Imperfect Inspection of a System with Unrevealed Failure and an Unrevealed Defective State

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***Abstract*— This paper proposes a model of inspection of a protection system in which the inspection outcome provides imperfect information of the state of the system. The system itself is required to operate on demand typically in emergency situations. The purpose of inspection is to determine the functional state of the system and consequently whether the system requires replacement. The system state is modeled using the delay time concept in which the failed state is preceded by a defective state. Imperfect inspection is quantified by a set of probabilities that relate the system state to the outcome of the inspection. The paper studies the effect of these probabilities on the efficacy of inspection. The analysis indicates that preventive replacement mitigates low quality inspection and that inspection is cost-effective provided the imperfect-inspection probabilities are not too large. Some derivative policies in which replacement is "postponed" following a positive inspection are also studied. An isolation valve in a utility network motivates the modeling.**

***Index Terms*— Preventive maintenance; replacement; quality of service; protection system; delay-time model**

NOTATION

T, T^*	The inspection interval (a decision variable) and its optimum value
M, M^*	Number of inspections until preventive replacement (a decision variable) and its optimum value
X	System age at defect arrival with *s*-density, *s*-distribution and reliability functions f_X , F_X , $\bar{F}_X$
Y	Delay-time from defect arrival to subsequent failure (time in defective state) with *s*-density, *s*-distribution and reliability functions f_Y , F_Y , $\bar{F}_Y$
G, D, F	System states: good, defective, failed, respectively
P, N	Inspection outcomes: positive, negative
α	Imperfect inspection probability $\Pr(\mathrm{P} \mid \mathrm{G})$
β_1	Imperfect inspection probability $\Pr(\mathrm{N} \mid \mathrm{D})$
β_2	Imperfect inspection probability $\Pr(\mathrm{N} \mid \mathrm{F})$
λ	Mean of exponential delay-time distribution
γ	Characteristic life parameter of Weibull defect arrival distribution
δ	Shape parameter of Weibull defect arrival distribution
c_{I}	cost of an inspection
c_{R}	cost of a replacement
c_{F}	downtime cost-rate
U	cost of a renewal cycle
W	downtime in a renewal cycle
V	length of a renewal cycle
Q	long-run total cost per unit time, cost-rate (objective function)

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I. INTRODUCTION

THIS paper studies a protection or preparedness system subject to imperfect inspection. This system is required to operate on demand typically in emergency situations. Such protection systems include military defense systems, medical equipment (e.g. defibrillators), automobile airbags, isolation valves, fire suppressors and alarms, secondary power supplies, and flood defenses. The Thames barrier [1] is an example of the latter. If this system fails to operate when the water level of the river is predicted to flood London then estimates of the cost of such a failure are tens of billions of pounds. These systems are inspected or tested on a regular basis to determine their functional state. Thus, isolation valves are closed and opened, cold-standby pumps are started, and the Thames barrier is raised. Such "inspections" incur significant costs. Therefore, system owners wish to know how often inspections should be performed and if inspection is effective.

In the proposed model, inspection is imperfect, so that the true functional state of the system cannot be known with certainty. The efficacy of inspection is then suspect, and there may exist circumstances in which inspection is not sufficiently effective to be economically justified. Such imperfect testing has been considered for critical systems (e.g. [2-4]), and for protection systems (e.g. [5,6]). These latter works are extended in this paper by supposing that a protection system is subject to a three-state failure process and inspection is imperfect. In the three-state failure process, a failure is preceded by the defective state and sojourns in the good and defective states are random variables [7,8]. This is the delay-time concept, developed initially by Christer [9], and later extended by many others for protection systems (e.g. [3,6-8,10,11]) and for critical systems (e.g. [12-17]). The sojourn in defective state is the delay-time. For a critical system, failure is self-announcing and the object of inspection is failure prevention. For a protection system, failure is not self-announcing and the object of inspection is to reveal the functional state of the system—that is, to determine whether the protection system will operate in the event of a demand for its function.

Others have extended the delay-time concept to critical systems with minor and major defect states, to model real systems more closely. However, imperfect inspection is modelled in a more restrictive way than we consider here in this paper. In [18,19], the minor-defect state may be missed at an inspection, whereas here is this paper inspection may misclassify both the defective and the failed states, albeit with lower probabilities in the latter case. In [20], inspection is perfect but replacements may be delayed. This is a different idea.

The possibility of the defective state itself can explain inspection errors. For example, an isolation valve (see e.g. [10]) that is either good or failed may be clearly indicated as such on inspection, but one that is defective may be more difficult to correctly classify as operational. This issue also arises in medical screening tests, whereby early disease stages are undetectable and the screening error-rate decreases as the disease develops [21]. Furthermore, degradation may be more likely to be overlooked in its early stages than in more advanced stages. This may be the result of perception of a maintainer that low degradation implies an insignificant risk of failure. Of course, in reality better testing-systems may provide better information about the states of systems and sub-systems. Nonetheless, it is important to study, in an idealized situation (the model), the effect of imperfect inspection upon the efficacy and efficiency of protection systems with a defective state. This can inform maintenance policy and decision making for real systems [22], in order to mitigate the serious consequences of an unmet demand. The approach taken in the paper is related to the notion of quality of maintenance [23], and there is a growing literature concerned with mistakes of perception [24,25], demonstrating increasing concern about human influence on the performance of a system.

The proposed model supposes that the outcome of an inspection provides imperfect information about the true condition (state) of the protection system. The protection system is subject to periodic inspection and the outcome of the inspection determines whether the system is replaced. The cost-rate (long-run total cost per unit time of maintenance and downtime due to failure) and availability of the protection system are determined. The paper then studies the effect of the model parameters on the behavior of these criteria. The paper also proposes a further policy in which the maintainer postpones action (replacement) either until a succession of positive inspections has occurred or for a fixed time period, in order to quantify the consequences of postponement. An isolation valve in a utility network motivates the numerical example that is described.

In the next section, the model of the principal policy is specified and expressions for the cost-rate and the availability are developed. Then the numerical example and study the policy behavior are presented. Postponement-type policies are then described in a similar fashion. The paper finishes with conclusions: a summary of findings and a discussion of limitations, potential developments and implications for the management of maintenance.

II. The Model

A. Model Specification

In what follows, the system is a single, non-repairable component and a socket that together performs an operational function [26] on demand.

This system deteriorates over time, but also may be subject to external shocks (e.g. a dredger crashed into a pier of the Thames barrier, sank, and damaged a gate and the flood defense system was not operational for a period). The failure process is modeled using the delay-time model [9,27], whereby the system may be in one of three states: good (G), defective (D); and failed (F). Times in the good and the defective state are random variables that are themselves mutually *s*-independent.

It is assumed that:

1. the system will operate on demand if it is in state G or D, but not if it is in state F;
2. inspections are scheduled at system ages kT, $k=1,\dots,M$, and replacement is scheduled at system age MT regardless of the system state at MT.
3. the purpose of inspection is to determine if the system will operate in the event of a demand;
4. an inspection outcome is either positive, P (the inspection test indicates the system would not operate on demand), or negative, N (the inspection test indicates the system would operate on demand);
5. the inspection outcome is related to the system state through the probabilities specified in Table I;

TABLE I
Imperfect Inspection Probabilities

		system state G	D	F
inspection outcome	N	$1-\alpha$	β_1	β_2
	P	α	$1-\beta_1$	$1-\beta_2$

6. if the inspection outcome is P then the system is replaced, and if it is N the system is not replaced;
7. replacement and renewal are synonymous;
8. the times taken to carry out inspection and replacement are negligible;
9. when the system is in state F, a downtime penalty cost with rate c_{F} is incurred; this in a sense is what the decision-maker is prepared to pay per unit of time to prevent the consequences of the event against which the system provides protection [28,29];
10. the cost of an inspection is c_{I} and the cost of a replacement is c_{R}.

Notice that assumptions 3), 4) and 6) imply that the outcome of inspection effectively determines whether the system is replaced. Assumption 5) implies that inspection does not determine the system state. An inspection outcome that classifies system state (as G, D or F), albeit with imprecision, leads to a different model to that is not studied in this paper.

Inspection alone cannot guarantee high availability of the system because inspection is imperfect, and the extent of the imperfection (and the cost) will determine whether inspection is effective. Consequently, the purpose of the model is to analyze circumstances in which inspection is effective, whence $M^* > 1$, and in which it is not, whence $M^* = 1$.

Inspection models in the literature are broadly of two types. The first type models the idea that inspection of a hot-system (or critical system) reveals a state that precedes failure. This is the delay-time model [9,27]. The purpose of this model is to plan inspections. The second type models the idea that inspection reveals the functional state of a cold-system (a protection system with unrevealed failure) [28,29]. The purpose is the same: to plan inspections. For inspection models of the first type, imperfect testing has been modeled in [30,31]. There, the inspection outcome may misclassify the underlying state of the system. For inspection models of the second type, imperfect inspection has also been studied [5,6,32-34], and again therein inspection may misclassify the system state. This paper conflates these types: the system in the model is a protection system (cold-system) that can be in a defective state. Thus, the novelty of the approach is to model imperfect inspection of a system with unrevealed failure and an unrevealed defective state, and to do so by stochastically relating the inspection outcome to the un-observed state of the (degrading) system.

The model is motivated by an isolation valve in a network used to transport a dangerous product. The valve is a protection system that is required to operate on demand. For example, the valve is normally open and in the event of damage to a part of the network, shutting the valve isolates the damaged part of the network and prevents contamination of the environment by the product. Such isolation valves deteriorate with age, are inspected, and replacement of a failed valve is important.

Inspection corresponds to shutting the valve and measuring the downstream flow-rate, *R*. The inspection outcome is regarded as positive if $R > r_{\mathrm{P}}$, and negative otherwise. In the good state, G, the actual flow rate through the shut valve (leakage) is small (e.g. < 0.1% of normal flow). In the defective state, D, the leakage is moderate, and in the failed state, F, the leakage is large (e.g. > 2% of normal flow). The measured flow-rate, *R*, through the shut valve may be related to leakage (and hence the state of the valve) by the imperfect inspection probabilities: $\Pr(R > r_{\mathrm{P}} | \mathrm{G}) = \alpha$, $\Pr(R \le r_{\mathrm{P}} | \mathrm{D}) = \beta_1$ and $\Pr(R \le r_{\mathrm{P}} | \mathrm{F}) = \beta_2$. Error in the measurement of R underlies the imperfection of inspection. This example illustrates two points in the model. Firstly, the inspection outcome and the system state are stochastically related. Secondly, it is natural that $\beta_1 > \beta_2$ (although this is not a requirement of the model), since the measured flow rate is less likely to be small when the leakage is large than when it is moderate. Thus, the valve may fail the inspection test (test positive) when it is defective, but it is less likely to do so than when it is failed. To the knowledge of the authors, these two types of false negative probabilities, β_1 and β_2, which relate inspection outcome to the underlying state of a system with unrevealed failure, have been not previously modeled in the literature.

This inspection process has similarities to destructive testing (e.g. [35]), whereby the destructive testing of an item provides imperfect information about the state other stochastically identical items.

In a special case one might suppose $\beta_2 = 0$, so that when the system is failed the test reveals the true operational state, and that when the system is defective the inspection does not.

If instead the inspection outcome can be G, D or F (imperfectly), then other models may be considered. A maintainer may wish to take an action that follows a D (inspection says component is defective) that is different to the action that follows an F (inspection says component is failed).

Thus, suppose the system is inspected at some time kT, and the outcome is D. Then, the decision maker may wish to take immediate action or to postpone action until new information or an opportunity (see [31] and the references therein) becomes available. Given $\alpha > 0$, this D may be a false positive, and given that the system can perform its operational function when defective anyway, the action might be not to replace but to inspect at $(k+1)T$. However, this is a different model to the one studied here. Nonetheless, there may exist circumstances in which the maintainer does not take immediate action following a positive inspection, either deferring a decision to the next inspection, say, or postponing replacement. Policies that postpone action are the subject of section IV.

B. *Development of the Cost-Rate*

Consider then the policy introduced in section II.A: schedule inspections at ages kT, $(k = 1, \ldots, M)$, and replace the system if an inspection outcome is P. If the system reaches age MT, replace the system regardless of whether the inspection outcome is P or N; this is preventive replacement. The cost-rate, $Q(M,T)$, is derived so that the cost-optimal policy (M^*, T^*) may be determined. Also, the properties of

$Q(M,T)$ and (M^*,T^*) with respect to the parameters, most notably the inspection parameters, may be studied.

Let K be the number of inspections until renewal.

Now $\Pr(K=1)$ depends on whether $M=1$ or $M>1$. If $M=1$ then $\Pr(K=1)=1$ because renewal must occur at time T. When $M>1$, it follows that

$$\Pr(K=1)=(1-\beta_2)\int_0^T F_Y(T-x)\,f_X(x)\mathrm{d}x$$
$$+(1-\beta_1)\int_0^T \bar{F}_Y(T-x)\,f_X(x)\mathrm{d}x+\alpha\bar{F}_X(T). \qquad (1)$$

The first term is the probability of failure before T and the outcome of inspection is P given the system is failed (this is the $(1-\beta_2)$ in the term). The second term is the probability that a defect arises before T, does not fail by T, and the outcome of inspection is P given the system is defective (this is the $(1-\beta_1)$ in the term). The third term is the probability of no defect by T and the outcome of inspection is P given the system is good (this is the α in the term). The events corresponding to three terms are pictorially represented in Fig.1

Thus, there is a careful distinction between the inspection outcome and the system state. The system state is unknown and unobserved. The inspection outcome is not an observation of the system state. If inspection is N for example, the system state remains unknown. Only a demand for the operation of the system can reveal the state of the system. But in the model there are no demands. Instead, a cost is incurred for the time that the system is F. It is not known for how long the system is in state F. But the expectation of this quantity is known, conditional on renewal at a particular inspection.

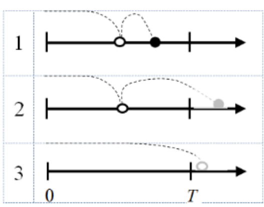


Fig. 1. Possible system states at first inspection. ○ defect arrival, ● failure, ● failure prevented by inspection

Thus, for example, if on inspection a flood barrier rises then the inspection outcome is N. But that does not mean that the state of the barrier is G (or even G or D). It could be F, because in the event of a real demand the barrier may not operate, perhaps because the conditions of the test and the conditions of the demand event (flood) are different. An inspection arguably can never reproduce exactly the conditions that exist at the time of a real demand (c.f. fire safety drills). If it did, then $\alpha=\beta_1=\beta_2=0$. For the case of the barrier one would hope that these inspection error probabilities are very close to zero. At Fukushima [36], protection systems (to supply power in the event of a flood) would have been tested on a regular basis and would have been found to be operational. If not the plant would have been shut down. Nonetheless, when the ultimate flood occurred there was no power from any system available to shut down the reactors.

Consider now $K=2$.

When $M>2$, Fig.2 shows six cases, or more precisely three sets of cases (system in failed state at $2T$, system in defective state at $2T$, and system in good state at $2T$). In the first set (that the system is in the failed state at $2T$) the defect can arise either in the first inspection interval or the second and the failure in the same inspection interval or if possible the subsequent, and in the second set, the defect can arise either in the first inspection interval or the second.

Thus

$$\Pr(K=2,M>2)=\beta_2(1-\beta_2)\int_0^T F_Y(T-x)f_X(x)\mathrm{d}x$$
$$+\beta_1(1-\beta_2)\int_0^T\{F_Y(2T-x)-F_Y(T-x)\}f_X(x)\mathrm{d}x$$
$$+(1-\alpha)(1-\beta_2)\int_T^{2T}F_Y(2T-x)f_X(x)\mathrm{d}x$$
$$+\beta_1(1-\beta_1)\int_0^T\bar{F}_Y(2T-x)\,f_X(x)\mathrm{d}x$$
$$+(1-\alpha)(1-\beta_1)\int_T^{2T}\bar{F}_Y(2T-x)f_X(x)\mathrm{d}x$$
$$+\alpha(1-\alpha)\bar{F}_X(2T). \qquad (2)$$

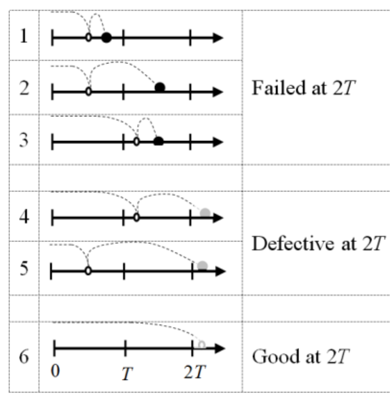


Fig. 2. Possible system states at second inspection given no replacement at first inspection.

When $M=2$, $K=2$ if an only if the system is not renewed at the first inspection. Therefore only events in the first interval (Fig.3) are of concern and the first inspection is itself $\mathrm{N}|\mathrm{F}$ (with probability β_2) or $\mathrm{N}|\mathrm{D}$ (with probability β_1) or $\mathrm{N}|\mathrm{G}$ (with probability $1-\alpha$).

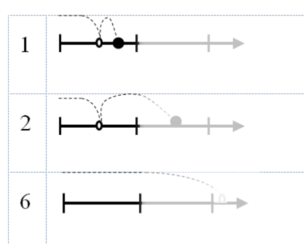


Fig. 3. Replacement at second inspection, considering events arising in the first inspection.

Thus

$$\Pr(K=2, M=2) = \beta_2 \int_0^T F_Y(T-x) f_X(x)\mathrm{d}x$$
$$+ \beta_1 \int_0^T \bar{F}_Y(T-x) f_X(x)\mathrm{d}x + (1-\alpha)\bar{F}_X(T). \quad (3)$$

Proceeding to the general case $K=k$, for $M>k$ there are three cases again:

- the system is in the failed state at kT, and the defect arose in any interval $i=1,...,k$ and the consequent failure in any interval $j=i,...,k$, and the inspection is $\mathrm{P}|\mathrm{F}$;
- the system is in the defective state at kT, and the defect arose in any interval $i=1,...,k$, and the inspection is $\mathrm{P}|\mathrm{D}$;
- and the system is in the good state at kT and the inspection is P.

Thus for $k=2,...,M-1$ ($M>2$), it follows that

$$\Pr(K=k) =$$
$$(1-\beta_2)\sum_{i=1}^{k}(1-\alpha)^{i-1}\beta_2^{k-i}\int_{(i-1)T}^{iT} F_Y(iT-x) f_X(x)\mathrm{d}x$$
$$+(1-\beta_2)\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1-\alpha)^{i-1}\beta_1^{j-i}\beta_2^{k-j}$$
$$\times\left\{\int_{(i-1)T}^{iT}\left\{F_Y(jT-x)-F_Y((j-1)T-x)\right\} f_X(x)\mathrm{d}x\right\}$$
$$+(1-\beta_1)\sum_{i=1}^{k}(1-\alpha)^{i-1}\beta_1^{k-i}\int_{(i-1)T}^{iT}\bar{F}_Y(kT-x) f_X(x)\mathrm{d}x$$
$$+\alpha(1-\alpha)^{k-1}\bar{F}_X(kT). \quad (4)$$

In this expression, the first two terms correspond to the case in which the system is in the failed state at kT. The first of these terms corresponds to the defect arising in the i-th inspection interval and the failure occurring in the same interval, with this failure being undetected until kT (this is the factor β_2^{k-i}). The second term corresponds to the defect arising in the i-th inspection interval and the failure occurring in a later interval, with imperfect inspections, $\mathrm{N}|\mathrm{D}$, occurring at the intervening inspections (this is the factor β_1^{j-i}) and the failure being undetected until kT (this is the factor β_2^{k-j}). In both terms the factor $(1-\alpha)^{i-1}$ is the probability of $\mathrm{N}|\mathrm{G}$ at each inspection prior to the defect arrival, and this must be the case otherwise the system would have been renewed earlier. The third term corresponds to the second case in the bullets above and the last term to the third case.

For $k=M$ ($M>2$), noting that replacement occurs at MT regardless of whether the inspection outcome is P or N, it follows that

$$\Pr(K=M) =$$
$$\sum_{i=1}^{M-1}(1-\alpha)^{i-1}\beta_2^{M-i}\int_{(i-1)T}^{iT} F_Y(iT-x) f_X(x)\mathrm{d}x$$
$$+\sum_{i=1}^{M-2}\sum_{j=i+1}^{M-1}(1-\alpha)^{i-1}\beta_1^{j-i}\beta_2^{M-j}$$
$$\times\left\{\int_{(i-1)T}^{iT}\left\{F_Y(jT-x)-F_Y((j-1)T-x)\right\} f_X(x)\mathrm{d}x\right\}$$
$$+\sum_{i=1}^{M-1}(1-\alpha)^{i-1}\beta_1^{M-i}\int_{(i-1)T}^{iT}\bar{F}_Y((M-1)T-x) f_X(x)\mathrm{d}x$$
$$+(1-\alpha)^{M-1}\bar{F}_X((M-1)T).$$

The first term in this expression corresponds to the case when a defect arises in the i-th inspection interval and causes a failure in the same interval and all subsequent inspections at least as far as the $M-1$th are negative. The second term (double sum) corresponds to a defect arising in the i-th inspection interval and causing a failure in a later interval but no later than the $M-1$th and all subsequent inspections at least as far as the $M-1$th are negative. The third term corresponds a defect arising in the i-th inspection interval and no failure occurring until at least the $M-1$th inspection. Notice further if $\beta_1=\beta_2=0$ in this expression, then immediately this reduces to

$$\Pr(K=M) = (1-\alpha)^{M-1}\bar{F}_X((M-1)T)$$

as required because in this case, for renewal to occur at MT, the first $M-1$ inspections must each be $\mathrm{N}|\mathrm{G}$ and no defect can have arisen by $(M-1)T$.

Then letting V_M be the length of a renewal cycle, it follows that

$$E(V_M) = \sum_{k=1}^{M} kT\Pr(K=k).$$

The calculation of the costs and the cost of a renewal cycle, U_M, proceeds as follows.

First denote the downtime in a cycle by W. Then note carefully that downtime occurs if and only if the system fails, and that failures are not self-announcing and the true system state is observed neither at failures nor at inspections. In reality, failure is only observed at external demands for the system function that occur when the system is failed. However, the model considers these demands only in the standard way [28,29] through a downtime cost-rate that is equivalent to the notion that demands arise according to a Poisson process with a fixed rate and severity.

Define the event F_k that the system fails and the system is renewed at kT. Then, when F_k occurs, the downtime is

$$W_k = kT - X - Y.$$

Let I_k be an indicator function for the event F_k. Observe that $I_k=1$ if and only if $I_j=0$ $j\neq k=1,...,M$. It therefore follows that

$$W = \sum_{k=1}^{M} W_k \times I_k.$$

Therefore

$$E(W) = \sum_{k=1}^{M} E(W_k \times I_k), \quad (5)$$

and for $k=1$ ($M>1$)

$$E(W_1\times I_1) = (1-\beta_2)\int_0^T\int_0^{T-x}(T-x-y) f_Y(y) f_X(x)\mathrm{d}y\mathrm{d}x,$$

and for $M=1$

$$E(W_1\times I_1) = \int_0^T\int_0^{T-x}(T-x-y) f_Y(y) f_X(x)\mathrm{d}y\mathrm{d}x, \quad (5b)$$

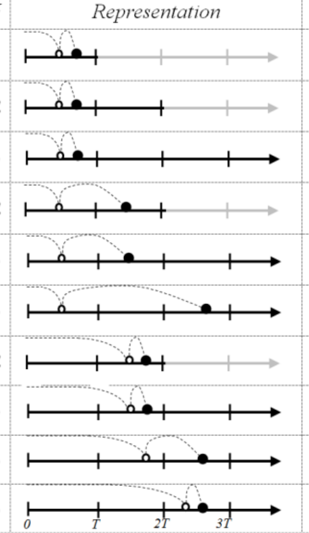
and for $k=2,...,M-1$ ($M>2$)

$$E(W_k \times I_k) = (1-\beta_2)\sum_{i=1}^{k}(1-\alpha)^{i-1}\beta_2^{k-i} \times\left\{\int_{(i-1)T}^{iT}\int_0^{iT-x}(kT-x-y)f_Y(y)f_X(x)\mathrm{d}y\mathrm{d}x\right\} + (1-\beta_2)\sum_{i=1}^{k-1}\sum_{j=i+1}^{k}(1-\alpha)^{i-1}\beta_1^{j-i}\beta_2^{k-j} \times\left\{\int_{(i-1)T}^{iT}\int_{(j-1)T-x}^{jT-x}(kT-x-y)f_Y(y)f_X(x)\mathrm{d}y\mathrm{d}x\right\},$$

and for $k = M$ ($M > 1$)

$$E(W_M \times I_M) = \sum_{i=1}^{M}(1-\alpha)^{i-1}\beta_2^{M-i} \times\left\{\int_{(i-1)T}^{iT}\int_0^{iT-x}(MT-x-y)f_Y(y)f_X(x)\mathrm{d}y\mathrm{d}x\right\} + \sum_{i=1}^{M-1}\sum_{j=i+1}^{M}(1-\alpha)^{i-1}\beta_1^{j-i}\beta_2^{M-j} \times\left\{\int_{(i-1)T}^{iT}\int_{(j-1)T-x}^{jT-x}(MT-x-y)f_Y(y)f_X(x)\mathrm{d}y\mathrm{d}x\right\}.$$

Explaining these expressions a little, in the formula for $E(W_k \times I_k)$, for $k = 2,...,M-1$ ($M > 2$), for example, two terms can be distinguished. In the first term the defect and the consequent failure arise in the same interval, and the preceding inspections are each $\mathrm{N}|\mathrm{G}$ with probability $(1-\alpha)^{i-1}$, and the subsequent inspections are $\mathrm{N}|\mathrm{F}$ with probability β_2^{k-i}, and the ultimate inspection, where renewal occurs, is $\mathrm{P}|\mathrm{F}$ with probability $(1-\beta_2)$. In the second term the defect and the consequent failure arise in the different intervals and the intervening inspections are each $\mathrm{N}|\mathrm{D}$ with probability β_1^{j-i}. Some cases are illustrated for $k = 1,2,3$ ($M > 3$) in Fig. 4.



i	j	k	Representation	Inspections	D
1	1	1		$(1-\beta_2)$	$T-x-y$
1	1	2		$\beta_2(1-\beta_2)$	$2T-x-y$
1	1	3		$\beta_2^2(1-\beta_2)$	$3T-x-y$
1	2	2		$\beta_1(1-\beta_2)$	$2T-x-y$
1	2	3		$\beta_1\beta_2(1-\beta_2)$	$3T-x-y$
1	3	3		$\beta_1^2(1-\beta_2)$	$3T-x-y$
2	2	2		$(1-\alpha)(1-\beta_2)$	$2T-x-y$
2	2	3		$(1-\alpha)\beta_2(1-\beta_2)$	$3T-x-y$
2	3	3		$(1-\alpha)\beta_1(1-\beta_2)$	$3T-x-y$
3	3	3		$(1-\alpha)^2(1-\beta_2)$	$3T-x-y$

Fig.4. Some cases that illustrate the calculation of the downtime.

When $M = 1$, and downtime occurs, the defect and the failure arise in the first and only interval, there are no inspections and so no inspection related probabilities.

When $k = 1$ ($M > 1$), and downtime occurs, then the failure must have occurred in the first interval and the first inspection must be $\mathrm{P}|\mathrm{F}$.

The expected cost of a renewal cycle is the sum of the cost of inspections, the cost of downtime, and the cost of renewal (which itself occurs with probability 1), so that

$$E(U_M) = c_\mathrm{I}\sum_{k=1}^{M-1}k\Pr(K=k) + (M-1)c_\mathrm{I}\Pr(K=M) + c_\mathrm{F}E(W) + c_\mathrm{R}, \quad (M > 1),$$

$$E(U_M) = c_\mathrm{F}E(W) + c_\mathrm{R}. \quad (M = 1).$$

Further notice that the model arbitrarily chooses not to incur the inspection cost at MT. The rationale for this or otherwise has been discussed at length in [5]. The formulae above are altered in a small way if it is assumed otherwise:

$$E(U_M) = c_\mathrm{I}\sum_{k=1}^{M}k\Pr(K=k) + c_\mathrm{F}E(W) + c_\mathrm{R}, \quad (M > 1),$$

$$E(U_M) = c_\mathrm{I} + c_\mathrm{F}E(W) + c_\mathrm{R}, \quad (M = 1).$$

Finally, the long-run cost per unit time or cost-rate by the renewal-reward theorem [37] is $Q(M,T) = E(U_M)/E(V_M)$, and the availability is $A(M,T) = 1 - E(W)/(T \times E(K))$.

When M is not finite (pure inspection policy), the expected cost per cycle and the expected cycle length are

$$E(U_\infty) = c_\mathrm{I}\sum_{k=1}^{\infty}k\Pr(K=k) + c_\mathrm{F}E(W_\infty) + c_\mathrm{R},$$

$$E(V_\infty) = \sum_{k=1}^{\infty}kT\Pr(K=k),$$

where

$$E(W_\infty) = \lim_{M\to\infty}E(W) = \lim_{M\to\infty}\sum_{k=1}^{M}E(W_k \times I_k)$$

in (5) and $\Pr(K=k)$ is given by (4), and the cost-rate is $Q(\infty,T) = E(U_\infty)/E(V_\infty)$, and the availability is $A(\infty,T) = 1 - E(W_\infty)/(T \times E(K))$.

Notice that $E(U_\infty) = \lim_{M\to\infty}E(U_M)$ and $E(V_\infty) = \lim_{M\to\infty}E(V_M)$. Therefore the pure inspection policy appears as a special case of the policy with preventive replacement when $M \to \infty$.

III. Numerical Example

In this study, the unit of cost is set equal to the cost of a replacement, so that $c_\mathrm{R} = 1$. The inspection cost and the downtime cost-rate are specified as $c_\mathrm{I} = 0.05$ and $c_\mathrm{F} = 5$, respectively. For the isolation valve example discussed in the introduction, suppose that the demand rate is 0.1 per year (1 loss of product every 10 years) and the cost of a contamination event is \$100,000. Then the cost-rate of unmet demands is \$10,000 per year. This in turn suggests a cost of renewal (of the valve mechanism) of \$2,000 and an inspection cost of \$100.

The time until a defect occurs is assumed have a Weibull distribution, thus $\bar{F}_X = \exp\{-(x/\gamma)^\delta\}$, with characteristic life $\gamma = 10$ in an arbitrary time unit and shape $\delta = 3$ (noting that valve-mechanism life of 10 years would seem reasonable).

The delay-time is assumed to be exponential, $\bar{F}_Y = \exp(-x/\lambda)$, with mean $\lambda = 1$. This assumption is considered for the numerical results but is not a restriction of the model.

Inspection parameters are set to $0.2 = \beta_1 > \beta_2 = 0.1$ and $\alpha = 0.1$.

This set of parameters values is called the base case. Table II presents the cost-optimal policy for this base case (case 2, shaded), and for other cases in which parameter values are varied. The (M,T) policy is considered along with two special cases, $M = 1$ (no inspection and thus age-based replacement) and $M = \infty$ (pure inspection).

Firstly it can be seen that as δ decreases, inspections become more frequent to compensate for the greater variance in the time to defect arrival, to the extent that when δ is smallest, pure inspection is near cost-optimal, and when δ is largest, age-based replacement is cost-optimal. Here, the cost-rate increases by 42% and the availability decreases accordingly. In addition, Figure 5 shows that in early life ($x < 7$) the hazard rate of a defect arrival decreases with δ. The reverse is true in later life. Thus, the optimum inspection interval appears to be adapted to the initial behavior of the hazard rate, a point noted in [38] which proposes a two-phase inspection policy that has lower costs and greater availability than the single-phase inspection policy. An extension of the *(M,T)* policy to a two-phase policy (M_1, T_1, M_2, T_2) could be analysed in a further study.

When M is finite and α, β_1, or β_2 increase, then *T** increases. However the corresponding *M** decreases and so does *M*T**. Thus, inspection is relaxed due to its decreasing quality, but this is mitigated by earlier preventive maintenance. When the pure inspection policy is considered ($M = \infty$), then the same behavior with α is observed but the situation is just the opposite (*T** decreases) when β_1 or β_2 increases. In this case, because there is no preventive maintenance, more frequent inspection is the best means to avoid defects or failures that remain undetected due to low quality inspections.

TABLE II
RESULTS

								(M,T) policy					No inspection, $M=1$			Pure inspection, $M=\infty$		
Case	δ	λ	β_1	β_2	α	c_I	c_F	T^*	M^*	T^*M^*	Q^*	A^*	T^*	Q^*	A^*	T^*	Q^*	A^*
1	2	1	0.2	0.1	0.1	0.05	5	1.01	10	10.0	0.303	0.985	3.7	0.397	0.977	0.9	0.307	0.985
2	3	1	0.2	0.1	0.1	0.05	5	1.61	4	6.4	0.268	0.989	4.7	0.288	0.987	0.9	0.292	0.986
3	5	1	0.2	0.1	0.1	0.05	5	6.00	1	6.0	0.214	0.994	6.0	0.214	0.994	0.9	0.280	0.987
4	3	0.5	0.2	0.1	0.1	0.05	5	1.50	4	6.0	0.290	0.987	4.4	0.309	0.986	0.8	0.261	0.984
5	3	2	0.2	0.1	0.1	0.05	5	1.77	4	7.1	0.243	0.990	5.1	0.263	0.989	1.1	0.322	0.988
6	3	1	0	0	0	0.05	5	0.85	12	10.2	0.212	0.993	4.7	0.288	0.987	0.7	0.216	0.992
7	3	1	0.2	0	0.1	0.05	5	1.45	5	7.2	0.260	0.989	4.7	0.288	0.987	1.0	0.277	0.987
8	3	1	0.2	0.2	0.1	0.05	5	1.91	3	5.7	0.274	0.988	4.7	0.288	0.987	0.9	0.309	0.985
9	3	1	0.1	0.1	0.1	0.05	5	1.42	5	7.1	0.264	0.989	4.7	0.288	0.987	1.0	0.283	0.987
10	3	1	0.4	0.1	0.1	0.05	5	1.91	3	5.7	0.275	0.988	4.7	0.288	0.987	0.9	0.310	0.984
11	3	1	0.2	0.1	0	0.05	5	0.79	11	8.7	0.231	0.991	4.7	0.288	0.987	0.6	0.243	0.990
12	3	1	0.2	0.1	0.2	0.05	5	2.03	3	6.1	0.286	0.988	4.7	0.288	0.987	1.2	0.327	0.984
13	3	1	0.2	0.1	0.1	0.03	5	1.24	6	7.4	0.255	0.990	4.7	0.284	0.987	0.9	0.270	0.988
14	3	1	0.2	0.1	0.1	0.1	5	1.98	3	6.0	0.288	0.988	4.7	0.299	0.987	1.0	0.343	0.982
15	3	1	0.2	0.1	0.1	0.05	2.5	1.89	4	7.6	0.231	0.980	5.5	0.246	0.977	1.2	0.248	0.977
16	3	1	0.2	0.1	0.1	0.05	10	1.20	5	6.0	0.310	0.994	4.0	0.336	0.993	0.7	0.344	0.992

Unit cost is the cost of preventive replacement, c_R ; characteristic life of defect arrivals γ=10 time units. Base case is shaded, and parameter variations from base case shaded.

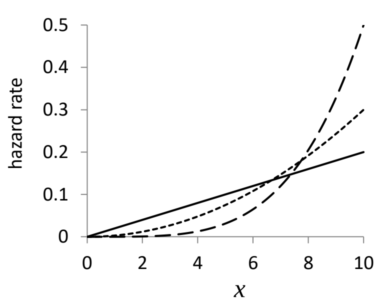


Fig. 5. Hazard rate of the Weibull distribution of defect arrival for $\gamma = 10$, δ=2 (solid line), δ=3 (dotted line), δ=5 (dashed line).

In both the *(M,T)* policy and the pure inspection policy, availability decreases as α or β_2 increase. The availability of the pure inspection policy decreases as β_1 increases across its entire range, but the availability of the *(M,T)* policy increases initially with β_1 but is insensitive to further increase. The pure replacement policy is by definition insensitive to the imperfect-inspection parameters because there is no inspection.

The (M,T) policy is cost-optimal over the range of values of the mean delay-time, λ, considered, and *T* increases with increasing λ and M does not vary with λ.

Secondly, comparing case 6 to case 2, it can be seen that the marginal increased cost of imperfect inspection is 26%. Reduction in $\Pr(\mathrm{P}|\mathrm{G})$ offers the greatest cost-benefit (the reduction in Q^* relative to case 2 is smaller in case 11 than in cases 7 or 9). This also benefits availability. Thus, to increase the availability of protection one should perform more inspections but only if they do not report positives when the system is D or F.

Finally, Inspection is cost-effective for a range of inspection costs (cases 13,2,14), and the superiority of the (M,T) policy increases with increasing downtime cost-rate, c_{F} (cases 15, 2, 16). The percentage increased cost of age-based replacement over the optimal policy is 6.5, 7.5 and 8.4% as c_{F} increases from 2.5 to 5 to 10, and correspondingly 7.4, 9.0, and 11.0% for pure inspection. Further, it can be seen that as c_{F} increases the age limit for replacement decreases (7.6 to 6.4 to 6.0 years), and inspection becomes more frequent. A consequence of this increasing frequency of maintenance is that the availability increases substantially (0.980 to 0.989 to 0.994).

As c_{I} increases inspection is less frequent and the availability decreases marginally. This is the opposite behavior to when c_{F} increases, whereby the inspection frequency and the availability both increase. As the inspection interval decreases so does the downtime as defects and failures are more likely to be detected.

IV. Other Inspection Models

A. Repeated Inspection

If inspections are frequent and the mean delay-time is large, then one might react to the first positive inspection by postponing a replacement decision until the subsequent inspection. A sensible policy might then be inspect at times kT, $k=1,2,\ldots$, and replace the system when the L-th consecutive inspection is positive.

However, difficulties with calculations arise because runs of positive inspections less than length L may precede the final renewal triggered by L consecutive positive inspections. Then it is necessary to consider the type 1 binomial distribution of order l [39] (the number of occurrences of l consecutive successes in a Bernoulli process). This allows one to determine $\Pr(Z=0)$ for a finite Bernoulli sequence of length n, $X_1,\ldots,X_n$, with $\Pr(X_i=1)=p$ and moving product of length L, $Z_i=\prod_{j=0}^{L-1}X_{i+j}$, and sum $Z=\sum_{i=1}^{n-L+1}Z_i$ (i.e. in a finite Bernoulli sequence the probability that there is no run of 1s of length L). This distribution has been used in reliability [40,41].

Nonetheless, there is the further added problem that if a defect arises in the i-th inspection interval then there arises a Bernoulli sequence in which p changes part way through. Setting $\beta_1=\beta_2=0$ avoids this difficulty, but this is not pursued.

B. Repeated Inspection $\alpha=0$

The combinatorial problem simplifies when $\alpha=0$ and when the policy replaces the system after the occurrence of L positive inspections that are not necessarily consecutive. This policy is now investigated for the imperfect inspection parameters defined in Table III.

In reality it may make sense that $\alpha=0$ because the recognition of faults (defects or failures) when they are present is arguably a more important issue than the contrary, because a false negative (potentially an unmet demand) may have much greater consequence than a false positive (replacement of a good valve).

TABLE III
IMPERFECT INSPECTION PROBABILITIES

		system state Z		
		G	D	F
inspection outcome	N	1	β_1	β_2
	P	0	$1-\beta_1$	$1-\beta_2$

The formulae that follow are valid for $L>1$. If $L=1$ then one uses the formulae in section II.B with $\alpha=0$.

For further simplicity, the model supposes that preventive replacement is not scheduled, so that $M=\infty$.

Let K be the number of inspections until renewal as before. For the (L,T) policy, $K=L,L+1,L+2,\ldots$ and

$$\Pr(K=L)=(1-\beta_2)^L\int_0^T F_Y(T-x)\,f_X(x)\mathrm{d}x$$
$$+\sum_{j=2}^{L}(1-\beta_1)^{j-1}(1-\beta_2)^{L-j+1}\int_0^T\left(\int_{(j-1)T-x}^{jT-x}f_Y(y)\mathrm{d}y\right)f_X(x)\mathrm{d}x$$
$$+(1-\beta_1)^L\int_0^T\bar{F}_Y(LT-x)\,f_X(x)\mathrm{d}x.$$

This is because when $K=L$ the defect must arise in the first interval. Then the first term corresponds to the defect and the failure arising in the first interval and the following inspections are all positive (with probability $(1-\beta_2)^L$). The second term corresponds to failure arising in the second or third,…, or L-th interval (hence the summation with these limits). Inspections that precede the failure are P with probability $1-\beta_1$ in each case; inspections that follow the failure are P with probability $1-\beta_2$. The final term corresponds to no failure arising before LT and each inspection is therefore $\mathrm{N}|\mathrm{D}$.

Consider now the remaining cases. When $K=L+k$, $k=1,2,\ldots$, a defect cannot arise later than in the interval $(kT,(k+1)T)$. Otherwise renewal would occur before $(L+k)T$. (For example, if L=2 and there are 5 inspections (k=3) a defect cannot appear later than $4T$.) The following formula distinguishes various cases:

$$\Pr(K=L+k)=\sum_{i=1}^{k+1}\binom{L+k-i}{L-1}(1-\beta_2)^L\beta_2^{k-i+1}\times$$
$$\int_{(i-1)T}^{iT}F_Y(iT-x)\,f_X(x)\mathrm{d}x$$
$$+\sum_{i=1}^{k+1}\sum_{j=i}^{k+L-1}\sum_{m=t}^{s}\binom{j-i+1}{m}(1-\beta_1)^m\beta_1^{j-i+1-m}\times$$
$$\binom{L+k-j-1}{L-m-1}\beta_2^r(1-\beta_2)^{L-m}\int_{(i-1)T}^{iT}\int_{jT-x}^{(j+1)T-x}f_Y(y)\mathrm{d}y\,f_X(x)\mathrm{d}x$$
$$+\sum_{i=1}^{k+1}\binom{L+k-i}{L-1}\beta_1^{k-i+1}(1-\beta_1)^L\times$$
$$\int_{(i-1)T}^{iT}\bar{F}_Y((L+k)T-x)\,f_X(x)\mathrm{d}x,$$

with $s = \min\{L-1, j-i+1\}$, $t = \max\{0, j-k\}$ and $r = \max\{0, k-j+m\}$.

The first summation in this expression corresponds to the case in which defect and failure occur in the same interval. If so, a defect cannot occur later than in (kT, $(k+1)T$). In the second summation, defect and failure occur in different intervals and a defect cannot occur later than in (kT, $(k+1)T$). The third summation considers the case when a defect occurs but there is no failure.

The expected number of inspections is given by

$$E(K) = L + \sum_{k=1}^{\infty} k \Pr(K = L+k),$$

which can be alternatively written as

$$E(K) = L + \sum_{k=1}^{\infty} \Pr(K \geq L+k).$$

The downtime calculation proceeds as follows. Let I_k be an indicator function for the event that a failed system is renewed at the $(L+k)$-th inspection. Observe that $I_k = 1$ if and only if $I_j = 0 \quad j \neq k$. It therefore follows that the downtime is given by

$$W = \sum_{k=0}^{\infty} W_{L+k} \times I_k,$$

where W_{L+k} is the downtime incurred when the system is renewed at the $(L+k)$-th inspection.

For $k = 0$, it follows that

$$\begin{aligned} &E(W_L \times I_0) = \\ &(1-\beta_2)^L \int_0^T \int_0^{T-x} (LT - x - y) f_Y(y) \mathrm{d}y f_X(x) \mathrm{d}x \\ &+ \sum_{j=2}^{L} (1-\beta_1)^{j-1} (1-\beta_2)^{L-j+1} \times \\ &\int_0^T \left(\int_{(j-1)T-x}^{jT-x} (LT - x - y) f_Y(y) \mathrm{d}y \right) f_X(x) \mathrm{d}x, \end{aligned}$$

and for $k > 0$,

$$\begin{aligned} &E(W_{L+k} \times I_k) = \\ &\sum_{i=1}^{k+1} \binom{L+k-i}{L-1} (1-\beta_2)^L \beta_2^{k-i+1} \times \\ &\int_{(i-1)T}^{iT} \int_0^{iT-x} ((L+k)T - x - y) f_Y(y) \mathrm{d}y f_X(x) \mathrm{d}x \\ &+ \sum_{i=1}^{k+1} \sum_{j=i}^{k+L-1} \sum_{m=t}^{s} \binom{j-i+1}{m} (1-\beta_1)^m \beta_1^{j-i+1-m} \times \\ &\binom{L+k-j-1}{L-m-1} \beta_2^r (1-\beta_2)^{L-m} \times \\ &\int_{(i-1)T}^{iT} \int_{jT-x}^{(j+1)T-x} ((L+k)T - x - y) f_Y(y) f_X(x) \mathrm{d}y \mathrm{d}x. \end{aligned}$$

The expected downtime in a renewal cycle is then

$$E(W) = \sum_{k=0}^{\infty} E(W_{L+k} \times I_k),$$

and the cost-rate is

$$Q(L,T) = \{c_\mathrm{I} E(K) + c_\mathrm{F} E(W) + c_\mathrm{R}\} / (T \times E(K)).$$

The availability, or uptime, is given by

$$\begin{aligned} A(L,T) &= 1 - \frac{E(W)}{(T \times E(K))} \\ &= 1 - \frac{\sum_{k=0}^{\infty} E(W_{L+k} \times I_k)}{\sum_{k=0}^{\infty} T(L+k) \Pr(K = L+k)}. \end{aligned}$$

The repeated inspection policy may be justified when the maintainer wants to extend system lifetime. Thus the maintainer is inclined to consider that a positive inspection is the result of a system that is defective rather than failed.

Also, it may be interesting to determine the cost of a repeated inspection policy in these circumstances in order to understand the cost of "ignorance", whereby a maintainer uses a policy (repeated inspection) that is necessarily cost-sub-optimal. In practice one would wish to make a maintainer aware of the cost of procrastination. If a maintainer does not seek immediate replacement, then postponement of replacement may be preferred. This policy is considered in the next section. But first some numerical results for the repeated inspection policy are considered briefly.

Again it is assumed that $\alpha = 0$ and the parameter values as in section III are used. Table IV briefly shows some results and it can be seen that in each case $L^* = 1$ as expected. Regarding the cost of "ignorance", the marginal increased cost of repeated inspections can be calculated. Therein, repeated inspection leads to greater cost and lower availability with increasing L. The marginal increased cost of repeated inspection is greatest when the mean delay-time is smallest (39% for L=2 when $\lambda = 2$ and 44% for L=2 when $\lambda = 0.5$). Also, as L increases, T^* decreases (more frequent inspection) but not so much that LT^* remains constant. Thus, increasing the inspection frequency does not compensate for repeated inspection, presumably because of the imperfect inspection. Indeed, for larger β_1 or β_2, LT^* increases with L more rapidly than for smaller β_1 or β_2.

C. *Postponed Replacement,* $\alpha = 0$

The inspection parameters are assumed as in Table III. Once a positive inspection has occurred, at kT say, it is supposed that: the maintainer decides to postpone replacement for a time τ; during this period of postponement $(kT, kT+\tau)$ there are no further inspections The rationale is that the maintainer seeks to extend the system life with a minimal cost, taking advantage of the delay-time, the time for which the system is defective but functional. Furthermore, the maintainer is aware that a problem exists and new inspections would incur an extra cost for a system which is close to replacement. Note, the cost-rate can be developed for $\alpha > 0$, but since this policy follows naturally from the previous (repeated inspection), the supposition that $\alpha = 0$ is continued.

Another aspect already mentioned is that a $\mathrm{N}|\mathrm{D}$ or $\mathrm{N}|\mathrm{F}$ inspection may be of greater concern that a $\mathrm{P}|\mathrm{G}$ inspection.

TABLE IV
RESULTS FOR REPEATED INSPECTION POLICY

					$L=1$			$L=2$			$L=3$		
Case	λ	β_1	β_2	c_I	T^*	Q^*	A^*	T^*	Q^*	A^*	T^*	Q^*	A^*
1	0.5	0.2	0.1	0.05	0.51	0.271	0.987	0.29	0.391	0.977	0.24	0.491	0.965
2	1	0.2	0.1	0.05	0.60	0.243	0.990	0.34	0.342	0.982	0.26	0.423	0.974
3	2	0.2	0.1	0.05	0.72	0.217	0.992	0.42	0.296	0.985	0.31	0.362	0.980
4	1	0.1	0.1	0.05	0.63	0.235	0.990	0.36	0.332	0.982	0.27	0.411	0.975
5	1	0.4	0.1	0.05	0.54	0.260	0.988	0.32	0.364	0.979	0.25	0.451	0.970
6	1	0.2	0.05	0.05	0.63	0.237	0.990	0.35	0.335	0.983	0.27	0.414	0.975
7	1	0.2	0.2	0.05	0.54	0.256	0.989	0.32	0.357	0.980	0.25	0.442	0.972
8	1	0.2	0.1	0.02	0.43	0.185	0.994	0.26	0.240	0.989	0.22	0.297	0.980
9	1	0.2	0.1	0.1	0.78	0.315	0.984	0.45	0.468	0.971	0.34	0.590	0.961

Unit cost is the cost of preventive replacement, c_R ; $c_F = 5$; characteristic life of defect arrivals $\gamma = 10$ time units, $\delta = 3$.

Let K be the number of inspections until renewal, $K = 1, 2, \ldots$ In this model K is the number of inspections up to an including the first positive inspection, and it follows that $\Pr(K=1)$, $\Pr(K=2)$, and $\Pr(K=k)$ are given by equations (1), (2) and (4) respectively but with $\alpha = 0$. Thus K has the same distribution as the policy in section II.B (policy 1) with $M = \infty$. Furthermore, when $\tau = 0$ policy 1 is obtained as a special case with $\alpha = 0$.

The cycle length for this postponed replacement policy has the modification for the additional period of postponement. Thus the expected cycle length is

$$E(V_\tau) = \tau + \sum_{k=1}^{\infty} kT \Pr(K = k) .$$

The downtime is different to policy 1, but in principle the derivation is similar. Thus, consider the event S_k : inspection at kT is positive and the defect arises at time x and the failure y time units later. The downtime conditional on S_k is $\Delta_{xy} = kT + \tau - x - y$, and the expected downtime is (for $\tau > 0$)

$$\begin{aligned} &E(W_\tau) \\ &= \sum_{i=1}^{\infty} (1-\beta_2) \left\{ \sum_{k=i}^{\infty} \beta_2^{k-i} \int_{(i-1)T}^{iT} \left\{ \int_0^{iT-x} \Delta_{xy} f_Y(y) \mathrm{d}y \right\} f_X(x) \mathrm{d}x \right. \\ &\left. + \sum_{j=i}^{\infty} \sum_{k=j+1}^{\infty} \beta_1^{j-i+1} \beta_2^{k-j-1} \int_{(i-1)T}^{iT} \left\{ \int_{jT-x}^{(j+1)T-x} \Delta_{xy} f_Y(y) \mathrm{d}y \right\} f_X(x) \mathrm{d}x \right\} \\ &+ \sum_{i=1}^{\infty} \sum_{k=i}^{\infty} (1-\beta_1) \beta_1^{k-i} \int_{(i-1)T}^{iT} \left\{ \int_{kT-x}^{kT+\tau-x} \Delta_{xy} f_Y(y) \mathrm{d}y \right\} f_X(x) \mathrm{d}x. \end{aligned}$$

Here, in the first term the defect and failure occur in the same interval $((i-1)T, iT)$ and the failure is detected at $kT, k > i$. In the second term the failure occurs in the interval $((j-1)T, jT)$ subsequent to that of the defect and the failure is detected at $kT, k > j+1$. In both cases the positive inspection is due to a failure so it is a true positive. In the final term, a defect is detected at kT and the failure occurs during the interval of postponement $(kT, kT+\tau)$.

The expected cost of a cycle is then

$$E(U_\tau) = c_I \sum_{k=1}^{\infty} k \Pr(K = k) + c_F E(W_\tau) + c_R .$$

For the parameter values in the cases in Table II, it follows that $\tau^* = 0$ always, and so for brevity these results are omitted. The optimality of $\tau^* = 0$ is contrary to the examples in [31] where $\alpha \neq 0$ and the possibility of opportunity-based maintenance means $\tau^* > 0$ is optimum.

Nonetheless it is interesting to consider the cost-rate if the maintainer acts sub-optimally and postpones replacement. Indeed Figure 6 indicates that postponement is not a good policy, because of the possibility that the system is failed at a positive inspection and the consequent downtime is costly. Moreover, postponement is less appropriate when β_2 is larger.

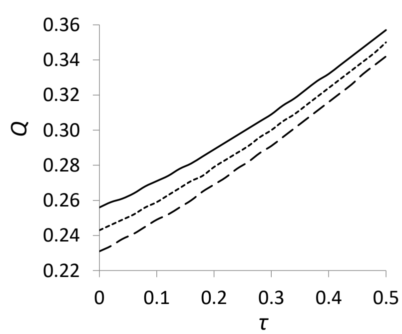


Fig.6. Cost-rate, Q, as a function of the length of postponement, τ , for $\beta_2 = 0$ (dash line), $\beta_2 = 0.1$ (dotted), $\beta_2 = 0.2$ (solid), and with T at its optimal value for the respective β_2 and other parameters as base case (case 2 in Table II).

However, when $\beta_2 = 0$ the cost rises more rapidly than for $\beta_2 > 0$ which is curious. This is perhaps because T is held at its optimum value for $\tau = 0$, and $\tau > 0$ may imply a smaller T^*. Nonetheless, for $\beta_2 = 0$ and a large mean delay-time, it might be expected that postponement to be optimal.

Finally, a policy in which the first positive inspection triggers a deeper, more costly inspection that verifies the state of the system can be considered. Then postponement only occurs if the system is defective (noting that because $\alpha = 0$ the system cannot be G). However, consideration of such a two stage inspection policy is beyond the scope of this paper.

Other related analyses are also possible. For example, if two inspection tests were available, with costs c_{I1} and c_{I2} such that the cheaper inspection was less effective, then one could ask which test is preferred. Alternatively, one might consider what is an appropriate investment to improve inspection-test effectiveness.

V. Conclusions

This paper studies imperfect inspection of a protection system. This system is subject to a three state (G, D, F) failure process, and sojourns in the G and D states are random variables. The inspection outcome provides imperfect information about the system state that is quantified through a set of probabilities that are parameterized in the model. Given then a level of ignorance about the state of the protection system following an inspection, the maintainer must decide whether to replace the system. At a higher level, the maintainer must decide whether to inspect. These decisions are studied by developing the cost-rate of an inspection and replacement policy that is natural in this context.

The novelty of the paper is the consideration of imperfect inspection for a protection system subject to a state (defective) that lies between the good and the failed states. Imperfect inspections can occur in both states although is less likely when the system is failed than defective. This mimics inspection of systems in real life. Thus the benefit of modeling the defective state is that this may better represent the reality in which inspection provides imperfect information about the true underlying state of the protection system. Given this uncertainty, the maintainer has to decide if inspection is an effective strategy. Further, interest in modeling the defective state also emerges if the duration of use on-demand is non-negligible, so that there is the possibility of failure during the demand period when the system is defective at the start of the demand period. However, this would be another study.

The analysis in this paper shows firstly that, since inspection may not be effective, it is natural that a maintainer would in ignorance replace the system at a particular age. The cases analyzed in the numerical example show that this policy is effective not only in terms of cost but also concerning availability. Thus preventive maintenance at *MT* is protection against low quality inspections. Then, secondly, the analysis shows that inspection is cost-effective provided the imperfect-inspection probabilities are not too large. Therein, the most important (to the cost-rate) is $\alpha = \Pr(\mathrm{P}|\mathrm{G})$. Finally, it is shown that there exist circumstances in which a pure inspection policy is near-cost-optimal. However, even when inspection is perfect, the ageing of the system implies that preventive replacement at *MT* remains a sensible policy. A two-stage policy that is an adaptation to the increasing hazard-rate of an ageing system may provide further cost-benefit. This would be another study.

The inclusion in the model of an additional imperfect-inspection probability, β_2, adds another level of complexity to the cost-rate function. Thus the expressions for the cost-rate as well as its derivative are rather complicated. This leads to an empirical study with no analytical results. Nevertheless since inspection aims to detect defective and failed states, only small and medium values of *T* constitute the region of interest. The results in Tables II and IV present the global optimum in that region at least.

For the repeated inspection policy, the imperfect-inspection probabilities are simplified in order to calculate the cost-rate and availability. Then it is found that repeated inspection leads to high cost and downtime, and postponement of replacement is not a good decision. However, this sub-optimality is in part due to the simplification (because it is likely that postponement would be justified when $\alpha > 0$). Corresponding calculations in the general case (with a full set of imperfect-inspection probabilities) would make an interesting and challenging study and may determine circumstances in which repeated inspection is preferable.

It would be interesting to consider imperfection in inspection when inspection reports the system state (G, D or F) rather than the functionality of the system (N or P). This is a new, different model worthy of future investigation.

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