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The number of configurations in lattice point counting I

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Abstract. When a strictly convex plane set S moves by translation, the set J of points of the integer lattice that lie in S changes. The number K of equivalence classes of sets J under lattice translations (configurations) is bounded in terms of the area of the Brunn-Minkowski difference set of S. If S satisfies the Triangle Condition, that no translate of S has three distinct lattice points in the boundary, then K is asymptotically equal to the area of the difference set, with an error term like that in the corresponding lattice point problem. If S satisfies a Smoothness Condition but not the Triangle Condition, then we obtain a lower bound for K, but not of the right order of magnitude.

The case when S is a circle was treated in our earlier paper by a more complicated method. The Triangle Condition was removed by considerations of norms of Gaussian integers, which are special to the circle.

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1 Introduction

A computer screen can be regarded as a finite part of a square lattice of lamps (Pick elements or "pixels"). A shape S is displayed by illuminating the lattice points which are computed to be inside the shape. On the length scale induced by the square lattice, the number of pixels lit is approximately A, the area of S, as in the Gauss circle problem. When the shape moves slowly across the screen, pixels are lit on the leading edge and others are extinguished on the trailing edge. Let $S(P)$ or $S(u, v)$ be the set obtained by translating the set S by the vector (u, v), so the origin (an interior point of S) moves to a point P, (u, v). Let $J(P)$ be the set of lattice points in $S(P)$, and let $N(P)$ be the number of points in $J(P)$.

The problem of machine vision is to "recognise" S from the set $J(P)$. Two sets $J(P)$ and $J(Q)$ are called equivalent if $J(Q)$ is a translation of $J(P)$ by a lattice vector. The number of inequivalent configurations $K(S)$ is a measure of the difficulty of the recognition problem. We assume that the whole of S is on-screen; another problem is to recognise S when the set $J(P)$ is partly off-screen. For a rectangle with sides parallel to the axes of the square lattice, the number of configurations is 4 or 5. We consider an "oval" S which is strictly

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convex, and may satisfy piecewise some differentiability conditions. For the circle radius R, the number of configurations $K(S)$ is asymptotic to $4\pi R^2$ [8]. We generalise this result to ovals satisfying the following condition.

Triangle Condition. There are no translates of lattice triangles in the boundary of S. The Triangle Condition is false for the circle, but in [8] we estimated the number of translates of lattice triangles directly as a contribution to the error term.

There is a further configuration number question, when the set S is allowed to move away from or towards the viewer, so a point (x, y) of S has a screen image $\lambda(x, y) + (u, v)$, where $\lambda > 0$ gives the change in screen size due to perspective. We write $S(\lambda, u, v)$, $J(\lambda, u, v)$ and $N(\lambda, u, v)$. We consider this question in the next paper [10] in this series.

To distinguish the effects of shape and size, we suppose that $S = RS_0$ where S_0 is a bounded strictly convex set containing the origin as an interior point, which determines the shape of S, and R is a scaling factor giving the size of S.

We summarize the known results about $N(P)$ for convex sets S (polygons as well as ovals) from Gauss [1] to Huxley ([6] Theorem 5).

Proposition 1. *Let S be a convex plane region of area A bounded by arcs G_i of lengths $l_i \geq 1$. An arc G_i is called* good *if there is a length scale $R_i \geq 2$ such that when we treat the radius of curvature ρ as a function of the tangent angle ψ on G_i, then ρ is continuously differentiable with*

$$c_1 R_i \leq \rho \leq c_2 R_i, \qquad \left|\frac{d\rho}{d\psi}\right| \leq c_3 R_i;$$

otherwise G_i is called bad*. Then the number of integer points in S is*

$$A + \mathbb{O}\left(\sum_i^{(1)} R_i^{\kappa}(\log R_i)^{\mu}\right) + \mathbb{O}\left(\sum_i^{(2)} l_i\right), \tag{1.1}$$

with $\kappa = 131/208$, $\mu = 18624/8320$. The sum $\sum^{(1)}$ is over good arcs and the sum $\sum^{(2)}$ is over bad arcs. The positive dimensionless constants c_1, c_2, c_3 may be chosen arbitrarily, but the implied constant in the first $\mathbb{O}$-sign is constructed from c_1, c_2 and c_3.

We can now give the appropriate differentiability condition.

Smoothness Condition. The set S is bounded by arcs G_i of lengths $l_i \geq 1$. At interior points of each arc G_i, the radius of curvature is a continuously differentiable function with

$$c_1 R \leq \rho \leq c_2 R, \qquad \left|\frac{d\rho}{d\psi}\right| \leq c_3 R \tag{1.2}$$

for some positive constants c_1, c_2 and c_3.

From Proposition 1 we see that if $S = RS_0$ is any convex plane region, and if $R \geq 2$, then

$$N(P) = A + \mathbb{O}(R) = A_0 R^2 + \mathbb{O}(R),$$

where A_0 is the area of S_0. If S_0 (and so S also) satisfies the Smoothness Condition, then

$$N(P) = A + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right).$$

If the set S is presented in polar coordinates as $r \leq f(\theta)$, then a sufficient set of conditions for the Smoothness Condition to hold is that there is a constant $c > 1$ for which

$$\frac{1}{c} \leq f(\theta) \leq c, \tag{1.3}$$

$$\frac{f^{(r)}(\theta)}{f(\theta)} \leq c^{r+1} \tag{1.4}$$

for $r = 1, 2, 3$, and

$$-c^3 \leq \frac{f''(\theta)}{f(\theta)} \leq 1 - \frac{1}{c^3} \tag{1.5}$$

hold piecewise.

To state our result for $K(S)$, the number of configurations, we introduce the Brunn-Minkowski difference set of S,

$$T = \{\mathbf{x} - \mathbf{y} \mid \mathbf{x}, \mathbf{y} \in S\}.$$

Let B be the area of T. There are scaling relations: let S_0 have difference set T_0 of area B_0. Then

$$T = RT_0, \qquad B = B_0 R^2. \tag{1.6}$$

Theorem 1. *Let S be a strictly convex plane set. Then*

$$K(S) \leq B + \mathcal{O}(R). \tag{1.7}$$

If S satisfies the Triangle Condition, then

$$K(S) = B + \mathcal{O}(R). \tag{1.8}$$

If S satisfies the Smoothness Condition, then (1.7) *can be sharpened to*

$$K(S) \leq B + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right). \tag{1.9}$$

If S satisfies the Smoothness Condition and the Triangle Condition, then

$$K(S) = B + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right). \tag{1.10}$$

The asymptotics (1.10) *are also true when S is a circle, with $B = 4A$.*

Theorem 2. *Let S satisfy the Smoothness Condition. Let C be the boundary curve of S, of length L. Let $\rho(\alpha)$ denote the radius of curvature of S at the point where the tangent (oriented anticlockwise) is at an angle α to the x-axis. Let $D(\alpha)$ denote the width of S measured between the two parallel tangents at an angle α. Then the area of the difference set T is*

$$(1.11)\qquad B = \int_0^{2\pi} D(\psi)\rho(\psi)\,d\psi,$$

lying in the range

$$(1.12)\qquad 4A \le B \le L^2/\pi.$$

If S is centrally symmetric ($F(\theta+\pi) = F(\theta)$ in the notation of (1.3)*), then $B = 4A$. If C has the property that $\rho(\psi+\pi)+\rho(\psi)$ is constant, then $B = L^2/\pi$.*

In an interval $R' \le R \le R'+1$, there are at most $\mathbb{O}(R'^3)$ values of R for which the Triangle Condition is false. Theorem 1 predicts about 314 inequivalent configurations when S is the circle radius 5. In fact there are 256 configurations, and 1320 translates of lattice point triangles. For the circles radii 4.999 and 5.0005, which do satisfy the Triangle Condition, there are respectively 304 and 316 configurations.

Theorem 3. *Let S satisfy the Smoothness Condition and* (1.3)*,* (1,4) *and* (1.5)*. Then for*

$$(1.13)\qquad R \ge 2^{17}3^2c^{14}$$

there are at least

$$(1.14)\qquad \frac{A_0R^{3/2}}{768\sqrt{2}c^8}$$

inequivalent configurations of integer points.

For a particular shape such as the circle, the constants in Theorem 3 can be greatly improved by calculating directly from the equation of the boundary curve C.

Theorem 4. *Let S be a strictly convex plane set. For $k \ge 3$, let $H(k)$ be the number of distinct points P modulo the integer lattice for which $S(P)$ has exactly k integer points in the boundary. Then*

$$(1.15)\qquad \sum_{k\ge 3}(k-1)(k-2)H(k) = \mathbb{O}\left(R^2\right).$$

If the upper bound in Theorem 4 could be sharpened, then in Theorem 1 we would find $K(S)$ asymptotically equal to B without assuming the Triangle Condition. It is a classical result that k must be $\mathbb{O}(R^{2/3})$ in Theorem 4 [11], with improvements [13, 5, 7] under various smoothness conditions.

In the proofs we consider the domain of a configuration J,

$$D(J) = \{P \mid J(P) = J\},$$

and establish a correspondence between vertices of the domains and integer points in the difference set T. In the first version of this paper, the correspondence was with integer points in a four-dimensional region, as in our earlier paper [8]. We thank Professor Kolountzakis for identifying B as the volume of the difference set. This observation led to the present simpler proof.

Our arguments do not determine whether the domains $D(J)$ obey a distribution law within the unit square (probably uniform, but possibly depending on the shape of S). The centres of mass of configurations should show the same distribution, although the centre of mass of J may not be a point of $D(J)$.

This work forms part of INTAS research project 03-51-5070 on ‘Analytic and Combinatorial Methods in Number Theory and Geometry’; the project title provides a good description of this paper.

2 Domains and the difference set

An S-oval $S(P)$ is the translate of S by the vector $\overrightarrow{OP}$. Let S' be the set S rotated through 180 degrees. An S'-oval $S'(M)$ is the translate of S' by the vector $\overrightarrow{OM}$. The integer point M lies in the S-oval $S(P)$ if and only if the point P lies in the S'-oval $S'(M)$. The configuration $J(P)$ is the set of integer points M for which P lies in $S'(M)$.

The domain $D(J)$ of a configuration J is the set of points P for which $J(P) = J$. The boundary of $D(J)$ is composed of arcs of the boundary curves $C'(M)$ of certain S'-ovals $S'(M)$, convex arcs when the integer point M is in J, concave arcs when M is not in J.

We regard the boundary curves $C'(M)$ as forming a planar graph, with vertices at the intersections of two or more boundary curves. All vertices have even valency. If the Triangle Condition holds for S, then it holds for S', so three boundaries of S'-ovals with different integer point centres cannot meet. Under the Triangle Condition all vertices have valency 4.

There is a partial correspondence between regions in the planar graph and domains of configurations. However some domains consist of a single point, a vertex, and some domains are disconnected. To explain this, we replace S' by $\lambda S'$ and consider the S'-ovals of varying size,

$$S'(\lambda, M) = \lambda S' + \overrightarrow{OM},$$

as λ increases. The boundary curves $C'(\lambda, M)$ move outwards as λ increases, and the intersection points of two boundaries $C'(\lambda, L)$ and $C'(\lambda, M)$ move around the boundaries as they expand. When the intersection point crosses a third boundary, one region shrinks and disappears, and another region appears on the opposite side of the intersection point. Even when four or more boundaries meet at a particular value of λ, for each region that disappears, another region appears on the opposite side of all the intersecting boundaries.

For counting purposes we identify each region which disappears at a multiple intersection with the region that appears opposite. The configuration $J(P)$ associated with the region

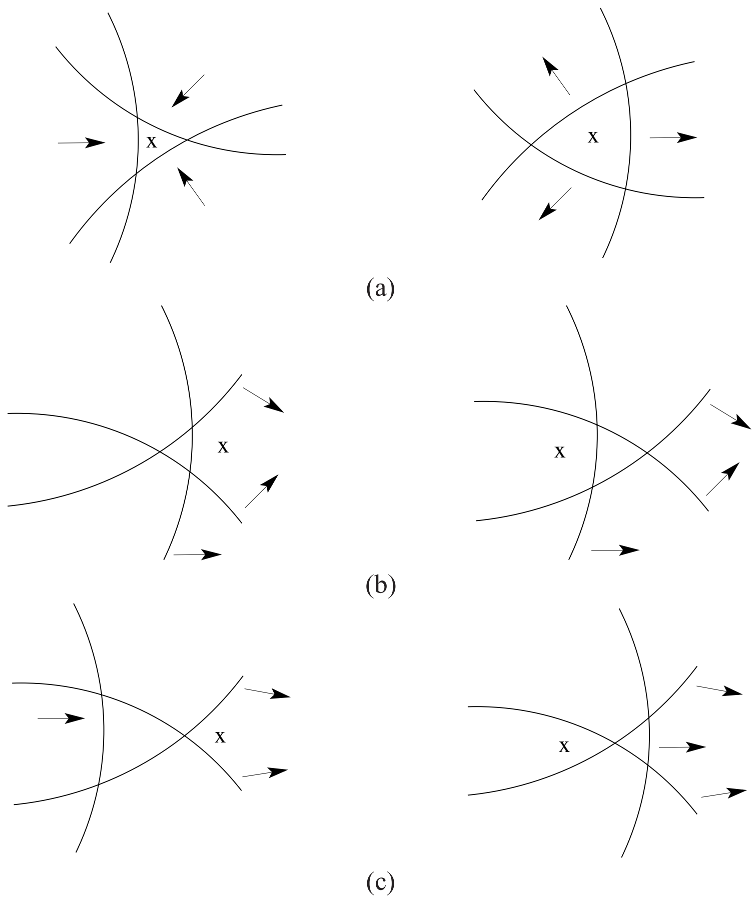

Figure 1 Reidemeister move: before and after a meeting of three boundaries.

changes. Usually $N(P)$ increases (cases (a) and (b) in Figure 1), but $N(P)$ can decrease (case (c) in Figure 1).

A rarer phenomenon as λ increases is that two disjoint S'-ovals $S'(\lambda, L)$ and $S'(\lambda, M)$ come to intersect. If no third boundary of an S'-oval is involved, there is a new configuration in the region of overlap, and the domain which was bounded (in part) by $C'(\lambda, L)$ and $C'(\lambda, M)$ is disconnected (Figure 2). So the number of regions increases by two.

Lemma 2.1 (multiple intersections). *Suppose there is a multiple intersections of k boundaries of S'-ovals at P, with l pairs of boundaries each meeting the other along a common tac-line. Then for $\lambda > 1$ and λ sufficiently close to 1, there are*

$$\frac{1}{2}(k-1)(k-2)+l$$

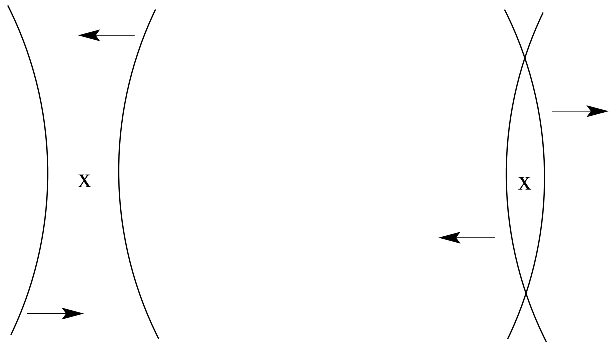

Figure 2 Reidemeister move: before and after an overlapping of two S'-ovals.

more regions for $\lambda S'$-ovals in a neighbourhood of P, whilst for $\lambda < 1$ and λ sufficiently close to 1*, there are*

$$\frac{1}{2}(k-1)(k-2)-l$$

more regions for $\lambda S'$-ovals in a neighbourhood of P.

Proof. When k lines cross at a point, they separate $2k$ regions. Consider k lines (not necessarily straight), added one at a time, with the n-th line cutting the previous $n-1$ lines in distinct points. With no lines, there is one region. The n-th line adds n extra regions. So the number of missing regions at a k-fold crossing is

$$1+\sum_{n=1}^{k} n-2k=1+\frac{k(k+1)}{2}-2k=\frac{k^2-3k+2}{2}.$$

If we allow a pair of lines to be tangent at $\lambda = 1$, then they do not meet for $\lambda < 1$, but they cut twice for $\lambda > 1$. Hence for each pair of boundaries tangent to one another for $\lambda = 1$, there is one less region for $\lambda < 1$, and one more region for $\lambda > 1$. □

We call λR a special value if the boundaries $C'(\lambda, L)$ and $C'(\lambda, M)$ of two S'-ovals with integer points L, M as centres have exactly one point in common, or if the boundaries of three or more S'-ovals with integer point centres have a common point. As λR increases through a value where two S'-ovals meet at a point P, there is a configuration $J(\lambda, P)$, containing both centres L and M of the S'-ovals, which first appears at the point P at this value of λ. When three boundaries $C'(\lambda, K)$, $C'(\lambda, L)$ and $C'(\lambda, M)$ meet at a point, in cases (b) and (c) of Figure 1 a configuration containing K, L and M already exists, but in case (a) such a configuration appears at the intersection point P as $J(\lambda, P)$ for this value of λ, replacing a configuration containing neither K, L nor M. So if R itself is a special value, configurations whose domain is a single point must occur where two S'-ovals have exactly one point in common, and may occur where three or more S'-ovals have a common boundary point.

For two given integer points L and M, there is one special value λR for which $S'(\lambda, L)$ and $S'(\lambda, M)$ have one point in common. It is also true that for three integer points K, L and M, there is at most one special value for which $S'(\lambda, K)$, $S'(\lambda, L)$ and $S'(\lambda, M)$ have a common boundary point. This is because a strictly convex set cannot have two triangles KLM, $K'L'M'$ of boundary points in the same cyclic order with corresponding sides parallel. Argument of this type are central to the second paper [10]. If $R' \leq \lambda R \leq R' + 1$, then the points K, L, M lie within a distance $\mathbb{O}(1)$ of the boundary of S, so there are $\mathbb{O}(R')$ such points and $\mathbb{O}(R'^3)$ special values of λR, as asserted in the Introduction. □

Lemma 2.2 (regions and the difference set). *For R sufficiently large, all configurations $J(P)$ are non-empty, and their domains are bounded. Let F be the number of integer points in the difference set T of S. Then for R sufficiently large and not a special value, the number of equivalence classes of regions in the domains diagram is $F - 1$.*

Proof. The domain of a non-empty configuration J is a subset of the intersection of the sets $S'(m)$ for integer points M in J, which is a bounded set. The origin O is an interior point of S_0, so both S_0 and S'_0, its rotation by 180º about O, contain a neighbourhood of the origin. For R sufficiently large, $S'(O) = RS'_0$ contains every point P in the unit square, and $S(P)$ contains the integer point O. Every point Q is equivalent to some point P in the unit square by

$$\overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{OM},$$

where $\overrightarrow{OM}$ is an integer vector, so $S(Q)$ contains the integer point M.

The plane modulo the integer lattice is the unit square with opposite sides identified, which is a torus. For λR small, there are two equivalence classes of configurations. Configurations of one point have simply connected domains, but the empty configuration has a multiply connected domain. The only integer point in λT is the origin. As λR increases through a value at which two S'-ovals $S'(\lambda, L)$ and $S'(\lambda, M)$ first acquire a point P in common, there are two new regions locally, and the two integer points L and M in $S(\lambda, P)$ correspond to two vectors $\overrightarrow{LM}$ and $\overrightarrow{ML}$ which give integer points N and N' in the difference set λT. Since N and N' do not lie in $\lambda' T$ for any $\lambda' < \lambda$, they are boundary points of λT.

Conversely, an integer point N on the boundary of λT corresponds to a chord UV of $S(\lambda, O)$ whose length is maximal among chords parallel to UV. At each point of the boundary of a strictly convex set there is a tac-line, a line l which has one point in the convex set, and all other points of the convex set lie on the same side of l. If every tac-line at U meets every tac-line at V on the same side of UV, then there is a chord $U'V'$, parallel to UV but longer than UV, on the opposite side of UV to the intersection of the tac-lines.

Since UV is maximal, there must be tac-lines l at U and m at V which are parallel. We chose P so that $\overrightarrow{OU} + \overrightarrow{OP}$ is an integer vector $\overrightarrow{OL}$. Since $\overrightarrow{UV} = \overrightarrow{ON}$, an integer vector, $\overrightarrow{OV} + \overrightarrow{OP}$ is an integer vector $\overrightarrow{OM}$. Now L and M are integer points on the boundary of $S(\lambda, P)$, with tac-lines parallel to l. Thus $S'(L)$, $S'(M)$ meet in one point at P, with a common tac-line parallel to l.

The $F - 1$ non-zero integer points in T form pairs symmetric in the origin. Each pair corresponds to two new regions locally. The domain for J empty contains two non-trivial circuits

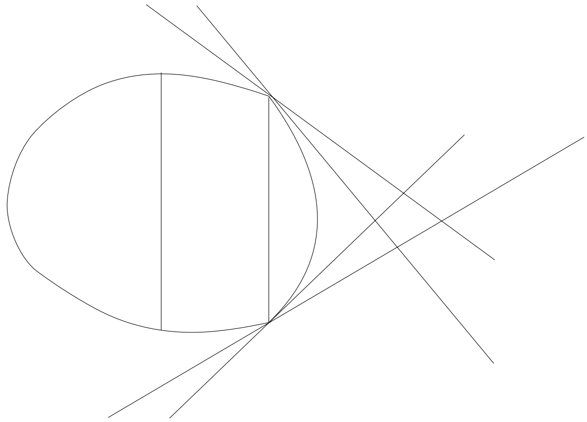

Figure 3 Tac-lines intersecting on the same side of a chord.

on the torus. For large R this multiply-connected region has disappeared, but we use up two integer points breaking the circuits. For large R the number of regions is therefore $F - 1$. □

Proof of Theorem 4. If R is not a special value, then $H(k)$ is zero for $k \geq 3$. If R is a special value, we consider λT for $\lambda > 1$, but λ close to 1. The set λT has area $\mathcal{O}(R^2)$. By Proposition 1 (with all arcs bad, so the error term is $\mathcal{O}(R)$), the number of integer points in λT is also $\mathcal{O}(R^2)$. Hence the number of regions in the domains diagram for λS is $\mathcal{O}(R^2)$, by Lemma 2.2.

We interpret $H(k)$ in Theorem 4 as the number of inequivalent points P at which the boundaries of k different S'-ovals meet. Let I be the number of inequivalent pairs of S'-ovals that meet at exactly one point. By Lemma 2.1 the number of regions in the domains diagram for S is smaller by

$$I + \sum_{k \geq 3} \frac{1}{2}(k-1)(k-2)H(k).$$

We deduce the inequality (1.15). □

Lemma 2.3 (size of domains). *Let S_0 be a bounded strictly convex plane set containing the origin as an interior point. Let $S = RS_0$ with R sufficiently large. Suppose that P and Q are points with $J(P) = J(Q)$. Then the distance PQ is bounded by a constant depending only on S_0, not on R.*

Proof. Since $J(P) = J(Q)$, all the integer points in $S(P)$ lie in $S(P) \cap S(Q)$. Let X be a point of intersection of the boundaries $C(P)$ and $C(Q)$. There is a point X_1 on $C(P)$ and a point X_2 on $C(Q)$ such that X_1X and XX_2 are equal and parallel to PQ. A strictly convex set has at most one other chord equal and parallel to a given chord. So $C(P)$ and $C(Q)$ intersect in two points X and Y. Let U_1 (near X) and V_1 (near Y) be the points on $C(P)$ where there is a

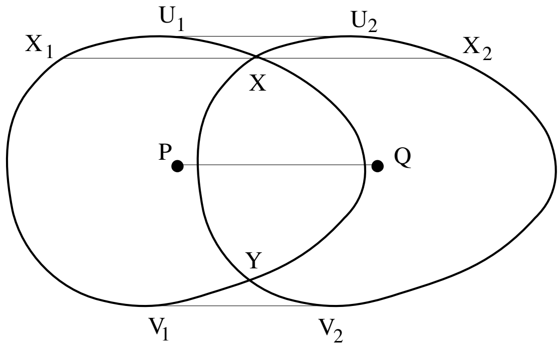

Figure 4 Overlapping S-ovals.

tac-line parallel to PQ, and let U_2 and V_2 be the corresponding points on $C(Q)$. Then U_1U_2 and V_1V_2 are common tac-lines to $S(P)$ and $S(Q)$ (Figure 4).

Let δ be the distance PQ, and let D be the distance between parallel lines U_1U_2 and V_1V_2. The region bounded by the line segment U_2U_1, the arc $U_1X_1V_1$ of $C(P)$, the line segment V_1V_2 and the arc V_2YXU_2 of $C(Q)$ has area δD. This region consists of the part of $S(P)$ not in $S(Q)$ together with two small regions, one by the line segment U_1U_2 and arcs XU_1 of $C(P)$, U_2X of $C(Q)$, and the other bounded by the line segment V_1V_2 and arcs V_1Y of $C(P)$, YV_2 of $C(Q)$. Hence the area of $S(P) \cap S(Q)$ is

$$A - \delta D + \mathbb{O}(\delta^2).$$

By Proposition 1 (with all arcs counted as bad) the number of integer points in $S(P)$ is

$$(2.1) \qquad N = A + \mathbb{O}(R)$$

and in $S(P) \cap S(Q)$ is

$$(2.2) \qquad N = A - \delta D + \mathbb{O}(\delta^2) + \mathbb{O}(R).$$

By substraction

$$(2.3) \qquad \delta D = \mathbb{O}(R) + \mathbb{O}(\delta^2).$$

Since S_0 contains a circle around the origin of some radius C_0,

$$(2.4) \qquad D \geq 2C_0R,$$

so in (2.3)

$$\delta = \mathbb{O}(1),$$

which establishes the lemma. □

Lemma 2.4 (disconnected domains). *Let S_0 be a bounded strictly convex set containing the origin as an interior point. Let $S = RS_0$, with R sufficiently large. Let E be the excess in the number of regions in the domain diagram, that is, the total number of inequivalent regions modulo the integer lattice minus the number of domains of inequivalent configurations modulo the integer lattice which are regions, not single points. Then*

$$E = \mathbb{O}(R). \tag{2.5}$$

Proof. We have to show that all pairs of disconnected domains arise through overlapping pairs of S'-ovals. Let J be a configuration whose domain is disconnected. The domain of J is a subset of the region

$$U = \bigcap_{L \in J} S'(L).$$

Let P and Q be points in different components D_1 and D_2 of the domain of J. The domain boundaries which cut the line PQ are arcs of boundaries $C'(M_i)$ of S'-ovals whose centre is an integer point M_i not in J. To explain the combinatorics of the domains diagram, we replace each S'-oval $S'(M_i)$ by $S'(\lambda_i, M_i)$ with independent scale factors λ_i. If we decrease the factors λ_i until no two S'-ovals $S'(\lambda_i, M_i)$ overlap, then the line PQ can be deformed to avoid all the S'-ovals $S'(\lambda_i, M_i)$, and D_1 and D_2 become part of the same connected component of the domain of J. Thus each disconnection of a domain is the result of the overlapping of some pair of S'-ovals.

When we return to consider λ increasing independently of the centre M_i, then the domain that is disconnected when two S'-ovals $S'(\lambda, M)$, $S'(\lambda, M')$ first come to meet may not be that of J itself, because another boundary $C'(\lambda, K)$ may sweep across U, adding the point K to J or removing it from J. But each overlapping pair of S'-ovals disconnects at most one domain, and the extra components which are contained in E can be assigned a different pair of overlapping S'-ovals $S'(M)$, $S'(M')$. Two integer vectors $\overrightarrow{MM'}$ and $\overrightarrow{M'M}$ give a pair of opposite points in the difference set T.

Next we show that disconnected domains correspond to integer points close to the boundary of T. The S'-ovals $S'(M)$ and $S'(M')$ cut the line PQ. There is a value λ_1 at which neither $S'(\lambda_1, M)$ nor $S'(\lambda_1, M')$ crosses the line PQ, and one of them, $S'(\lambda_1, M)$ say, meets PQ in one point W. The value λ_2 at which $S'(\lambda_2, M)$ and $S'(\lambda_2, M')$ meet in one point has $\lambda_2 \geq \lambda_1$. We estimate λ_2. By Lemma 2.3, PQ has bounded length, and PQ cuts $S'(M)$ in a bounded interval XY. Since W lies between X and Y, XW is bounded. As X lies on the boundary $C'(M)$ and W' lies on $C'(\lambda_1, M)$ we have

$$\lambda R \geq \lambda_2 R \geq \lambda_1 R = \lambda R - \mathbb{O}(1). \tag{2.6}$$

The constant $\mathbb{O}(1)$ in (2.6) depends on the position of the origin within S_0, but not on R. Hence there is a number λ_0 with

$$\lambda_0 = \lambda R - \mathbb{O}(1) \tag{2.7}$$

such that all the integer points in T which correspond to pairs of components of domains in the domains diagram for S-ovals must lie on the boundary of some $\lambda_2 T$ with $\lambda_2 > \lambda_0$.

Let F and F_0 be the numbers of integer points in the difference set T and in $\lambda_0 T$ respectively. By Proposition 1 (with all arcs treated as bad)

$$F = B + \mathbb{O}(R) = B_0 R^2 + \mathbb{O}(R), \tag{2.8}$$

$$F_0 = B_0 \lambda_0^2 R^2 + \mathbb{O}(R). \tag{2.9}$$

The excess E of the lemma is the number of disconnection events. They correspond to pairs of integer points connected in $F - F_0$. Hence

$$E \leq \frac{1}{2}(F - F_0) = B_0(1 - \lambda_0^2)R^2 + \mathbb{O}(R) = \mathbb{O}(R). \tag{2.10}$$ □

Proof of Theorem 1 without the Smoothness Condition. If R is not a special value, then by Lemma 2.2 the number of inequivalent regions in the domains diagram is $F - 1$. The definition of E in Lemma 2.4 gives

$$K(S) = F - E - 1 = B + \mathbb{O}(R), \tag{2.11}$$

where we have used the estimates (2.5) and (2.8) of Lemma 2.4.

If R is a special value, then there may be configurations whose domain is a single point P. The point P is either a multiple point, where three or more boundaries of S'-ovals with integer point centres meet, or a tangent point, where the set $\{P\}$ is the intersection of two S'-ovals, and no other boundary of an S'-oval passes through P.

Under the Triangle Condition there are no multiple points. We compare the domains diagram for ovals $S(P)$ with the domains diagram for smaller ovals $S(\lambda, P)$ with $\lambda < 1$, and λ so close to 1 that $\lambda R > R'$ for each special value R' with $R' < R$. A tangent point P in the domains diagram for S-ovals adds one configuration $J(P)$ to $K(S)$. and adds one to the excess E for the existing domain which becomes separated. Thus

$$K(S) = K(\lambda S) = \lambda^2 B + \mathbb{O}(R),$$

and (2.11) still holds if λ is sufficiently close to 1.

If R is a special value, and the Triangle Condition does not hold, then we compare the domains diagram for ovals $S(P)$ with the domains diagram for larger ovals $S(\lambda, P)$ with $\lambda > 1$, and λ so close to 1 that $\lambda R < R'$ for each special value R' with $R' > R$. Each configuration $J(P)$ of S-ovals also occurs as $J(\lambda, Q)$ for some point Q, but there may be configurations $J = J(\lambda, Q)$ whose domain, as λ decreases to 1, shrinks to a multiple point P not in the domain of J. For λ sufficiently close to 1

$$K(S) \leq K(\lambda S) = \lambda^2 B + \mathbb{O}(R) = B + \mathbb{O}(R), \tag{2.12}$$

and we have established (1.7) and (1.8) of Theorem 1. □

3 Analysis of the difference set

We suppose that S satisfies the Smoothness Condition, and we compute the area and radius of curvature of the difference set T. Some of the results make sense under weaker differentiability conditions on S, and can be extended by approximating such sets S by sets which satisfy the full Smoothness Condition.

Lemma 3.1 (area of the difference set). *The area of the difference set T is given by* (1.11) *of Theorem 2.*

Proof. Let PQ be a maximal chord of S in the direction θ, of length $G(\theta)$, say. The tangents at P and Q must be parallel and oppositely oriented, at angles α and $\pi + \alpha$, say.

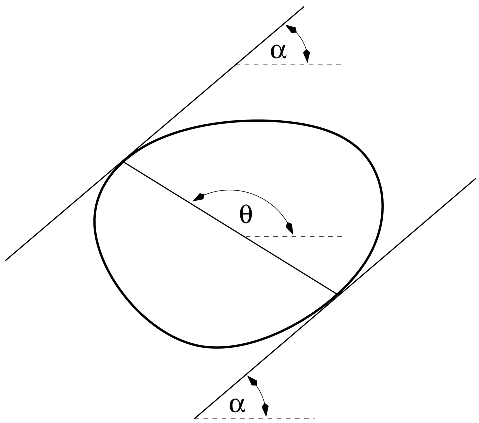

Figure 5 A maximal chord of S.

We write the vector PQ as a complex number

$$G(\theta)e^{i\theta} = z(\alpha) = \int_{\alpha}^{\alpha+\pi} \rho(\psi)e^{i\psi}\,d\psi.$$

The polar equation of the difference set T is $r = G(\theta)$, so the area of T is

$$\begin{aligned}
\frac{1}{2}\int_0^{2\pi} G(\theta)^2\,d\theta &= \frac{1}{2}\int_{\alpha=0}^{2\pi} |z(\alpha)|^2\,\frac{dz(\alpha)}{iz(\alpha)} \\
&= \frac{1}{2i}\int_0^{2\pi} \overline{z(\alpha)}(\rho(\alpha+\pi)e^{i\alpha+i\pi} - \rho(\alpha)e^{i\alpha})\,d\alpha \\
&= -\frac{1}{2i}\int_0^{2\pi} G(\theta)e^{i\alpha-i\theta}(\rho(\alpha+\pi)+\rho(\alpha))\,d\alpha \\
&= -\frac{1}{2i}\int_0^{2\pi} G(\theta)\rho(\alpha)e^{i\alpha-i\theta}\,d\alpha,
\end{aligned}$$

since $G(\theta) = G(\theta + \pi)$. We know that the area is real, so it is

$$\int_0^{2\pi} G(\theta)\rho(\alpha)\cos(\alpha - \theta + \frac{\pi}{2})\,d\alpha = \int_0^{2\pi} D(\alpha)\rho(\alpha)\,d\alpha,$$

where $D(\alpha)$ is the width defined in Theorem 2. □

Lemma 3.2 (curvature of the difference set). *The difference set T of S is obtained by symmetrising the radius of curvature at pairs of points with parallel tangents, then doubling in size. In symbols, let $\rho(\psi)$ and $\sigma(\psi)$ be the radii of curvature of the boundaries of S and T at points where the respective tangents make an angle ψ with the x-axis. Then*

$$\sigma(\psi) = \rho(\psi) + \rho(\psi + \pi), \quad \sigma'(\psi) = \rho'(\psi) + \rho'(\psi + \pi). \tag{3.1}$$

Corollary. *The difference set T also satisfies the Smoothness Condition for R sufficiently large.*

Proof. We start from the formula for the radius of curvature of a curve given parametrically by the coordinates $x(t)$, $y(t)$, so

$$\tan\psi = \frac{\dot y}{\dot x}, \quad \sec^2\psi\,\dot\psi = \frac{\ddot y}{\dot x} - \ddot x\dot y, \quad \rho = \frac{(\dot x^2 + \dot y^2)^{3/2}}{\dot x\ddot y - \ddot x\dot y},$$

where dots indicate derivatives with respect the "time" t. Another differentiation gives

$$\dot\rho = \frac{3(\dot x\ddot x + \dot y\ddot y)(\dot x^2 + \dot y^2)^{1/2}}{\dot x\ddot y - \ddot x\dot y} - \frac{(\dot x\dddot y - \dddot x\dot y)(\dot x^2 + \dot y^2)^{3/2}}{(\dot x\ddot y - \ddot x\dot y)^2},$$

$$\frac{d\rho}{d\psi} = \frac{\dot\rho}{\dot\psi} = \frac{3(\dot x\ddot x + \dot y\ddot y)(\dot x^2 + \dot y^2)^{3/2}}{(\dot x\ddot y - \ddot x\dot y)^2} - \frac{(\dot x\dddot y - \dddot x\dot y)(\dot x^2 + \dot y^2)^{5/2}}{(\dot x\ddot y - \ddot x\dot y)^3}.$$

We change to a complex variable:

$$x = \frac{z + \bar z}{2}, \quad y = \frac{z - \bar z}{2i}, \quad \dot x = \frac{\dot z + \bar{\dot z}}{2}, \quad y = \frac{\dot z - \bar{\dot z}}{2},$$

so

$$\begin{aligned}
\dot x^2 + \dot y^2 &= |\dot z|^2 = \dot z\bar{\dot z},\\
\dot x\ddot y - \ddot x\dot y &= \frac{1}{2i}(\bar{\dot z}\ddot z - \dot z\bar{\ddot z}),\\
\dot x\dddot y - \dddot x\dot y &= \frac{1}{2i}(\bar{\dot z}\dddot y - \dot z\overline{\dddot y}),\\
\dot x\ddot x + \dot y\ddot y &= \frac{1}{2}(\dot z\bar{\ddot z} + \ddot z\bar{\dot z}).
\end{aligned}$$

Hence

$$\rho = \frac{2i|\dot{z}|^3}{\bar{\dot{z}}\dddot{y} - \dot{z}\overline{\dddot{y}}}, \tag{3.2}$$

$$\frac{d\rho}{d\psi} = \frac{4(\bar{\dot{z}}\dddot{y} - \dot{z}\overline{\dddot{y}})|\dot{z}|^5}{(\bar{\dot{z}}\ddot{z} - \dot{z}\bar{\ddot{z}})^3} - \frac{6(\dot{z}\bar{\ddot{z}} + \ddot{z}\bar{\dot{z}})|\dot{z}|^3}{(\bar{\dot{z}}\ddot{z} - \dot{z}\bar{\ddot{z}})^2}. \tag{3.3}$$

We apply these formulae to the difference set T, with α as the time parameter, and

$$\dot{z} = \frac{dz}{d\alpha} = \rho(\alpha + \pi)e^{i\alpha + i\pi} - \rho(\alpha)e^{i\alpha} = -e^{i\alpha}\sigma(\alpha),$$

so

$$\begin{aligned} \ddot{z} &= -e^{i\alpha}\sigma'(\alpha) - ie^{i\alpha}\sigma(\alpha), \\ \dddot{y} &= -e^{i\alpha}\sigma''(\alpha) - 2ie^{i\alpha}\sigma'(\alpha) + e^{i\alpha}\sigma(\alpha), \end{aligned}$$

with

$$\begin{aligned} |\dot{z}| &= \sigma(\alpha), \\ \bar{\dot{z}}\ddot{z} - \dot{z}\bar{\ddot{z}} &= 2\,i\,\sigma(\alpha)^2, \\ \dot{z}\bar{\ddot{z}} + \ddot{z}\bar{\dot{z}} &= 2\sigma(\alpha)\sigma'(\alpha), \\ \bar{\dot{z}}\dddot{y} - \dot{z}\overline{\dddot{y}} &= 4\,i\,\sigma(\alpha)\sigma'(\alpha). \end{aligned}$$

We substitute (3.2) and (3.3) and simplify to obtain (3.1).

In the Corollary, if ρ satisfies (1.1) piecewise with constants c_1, c_2, and c_3, then σ satisfies (1.1) piecewise with constants $2c_1$, $2c_2$, and $2c_3$. □

Lemma 3.3 (Fourier expansion for the radius of curvature). *The radius of curvature ρ of S has a Fourier series*

$$\rho(\psi) = \sum_{n=-\infty}^{\infty} r(n)e^{in\psi} \tag{3.4}$$

in which

$$r(-n) = \overline{r(n)}, \quad r(-1) = r(1) = 0. \tag{3.5}$$

Corollary. *The radius of curvature of the difference set T has the Fourier series*

$$\sigma(\psi) = 2 \sum_{\substack{n=-\infty \\ n \text{ even}}}^{\infty} r(n)e^{in\psi}. \tag{3.6}$$

Proof. The continuous real-valued function ρ is periodic in ψ, so it must have a convergent Fourier series of the form (3.4). We obtain a Cartesian coordinate system by

$$(3.7)\qquad x = \int_0^{\psi} \rho \cos\psi \, d\psi = \frac{1}{2} \sum_{-\infty}^{\infty} r(n) \int_0^{\psi} \left(e^{i(n+1)\psi} + e^{i(n-1)\psi}\right) d\psi$$

$$= \frac{1}{2} \sum_{-\infty}^{\infty} (r(n-1) + r(n+1)) \int_0^{\psi} e^{in\psi} \, d\psi$$

$$= \frac{(r(1) + r(-1))\psi}{2} + \sum_{m \neq 0} \frac{(r(m+1) + r(m-1))(e^{im\psi} - 1)}{2\, i\, m},$$

$$(3.8)\qquad y = \int_0^{\psi} \rho \sin\psi \, d\psi = \frac{1}{2i} \sum_{-\infty}^{\infty} (r(m-1) + r(m+1)) \int_0^{\psi} e^{im\psi} \, d\psi$$

$$= \frac{(r(-1) - r(1))\psi}{2i} + \sum_{m \neq 0} \frac{(r(m+1) - r(m-1))(e^{im\psi} - 1)}{2m}.$$

Since x and y must also be periodic functions, we have $r(1) = 0$, $r(-1) = 0$. The Fourier series for $\sigma(\psi)$ follows by Lemma 5.2. □

Lemma 3.4 (area in intrinsic coordinates). *The area A of S can be expressed in terms of the Fourier expansion of Lemma 3.3 by*

$$(3.9)\qquad A = \pi \sum_{n=-\infty}^{\infty} \frac{|r(n)|^2}{1 - n^2}.$$

Corollary 1. *In terms of the same Fourier expansion, the area B of the difference set T is*

$$(3.10)\qquad B = 4\pi \sum_{n \text{ even}} \frac{|r(n)|^2}{1 - n^2}.$$

Corollary 2. *We have the inequality of Theorem 2,*

$$(3.11)\qquad 4A \leq B \leq \frac{L^2}{\pi}.$$

Proof. We have the boundary integrals

$$(3.12)\qquad A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \int_0^{2\pi} \left(x \frac{dy}{d\psi} - y \frac{dx}{d\psi} \right) d\psi.$$

We use the Fourier series (3.7) and (3.8) of Lemma 3.3, but we shift the origin from the point $\psi = 0$ on the boundary curve C to the centre of mass of C. We write

$$(3.13)\qquad x = \sum_{m=-\infty}^{\infty} \frac{u(m)}{i\, m} e^{im\psi}, \qquad y = \sum_{m=-\infty}^{\infty} \frac{v(m)}{i\, m} e^{im\psi},$$

with

$$(3.14)\quad u(n)=\frac{r(n-1)+r(n+1)}{2},\qquad v(n)=\frac{r(n-1)-r(n+1)}{2i},$$

so that $u(0)=v(0)=0$. The derivatives are

$$(3.15)\quad \frac{dx}{d\psi}=\rho(\psi)\cos\psi=\sum_{n=-\infty}^{\infty}u(n)e^{in\psi},$$

$$(3.16)\quad \frac{dy}{d\psi}=\rho(\psi)\sin\psi=\sum_{n=-\infty}^{\infty}v(n)e^{in\psi}.$$

Substituting the expressions (3.13), (3.15) and (3.16) into (3.12) and integrating, we have

$$(3.17)\quad A=\pi\sum_{n=-\infty}^{\infty}\frac{u(n)v(-n)-v(n)u(-n)}{i\,n}.$$

By (3.14) and (3.5)

$$\begin{aligned}u(n)v(-n)-v(n)u(-n)&=\frac{(r(n-1)+r(n+1))(\overline{r(n-1)}-\overline{r(n+1)})}{-4i}\\&\quad-\frac{(r(n-1)-r(n+1))(\overline{r(n-1)}+\overline{r(n+1)})}{4i}\\&=\frac{i}{4}(2\,|r(n-1)|^2-2\,|r(n+1)|^2).\end{aligned}$$

Hence in (3.17) we have

$$A=\frac{\pi}{2}\sum_n\frac{|r(n-1)|^2}{n}-\frac{\pi}{2}\sum_n\frac{r(n+1)^2}{n}=\frac{\pi}{2}\sum_m|r(m)|^2\left(\frac{1}{m+1}-\frac{1}{m-1}\right),$$

which gives the result (3.9) of the lemma. The result (3.10) of Corollary 1 is immediate by the Corollary to Lemma 3.3. For Corollary 2 we note that the length of C is

$$L=\int_0^{2\pi}\rho(\psi)\,d\psi=2\pi r(0),$$

and that in (3.9) and (3.10) all the terms with $n\neq 0$ contribute negatively. □

Lemma 3.5 (size of domains with the Smoothness Condition). *Let S_0 be a bounded strictly convex plane set containing the origin as an interior point. Let $S=RS_0$, with R sufficiently large, satisfy the Smoothness Condition. Let P and Q be points with $J(P)=J(Q)$. Then the distance δ from P to Q satisfies*

$$(3.18)\quad \delta=\mathcal{O}\left(R^{\kappa-1}(\log R)^{\mu}\right),$$

where κ and μ are the exponents in Proposition 1.

Proof. The sets $S(P)$ and $S(P) \cap S(Q)$ both satisfy the hypotheses of Proposition 1, so in the proof of Lemma 2.3 we may replace $\mathcal{O}(R)$ by

$$\mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right)$$

in (2.1), (2.2), and (2.3). Comparing (2.4) with the sharpened version of (2.3) establishes the lemma. □

Lemma 3.6 (disconnected domains with the Smoothness Condition). *Let S_0 be a bounded strictly convex plane set containing the origin as an interior point. Let $S = RS_0$, with R sufficiently large, satisfy the Smoothness Condition. Then in Lemma 2.4 the excess E satisfies*

$$E = \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right), \tag{3.19}$$

where κ and μ are the exponents of Proposition 1.

Proof. The better estimate for δ in (3.18) of Lemma 3.5 lets us replace the error term $\mathcal{O}(1)$ in (2.6) and (2.7) by

$$\mathcal{O}\left(R^{\kappa-1}(\log R)^{\mu}\right).$$

For R sufficiently large, the Corollary of Lemma 3.2 lets us apply Proposition 1 to the difference sets T and $\lambda_0 T$ with all arcs good. We replace the error terms $\mathcal{O}(R)$ in (2.8) and (2.9) by

$$\mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right).$$

Then in (2.10) we have

$$E \le \frac{1}{2}(F - F_0) = B_0(1-\lambda_0^2)R^2 + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right) = \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right),$$

which establishes the lemma. □

If S has a unique tac-line at each boundary point in addition to satisfying the Smoothness Condition, then we can sharpen the error term $\mathcal{O}(1)$ in (2.6) to $\mathcal{O}(\delta^2/R)$. This means that the integer points corresponding to disconnections of domains lie very close to the boundary of the difference set T. After changing to local Cartesian coordinates, we can use Theorem 1 of [5] and obtain the bound (3.19) with the smaller exponents $\kappa = 3/5$, $\mu = 1/10$. The improved estimate for E alone does not improve the remainder term in Theorem 1.

Proof of Theorem 1 with the Smoothness Condition. As in Lemma 3.5 we reduce the error term in (2.8) from $\mathcal{O}(R)$ to

$$\mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right),$$

so by Lemma 3.6, in the proof of Theorem 1 (2.11) becomes

$$K(S) = F - E - 1 = B + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right),$$

and (2.10) becomes

$$K(S) \leq K(\lambda S) = \lambda^2 B + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right) = B + \mathcal{O}\left(R^{\kappa}(\log R)^{\mu}\right),$$

for λ sufficiently close to 1. These are (1.9) and (1.10) of Theorem 1. □

4 Moments of configurations

We represent a configuration J by its moments M_1, M_2 defined by

(4.1) $$M_1(P) = \sum_{(m,n)\in J(P)} m, \qquad M_1(P) = \sum_{(m,n)\in J(P)} n,$$

plotted as a point in two-dimensional space. Different configurations contain different subsets of the critical points.

Our first lemma is a modification of Lemma 2 of [5].

Lemma 4.1 (an integer point close to a curve). *Let $g(x)$ be a real function, twice continuously differentiable on an interval I of length $M \geq 4$, with*

(4.2) $$g'(x_0) = 0$$

for some point x_0 on I, and

(4.3) $$\frac{8}{M^2} \leq \frac{\Delta}{b} \leq |g''(x)| \leq b\Delta$$

for each x on I, where $b \geq 1$ and $\Delta \leq 1/2$ are real parameters. Then there is an integer n and a closed subinterval J of length 2 in I with

(4.4) $$|g(x) - n| \leq \delta = \sqrt{8b^3\Delta}$$

for each x on J.

Corollary. *For each u in $0 \leq u \leq 1$, there is an integer m with*

(4.5) $$|g(x) - m| \leq \delta$$

for

(4.6) $$m + u - \frac{1}{2} \leq x \leq m + u + \frac{1}{2}.$$

Proof. Since $g''(x)$ is continuous, $g''(x)$ has constant sign on I. Without loss of generality, we suppose that $g''(x) > 0$.

For x in I, by the mean value theorem there are ξ and η between x_0 and x with

$$g(x) = g(x_0) + \frac{1}{2}(x - x_0)^2 g''(\xi), \tag{4.7}$$

$$g'(x) = (x - x_0)g'(\eta). \tag{4.8}$$

The lower bound for M in (4.3) ensures that for some choice of the $\pm$ sign, the point

$$x_1 = x_0 \pm \sqrt{\frac{2b}{\Delta}} \tag{4.9}$$

lies in I. Without loss of generality, we can take the $+$ sign in (4.9). By (4.7)

$$g(x_1) \geq g(x_0) + 1.$$

Let n be an integer in $[g(x_0), g(x_1)]$. Then $n = g(x_2)$ for some x_2 in $[x_0, x_1]$, an interval of length at least 2. By (4.8), for x in $[x_0, x_1]$ we have

$$|g'(x)| \leq (x - x_0)b\Delta \leq b\Delta\sqrt{\frac{2b}{\Delta}} = \sqrt{2b^3\Delta} = \frac{\delta}{2}.$$

Let J be a subinterval of $[x_0, x_1]$ of length 2, containing x_2. By the mean value theorem, for x in J there is some ζ between x and x_2 with

$$g(x) = g(x_2) + (x - x_2)g'(\zeta),$$

so

$$|g(x) - n| \leq 2|g'(\zeta)| \leq \delta.$$

To obtain the Corollary, we let J be the interval $[x_3, x_4]$, and we choose m to be the integer for which

$$x_3 + \frac{1}{2} \leq m + u \leq x_3 + \frac{3}{2} = x_4 - \frac{1}{2}.$$

□

Lemma 4.2 (distinct configurations). *Suppose that*

$$R > 2^{15}c^{14}. \tag{4.10}$$

Then if (u, v), (u', v') *are points in the unit square with*

$$v' - v > 128c^7\sqrt{\frac{2}{R}}, \tag{4.11}$$

the configurations $J(u, v)$ *and* $J(u', v')$ *are different.*

Proof. We consider the curve $C((u+u')/2,(v+v')/2)$, with

$$x = r\cos\theta + \frac{u+u'}{2}, \qquad y = r\sin\theta + \frac{v+v'}{2},$$

so

$$\frac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta, \qquad \frac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta.$$

Let β be the angle in $0 < \beta < \pi/2$ with

$$\tan\beta = \frac{1}{2c^2}.$$

For

$$\theta = \frac{\pi}{2} + \phi, \qquad -\beta \le \phi \le \beta$$

we have

$$\frac{dx}{d\theta} = -f'(\theta)\sin\phi - f(\theta)\cos\phi,$$

so by (1.2)

$$\begin{aligned} \left|\frac{dx}{d\theta}\right| &\le f(\theta)(\cos\phi + c^2|\sin\phi|) \le f(\theta)(1 + c^2\sin\beta) \\ &\le f(\theta)\left(1 + \frac{c^2}{\sqrt{4c^2+1}}\right) \le \frac{3}{2}f(\theta), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \left|\frac{dx}{d\theta}\right| &\ge f(\theta)(\cos\phi - c^2|\sin\phi|) \ge f(\theta)(\cos\beta - c^2\sin\beta) \\ &\ge f(\theta)\left(\frac{2c^2}{\sqrt{4c^2+1}} - \frac{c^2}{\sqrt{4c^2+1}}\right) \ge \frac{f(\theta)}{\sqrt{5}} \ge \frac{3}{7}f(\theta). \end{aligned} \tag{4.13}$$

We can now express the Cartesian second derivative by

$$\left|\frac{d^2y}{dx^2}\right| = \frac{1}{\rho}\left|\frac{ds}{dx}\right|^{3/2} = \frac{|ds/d\theta|^{3/2}}{\rho\,|dx/d\theta|^{3/2}} = \frac{f^2 + 2f'^2 - ff''}{|dx/d\theta|^{3/2}}.$$

The upper bound conditions (1.3) and (1.4) of Theorem 1 give

$$\begin{aligned} &f^2 \le f^2 + f'^2 \le (1+c^2)f^2 \le 2c^2f^2, \\ &f^2 + 2f'^2 - ff'' \le f^2(1 + 2c^4 + c^3) \le 4c^4f^2. \end{aligned}$$

Hence by (4.13) we have

$$\left|\frac{d^2y}{dx^2}\right| \le \frac{4c^4f^2}{|dx/d\theta|^{3/2}} \le 4c^4f^2\left(\frac{7}{3f}\right)^{3/2} \le 4\left(\frac{7}{3}\right)^{3/2}\frac{c^5}{R} \le \frac{16c^5}{R}.$$

The upper bound condition (1.5) of Theorem 1 gives

$$f^2+2f'^2-ff'' \ge f^2-ff'' \ge \frac{f^2}{c^3},$$

and by (4.12)

$$\left|\frac{d^2y}{dx^2}\right| \ge \frac{f^2/c^3}{|dx/d\theta|^{3/2}} \ge \frac{f^2}{c^3}\left(\frac{2}{3f}\right)^{3/2} \ge \left(\frac{2}{3}\right)^{3/2}\frac{1}{c^4R} \ge \frac{1}{4c^4R}.$$

We put

$$\Delta = \frac{2\sqrt{c}}{R}, \qquad b = 8c^{9/2}.$$

Then

$$\frac{1}{4c^4R} = \frac{\Delta}{b} \le \left|\frac{d^2y}{dx^2}\right| \le \frac{16c^5}{R} = b\Delta.$$

We apply Lemma 4.1 to the interval

$$I = \left[f\left(\frac{\pi}{2}+\beta\right)\cos\left(\frac{\pi}{2}+\beta\right)+\frac{u+u'}{2}, f\left(\frac{\pi}{2}-\beta\right)\cos\left(\frac{\pi}{2}-\beta\right)+\frac{u+u'}{2}\right],$$

an interval of length

$$\begin{aligned} M &= \left(f\left(\frac{\pi}{2}+\beta\right)+f\left(\frac{\pi}{2}-\beta\right)\right)\sin\beta \ge \frac{2R}{c}\sin\beta \\ &\ge \frac{2R}{c\sqrt{4c^2+1}} \ge \frac{2R}{\sqrt{5}c^2}. \end{aligned}$$

We have $M \ge 4$ for

(4.14) $\quad R \ge \sqrt{20}c^2,$

and the left hand inequality of (4.3) of Lemma 4.1 holds for

$$M^2 \ge 32c^4R,$$

which requires the stronger condition

(4.15) $\quad R \ge 40c^8.$

We apply the Corollary to Lemma 4.1 with u replaced by $(u+u')/2$, noting that $m+u$ and $m+u'$ both lie in the interval

$$\left[m+\frac{u+u'}{2}-\frac{1}{2}, m+\frac{u+u'}{2}+\frac{1}{2}\right].$$

So, for $v'-v>2\delta$, we have

$$g(m+u)\le n+\delta<n+\frac{v'-v}{2},$$

so the point (m,n) lies outside $B(u,v)$, whilst

$$g(m+u')\ge n-\delta>n-\frac{v'-v}{2},$$

so the point (m,n) lies inside $B(u',v')$.

The value of δ is

$$\delta=\sqrt{8b^3\Delta}=64c^7\sqrt{\frac{2}{R}}.$$

We have $\delta<1/2$ if

$$R>2^{15}c^{14},$$

which is the condition (4.10) of the lemma, implying the conditions (4.14) and (4.15). □

Proof of Theorem 3. We assume the stronger condition

(4.16) $\quad R\ge 2^{17}3^2c^{14}$,

so that the condition (4.11) of Lemma 4.2 becomes

$$v'-v>2\delta=128c^7\sqrt{\frac{2}{R}},$$

where

$$\delta\le\frac{1}{12}.$$

Let k be the integer

$$k=\left[\frac{1}{2\delta}\right]-2\ge\frac{1}{2\delta}-3\ge\frac{1}{4\delta}=\frac{\sqrt{R}}{256\sqrt{2}c^7},$$

so that

$$\frac{1}{k+1} > 2\delta.$$

We give v the values $v_i = i/k$ for $i = 1, 2, \ldots, k$. For any u, u' the configurations $J(u, v_i)$ and $J(u', v_{i+1})$ are different by Lemma 4.2, and $J(u, v_k)$ is not equivalent to $J(u', v_i)$.

The configuration $J(1, v_i)$ is equivalent to $J(0, v_i)$ with every integer point shifted by the vector $(1, 0)$, so

$$M_1(1, v_i) = M_1(0, v_i) + N(0, v_i). \tag{4.17}$$

We consider paths $D_1, \ldots, D_k$ from the line $x = 0$ to the line $x = 1$. The path D_i consists of the line $y = v_i$ with semicircular indentations about any possible points on the line $y = v_i$ where three or more regions meet. The radius of indentations is

$$\leq \frac{1}{2k(k+1)},$$

so that points on adjacent paths D_i and D_{i+1} have y-coordinates differing by

$$\geq \frac{1}{k} - \frac{2}{2k(k+1)} = \frac{1}{k+1} > 2\delta.$$

Again, by Lemma 4.2, points on different paths D_i and D_{i+1} represent different configurations, and paths on D_1 and D_k represent inequivalent configurations.

To estimate $N(0, v_i)$ in (4.17), we use an explicit form of Proposition 1 with all arcs treated as bad, the elementary estimate

$$|N(u, v) - aR^2| \leq 2D(0) + 2D\left(\frac{\pi}{2}\right) + 4$$

(see Hardy and Wright [2]). We have

$$aR^2 \geq \pi \left(\frac{R}{c}\right)^2,$$

and by (4.16)

$$2D(0) + 2D\left(\frac{\pi}{2}\right) + 4 \leq 8cR + 4 \leq 3\pi cR \leq \frac{\pi}{2}\left(\frac{R}{c}\right)^2 \leq \frac{aR^2}{2},$$

so that

$$N(0, v_i) \geq \frac{aR^2}{2}.$$

Each time the path D_i enters a new region, the configuration $J(u, v)$ changes by gaining or losing just one point, so the moment $M_1(u, v)$ changes at most $cR + 1$ by (1.2). Hence points on D_i represent at least

$$\frac{N(0, v_i)}{cR + 1}$$

configurations, of which at least

$$\frac{N(0, v_i)}{cR + 1} - 1 \geq \frac{aR^2}{2c(R + 1)} \geq \frac{aR}{3c}$$

are inequivalent. The k paths give

$$\geq \frac{aRk}{3c} \geq \frac{aR^{3/2}}{768\sqrt{2}c^8}$$

inequivalent configurations. □

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