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Integer matrix factorisations, superalgebras and the quadratic form obstruction

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ABSTRACT

We identify and analyse obstructions to factorisation of integer matrices into products $N^T N$ or N^2 of matrices with rational or integer entries. The obstructions arise as quadratic forms with integer coefficients and raise the question of the discrete range of such forms. They are obtained by considering matrix decompositions over a superalgebra. We further obtain a formula for the determinant of a square matrix in terms of adjugates of these matrix decompositions, as well as identifying a *co-Latin* symmetry space.

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1. Introduction

The question whether a given square integer matrix M can be factorised into a product of two integer matrices, either in the form of a square $M = N^2$ or (in case of a symmetric positive definite matrix) in the form $M = N^T N$, has a long history in number theory. It is known that if $n \leq 7$ and M is an $n \times n$ symmetric positive definite matrix with integer entries and determinant 1, then a factorisation $M = N^T N$ with N an $n \times n$ matrix with integer entries exists. However, there are examples of such matrices with dimension $n = 8$ which cannot be factorised in this way. This result is mentioned by Taussky (see [14, p. 812]) and goes back to Hermite, Minkowski, and Mordell [11].

The number theoretic properties relating to the factorisation of symmetric positive definite $n \times n$ integer matrices M with fixed determinant have classical connections to the theory of positive definite quadratic forms in n variables, see e.g. [12] and the above references.

In particular, Mordell considered the similarity classes of $n \times n$ matrices with determinant 1, where two such matrices L, M are in the same class if there exists a unimodular integral matrix N such that $M = N^T L N$. The number of such similarity classes is denoted by h_n. Then a matrix M is in the similarity class of I_n (the identity matrix) if and only if there exists a factorisation $M = N^T N$ with an integer matrix N. This implies that the quadratic form classically associated with the symmetric matrix M, $q(x) = x^T M x$, can be written as

$$q(x) = x^T M x = x^T N^T N x = y^T y = \sum_{j=1}^{n} y_j^2, \tag{1.1}$$

where $y = Nx$, and N has determinant 1. Thus, the factorisation can be used to write the quadratic form $q(x)$ as a sum of squares of n linear factors. When $n = 8$, such a factorisation may not exist, as Minkowski proved in 1911 that $h_n \geq [1 + n/8]$, so $h_n \geq 2$ if $n = 8$. Mordell showed that $h_8 = 2$ [12], and Ko showed that $h_9 = 2$ as well [9].

In the present paper, we revisit the question of integer matrix factorisation in the light of recent general results on matrix decompositions [7], [8]. We establish in Corollary 3.1 that the existence of integer solutions to a certain quadratic equation is a necessary condition for a matrix factorisation of the type $M = N^2$ or $M = N^T N$ (for symmetric positive definite M) to exist. It is interesting to note that solutions to this new type of quadratic equation associated with a given integer matrix M can also lead to rational matrix factors N with entries in $\frac{1}{n^2}\mathbb{Z}$.

Throughout the paper, we use the classical example of the Wilson matrix [4], [6], [10], [13]

$$W = \begin{pmatrix} 5 & 7 & 6 & 5 \\ 7 & 10 & 8 & 7 \\ 6 & 8 & 10 & 9 \\ 5 & 7 & 9 & 10 \end{pmatrix} \tag{1.2}$$

to demonstrate the methodologies under consideration. This integer matrix has determinant 1 and hence an integer inverse matrix, but is moderately ill conditioned, despite its small size and entries. It has the integer factorisation $W = Z^T Z$ discovered in [6] with

$$Z = \begin{pmatrix} 2 & 3 & 2 & 2 \\ 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \tag{1.3}$$

We note that the entries of Z are nonnegative and, although the matrix is not triangular, it has a block upper triangular structure and can be thought of as a block Cholesky factor of W.

The quadratic form associated with the Wilson matrix can be written, using (1.1) and (1.3), as a sum of four squares:

$$\begin{aligned} q(x) = x^T W x = (x_3 + x_4)^2 + (x_3 + 2x_4)^2 + (x_1 + x_2 + 2x_3 + x_4)^2 \\ + (2x_1 + 3x_2 + 2x_3 + 2x_4)^2. \end{aligned} \tag{1.4}$$

As Z is a unimodular integer matrix and hence has an integer inverse, it follows by Lagrange's four-square theorem that the quadratic form q generated by the Wilson matrix is universal [1] in the sense that it generates all positive integers as x ranges over $\mathbb{Z}^4$.

This shows that integer matrix factorisation is a valuable tool in studying the quadratic form generated by an integer matrix. As a result of our considerations in Section 3 below, the question of factorising the Wilson matrix in the form $W = Z^T Z$ is associated with the solutions of the quadratic equation

$$2w^2 + x_1^2 + x_1 x_2 + x_1 x_3 + x_2^2 + x_2 x_3 + x_3^2 = 952. \tag{1.5}$$

Indeed, a necessary (but not sufficient) condition for W to factorise is that integer solutions (w, x_1, x_2, x_3) to the quadratic equation (1.5) exist. Thus this equation can be considered a quadratic form obstruction to integer factorisability. This approach to integer matrix factorisation was briefly alluded to, but not fully worked out in [6].

In Section 2 we derive a useful S+V decomposition of square matrices, first identified in [8], and give explicit formulas for constructing it that were not given in [8]. In Section 3, we use the concept of matrix weight, associated with the S factor, to establish the quadratic form obstruction to integer matrix factorisation and show how a solution of the corresponding quadratic equation can be used to calculate the matrix factors. In Section 4, we discuss adjugate matrices in view of the matrix decomposition and in particular show that the type S part of a matrix is characterised by having an adjugate with all equal entries. Finally, in Section 5 we identify the type V part of a matrix as belonging to a space of co-Latin squares, defined as square matrices with constant sum over all entries carrying the same symbol in any Latin square.

2. The S + V decomposition of matrices

The following symmetries of $n \times n$ matrices were considered in [8].

(S) A matrix $M = (m_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ has the *constant sum property* (or is of type S) if there is a number $w \in \mathbb{R}$, called the *weight* of the matrix, such that

$$\sum_{j=1}^n m_{i,j} = \sum_{j=1}^n m_{j,i} = nw \qquad (i \in \{1, \dots, n\}).$$

The vector subspace of $\mathbb{R}^{n\times n}$ of matrices having the constant sum property with some weight is denoted by S_n and can be characterised as

$$S_n = \left\{ M \in \mathbb{R}^{n\times n} : 1_n^T M u = 0 = u^T M 1_n \ (u \in \{1_n\}^\perp) \right\},$$

where $1_n \in \mathbb{R}^n$ is the column vector with all entries equal to 1 and orthogonality is with respect to the standard inner product, $\{1_n\}^\perp = \{u \in \mathbb{R}^n : u^T 1_n = 0\}$ (cf. [8, Thm. 2.6 (a)]).

(V) A matrix $M = (m_{i,j})_{i,j=1}^n \in \mathbb{R}^{n\times n}$ has the *vertex cross sum property* (or is of type V) if

$$m_{i,j} + m_{k,l} = m_{i,l} + m_{k,j} \quad (i, j, k, l \in \{1, \dots, n\})$$

and the matrix entries sum to zero, $\sum_{i,j=1}^n m_{i,j} = 0$. The vector subspace of $\mathbb{R}^{n\times n}$ of matrices having the vertex cross sum property is denoted by V_n and can be characterised as

$$V_n = \left\{ M \in \mathbb{R}^{n\times n} : u^T M v = 0 \ (u, v \in \{1_n\}^\perp),\ 1_n^T M 1_n = 0 \right\} \tag{2.1}$$

(cf. [8, Thm. 2.6 (e)]). (We derive a surprising alternative characterisation of this space in Section 5.)

The spaces S_n and V_n only have the null matrix in common; in fact, they complement each other and give $\mathbb{R}^{n\times n}$ a superalgebra structure in the following way (cf. [8, Thm. 2.5 (a)]).

Theorem 2.1. $\mathbb{R}^{n\times n} = S_n \oplus V_n$. *Moreover, S_n is a subalgebra of $\mathbb{R}^{n\times n}$, and*

$$S_n S_n \subset S_n, \quad S_n V_n \subset V_n, \quad V_n S_n \subset V_n, \quad V_n V_n \subset S_n.$$

We show below in Corollary 2.1 that this decomposition of the matrix algebra is orthogonal.

We begin by showing in Theorem 2.2 that every element M of V_n can be written in the form $M = a 1_n^T + 1_n b^T$, before deriving a formula for obtaining these vectors a and b in Theorem 2.3.

Theorem 2.2. *A matrix $M \in \mathbb{R}^{n\times n}$ is an element of V_n if and only if there exist vectors $a, b \in \{1_n\}^\perp$ such that $M = a1_n^T + 1_n b^T$.*

Proof. Let $M \in V_n$. Then for any $v \in \{1_n\}^\perp$, we have $Mv \in \{1_n\}^{\perp\perp} = \mathbb{R}\,1_n$; hence there is a linear form $f : \{1_n\}^\perp \to \mathbb{R}$ such that $Mv = 1_n f(v)$ $(v \in \{1_n\}^\perp)$. By the Riesz representation theorem, there is a vector $b \in \{1_n\}^\perp$ such that $f(v) = b^T v$ $(v \in \{1_n\}^\perp)$, and consequently $Mv = 1_n b^T v$ $(v \in \{1, n\}^\perp)$. Now every $x \in \mathbb{R}^n$ can be written in the form $x = \alpha 1_n + v$ with suitable $v \in \{1, n\}^\perp$ and $\alpha \in \mathbb{R}$, so

$$\begin{aligned} Mx &= \alpha M 1_n + Mv = \alpha b 1_n^T 1_n + 1_n b^T v = (a1_n^T + 1_n b^T)(\alpha 1_n + v) \\ &= (a1_n^T + 1_n b^T)x, \end{aligned}$$

where $a := \frac{1}{n} M 1_n$, bearing in mind that $1_n^T 1_n = n$. It follows from the last equation in (2.1) that $1_n^T a = 0$. Conversely, for any $a, b \in \{1_n\}^\perp$ the matrix $a1_n^T + 1_n b^T \in V_n$. □

Clearly $M \in V_n$ is symmetric if and only if $a = b$ in the above representation. Theorem 2.2 shows that $\dim V_n = 2n - 2$, so $\dim S_n = n^2 - 2n + 2$ (see also [8, p. 14]), and that the rank of elements of V_n cannot exceed 2. Moreover, this theorem makes the spectral decomposition of any matrix $M \in V_n$ very transparent. Indeed, the range of M is spanned by the orthogonal vectors a and 1_n, so any eigenvector for a non-zero eigenvalue must be of the form $u = \alpha a + \beta 1_n$ with numbers $\alpha, \beta \in \mathbb{C}$. Then, bearing in mind that $1_n^T a = 0$ and $b^T 1_n = 0$, and denoting the eigenvalue by λ, we find

$$\lambda u = Mu = (a1_n^T + 1_n b^T)(\alpha a + \beta 1_n) = n\beta a + \alpha b^T a 1_n,$$

giving $n\beta = \lambda\alpha$ and $b^T a\,\alpha = \lambda\beta$. Hence any nonzero eigenvalue of M is an eigenvalue of the 2×2 matrix $\begin{pmatrix} 0 & n \\ b^T a & 0 \end{pmatrix}$ and vice versa. This gives the characteristic polynomial for M,

$$\chi_M(\lambda) = \lambda^{n-2}(\lambda^2 - nb^T a)$$

and furthermore the eigenvectors for M as $v_\pm = \sqrt{n}a \pm \sqrt{b^T a}1_n$ for eigenvalues $\lambda_\pm = \pm\sqrt{nb^T a}$ if $b^T a > 0$, $v_\pm = \sqrt{n}a \pm i\sqrt{-b^T a}1_n$ for eigenvalues $\lambda_\pm = \pm i\sqrt{-nb^T a}$ if $b^T a < 0$ and the single eigenvector a for eigenvalue 0 if $b^T a = 0$ (and $a \neq 0$; otherwise any non-null vector orthogonal to b will do).

Finally, Theorem 2.2 easily yields the orthogonality with respect to the Frobenius inner product of the direct sum decomposition of the space of $n \times n$ matrices into S_n and V_n. The Frobenius inner product of two matrices $A, B \in \mathbb{R}^{n\times n}$ is defined as $\langle A, B\rangle = \operatorname{tr} A^T B$.

Corollary 2.1. *The decomposition $\mathbb{R}^{n\times n} = S_n \oplus V_n$ is orthogonal with respect to the Frobenius inner product.*

Proof. Let $S \in S_n$, $V \in V_n$; then by Theorem 2.2 there are vectors $a, b \in \{1_n\}^\perp$ such that $V = a1_n^T + 1_n b^T$. Hence

$$\begin{aligned}\langle S, V\rangle &= \operatorname{tr} S^T V = \operatorname{tr}(S^T a 1_n^T + S^T 1_n b^T) = 1_n^T S^T a + b^T S^T 1_n \\ &= a^T S 1_n + b^T S^T 1_n = 0,\end{aligned}$$

since $S1_n = S^T 1_n = n w_S 1_n$, where w_S is the weight of the matrix S. □

We can calculate the weight of any matrix in S_n as the mean of all matrix entries. In fact, it is meaningful to define the linear form

$$\operatorname{wt} : \mathbb{R}^{n\times n} \to \mathbb{R}, \quad \operatorname{wt} M = \frac{1}{n^2}\sum_{i,j=1}^{n} m_{i,j}$$

giving the weight of any $n \times n$ matrix $M = (m_{i,j})_{i,j=1}^n$. Theorem 2.2 immediately shows that $\operatorname{wt} M = 0$ for all $M \in V_n$.

Any $M \in S_n$ can be uniquely decomposed into

$$M = M_0 + (\operatorname{wt} S)\,\mathcal{E}_n,$$

where $M_0 \in S_n$ has weight 0 and $\mathcal{E}_n = 1_n 1_n^T \in S_n$ is the $n \times n$ matrix with all entries equal to 1. In conjunction with the previous observations, this gives rise to the following unique decomposition of general $n \times n$ matrices, including an explicit formula for the calculation of the parts.

Theorem 2.3. *Let $M = (m_{i,j}) \in \mathbb{R}^{n\times n}$. Then there is a unique decomposition*

$$M = M_V + M_0 + (\operatorname{wt} M)\,\mathcal{E}_n,$$

where $M_0 \in S_n$ with weight 0 and $M_V = a1_n^T + 1_n b^T \in V_n$, and the entries of the vectors a and b are given by

$$a_i = \frac{1}{n}\sum_{j=1}^{n} m_{i,j} - \operatorname{wt} M \quad (i \in \{1,\dots,n\}),$$

$$b_j = \frac{1}{n}\sum_{i=1}^{n} m_{i,j} - \operatorname{wt} M \quad (j \in \{1,\dots,n\}).$$

In particular, if M is an integer matrix then the vectors $n^2 a$ and $n^2 b$ have integer entries.

Proof. Let $\{\varepsilon_j : j \in \{1,\dots,n-1\}\}$ be an orthonormal basis of $\{1_n\}^\perp$. Then

$$\left\{\frac{1}{\sqrt{n}}\varepsilon_j 1_n^T, \frac{1}{\sqrt{n}} 1_n \varepsilon_j^T : j \in \{1,\dots,n-1\}\right\}$$

is an orthonormal basis, with respect to the Frobenius inner product, of the ($2n-2$-dimensional) space V_n. Indeed, for $j,k \in \{1,\dots,\ n\ -\ 1\}$ we have

$$\left\langle \varepsilon_j 1_n^T, 1_n \varepsilon_k^T \right\rangle = \operatorname{tr} 1_n \varepsilon_j^T 1_n \varepsilon_k^T = 0$$

since $\varepsilon_j^T 1_n = 0$, and

$$\left\langle \varepsilon_j 1_n^T, \varepsilon_k 1_n^T \right\rangle = \operatorname{tr} 1_n \varepsilon_j^T \varepsilon_k 1_n^T = \delta_{j\,k}\ \operatorname{tr} 1_n 1_n^T = n\delta_{j\,k}$$

and analogously for $\left\langle 1_n \varepsilon_j^T, 1_n \varepsilon_k^T \right\rangle$.

These observations enable us to find M_V, the V_n part of $M \in \mathbb{R}^{n\times n}$ by orthogonal projection using the above orthonormal basis. We have

$$\begin{aligned} M_V &= \sum_{j=1}^{n-1} \left(\left\langle \frac{1}{\sqrt{n}} \varepsilon_j 1_n^T, M \right\rangle \frac{1}{\sqrt{n}} \varepsilon_j 1_n^T + \left\langle \frac{1}{\sqrt{n}} 1_n \varepsilon_j^T, M \right\rangle \frac{1}{\sqrt{n}} 1_n \varepsilon_j^T \right) \\ &= \frac{1}{n} \sum_{j=1}^{n-1} \left(\operatorname{tr}(1_n \varepsilon_j^T M) \varepsilon_j 1_n^T + \operatorname{tr}(\varepsilon_j 1_n^T M) 1_n \varepsilon_j^T \right) \\ &= \frac{1}{n} \sum_{j=1}^{n-1} (\varepsilon_j^T M 1_n) \varepsilon_j 1_n^T + \frac{1}{n} \sum_{j=1}^{n-1} 1_n ((1_n^T M \varepsilon_j) \varepsilon_j)^T = a_M 1_n^T + 1_m b_M^T \end{aligned}$$

with

$$a_M = \frac{1}{n} \sum_{j=1}^{n-1} (\varepsilon_j^T M 1_n) \varepsilon_j, \qquad b_M = \frac{1}{n} \sum_{j=1}^{n-1} (1_n^T M \varepsilon_j) \varepsilon_j.$$

Observing that $\{\varepsilon_j : j \in \{1,\dots,n-1\}\} \cup \{\frac{1}{\sqrt{n}} 1_n\}$ is an orthonormal basis of $\mathbb{R}^n$, we find

$$\sum_{j=1}^{n-1} (\varepsilon_j^T v) \varepsilon_j = v - \frac{1}{n} (1_n^T v) 1_n \qquad (v \in \mathbb{R}^n).$$

Hence

$$a_M = \frac{1}{n} \left(M 1_n - \frac{1}{n} (1_n^T M 1_n) 1_n \right),$$

and

$$b_M = \frac{1}{n} \left(M^T 1_n - \frac{1}{n} (1_n^T M^T 1_n) 1_n \right). \quad \square$$

Example: the Wilson matrix. For the Wilson matrix W this decomposition takes the form

$$16W = 1_4 \begin{pmatrix} -27 \\ 9 \\ 13 \\ 5 \end{pmatrix}^T + \begin{pmatrix} -27 \\ 9 \\ 13 \\ 5 \end{pmatrix} 1_4^T + \begin{pmatrix} 15 & 11 & -9 & -17 \\ 11 & 23 & -13 & -21 \\ -9 & -13 & 15 & 7 \\ -17 & -21 & 7 & 31 \end{pmatrix} + 119\mathcal{E}_4 \tag{2.2}$$

and for the integer matrix factor Z of eq. (1.3)

$$16Z = 1_4 \begin{pmatrix} 17 \\ 1 \\ -7 \\ -11 \end{pmatrix}^T + \begin{pmatrix} -7 \\ -3 \\ 5 \\ 5 \end{pmatrix} 1_4^T + \begin{pmatrix} 3 & 15 & -9 & -9 \\ 3 & -1 & 7 & -9 \\ -5 & -9 & -1 & 15 \\ -1 & -5 & 3 & 3 \end{pmatrix} + 19\mathcal{E}_4. \tag{2.3}$$

The last terms on the right-hand side of these formulae correctly yield $\operatorname{wt} W = \frac{119}{16}$ and $\operatorname{wt} Z = \frac{19}{16}$.

3. Integer factorisation of matrices and the quadratic form obstruction

On the basis of the matrix decomposition established in the preceding section, we now derive the quadratic equation arising from balancing the weights in a matrix factorisation.

Theorem 3.1. *Let $M, N \in \mathbb{R}^{n\times n}$, and let $M = M_V + M_0 + (\operatorname{wt} M)\,\mathcal{E}_n$ and $N = N_V + N_0 + (\operatorname{wt} N)\,\mathcal{E}_n$ be their decompositions as in Theorem 2.3, with $N_V = a1_n^T + 1_n b^T$.*

(i) If $M = N^T N$ then

$$\operatorname{wt} M = |a|^2 + n\,(\operatorname{wt} N)^2, \tag{3.1}$$

$M_V = y1_n^T + 1_n y^T$ with $y = N_0^T a + n\,(\operatorname{wt} N)b$, and $M_0 = N_0^T N_0 + n\,bb^T$.

(ii) If $M = N^2$ then

$$\operatorname{wt} M = b^T a + n\,(\operatorname{wt} N)^2, \tag{3.2}$$

$M_V = y1_n^T + 1_n z^T$ with $y = N_0^T a + n(\operatorname{wt} N)a$ and $z = N_0^T b + (\operatorname{wt} N)b$, and $M_0 = N_0^2 + n\,ab^T$.

Proof. (i) Since $N_0\mathcal{E}_n = N_0^T\mathcal{E}_n = \mathcal{O}_n$, we have

$$\begin{aligned} N^T N = {} & (N_V^T N_V + N_0^T N_0) \\ & + (N_V^T N_0 + N_0^T N_V + (\operatorname{wt} N) N_V^T \mathcal{E}_n + (\operatorname{wt} N)\mathcal{E}_n N_V) + n(\operatorname{wt} N)^2 \mathcal{E}_n, \end{aligned}$$

where, by the superalgebra property, the matrices in the first bracket lie in S_n, the matrices in the second bracket in V_n. Writing $N_V = a1_n^T + 1_n b^T$, we find $N_V^T N_V = a^T a\mathcal{E}_n + nbb^T$, $N_V^T N_0 = 1_n a^T N_0$, $N_0^T N_V = N_0^T a 1_n^T$, $N_V^T \mathcal{E}_n = nb1_n^T$ and $\mathcal{E}_n N_V = n1_n b^T$.

Hence the uniqueness of the decomposition of M by Theorem 2.3 gives the claimed identities.

(ii) Similarly, we find

$$N^2 = (N_V^2 + N_0^2) + (N_V N_0 + N_0 N_V + (\operatorname{wt} N) N_V \mathcal{E}_n + (\operatorname{wt} N) \mathcal{E}_n N_V) \\ + n(\operatorname{wt} N)^2 \mathcal{E}_n,$$

and, setting $N_V = a1_n^T + 1_n b^T$, note that $N_V^2 = b^T a \mathcal{E}_n + nab^T$, $N_V N_0 = 1_n b^T N_0$, $N_0 N_V = N_0 a 1_n^T$, $N_V \mathcal{E}_n = na1_n^T$ and $\mathcal{E}_n N_V = n1_n b^T$. This gives the claimed identities by uniqueness of decomposition. □

The above theorem gives the following quadratic form obstructions to the factorisation of an integer matrix M into either $M = N^T N$ or $M = N^2$ with an integer matrix N.

Corollary 3.1. *Given a matrix $M \in \mathbb{Z}^{n\times n}$, it is necessary for the existence of a factorisation $M = N^T N$ with $N \in \mathbb{Z}^{n\times n}$ that $n^2 \operatorname{wt} N \in \mathbb{Z}$ and that the vector components $n^2 a_j, n^2 b_j \in \mathbb{Z}$ ($j \in \{1, \dots, n\}$), where $N = a1_n^T + 1_n b^T + N_0 + w_N \mathcal{E}_n$ is the decomposition of N as in Theorem 2.3, form a solution of the quadratic equation*

$$n^4 \operatorname{wt} M = n\,(n^2 \operatorname{wt} N)^2 + \sum_{j=1}^{n} (n^2 a_j)^2.$$

Similarly, for the decomposition $M = N^2$, the quadratic equation is

$$n^4 \operatorname{wt} M = n\,(n^2 \operatorname{wt} N)^2 + \sum_{j=1}^{n} (n^2 a_j)(n^2 b_j).$$

Suppose we are given a symmetric integer matrix $M \in \mathbb{Z}^{n\times n}$ and have found a solution $(w_N, a_1, \dots, a_n) \in \frac{1}{n^2}\mathbb{Z}^{n+1}$ of the quadratic equation associated with the factorisation $M = N^T N$,

$$n^4 \operatorname{wt} M = n^5 w_N^2 + n^4 \sum_{j=1}^{n} a_j^2.$$

To complete the factorisation, we need to identify a vector b and a matrix N_0 satisfying the equations in Theorem 3.1 (i). Using the decomposition $M = y1_n^T + 1_n y^T + M_0 + (\operatorname{wt} M)\,\mathcal{E}_n$, we have $b = \frac{1}{nw_N}\,(y - N_0^T a)$, so we only need to find a solution N_0 of the quadratic matrix equation

$$N_0^T (aa^T + nw_N^2 \mathcal{I}_n) N_0 - N_0^T a y^T - y a^T N_0 = nw_N^2 M_0 - yy^T \tag{3.3}$$

or, setting $N_0 = L - \frac{1}{a^T a + nw_N^2}\, ay^T$, the simpler quadratic

$$L^T(aa^T + nw_N^2\mathcal{I}_n)L = nw_N^2\Big(M_0 + \frac{1}{a^Ta + nw_N^2}yy^T\Big). \tag{3.4}$$

The right-hand side and the middle factor on the left-hand side of these equations are determined in terms of the S+V decomposition of the given matrix M and the particular solution of the factorisation quadratic form considered. Although determining N_0 or L from these equations is a factorisation problem of similar type to the original equation $M = N^TN$, we found that their solution was computationally more effective.

Remark 1. We remark in passing that the matrix $aa^T + nw_N^2$ appearing as a middle factor on the left-hand side of eq. (3.4) is symmetric positive definite and can be written as A^TA, where $A = \sqrt{nw_N^2} + \frac{\sqrt{a^Ta+nw_N^2}-\sqrt{nw_N^2}}{a^Ta}aa^T$ and $A^{-1} = \frac{1}{\sqrt{nw_N^2}} + \frac{1}{a^Ta}\left(\frac{1}{\sqrt{a^Ta+nw_N)N^2}} - \frac{1}{\sqrt{nw_N^2}}\right)aa^T$. Thus $N_0 = A^{-1}L - \frac{1}{a^Ta+nw_N^2}ay^T$, where

$$L^TL = nw_N^2\Big(M_0 + \frac{1}{a^Ta + nw_N^2}yy^T\Big),$$

but due to the presence of square roots this is unlikely to give a rational solution N_0.

Example: the Wilson matrix. The quadratic equation arising from balancing the weights in the assumed factorisation of the Wilson matrix $W = Z^TZ$ is

$$\begin{aligned}\operatorname{wt} W = \frac{119}{16} &= a_1^2 + a_2^2 + a_3^2 + a_4^2 + 4w_Z^2\\ &= 2(a_1^2 + a_2^2 + a_3^2 + a_1a_2 + a_1a_3 + a_2a_3) + 4\,w_Z^2,\end{aligned}$$

as $a_4 = -a_1 - a_2 - a_3$. Multiplying the equation by 2^7 and setting $x_i = 16a_i$, $w = 16w_Z$, we find that a necessary condition for the integer factorisation of the Wilson matrix is that there are integer solutions to the quadratic equation (1.5)

$$2w^2 + x_1^2 + x_1x_2 + x_1x_3 + x_2^2 + x_2x_3 + x_3^2 = 952.$$

Solving this equation for w, x_1, x_2, x_3 in Mathematica 11.0 on a PC with a Intel Core i7 6500CPU, gave the 1728 solutions in just under 6 seconds. Exactly one third (576) of these solutions lead to rational matrix factorisations $W = Z^TZ$ with $Z \in \frac{1}{16}\mathbb{Z}^{4\times 4}$.

The process of converting solutions into matrix factors, i.e. of finding suitable vectors b and matrices N_0 satisfying the equations of Theorem 3.1 (i) as outlined after Corollary 3.1, took considerably longer at 34 minutes. Our approach involved utilising (3.3), in which the vector b is eliminated and the right-hand-side completely determined for a given factor weight w_N. Substituting potential solutions for the vector a and weight w_N thus reduces the general problem of finding the matrix N_0 to that of an $(n-1)\times(n-1)$ unknown matrix. For the Wilson matrix this is a 9-dimensional problem, and then for each of the 9-dimensional solutions for N_0, the vector b can be quickly constructed.

Adding together the elements of the decomposition we then recover the rational matrix factors N.

To a large part, the multiplicity of solutions to the factorisation problem is expected. Indeed, it is clear that if $M = N^T N$ and U is an orthogonal matrix, then UN is another solution of the factorisation problem; conversely, if $\det M = 1$ and N and N' are solutions with integer entries, then $N' = UN$, where $U = N'N^{-1}$ is an integer orthogonal matrix. It is also not hard to see that any integer orthogonal matrix is a signed permutation matrix, i.e. a matrix which has exactly one non-zero entry, either 1 or -1, in each row and in each column.

It is therefore natural to classify the factorisation matrices (integer or rational) modulo left multiplication with integer orthogonal matrices. For the factorisations of Wilson's matrix obtained through the above procedure, this gives three distinct classes, represented by the matrix Z of (1.3) and the further two matrices

$$Z' = \begin{pmatrix} \frac{1}{2} & 1 & 0 & 1 \\ \frac{3}{2} & 2 & 3 & 3 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & 2 & 1 & 0 \end{pmatrix}, \quad Z'' = \begin{pmatrix} \frac{3}{2} & 2 & 2 & 2 \\ \frac{3}{2} & 2 & 2 & 1 \\ \frac{1}{2} & 1 & 1 & 2 \\ -\frac{1}{2} & -1 & 1 & 1 \end{pmatrix}. \tag{3.5}$$

The three factorisations $W = Z^T Z = Z'^T Z' = Z''^T Z''$ correspond to the solutions $(w, x_1, x_2, x_3) = (19, 17, 1, -7)$, $(w, x_1, x_2, x_3) = (18, -8, 20, -12)$ and $(w, x_1, x_2, x_3) = (19, 11, 7, -1)$ of eq. (1.5), respectively. Note that Z' and Z'' are not integer matrices, so the equivalence class of Z comprises all integer factorisations of the Wilson matrix.

4. Determinants, decompositions and adjugates

In this section we consider how the matrix decomposition of Theorem 2.3 can be used to obtain expressions for the determinant of a matrix. We begin by recalling that the *adjugate* of an $n \times n$ matrix M is defined by

$$\operatorname{adj} M = ((-1)^{i+j} \det(M_{j,i}))_{i,j=1}^n,$$

where $M_{j,i}$ denotes the submatrix of M obtained by deleting row j and column i; $\operatorname{adj} M$ is the transpose of the matrix of cofactors of M. If M is nonsingular, then the adjugate can be written as $\operatorname{adj} M = (\det M)M^{-1}$.

The adjugate of a matrix appears naturally when calculating the determinant of a rank-1 perturbation of a matrix, as shown in the following lemma (cf. [5]).

Lemma 4.1. *Let $M \in \mathbb{C}^{n\times n}$ and $u, v \in \mathbb{C}^n$. Then*

$$\det\left(M + uv^T\right) = \det(M) + v^T \operatorname{adj}(M)\, u.$$

Proof. Suppose M is a nonsingular $n \times n$ matrix. Then

$$\begin{aligned}\det(M + uv^T) &= \det(M)\det(I + M^{-1}uv^T) = \det(M)(1 + v^T M^{-1}u) \\ &= \det(M) + v^T(\det(M)M^{-1})u.\end{aligned}$$

This gives the stated identity for regular matrices. The general case follows from the facts that the set of regular matrices is dense in $\mathbb{C}^{n\times n}$ and that the determinant and adjugate are continuous functions of the matrix. □

Corollary 4.1. *Let $M_0 \in S_n$ with weight 0, and $M = M_0 + w_M\mathcal{E}_n$ with some $w_M \in \mathbb{C}$. Then*

$$\det M = w_M\, 1_n^T(\mathrm{adj} M_0)1_n = n^2(\mathrm{wt}\, M)(\mathrm{wt}\,\mathrm{adj}\, M_0).$$

Proof. Writing $\mathcal{E}_n = 1_n 1_n^T$, we have $M = M_0 + w_M 1_n 1_n^T$, and applying Lemma 4.1 gives $\det M = \det M_0 + w_m 1_n^T(\mathrm{adj} M_0)1_n$. As $M_0 1_n = 0$, $\det M_0 = 0$. □

Remark 2. If $M = a1_n^T + 1_n b^T + M_0 + (\mathrm{wt}\, M)\mathcal{E}_n$ is the decomposition of $M \in \mathbb{C}^{n\times n}$ as in Theorem 2.3, where $a, b \in \{1_n\}^\perp$ and $M_0 \in S_n$ with weight 0, then Lemma 4.1 and Corollary 4.1 can be used to derive the formula

$$\begin{aligned}\det M = {}&(\mathrm{wt}\, M - 1)\, 1_n^T\, \mathrm{adj}\left(M_0 - ab^T\right) 1_n \\ &+ (b + 1_n)^T\, \mathrm{adj}\left(M_0 - ab^T + (\mathrm{wt}\, M - 1)\mathcal{E}_n\right)(a + 1_n).\end{aligned}$$

Example: the Wilson matrix. For the Wilson matrix we have $\det(W_S) = \frac{357}{8}$, and $\mathrm{adj}(W_0) = \frac{3}{8}\mathcal{E}_4$, where

$$\begin{aligned}W_S = W_0 + w_W &= \frac{1}{8}\begin{pmatrix} 67 & 65 & 55 & 51 \\ 65 & 71 & 53 & 49 \\ 55 & 53 & 67 & 63 \\ 51 & 49 & 63 & 75 \end{pmatrix} \\ &= \frac{1}{16}\begin{pmatrix} 15 & 11 & -9 & -17 \\ 11 & 23 & -13 & -21 \\ -9 & -13 & 15 & 7 \\ -17 & -21 & 7 & 31 \end{pmatrix} + \frac{119}{16}\mathcal{E}_4.\end{aligned}$$

By Corollary 4.1, $\det(W_S) = 16(\mathrm{wt}\, W)(\mathrm{wt}\,\mathrm{adj}\, W_0)$, and as $\mathrm{adj}\, W_0 = \frac{3}{8}\mathcal{E}_4$, this equates to $16\,\frac{119}{16}\,\frac{3}{8} = \frac{357}{8}$.

Thus the adjugate of the weightless type S part of the Wilson matrix is a multiple of $\mathcal{E}_4$. As we show in the following theorem, the adjugate of a weightless type S matrix is in fact always a scalar multiple of $\mathcal{E}_n$. Moreover, the converse holds in the sense that a

matrix whose adjugate is a *non-zero* multiple of $\mathcal{E}_n$ must be a weightless type S matrix. Of course any matrix of rank $n-2$ or less has adjugate $0 = 0\mathcal{E}_n$.

Theorem 4.1. *(a) Let $M \in S_n$ with weight 0. Then there is a constant $w \in \mathbb{C}$ such that* $\operatorname{adj} M = w\,\mathcal{E}_n$.

(b) Let $M \in \mathbb{C}^{n\times n}$ such that $\operatorname{adj} M = w\,\mathcal{E}_n$ *with some $w \in \mathbb{C} \setminus \{0\}$. Then $M \in S_n$,* $\operatorname{wt} M = 0$ *and* $\operatorname{rank} M = n-1$.

Proof. (a) Let $j \in \{1,\dots,n\}$ and denote by $m_1,\dots,m_n \in \mathbb{R}^{n-1}$ the columns of the matrix M with the j-th row omitted. Let $k \in \{1,\dots,n-1\}$. Then

$$m_k = -\sum_{l\in\{1,\dots,n\},l\neq k} m_l,$$

as the row sums of M vanish; hence

$$\begin{aligned}
&\det(m_1,\dots,m_{k-1},m_k,m_{k+2},\dots,m_n)\\
&\qquad = \det(m_1,\dots,m_{k-1},-\sum_{l\neq k} m_l, m_{k+2},\dots,m_n)\\
&\qquad = -\sum_{l\neq k}\det(m_1,\dots,m_{k-1},m_l,m_{k+2},\dots,m_n)\\
&\qquad = -\det(m_1,\dots,m_{k-1},m_{k+1},m_{k+2},\dots,m_n),
\end{aligned}$$

noting that all the determinants in the sum vanish except for the term $l = k+1$. This shows that the $(j,k+1)$ entry and the (j,k) entry of $\operatorname{adj} M$ are equal. Since this holds for all $k \in \{1,\dots,n-1\}$ and for all $j \in \{1,\dots,n\}$, it follows that $\operatorname{adj} M$ has constant columns.

Applying the same argument to M^T (which also has row sums 0), we find that $\operatorname{adj} M$ also has constant rows, and hence $\operatorname{adj} M = w\mathcal{E}_n$ for some $w \in \mathbb{C}$.

(b) Let $j \in \{1,\dots,n\}$ and let $m_1,\dots,m_n$ be as in (a). Since

$$\det(m_1,\dots,m_{n-1}) = (-1)^{n+j} w \neq 0, \tag{4.1}$$

the vectors $m_1,\dots,m_{n-1}$ form a basis of $\mathbb{C}^{n-1}$, so

$$m_n = \sum_{l=1}^{n-1} \alpha_l m_l$$

for suitable $\alpha_1,\dots,\alpha_{n-1} \in \mathbb{C}$. Now for any $k \in \{1,\dots,n-1\}$ we have

$$\begin{aligned}
w &= (-1)^{k+j}\det(m_1,\dots,m_{k-1},m_{k+1},\dots,m_{n-1},m_n)\\
&= (-1)^{k+j+n-1-k}\det(m_1,\dots,m_{k-1},m_n,m_{k+1},\dots,m_{n-1}).
\end{aligned}$$

Combining this with eq. (4.1), we find that $0 = \det(m_1, \dots, m_{k-1}, m_k + m_n, m_{k+1}, \dots, m_{n-1})$. Hence, for suitable $\beta_l \in \mathbb{C}(l \in \{1, \dots, n-1\} \setminus \{k\})$ we have

$$\sum_{l \neq k}^{n-1} \beta_l m_l = m_k + m_n = m_k + \sum_{l=1}^{n-1} \alpha_l m_l = (1 + \alpha_k) m_k + \sum_{l \neq k}^{n-1} \alpha_l m_l.$$

As $m_1, \dots, m_{n-1}$ are linearly independent, it follows that $1 + \alpha_k = 0$, i.e. that $\alpha_k = -1$.

Since this holds for all $k \in \{1, \dots, n-1\}$, we can conclude that $\sum_{l=1}^{n} m_l = 0$, i.e. that the rows of M, except for the jth row, add up to 0. Since $j \in \{1, \dots, n\}$ was arbitrary, this holds in fact for all rows.

Applying the above reasoning to the transpose of M, we find that its columns also add up to 0. □

5. Co-Latin matrices

An $n \times n$ Latin square (or Latin matrix) is an $n \times n$ matrix with entries from $\{1, \dots, n\}$ such that the entries of each row and of each column are distinct, i.e. each number in $\{1, \dots, n\}$ appears exactly once in each row and in each column. Evidently any Latin matrix is of type S. If $L = (\ell_{p,q})_{p,q=1}^n$ is a Latin square and $k \in \{1, \dots, n\}$, we define $L^{(-1)}(k) = \{(p, q) \in \{1, \dots, n\}^2 : \ell_{p,q} = k\}$.

Definition 1. A matrix $M = (m_{i,j})_{i,j=1}^n \in \mathbb{R}^{n \times n}$ has the *co-Latin* property if

$$\sum_{(p,q) \in L^{(-1)}(k)} m_{p,q} = 0$$

for all $k \in \{1, \dots, n\}$ and all $n \times n$ Latin squares L.

Clearly the set of $n \times n$ co-Latin matrices forms a subspace of $\mathbb{R}^{n \times n}$. The co-Latin property can be considered an extreme opposite of the weightless type S property. A matrix in S_n with weight 0 has the property that its entries in any one row or column add to 0; a co-Latin matrix has the property that any selection of n entries such that no two selected entries lie in the same row or the same column add to 0. We show in the following that these properties are indeed complementary in the sense of unique decomposability of any given weightless $n \times n$ matrix into a weightless type S matrix and a co-Latin matrix; this is a direct consequence of the following theorem, which identifies co-Latin matrices with V_n in (2.1).

Theorem 5.1. *Let $n \in \mathbb{N}$. The space of all $n \times n$ co-Latin matrices is equal to V_n.*

To prepare the proof of Theorem 5.1, we first show that there exists a Latin square which has the entries 1 and 2 pairwise on the diagonally opposite corners of a rectangle.

By suitable row and column permutations, we can assume without loss of generality that this rectangle is the top left 2×2 square. The existence of such a Latin square is not trivial; in fact there is none in dimension 3, as the arrangement

$$\begin{pmatrix} 1 & 2 & * \\ 2 & 1 & * \\ * & * & * \end{pmatrix}$$

enforces two 3s in the third column and the third row, violating the defining condition of a Latin square.

Lemma 5.1. *Let $n \in \mathbb{N} \setminus \{1, 3\}$. Then there exists an $n \times n$ Latin square $L = (\ell_{ij})$, such that $\ell_{1\,1} = \ell_{2\,2} = 1$ and $\ell_{1\,2} = \ell_{2\,1} = 2$.*

Proof. We consider the cases of even and odd n separately.

(i) *Even $n = 2m$.* Consider the Hankel Latin square $\tilde{L} = (\tilde{\ell}_{i,j})_{i,j=1}^{m}$, where $\tilde{\ell}_{i,j} = 1 + ((i + j - 2) \bmod m)$ $(i, j \in \{1, \ldots, m\})$, e.g. for $m = 4$

$$\tilde{L} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{pmatrix},$$

and then replace each (i, j) entry in this matrix with the 2×2 block

$$L = \begin{pmatrix} 2\tilde{\ell}_{ij} - 1 & 2\tilde{\ell}_{ij} \\ 2\tilde{\ell}_{ij} & 2\tilde{\ell}_{ij} - 1 \end{pmatrix}$$

to create the Latin square L.

(ii) *Odd n.* Here we start off the first 3 antidiagonals of L thus:

$$\begin{pmatrix} 1 & 2 & 3 & \\ 2 & 1 & & \\ 3 & & & \end{pmatrix},$$

and then fill up the antidiagonals up to and including the main antidiagonal in the Hankel Latin manner described above. We then fill the next three antidiagonals with

$$\begin{array}{ccc} 3\ 1\ 2\ 1\ 2 & \cdots & 1\ 2\ 1\ 2\ 3, \\ 2\ 3\ 3\ 3 & \cdots & 3\ 3\ 3\ 1, \\ 1\ 2\ 1 & \cdots & 2\ 1\ 2, \end{array}$$

respectively, and complete the remaining antidiagonals in the standard Hankel Latin manner. For example, for $n = 9$ this gives the Latin square

$$L = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 1 & 4 & 5 & 6 & 7 & 8 & 9 & 3 \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 1 \\ 4 & 5 & 6 & 7 & 8 & 9 & 1 & 3 & 2 \\ 5 & 6 & 7 & 8 & 9 & 2 & 3 & 1 & 4 \\ 6 & 7 & 8 & 9 & 1 & 3 & 2 & 4 & 5 \\ 7 & 8 & 9 & 2 & 3 & 1 & 4 & 5 & 6 \\ 8 & 9 & 1 & 3 & 2 & 4 & 5 & 6 & 7 \\ 9 & 3 & 2 & 1 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}$$

This construction works from $n = 5$ onwards. $\square$

We can now prove the following.

Lemma 5.2. *Let $n \in \mathbb{N}$, and let $M = (m_{i,j})_{i,j=1}^n$ be an $n \times n$ co-Latin square. Then $M \in V_n$.*

Proof. The statement is trivial in the case $n = 1$.

Now consider $n \in \mathbb{N} \setminus \{1, 3\}$. Let $i, j, k, l \in \{1, \dots, n\}$ such that $i \neq k$ and $j \neq l$. By suitable permutation of the rows and columns of the Latin square constructed in Theorem 5.1, there exists a Latin square $L = (\ell_{p,q})_{p,q=1}^n$ such that $\ell_{i,j} = \ell_{k,l} = 1$ and $\ell_{i,l} = \ell_{k,j} = 2$. By the co-Latin property, we know that

$$\sum_{(p,q) \in L^{(-1)}(1)} m_{p,q} = 0,$$

so

$$m_{i,j} + m_{k,l} = - \sum_{(p,q) \in L^{(-1)}(1) \setminus \{(i,j),(k,l)\}} m_{p,q}. \tag{5.1}$$

Now consider the matrix $L' = (\ell'_{p,q})_{p,q=1}^n$ which arises from L by keeping all the same entries except that $\ell'_{i,j} = \ell'_{k,l} = 2$ and $\ell'_{i,l} = \ell'_{k,j} = 1$. Then L' is still a Latin square, and by the co-Latin square property we find that

$$\sum_{(p,q) \in L'^{(-1)}(1)} m_{p,q} = 0,$$

so

$$m_{i,l} + m_{k,j} = - \sum_{(p,q) \in L'^{(-1)}(1) \setminus \{(i,l),(k,j)\}} m_{p,q}. \tag{5.2}$$

Noting that the index sets of the sums in (5.1) and (5.2), and hence the values of these sums, are the same, we conclude that $M \in V_n$.

Finally, to see that the statement holds true in the case $n = 3$, we note that, up to permutations of the symbols $\{1, 2, 3\}$, there are only two different 3×3 Latin squares, namely either of the form

$$\begin{pmatrix} A & B & C \\ B & C & A \\ C & A & B \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} D & E & F \\ F & D & E \\ E & F & D \end{pmatrix}.$$

To show that a 3×3 co-Latin square M has the vertex cross sum property, without loss of generality we can consider the case $i = j = 1$, $k = l = 2$ (as the other cases can be reduced to this by suitable row and column permutations). Then the first of the above Latin matrices shows that $m_{2,1} + m_{1,2} + m_{3,3} = 0$, the second Latin matrix shows that $m_{1,1} + m_{2,2} + m_{3,3} = 0$, and it follows that $m_{1,1} + m_{2,2} = m_{2,1} + m_{1,2}$. $\square$

We can now complete the proof of Theorem 5.1.

Proof of Theorem 5.1. By Lemma 5.2, any $n \times n$ co-Latin matrix is an element of V_n.

Conversely, by Theorem 2.2 any element of V_n is of the form $a1_n^T + 1_n b^T$, with suitable $a, b \in \{1_n\}^\perp$. Since a Latin square takes each value in $\{1, \ldots, n\}$ exactly once in each row and each column, it is hence straightforward to see that $a1_n^T$ and $1_n b^T$ are co-Latin squares. $\square$

Example: the Wilson matrix. Using the V part of our integer factorisation matrix Z_0, given by Z_V in (2.3), we have

$$Z_V = \frac{1}{8} \begin{pmatrix} 5 & 7 & 11 & 11 \\ -3 & -1 & 3 & 3 \\ -7 & -5 & -1 & -1 \\ -9 & -7 & -3 & -3 \end{pmatrix}.$$

It can be easily verified numerically that this type V matrix satisfies all $4! = 24$ Latin selections summing to 0, and so is a co-Latin matrix.

6. Conclusions

Matrices with integer entries play an important role in many modern applications of mathematics. In numerical analysis, for example, they make convenient test matrices because they are exactly representable. Families of matrices with bounded integer entries have recently been termed Bohemian matrices and various aspects of them have been studied [2], [3], [4]. The factorisation of integer matrices in the form $M = N^T N$ with integer N is related to a classical topic in number theory, but there has not been much work on finding such factorisations. In this work we have provided some new ideas to

determine under what circumstances integer factorisations exist and to develop an approach to computing them. Our results are founded upon the orthogonal decomposition of the algebra of square matrices into two parts, of which one part is the subalgebra of constant (row and column) sum matrices, while we identified the other part as a space of co-Latin square matrices with symmetries determined by the properties of Latin squares.

Declaration of competing interest

There is no competing interest.

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