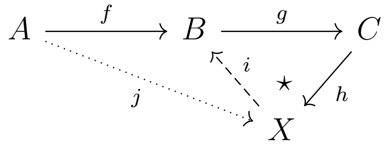
$$B \xrightarrow{g} C \xrightarrow{h} X \xrightarrow{i} B[1]. \tag{2.10}$$

Then for every right Postnikov system



(2.11)

there exists a map $A[1] \xrightarrow{j} X$ such that



(2.12)

is a Postnikov system whose convolution is isomorphic to that of (2.11). Indeed, let $X' \xrightarrow{t} X$ be an isomorphism which identifies the exact triangles in (2.12) and in (2.11) and set $j = t \circ j'$.

Thus the convolutions of all right Postnikov systems associated to (2.1) are isomorphic to the cones of all possible maps $j\colon A[1] \to X$ with $f = i \circ j$. To show that all convolutions are isomorphic, it would suffice to show that $i \circ (-)$ is injective.

Now consider the following fragment of the long exact sequence obtained by applying $\operatorname{Hom}^{\bullet}_{\mathcal{D}}(A, -)$ to the exact triangle (2.10):

$$\cdots \to \operatorname{Hom}^{-1}_{\mathcal{D}}(A, B) \xrightarrow{g\circ(-)} \operatorname{Hom}^{-1}_{\mathcal{D}}(A, C) \xrightarrow{h\circ(-)} \to \operatorname{Hom}^{-1}_{\mathcal{D}}(A, X) \xrightarrow{i\circ(-)} \operatorname{Hom}^{0}_{\mathcal{D}}(A, B) \to \dots$$

Since the sequence is exact, $g \circ (-)$ being surjective is equivalent to $h \circ (-) = 0$ which in turn is equivalent to $i \circ (-)$ being injective. The claim follows. □

Remark 2.3.
1. Note, in particular, that if $\operatorname{Hom}^{-1}_{\mathcal{D}}(A, C)$ is zero then both the criteria in Lemma 2.2 above are automatically fulfilled. Thus these criteria each refine that of $\operatorname{Hom}^{-1}_{\mathcal{D}}(A, C)$ vanishing.
2. In view of Lemma 2.1 if either of the criteria in Lemma 2.2 holds then the convolutions of all right and all left Postnikov systems associated to (2.1) are isomorphic.

2.2. **Enhanced triangulated categories and twisted complexes.** For technical details on twisted complexes, pretriangulated categories and DG-enhancements see [AL17, §3], [BK90], [LO10, §1]. Below we give a brief overview.

A *DG-category* is a category $\mathcal{C}$ enriched over the category $\mathbf{Mod}\text{-}k$ of differentially graded complexes over k. Its morphism complexes are complexes of k-modules. Truncating each complex to its degree zero cohomology produces an ordinary category which is called the *homotopy category* $H^0(\mathcal{C})$ of $\mathcal{C}$. Examples to keep in mind are a DG-algebra and the DG-category of complexes of objects of an abelian category.

Enhanced triangulated categories were originally introduced in [BK90]. Roughly, an enhanced triangulated category is a triangulated category T together with the data of a DG-category $\mathcal{C}$ that truncates to it. More precisely, a DG-category $\mathcal{C}$ is *pretriangulated* if $H^0(\mathcal{C})$ is a triangulated subcategory of $H^0(\mathbf{Mod}\text{-}\mathcal{C})$. A pretriangulated DG-category $\mathcal{C}$ is a *DG-enhancement* of a triangulated category T if there is an exact equivalence $T \simeq H^0(\mathcal{C})$. An example to keep in mind is that the bounded derived category of an abelian category is usually enhanced by the DG-categories of complexes of injective or projective objects, respectively.

DG-enhancements are considered up to *quasi-equivalences,* the DG-functors whose H^0-truncation is an equivalence of the underlying triangulated categories. These play a role analogous to that of quasi-isomorphisms between complexes of objects in an abelian category. An *enhanced triangulated category* is a quasi-equivalence class of pretriangulated DG-categories. It is defined by specifying a pretriangulated DG-category $\mathcal{C}$. The underlying triangulated category is then $H^0(\mathcal{C})$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two enhanced triangulated categories. Let $D(\mathcal{A}\text{-}\mathcal{B})$ be the derived category of $\mathcal{A}$-$\mathcal{B}$-bimodules. The enhanceable exact functors $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ are in one-to-one correspondence with the isomorphism classes in $D^{\mathcal{B}\text{-}qr}(\mathcal{A}\text{-}\mathcal{B})$, the full subcategory of $D(\mathcal{A}\text{-}\mathcal{B})$ consisting of $\mathcal{B}$-quasi-representable bimodules [Toë07]. If the underlying triangulated categories are Karoubi-complete, we can use the Morita framework where $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalence classes of small DG-categories. The underlying triangulated categories are the full subcategories $D_c(\mathcal{A})$ and $D_c(\mathcal{B})$ of the compact objects in $D(\mathcal{A})$ and $D(\mathcal{B})$. The enhanceable exact functors are in one-to-one correspondence with the isomorphism classes in $D^{\mathcal{B}\text{-}\mathcal{P}erf}(\mathcal{A}\text{-}\mathcal{B})$, the full subcategory of $D(\mathcal{A}\text{-}\mathcal{B})$ consisting of $\mathcal{B}$-perfect bimodules [Toë07]. Either way, this shows that to make our results applicable to any pair of adjoint enhanceable exact functors between two enhanced triangulated categories it suffices to work with homotopy adjoint DG-bimodules.

Let $\mathcal{C}$ be a pretriangulated DG-category. Let $\text{Pre-Tr}(\mathcal{C})$ be the DG-category of *one-sided twisted complexes* (E_i, q_{ij}) over $\mathcal{C}$. Here *one-sided* means that $q_{ij} = 0$ for $i - j \leq 0$. The category $H^0(\text{Pre-Tr}(\mathcal{C}))$ has a natural triangulated structure: it is the triangulated hull of $H^0(\mathcal{C})$ in $H^0(\mathbf{Mod}\text{-}\mathcal{C})$. Since $\mathcal{C}$ is pretriangulated, the natural embedding $H^0(\mathcal{C}) \to H^0(\text{Pre-Tr}(\mathcal{C}))$ is an equivalence, see [BK90, §3], [Dri04, §2.4], [Kel06, §4.5]. Fix its quasi-inverse $H^0(\text{Pre-Tr}(\mathcal{C})) \to H^0(\mathcal{C})$. We refer to it as the *convolution functor* and write $\{E_i, q_{ij}\}$ for the convolution in $H^0(\mathcal{C})$ of the twisted complex (E_i, q_{ij}). We think of $\mathcal{C}$ as a DG-enhancement of the triangulated category $H^0(\mathcal{C})$ and of $\text{Pre-Tr}(\mathcal{C})$ as an enlargement of $\mathcal{C}$ to a bigger DG-enhancement of $H^0(\mathcal{C})$ which allows for the calculus of twisted complexes described below.

Any one-sided twisted complex (E_i, q_{ij}) over $\mathcal{C}$ defines an ordinary differential complex

$$\dots \xrightarrow{q_{i-2,i-1}} E_{i-1} \xrightarrow{q_{i-1,i}} E_i \xrightarrow{q_{i,i+1}} E_{i+1} \xrightarrow{q_{i+1,i+2}} \dots \tag{2.13}$$

in $H^0(\mathcal{C})$. This is because by the definition of a twisted complex all $q_{i,i+1}$ are closed of degree 0 and we have $q_{i,i+1} \circ q_{i-1,i} = (-1)^i d q_{i-1,i+1}$. The data of the higher twisted differentials of (E_i, q_{ij}) defines a number of Postnikov systems for (2.13) in $H^0(\mathcal{C})$ whose convolutions are all isomorphic to $\{E_i, q_{ij}\}$, see [AL20, Cor. 2.9]. Below we describe this in detail for two- and three-term twisted complexes.

A two-term one-sided twisted complex concentrated in degrees $-1, 0$ is the data of

$$A \xrightarrow{f} \underset{\text{deg.0}}{B} \tag{2.14}$$

where $A, B \in \mathcal{C}$ and f is a degree 0 closed map in $\mathcal{C}$. The corresponding complex in $H^0(\mathcal{C})$ is

$$A \xrightarrow{f} B \tag{2.15}$$

A Postnikov system for (2.15) is an exact triangle incorporating f. The triangle defined by (2.14) is

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle l}\nwarrow & & \swarrow{\scriptstyle k} \\ & \{A \xrightarrow{f} \underset{\text{deg.0}}{B}\} & \end{array} \tag{2.16}$$

where l and k are the images in $H^0(\mathcal{C})$ of the following maps of twisted complexes:

$$\bar{l}: \quad \begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow^{\text{Id}} & \\ & & \underset{\text{deg.0}}{A} \end{array} \qquad\qquad \bar{k}: \quad \begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \text{Id}} \\ A & \xrightarrow{f} & \underset{\text{deg.0}}{B} \end{array}$$

A three-term one-sided twisted complex concentrated in degrees -2, -1, 0 is the data of

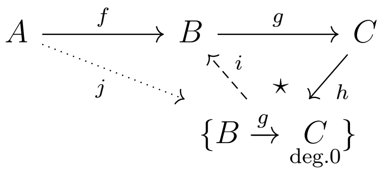
$$A \xrightarrow{f} B \xrightarrow{g} \underset{\text{deg.0}}{C}, \quad x: A \dashrightarrow C \tag{2.17}$$

where $A, B, C \in \mathcal{C}$, f and g are closed maps of degree 0 in $\mathcal{C}$ and x is a degree -1 map in $\mathcal{C}$ such that that $dx = -g \circ f$. The corresponding complex in $H^0(\mathcal{C})$ is

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{2.18}$$

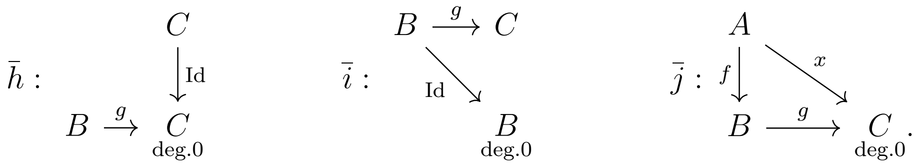
with the composition $g \circ f$ being zero in $H^0(\mathcal{C})$ as it is explicitly a boundary dx in $\mathcal{C}$. Note that the datum of the degree -1 map x is not visible in the triangulated category $H^0(\mathcal{C})$, it exists only in its DG-enhancement $\mathcal{C}$. The twisted complex (2.17) in $\mathcal{C}$ specifies not only the ordinary complex (2.18) in $H^0(\mathcal{C})$, but also its convolution. Specifically, the datum of the map x in $\mathcal{C}$ defines a right and a left Postnikov system for the complex (2.18) in $H^0(\mathcal{C})$:

Definition 2.4. The *right Postnikov system induced by the twisted complex* (2.17) is

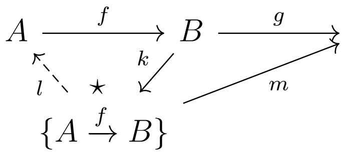


(2.19)

where the maps h, i, j are the images in $H^0(\mathcal{C})$ of the following maps of twisted complexes:

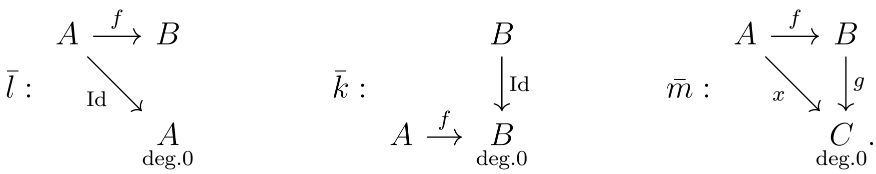


The *left Postnikov system induced by the complex* (2.17) is



(2.20)

where the maps l, k, m are the images in $H^0(\mathcal{C})$ of the respective maps:



Lemma 2.5. *For any twisted complex* (2.17) *the convolutions of its left and right Postnikov systems are isomorphic in* $H^0(\mathcal{C})$ *to the convolution of the twisted complex itself.*

Proof. By definition the convolutions of (2.19) and (2.20) are Cone(j) and Cone(m), respectively. As we've seen, given a map in $\mathcal{C}$ the cone of its image in $H^0(\mathcal{C})$ is its convolution as a two-term twisted complex over $\mathcal{C}$. In case of $\bar{j}$ and $\bar{m}$, the objects of this twisted complex are themselves convolutions of twisted complexes. The double convolution of a twisted complex of twisted complexes is isomorphic to the convolution of its total complex [BK90, §2]. In case of both $\bar{j}$ and $\bar{m}$ these total complexes coincide with (2.17), whence the result. □

The conceptual explanation for Lemma 2.1 is as follows. Any Postnikov system for a given complex in $H^0(\mathcal{C})$ lifts (non-uniquely) to a twisted complex over $\mathcal{C}$. In Lemma 2.6 we prove this for three-term complexes. This twisted complex can then be used to induce a Postnikov system of any given type whose convolvution is isomorphic to the convolution of the original Postnikov system. In Lemma 2.5 we prove this for three-term complexes. The general case can be proved in a similar way but with a more convoluted notation.

Lemma 2.6. *Any right or left Postnikov system for a differential complex*

$$A \xrightarrow{\;f\;} B \xrightarrow{\;g\;} C \tag{2.21}$$

in $H^0(\mathcal{C})$ is induced up to an isomorphism by some lift of (2.21) to a three-term twisted complex over $\mathcal{C}$.

Proof. We prove the claim for left Postnikov systems, the other case is analogous. Let the following be a left Postnikov system for (2.21):

$$\begin{array}{ccccc} A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C \\ {\scriptstyle l'}\nwarrow & \star & \swarrow{\scriptstyle k'} & \nearrow{\scriptstyle m'} & \\ & Y & & & \end{array} \tag{2.22}$$

Let $\bar f$ and $\bar g$ be some lifts of f and g from $H^0(\mathcal{C})$ to $\mathcal{C}$.

Since any two exact triangles incorporating f are isomorphic, the exact triangle in (2.22) is isomorphic to the exact triangle

$$A \xrightarrow{f} B \xrightarrow{k} \left\{ A \xrightarrow{\bar f} \underset{\text{deg.}0}{B} \right\} \xrightarrow{l} A[1],$$

where k and l are the images in $H^0(\mathcal{C})$ of the twisted complex maps

$$\bar l : \begin{array}{ccc} A & \xrightarrow{\bar f} & B \\ & \searrow^{\text{Id}} & \\ & & \underset{\text{deg.}0}{A} \end{array} \qquad\qquad \bar k : \begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \text{Id}} \\ A & \xrightarrow{\bar f} & \underset{\text{deg.}0}{B} \end{array}.$$

Hence (2.22) is isomorphic to the Postnikov system

$$\begin{array}{ccccc} A & \xrightarrow{\;f\;} & B & \xrightarrow{\;g\;} & C \\ {\scriptstyle l}\nwarrow & \star & \swarrow{\scriptstyle k} & \nearrow{\scriptstyle m} & \\ & \left\{ A \xrightarrow{\bar f} B \right\} & & & \end{array} \tag{2.23}$$

for some map m. It remains to show that there exists a degree -1 map $x \colon A \to C$ in $\mathcal{C}$ such that m equals the image in $H^0(\mathcal{C})$ of the twisted complex map

$$\bar m : \begin{array}{ccc} A & \xrightarrow{\bar f} & B \\ & \searrow^{x} & \downarrow{\scriptstyle \bar g} \\ & & \underset{\text{deg.}0}{C} \end{array}.$$

Since the convolution functor is an equivalence we can lift m to some closed degree 0 map of twisted complexes

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ & \overset{x'}{\searrow} & \Big\downarrow{\scriptstyle g'} \\ & & \underset{\text{deg.0}}{C} \end{array}. \tag{2.24}$$

We have $dx' + g' \circ f = 0$ as the map is closed. By definition of a Postnikov system $m \circ k = g$ in $H^0(\mathcal{C})$ and thus $\bar{g} - g' = d\alpha$ for some degree -1 map α. We then have

$$\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ & \overset{x'}{\searrow} & \Big\downarrow{\scriptstyle g'} \\ & & \underset{\text{deg.0}}{C} \end{array} + d\left(\begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ & & \Big\downarrow{\scriptstyle \alpha} \\ & & \underset{\text{deg.0}}{C} \end{array}\right) = \begin{array}{ccc} A & \xrightarrow{\bar{f}} & B \\ & \overset{x'-\alpha\circ\bar{f}}{\searrow} & \Big\downarrow{\scriptstyle \bar{g}} \\ & & \underset{\text{deg.0}}{C} \end{array}. \tag{2.25}$$

Since the RHS of (2.25) differs from (2.24) by a boundary, its image in $H^0(\mathcal{C})$ is also m. Set $x = x' - \alpha \circ \bar{f}$. The left Postnikov system induced by the twisted complex

$$A \xrightarrow{\bar{f}} B \xrightarrow{\bar{g}} \underset{\text{deg.0}}{C}, \qquad x\colon A \dashrightarrow C$$

is precisely (2.23), and thus isomorphic to the original left Postnikov system (2.22), as desired. □

2.3. **$\mathbb{P}^n$-functors.** Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced triangulated categories.

Definition 2.7 ([Add16])**.** A *split* $\mathbb{P}^n$*-functor* is a functor $F\colon \mathcal{A} \to \mathcal{B}$ which has left and right adjoints $L, R\colon \mathcal{B} \to \mathcal{A}$ such that:

(1) For some autoequivalence H of $\mathcal{A}$ there exists an isomorphism
$$H^n \oplus H^{n-1} \oplus \cdots \oplus H \oplus \mathrm{Id} \xrightarrow{\gamma} RF. \tag{2.26}$$
(2) (The strong monad condition) In the monad structure on $H^n \oplus H^{n-1} \oplus \cdots \oplus H \oplus \mathrm{Id}$ induced by γ^{-1} from the adjunction monad RF the left multiplication by H acts on
$$H^n \oplus H^{n-1} \oplus \cdots \oplus H \tag{2.27}$$
as an upper triangular matrix with Id's on the main diagonal.
(3) (The weak adjoints condition) $R \simeq H^n L$.

Note that as the matrix in the strong monad condition is evidently invertible the resulting endomorphism of (2.27) is necessarily an isomorphism.

Let ψ be the composition:

$$FHR \hookrightarrow FH^nR \oplus \cdots \oplus FHR \oplus FR \underset{\sim}{\xrightarrow{\gamma}} FRFR \xrightarrow{FR\epsilon - \epsilon FR} FR.$$

The $\mathbb{P}^n$-*twist* P_F was defined in [Add16, §3.3] as the convolution of

$$FHR \xrightarrow{\psi} FR \xrightarrow{\epsilon} \mathrm{Id} \tag{2.28}$$

given by a certain canonical right Postnikov system associated to it. Addington noted that such system is no longer unique but provided a canonical choice of one.

As mentioned in the Introduction, in his original definition in [Add16] Addington simply called these objects $\mathbb{P}^n$-*functors*. We propose to call them *split* $\mathbb{P}^n$-*functors* instead, since the monad RF splits into a direct sum of Id and powers of H. We then propose the following more general definition which allows RF to be a repeated extension:

Definition 2.8. Let H be an endofunctor of $\mathcal{A}$. A *cyclic extension of* Id *by* H *of degree* n is a repeated extension Q_n of the form

$$\begin{array}{ccccccccccc} \mathrm{Id} & \xrightarrow{\iota_1} & Q_1 & \xrightarrow{\iota_2} & Q_2 & \to \dots \to & Q_{n-2} & \xrightarrow{\iota_{n-1}} & Q_{n-1} & \xrightarrow{\iota_n} & Q_n. \\ \nwarrow & \star \swarrow{\scriptstyle \mu_1} & \nwarrow & \star \swarrow{\scriptstyle \mu_2} & \nwarrow & \swarrow & \nwarrow & \star \swarrow{\scriptstyle \mu_{n-1}} & \nwarrow & \star \swarrow{\scriptstyle \mu_n} & \\ & H & \dashleftarrow & H^2 & \dashleftarrow & \dots & \dashleftarrow & H^{n-1} & \dashleftarrow & H^n & \end{array} \tag{2.29}$$

Here all starred triangles are exact, all the remaining triangles are commutative, and all the dashed arrows denote maps of degree 1. We further write ι for the map $\mathrm{Id} \xrightarrow{\iota_n \circ \dots \circ \iota_1} Q_n$.

Note that if we replace each H^i by $H^i[-i]$ and adjust degrees of all the maps accordingly, the bottom row of (2.29) becomes a differential complex, while the rest of (2.29) becomes a Postnikov system on that complex. Thus, equivalently, a cyclic extension of Id by H of degree n is any convolution of a one-sided twisted complex of the form

$$H^n[-n] \longrightarrow H^{n-1}[-(n-1)] \longrightarrow \dots \longrightarrow H[-1] \longrightarrow \underset{\mathrm{deg}.0}{\mathrm{Id}}.$$

The maps $\mathrm{Id} \xrightarrow{\iota} Q_n$ and $Q_n \xrightarrow{\mu_n} H^n$ are the inclusion of the degree 0 term and the projection on the degree $-n$ term, respectively.

Definition 2.9. A $\mathbb{P}^n$-*functor* is a functor $F\colon \mathcal{A} \to \mathcal{B}$ with left and right adjoints $L, R\colon \mathcal{B} \to \mathcal{A}$ such that

(1) There exists an isomorphism

$$Q_n \xrightarrow{\gamma} RF$$

where Q_n is a cyclic extension of $\mathrm{Id}_{\mathcal{A}}$ by an autoequivalence H of $\mathcal{A}$ with $H(\mathrm{Ker}\, F) = \mathrm{Ker}\, F$. Moreover, this isomorphism intertwines $\mathrm{Id} \xrightarrow{\eta} RF$ and $\mathrm{Id} \xrightarrow{\iota} Q_n$, that is, the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Id} & \xrightarrow{\iota} & Q_n \\ & \searrow^{\eta} & \downarrow{\scriptstyle\gamma} \\ & & RF. \end{array} \tag{2.30}$$

Now, observe that as $F \xrightarrow{F\eta} FRF$ is a retract, so is $F \xrightarrow{F\iota} FQ_q$ by (2.29). Hence the exact triangle $FR \to FQ_1R \to FHR$ is also split. Choose any splitting $FHR \to FQ_1R$ and denote by ϕ the composition

$$FHR \to FQ_1R \xrightarrow{\iota_n \circ \ldots \circ \iota_2} FQ_nR \xrightarrow{F\gamma R} FRFR. \tag{2.31}$$

Define the map $FHR \xrightarrow{\psi} FR$ to be the composition

$$FHR \xrightarrow{\phi} FRFR \xrightarrow{FR\epsilon - \epsilon FR} FR.$$

Any choice of the splitting $FHR \to FQ_1R$ in the definition of ϕ produces the same ψ since the following composition is zero:

$$FR \xrightarrow{F\eta R} FRFR \xrightarrow{FR\epsilon - \epsilon FR} FR$$

(2) (The monad condition) The following is an isomorphism:

$$FHQ_{n-1} \xrightarrow{FH\iota_{n-1}} FHRF \xrightarrow{\psi F} FRF \xrightarrow{F\kappa} FC[1], \tag{2.32}$$

here C is the spherical cotwist of F defined by an exact triangle

$$C \to \mathrm{Id} \xrightarrow{\eta} RF \xrightarrow{\kappa} C[1].$$

(3) (The adjoints condition) The following is an isomorphism:

$$FR \xrightarrow{FR\eta} FRFL \xrightarrow{\mu_n L} FH^nL. \tag{2.33}$$

(4) (The highest degree term condition) There is an isomorphism that makes the diagram commute:

$$\begin{array}{ccccccc} FHQ_{n-1}L & \xrightarrow{FH\iota_nL} & FHRFL & \xrightarrow{\psi FL} & FRFL & \xrightarrow{F\mu_nL} & FH^nL \\ \| & & & & & & \downarrow \\ FHQ_{n-1}L & \xrightarrow{FH\iota_nL} & FHRFL & \xrightarrow{FHR\psi'} & FHRFH'L & \xrightarrow{FH\mu_nH'L} & FHH^nH'L, \end{array}$$

where H' is the left adjoint of H and $\psi' \colon FL \to FH'L$ is obtained from $\psi \colon FHR \to FR$ via adjunctions.

In the split case treated by Addington the objects FHQ_{n-1} and $FC[1]$ are both isomorphic to

$$FH^n \oplus \ldots \cdots \oplus FH.$$

The map (2.32) is the image under F of the left multiplication by H in the RF monad structure minus a strictly upper triangular matrix. Our monad condition asks for (2.32) to be invertible, while the one in [Add16] asks for the left multiplication by H to be upper triangular with Id's on the main diagonal. The precise non-split analogue of this would be requesting the map (2.32) to come from a one-sided map of twisted complexes whose vertical arrows are homotopy equivalences. This stronger condition implies our highest degree term condition and implies that $RF \xrightarrow{R\eta L} RFLF \xrightarrow{\mu_n LF} H^n LF$ is an isomorphism [AL19, Lemmas 5.16 and 5.13]. That, in turn, means that the existence of any isomorphism $FR \simeq FH^nL$ implies our adjoints condition above [AL19, Prop. 5.14]. Thus, even the non-split analogue of a $\mathbb{P}^n$-functor in the sense of [Add16] satisfies our definition.

Definition 2.10. The $\mathbb{P}$-*twist* P_F of a $\mathbb{P}^n$-functor F is the unique convolution of the complex

$$FHR \xrightarrow{\psi} FR \xrightarrow{\epsilon} \mathrm{Id}\,. \tag{2.34}$$

The uniqueness of the convolution is the main result of this paper, see Theorem 3.2 and Theorem 3.1. In [AL19] we prove that this $\mathbb{P}$-twist is indeed an autoequivalence of $\mathcal{B}$.

3. Uniqueness of ℙ-twists

3.1. **An approach via triangulated categories.** Let Z and X be separated schemes of finite type over a field k. We work with Fourier-Mukai kernels using the functorial notation: e.g. for any Fourier-Mukai kernels $F \in D(Z \times X)$ and $G \in D(X \times Z)$ of exact functors $D(Z) \xrightarrow{f} D(X)$ and $D(X) \xrightarrow{g} D(Z)$ we write FG for the Fourier-Mukai kernel of $f \circ g$ given by the standard Fourier-Mukai kernel composition:

$$\pi_{13*}(\pi_{12}^* G \overset{\mathbf{L}}{\otimes} \pi_{23}^* F) \in D(X \times X).$$

Here π_{ij} are projections from $X \times Z \times X$ to the corresponding partial products. We further write $\mathrm{Id}_Z \in D(Z \times Z)$ and $\mathrm{Id}_X \in D(X \times X)$ for the structure sheafs of the diagonals.

Let $F \in D(Z \times X)$ and $R \in D(X \times Z)$ be Fourier-Mukai kernels and let maps $FR \xrightarrow{\epsilon} \mathrm{Id}_X$ and $\mathrm{Id}_Z \xrightarrow{\eta} RF$ define a 2-categorical adjunction of F and R, i.e. the following compositions are identity maps:

$$F \xrightarrow{F\eta} FRF \xrightarrow{\epsilon F} F,$$

$$R \xrightarrow{\eta R} RFR \xrightarrow{R\epsilon} R.$$

In other words, consider adjoint exact functors $(f, r)\colon D(Z) \rightleftarrows D(X)$ with a fixed lift to 2-categorically adjoint Fourier-Mukai kernels (F, R). Let G be a Fourier-Mukai kernel of an exact functor $g : D(X) \to D(Z)$.

Theorem 3.1. *For any $FG \xrightarrow{f} FR$ with $\epsilon \circ f = 0$ all convolutions of the following complex are isomorphic:*

$$FG \xrightarrow{f} FR \xrightarrow{\epsilon} \mathrm{Id}_X\,. \tag{3.1}$$

Proof. By Lemma 2.1 it suffices to show that the convolutions of all right Postnikov systems associated to (3.1) are isomorphic, since for any left Postnikov system there exists a right Postnikov system with an isomorphic convolution. Then by Lemma 2.2 it suffices to show that the natural map

$$\operatorname{Hom}^{-1}_{D(X\times X)}(FG, FR) \xrightarrow{\epsilon\circ(-)} \operatorname{Hom}^{-1}_{D(X\times X)}(FG, \operatorname{Id}_X)$$

is surjective. The idea is: by the 2-categorical adjunction of F and R it suffices to show the surjectivity of

$$\operatorname{Hom}^{-1}_{D(X\times Z)}(G, RFR) \xrightarrow{R\epsilon\circ(-)} \operatorname{Hom}^{-1}_{D(X\times Z)}(G, R),$$

whereupon we note that $R\epsilon$ has a right quasi-inverse $\eta R\colon R \to RFR$.

Indeed, let $\phi \in \operatorname{Hom}^{-1}_{D(X\times X)}(FG, \operatorname{Id}_X)$ be any element. Let

$$\psi \in \operatorname{Hom}^{-1}_{D(X\times X)}(FG, FR)$$

be the composition

$$FG \xrightarrow{F\eta G} FRFG \xrightarrow{FR\phi} FR.$$

Then $\epsilon\circ\psi$ is the composition

$$FG \xrightarrow{F\eta G} FRFG \xrightarrow{FR\phi} FR \xrightarrow{\epsilon} \operatorname{Id}_X .$$

Since the composition of Fourier-Mukai kernels is functorial the following two compositions are equal:

$$FRFG \xrightarrow{FR\phi} FR \xrightarrow{\epsilon} \operatorname{Id}_X \quad \text{and} \quad FRFG \xrightarrow{\epsilon FG} FG \xrightarrow{\phi} \operatorname{Id}_X .$$

Thus $\epsilon\circ\psi$ equals the composition

$$FG \xrightarrow{F\eta G} FRFG \xrightarrow{\epsilon FG} FG \xrightarrow{\phi} \operatorname{Id}_X$$

which is just ϕ since $(\epsilon FG)\circ(F\eta G) = \operatorname{Id}$. We conclude that $\epsilon\circ(-)$ is surjective as desired. □

3.2. **An approach via DG-enhancements.** Let $\mathcal{A}$ and $\mathcal{B}$ be two small DG categories. Let $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$, $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$, $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ be the bar categories of $\mathcal{A}$-$\mathcal{B}$-, $\mathcal{B}$-$\mathcal{A}$-, $\mathcal{A}$-$\mathcal{A}$- and $\mathcal{B}$-$\mathcal{B}$-bimodules [AL20]. These could be replaced by any other DG enhancements of the derived categories of bimodules equipped with (homotopy) unital tensor bifunctors $(-)\otimes_{\mathcal{A}}(-)$ and $(-)\otimes_{\mathcal{B}}(-)$ which descend to the bifunctors $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{A}}(-)$ and $(-)\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}(-)$ between the derived categories. For example, one can take h-projective or h-injective enhancements. The advantage of bar categories is that any adjunction of enhanceable functors can be lifted to a pair of homotopy adjoint bimodules described in the next paragraph, cf. [AL20, §5.2]

Let $M \in \mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ and $N \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ be *homotopy adjoint*, that is, there exist maps

$$\epsilon : N \otimes_{\mathcal{A}} M \to \mathcal{B} \qquad \eta : \mathcal{A} \to M \otimes_{\mathcal{B}} N$$

in $\mathcal{A}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ such that

$$M \xrightarrow{\eta \otimes \mathrm{Id}} M \otimes_{\mathcal{B}} N \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{Id} \otimes \epsilon} M \tag{3.2}$$

$$N \xrightarrow{\mathrm{Id} \otimes \eta} N \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} N \xrightarrow{\epsilon \otimes \mathrm{Id}} N \tag{3.3}$$

are homotopic to Id_M and Id_N. Thus there exists a degree -1 map $\zeta : M \to M$ such that the composition in (3.2) equals $\mathrm{Id}_M + d\zeta$.

Let $X \in \mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{A}$ and let $X \otimes_{\mathcal{A}} M \xrightarrow{f} N \otimes_{\mathcal{A}} M$ be any map such that the following is a differential complex in $D(\mathcal{B}\text{-}\mathcal{B}) \simeq H^0(\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B})$:

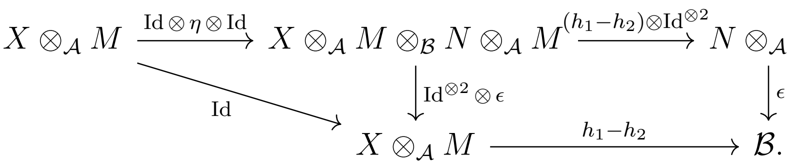
$$X \otimes_{\mathcal{A}} M \xrightarrow{\quad f \quad} N \otimes_{\mathcal{A}} M \xrightarrow{\quad \epsilon \quad} \mathcal{B}. \tag{3.4}$$

Proposition 3.1. *Any two lifts of* (3.4) *to a twisted complex over* $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ *are homotopy equivalent.*

Proof. Any lift of (3.4) in $D(\mathcal{B}\text{-}\mathcal{B})$ to a twisted complex over $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$ is readily seen to be homotopy equivalent to a twisted complex which lifts f to f and ϵ to ϵ. The latter is simply a choice of the degree -1 map $h \colon X \otimes_{\mathcal{A}} M \to \mathcal{B}$ with $\epsilon \circ f + dh = 0$. Let h_1 and h_2 be any two such maps. Define ξ to be the composition

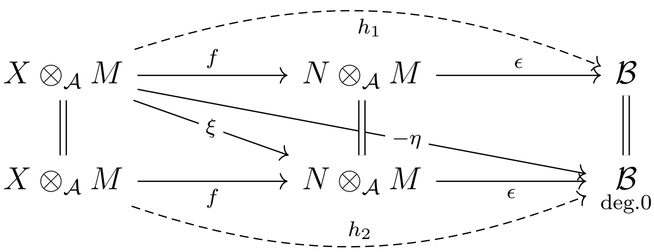
$$X \otimes_{\mathcal{A}} M \xrightarrow{\mathrm{Id} \otimes \eta \otimes \mathrm{Id}} X \otimes_{\mathcal{A}} M \otimes_{\mathcal{B}} N \otimes_{\mathcal{A}} M \xrightarrow{(h_1 - h_2) \otimes \mathrm{Id}^{\otimes 2}} N \otimes_{\mathcal{A}} M.$$

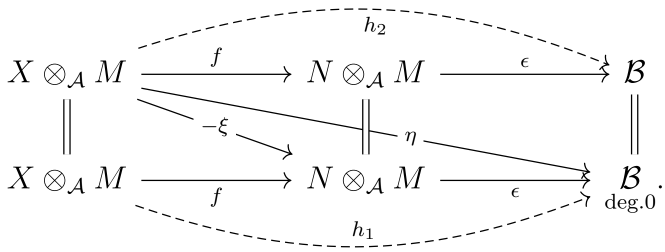
Then $d\xi = 0$. Consider the following diagram:



(3.5)

It descends to a commutative diagram in $D(\mathcal{B}\text{-}\mathcal{B})$, thus it commutes up to a homotopy in $\mathcal{B}\text{-}\overline{\mathbf{Mod}}\text{-}\mathcal{B}$. Let η be the homotopy up to which it commutes, so that $d\eta = \epsilon \circ \xi - h_1 + h_2$. Then the following are two mutually inverse isomorphisms of twisted complexes:





□

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be enhanced triangulated categories. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor with a right adjoint R. Let $\epsilon\colon FR \to \mathrm{Id}_{\mathcal{B}}$ be the adjunction counit. Let $G\colon \mathcal{B} \to \mathcal{A}$ be any exact functor and $f\colon FG \to FR$ be any natural transformation with $f \circ \epsilon = 0$. Finally, assume that all these are also enhanceable.*

Then all convolutions of the following three-term complex are isomorphic:

$$FG \xrightarrow{f} FR \xrightarrow{\epsilon} \mathrm{Id}_{\mathcal{B}}\,. \tag{3.6}$$

Proof. As at the beginning of this section we can lift F and R to a pair of homotopy adjoint bimodules M and N and we can lift G to an bimodule X. Then by Prop. 3.1 any two lifts of (3.6) to a twisted complex are homotopy equivalent. By Lemmas 2.5 and 2.6 every convolution of (3.6) is isomorphic in $D(\mathcal{B}\text{-}\mathcal{B})$ to the convolution of some twisted complex lifting it. It follows that all convolutions of (3.6) are isomorphic. □

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Email address: ranno@math.ksu.edu

Department of Mathematics, Kansas State University, 138 Cardwell Hall, Manhattan, KS 66506, USA

Email address: LogvinenkoT@cardiff.ac.uk

School of Mathematics, Cardiff University, Senghennydd Road,, Cardiff, CF24 4AG, UK