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# Virtual series-system models of imperfect repair

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**Abstract:** Novel models of imperfect repair are fitted to classic reliability datasets. The models suppose that a virtual system comprises a component and a remainder in series. On failure of the component, the component is renewed, and on failure of the remainder, the component is renewed and the remainder is minimally repaired. It follows that the repair process is a counting process that is the superposition of a renewal process and a Poisson process. The repair effect, that is, the extent to the system is repaired by renewal of the component, depends on the relative intensities of the superposed processes. The repair effect may be negative, when the intensity of the part that is a renewal process is a decreasing function. Other special cases of the model exist (renewal process, Poisson process, superposed renewal process and homogeneous Poisson process). Model fit is important because the nature of the model and corresponding parameter values determine the effectiveness of maintenance, which we also consider. A cost-minimizing repair policy may be determined provided the cost of preventive-repair is less than the cost of corrective-repair and the repairable part is ageing. If the remainder is ageing, then policy needs to be adapted as it ages.

**Keywords:** renewal process; counting process; reliability; maintenance; multi-component system

## 1. Introduction

Much research has sought to model imperfect repair of a multi-component system in a way that captures a repair effect that lies somewhere between “as good as new” (perfect repair) and “as bad as old” (minimal repair). At the ends of this spectrum lie the renewal process (Ross, 1996) and the non-homogeneous Poisson process (Cox and Lewis, 1966). In between, and thus capturing imperfect repair, lie **mixed processes** (Brown and Proschan, 1983; Kallen, 2011; Doyen, 2014) virtual age-reduction models (Kijima and Sumita, 1986; Kijima, 1989; Jack, 1998; Tanwar et al., 2014), geometric processes (Lam, 1988; Wang and Pham, 1996; Wu and Clements-Croome, 2006; Wu and Scarf, 2015; Zhang et al., 2015; Wu and Wang, 2018), hazard-reduction models (Doyen and Gaudoin, 2004; Percy and Alkali, 2006; Percy et al., 2010), virtual component models (Wu and Scarf, 2017); and generalizations of these (Doyen et al., 2017). All these attempt to model multi-component systems parsimoniously.

Models with few parameters are desirable because failure data are scarce (Scarf, 1997) and often non-existent (e.g. Desa and Christer, 2001). As a consequence, maintenance policy is imprecisely specified (Baker and Scarf, 1995). While a component-wise model of imperfect repair of multi-component systems, in which repair corresponds to renewal of a particular component, is conceptually simple, such a model requires knowledge of individual component reliabilities.

Component dependencies (e.g. Do et al., 2018) add further difficulties. Many components have never failed and most rarely fail (e.g. Akbarov et al., 2008). Thus, research that seeks to model the repair-effect, either directly or indirectly, with as few parameters as possible is important. This paper continues this tradition.

A focus of these parsimonious models is estimation of the repair effect. This is directly parameterized within the age- and hazard-reduction models, and indirectly parameterized in the models described in this paper. Estimation of the repair effect is challenging because data are typically scarce and subjective elicitation from maintenance engineers who carry out repairs is compromising (Zhang and Wang, 2014). It may be better therefore for repair models to indirectly specify the repair effect, so that it is a property of the model (a function of parameters) rather than a parameter itself. Then it can be indirectly estimated. Thus, one of the objectives of this paper is the indirect estimation of the repair effect. **We also allow that this repair effect may be negative, whereby, for example, a poor-quality spare-part is used or a fault is induced (Scarf and Cavalcante, 2012) at preventive maintenance.**

The repair effect is estimated using a model that we outline here. **Conceptionally, the model posits a series system with an unknown number  $m$  of components. By unknown here, we mean that  $m$  is a parameter about which we have no information.** Each component generates events that occur in time according to a counting process (Cox and Lewis, 1966). These events are typically failures but need not be so in general. This is convenient because failures are often ambiguously defined and what is observed are the times of maintenance interventions, which we term repairs. At each event, the system is repaired, so that the events are triggers for repair. On repair of the system, we suppose that a component is replaced with a new (age-zero) identical component, but the component replaced does not necessarily correspond to the component that fails, or more generally that generates the repair event. If these components are indeed different, then the component that generates the repair event is unaffected by the event. If they are one and the same, then it is replaced. In this way, we suppose the system is a virtual system consisting of a virtual component, which is the component that is always replaced on repair, and a virtual sub-system, which is the collection of the other virtual components in the system. The latter is the remainder of the system. We use this virtual-system terminology to distinguish the system from a 1-out-of- $m$  series system—models of repair of series systems (e.g. Smith and Dekker, 1997) usually assume that the component that fails is replaced. Further, we suppose that each virtual component corresponds to a counting process.

The fundamental ideas for this type of model are described in Wu and Scarf (2017). In our paper here we take a number of further steps. First, we show that the virtual series-system model can be formulated as a superposition of a renewal process and a nonhomogeneous Poisson process. Second, we fit the model (and special cases of it) to a number of classic datasets from the reliability literature. In so doing, we consider carefully the goodness-of-fit of the models, a critical activity given the scarcity of data in real situations. We then use the models to show how the repair effect can be specified and estimated. Finally, we show how the models can be used to inform maintenance policy decision-making and discuss their implications for practice.

The layout of the paper then corresponds to the order of these developments. However, first are some preliminaries about counting processes that will be needed for the model specification in Section 3. The subsections in Section 3 present the models and their properties. The final subsection of Section 3 describes parameter estimation for the model by the method of maximum

likelihood. Section 4 describes the classic datasets we use, the parameterization of the renewal process and Poisson process elements of the model, and the results of the model fitting. Section 5 is concerned with maintenance policy and the repair effect. We finish with conclusions in the final section.

## 2. Preliminaries

First we state some preliminary definitions and results that are well-known (Cox and Lewis, 1966). The number of events that arise in an interval  $[0, \tau]$  is in general a counting process  $N(\tau)$  whose behaviour is determined by the intensity function

$$\lambda(t|\mathcal{F}_{t-}) = \lim_{\delta t \rightarrow 0} \frac{\Pr(N(t + \delta t) - N(t) \geq 1 | \mathcal{F}_{t-})}{\delta t},$$

where  $\mathcal{F}_{t-}$  is the history (filtration) of the process up to time  $t$ .  $\lambda(t|\mathcal{F}_{t-})$  is an intensity function of a counting process if and only if  $\Lambda(t) = \int_0^t \lambda(u|\mathcal{F}_{u-}) du$  exists for all  $t \in [0, \tau]$ , is positive and is an increasing function in  $t$ . Essentially this means  $\lambda(\cdot)$  must be positive and defined everywhere on the positive real line.  $\Lambda(t)$  is called the cumulative intensity function (CIF).

If the intensity function does not depend on the history ( $\lambda(t|\mathcal{F}_{t-}) = \lambda(t)$ ) and at most one event can occur at any time  $t$ , then the counting process is a Poisson process. If the intensity function is a constant ( $\lambda(t) = \lambda$ ) then the counting process is a homogeneous Poisson process. If the intensity function depends only on the time since the last event ( $\lambda(t|\mathcal{F}_{t-}) = h(t - T_{N(t-)})$ ) then the counting process is a renewal process. Consequently, the renewal process is characterized by the distribution  $F_X$  of the time since the last event,  $X = t - T_{N(t-)}$ , and  $h$  is the hazard function of this distribution ( $h(x) = F'_X(x) / \{1 - F_X(x)\}$ ).

The homogeneous Poisson process is a renewal process and  $X$  is exponentially distributed with mean  $1/\lambda$ , and  $h_X(x) = F'_X(x) / \{1 - F_X(x)\} = \lambda \exp(-\lambda x) / \exp(-\lambda x) = \lambda$ .

Our next preliminary point concerns the addition (superposition) of counting processes. Let  $N_1(\tau)$  and  $N_2(\tau)$  be counting processes. Then their superposition  $N_S(\tau) = N_1(\tau) + N_2(\tau)$  is a counting process, and if  $\lambda_1(t|\mathcal{F}_{1,t-})$  does not depend on  $\mathcal{F}_{2,t-}$  and vice versa (independence), then  $N_S(\tau)$  has intensity function  $\lambda_1(t|\mathcal{F}_{1,t-}) + \lambda_2(t|\mathcal{F}_{2,t-})$ .

## 3. Virtual series-system models

### 3.1. Partially renewing process

We describe two models. In the first, it is always the oldest component in the virtual system that is replaced on repair. In the second model, notionally, it is the youngest.

The virtual system (VS) is 1-out-of- $m$  system with  $m$  identical and independent **virtual** components. Each **virtual** component corresponds to a counting process. **We do not attempt to make a correspondence between these  $m$  virtual components of the virtual system and  $m$  components of a real, multi-component system. Thus, the virtual system idea is conceptual. We use the term *virtual* to qualify *component* and *system* deliberately, in a way that is analogous to using the term *virtual* in the virtual age models of imperfect repair.**

**The 1-out-of- $m$  system structure implies that** the virtual system is the superposition of these  $m$  independent, identical counting processes. Suppose events (repairs) have occurred at times  $t_1, t_2, \dots, t_{i-1}, \dots$  ( $i > m$ ). Let  $h_C(\cdot)$  be a hazard function.



*Proposition 1. The function*

$$\lambda_S(t) = \begin{cases} mh_C(t), & t < t_1, \\ \sum_{j=1}^{i-1} h_C(t-t_j) + (m-i+1)h_C(t), & t_{i-1} \leq t < t_i, \quad 1 < i \leq m, \\ \sum_{j=i-m}^{i-1} h_C(t-t_j), & t_{i-1} \leq t < t_i, \quad m < i, \end{cases} \quad (1)$$

*is the intensity function of a counting process.*

*Proof.* For all  $t > 0$ ,  $\Lambda_S(t) = \int_0^t \lambda_S(u) du$  is defined, positive and increasing in  $t$  because  $h_C(t)$  is defined and positive because it is a hazard function. *End of proof.*

The intensity function  $\lambda_S(t)$  characterises the virtual system, but the proof is not instructive, so we make some remarks. The construction of  $\lambda_S(t)$  is such that before the first repair (event) ( $t < t_1$ ) the  $m$  virtual components are each aged  $t$  with identical hazards  $h_C(t)$ . After the first repair, and at a time when the number of virtual components is at least as big as the number of repairs (second line on right hand side), then some virtual components have been replaced (renewed) and some have not. The latter collectively contribute  $(m-i+1)h_C(t)$  to the system intensity, the former  $h_C(t-t_1) + h_C(t-t_2) + \dots$ . When the number of repairs exceeds the number of virtual components ( $m < i$ ), the ages of the  $m$  virtual components (arguments of the hazards) correspond to the times since the  $m$  most recent repairs. In this way, we interpret the model as one in which the oldest component in the virtual system is replaced at a repair.

At steady-state ( $t \rightarrow \infty$ ), if  $h_C(t)$  is increasing with  $t$  (IFR) (Barlow et al., 1996), then  $\lambda_S(t) < mh_C(x_m)$ , where  $x_m$  is the time since the  $m$ th most recent repair, and so the counting process is stationary (in the sense that  $\lambda_S(t)$  never increases without limit). We might call this a *partially renewing process*. The size of  $m$  determines the extent to which the system is partially renewed at each repair, and the system is always somewhat better than old on repair. If  $h_C(t)$  is decreasing with  $t$  (DFR), then again the counting process is stationary at steady-state, but here on repair the system will tend to be worse than old.

### 3.2. Superposition of a renewal process and a Poisson process

Consider now the notion that the youngest component is replaced on repair. Then the same component is always replaced on repair. The next proposition describes the system intensity function.

*Proposition 2. Let  $t = t_0 = 0$ . Then the function*

$$\lambda_S(t) = h_C(t-t_{i-1}) + (m-1)h_C(t), \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \dots, \quad (2)$$

*is the intensity function of a counting process.*

*Proof.* For all  $t > 0$ ,  $\Lambda_S(t) = \int_0^t \lambda_S(u) du$  is defined, positive and increasing in  $t$  because  $h_C(t)$  is defined and positive because it is a hazard function. *End of proof.*

Here the same virtual component is replaced at every repair; this is the renewing hazard  $h_C(t-t_{i-1})$ . The other  $m-1$  virtual components have ages that are unchanged on repair; this is the non-renewing hazard  $(m-1)h_C(t)$ .

If  $h_C(\cdot)$  is a decreasing function (DFR), so that a repair returns the system to a state that is “worse than old”, then  $\lambda_S(t) \rightarrow h_C(x) + \lambda$  as  $t \rightarrow \infty$  ( $\lim_{t \rightarrow \infty} h_C(t) = \lambda \geq 0$ ), where  $x$  is the time

since last repair, and so the counting process is a renewal process in the limit (at steady-state). If  $\lambda > 0$ , then the limiting process with intensity (2) is the superposition of a renewal process (RP) and a homogeneous Poisson process (HPP) with intensity  $\lambda$ . When  $h_C(\cdot)$  is an increasing function (IFR), if it increases without limit ( $h_C(t) \rightarrow \infty$  as  $t \rightarrow \infty$ ), then  $\lambda_S(t)$  increases without limit and the process is non-stationary. If  $h_C(t) \rightarrow \lambda$  as  $t \rightarrow \infty$ , then  $\lambda_S(t) \rightarrow h_C(x) + \lambda$ , as  $t \rightarrow \infty$ , and again at steady-state the virtual system process is the superposition of a renewal process and a homogeneous Poisson process. This then suggests a modification to the model in which **we set**

$$\lambda_{S,2}(t) = h_C(t - t_{i-1}) + \lambda_R(t), \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \dots, \quad (3)$$

corresponding to a virtual system comprising a virtual, renewing component with hazard  $h_C(\cdot)$  and the remainder of the system with intensity function  $\lambda_R(t)$ . This model is introduced as a failure model in Wu and Scarf (2017), wherein when the component fails it is replaced, and when the remainder fails, the component is replaced and the remainder is minimally repaired. The virtual, renewing component is called the virtual component (VC). The remainder, which is minimally repaired, is called the virtual sub-system (VSS). The VC and the VSS form the virtual system (VS). We shall use these terms in this paper. **The model (3) is a superposition of a renewal process (RP) and a nonhomogeneous Poisson process (NHPP).**

In the special case of (3) with

$$\lambda_{S,1}(t) = h_C(t - t_{i-1}) + \lambda, \quad t_{i-1} \leq t < t_i, \quad i = 1, 2, \dots, \quad (4)$$

the VC is a renewal process (RP) and the VSS is a homogeneous Poisson process (HPP), and so the VS is an RP—the superposition of an RP and an HPP is another RP. The partially renewing process (1) is a simplification of a second model introduced in Wu and Scarf (2017).

For the model  $\lambda_{S,2}(t)$ , equation (3), if  $h_C(t)$  is DFR, then repairs return the system to a state that is worse than just before repair (worse than old). If  $h_C(t)$  is IFR, then repairs return the system to a state that is better than just before repair (better than old but not as good as new). The repair effect—the extent to which repair improves (or worsens) the system—depends on  $h_C(t)$ . If  $\lambda_R(t)$  is large, repairs are frequent but the reduction (if  $h_C(t)$  is IFR) in intensity at each repair will be small. When  $\lambda_R(t)$  is small, repairs are infrequent but the reduction (if  $h_C(t)$  is IFR) in intensity at each repair will be large.

### 3.3. Further remarks

The partially renewing process (1) might be generalized so that  $r$  from  $m$  **virtual** components are renewed on failure, although data are unlikely to distinguish a “1-from-3” model from a “2-from-6” model. Nonetheless the hazard function can be formulated in this case.

Replacement of the youngest **virtual** component always on repair when  $h_C(\cdot)$  is DFR may be an interesting model. Software reliability might follow such a process. On release (repair), a new version has new code that is the least reliable part of the software and the remainder of the code **has been** minimally repaired (debugged).

When  $h_C(\cdot) = h$  (a constant, non-time-dependent), then  $\lambda_S(t)$  in (1) is a constant intensity and the repair process is a homogeneous Poisson process (HPP). When  $h_C(\cdot) = h$ ,  $\lambda_{S,2}(t)$  in (3) is the intensity of a superposition of an HPP and an NHPP, which is an NHPP. This suggests that  $\lambda_{S,2}(t)$  provides a more flexible model than  $\lambda_S(t)$  in (1). Also, the latter partially renewing process is likely to be difficult to distinguish from an HPP (at estimation) because of its stationarity. Further, the

non-stationarity of (2)—its ageing behavior—is linked to the hazard of the VC. This essentially makes the model considerably less flexible than (3).

So, in the remainder of the paper, we focus the model  $\lambda_{S,2}(t)$ , **equation (3)**, and its special case  $\lambda_{S,1}(t)$ , equation (4).

### 3.4. Parameterization

When  $h_C(\cdot)$  is specified with two parameters (e.g. Weibull hazard  $h(t) = (\beta_1 / \alpha_1)(t / \alpha_1)^{\beta_1-1}$  (Lawless, 1983),  $\lambda_{S,1}(t)$  (equation 4) has three parameters.

A natural parameterization for remainder of the system (the non-renewing virtual sub-system, VSS) is  $\lambda_R(t) = (\beta_2 / \alpha_2)(t / \alpha_2)^{\beta_2-1}$ , which corresponds to the power-law non-homogeneous Poisson process (e.g. Rigdon and Asit, 1989), and can model degradation of e.g. mechanical systems ( $\beta_2 > 1$ ) and improvement of e.g. software systems ( $\beta_2 < 1$ ). Then  $\lambda_{S,2}(t)$  (equation 3) has four parameters. When  $\lambda_R(t) = 0$  for all  $t$ , then the model reduces to a renewal process (RP) (2 parameters)

Notice that intensity (2) can model an ageing VS with three parameters ( $m$  plus two for the RP).

### 3.5. Mean time to next repair and mean time between repairs

**In this sub-section we derive the mean time to next repair of the virtual system, and, when it exists, the mean time between repairs.**

First consider the model with intensity function  $\lambda_{S,2}(t)$ , **equation (3)**. Given a repair at time  $t^*$ , the time to next repair  $Z = \min(X, Y)$ , where  $X$  is the random variable with hazard function  $h_C(z)$  and  $Y$  is the random variable with hazard function  $\lambda_R(t^* + z)$ . As an aside, we can see that the model is related to a competing risks model (e.g. Crowder, 2012).

Then, the mean time to the next repair,  $E\{\min(X, Y)\}$  can be obtained as follows. Denote the probability density and reliability (survival) functions of  $X$  by  $f_C(x)$  and  $S_C(x)$  and the probability density and reliability functions of  $Y$  by  $f_{R,t^*}(y)$  and  $S_{R,t^*}(y)$ . Then the probability density function of  $Z = \min(X, Y)$  is given by

$$f_Z(z) = f_C(z)S_{R,t^*}(z) + f_{R,t^*}(z)S_C(z). \quad (5)$$

This is because  $\Pr\{\min(X, Y) \leq z\} = 1 - \Pr(X > z)\Pr(Y > z)$  and differentiating both sides with respect to  $z$  gives (5). Then, given a repair at  $t^*$ , the mean time to the next repair is

$$E\{\min(X, Y)\} = \int_0^\infty z\{f_C(z)S_{R,t^*}(z) + f_{R,t^*}(z)S_C(z)\}dz.$$

When the VSS is HPP,  $Y \sim \text{Ex}(\lambda)$  (exponentially distributed) and  $\lambda$  does not depend on  $t^*$ , **and so the mean time between repairs (MTBR) exists and is given by is a constant:**

$$\text{MTBR} = E\{\min(X, Y_{\text{exp}})\} = \int_0^\infty z\{f_C(z) + \lambda S_C(z)\}\exp(-\lambda z)dz. \quad (6)$$

Finally, if  $X = T$  is a constant and  $\lambda_R(t) = \lambda$ , then  $Z = \min(T, Y_{\text{exp}})$  and the repair process looks like that arising from an age based replacement policy (Barlow and Hunter, 1960) that is applied to a non-ageing system.

### 3.6. Parameter estimation

*Proposition 4.* Suppose **that** the repair times are  $t_1, \dots, t_n$ , the virtual system is new (all virtual components are new) at  $t = t_0 = 0$ , the virtual system is observed over  $[0, t_{n+1}]$ , and the intensity function  $\lambda_{S,2}(t; \theta)$  is parameterized by  $\theta$ . Then the log-likelihood function for  $\theta$  is given by

$$l(\theta, t_1, \dots, t_n) = \sum_{k=1}^n \log\{h_C(t_k - t_{k-1}) + \lambda_R(t_k)\} - \sum_{k=1}^{n+1} \int_0^{t_k - t_{k-1}} h_C(t) dt - \int_0^{t_{n+1}} \lambda_R(t) dt. \quad (7)$$

*Proof.* The likelihood function for a counting process with intensity  $\gamma(t)$ , and failure times as defined, is given by  $\prod_{i=1}^n \gamma(t_i) \exp\{-\int_0^{t_{n+1}} \gamma(t) dt\}$  in general (Andersen et al., 1993). Hence (7) follows on substitution of  $\lambda_{S,2}(t)$ . *End of proof.*

Notice that when  $\lambda_R(t) = \lambda$  (RP+HPP model with intensity  $\lambda_{S,1}(t)$ ), the log-likelihood in (7) simplifies to

$$l(\theta, t_1, \dots, t_n) = \sum_{k=1}^n \log\{h_C(t_k - t_{k-1})\} + n\lambda - \sum_{k=1}^{n+1} \int_0^{t_k - t_{k-1}} h_C(t) dt - \lambda t_{n+1}. \quad (8)$$

Setting  $\lambda = 0$  gives the likelihood for the RP. When  $h_C(\cdot) = 0$ , so the model is the HPP, **expression** (8) reduces to  $l = n \log \lambda - \lambda t_n$ . Equivalently,  $h_C(\cdot) = h$  (constant) and  $\lambda = 0$  gives the same model (HPP).

Competing models can be compared using AIC (Akaike, 2011). Model fit can be checked by plotting the cumulative number of repairs and the fitted cumulative intensity function, and by comparing the MTBR with that implied under the fitted model, **provided it exists**.

## 4. Model fitting

### 4.1. Classic datasets

Now we fit models with intensities  $\lambda_{S,1}(t)$  and  $\lambda_{S,2}(t)$  to well-known datasets and compare with HPP and RP in each case, on basis of AIC. We cite the originator of the dataset in each case. The datasets are given in Table 1, in order of analysis: photocopier (Baker, 1996), Halfbeak (Ascher and Feingold, 1984), Grampus (Lee, 1980), 180t dump-truck (Coetzee, 1996), air-conditioners 7909, 7912, 7913, 7914 (Proshan, 1963). These datasets have been analyzed by many (e.g. Akman and Raftery, 1986; Baker, 1991; Vaurio, 1999; Phillips, 2000, 2001; Pulcini, 2001; Hong and Pai, 2006; Caroni, 2010; Fuqing and Kumar U, 2012; Gámiz and Lindqvist, 2016; Doyen et al., 2017; Syamsundar and Kumar, 2017), and feature in classic analyses in text books (e.g. Crowder et al., 1991; Meeker and Escobar, 1998; Blischke and Murthy, 2000).

### 4.2. Specification of the hazard functions

We now specify the hazard functions of the VC and the VSS. We use a Weibull hazard for both the VC, so that  $h_C(t) = (\beta_1 / \alpha_1)(t / \alpha_1)^{\beta_1 - 1}$ , and the VSS, so that  $\lambda_R(t) = (\beta_2 / \alpha_2)(t / \alpha_2)^{\beta_2 - 1}$ . In this way the lifetime of the VC (which is always renewed on repair) is Weibull distributed, and failures of the VSS, which are minimally repaired, arise according to the power-law non-homogeneous Poisson process. The full model, which we denote “RP+NHPP”, has 4 parameters.

The reduced model with  $\beta_2 = 1$ , so that  $\lambda_R(t) = \lambda = 1 / \alpha_2$  (constant), which we denote “RP+HPP”, has three parameters and **its** mean time between repairs (MTBR) is



$$\text{MTBR} = \int_0^\infty z \left\{ \frac{\beta_1}{\alpha_1} \left( \frac{z}{\alpha_1} \right)^{\beta_1-1} + \frac{1}{\alpha_2} \right\} \exp \left\{ - \left( \frac{z}{\alpha_1} \right)^{\beta_1} - \frac{z}{\alpha_2} \right\} dz. \quad (9)$$

When the virtual system is just the VC ( $\lambda = 1/\alpha_2 = 0$ ), the model is a renewal process, has 2 parameters, and the MTBR is the mean of a Weibull distribution:  $\alpha_1 \Gamma(1 + 1/\beta_1)$ .

When the virtual system is just the VSS (no renewing component) and the VSS is HPP, the model has one parameter ( $\alpha_2$ ) and the MTBR is  $\alpha_2$ .

Suppose the VS is subject to the maintenance policy and the VSS is HPP: on failure of the VSS, minimally repair the VSS and replace the VC; and when the VC reaches age  $\alpha_1$ , then replace the VC. Then the MBTR is  $E\{\min(\alpha_1, Y)\}$ , where  $Y \sim \text{Ex}(\lambda)$ . Thus

$$\text{MTBR} = \int_0^{\alpha_1} \exp\{-z/\alpha_2\} dz.$$

Table 1. Times of repairs; photocopier in days; others in hours

photocopier			Halfbeak		Grampus		180t dump-truck				7909	7912	7913	7914
7	411	768	1382	21603	860	11511	78	13265	17968	20432	90	23	97	50
8	419	831	2990	21658	1258	11575	158	13508	17984	20433	100	284	148	94
9	461	868	4124	21688	1317	12100	331	13673	18175	20434	160	371	159	196
58	470	875	6827	21750	1442	12126	381	13780	18443	20698	346	378	163	268
84	475	925	7472	21815	1897	12368	523	14443	18458	21460	407	498	304	290
86	482	937	7567	21820	2011	12681	620	14501	18667	21543	456	512	322	329
98	505	940	8845	21822	2122	12795	664	14656	18669	21584	470	574	464	332
104	509	943	9450	21888	2439	13399	1805	14906	18701	21602	494	621	532	347
104	527	946	9794	21930	3203	13668	1817	14983	18723	21645	550	846	609	544
112	533	946	10848	21943	3298	13780	2068	15004	18822	21706	570	917	689	732
113	552	952	11993	21946	3902	13877	3253	15062	18860	21762	649	1163	690	811
119	555	954	12300	22181	3910	14007	4489	15072	18922	21867	733	1184	706	899
121	561	957	15413	22311	4000	14028	4725	15136	18935	21912	777	1226	812	945
127	561	993	16497	22634	4247	14035	4961	15206	18945	21914	836	1246	1018	950
127	575	1013	17352	22635	4411	14173	5138	15247	18960	21937	865	1251	1100	955
194	587	1077	17632	22669	4456	14173	5200	15700	18961	21938	983	1263	1154	991
195	603	1099	18122	22691	4517	14449	5278	15714	18979	21939	1008	1383	1185	1013
212	622	1108	19067	22846	4899	14587	5711	15972	19013	21951	1164	1394	1401	1152
216	630	1125	19172	22947	4910	14610	6400	16186	19032	21954	1474	1397	1447	1362
229	635	1135	19299	23149	5676	15070	6444	16284	19034	21982	1550	1411	1558	1459
229	639	••••	19360	23305	5755	•••••	6677	16329	19169	•••••	1576	1482	1597	1489
230	646		19686	23491	6137		7999	16425	19184		1620	1493	1660	1512
266	651		19940	23526	6221		8001	16605	19201		1643	1507	1678	1525
267	651		19944	23774	6311		8489	16723	19416		1705	1518	1869	1539
279	673		20121	23791	6613		9000	16731	19455		1835	1534	1887	••••
292	684		20132	23822	6975		9086	16797	19525		2043	1624	2050	
300	692		20431	24006	7335		10262	16859	19595		2113	1625	2074	
301	693		20525	24286	8158		10817	17090	19601		2214	1641	••••	
308	695		21057	25000	8498		11062	17305	19613		2422	1693		
317	698		21061	25010	8690		11082	17484	19643		••••	1788		
324	709		21309	25048	9042		11086	17510	19671			••••		
335	712		21310	25268	9330		11122	17511	19713					
337	714		21378	25400	9394		11534	17536	19785					
352	722		21391	25500	9426		12031	17621	19801					
384	731		21456	25518	9872		12339	17703	19937					
393	742		21461	•••••	10191		12733	17809	19990					

### 4.3. Results

Parameter estimates and AICs are shown in Table 2. Where the RP+NHPP is best model so far, we also fit the NHPP. The best models on the basis of minimum AIC are shaded. Observed cumulative number of failures and fitted cumulative intensity functions (CIF) are shown in Figures 1-5. Where there is some interesting behavior the CIF is shown “zoomed-in”.

Table 2. Maximum likelihood estimates and standard errors, all datasets, and AIC and fitted (model) mean time between repairs (MTBR).

		$\alpha_1$	s.e.	$\beta_1$	s.e.	$\alpha_2$	s.e.	$\beta_2$	s.e.	AIC	Fitted MTBR
<b>photocopier</b> $n = 92$ $MTBR = 12.3$	RP	11.781	1.418	0.914	0.072	-	-	-	-	649.5	12.3
	HPP	-	-	-	-	12.33	1.29	-	-	648.3	12.3
	RP+HPP	66.19	1.23	46.96	29.15	12.62	1.34	-	-	647.4	12.3
	RP+NHPP	0.919	0.10	44.45	24.40	8.49	4.51	66.1	*	648.8	
<b>Halfbeak</b> $n = 71$ $MTBR = 359.4$	RP	251.1	49.9	0.631	0.06	-	-	-	-	949.3	355.3
	HPP	-	-	-	-	359.4	42.68	-	-	979.6	359.4
	RP+HPP	245.7	50.2	0.618	0.06	5.073e6	8388	-	-	935.8	354.2
	RP+NHPP	1599	443	1.499	0.48	1.201e4	2264	5.384	1.25	918.7	
	NHPP	-	-	-	-	5443.3	1010	2.759	0.32	927.9	
<b>Grampus</b> $n = 56$ $MTBR = 269.1$	RP	265.8	38.43	0.971	0.103	-	-	-	-	742.6	269.3
	HPP	-	-	-	-	269.1	36.0	-	-	740.7	269.1
	RP+HPP	1320	19.5	69.00	73.9	273.7	*	-	-	741.2	271.4
	RP+NHPP	1100	1433	0.991	0.703	730.1	305.4	1.24	*	744.8	
<b>180t dump-t</b> $n = 128$ $MTBR = 171.7$	RP	132.4	17.8	0.693	0.047	-	-	-	-	1541.6	169.1
	HPP	-	-	-	-	171.7	15.2	-	-	1575.4	171.7
	RP+HPP	135.2	18.5	0.690	0.047	9783	4212	-	-	1543.7	168.7
	RP+NHPP	333.9	92.8	0.727	0.086	9232	1951	4.743	1.041	1515.1	
	NHPP	-	-	-	-	1004	279	1.572	0.139	1554.7	
<b>7909</b> $n = 29$ $MTBR = 83.5$	RP	90.90	13.83	1.293	0.18	-	-	-	-	315.7	84.0
	HPP	-	-	-	-	83.52	15.51	-	-	316.7	83.5
	RP+HPP	310.1	2.297	124.7	129.4	72	*	-	-	316.5	87.3
	RP+NHPP	99.74	18.23	1.324	0.20	503	*	0.718	0.55	319.6	
<b>7912</b> $n = 30$ $MTBR = 59.6$	RP	54.62	12.37	0.853	0.12	-	-	-	-	307.8	59.3
	HPP	-	-	-	-	59.60	10.88	-	-	307.3	59.6
	RP+HPP	261.0	0.002	1.357e05	*	57.45	10.0	-	-	292.7	56.8
	RP+NHPP	261	0.000	1.000e06	*	266.6	99.3	1.745	0.32	285.3	
	NHPP	-	-	-	-	186.7	80.3	1.505	0.28	304.9	
<b>7913</b> $n = 27$ $MTBR = 76.8$	RP	79.91	14.37	1.123	0.17	-	-	-	-	291.9	76.6
	HPP	-	-	-	-	76.83	14.79	-	-	290.4	76.8
	RP+HPP	213.7	5.14	46.99	60.6	82.40	16.8	-	-	288.3	76.0
	RP+NHPP	213.6	42.5	45.10	0.2	70.00	*	0.952	49.6	290.2	
<b>7914</b> $n = 24$ $MTBR = 64.1$	RP	64.80	13.63	1.025	0.16	-	-	-	-	251.7	64.1
	HPP	-	-	-	-	64.14	13.10	-	-	249.7	64.1
	RP+HPP	210.0	0.002	1.229e5	1.68e04	69.70	15.2	-	-	234.7	66.3
	RP+NHPP	210.0	0.002	7.318e6	*	143.0	50.4	1.200	0.22	225.9	
	NHPP	-	-	-	-	82.91	51.82	1.088	0.22	251.6	

\* standard error not estimable

We now comment on the fitted models and parameter estimates, taking each dataset in turn.

For the photocopier data, we might select RP+HPP as the best model here, and Figure 1 shows the cumulative number of failures against time and the fitted cumulative intensity function for this model. However, for the RP+HPP,  $\hat{\beta}_1$  is very large, and so  $\hat{\alpha}_1$  is somewhat close to the boundary of the parameter space,  $\max_{i=1,\dots,n}(x_i)$ , the largest inter-failure time. We can see this inter-failure time in Figure 1b, and the rapidly increasing hazard associated with it. This dataset is also complicated by frequent zero-valued inter-failure times, most likely due to imprecision in the timing of failure. We assumed the inter-failure times were 0.5 (half a day) in this case. So, some very short

inter-failure times, which suggest hyper-exponentiality ( $\hat{\beta}_1 < 1$  in RP), and an absence of very long inter-failure times, which would account for the very large  $\hat{\beta}_1$  in the RP+HPP, make drawing conclusions from these results difficult.

The results for the Halfbeak data are clearer cut. The AIC and the standard errors of the best fitting model support this claim. The RP+NHPP is the best model, suggesting that the system comprises a renewing part that is IFR ( $\hat{\beta}_1 = 1.5$ ) and remainder that is ageing ( $\hat{\beta}_2 = 5.4$ ) and rapidly so in later life (Figure 2a). The renewing IFR component is apparent in the plotted cumulative intensity (Figure 2b).

For the Grampus, failures appear completely random and the system is not ageing (HPP). This is confirmed to an extent by  $\hat{\beta}_1 \approx 1$  in the RP model. Here the MLE procedure for RP+HPP is encountering difficulty (very flat likelihood function) because this model is over-parameterized in these circumstances. Essentially, when the data arise from a Poisson process, parameters in a model that is a superposition of processes will be difficult to resolve.

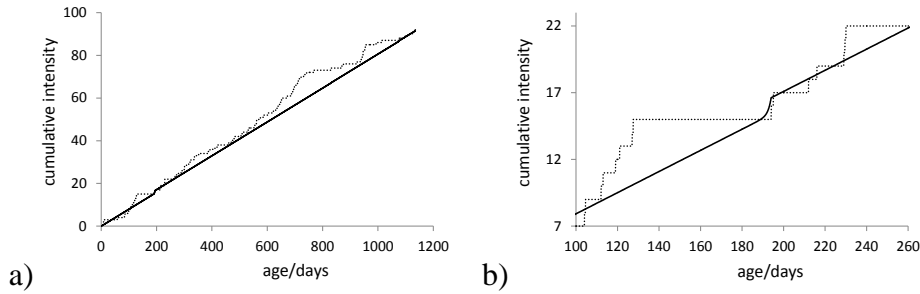


Figure 1. Photocopier repair data: cumulative number of repairs (dotted line) and fitted (RP+HPP) cumulative intensity function (solid line). a) all data; b) zoomed in.

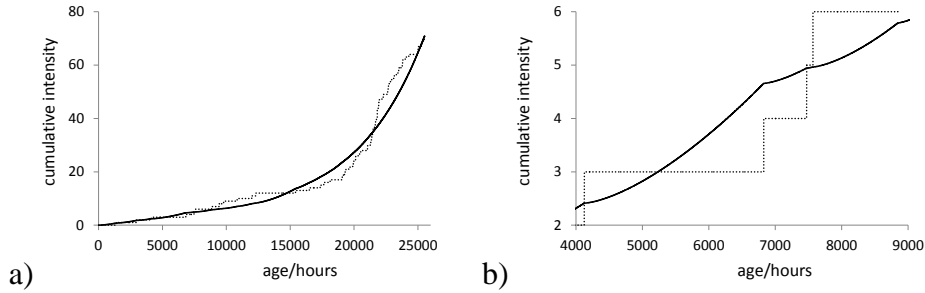


Figure 2. Halfbeak repair data: cumulative number of repairs (dotted line) and fitted (RP+NHPP) cumulative intensity function (solid line). a) all data; b) zoomed in.

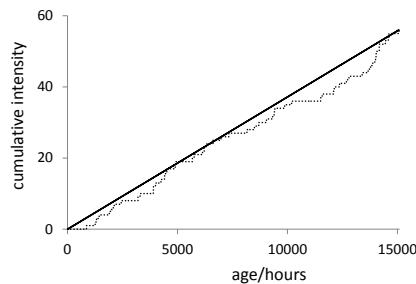


Figure 3. Grampus repair data: cumulative number of repairs (dotted line) and fitted (HPP) cumulative intensity function (solid line).

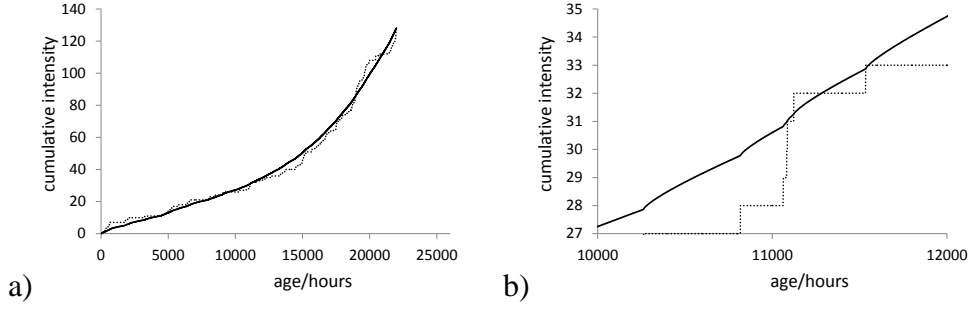


Figure 4. 180t dump-truck repair data: cumulative number of repairs (dotted line) and fitted (RP+NHPP) cumulative intensity function (solid line). a) all data; b) zoomed in.

The results for the 180t dump-truck are clear, like the Halfbeak. However, here the renewing part is DFR ( $\hat{\beta}_1 = 0.69$ ). This is apparent in Figure 4b, so that the system is at highest risk of failure immediately following a repair. The remainder of the system is also clearly ageing (Figure 4a).

The air-conditioner datasets we discuss collectively (Figure 5). On the whole these datasets are much smaller than the others, and therefore our conclusions should be more cautious. 7912 and 7914 show effects that are similar to the photocopier data:  $\hat{\beta}_1$  very large for the minimum AIC model in each case, suggesting irregularity in the estimation (e.g. Cheng and Traylor, 1995). 7909 (an RP which is IFR,  $\hat{\beta}_1 = 1.3$ ) is straightforward. The results for 7914 (the smallest dataset) for the models that are superpositions are suspect, and so in these circumstances we select the HPP (purely random repairs) as the best model.

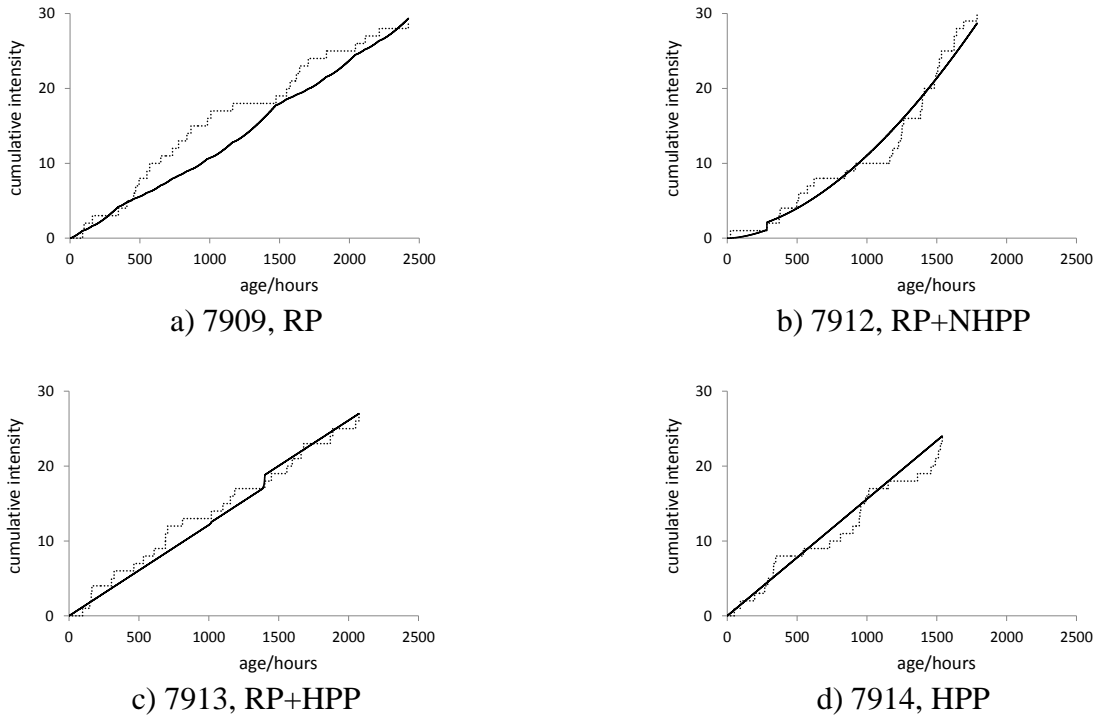


Figure 5. Air-conditioner repair data: cumulative number of repairs (dotted line) and fitted cumulative intensity function (solid line).

Throughout these analyses we use the mean time between repairs (MTBR) as a check on model fit, while noting that a fitted MTBR does not exist for a non-stationary model (NHPP or

RP+NHPP). We can see that when estimation is irregular there is a tendency for the fitted MTBR to disagree with the observed MTBR.

Overall, we might conclude that 30 failures are too few to fit models with more than two parameters. The RP+HPP has particular difficulties in this regard because it is stationary. Where the RP+NHPP fits well (Halfbeak and 180t dump-truck), the NHPP alone does not fit well. This supports the case for the usefulness of the models developed in this paper.

## 5. Maintenance policy

As we have so far not distinguished, both in the repair process modelling and the data fitting, preventive repairs from corrective repairs, we frame a maintenance policy in two ways: i) using a marginal-cost approach, and ii) as an age-based replacement policy (Barlow and Hunter, 1960).

### 5.1. Marginal cost policy

We consider a maintenance policy first in the context of the stationary model (4). Let us suppose that  $h_C(\cdot)$  is IFR. Then it is natural to subject the VS to age-based repair: that is, to *preventively* repair the VS (and hence the VC) when the time since last repair reaches some control limit,  $T$ , say. We call this the  $T$ -policy. Then we will notionally regard the efficacy of age-based repair as the repair effect. We use the notation  $Z_T$  for the time between repairs under the  $T$ -policy, and note that  $Z_T$  does not depend on the time of the last repair (because the VS is an RP).  $X$  is the lifetime of the VC (with hazard function  $h_C(x)$ ).  $Y \sim \text{Ex}(\lambda)$  is the lifetime of the VSS. Then

$$Z_T = \min(T, X, Y).$$

*Definitions.* (i) The marginal cost-rate  $Q_T$  of the  $T$ -policy is  $c_p \phi_p(T)$  where  $c_p$  is the cost of an additional, preventive repair and  $\phi_p(T)$  is the rate of occurrence of preventive repairs (preventive-repair-rate) under the  $T$ -policy. (ii) The preventive-repair effect is  $\delta = 100 \times \phi_p(T) / \phi$ , where  $\phi$  is the rate of occurrence of (all) repairs (repair-rate).

The justification of these is as follows. A repair is either preventive or not preventive. Let us for this explanation call the latter event a failure. Then the preventive-repair effect is the percentage of failures prevented by preventive repair. If  $T$  is small, most repairs will be preventive and so  $\phi_p(T) / \phi$  will be close to 1, and the preventive-repair effect will be nearly 100%. To counterbalance this, the marginal cost of the small  $T$ -policy will be high.

*Proposition 4.* Under the  $T$ -policy, (i)  $\phi_p(T) = \Pr\{\min(X, Y) > T\} / E\{\min(T, X, Y)\}$  and (ii)  $\phi = 1 / E\{\min(T, X, Y)\}$ .

*Corollaries.*  $\delta = 100 \times \Pr\{\min(X, Y) > T\}$  and  $\delta$  decreases as  $T$  increases.

*Proofs.* The corollaries follow immediately from the definitions and Proposition 4. Proposition 4(i) is true because the  $T$ -policy defines a renewal policy with renewal at the time (since last renewal) of the preventive repair,  $T$ , or “failure” of the VC,  $X$ , or “failure” of the VSS,  $Y$ , whichever occurs soonest, that is, at  $Z_T = \min(T, X, Y)$ . Therefore, the expected cycle length is  $E\{Z_T\}$  and the probability that a renewal cycle ends with preventive-repair is  $\Pr\{\min(X, Y) > T\}$ . Defining  $N_p(\tau; T)$  as the number of preventive repairs in  $[0, \tau]$  under the  $T$ -policy, it follows that

$$\phi_p(T) = \lim_{\tau \rightarrow \infty} \frac{N_p(\tau; T)}{\tau} = \frac{\Pr\{\min(X, Y) > T\}}{E\{Z_T\}}, \quad (11)$$



by the renewal-reward theorem. Proposition 4(ii) follows by a similar argument in respect the number of repairs in  $[0, \tau]$  under the  $T$ -policy,  $N(\tau; T)$ . *End of Proof.*

For calculation of the terms in (11), we note that

$$\Pr\{\min(X, Y) > T\} = \Pr(X > T) \Pr(Y > T) = S_X(T) S_Y(T), \quad (12)$$

and

$$\begin{aligned} E\{Z_T\} &= E\{\min(T, X, Y)\} = \int_0^T \{1 - F_{\min(X, Y)}(u)\} du \\ &= \int_0^T \Pr\{\min(X, Y) > u\} du \\ &= \int_0^T S_X(u) S_Y(u) du. \end{aligned} \quad (13)$$

### 5.2. Cost-rate policy

An alternative to the marginal cost approach is a total-cost approach. This may be used if the cost of preventive-repair,  $c_P$ , and the cost of corrective repair,  $c_F$ , are specified. Then, because the VS is an RP, the optimum  $T$ -policy is that which minimizes the long-run cost per unit time (cost-rate)

$$Q_{ABR}(T) = \frac{c_P \Pr\{\min(X, Y) > T\} + c_F \Pr\{\min(X, Y) \leq T\}}{E\{Z_T\}}, \quad (14)$$

and there exists an optimum policy, the  $T^*$ -policy, provided  $h_C(\cdot)$  is IFR and  $c_P < c_F$ . This is just the age-based replacement policy when the random variable that describes the time to failure is  $\min(X, Y)$ .

Next we comment on the case when the VC is RP+NHPP (non-stationary). Our suggestion is to suppose that at some time  $t'$ , at which the maintenance policy is being planned,  $\lambda_R(t)$  is fixed at least ~~as far as~~ **until** the next preventive-repair (quasi-stationary), and then to use either of the approaches above, planning preventive repairs one step at a time. In this way, effectively, a varying  $\lambda_R(t)$  is approximated by a step function (but is not fitted to data as such).

### 5.3. Results

We now implement this idea for the Halfbeak because RP+NHPP is the best model for this system and in this model  $\beta_1 > 1$ , so that preventive repair has a positive benefit (the system is better-than-old after repair). Note, when  $\beta_1 < 1$ , preventive repair has a negative benefit (the system is worse-than-old after repair).

For the Halfbeak,  $X \sim \text{We}(\alpha_1, \beta_1)$  so that  $S_X(x) = \exp(-(x/\alpha_1)^{\beta_1})$ , and  $Y \sim \text{Ex}(\lambda_R)$  so that  $S_Y(y) = \exp(-\lambda_R y)$ . Under the quasi-stationary idea, we set  $\lambda_R = (\beta_2/\alpha_2)(t'/\alpha_2)^{\beta_2-1}$ . From (12) and (13) we then obtain that at time  $t'$

$$\Pr\{\min(X, Y) > T\} = \exp\{-(T/\alpha_1)^{\beta_1} - \lambda_R T\} = \exp\{-(T/\alpha_1)^{\beta_1} - T(\beta_2/\alpha_2)(t'/\alpha_2)^{\beta_2-1}\}$$

and

$$E\{Z_T\} = \int_0^T \exp\{-(u/\alpha_1)^{\beta_1} - u(\beta_2/\alpha_2)(t'/\alpha_2)^{\beta_2-1}\} du.$$

The parameters  $\alpha_1, \alpha_2, \beta_1, \beta_2$  are set at their maximum likelihood estimates. We suppose that  $c_P = 1$  and  $c_F = 5$ .

In Figure 6a we present the preventive repair effect  $\delta$ , as we define it (Definition (ii) above), versus  $T$  under the notion of quasi-stationarity. Thus we consider, hypothetically, what would be appropriate preventive-repair intervals when the system is aged  $t'$  hours, that is, at the time at which the system has been operating for  $t'$  hours, for  $t'=10000$ ,  $t'=20000$ , and  $t'=30000$ . These “ages” correspond roughly to early life, mature life, and beyond-useful operating life. The corresponding marginal cost-rates (assuming a unit cost for a preventive repair) are shown in Figure 6b. Figure 7 presents the total cost-rate, equation (14). Separate graphs are required here because the total cost-rates for the various  $t'$  are on very different scales. The total cost-rate at the optimum policy increases 5-fold at each age-step.

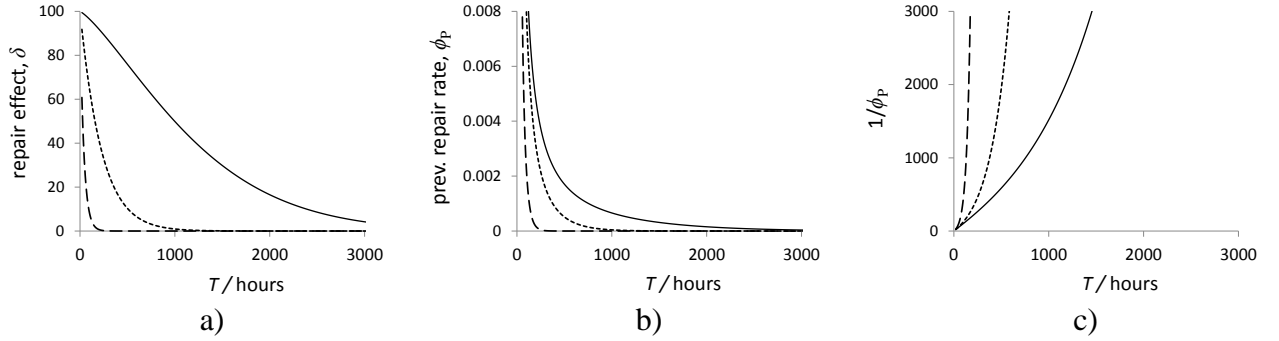


Figure 6. Halfbeak: a) repair effect  $\delta$  versus  $T$ , b) preventive-repair rate  $\phi_p(T)$  versus  $T$ , c) mean time between preventive repairs  $1/\phi_p(T)$  versus  $T$ , under quasi-stationarity for  $t'=10000$  (solid line),  $t'=20000$  (dotted line),  $t'=30000$  (dashed line).

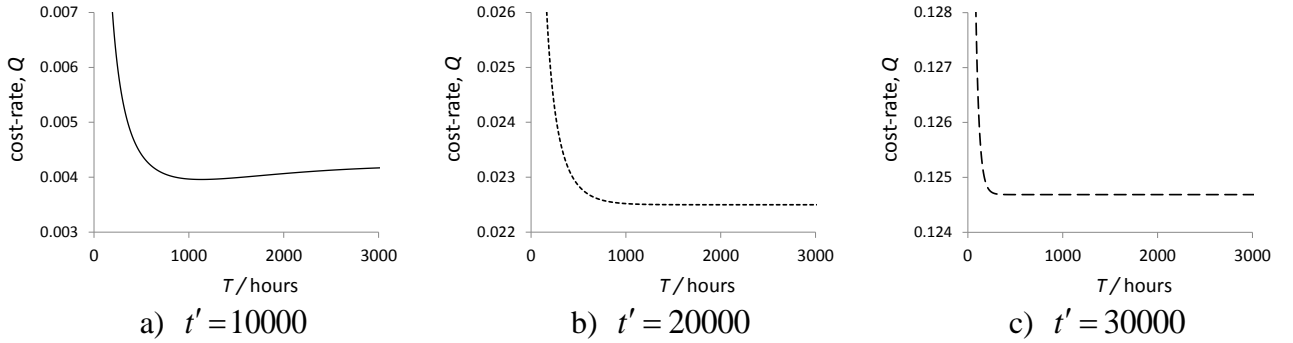


Figure 7. Halfbeak: cost-rate  $Q_{ABR}(T)$  versus  $T$  under a quasi-stationarity for a)  $t'=10000$  (solid line), b)  $t'=20000$  (dotted line); c)  $t'=30000$  (dashed line).

In Figure 6a) we can observe that, in the early life of the system, preventive-repair (in which the part that corresponds to the VC is renewed) at a 1000 operating-hours interval is approximately 50% effective. When the ratio of the cost-of-failure to cost of preventive-repair is 5, the cost-rate is minimized at this interval (Figure 7a). In later life, because the part of the system that is not affected by preventive-repair is failing more frequently, preventive repair is now less effective (<20% at  $T=500$  hours) and preventive-repair is cost-inefficient (Figure 7b). Projecting beyond the actual lifetime of the system, we can see (Figure 6a and Figure 7c) that preventive maintenance is now hopeless in preventing failure. Figure 6c) confirms this analysis. When  $t'=10000$ , for small  $T$

(<1000 hours) nearly all repairs are preventive repair (noting that if  $1/\phi_p(T) = T$  then every repair is a preventive repair). For  $t' = 30000$ , a  $T$ -policy even for a very small  $T$ —which would have a very large cost—is almost completely ineffective.

#### 5.4. *Implications for practice*

It is understood that in reality repair cannot make a system like-new, and industry has sought to quantify the effect of repair on a system. The models studied in this paper demonstrate partial-renewal, how the repair-effect can be quantified, and that repairs become less effective as a system ages, to a point where scrappage or complete overhaul is the only viable option. The models also quantify these notions, albeit once the system has been under observation for an extended period of operation. One might argue then this methodology is only useful for obsolete systems, and therefore never useful. We counter argue that the models will provide an environment (the mathematical world) in which to explore the behavior of new systems, for which data are not yet available, that are similar to those studied here. This paper then quantifies not only age-related partial-renewal but also shows—by considering a number of different systems—that systems differ in their ageing rates and in their balance of preventable to unpreventable failures and that maintenance needs to be adapted accordingly. The paper also indicates the typical quantity of objective data that is required for useful conclusions about system behavior to be drawn.

### 6. Conclusions

This paper describes some models of imperfect repair that are alternatives to age-reduction, hazard-reduction, and geometric-type processes. The models use the concept of a virtual series system in which a virtual component is renewed whenever the system is repaired. Repairs may occur preventively or on failure of the virtual component or on failure of the remainder of the system. The system failure process is the superposition of the failure process of the virtual component and the failure process of the remainder. When the remainder fails it is minimally repaired (and the virtual component is replaced). This then models the notion that routine maintenance is carried out at every repair and that this routine maintenance is homogeneous and only marginally effective. The models developed can quantify the extent to which maintenance is effective and therefore be used in inform decision making about how often it should be scheduled.

The failure processes of the virtual component and the remainder of the system are modelled with counting processes, a renewal process for the former and a nonhomogeneous Poisson process for the latter. Two parameters is the minimum specification for each of these, making four in total for the full model. This is considerably less than would be required for a multi-component model. This (parsimonious parameterization) is the nature of the class of models that includes these virtual system models. It lends them to fitting to failure data, and we demonstrate this for a range of classic failure datasets. Therein, we find that reduced models are often best-fitting. Reduced models include the superposition of a renewal process and a homogeneous Poisson process corresponding to a renewing part and a non-ageing part (three parameters), a nonhomogeneous Poisson process corresponding to a system with no renewing part (two parameters), a renewal process corresponding to a renewing system (two parameters), and a homogeneous Poisson process corresponding to a non-ageing system with no renewing part (one parameter). The best-fitting model then indicates which particular maintenance policy is appropriate.

A model with few parameters is important because failures are rare. Indeed extant systems for which the maintenance requirements must be specified may not have failed ever. Nonetheless, engineers can make judgements about whether extant systems are similar in their failure behaviour to legacy systems and use the modelling of legacy systems to inform decisions. Where some failures have been observed this suggests that estimation of the models described in this paper might proceed through a Bayesian approach, whereby fitted models for legacy systems are used to encode prior information about parameters. We will consider the efficacy of such an approach in further development of this work. Another development that can be explored in future is fitting the models to data with right censoring (survivals). The log-likelihood (7) lends itself directly to this. Analysis of model for software reliability would also be interesting.

The implications for practice are as follows. Firstly, these models can be used to estimate repair-effects indirectly and objectively. Secondly, if repair is such that its effect is to renew one ageing part of a system, then more spending on repairs is beneficial, and if a particular benefit (preventive-repair-effect) is desired then its cost may be determined. Alternatively, a cost-minimizing repair policy may be determined provided the cost of preventive-repair is less than the cost of corrective repair and the repairable part is ageing. If the remainder is ageing, then the maintenance policy needs to be adapted as it ages.

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