Large deviations for a class of tempered subordinators and their inverse processes*

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Abstract

We consider a class of tempered subordinators, namely a class of subordinators with one-dimensional marginal tempered distributions which belong to a family studied in [3]. The main contribution in this paper is a non-central moderate deviations result. More precisely we mean a class of large deviation principles that fill the gap between the (trivial) weak convergence of some non-Gaussian identically distributed random variables to their common law, and the convergence of some other related random variables to a constant. Some other minor results concern large deviations for the inverse of the tempered subordinators considered in this paper; actually, in some results, these inverse processes appear as random time-changes of other independent processes.

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1 Introduction

Several non-standard stochastic processes in the literature are defined by $\{X(T(t)) : t \geq 0\}$, where $\{T(t) : t \geq 0\}$ is an independent random time-change of a standard stochastic process $\{X(t) : t \geq 0\}$. An important class of random time-changes is given by subordinators, i.e. nondecreasing Lévy processes (see e.g. [4] and [25] as references on these processes); however, in several recent references the process $\{T(t) : t \geq 0\}$ is the inverse of a subordinator.

The family of (positive) stable subordinators is widely studied. An important feature of these processes is that their finite dimensional distributions do not have finite moments. In some situations this could be a problem and this explains the increasing popularity of the tempered version of stable subordinators. In fact these tempered processes have finite dimensional distributions with finite moments, and they keep some other properties of the stable subordinators themselves (for instance in both cases the finite dimensional distributions are self-decomposable, and therefore infinite divisible). Here we recall [23], [9], [27], [28], [17], [21] as references on tempered stable processes, tempered stable subordinators and, in some cases, on inverse of stable subordinators; other more recent references are [10], [20], [18], [13] and [19]. We also recall [12] as a brief survey on tempered stable distributions and their associated Lévy processes.

The interest of the processes studied in this paper is motivated by their connections with important research fields as, for instance, the theory of fractional differential equations (see e.g.

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[15]; see also [2] for the tempered case) and the theory of processes with long-range dependence (see e.g. [24]).

In this paper we consider a 4-parameter family of infinitely divisible distributions introduced in [3] (Section 3), which is inspired by some ideas in [16]. This family, which generalizes the Tweedie distribution (case $\delta = 0$) and the positive Linnik distribution (case $\theta = 0$), is constructed by considering the randomization of the parameter λ with a Gamma distributed random variable. Actually, in our results, we often have to restrict the analysis on the case $\delta = 0$.

The asymptotic results presented in this paper concern the theory of large deviations; see e.g. [7] as a reference on this topic. This theory gives asymptotic computations of small probabilities on an exponential scale. Here we also recall [14] as a reference on large deviations and the averaging theory.

We start with some preliminaries in Section 2. Section 3 is devoted to the main contribution in this paper, i.e. a class of large deviation principles that can be seen as a result of *non-central moderate deviations* with respect to θ (see Proposition 3.3 as $\theta \to \infty$; see also Remark 3.4 for the case $\theta \to 0$). These large deviation principles fill the gap between two asymptotic regimes:

- a weak convergence to a non-Gaussian distribution; actually we have a family of identically distributed random variables, which converge weakly to their common law;
- the convergence to a constant of some other related random variables.

This is illustrated in detail in Remark 3.1. Obviously, we use the term "non-central" because the weak limit in the first item is not Gaussian.

In Section 4 we present some other minor large deviation results for the inverse of the subordinators studied in this paper; this will be done by applying the results in [8]. Some other minor results are presented in Section 5, where the inverse of the subordinators studied in this paper are random time-changes of other independent processes.

2 Preliminaries and some remarks

In this section we present some preliminaries on large deviations and on the family of tempered distributions introduced in [3].

2.1 Preliminaries on large deviations

Here we recall some preliminaries on the theory of large deviations; see e.g. the definitions in [7], pages 4-5. Let $\mathcal{Y}$ be a topological space, and let $\{Y_r\}_r$ be a family of $\mathcal{Y}$-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$; then $\{Y_r\}_r$ satisfies the large deviation principle (LDP from now on), as $r \to r_0$ (possibly $r_0 = \infty$), with speed v_r and rate function I if: $v_r \to \infty$ as $r \to r_0$, $I : \mathcal{Y} \to [0, \infty]$ is a lower semicontinuous function, and the inequalities

$$\liminf_{r \to r_0} \frac{1}{v_r} \log P(Y_r \in O) \geq -\inf_{y \in O} I(y) \text{ for all open sets } O$$

and

$$\limsup_{r \to r_0} \frac{1}{v_r} \log P(Y_r \in C) \leq -\inf_{y \in C} I(y) \text{ for all closed sets } C$$

hold. A rate function is said to be good if $\{\{y \in \mathcal{Y} : I(y) \leq \eta\} : \eta \geq 0\}$ is a family of compact sets.

We essentially deal with cases where $\mathcal{Y} = \mathbb{R}^h$ for some integer $h \geq 1$, and we often use the Gärtner Ellis Theorem (see e.g. Theorem 2.3.6 in [7]). Here we briefly recall the statement of that theorem and, in view of what follows, throughout this paper we use the notation $\langle \cdot, \cdot \rangle$ for the inner product in $\mathbb{R}^h$. Assume that there exists

$$\lim_{r \to r_0} \frac{1}{v_r} \log \mathbb{E}[e^{v_r \langle y, Y_r \rangle}] = \Lambda(y) \text{ (for all } y \in \mathbb{R}^h)$$

as an extended real number; moreover assume that the function Λ is finite in a neighborhood of the origin (i.e. $y = 0$, where $0 \in \mathbb{R}^h$ is the null vector), and it is lower semicontinuous and essentially smooth according to Definition 2.3.5 in [7]. Then $\{Y_r\}_r$ satisfies the LDP with speed v_r and good rate function Λ^* defined by

$$\Lambda^*(x) := \sup_{y \in \mathbb{R}^h} \{\langle x, y\rangle - \Lambda(y)\}.$$

The function Λ^* is called Fenchel-Legendre transform of the function Λ.

Remark 2.1. *Let us consider the above setting of the Gärtner Ellis Theorem and, for simplicity, we consider the case $r_0 = \infty$. Moreover we consider the closed set $C_\delta := \{x \in \mathbb{R} : \|x - \nabla\Lambda(0)\| \geq \delta\}$ for some $\delta > 0$; then, since $\Lambda^*(x) = 0$ if and only if $x = \nabla\Lambda(0)$, we have $\Lambda^*(C_\delta) := \inf_{y \in C_\delta} \Lambda^*(y) > 0$. We want to consider the LDP upper bound for the closed set C_δ. Then, for all $\varepsilon > 0$ small enough, there exists r_ε such that*

$$P(|Y_r - \nabla\Lambda(0)| \geq \delta) \leq e^{-v_r(\Lambda^*(C_\delta)-\varepsilon)} \text{ for all } r > r_\varepsilon.$$

Thus Y_r converges to $\nabla\Lambda(0)$ in probability. Moreover it is possible to check the almost sure convergence along a sequence $\{r_n : n \geq 1\}$ such that $r_n \to \infty$; in fact, by a standard application of Borel Cantelli Lemma, we can say that Y_{r_n} converges to $\nabla\Lambda(0)$ almost surely if

$$\sum_{n \geq 1} e^{-v_{r_n}(\Lambda^*(C_\delta)-\varepsilon)} < \infty; \tag{1}$$

for instance, when $v_r = r$ (we have this situation in Sections 4 and 5), condition (1) holds with the sequence $r_n = n$.

Here we also recall the contraction principle (see e.g. Theorem 4.2.1 in [7]), that will be used in Remark 3.3. Let $\{Y_r\}_r$ be a family of $\mathcal{Y}$-valued random variables defined on the same probability (as above), and assume that $\{Y_r\}_r$ satisfies the LDP, as $r \to r_0$, with speed v_r and good rate function I. Then, if we consider a continuous function $f : \mathcal{Y} \to \mathcal{Z}$, where $\mathcal{Z}$ is another topological space, the family of $\mathcal{Z}$-valued random variables $\{f(Y_r)\}_r$ satisfies the LDP, as $r \to r_0$, with speed v_r and good rate function J defined by

$$J(z) := \inf\{I(y) : y \in \mathcal{Y},\ f(y) = z\}.$$

2.2 Preliminaries on the tempered distributions in this paper

We consider a family of subordinators $\{S_{(\gamma,\lambda,\theta,\delta)}(t) : t \geq 0\}$, where the parameters $(\gamma, \lambda, \theta, \delta)$ belong to a suitable set $\mathcal{P} := \mathcal{P}_1 \cup \mathcal{P}_2$, i.e.

$$\mathcal{P}_1 = (-\infty, 0) \times (0, \infty) \times (0, \infty) \times [0, \infty) \text{ and } \mathcal{P}_2 = (0, 1) \times (0, \infty) \times [0, \infty) \times [0, \infty);$$

actually other cases could be allowed ($\gamma = 0$ when $(\gamma, \lambda, \theta, \delta) \in \mathcal{P}_1$ and $\gamma = 1$ when $(\gamma, \lambda, \theta, \delta) \in \mathcal{P}_2$) but they will be neglected because they give rise to deterministic random variables. Then, for each $(\gamma, \lambda, \theta, \delta) \in \mathcal{P}$ and for all $t \geq 0$, we consider the moment generating function

$$\mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(t)}] = \exp(t\kappa_{(\gamma,\lambda,\theta,\delta)}(y)) \text{ (for all } y \in \mathbb{R}),$$

where

$$\kappa_{(\gamma,\lambda,\theta,\delta)}(y) := \log \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(1)}].$$

We remark that we are setting $y = -s$, where $s > 0$ is the argument of the Laplace transforms in [3] and, moreover, we have $\mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(1)}] = \infty$ for some $y > 0$. Furthermore, in view of the applications of the Gärtner Ellis Theorem, it is useful to introduce the Fenchel-Legendre transform of the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$, i.e. the function $\kappa^*_{(\gamma,\lambda,\theta,\delta)}$ defined by

$$\kappa^*_{(\gamma,\lambda,\theta,\delta)}(x) := \sup_{y\in\mathbb{R}}\{xy - \kappa_{(\gamma,\lambda,\theta,\delta)}(y)\}. \tag{2}$$

We remark that, when we deal with $\{S_{(\gamma,\lambda,\theta,\delta)}(t) : t \geq 0\}$, the Gärtner Ellis Theorem can be applied only when $\theta > 0$; in fact in this case the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$ is finite in a neighborhood of the origin $y = 0 \in \mathbb{R}$.

Case $\delta = 0$. This is the case of Tweedie distribution (see Section 2.2 in [3] and the references cited therein). We have

$$\kappa_{(\gamma,\lambda,\theta,0)}(y) := \log \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,0)}(1)}] = \lambda\mathrm{sgn}(\gamma)(\theta^\gamma - (\theta-y)^\gamma)$$

if $y \leq \theta$, and equal to infinity otherwise. Note that

$$\kappa_{(\gamma,\lambda,\theta,0)}(y) = \lambda\kappa_{(\gamma,1,\theta,0)}(y).$$

Thus we have the two following cases:

$$\text{if } \gamma \in (-\infty,0), \quad \kappa_{(\gamma,\lambda,\theta,0)}(y) := \log \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,0)}(1)}] = \begin{cases} \frac{\lambda}{\theta^{-\gamma}}\left(\left(\frac{\theta}{\theta-y}\right)^{-\gamma} - 1\right) & \text{if } y < \theta \\ \infty & \text{otherwise,} \end{cases}$$

that is a compound Poisson distribution with Gamma distributed jumps;

$$\text{if } \gamma \in (0,1), \quad \kappa_{(\gamma,\lambda,\theta,0)}(y) := \log \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,0)}(1)}] = \begin{cases} \lambda(\theta^\gamma - (\theta-y)^\gamma) & \text{if } y \leq \theta \\ \infty & \text{otherwise,} \end{cases}$$

that is the tempered positive Linnik distribution (actually we have the tempered case if $\theta > 0$). In view of the application of the Gärtner Ellis Theorem, we can get the full LDP if the function $\kappa_{(\gamma,\lambda,\theta,0)}$ is steep; then, in both cases $\gamma \in (0,1]$ and $\gamma \in (-\infty,0)$, we need to check the condition $\lim_{y\to\theta^-} \kappa'_{(\gamma,\lambda,\theta,0)}(y) = \infty$ and this can be easily done (the details are omitted).

Case $\delta > 0$. We construct this case starting from the previous one and by considering a Gamma subordination, i.e. a randomization of the parameter λ with a Gamma distributed random variable $G_{\delta,\lambda}$ such that

$$\mathbb{E}[e^{yG_{\delta,\lambda}}] = (1-\lambda\delta y)^{-1/\delta} = \left(\frac{(\lambda\delta)^{-1}}{(\lambda\delta)^{-1} - y}\right)^{1/\delta}$$

if $y < (\lambda\delta)^{-1}$, and equal to infinity otherwise. Then, by taking into account the moment generating function of the random variable $G_{\delta,\lambda}$, we have

$$\mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(1)}] := \mathbb{E}[e^{\kappa_{(\gamma,1,\theta,0)}(y)G_{\delta,\lambda}}] = (1-\delta\kappa_{(\gamma,\lambda,\theta,0)}(y))^{-1/\delta} = (1-\lambda\delta\,\mathrm{sgn}(\gamma)(\theta^\gamma - (\theta-y)^\gamma))^{-1/\delta}$$

if $y \leq \theta$ and $\mathrm{sgn}(\gamma)(\theta^\gamma - (\theta-y)^\gamma) < (\lambda\delta)^{-1}$, and equal to infinity otherwise. Thus, for the same values of y, the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$ is defined by

$$\kappa_{(\gamma,\lambda,\theta,\delta)} := -\frac{1}{\delta}\log(1-\lambda\delta\,\mathrm{sgn}(\gamma)(\theta^\gamma - (\theta-y)^\gamma)).$$

Moreover, let y_0 be the abscissa of convergence of the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$, and therefore we have $\kappa_{(\gamma,\lambda,\theta,\delta)}(y) < \infty$ for $y < y_0$ and $\kappa_{(\gamma,\lambda,\theta,\delta)}(y) < \infty$ for $y > y_0$. We remark that $y_0 \in [0,\theta]$. Then we can easily check the steepness of $\kappa_{(\gamma,\lambda,\theta,\delta)}$, i.e.

$$\kappa'_{(\gamma,\lambda,\theta,\delta)}(y) = \frac{\kappa'_{(\gamma,\lambda,\theta,0)}(y)}{1-\delta\kappa_{(\gamma,\lambda,\theta,0)}(y)} \to \infty \text{ as } y \to y_0^-.$$

Finally, as pointed out in [3] (see just after equation (11)), we note that

$$\lim_{\delta\to 0^+} \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(1)}] = \lambda \mathrm{sgn}(\gamma)(\theta^\gamma - (\theta-y)^\gamma)$$

for all $y \leq \theta$, and equal to infinity otherwise. Therefore we recover the case $\delta = 0$ by taking the limit as $\delta \to 0^+$; in fact the random variable $G_{\delta,\lambda}$ converges weakly to the constant λ as $\delta \to 0^+$.

Some further comments on both cases $\delta = 0$ and $\delta > 0$, for $\theta > 0$. Let $\theta > 0$ be fixed. Then, for all $\delta \geq 0$, $\frac{S_{(\gamma,\lambda,\theta,\delta)}(t)}{t} \to \kappa'_{(\gamma,\lambda,\theta,\delta)}(0) = \lambda\gamma\theta^{\gamma-1}$ as $t \to \infty$ because the rate function $\kappa^*_{(\gamma,\lambda,\theta,\delta)}(x)$ uniquely vanishes at $x = \kappa'_{(\gamma,\lambda,\theta,\delta)}(0)$. Thus, since the limit value $\kappa'_{(\gamma,\lambda,\theta,\delta)}(0)$ does not depend on δ, it is interesting to see how the rate function $\kappa^*_{(\gamma,\lambda,\theta,\delta)}(x)$ varies with δ around the limit value. In fact, if there exists $\rho > 0$ small enough such that

$$\kappa^*_{(\gamma,\lambda,\theta,\delta_1)}(x) > \kappa^*_{(\gamma,\lambda,\theta,\delta_2)}(x) \text{ for } 0 < |x - \lambda\gamma\theta^{\gamma-1}| < \rho,$$

we can say that $\frac{S_{(\gamma,\lambda,\theta,\delta_1)}(t)}{t}$ converges faster than $\frac{S_{(\gamma,\lambda,\theta,\delta_2)}(t)}{t}$.

In view of what follows, we remark that

$$\kappa''_{(\gamma,\lambda,\theta,\delta)}(0) = \frac{\kappa''_{(\gamma,\lambda,\theta,0)}(0)(1-\delta\kappa_{(\gamma,\lambda,\theta,0)}(0)) + \delta(\kappa'_{(\gamma,\lambda,\theta,0)}(0))^2}{(1-\delta\kappa_{(\gamma,\lambda,\theta,0)}(0))^2} = \kappa''_{(\gamma,\lambda,\theta,0)}(0) + \delta(\kappa'_{(\gamma,\lambda,\theta,0)}(0))^2$$

for all $\delta > 0$; actually we can also take $\delta = 0$ as a trivial equality. Thus, for $0 \leq \delta_1 < \delta_2$, we have

$$\kappa''_{(\gamma,\lambda,\theta,\delta_1)}(0) < \kappa''_{(\gamma,\lambda,\theta,\delta_2)}(0)$$

and the above local inequality between $\kappa^*_{(\gamma,\lambda,\theta,\delta_1)}(x)$ and $\kappa^*_{(\gamma,\lambda,\theta,\delta_2)}(x)$ holds by some properties of the Fenchel-Legendre transform. We also remark that our conclusion has some relationship with the Gamma subordination explained above for the case $\delta > 0$; in fact we have $\mathrm{Var}[G_{\delta,\lambda}] = \lambda^2\delta$, and therefore $\delta_1 < \delta_2$ yields $\mathrm{Var}[G_{\delta_1,\lambda}] < \mathrm{Var}[G_{\delta_2,\lambda}]$.

Finally we note that, if we take $\delta_2 > 0$, we get

$$\kappa_{(\gamma,\lambda,\theta,\delta_2)}(y) = -\frac{1}{\delta_2}\log(1-\delta_2\kappa_{(\gamma,\lambda,\theta,0)}(y)) \geq \kappa_{(\gamma,\lambda,\theta,0)}(y)$$

for all y such that $\kappa_{(\gamma,\lambda,\theta,0)}(y) < \frac{1}{\delta_2}$. Then, if $\delta_1 = 0$, the above local inequality between $\kappa^*_{(\gamma,\lambda,\theta,\delta_1)}(x)$ and $\kappa^*_{(\gamma,\lambda,\theta,\delta_2)}(x)$ holds for all $x \neq \lambda\gamma\theta^{\gamma-1}$.

2.3 Some remarks

Here we present some other minor remarks on the family of subordinators studied in this paper.

Remark 2.2 (Composition of independent processes)**.** *It is well-known (and it is easy to check) that, if we consider h independent subordinators $\{\{S_{(\gamma_i,\lambda_i,\theta_i,\delta_i)}(t) : t \geq 0\} : i \in \{1,\dots,h\}\}$, the process $\{S(t) : t \geq 0\}$ defined by*

$$S(t) := S_{(\gamma_1,\lambda_1,\theta_1,\delta_1)} \circ \cdots \circ S_{(\gamma_n,\lambda_n,\theta_n,\delta_h)}(t)$$

is a subordinator and, moreover, for all $t \geq 0$ we have

$$\mathbb{E}[e^{yS(t)}] = e^{t\kappa_S(y)}, \text{ where } \kappa_S(y) := \kappa_{(\gamma_h,\lambda_h,\theta_h,\delta_h)} \circ \cdots \circ \kappa_{(\gamma_1,\lambda_1,\theta_1,\delta_1)}(y).$$

A natural question is whether, in some cases, the composition of independent processes in this family still belongs to this family. One can check that this is possible only in a very particular case, i.e.

$$(\gamma_i,\lambda_i,\theta_i,\delta_i) = (\gamma_i,1,0,0) \text{ with } \gamma_i \in (0,1), \text{ for all } i \in \{1,\dots,h\},$$

and we have

$$\kappa_{(\gamma_h,1,0,0)} \circ \cdots \circ \kappa_{(\gamma_1,1,0,0)}(y) = \kappa_{(\gamma_1\cdots\gamma_h,1,0,0)}.$$

Remark 2.3 (Generalization of the mixtures in [13]). *We consider h independent subordinators $\{\{S_{(\gamma_i,\lambda_i,\theta_i,\delta_i)}(t) : t \geq 0\} : i \in \{1, \dots, h\}\}$ and, for some $c_1, \dots, c_h > 0$, let $\{S(t) : t \geq 0\}$ be the process defined by*

$$S(t) := \sum_{i=1}^{h} S_{(\gamma_i,\lambda_i,\theta_i,\delta_i)}(c_i t).$$

Note that this kind of processes is a generalization of the mixtures studied in [13]; actually in that reference the authors require some unnecessary restrictions on the parameters (in particular the condition $c_1 + \cdots + c_h = 1$ that explains the term mixture *used in [13]). For all $t \geq 0$ we have*

$$\mathbb{E}[e^{yS(t)}] = e^{t\kappa_S(y)}, \text{ where } \kappa_S(y) := \sum_{i=1}^{h} c_i \kappa_{(\gamma_i,\lambda_i,\theta_i,\delta_i)}(y).$$

A natural question is whether, in some cases, the generalized mixture of processes in this family (according to the terminology here) still belongs to this family. One can check that this is possible in a very particular case, i.e.

$$(\gamma_i, \lambda_i, \theta_i, \delta_i) = (\gamma, \lambda_i, 0, 0) \text{ for all } i \in \{1, \dots, h\}, \text{ for some } \gamma \in (0, 1),$$

and we have

$$\kappa_S(y) = \sum_{i=1}^{h} c_i \kappa_{(\gamma,\lambda_i,0,0)}(y) = \sum_{i=1}^{h} c_i \lambda_i \kappa_{(\gamma,1,0,0)}(y).$$

3 Non-central moderate deviations (for $\delta = 0$)

The term moderate deviations is used in the literature for a suitable class of LDPs governed by the same rate function; moreover, in some sense, moderate deviations fill the gap between a convergence to a constant and a weak convergence to a Gaussian distribution (see e.g. Theorem 3.7.1 in [7] which concerns the case of empirical means of i.i.d. random vectors, and we can refer to the Law of Large Numbers and to the Central Limit Theorem).

In this section we study a non-central moderate deviation regime for $\{S_{(\gamma,\lambda,\theta,0)}(t) : t \geq 0\}$ with respect to θ; as we said above, we use the term non-central because we deal with a non-Gaussian weak limit. Here we deal with finite families of increments of the subordinator; however, as we shall explain in Remark 3.3, it is also possible to present analogue results for the finite dimensional distributions of the subordinator.

We start with by considering a family of identically distributed random variables (and therefore they are trivially weak convergent).

Proposition 3.1. *Let $m \geq 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m$ be arbitrarily fixed. Then, for all $\theta > 0$, the random vector $(\theta S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^\gamma) - \theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^\gamma))_{i=1,\dots,m}$ is distributed as $(S_{(\gamma,\lambda,1,0)}(t_i) - S_{(\gamma,\lambda,1,0)}(t_{i-1}))_{i=1,\dots,m}$. Thus*

$$\{(\theta S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^\gamma) - \theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^\gamma))_{i=1,\dots,m} : \theta > 0\}$$

is a family of identically distributed random vectors.

Proof. By taking into account the independence and the distribution of the increments, for all

$\theta > 0$ we have

$$\log \mathbb{E}\left[\exp\left(\sum_{i=1}^m y_i(\theta S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^\gamma) - \theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^\gamma))\right)\right]$$
$$= \sum_{i=1}^m \log \mathbb{E}\left[e^{\theta y_i (S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^\gamma) - S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^\gamma))}\right] = \sum_{i=1}^m \frac{t_i - t_{i-1}}{\theta^\gamma}\kappa_{(\gamma,\lambda,\theta,0)}(\theta y_i)$$
$$= \begin{cases} \sum_{i=1}^m \frac{t_i - t_{i-1}}{\theta^\gamma}\lambda \operatorname{sgn}(\gamma)(\theta^\gamma - (\theta - \theta y_i)^\gamma) & \text{if } \theta y_1, \dots, \theta y_m \leq \theta \\ \infty & \text{otherwise} \end{cases}$$
$$= \begin{cases} \sum_{i=1}^m (t_i - t_{i-1})\lambda \operatorname{sgn}(\gamma)(1 - (1 - y_i)^\gamma) & \text{if } y_1, \dots, y_m \leq 1 \\ \infty & \text{otherwise} \end{cases} = \sum_{i=1}^m (t_i - t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i).$$

This completes the proof. □

The result stated in Proposition 3.1 allows to consider different kind of weak convergence. Here we mainly consider the case $\theta \to \infty$; the case $\theta \to 0$ will be briefly discussed in Remark 3.4.

Proposition 3.2. *Let $m \geq 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m$ be arbitrarily fixed. Moreover let $g(\gamma), h(\gamma) \in \mathbb{R}$ be such that $\gamma - h(\gamma) = 1 - g(\gamma) > 0$. Then the family of random vectors*

$$\{(\theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)}) - \theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))_{i=1,\dots,m} : \theta > 0\}$$

satisfies the LDP with speed $\theta^{\gamma - h(\gamma)}$, or equivalently $\theta^{1-g(\gamma)}$, and good rate function $I_{t_1,\dots,t_m}$ defined by

$$I_{t_1,\dots,t_m}(x_1, \dots, x_m) = \sum_{i=1}^m (t_i - t_{i-1})\kappa^*_{(\gamma,\lambda,1,0)}\left(\frac{x_i}{t_i - t_{i-1}}\right).$$

Proof. We want to apply the Gärtner Ellis Theorem. Firstly, we have

$$\frac{1}{\theta^{\gamma-h(\gamma)}} \log \mathbb{E}\left[\exp\left(\theta^{\gamma-h(\gamma)} \sum_{i=1}^m y_i(\theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)}) - \theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))\right)\right]$$
$$= \frac{1}{\theta^{\gamma-h(\gamma)}} \sum_{i=1}^m \log \mathbb{E}\left[e^{\theta^{\gamma-h(\gamma)+g(\gamma)} y_i (S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)}) - S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))}\right]$$
$$= \sum_{i=1}^m \frac{t_i - t_{i-1}}{\theta^{\gamma-h(\gamma)+h(\gamma)}}\kappa_{(\gamma,\lambda,\theta,0)}(\theta^{\gamma-h(\gamma)+g(\gamma)} y_i) = \sum_{i=1}^m \frac{t_i - t_{i-1}}{\theta^\gamma}\kappa_{(\gamma,\lambda,\theta,0)}(\theta y_i).$$

Moreover, by taking into account some computations in the proof of Proposition 3.1, the final expression does not depend on θ, and we have

$$\sum_{i=1}^m \frac{t_i - t_{i-1}}{\theta^\gamma}\kappa_{(\gamma,\lambda,\theta,0)}(\theta y_i) = \sum_{i=1}^m (t_i - t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i) \text{ for all } \theta > 0. \tag{3}$$

Then, for all $(y_1, \dots, y_m) \in \mathbb{R}^m$, we have

$$\lim_{\theta\to\infty} \frac{1}{\theta^{\gamma-h(\gamma)}} \log \mathbb{E}\left[\exp\left(\theta^{\gamma-h(\gamma)} \sum_{i=1}^m y_i(\theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)}) - \theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))\right)\right]$$
$$= \sum_{i=1}^m (t_i - t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i).$$

So we can apply the Gärtner Ellis Theorem and the desired LDP holds with good rate function $I_{t_1,\dots,t_m}$ defined by

$$I_{t_1,\dots,t_m}(x_1,\dots,x_m)=\sup_{(y_1,\dots,y_m)\in\mathbb{R}^m}\left\{\sum_{i=1}^m y_ix_i-\sum_{i=1}^m(t_i-t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i)\right\}.$$

Finally, one can check that the rate function $I_{t_1,\dots,t_m}$ defined here coincides with the one in the statement of the proposition; this can be done with some standard computations (for instance one can follow the lines of the proof of Lemma 5.1.8 in [7]). □

For completeness we discuss the convergence of the random variables in Proposition 3.2 to a constant vector. Firstly, for θ large enough which depends on $y_1,\dots,y_m$ (otherwise the moment generating function below is equal to infinity), we have

$$\begin{aligned}\log\mathbb{E}&\left[\exp\left(\sum_{i=1}^m y_i(\theta^{g(\gamma)}S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)})-\theta^{g(\gamma)}S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))\right)\right]\\&=\sum_{i=1}^m\frac{t_i-t_{i-1}}{\theta^{h(\gamma)}}\kappa_{(\gamma,\lambda,\theta,0)}(\theta^{g(\gamma)}y_i)=\sum_{i=1}^m\frac{t_i-t_{i-1}}{\theta^{h(\gamma)}}\lambda\mathrm{sgn}(\gamma)(\theta^\gamma-(\theta-\theta^{g(\gamma)}y_i)^\gamma)\\&=\sum_{i=1}^m(t_i-t_{i-1})\lambda\mathrm{sgn}(\gamma)\theta^{\gamma-h(\gamma)}\left(1-\left(1-\frac{y_i}{\theta^{1-g(\gamma)}}\right)^\gamma\right).\end{aligned}$$

Then

$$\begin{aligned}\lim_{\theta\to\infty}\log\mathbb{E}&\left[\exp\left(\sum_{i=1}^m y_i(\theta^{g(\gamma)}S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)})-\theta^{g(\gamma)}S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/\theta^{h(\gamma)}))\right)\right]\\&=\sum_{i=1}^m y_i(t_i-t_{i-1})\lambda\mathrm{sgn}(\gamma)\gamma,\end{aligned}$$

so that the random variables in Proposition 3.2 converge (as $\theta\to\infty$) to the vector $(x_1(\gamma),\dots,x_m(\gamma))$ defined by

$$x_i(\gamma)=(t_i-t_{i-1})\lambda\mathrm{sgn}(\gamma)\gamma\text{ (for all }i\in\{1,\dots,m\}).$$

Moreover, as one can expect, $I_{t_1,\dots,t_m}(x_1,\dots,x_m)=0$ if and only if

$$\begin{aligned}(x_1,\dots,x_m)&=\left(\frac{\partial}{\partial y_i}\sum_{i=1}^m(t_i-t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i)\Big|_{(y_1,\dots,y_m)=(0,\dots,0)}\right)_{i=1,\dots,m}\\&=((t_i-t_{i-1})\kappa'_{(\gamma,\lambda,1,0)}(0))_{i=1,\dots,m}=(x_1(\gamma),\dots,x_m(\gamma)).\end{aligned}$$

Now we are ready to present the non-central moderate deviation result (as $\theta\to\infty$); see also Remark 3.1.

Proposition 3.3. *Let $m\geq 1$ and $0=t_0<t_1<t_2<\cdots<t_m$ be arbitrarily fixed. Moreover let $g(\gamma),h(\gamma)$ be such that $\gamma-h(\gamma)=1-g(\gamma)>0$ (as in Proposition 3.2). Then, for all families of positive numbers $\{a_\theta:n\geq 1\}$ such that*

$$a_\theta\to 0\text{ and }\theta^{\gamma-h(\gamma)}a_\theta=\theta^{1-g(\gamma)}a_\theta\to\infty\text{ (as }\theta\to\infty),\tag{4}$$

the family of random vectors

$$\{(a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_i/(a_\theta\theta^\gamma))-a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/(a_\theta\theta^\gamma)))_{i=1,\dots,m}:\theta>0\}$$

satisfies the LDP with speed $1/a_\theta$ and good rate function $I_{t_1,\dots,t_m}$ presented in Proposition 3.2.

Proof. We want to apply the Gärtner Ellis Theorem. For all $\theta > 0$, by taking into account equation (3) for the last equality below, we get

$$\frac{1}{1/a_\theta}\log\mathbb{E}\left[\exp\left(\frac{1}{a_\theta}\sum_{i=1}^m y_i(a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_i/(a_\theta\theta^\gamma)) - a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/(a_\theta\theta^\gamma)))\right)\right]$$
$$= a_\theta\sum_{i=1}^m\log\mathbb{E}\left[e^{\theta y_i\{S_{(\gamma,\lambda,\theta,0)}(t_i/(a_\theta\theta^\gamma)) - S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/(a_\theta\theta^\gamma))\}}\right]$$
$$= a_\theta\sum_{i=1}^m\frac{t_i - t_{i-1}}{a_\theta\theta^\gamma}\kappa_{(\gamma,\lambda,\theta,0)}(\theta y_i) = \sum_{i=1}^m\frac{t_i - t_{i-1}}{\theta^\gamma}\kappa_{(\gamma,\lambda,\theta,0)}(\theta y_i) = \sum_{i=1}^m(t_i - t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i);$$

so, for all $(y_1,\ldots,y_m)\in\mathbb{R}^m$, we have

$$\lim_{\theta\to\infty}\frac{1}{1/a_\theta}\log\mathbb{E}\left[\exp\left(\frac{1}{a_\theta}\sum_{i=1}^m y_i(a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_i/(a_\theta\theta^\gamma)) - a_\theta\theta S_{(\gamma,\lambda,\theta,0)}(t_{i-1}/(a_\theta\theta^\gamma)))\right)\right]$$
$$= \sum_{i=1}^m(t_i - t_{i-1})\kappa_{(\gamma,\lambda,1,0)}(y_i).$$

We conclude the proof by considering the same application of the Gärtner Ellis Theorem presented in the proof of Proposition 3.2. □

We conclude with some remarks.

Remark 3.1. *The class of LDPs in Proposition 3.3 fill the gap between the following asymptotic regimes:*

- *the convergence of the random variables in Proposition 3.2 to* $(x_1(\gamma),\ldots,x_m(\gamma))$*;*
- *the weak convergence of the random variables in Proposition 3.1 that trivially converge to their common law, and therefore the law of the random vector* $(S_{(\gamma,\lambda,1,0)}(t_i) - S_{(\gamma,\lambda,1,0)}(t_{i-1}))_{i=1,\ldots,m}$.

In some sense these two asymptotic regimes can be recovered by considering two extremal choices for a_θ *in Proposition 3.3, i.e.* $a_\theta = \frac{1}{\theta^{\gamma-h(\gamma)}} = \frac{1}{\theta^{1-g(\gamma)}}$ *and* $a_\theta = 1$*, respectively. Note that, in both cases, one condition in* (4) *holds and the other one fails.*

Remark 3.2. *The rate function* $I_{t_1,\ldots,t_m}$ *in Propositions 3.2 and 3.3 has some connections with the two asymptotic regimes as* $\theta\to\infty$ *presented in Remark 3.1.*

- *The rate function* $I_{t_1,\ldots,t_m}$ *uniquely vanishes at* $(x_1(\gamma),\ldots,x_m(\gamma))$ *and, as already remarked, this vector is the limit of the random variables in Proposition 3.2 as* $\theta\to\infty$.
- *The Hessian matrix* $\left(\frac{\partial^2}{\partial x_i\partial x_j}I_{t_1,\ldots,t_m}(x_1,\ldots,x_m)\Big|_{(x_1,\ldots,x_m)=(x_1(\gamma),\ldots,x_m(\gamma))}\right)_{i,j=1,\ldots,m}$ *has some connections with the law of the random vector* $(S_{(\gamma,\lambda,1,0)}(t_i) - S_{(\gamma,\lambda,1,0)}(t_{i-1}))_{i=1,\ldots,m}$ *that appears in Proposition 3.1. More precisely it is a diagonal matrix by the independence of the increments and, as far as the diagonal entries are concerned, we have*

$$\frac{\partial^2}{\partial x_i^2}I_{t_1,\ldots,t_m}(x_1,\ldots,x_m)\Big|_{(x_1,\ldots,x_m)=(x_1(\gamma),\ldots,x_m(\gamma))}$$
$$= \frac{1}{(t_i - t_{i-1})\mathrm{Var}[S_{(\gamma,\lambda,1,0)}(1)]} = \frac{1}{\mathrm{Var}[S_{(\gamma,\lambda,1,0)}(t_i) - S_{(\gamma,\lambda,1,0)}(t_{i-1})]}\ (\textit{for all } i\in\{1,\ldots,m\}).$$

Remark 3.3. *The results presented in this section concern the increments of the process. However we can derive analogue results for the finite dimensional distributions of the subordinator. The idea is to combine the above propositions and a suitable transformation of the involved random vectors with the continuous function*

$$(x_1, \ldots, x_m) \mapsto f(x_1, \ldots, x_m) := \left(x_1, x_1 + x_2, \ldots, \sum_{i=1}^{m} x_i\right).$$

In particular, as far as Propositions 3.2 and 3.3 are concerned, we can apply the contraction principle recalled in Section 2.1. Then we have the following statements.

- *For all $\theta > 0$, the random vector $\theta S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{\gamma}))_{i=1,\ldots,m}$ is distributed as $S_{(\gamma,\lambda,1,0)}(t_i))_{i=1,\ldots,m}$; therefore $\{(\theta S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{\gamma}))_{i=1,\ldots,m} : \theta > 0\}$ is a family of identically distributed random vectors.*

- *The family of random vectors $\{(\theta^{g(\gamma)} S_{(\gamma,\lambda,\theta,0)}(t_i/\theta^{h(\gamma)}))_{i=1,\ldots,m} : \theta > 0\}$ satisfies the LDP with speed $\theta^{\gamma-h(\gamma)}$, or equivalently $\theta^{1-g(\gamma)}$, and good rate function $J_{t_1,\ldots,t_m}$ defined by*

$$\begin{aligned} J_{t_1,\ldots,t_m}(z_1, \ldots, z_m) &= \inf\{I_{t_1,\ldots,t_m}(x_1, \ldots, x_m) : f(x_1, \ldots, x_m) = (z_1, \ldots, z_m)\} \\ &= I_{t_1,\ldots,t_m}(z_1, z_2 - z_1, \ldots, z_m - z_{m-1}) = \sum_{i=1}^{m} (t_i - t_{i-1}) \kappa^*_{(\gamma,\lambda,1,0)}\left(\frac{z_i - z_{i-1}}{t_i - t_{i-1}}\right), \end{aligned}$$

  where $z_0 = 0$ in the last equality.

- *If condition (4) holds, then the random vectors $\{(a_\theta \theta S_{(\gamma,\lambda,\theta,0)}(t_i/(a_\theta \theta^{\gamma})))_{i=1,\ldots,m} : \theta > 0\}$ satisfy the LDP with speed $1/a_\theta$ and good rate function $J_{t_1,\ldots,t_m}$ defined in the item above.*

Remark 3.4. *All the results above and Remark 3.3 concern the case $\theta \to \infty$. In order to obtain the analogue versions for the case $\theta \to 0$ some changes are needed. Proposition 3.1 is essentially without changes because the involved random variables are identically distributed. The condition $\gamma - h(\gamma) = 1 - g(\gamma) > 0$ in Propositions 3.2 and 3.3 (and in Remark 3.3) has to be replaced with $\gamma - h(\gamma) = 1 - g(\gamma) < 0$. The speed function is again $\theta^{\gamma-h(\gamma)} = \theta^{1-g(\gamma)}$, which tends to infinity as $\theta \to 0$ because $\gamma - h(\gamma) = 1 - g(\gamma) < 0$. Condition (4) in Proposition 3.3 has to be replaced with*

$$a_\theta \to 0 \text{ and } \theta^{\gamma-h(\gamma)} a_\theta = \theta^{1-g(\gamma)} a_\theta \to \infty \text{ (as } \theta \to 0); \tag{5}$$

note that, in both conditions (4) and (5), one requires that a_θ tends to zero slowly. We can also present a version of Remark 3.1. Firstly Proposition 3.3 for the case $\theta \to 0$ provides a class of LDPs which fill the gap between a convergence to $(x_1(\gamma), \ldots, x_m(\gamma))$, and a trivial weak convergence as $\theta \to 0$ for a family of identically distributed random variables (they can be derived by Propositions 3.2 for the case $\theta \to 0$, and Proposition 3.1). Moreover the convergence to a constant and the trivial weak convergence correspond to the cases $a_\theta = \theta^{-(\gamma-h(\gamma))} = \theta^{-(1-g(\gamma))}$ and $a_\theta = 1$; so, in both cases, one condition in (5) holds and the other one fails. Finally, since the rate functions in Propositions 3.2 and 3.3 (and in Remark 3.3) for $\theta \to 0$ coincide with the ones for the case $\theta \to \infty$, we can repeat the comments in Remark 3.2 without any changes.

Remark 3.5. *It is possible to consider a more general version of Proposition 3.1 by replacing the process $\{S_{(\gamma,\lambda,\theta,0)}(t) : t \geq 0\}$ with a more general self-similar process $\{S(t) : t \geq 0\}$ with having independent and stationary increments. More precisely let $\{S(t) : t \geq 0\}$ be a self-similar process with index $H > 0$ and, again, let $m \geq 1$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m$ be arbitrarily fixed. Then, for all $\theta > 0$, the random vector $(\theta S(t_i/\theta^{1/H}) - \theta S(t_{i-1}/\theta^{1/H}))_{i=1,\ldots,m}$ is distributed as $(S(t_i) - S(t_{i-1}))_{i=1,\ldots,m}$. Furthermore a weaker version of this result, with $m = 1$ only, could be considered if we do not require any hypotheses on the increments.*

4 Large deviations for inverse processes

In this section we consider the inverse of $\{S_{(\gamma,\lambda,\theta,\delta)}(t) : t \geq 0\}$, i.e. the process $\{T_{(\gamma,\lambda,\theta,\delta)}(t) : t \geq 0\}$ defined by

$$T_{(\gamma,\lambda,\theta,\delta)}(t) := \inf\{u > 0 : S_{(\gamma,\lambda,\theta,\delta)}(u) > t\}.$$

Remark 4.1. *Assume that $\delta = 0$. Then $\{S_{(\gamma,\lambda,\theta,0)}(t) : t \geq 0\}$ and $\{S_{(\gamma,1,\theta,0)}(\lambda t) : t \geq 0\}$ have the same finite-dimensional distributions. Thus $\{T_{(\gamma,\lambda,\theta,0)}(t) : t \geq 0\}$ and $\left\{\frac{T_{(\gamma,1,\theta,0)}(t)}{\lambda} : t \geq 0\right\}$ also have the same finite-dimensional distributions; however, in what follows, we only need to consider their one-dimensional distributions.*

Our aim is to illustrate an application of the results for inverse processes in [8]; actually we always consider the simple case in which the u, v, w in that reference are the identity function. Moreover, since the speed for the LDPs in this section is always $v_t = t$, we omit this detail.

A naive approach is to consider the application of the Gärtner Ellis Theorem to $\{T_{(\gamma,\lambda,\theta,\delta)}(t)/t : t > 0\}$ as $t \to \infty$; in other words, if there exists (for all $y \in \mathbb{R}$)

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{yT_{(\gamma,\lambda,\theta,\delta)}(t)}] = \Lambda_{(\gamma,\lambda,\theta,\delta)}(y) \tag{6}$$

as an extended real number, and the function $\Lambda_{(\gamma,\lambda,\theta,\delta)}$ satisfies some conditions, we can say that $\{T_{(\gamma,\lambda,\theta,\delta)}(t)/t : t > 0\}$ satisfies the LDP with good rate function $\Lambda^*_{(\gamma,\lambda,\theta,\delta)}$ defined by

$$\Lambda^*_{(\gamma,\lambda,\theta,\delta)}(x) := \sup_{y\in\mathbb{R}}\{xy - \Lambda_{(\gamma,\lambda,\theta,\delta)}(y)\}.$$

Unfortunately, in general, the moment generating function $\mathbb{E}[e^{yT_{(\gamma,\lambda,\theta,\delta)}(t)}]$ is not available.

The approach based on the application of the results in [8] allows to overcome this problem. In order to do that we consider the LDP of $\{S_{(\gamma,\lambda,\theta,\delta)}(t)/t : t > 0\}$ as $t \to \infty$, and this can be done by considering an application of the Gärtner Ellis Theorem because the moment generating function $\mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(t)}]$ is available. In fact we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{yS_{(\gamma,\lambda,\theta,\delta)}(t)}] = \kappa_{(\gamma,\lambda,\theta,\delta)}(y) \text{ (for all } y \in \mathbb{R}), \tag{7}$$

where the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$ has been introduced in Section 2.2; moreover, if the function $\kappa_{(\gamma,\lambda,\theta,\delta)}$ satisfies some conditions (see the case $\theta > 0$ below), the LDP holds with good rate function $\kappa^*_{(\gamma,\lambda,\theta,\delta)}$ defined by (2). Then we can apply the results in [8] and we have the following claims.

Claim 4.1. *By Theorem 1(i) in [8], $\{T_{(\gamma,\lambda,\theta,\delta)}(t)/t : t > 0\}$ satisfies the LDP with good rate function $\Psi_{(\gamma,\lambda,\theta,\delta)}$ defined by*

$$\Psi_{(\gamma,\lambda,\theta,\delta)}(x) = x\kappa^*_{(\gamma,\lambda,\theta,\delta)}(1/x)$$

for $x > 0$, $\Psi_{(\gamma,\lambda,\theta,\delta)}(0) = \lim_{x\to 0^+} \Psi_{(\gamma,\lambda,\theta,\delta)}(x)$, and $\Psi_{(\gamma,\lambda,\theta,\delta)}(x) = \infty$ for $x < 0$.

Claim 4.2. *By Theorem 3(ii) in [8] (note that the function I in that reference coincides with $\kappa^*_{(\gamma,\lambda,\theta,\delta)}$ in this paper) condition (6) holds for $y < \kappa^*_{(\gamma,\lambda,\theta,\delta)}(0)$ and we have*

$$\Lambda_{(\gamma,\lambda,\theta,\delta)}(y) = \sup_{x\in\mathbb{R}}\{xy - \Psi_{(\gamma,\lambda,\theta,\delta)}(x)\}; \tag{8}$$

moreover we have

$$\kappa^*_{(\gamma,\lambda,\theta,\delta)}(0) = -\lim_{y\to-\infty} \kappa_{(\gamma,\lambda,\theta,\delta)}(y) = \begin{cases} \infty & \text{if } \gamma \in (0,1) \text{ and } \delta \geq 0 \\ \frac{1}{\delta}\log(1 + \lambda\delta\theta^\gamma) & \text{if } \gamma \in (-\infty, 0) \text{ and } \delta > 0 \\ \lambda\theta^\gamma & \text{if } \gamma \in (-\infty, 0) \text{ and } \delta = 0. \end{cases}$$

We also recall that, for all $\delta \geq 0$, we have $\kappa^*_{(\gamma,\lambda,\theta,\delta)}(x) = 0$ if and only if $x = \kappa'_{(\gamma,\lambda,\theta,\delta)}(0) = \lambda\text{sgn}(\gamma)\gamma\theta^{\gamma-1}$; thus, by the definition of $\Psi_{(\gamma,\lambda,\theta,\delta)}$, we have $\Psi_{(\gamma,\lambda,\theta,\delta)}(x) = 0$ if and only if $x = (\kappa'_{(\gamma,\lambda,\theta,\delta)}(0))^{-1}$.

We shall discuss the case $\theta = 0$ in Section 4.1 and the case $\theta > 0$ in Section 4.2. Finally, in Section 4.3, we shall compute the function $\Lambda_{(\gamma,\lambda,\theta,\delta)}$ in (8) when $\delta = 0$.

4.1 Case $\theta = 0$

In this case we cannot apply the Gärtner Ellis Theorem to obtain the LDP of $\{S_{(\gamma,\lambda,\theta,\delta)}(t)/t : t > 0\}$ as $t \to \infty$; in fact $\kappa_{(\gamma,\lambda,0,\delta)}$ is not finite in a neighborhood of the origin. We recall that we only have $\gamma \in (0,1)$ when $\theta = 0$. We can obtain the LDP of $\{T_{(\gamma,\lambda,0,\delta)}(t)/t : t > 0\}$ as $t \to \infty$ only if $\delta = 0$. In fact, if we consider the Mittag-Leffler function $E_\gamma(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\gamma k+1)}$, we have

$$\mathbb{E}[e^{yT_{(\gamma,\lambda,0,0)}(t)}] = \mathbb{E}[e^{yT_{(\gamma,1,0,0)}(t)/\lambda}] = E_\gamma\left(\frac{y}{\lambda}t^\gamma\right)$$

by Remark 4.1 and by a well-known result in the literature for $\lambda = 1$ (see e.g. eq. (24) in [22], or eq. (16) in [5] for the case $y \leq 0$). Then, by taking into account the asymptotic behavior of the Mittag-Leffler function as its argument tends to infinity (see e.g. eq. (1.8.27) in [15]), we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{yT_{(\gamma,\lambda,0,0)}(t)}] = \begin{cases} (y/\lambda)^{1/\gamma} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases} =: \Lambda_{(\gamma,\lambda,0,0)}(y). \tag{9}$$

So, in this case, we can consider the naive approach discussed above (see the sentence with equation (6)). Then the Gärtner Ellis Theorem yields the LDP of $\{T_{(\gamma,\lambda,0,0)}(t)/t : t > 0\}$ as $t \to \infty$ with good rate function $\Psi_{(\gamma,\lambda,0,0)} := \Lambda^*_{(\gamma,\lambda,0,0)}$, i.e.

$$\Psi_{(\gamma,\lambda,0,0)}(x) := \sup_{y\in\mathbb{R}}\{xy - \Lambda_{(\gamma,\lambda,0,0)}(y)\} = \begin{cases} \lambda^{1/(1-\gamma)}(\gamma^{\gamma/(1-\gamma)} - \gamma^{1/(1-\gamma)})x^{1/(1-\gamma)} & \text{if } x \geq 0 \\ \infty & \text{if } x < 0; \end{cases} \tag{10}$$

moreover, noting that $\Lambda'_{(\gamma,\lambda,0,0)}(0) = 0$, we have $\Lambda^*_{(\gamma,\lambda,0,0)}(x) = 0$ if and only if $x = 0$.

4.2 Case $\theta > 0$

In this case we can apply the Gärtner Ellis Theorem by considering the limit in (7); in fact the function $\kappa_{(\gamma,\lambda,\theta,\delta)}(y)$ is finite in a neighborhood of the origin. However, we cannot provide an explicit expression of $\kappa^*_{(\gamma,\lambda,\theta,\delta)}$ and $\Psi_{(\gamma,\lambda,\theta,\delta)}$ when $\delta > 0$. On the contrary, this is feasible when $\delta = 0$. In fact, after some easy computations, we get:

$$\kappa^*_{(\gamma,\lambda,\theta,0)}(x) = x\left(\theta - \left(\frac{x}{\lambda\mathrm{sgn}(\gamma)\gamma}\right)^{1/(\gamma-1)}\right) - \lambda\mathrm{sgn}(\gamma)\left(\theta^\gamma - \left(\frac{x}{\lambda\mathrm{sgn}(\gamma)\gamma}\right)^{\gamma/(\gamma-1)}\right) \text{ (for all } x > 0)$$

and $\kappa^*_{(\gamma,\lambda,\theta,\delta)}(0) = \infty$;

$$\Psi_{(\gamma,\lambda,\theta,0)}(x) = \theta - (\lambda\mathrm{sgn}(\gamma)\gamma x)^{1/(1-\gamma)} + \lambda\mathrm{sgn}(\gamma)x\left((\lambda\mathrm{sgn}(\gamma)\gamma x)^{\gamma/(1-\gamma)} - \theta^\gamma\right) \text{ (for all } x \geq 0). \tag{11}$$

Moreover, in particular, the right derivative of $\Psi_{(\gamma,\lambda,\theta,0)}$ at $y = 0$ is

$$\Psi'_{(\gamma,\lambda,\theta,0)}(0) = \begin{cases} -\lambda\theta^\gamma & \text{if } \gamma \in (0,1) \\ -\infty & \text{if } \gamma \in (-\infty,0). \end{cases} \tag{12}$$

We also remark that, when $\gamma \in (0,1)$, the expression of $\Psi_{(\gamma,\lambda,\theta,0)}$ in (11) yields

$$\Psi_{(\gamma,\lambda,\theta,0)}(x) = \theta + \Psi_{(\gamma,\lambda,0,0)}(x) - \lambda\theta^\gamma x \text{ (for all } x \geq 0), \tag{13}$$

where $\Psi_{(\gamma,\lambda,0,0)}$ is the function computed for the case $\theta = 0$ (see (10)).

4.3 The function $\Lambda_{(\gamma,\lambda,\theta,\delta)}$ in (8), for $\delta = 0$

We restrict the attention to the case $\delta = 0$ because we have an explicit expression for $\kappa^*_{(\gamma,\lambda,\theta,\delta)}$ and, obviously, also for $\Psi_{(\gamma,\lambda,\theta,\delta)}$. More precisely, by taking into account (13), we consider $\Psi_{(\gamma,\lambda,\theta,\delta)}$ in (11) for $\theta \geq 0$, with $\gamma \in (0,1)$ when $\theta = 0$.

So we have

$$\Lambda_{(\gamma,\lambda,\theta,0)}(y) = \sup_{x \geq 0}\{xy - \Psi_{(\gamma,\lambda,\theta,0)}(x)\}$$

where, if we consider the positive constant c_γ defined by

$$c_\gamma := \begin{cases} \gamma^{\gamma/(1-\gamma)} - \gamma^{1/(1-\gamma)} & \text{if } \gamma \in (0,1) \\ (-\gamma)^{\gamma/(1-\gamma)} + (-\gamma)^{1/(1-\gamma)} & \text{if } \gamma \in (-\infty,0), \end{cases}$$

we have

$$\Psi_{(\gamma,\lambda,\theta,0)}(x) = \begin{cases} \theta - \lambda\theta^\gamma x + \lambda^{1/(1-\gamma)} c_\gamma x^{\gamma/(1-\gamma)} & \text{if } \gamma \in (0,1) \\ \theta + \lambda\theta^\gamma x - \lambda^{1/(1-\gamma)} c_\gamma x^{\gamma/(1-\gamma)} & \text{if } \gamma \in (-\infty,0) \end{cases} \quad \text{for all } x \geq 0.$$

Then we can state some results in the following lemma. Note that the next formula (14) with $\theta = 0$ meets the expression of the limit in (9).

Lemma 4.3. *We have:*

$$\Lambda_{(\gamma,\lambda,\theta,0)}(y) = \begin{cases} -\theta & \text{if } y < -\lambda\theta^\gamma \\ \left(\theta^\gamma + \frac{y}{\lambda}\right)^{1/\gamma} - \theta & \text{if } y \geq -\lambda\theta^\gamma \end{cases} \quad \text{for } \gamma \in (0,1), \tag{14}$$

$$\Lambda_{(\gamma,\lambda,\theta,0)}(y) = \begin{cases} \left(\theta^\gamma - \frac{y}{\lambda}\right)^{1/\gamma} - \theta & \text{if } y < \lambda\theta^\gamma \\ \infty & \text{if } y \geq \lambda\theta^\gamma \end{cases} \quad \text{for } \gamma \in (-\infty,0), \tag{15}$$

and, in both cases,

$$\Psi_{(\gamma,\lambda,\theta,0)}(x) = \sup_{y \in \mathbb{R}}\{xy - \Lambda_{(\gamma,\lambda,\theta,0)}(y)\}.$$

Proof. All the results can be proved with some standard computations. The details are omitted. □

We conclude with another remark concerning both cases $\gamma \in (0,1)$ and $\gamma \in (-\infty,0)$. Note that equation (16) in the following remark has some analogies with equations (12)-(13) in [11] where the authors deal with counting processes, which are non-decreasing processes.

Remark 4.2. *For all $x \geq 0$ we have*

$$x\kappa^*_{(\gamma,\lambda,\theta,0)}(1/x) = x\sup_{y \leq \theta}\{y/x - \kappa_{(\gamma,\lambda,\theta,0)}(y)\} = \sup_{y \leq \theta}\{y - x\kappa_{(\gamma,\lambda,\theta,0)}(y)\};$$

moreover, if we consider the change of variable $z = -\kappa_{(\gamma,\lambda,\theta,0)}(y)$ and if we set

$$\mathcal{I} := (-\kappa_{(\gamma,\lambda,\theta,0)}(\theta), -\kappa_{(\gamma,\lambda,\theta,0)}(-\infty)),$$

then we get

$$x\kappa^*_{(\gamma,\lambda,\theta,0)}(1/x) = \sup_{z \in \mathcal{I}}\left\{\kappa^{-1}_{(\gamma,\lambda,\theta,0)}(-z) + xz\right\} = \sup_{z \in \mathcal{I}}\left\{xz - (-\kappa^{-1}_{(\gamma,\lambda,\theta,0)}(-z))\right\}.$$

Thus $\Psi_{(\gamma,\lambda,\theta,0)}$ can be seen as the Fenchel-Legendre transform of the function

$$z \mapsto \tilde{\Psi}(z) := -\kappa^{-1}_{(\gamma,\lambda,\theta,0)}(-z), \tag{16}$$

where z belongs to a suitable set where the inverse function is well-defined. In fact we have

$$\mathcal{I} = \begin{cases} (-\lambda\theta^\gamma, \infty) & \text{if } \gamma \in (0,1) \\ (-\infty, \lambda\theta^\gamma) & \text{if } \gamma \in (-\infty,0). \end{cases}$$

and, in both cases $\gamma \in (-\infty,0)$ and $\gamma \in (0,1)$, the interval $\mathcal{I}$ coincides with the set where the function $\Lambda_{(\gamma,\lambda,\theta,0)}$ is strictly increasing and finite (see (14) and (15)).

5 Large deviations for time-changes with inverse processes

The aim of this section is to present some applications of the Gärtner Ellis Theorem in order to obtain LDPs for $\{X(T_{(\gamma,\lambda,\theta,\delta)}(t))/t : t > 0\}$, when $\{X(t) : t \geq 0\}$ is some suitable $\mathbb{R}^h$-valued process (for some integer $h \geq 1$), and independent of $\{T_{(\gamma,\lambda,\theta,\delta)}(t) : t \geq 0\}$. Actually, since we want to refer to the contents of Sections 4.1 and 4.2, in this section we always restrict the attention to the case $\delta = 0$. Moreover all the LDPs stated in this section holds with speed t; therefore we always omit this detail (as we did in Section 4).

The simplest case is when $\{X(t) : t \geq 0\}$ is a Lévy process; in fact we have

$$\mathbb{E}[e^{\langle \eta, X(t)\rangle}] = e^{t\Lambda_X(\eta)} \text{ (for all } \eta \in \mathbb{R}^h\text{)}, \text{ where } \Lambda_X(\eta) := \log \mathbb{E}[e^{\langle \eta, X(1)\rangle}].$$

In this case the application of the Gärtner Ellis Theorem works well when the function Λ_X is finite in a neighborhood of the origin $\eta = 0 \in \mathbb{R}^h$; if $h = 1$ this means that all the random variables $\{X(t) : t \geq 0\}$ are light tailed distributed (see e.g. [1], Chapter I, Section 2).

A more general situation concerns additive functionals of Markov processes (here we recall [26] as a reference with results based on the Gärtner Ellis Theorem); however, for simplicity, we refer to the case of Markov additive processes (see e.g. [1], Chapter III, Section 4; actually the presentation in that reference concerns the case $h = 1$). We have a Markov additive process $\{(J(t), X(t)) : t \geq 0\}$ if, for some set E, it is a $E \times \mathbb{R}^h$-valued Markov process with suitable properties; in particular $\{J(t) : t \geq 0\}$ is a Markov process. We refer to the continuous time case with a finite state space E for $\{J(t) : t \geq 0\}$; see e.g. [1], page 55. We also assume that $\{J(t) : t \geq 0\}$ is irreducible and, for simplicity, that $\mathbb{E}[e^{\langle \eta, X(t)\rangle}] < \infty$ for all $\eta \in \mathbb{R}^h$. Then, as a consequence of Proposition 4.4 in Chapter III in [1], we have

$$\min_{i \in E} h_i(\eta) e^{t\Lambda_X(\eta)} \leq \mathbb{E}[e^{\langle \eta, X(t)\rangle}] \leq \max_{i \in E} h_i(\eta) e^{t\Lambda_X(\eta)}$$

where $e^{t\Lambda_X(\eta)}$ is a suitable simple and positive eigenvalue and $(h_i(\eta))_{i \in E}$ is a positive eigenvector (these items can be found by a suitable application of the Perron Frobenius Theorem).

Now we are ready to illustrate the applications of the Gärtner Ellis Theorem which provides the LDP for $\{X(T_{(\gamma,\lambda,\theta,\delta)}(t))/t : t > 0\}$ with rate function $H_{(\gamma,\lambda,\theta,0)}$, say. In particular we can have a trapping and delaying effect for $\theta = 0$ (see Remark 5.1), and a possible rushing effect for $\theta > 0$; we recall a recent reference with this kind of analysis for time-changed processes is [6], even if the approach in this paper is different from the one in that reference. We also give some comments on the behavior of $H_{(\gamma,\lambda,\theta,0)}(x)$ around the origin $x = 0$ for $h = 1$; this will be done for both cases $\theta = 0$ and $\theta > 0$, and we see that right and left derivatives at $x = 0$ (which will be denoted by $D_- H_{(\gamma,\lambda,\theta,0)}(0)$ and $D_+ H_{(\gamma,\lambda,\theta,0)}(0)$) can be different.

5.1 Case $\theta = 0$

Here we refer to the content of Section 4.1. We also recall that we only have $\gamma \in (0, 1)$ when $\theta = 0$. Then, after some standard computations (with a conditional expectation with respect to the independent random time-change), we get

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\langle \eta, X(T_{(\gamma,\lambda,0,0)}(t))\rangle}] &= \lim_{t\to\infty} \frac{1}{t} \log E_\gamma \left(\frac{\Lambda_X(\eta)}{\lambda} t^\gamma \right) \\ &= \left\{ \begin{array}{ll} (\Lambda_X(\eta)/\lambda)^{1/\gamma} & \text{if } \Lambda_X(\eta) \geq 0 \\ 0 & \text{if } \Lambda_X(\eta) < 0 \end{array} \right. = \Lambda_{(\gamma,\lambda,0,0)}(\Lambda_X(\eta)),\end{aligned}$$

where $\Lambda_{(\gamma,\lambda,0,0)}(\cdot)$ is the function in (9) (see also (14) with $\theta = 0$).

Then, under suitable hypotheses, by the Gärtner Ellis Theorem, $\{X(T_{(\gamma,\lambda,0,0)}(t))/t : t > 0\}$ satisfies the LDP with good rate function $H_{(\gamma,\lambda,0,0)}$ defined by

$$H_{(\gamma,\lambda,0,0)}(x) := \sup_{\eta \in \mathbb{R}^h} \{\langle \eta, x\rangle - \Lambda_{(\gamma,\lambda,0,0)}(\Lambda_X(\eta))\}.$$

We can say that $H_{(\gamma,\lambda,0,0)}(x) = 0$ if and only if $x = \Lambda'_{(\gamma,\lambda,0,0)}(\Lambda_X(0))\nabla\Lambda_X(0)$; thus, since $\Lambda_X(0) = 0$ and $\Lambda'_{(\gamma,\lambda,0,0)}(0) = 0$, we have $H_{(\gamma,1,0,0)}(x) = 0$ if and only if $x = 0$, whatever is $\nabla\Lambda_X(0)$.

Remark 5.1. *We can say that* $\frac{X(T_{(\gamma,\lambda,0,0)}(t))}{t}$ *converges to zero as* $t \to \infty$ *(at least in probability; see Remark 2.1 for a discussion on the almost sure of* $\frac{X(T_{(\gamma,\lambda,0,0)}(t_n))}{t_n}$ *along a sequence* $\{t_n : n \geq 1\}$ *such that* $t_n \to \infty$*), and this happens whatever is the limit* $\nabla\Lambda_X(0)$ *of* $\frac{X(t)}{t}$*. This is not surprising because random time-changes with* $\{T_{(\gamma,\lambda,0,0)}(t) : t \geq 0\}$ *typically give rise to a sort of trapping and delaying effect; a discussion on this aspect for random time-changes can be found in [6].*

We conclude with some statements for the case $h = 1$. In what follows we consider certain inequalities; however similar statements hold if we consider inverse inequalities. We assume that $\Lambda'_X(0) > 0$.

- If there exists $\eta_0 < 0$ such that $\Lambda_X(\eta_0) = 0$ (note that this condition can occur because Λ_X is convex and $\Lambda_X(0) = 0$), we can say that $D_-H_{(\gamma,\lambda,0,0)}(0) = \eta_0$ and $D_+H_{(\gamma,\lambda,0,0)}(0) = 0$.
- On the contrary, if Λ_X is strictly increasing (and therefore uniquely vanishes at $\eta = 0$), we have $H_{(\gamma,\lambda,0,0)}(x) = \infty$ for all $x < 0$ instead of $D_-H_{(\gamma,\lambda,0,0)}(0) = \eta_0$.

5.2 Case $\theta > 0$

Here we refer to the content of Section 4.2. We start with the same standard computations considered in Section 4.1 but here we cannot refer to (9). In fact in this case we refer to Claim 4.2 in order to have the limit (6) for all $y \in \mathbb{R}$; so, as stated in Claim 4.2, we take $\gamma \in (0, 1)$ in order to have $\kappa^*_{(\gamma,\lambda,\theta,\delta)}(0) = \infty$. Then we get

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[e^{\langle \eta, X(T_{(\gamma,\lambda,\theta,0)}(t))\rangle}] = \Lambda_{(\gamma,\lambda,\theta,0)}(\Lambda_X(\eta));$$

moreover $\Lambda_{(\gamma,\lambda,\theta,0)}(\cdot)$ is given by (14).

Then, under suitable hypotheses, by the Gärtner Ellis Theorem, $\{X(T_{(\gamma,\lambda,\theta,0)}(t))/t : t > 0\}$ satisfies the LDP with good rate function $H_{(\gamma,\lambda,\theta,0)}$ defined by

$$H_{(\gamma,\lambda,\theta,0)}(x) := \sup_{\eta\in\mathbb{R}^h} \{\langle \eta, x\rangle - \Lambda_{(\gamma,\lambda,\theta,0)}(\Lambda_X(\eta))\}.$$

We can say that $H_{(\gamma,\lambda,\theta,0)}(x) = 0$ if and only if $x = \Lambda'_{(\gamma,\lambda,\theta,0)}(\Lambda_X(0))\nabla\Lambda_X(0)$; thus, since $\Lambda_X(0) = 0$ and $\Lambda'_{(\gamma,\lambda,\theta,0)}(0) = \frac{\theta^{1-\gamma}}{\lambda\gamma}$ (note that $\Lambda'_{(\gamma,\lambda,\theta,0)}(0) = (\kappa'_{(\gamma,\lambda,\theta,0)}(0))^{-1}$ as one can expect), we have $H_{(\gamma,\lambda,\theta,0)}(x) = 0$ if and only if $x = \frac{\theta^{1-\gamma}}{\lambda\gamma}\nabla\Lambda_X(0)$. So $X(T_{(\gamma,\lambda,\theta,0)}(t))/t$ converges to a limit that depends on $\nabla\Lambda_X(0)$, and we have a possible rushing effect.

We conclude with some statements for the case $h = 1$. In what follows we consider certain inequalities; however similar statements hold if we consider inverse inequalities.

- If there exists $\eta_1 < \eta_2 < 0$ such that $\Lambda_X(\eta_1) = \Lambda_X(\eta_2) = -\lambda\theta^\gamma$ (and this happens if $\Lambda'_X(0) > 0$), then $D_-H_{(\gamma,\lambda,\theta,0)}(0) = \eta_1$ and $D_+H_{(\gamma,\lambda,\theta,0)}(0) = \eta_2$.
- On the contrary, if there exists a unique $\eta_0 < 0$ such that $\Lambda_X(\eta_0) = -\lambda\theta^\gamma$ (and this could happen if Λ_X is strictly increasing) we have $D_+H_{(\gamma,\lambda,\theta,0)}(0) = \eta_0$ and $H_{(\gamma,\lambda,\theta,0)}(x) = \infty$ for $x < 0$.

Remark 5.2. *Note that* $H_{(\gamma,\lambda,\theta,0)}$ *coincides with* $\Psi_{(\gamma,\lambda,\theta,0)}$ *when we have* $X(t) = t$ *for all* $t \geq 0$*. In such a case* $\Lambda_X(\eta) = \eta$ *for all* $\eta \in \mathbb{R}$ *and therefore we have* $\Lambda_X(\eta_0) = -\lambda\theta^\gamma$ *for* $\eta_0 = -\lambda\theta^\gamma < 0$*. Thus we get* $D_+H_{(\gamma,\lambda,\theta,0)}(0) = -\lambda\theta^\gamma$ *and this agrees with the right derivative of* $\Psi_{(\gamma,\lambda,\theta,0)}(y)$ *at* $y = 0$ *in* (12) *for* $\gamma \in (0, 1)$.

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