# A Closed Economy DSGE Model with Occasionally Binding Cash-In-Advance Constraints

by

Xingchen Li

A Thesis Submitted in Fulfilment of the Requirements for the Degree of Doctor of Philosophy of Cardiff University

Economics Section

Cardiff Business School

Cardiff University

Thesis Supervisors:

Prof. Huw Dixon (Primary)

Dr. Panayiotis Pourpourides (Secondary)

September 2020

# Abstract

This thesis is planned to investigate whether endogenously binding cash-in-advance (CIA) constraints helps to explain the persistent liquidity traps in the US since the 2008 financial crisis. It extends the theoretical general equilibrium analysis by Dixon and Pourpourides (2016) to build a closed economy DSGE model. The model is evaluated based on the calibration and is estimated by the method proposed by Guerrieri and Iacoviello (2015, 2017) using filtered data in the period of 1985Q1 to 2017Q4 for the United States. When money is the only asset, an increase in money supply followed by a technological innovation triggers a nonbinding CIA constraint, which can cause a consumption boom. The changes in the inflation rate are the main driving force for the nonbinding CIA constraints. The periods when the CIA constraint is slack follows the waves of quantitative easing policies. However, when capital and bonds are introduced, a nonbinding CIA constraint fails to generate a consumption boom. Two ways of modelling monetary policy, money growth rule and interest rate feedback rule with zero-lower bound (ZLB), are compared. Money, which serves as a safe asset, depresses output and its components. Things are even worse when the monetary policy is set via an interest rate feedback rule with ZLB as the nominal interest rate becomes the main driving force for nonbinding CIA constraints. When the ZLB already binds, an increase in money supply loses its ability to stimulate the economy and the liquidity traps are more persistent. The model when both occasionally binding CIA constraints and ZLB are included successfully matches the data, especially for the period after the 2008 global crisis. The Friedman rule is still optimal, but the monetary authority should consider subsidizing or taxing on specific sectors instead of conducting open market operations that raise the base money in the whole economy.

# Acknowledgements

My deepest gratitude goes first and foremost to my first supervisor Professor Huw Dixon for his excellent guidance, greatest patience and tremendous support both academically and personally during the completion of this thesis. I am truly thankful to his lovely wife Barbara for her good humour and precious companion.

I would also like to express my appreciation to my second supervisor Dr Panayiotis Pourpourides for his concise comments and helpful suggestions on the drafts throughout the stage of this thesis.

Thanks also to all the colleagues and support staff at Cardiff Business School for providing a friendly research and teaching environment during past years. I particularly thank Aigerim Rysbayeva, Yuji Fang, Zequn Xu, Ning Han and Yao Yao for sharing a relaxing office.

Special thanks to my friends, including among others Dr Jenyu Chou, Dingming Sun, Shangheng Gao, Jie Sun, Xin Wei, Wenchao Zou and Yingzi Chen. Their support and advice always calm me down, especially during the unfamiliar and anxious time of the COVID-19.

I am truly indebted to my parents, Guangsheng Li and Jun Chen, for their continuous support and encouragement. Their endless love has walked me through all tough stages of my PhD. Most of all, I wish to say a heartfelt thank you to my fantastic mum and all the people who have been putting their selfless efforts to fight with the coronavirus.

I dedicate this thesis to my supervisors, colleagues, friends and family.

# Contents

# List of Figures

# List of Tables

# Chapter 1 Introduction

Periods of persistent liquidity traps after the 2008 global crisis have raised a reconsideration of the role of increased money supply. Most macroeconomists believe that money is super neutral in the long run. From this perspective, an increase in money supply is unable to crowd out investment. Indeed, a change in the quantity of money results in a proportional change in the general level of price. Thus, an increase in money supply has no effect on real economic variables in the long run. In the short run, however, it is believed that an increase in money supply is correlated with an increase in output and its components. As a result, the economy can be stimulated by a money supply growth.

However, this view is challenged by recent researchers. They point out that the open market operations like Quantitative Easing (QE) programs under near-zero interest rate policies make very limited contributions to stimulate the economy but lead to a deeper liquidity trap and a slower recovery. According to the observed data for the US, which is shown in Figure 1-1, periods when the liquidity traps are persistent happen to coincide with substantial increases in the money base.

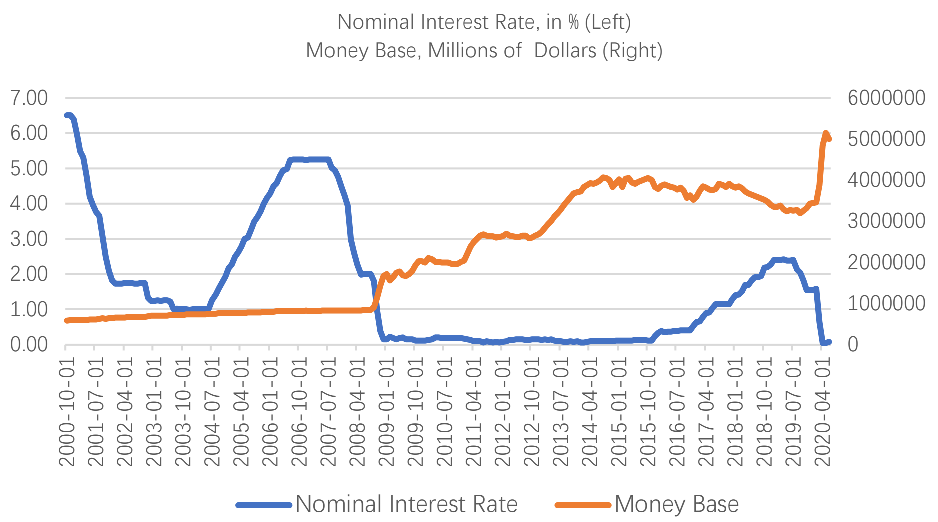


*Figure 1-1 Policy rates and money base in the US*

*Sources: Federal Reserve Bank of St. Louis*

Additionally, since central banks adopted near-zero interest rates policies and large-scale asset purchase programs, inflation rates have been predicted to be high by inflation hawks. In contrast, trends towards deflation, rather than inflationary pressures, have been seen as in Figure 1-2. The inflation rate of the US reached its most peak around September 2011 at 3.9 per cent and have fallen steadily since then. It reached its lowest value of -0.2 per cent in April 2015. More recently after the 2020 COVID-19 pandemic, the US inflation rate decreased sharply and has displayed a trend towards deflation. The inflation rate trends in the UK as well as the average trends of all OECD countries are quite similar to those of the US but are milder.

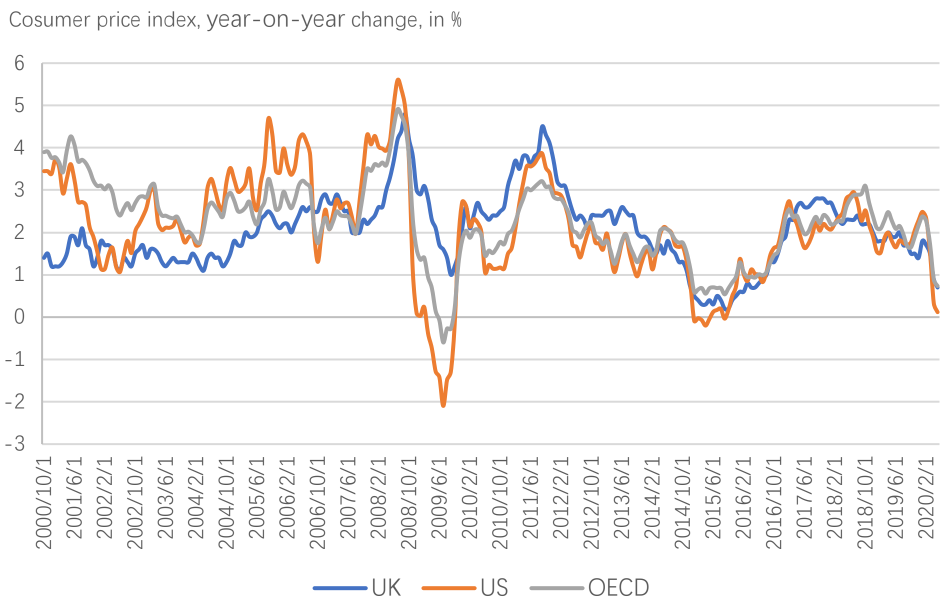


*Figure 1-2 Inflation rates trends*

*Sources: OECD Data*

More and more papers are trying to make reasonable explanation for the slow recovery and deflation trends in the persistent liquidity traps. Bernanke and Gertle (1995) introduce an external premium to explain the liquidity trap recession. Garcia-Schmidt and Woodford (2015) provide another explanation that forward-looking agents predict a low inflation target from a prolonged period of low policy rates, which leads to an expectation of low inflation and consequently materialises in low inflation. In contrast to the statement of Cochrane (2015) that the policy rate has to increase for the increase of inflation, Demary and Huther (2015) argue that it is the low equilibrium real interest rate combined with the zero-lower bound on nominal interest rate that constrains the effectiveness of monetary policy and results in a low inflation. Buera and Nicolini (2017) find that an excess demand of outside liquidity assets may lead to a deflation when the zero bound on nominal interest rate binds unless there is an increase in the nominal money supply. Ragot (2017) uses a model with heterogenous households to show that a tightened credit constraint increases real money demand and causes deflationary pressures as well as a liquidity trap. Bacchetta et al. (2017) also note that it is the heterogeneity of agents that results in the disappointing investment levels and output growth during the periods of liquidity traps. Di Tella (2017) analyses a flexible-price model with incomplete idiosyncratic risk sharing and finds that investment is too high during a boom and too low during a recession as money serves as a safe asset.

Money is always in the centre of the Macroeconomics. New monetarist economics, named by Williamson and Wright (2010, 2011), keep even deeper insights on the explicit models of money, liquidity and assets. The four general functions of money can be summarised as medium of exchange, unit of account, store of value and the source of deferred payment. Three major motives for agents to hold money are transactions, precautionary and speculative motives. Two popular ways of modelling the demand for money are money-in-utility function and cash-in-advance (CIA, henceforth) constraint. Two prevailing ways of modelling monetary policy are money growth rules and interest feedback rules. According to the comparison of these methods of modelling money demand and monetary policy by Bhattacharjee and Thoenissen (2007), the money-in-utility model closed by an interest feedback rule is the one comes furthest to the data while the CIA model does a better job no matter which monetary policy is chosen. The money-in-utility function is always doubted as money does not provide direct utility to agents. The CIA model, in contrast, not only considers the transaction and precautionary demand for money but also emphasises the role of money as a store of value even though money has no direct utility. Therefore, it is still popular to introduce a CIA constraint in investigating the role of money, especially during the periods of liquidity traps. Williamson (2013) provides some notes on liquidity traps with cash in advance. Buera and Nicolini (2017) impose a CIA constraint on households to study how changes in the outside supply of liquid assets will affect the economy. Di Tella (2017) adopts both money-in-utility function and CIA constraint to prove his result that money depresses the investment during liquidity traps. However, most of these literatures with CIA constraint generally assume that the constraint is always binding, while the others mainly focus on the general equilibriums of binding CIA case and nonbinding CIA case separately. This thesis is, therefore, designed to see whether endogenously binding cash-in-advance constraint in a DSGE model can explain the ineffectiveness of monetary policies and give some police implications for further recessions caused by COVID-19 in 2020.

Money is offered a value by its ability of providing liquidity services, while ordinary assets are given a value by providing dividends. The CIA constraint is, in this sense, served as a liquidity constraint and money can be seen as an asset. Going back to the methodological analysis of Svensson (1985), 'the liquidity services of money are endogenously determined by the value of relaxing the liquidity constraint — the shadow price of the liquidity constraint'. He is the very first to provide a full general equilibrium solution to consider the endogenously binding CIA constraint and discuss the causation between nonbinding CIA constraint and nominal interest rate.

However, to the best of my knowledge, no attempt has been to construct a DSGE model where the CIA constraint can be endogenously binding or nonbinding. This thesis firstly attempts to provide a closed economy DSGE model with occasionally binding CIA constraints following the standard structure of Galí (1999) and Smets and Wouters (2007). For the deeper insight of the Great Recession, the nonlinearity of zero-lower bound (ZLB, henceforth) on nominal interest rate is introduced in addition to the nonlinearity of CIA constraint. The attention of this thesis is focused on the empirical contributions to better understanding the role of money demand during liquidity traps.

Chapter 2 is a review of both theoretical and empirical literatures related to the Classical and New monetarist economics. It starts from the traditional monetary theories that demonstrate the functions of money and why the individuals are motivated to hold money. The review then extends to different methods of modelling money demand. A few attempts that include endogenously binding CIA constraint theoretically and empirically are discussed. Since the world is suffering from a deep recession after the COVID-19 without fully recovering from the Great Recession, the economy may remain in a longer liquidity trap. Although different stages of quantitative easing programs have been conducted, they seem have quite limited effects. A review of why there is a persistent liquidity trap and how to escape it is necessary. It is well known that the economy is in a liquidity trap when the nominal interest rate is zero or the ZLB constraint is binding. This is consistent with a nonbinding CIA constraint[1]. The interaction between ZLB and CIA constraint may shed light on the persistent liquidity traps. Monetary policies are widely believed to have effects on real variables when the markets are imperfectly competitive and there are some rigidities in prices and nominal wages. A short summary of modelling monopolistic competition and nominal rigidities is also provided.

In Chapter 3, a simple closed economy DSGE model with money as the only asset is displayed. Since money is the only asset, the model is closed by a money growth rule. The CIA constraint is the Clower-Lucas type that only consumption should be financed by cash. Whether the CIA constraint is binding or not depends on the changes in the expected inflation. Money is assumed to have no value intrinsically and its return is an inverse function of the expected inflation. If the inflation is always positive or is expected to rise, households are impossible to earn positive return on holding money and, hence, have no incentive to save in money. However, they are constrained by a CIA constraint, and thus tend to hold the exact amount of money for their consumption needs. This is the case when the CIA constraint is always binding. If the inflation happens to be negative or the households expect a reduction in the inflation, which is more relevant to the reality since 2008, households have a chance to earn positive return on money. Money, in this case, serves as a profit-earning asset. Hence, households are indifferent between holding money and spending and the CIA constraint is nonbinding. As long as money is the only asset, money is super neutral when the CIA

[1] See Svensson (1985) and derivations in Chapter 4 for more details.

constraint is slack. Since labour is the only input for production, households will demand and supply as they desire at the market prices and wages. The economy is working in an efficient way.

Four different cases are discussed separately. It starts from the case when both wage and price are flexible and extends step by step to the more comprehensive case when both wage and price are sticky and partially indexed to past inflation. The calibrated IRFs confirm that the production is more efficient when the CIA constraint is nonbinding than when it is always binding. Since changes in the expected inflation are the driving force for a nonbinding CIA constraint, an increase in money supply followed by a technological innovation triggers a nonbinding CIA constraint for some periods as a rise in productivity leads to an immediate decrease in the price level. As a result, there is a consumption boom when the CIA constraint is nonbinding, which is followed by a consumption decline because the households need time to accumulate real balances after the CIA constraint binds. The length of nonbinding CIA constraint periods increases with more frictions included.

The model with both wages and prices that are sticky and partially indexed to past inflation is estimated using filtered data in the period of 1985Q1 to 2017 Q4 for the United States by the adjusted Bayesian estimation proposed by Guerrieri and Iacoviello (2017). The estimation result suggests that the periods when the CIA constraint is nonbinding follow different stages of the open market operation programs (i.e. QE programs). It confirms the contributions of QE programs on stimulating the economy. The QE programs, in addition, show their ability in helping the economy to get rid of liquidity traps when the agents fully adjust their expectations on inflation. It also matches the data in that the CIA constraint is always binding before the Great Recession.

This simple DSGE model with money as the only asset is a successful attempt to prove that the CIA constraint does not always bind. But it is unrealistic for money to be the only asset and the model fails to see the whole picture after the 2008 global crisis. In Chapter 4, other assets, nominal bonds and capital, are introduced. The CIA constraint, hence, is the generalized

Stockman type that consumption and a fraction of investment should be financed by cash. Since the nominal bonds are included, another key component for monetary policy, nominal interest rate, is able to be modelled. The inclusion of nominal interest rate raises the consideration for an alternative way of modelling monetary policy, namely, an interest rate feedback rule, or more specifically, Taylor rule. The Federal Reserve adhered to the Taylor rule officially since 1995 and relies on it for the management of monetary policies. Nominal interest rate is viewed as the main tool of conventional monetary policy and has been cut to nearly zero during the period of the Great Recession and, more recently, the COVID-19 pandemic. Hence, besides the nonlinearity of the CIA constraint, another nonlinearity of ZLB constraint, which refers to the situation when the nominal interest rate cannot fall any further below zero, is also taken into account.

The results in Chapter 3 are greatly changed. If the model is still closed by the money growth rule, changes in the expected inflation remain the main causes for a nonbinding CIA constraint. However, if the model is closed by the Taylor rule with a ZLB, a combination of nominal interest rate and inflation makes contributions to the nonbinding CIA constraint. In particular, when the model is closed by the money growth rule, a negative inflation or a decrease in the expected inflation gives the households an incentive to hold money and the CIA constraint is, in turn, nonbinding. A nonbinding CIA constraint leads to a zero nominal interest rate or a binding ZLB. The economy, therefore, falls into a liquidity trap. This results in a circumstance when money, nominal bonds and capital are perfect substitutes. Money replaces bonds and capital as a safe store of value that prevents the interest rate from falling and crowds out investment. Since both labour and capital are inputs of production, depression in investment, which slows down the capital accumulation, in turn cause a limited growth in output. Things are even worse when the model is closed by the Taylor rule with ZLB. Decreases in nominal interest rate is stopped by both binding ZLB and nonbinding CIA constraint, which leads to an even larger depression in the growth of investment and output. The nonbinding CIA constraint is also a cause of binding ZLB. The interaction between nonbinding CIA constraint and binding ZLB triggers a much longer liquidity trap as well as a much slower recovery.

The estimation results for the model closed by the money growth rule are similar to those in Chapter 3. The only difference is that different stages of QE programs have quite limited ability to stimulate the economy. During normal times, an increase in money supply causes a lower interest rates and encourages consumption and investment. However, during unusual times of liquidity traps, the agents are indifferent to an increase in the money stock as the interest rate is prevented from falling further to negative. The extra money in their pocket is saved or held instead of being spent. The estimation for the model closed by Taylor rule with ZLB significantly match the data, especially for the periods after the 2008 global crisis. Unfortunately, the different stages of QE programs fail to help the economy escape liquidity traps. This may be because large increases in the money base have very limited impact on stimulating aggregate demand as long as the nominal interest rate is set to be at its lower bound. The increase in the expected inflation resulting from the increases in aggregate demand is too small to drive the real interest rate down if the consumers expect that the government is likely to commit to a low inflation.

Finally, Chapter 5 makes some conclusions about possible policy implications and discusses extensions of existing works. The COVID-19 pandemic forces the government to urgently come up with an idea of escaping liquidity traps and prepare for post-pandemic recovery. This thesis uses a relatively simple model to prove that the combination of lower policy rate and increase in money base is even less likely to help the economy recover from liquidity traps.

Uncertainties of further crisis may increase precautionary savings of households. When they are indifferent between saving and spending, they tend to hold money as a relatively safe asset to fight uncertainties. As a result, it is even harder to stimulate the aggregate demand. The governments are believed to always be succeed in implementing supply-side policies. However, the supply-side policies are unable to solve the fundamental problem of the shortage in aggregate demand. It is the time for the governments to maintain their credibility so that the public is more confident of fighting uncertainties. A responsible and transparent government tends to be a better solution of escaping liquidity traps. Instead of continuing target low interest rate, the central banks are better to target at a higher interest rate so that the inflation expectations of agents will rise. The inflation, hence, is the only driving force that can help the recovery.

Learning from the experiences from the Great Recession that open market operations which increases the money base of the whole economy has very limited impact on increasing economic growth, it may be a possible way to provide subsidies to specific sectors. Money, which serves as a safe asset, crowds out investment. A special subsidy on capital can be imposed to increase the demand for investment[2].

This thesis provides the first glance of how the endogenously binding CIA constraint can affect the dynamics of the economy. Further research can be done by introducing an appropriate financial sector so that the model is able to account for more important sources of aggregate fluctuations. Another interesting extension is to analyse how the credibility of the governments will influence the length of nonbinding CIA constraints.

[2] This is consistent with the idea of Di Tella (2017).

# Chapter 2 Literature reviews

## 2.1 Introduction

The demand of money is always in the centre of implementing monetary policy as it enables a small change in monetary aggregates to have huge and predictable influences on real variables such as output and employment. Thus, the field of monetary economics is trying to understand the relationship between nominal and real variables at aggregate level. Since the global financial crisis in 2008, the liquidity property of money has raised the attention of central banks, or more generally monetarists, to looking deep into the relationship between money, liquidity, interest rates, inflation and output. A steady trend of theoretical and empirical researches has been conducted around the world over the past several decades. Majority of the work was confined to theoretical analysis or general equilibrium models, while relatively fewer researches were focused on dynamic stochastic general equilibrium (DSGE) models. However, due to the concern among central banks on the impact of both conventional and unconventional monetary policy on real variables, the studies conducted via DSGE models have been increasing in recent years.

## 2.2 Functions of Money

Why monetarists love money? The use of money is considered to broader the world economic transactions. What is special about money? Simply, the special functions of money make money special. In modern society, money is served as the medium of exchange and the standard unit in which the value of goods and services can be measured. In general, money has four traditional functions: medium of exchange, unit of account, store of value, and the source of deferred payment. Clower (1967) reconsiders the micro foundations of money as the medium of exchange by comparing a money economy with a barter economy. In the money economy, 'money buys goods and goods buy money; but goods do not buy goods'. This is later being seen as the central theme of the theory of a money economy. He demonstrates that 'the conception of a money economy implicit in modern accounts of the general equilibrium theory of money and prices is formally equivalent to the Classical conception of a barter economy'. However, from the perspective of economic efficiency, the use of money minimises the time required to trade goods for goods in a barter economy. Money also helps to avoid double coincidence of wants of traders in a barter economy.

The function of money as unit of account can be explained as the standard unit in which prices and debt can be expressed. This function also promotes the economic efficiency by reducing the number of calculations required to exchange goods. In a barter economy, the number of calculations required to exchange goods increases with the number of goods. For example, consider an economy with only 3 goods. Three prices have to be posted to enable exchange. Now move to 4 goods. Six prices are needed as there are six pairs of goods to compare. In general, if the number of goods equal to $n$, $n(n-1)/2$ prices need to be posted in a barter economy while only $n$ prices are needed in a money economy. As the number of goods increases, the efficiency improved by the usage of money is more significant.

The function of money as a store of value can be explained as money transfers purchasing power over time. Since the income is not normally spent at the time it is received, timing of income and expenditure differs. Is money unique as a store of value? Obviously, any asset, such as durable goods, art, jewellery, financial assets and so on, might perform the function as a store of value. What is the advantage of money over other assets? It is the property of liquidity that makes money superior. Liquidity is defined as the speed at which the assets can be converted into a medium of exchange. Since money itself is the medium of exchange, it is most liquid among all assets by its own nature. All other assets usually involve a transaction costs or a liquidity premium for them to become liquid.

## 2.3 Money Demand Theories

Money demand elasticities are of great importance for the implementation and effectiveness of monetary policies. Hence, a stable demand for money is a crucial condition for the feasibility of monetary policies. According to Friedman (1956), money demand function is the most stable macroeconomic relation and also one of the most important relations in the analysis of economic behaviour, especially in selecting appropriate monetary implications. A predictable link between monetary aggregates and changes in the components in the money demand function is necessary for the success of monetary policy which relies on the control of monetary aggregates to affect real economic variables. Money demand theories have evolved over time. A comprehensive review of expanding literature on all money demand theories are far beyond the ambition of this thesis. Instead, a review on the history of two relevant branches of money demand theory, a cash-in-advance approach as well as money as an asset, is displayed.

### 2.3.1 Cash-In-Advance (CIA) Approach

A class of models that emphasises the transaction role played by money is the cash-in-advance models which are the base models for this thesis. The CIA models are equilibrium models which involves a specific restriction that goods purchase in a given period should be paid for by currency carried from the previous period. This specific sort of limitation is commonly known as cash-in-advance constraint or Clower constraint (Clower, 1967). It provides an alternative for money-in-utility models and offers a more intuitively appealing analytical tool to investigate the core question that why rational agents tend to hold money. In money-in-utility function models, households are assumed to attain utility not only from consuming goods and services but also from holding real balances (Patinkin, 1965). In this approach, money is assumed to yield a direct utility, which is not associate with other assets that derive only an indirect utility via the income they generate and the consumption goods they are used to purchased, to the agents. However, Brunner (1951) earlier recognizes the transaction role of money was not in the utility function but from the constraints faced by the agent when making decision on how much to supply and demand of each good. Money-in-utility approach is criticized by monetary economists, who argue that the approach should model the transaction process more explicitly. It is also subject to the criticism that the utility theory assumes that agents derive utility only from the commodities or services that are actually consumed, and the true utility does not include real balance. In contrast, the CIA constraint can create a transaction demand for money even though money provides no direct utility.[3] However, Feenstra (1986) studied the relationship between these two approaches and showed that these two approaches may be equivalent under certain regularity conditions. Bhattacharjee and Thoenissen (2007) also compare these two methods of modelling money as well as two ways of modelling monetary policy. They find that the cash-in-advance model closed by a money growth rule performs a better match of the data.

Lucas (1980) makes vital contributions in the development of CIA models by providing microfoundations for money demand and extending the theoretical support for the transaction motives of money demand. He includes the optimizing behaviour of agents, as discussed in Baumol (1952) and Tobin (1956), and CIA constraint in a macroeconomics equilibrium setting to analyse the transaction demand for money. In his frictionless model, the representative household chooses money balances, consumption and savings subject to the CIA constraint, $P_t c_t \leq M_t$, after observing the state of the economy. Consumers hold money only for consumption, and thus the CIA constraint will be always binding, $P_t c_t = M_t$, in this risk-free model. Other assets like bonds earn a positive rate of return while money has no nominal return and the real return is negative because of the inflation. The result of this frictionless CIA model is that 'the velocity of money is equal to one which is a constant' when the CIA constraint is always binding. This seems to capture the basic idea of the transaction demand for money. But he makes no attempt to understand the deeper frictions that cause this constraint. In the presence of frictions, the velocity of money can vary below one as the CIA constraint can be nonbinding due to the uncertainties faced by households.

Svensson (1985) adds some uncertainties into Lucas' model by assuming that the representative household chooses consumption, money balances and savings before observing the state of the economy. Lucas restricts his analysis to the situation where the nominal interest rates are always positive. With this positive interest rates households tend to hold exactly the amount of cash they need to buy consumption goods to avoid the interest losses on

3 This was the rationale behind the general formulation of the CIA constraint that firstly put forth by Grandmont and Younes (1972).

excess cash holdings. Hence, there is only transaction money demand but no precautionary money demand. However, in Svensson's model, consumers must make decision on their cash balances before they know the current state and thus before they know what they will consume. Therefore, all transaction, precautionary and speculative motives of demand for money are taken into account. When the economy is in good state or the interest rate is low, consumers tend to hold more precautionary balances, the CIA constraint becomes nonbinding, $P_t c_t < M_t$, and the velocity of money is less than one. It can be concluded that the velocity of money will change as a consequence of the state uncertainty, and it will depend on the interest rate as well.

Lucas and Stokey (1987) introduces a cash-credit goods model where households could use either cash or credit to buy consumption goods. In their model, there are two types of goods, cash goods which must be purchased with cash and credit goods which can be purchased with credit. Cash goods are subject to a CIA constraint while credit goods are non-market goods like leisure. Both cash and credit goods will be priced at the same nominal level in each period. Agents also choose their money holdings before they observe the state of the economy. Current money growth can affect the current allocation only through its value as a signal, that is, current money growth can have the influence on the expectation of households about future economy states. Securities are trade before the state is known but after some signal is announced or after money injections are made. If the signal predicts the state accurately, the price of one period nominal bond will be equal to the marginal rate of substitution between cash and credit goods. Since the securities are trade after some signal is announced, the stochastic behaviour of the interest rate depends heavily on the information available when securities are traded. Lucas (1989) extends the model toward realism and concluded that money moves in or out of security markets as a result of the changes in the current state. The price of a security, hence, depend not only on the income stream but also on the liquidity in the market when it is traded. Therefore, traders in security markets will carry cash balances over period if and only if the short-term interest rate is zero.

Cooley and Hansen (1989) and Yun (1996) incorporates money into a real business cycle (RBC) model by a CIA constraint to analyse the properties of the business cycle, which later researches of RBC models with a CIA constraint are based on. Cooley and Hansen (1989) assume flexible price while Yun (1996) introduce some nominal price rigidities. However, both models employed an exogenous money growth rule.

Since including uncertainty in the model allows taking all the transaction demand for money, precautionary demand for money and demand for money as a store of value into account, the CIA model puts a restriction in terms of timing and interval of transactions (Howitt, 1992). As the CIA constraint put an upper limit on purchases in a given period, the money demand tends to be less sensitive to the changes in the level of interest rate (McCallum and Goodfriend, 1987).

CIA constraint is firstly used to apply only to consumption goods (the Clower-Lucas type) and then apply to both consumption and investment goods (the Stockman type). Or in a more general way, it is applied to a fraction of investment goods as well as consumption goods (the generalised Stockman type). Stockman (1981) compares two types of the CIA model in a discrete time framework and focuses on the steady state while Abel (1985) examines the dynamic properties. Palivos et al (1993) explain the fluctuations in the velocity of money by extending the CIA constraint by letting investment and consumption be financed by cash. Miyazaki (2010) extends Chen and Guo (2008)'s work to characterizes models when money is required to purchase not only consumption goods but also all or some investment goods. Their statics analysis shows that an increase in inflation or a more binding CIA constraint lowers the capital stock level in the long run.

More recent literatures adopting CIA constraint are highly related to the liquidity traps where most economies suffer since the 2008 global crisis, which is discussed in the Section 2.4 and 2.5.

### 2.3.2 Money as An Asset Approach

The function of money as a store-of-value has been emphasized by treating money as an asset. These theories often view the money demand in the context of a portfolio choice problem. That is, the demand for money is interpreted as a problem of allocating wealth among an asset portfolio that includes money, and each asset can generate some mix of explicit (or pecuniary) and implicit (or non-pecuniary) flows. In terms of money, the pecuniary yield not only includes the ease of making transaction as implied by transaction model but also renders liquidity and safety (Judd and Scadding, 1982). Tobin (1958) demonstrates an alternative explanation for liquidity preference arising from the different expectations of future interest rates that the theory of risk-aversion behaviour of individuals also provided a basis for the liquidity preference. He postulates that an individual would have the incentive to hold a part of his/her wealth in money in the portfolio due to the reason that the return on holding money is more certain than return on holding earning assts. It is, hence, risker to hold other assets compared to holding money alone. The difference in the level of riskiness may be a result that the government bonds and equities are subject to the volatility in market price while money is not. It is the higher expected rate of return from these assets that gives the individual willingness to face risks. As a consequence, the risk-averse household may include a portion of money in an optimal portfolio for safety reasons. However, Fischer (1975) shows that the risk-aversion behaviour alone cannot provide a basis for holding money as money is not complete riskless since it is subject to the risk of inflation.

Recent researches especially those after the global financial crisis reconsider to treat money as an asset by emphasizing its function as a store of value. The asset function of money leads to the asset of portfolio approach where major concern is placed on risks and expect returns of assets. These models put more attention to the relationship between the interest rate and real money demand, and the importance of wealth and liquidity are seen as other key determinants of money demand. Since the return of money can be defined as an inverse function of inflation, the money can generate positive returns if the inflation is negative and hence money can be treated as an asset or a cheap collateral in this case. Le et al. (2014) assumed that money can provide cheap collateral in a model of banking and introduced the effect of M0 on the credit premium through its effect on the cost of transferring collateral into liquid assets. Bacchetta et al. (2017) introduces money in a model with liquid assets due to the lack of income pledgeability. Investors face two phases, investing and saving phases, and are subject to a credit constraint. They find that money can be held as a saving tool only when the nominal interest rate is zero. Di Tella (2017) also argues that it is money that serves as a safe asset that prevent interest rate from falling during recession and depresses investment. There is also a large literature that treats money as a bubble, an asset that does not pay dividend but have positive market value and earns the return. But almost all papers are in the context of overlapping generation models or incomplete risk sharing models [4]. Brunnermeier and Sannikov (2016) study the optimal inflation rate and describe money as a bubble which imposes a strong constraint on interest rates. In contrast, there is no bubble in this thesis and money gains its value from the liquidity services it provides, as measured by the relaxing of the liquidity constraints.

## 2.4 Occasionally Binding Cash-In-Advance Constraints

In money theory section, the original theories of cash-in-advance models has been listed. Cash-in-advance models has continued to be widely applied in monetary economics analysis. [5] In most papers, the CIA constraint is assumed to be always binding if the nominal interest rate is always positive. However, since the global financial crisis in 2008 and the COVID-19 pandemic, countries like United States have adopted zero interest-rate policy. Whether the CIA constraint is always binding has been challenged. Michel and Wigniolle (2005) study two regimes: one when the CIA constraint is binding and money is strictly dominated by capital, the other one is the constraint is relaxing and money has the same return as capital. The second regime is also called as the case when money works as a temporary bubble. Other works also find weak empirical evidence for the CIA constraints to be binding on consumption. The assumption that the CIA constraint on consumption is always binding is justified if the correlation between consumption and money are relatively high. Heer and Maußner (2015)

---

[4] See Samuelson (1958), Diamond (1965), Bewley et al (1983), Tirole (1985), Aiyagari (1994), Asriyan et al (2016).

[5] See Evans et al (2007), Devereux and Siu (2007), Díaz-Giménez et al (2008, 2013), Hromcová (2008), Alvarez et al. (2009), Giraud and Tsomocos (2010), Adão et al. (2011), Telyukova and Visschers (2011,2013), Heer and Maußner (2015).

suggest that 82% fraction of nominal consumption is subject to the CIA constraint by their calibration. However, the empirical evidence suggested that these correlations are relatively low.[6] Binding financial constraints for R&D investment may also provide week empirical support for binding CIA constraint on consumption. It is reasonable to expect an interaction between these two constraints. If the CIA constraint were binding, it is likely that the financial constraint for R&D investment would be binding as well. Hall and Lerner (2010) provide a summary for empirical literatures which showed inconclusive evidence of the linkage between R&D investment and financial constraints. Brown et al. (2013) also find little evidence of binding financial constraints on R&D investment.

Dixon and Pourpourides (2016) developed a general theorical framework to analysis endogenously occasionally binding CIA constraints. Though they are not the first to study the framework where the periods in which the CIA constraint binds and the periods in which it does not are determined endogenously, there are only a few papers which mostly focus on numerical simulations.[7] As is well known, whether the CIA constraint binds in a particular period or not depends on the expectations of risk-averse agents about the future return on money. Dixon and Pourpourides (2016) put forth that imperfect competition also affects the proportion of times when the CIA constraint is binding. When the occasionally binding CIA constraints is considered endogenously, money can show some real effects without requiring a portfolio choice. That is, money is non-neutral without the presence of other physical assets like capital or restrictions on how assets are transacted.[8] Although nominal wages and prices are fully flexible, prices can exhibit a sluggish response to a change in money supply. A simple real

[6] Ragot (2014) finds that the correlation between consumption and money is 0.3 for Italy.

[7] Devereux and Siu (2007), for example, propose a state-dependent model and find that prices respond more to positive shocks than they do to negative shocks. Alvarez et al (2009) allow the CIA constraint to be occasionally binding with exogenous production. In this economy, households are restricted from transferring money between interest-bearing accounts and consumption accounts.

[8] Chamley and Polemarchakis (1984) point out that the non-neutrality of money in general equilibrium lies in the existence of other real assets. A change in money supply firstly affects the level of price which in turn affects the relative return on money as an asset to the other physical assets. As a result, the effects of money are contributed to a portfolio choice problem. Heterogeneous beliefs or other frictions are required within this framework for money to have real effects.

business cycle model in Chapter 3 provides a proof for these results. When money is the only asset and the nominal wages and prices are fully flexible, there are some sluggish effects of a change in the expected inflation.[9] As the stickiness of nominal wages and prices are included, the level of sluggish effects increases.

Whether the current CIA constraint binds or not is associated with the current growth in the money supply and productivity growth via their effects on inflation. According to Cooley and Hansen (1989), "… the most important influence of money on the short-run fluctuations are likely to stem from the influence of the money supply process on expectation of relative prices". When the CIA constraint is nonbinding, the economy behaves in a Classical way as the households can demand and supply as much as they want to at the market prices and wages. Hence, output is higher, and money is neutral. This happens when the consumers' expected value of money for the next period is equal to its current value. In this case, consumers are indifferent between spending one unit of money now and holding it for the next period. However, when some particular events occur, the CIA constraint becomes binding when the households expect a decreasing trend in the relative value of money. This gives the agents an incentive to spend all their money holdings during the current period. This induces a leisure-consumption distortion that the households switch from consumption, which is constrained by the CIA constraint, to leisure. Here, the CIA constraint can be seen as a tax on consumption. With a binding CIA constraint, both working hours and consumption are less. In this case, money has real effects as the level of price has a direct effect on the demand of consumers through the CIA constraint. This transmission mechanism can be viewed as a Keynesian effective demand mechanism that the desired consumption can only be effective if there is enough cash to execute it.[10] Therefore, whether the CIA constraint binds or not depends on the relative value of money which is a function of the expected inflation. When the inflation rate is always non-negative, the CIA constraint is always binding. When the inflation rate is negative, the CIA constraint can be nonbinding, and the economy is at its

---

[9] The assumption that money is the only asset in the economy is not unusual in existing literature. See Lagos and Wright (2003) and Lagos and Rocheteau (2005).

[10] See Clower (1965), Leijonhufvud (1968), Benassy (1975), Malinvaud (1975).

efficient output.[11] The model in Chapter 3 shows that if there is an expansionary monetary policy followed by a positive technology shock the CIA constraint can be nonbinding for some periods. Since money is the only asset, there is an obvious consumption boom during the periods when the CIA constraint is nonbinding as the consumers behave in a standard way. The prior decision of money transfer may be optimal ex-ante but not optimal ex post. If there are some frictions in the economy, it may take longer for the economy to return back. In Chapter 4, a model including real assets, capital and bonds, provides a more intuitive implication for monetary policies. An increase in money supply may lose it ability to stimulate the economy. Instead it will trigger a deeper liquidity trap when the economy is already stuck at its zero-policy rate. Next Section discusses the linkage between liquidity traps and nonbinding CIA constraints.

## 2.5 Liquidity Traps

A liquidity trap is the condition where monetary policy is unable to stimulate economic activities during the periods when the nominal interest rates are near or at zero. This idea was pointed out originally by Keynes (1936) and Hicks (1937). The liquidity trap was said to exist for the first time during the period of Great Depression in the 1930's. There are literarily two approaches to assessing liquidity traps, namely, the Keynesian approach and monetary approach[12].

### 2.5.1 Keynesian Approach

The Keynesian approach of liquidity trap focuses on the role of the zero-lower bound on nominal interest rates. It follows the view of Keynes (1936) and Hicks (1937) that the only cause of a liquidity trap is zero nominal interest rate. Further increase in money supply to stimulate aggregate demand is ineffective when it is no way to set lower nominal interest rate than zero.

Krugman (1998) proposed a standard IS-LM model to extend the Keynesian view of liquidity traps. If the nominal interest rate is strictly positive, agents

[11] This has similarities to the Friedman (1969) that the optimum inflation rate is negative.
[12] See Brycz (2012) for more details.

will hold bonds instead of money. If the nominal interest rate could be negative, agents would hold money rather than bonds. Money and bonds are perfect substitutes only when interest rate is close to zero. In this circumstance, further increase in money supply fail to change neither output nor price level. He also incorporates a financial intermediation and provides a way to escape the liquidity trap that the monetary authorities should raise the expectations of agents on future price level.

Svensson (2001) develops an open economy model to analyse the relationship between output gap, inflation and real interest rate in a liquidity trap. As long as the monetary authority is not going to lower the real interest rate in the future, agents will expect no increase in current output in the future, assuming everything else will not rise. Eggertson and Woodford (2003) make a conclusion that assets with different maturities turn to be perfect substitutes in a condition of zero interest rate.

Market frictions in the financial intermediation sector also make contributions to the existence of liquidity traps. Bernanke and Gertler (1995) introduce an external finance premium to explain the liquidity trap recession. During the downturn, a rise in the premium makes the interest rate faced by agents higher even though the monetary authority sticks to a very low short-term interest rate. Kacperczyk and Schnabl (2010) and Martens and Raven (2011) show that credit channel in the US depresses the expectations of agents. Caballero and Farhi (2017) also show how pessimists can contribute to a demand recession.

However, this view has been challenged. Adam and Billi (2006, 2007) and Eggertsson (2008) argue that the central bank can stimulate the economy at liquidity traps by committing to a lower interest rate in the future when the economy has already recovered.

### 2.5.2 Monetary Approach

Monetary approach is referred to the Friedman’s helicopter drop, which is commonly believed as a solution of liquidity traps. If there is an unanticipatedly additional increase in the money stock, there will be an increase in spending because of the changes in the marginal utility relationships. Grandmont and Laroque (1976) state that “the demand for money may tend to infinity when the rate of interest goes to zero”. when the central bank issues fiat money via open market purchases, assets prices tend to go infinity. As long as agents expect that there is no rise in goods prices, the value of money will increase. When agents expect an increase in goods prices as well, the money value will go to zero.

This view is further supported by recent works on the effectiveness of monetary policy in a condition of liquidity traps. Gambacorta et al. (2014) survey macroeconomic data from advanced economics and find that the expansion of central bank’s balance sheet has stimulated output and inflation. DelNegro et al. (2017) sketch a New Keynesian model by introducing liquidity frictions and find that the central bank is able to stop the deflationary spiral by increasing its provision of liquidity. De Tella (2017) shares the same view but focuses on the frictionless aspects of liquidity traps. Panizza and Wyplosz (2018) investigate the effects of unconventional monetary policies in the euro area, Japan, the United Kingdom, and the United States. They confirm that asset purchase programs have stimulated output as well as inflation. Lhuissier et al (2020) use a Bayesian structural vector autoregression framework to show that the unconventional monetary policies work effectively even when the short-term interest rate is near its effective lower bound.

Other researches have done by estimating a negative shadow interest rate to prove the effectiveness of unconventional monetary policy[13]. Wu and Xia (2016) find that the shadow rate of US is likely to be as low as -4 percent. Debortoli, Galí, and Gambetti (2018) provide both direct and indirect evidence for the contributions of monetary policy in stabilizing the economy. Mouabbi and Sahuc (2019) construct a DSGE New Keynesian model with a shadow rate as a proxy for unconventional monetary policies and confirm

[13] According to the shadow rate mechanism proposed by Black (1995), when the observed short-term interest rate is at its lower bound and the long-term interest rates are constrained at the meantime, the shadow rate captures the movements in the overall yield curve. Hence, it can be seen as an unobserved short-term rate, which is consistent with the longer-term rates that would prevail when the lower bound on interest rate was nonbinding. So far, the shadow rates have been commonly interpreted as a single monetary policy measure that captures the effectiveness of unconventional monetary policy.

that monetary policy remained valid during liquidity traps.

However, some papers challenged the effectiveness of monetary policy at liquidity traps. Chung et al. (2012) stress that unconventional monetary policy is a poor substitute for conventional monetary policy and is much less effective than reducing the short-term interest rate in stimulating economic activities[14]. Campbell et al. (2012) argue that the effective lower bound is costly. Though unconventional monetary policy may work, there is diminishing returns of this policy when the long-term interest rate is likely to be at its effective bound. Other papers that challenges the effectiveness of unconventional monetary policy can be referred to those in Section 2.3.2 that money is treat as a cheap collateral or a bubble. Additional money supply will become reserves in the banks account instead of providing the credit to the economy. Asriyan et al. (2016) show that a bubble crash can be the cause of a long-lasting liquidity traps.

### 2.5.3 Nonbinding CIA Constraints and Liquidity Traps

Svensson (1985) provides a comprehensive study of the relationship between nonbinding CIA constraint and liquidity trap. He argues that the timing of information matters for the relationship between nonbinding CIA constraint and nominal interest rate. If the asset market opens after the state of the economy is known but before the goods market opens as in Lucas (1982), there are no possibility to change money holdings before going to the goods market. In this case, a nonbinding CIA constraint is consistent with zero interest rate. However, when the asset market opens before the state of economy is known as in Svensson (1985), positive nominal interest rate is possible when the CIA constraint is nonbinding for some periods. The timing of all transactions follows Svensson (1985) in this thesis. But, differing from the conclusions of Svensson (1985), if one-period of government bonds is considered as a liquidity asset, a nonbinding CIA constraint implies zero expected nominal interest rate.

[14] Similar results are also found by Boeckx et al. (2014), Borio and Gambacorta (2017), Borio and Hofmann (2017), Filardo and Nakajima (2018), Hesse et al. (2018) and Burriel and Galesi (2018).

Though the CIA constraint is commonly used when modelling money demand, almost all empirical papers focus on the case when it is assumed to be always binding and ignore the causation between the nonbinding CIA constraint and liquidity traps. Very few papers have a look at linkage between the CIA constraint and liquidity traps. Some of them adopt overlapping generation models. Asriyan et al. (2016) develop an overlapping generations model of bubbles by imposing a CIA constraint on all individuals as well as a credit constraint on entrepreneurs. The credit constraint they proposed, which can be seen as a modified CIA constraint on entrepreneurs, is the key restriction as money and credit may be used as a store of value when the economy is inside the liquidity traps. The others are restricted to a perfect foresight economy. Buera and Pablo Nicolini (2017) follow the work of Buera and Moll (2015) by adding a CIA constraint to study the effect of changes in the outside supply of money and bonds. They find that an excess demand of outside liquidity assets may lead to a deflation when the zero bound on nominal interest rate binds unless there is an increase in the nominal supply. This thesis extends the literature by constructing a DSGE model to let the CIA constraint and liquidity trap interact endogenously.

## 2.6 Monopolistic Competition

DSGE models are always believed to provide a microeconomic foundation for macroeconomic analysis and contribute to the integration of microeconomics and macroeconomics. Two major type of DSGE models are real business cycle models (RBC) and new Keynesian (NK) models, which differ in the assumptions of market structures. The RBC models assume perfect competition while the NK models assume imperfect or monopolistic competition. Since the assumptions of RBC models are perfect competition and fully flexible prices and wages, the monetary policies have no effect on real variables. However, if imperfect competition and nominal rigidities are added, which leads to the development of NK models, monetary policies can have effects on real variables. So far, the NK models have prevailed for the analysis of monetary policies as RBC models fail to compliant with real data. Hall (1986) and Basu and Fernald (1997) show that goods market is not perfectly competitive by providing the evidence of mark-ups on prices. Hornstein (1993) suggest that the effects of productivities shocks in models with imperfect competitions are weaker than in models with perfect competition while other shocks play a significant role in explaining economic dynamics.

Dixit and Stiglitz (1977) introduce an aggregator of household preferences to present monopolistic competition, which is then adopted by almost all DSGE models with the assumption of imperfect competition. This convenient aggregator that satisfies the property of the homothetic utility function and constant elasticity of substitution (CES) is always challenged by researches such as Benassy (1996) and Feenstra (2003). They suspect the micro foundations of CES and provide examples of non-CES homothetic demand systems. CES assumes all goods and factors share the same constant price elasticity of demand, which implies the relative demand for any goods or factors is independent of the prices of other goods or factors. Matsuyama and Ushchev (2017) propose three alternative classes of flexible homothetic demand systems to replace CES. However, those attempts have not changed the popularity of CES in modelling monopolistic competition.

Nominal rigidities are acknowledged to be a critical issue for macroeconomics, especially for monetary policy. It is proved to succeed in explaining the observed persistence in aggregate output and inflation and real effects of monetary shocks. Various models of nominal rigidities have been considered in the macroeconomic literatures. In general, it can be divided into two major types, state-dependent models and time-dependent models. State-dependent approaches focus on the attribution of nominal rigidities to fixed costs or menu costs[15]. It is suggested that the existence of a fixed cost leads to an inaction band for dependent variables. However, it has been difficult to make state-dependent models tractable enough to bring nominal rigidities into DSGE models.

Time-dependent models of pricing, as opposed to state-dependent models, well settle down in a dynamic framework. Taylor (1979) and Calvo (1983) develop two workhorse models for staggered wage and price setting. Taylor's model, which focus on the wage setting, assumes that wages are set for a

---

[15] See Akerlof and Yellen (1985), Mankiw and Parkin (1986), Khan (1997), and Fehr and Goette (2005).

particular period, 4 quarters for example. It is motivated by the institutional arrangements that are popular in the unionized manufacturing sector in the United States, but it does not provide an explanation why wage contracts last for a constant period. Calvo (1983) proposes a random duration model that based on a constant hazard rate model, i.e. the firm (union) has a given probability to reset its price (wage) each period. It is a widespread model to introduce nominal price rigidities in New Keynesian DSGE models. Indexation of wages or prices to past inflation becomes a popular extension because it typically results in more persistence in the inflation for monetary policy models. Yun (1996) develops a Calvo model with indexation for price setting. Erceg, Henderson, and Levin (2000) and Smets and Wouters (2003 and 2007) then use this idea to include wage changes. Although it is widely used in macroeconomic models because of its convenience in aggregation, it is criticized by the lack of microeconomic foundations. On the one hand, the probability of reset price or wage is assumed to be constant over time. On the other hand, Cogley and Sbordone (2008) and Dixon and Kara (2010) point out that the implication of indexed Calvo model that nominal wages and prices are reset every period is counterfactual. The Generalized Taylor and Generalized Calvo relax the assumption of constant wage contracts period and constant hazard rate by allowing the duration distributions to be the same as the distribution found in the real data[16].

It is also worth mentioning the development of the new Keynesian models with respect to labour market other than Taylor-type and Calvo-type wage stickiness. Pissarides (2011) point out search frictions and matching in DSGE models can explain the existence of involuntary unemployment as well as the dynamics of job creation and job destruction[17]. Another important development should be the efficiency wage model, which is epitomized in the paper by Shapiro and Stiglitz (1984). This model shows a linkage between wages and unemployment and provides an explanation of procyclical wages. A higher wage means greater labour productivity, and firms will be forced to

[16] See Taylor (1993), Coenen et al. (2007), Dixon and Kara (2011), Wolman (1999), Guerrieri (2006), Dixon (2009), and Dixon and Bihan (2011).

[17] Diamond (1982), Mortensen (1982), and Pissarides (1985) are earlier discussions of search and matching specification.

pay a higher wage when the unemployment rate is lower.

## 2.7 Conclusion

In this Chapter, the review traces back to the history of money, from four major functions of money and three motives of money demand to modern theories of money demand that money can be modelled by a CIA constraint or an asset function. In the last 10 years, the world is continuing suffering from a recession that challenges the effectiveness of various type of monetary policies. The central banks are struggling to find the cause of the recession and the way to recover. The failure of monetary policies in stimulating the aggregate demand has been studied by taking different kind of risks and shocks into account. But it is still unclear for the escape of liquidity traps. This thesis, instead of considering the complicated combinations of shocks, starts from the very beginning thought of why people hold money to see how the expectations of agents will affect their behaviours.

# Chapter 3 A Closed Economy Model without Bonds and Capital

## 3.1 Introduction

Here I present a DSGE model that is the basis for the rest of the study. It is a standard closed economy DSGE model with occasionally binding cash-in-advance (CIA) constraints following Galí (1999) and Dixon and Pourpourides (2016). So far, cash-in-advance models have continued to be widely used to analyse monetary economics. It is distinguished from money-in-utility model which includes money directly in utility instead of adding an extra constraint. In most of papers, the CIA constraint is assumed to be always binding if the nominal interest rate is set to be strictly positive. Since the Great Recession in 2009, the nominal short-term interest rate is pushed near to its lower limit zero. The assumption of strictly positive nominal interest rate seems to be violated. This gives an opportunity for the CIA constraint to be nonbinding rather than always binding. Dixon and Pourpourides (2016) suggest that an equilibrium that is obtained when there is a binding CIA constraint is always welfare inferior to an equilibrium that happens at a nonbinding CIA constraint for any given level of technology without the assumption of nominal rigidities. They also illustrate that the periods when CIA constraints bind or not can be endogenously determined. Cooley and Hansen (1989) argue that the short-run fluctuations of money are most likely to be influenced by the effects of the money supply rule on the consumers' expectations of relative prices. What makes the CIA constraints to be binding or not is the expectations of the risk-averse households with respect to the future relative value of money. On the one hand, the CIA constraint is nonbinding when money is neutral, that is, the expected value of money is equal to its current value. On the other hand, if the households' expectations of the relative value of money tomorrow are decreasing, they will rush to spend all their money today which will finally be constrained by the CIA constraints. Then the CIA constraint will become binding and there will be a unique equilibrium where money can have real effects instead of being neutral.

It is not unusual to allow money to be the only assets in this kind of model as money can have real effects without the existence of other physical assets. Money is introduced in the economy to reduce the transactions costs and serve as a store of value. Hence, money as an asset itself can have direct effects in the economy. To demonstrate the direct effects of money, it is better to keep the simple financial structure without any bother of other financial assets, such as bonds and capital. For example, if the interest rate of a nominal bond is low or even close to zero, this triggers a trade-off between holding money and holding bonds. In terms of capital, most literatures take into account precautionary money demand for both consumption and capital accumulations. However, the real effects of money may be vague due to the inclusion of investment. Because of the inclusion of investment in the CIA constraint, the inflation will show enough persistence without any nominal and real rigidities (Auray and Blas, 2007). These effects will have more or less distortions on the real effects of money and is analysed in Chapter 4. Here, only money is included in this model setup.

Money is assumed to play a role in the economy due to the precautionary demand for money. However, in addition to this precautionary consideration, money can still have a potential role when the determination of whether the current CIA binds or not is closely related to the order of money supply shock and productivity shock. If the transfer of money is chosen by the monetary authority before the realization of technology innovation, then this transfer may be welfare improving by making CIA constraints deviate from binding to nonbinding for certain periods as output and employment is higher when the CIA constraint does not bind.

Alvarez et al (2009) propose a CIA economy with exogenous production. The CIA constraint is allowed to be occasionally binding by restricting households from transferring money between interest-bearing accounts and consumption accounts every period. Differently, in the following models, output is modelled endogenously rather than a stochastic endowment process and there is no restriction on households other than a CIA constraint. Four different scenarios are discussed here. Firstly, both nominal wages and prices are assumed to be fully flexible, however, prices can still respond sluggishly to the growth in money supply and technology. Secondly, nominal prices follow Calvo setting while nominal wages remain flexible. Thirdly, nominal wage follows Calvo setting while nominal prices stay flexible. Lastly, both nominal wage and prices are sticky. It indicates that the more sluggishness is introduced the more likely the CIA constraint can be nonbinding.

The model is described in Section 3.2. Then Section 3.3 outlines a baseline calibration, followed by a comparison of the impulse response functions between different scenarios. Section 3.4 is the estimation results and corresponding policy implications are illustrated in Section 3.5.

## 3.2 The Model

A typical cash-in-advance model is outlined below, with endogenous production. What is a cash-in-advance constraint? It is assumed that consumers must pay for goods in cash before they can buy goods. Generally speaking, the purchase of goods in the current period must necessarily be paid for by the currency held over from the previous periods (Lucas, 1980a). This kind of model is usually used to distinguish a money economy from a barter economy. As four different scenarios are discussed, it is more convenient to start with the most comprehensive case, that is, when both nominal wages and prices are set with a Calvo style.

In this economy, there are a continuum of households who supply differentiated labour capital to labour packing firms or a labour union, a representative final goods firm which uses intermediate goods produced by monopolistically competitive firms to produce final goods. The final goods market is perfectly competitive. The government chooses the money supply process targeting at targeted inflation and makes money transfer to households. The price-taking consumer makes decision on consumption, money holdings for next period's consumption and time division between working and leisure in order to maximise her utility subject to a budget constraint and a cash-in-advance constraint. Here, money is the only asset to keep the financial structure simple and get rid of distortions from other real assets like bonds and capital. The monopolistic firms only hire workers to produce intermediate goods and can change their price during some periods. Thus, there are two nominal rigidities in this setup, namely, sticky wage and sticky price. The consumer is the shareholder of the monopolistic firms, that is, the profits made by these firms are all used up by consumers. The two uncertainties faced by the consumer are technology innovation and money supply shock which is transit to real variables through money transfer.

The aim here is to analyse the effects of occasionally binding cash-in-advance constraints under this standard setup. What really matters is the timing of all these transactions, which is always the key of a cash-in-advance economy. The consumer enters the current period with the money hold over from last period and receives a cash transfer from the government. This cash transfer is chosen by the government last period. The consumer learns technology innovation and supplies labour to monopolistic firms. After this, the goods market opens, where the consumer pays for consumption goods with her cash holdings which are consist of cash hold over from last period and cash transfer received this period. Then the goods market closes. The consumer receives her wages in cash and profits from firms which are owned by herself. Simply, if the money transfer decision is made before the technology innovation is realized, this money transfer may be optimal before but not optimal after. This creates a space for the CIA constraints to be occasionally binding.

## 3.2.1 Labour Union

The nominal rigidities in wages are introduced into the model by relaxing the assumption of a representative household and assuming there is a continuum of differentiated households which is indexed by $i \in [0,1]$. These households supply differentiated labour to a labour union or a labour packing firm. Then the union or labour packing firm bundles the differentiated labour capital into a homogenous labour input which is available for production, $h_t^d$. The bundling technology can be expressed as

$$h_t^d = \left(\int_0^1 \left(h_t(i)\right)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} di\right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}, \varepsilon_w > 1 \tag{3.1}$$

Here, $\varepsilon_w$ is the elasticity of substitution among different types of labour input, which is assumed to be greater than one, meaning that different types of labour input are substitutes.

Then the profit maximization problem for a labour union is

$$\max_{h_t(i)} W_t(\int_0^1 \left(h_t(i)\right)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} di)^{\frac{\varepsilon_w}{\varepsilon_w - 1}} - \int_0^1 W_t(i) h_t(i) di$$

$W_t$ is the aggregate nominal wage, while $W_t(i)$ denotes the nominal wage of household variety $i$.

This profit maximization problem will give the demand function for each variety of labour

$$h_t(i) = \left(\frac{W_t(i)}{W_t}\right)^{-\varepsilon_w} h_t^d \tag{3.2}$$

This downward sloping demand for labour variety $i$ depends on the aggregate labour demand and the relative wage of household $i$. Since this specification depends on the relative wage, it should be the same in terms of the nominal wage or the real wage ratio.

The aggregate wage index can be defined as

$$W_t h_t^d = \int_0^1 W_t(i) h_t(i) di \tag{3.3}$$

Given the demand function, this aggregate wage index can be simplified to be

$$W_t^{1-\varepsilon_w} = \int_0^1 (W_t(i))^{1-\varepsilon_w} di \tag{3.4}$$

Then the aggregate effective labour supply $h_t$, which is the sum of labour varieties, is equal to the actual labour employed $h_t^d$ times a wage markup $v_t^w$. For a given value of total labour employed, the effective labour supply is less as the labour is employed more unequally across firms.

$$h_t = h_t^d v_t^w \tag{3.5}$$

$h_t^d$, as defined above, refers to the actual labour used in production and is not necessarily equal to the aggregate labour supply. $v_t^w = \int_0^1 \left(\frac{W_t(i)}{W_t}\right)^{-\varepsilon_w} di$ is the wage markup, which measures the wage dispersion across different labour varieties. It is equal to unity if and only if all wages are equal, otherwise it is strictly greater than one, implying that the aggregate labour actually used in production is always smaller than the aggregate labour supply of households.

## 3.2.2 Firm Problem

Imperfect competition is assumed here. The production is broken up into two sectors, one is the intermediate goods sector and the other one is the final goods sector. The final goods sector is perfectly competitive, so it can be seen as a representative final goods firm. This firm does not use any factors for production but rather bundles intermediate goods into the final goods. The intermediate goods firms, which is indexed by $j \in [0,1]$, hire only workers to produce differentiated goods. All differentiated goods populate the unit interval.

#### 3.2.2.1 Final Goods Firm

The final good is an aggregate of intermediate goods with a constant elasticity of substitution. The production function is

$$y_t = \left[\int_0^1 y_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} dj\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \tag{3.6}$$

Here, $\varepsilon_p$ is a positive parameter and it governs the degree of substitution among intermediate goods. If it goes to infinity, all intermediate goods are perfect substitutes. If it goes to zero, all intermediate goods become perfect complements. When $\varepsilon_p = 1$, there is a unit elasticity of substitution and then the production function becomes Cobb-Douglas. For what follows assume that $\varepsilon_p > 1$.

The objective of the final goods firm is to maximize its profits, given a final good price, $P_t$, and the intermediate goods prices, $P_t(j)$

$$\max_{y_t(j)} \Pi_t = P_t \, [\int_0^1 y_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} dj]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} - \int_0^1 P_t(j) \, y_t(j) dj$$

Differentiating the profits with respect to $y(j)$ and setting it equal to zero, it will give us the demand curve for each intermediate good

$$y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} y_t \tag{3.7}$$

This is the relative demand for differentiated intermediate good $j$, which depends on its relative price, with $\varepsilon_p$ the price elasticity of demand.

As the final good firm is operating in a perfectly competitive environment, its profit is always zero. Using the demand specification, we can solve for the aggregate price index

$$P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon_p} dj\right]^{\frac{1}{1-\varepsilon_p}} \tag{3.8}$$

### 3.2.2.2 Intermediate Goods Firms

The next is the intermediate goods firms. Because money is the only asset in this model, the intermediate goods firms can only use labour to produce output with constant returns to scale production technology and labour. Production of firm $j$ is described by the following function

$$y_t(j) = A_t h_t^d(j) \tag{3.9}$$

$A_t$ is aggregate technology and is common among all intermediate goods firms. The log productivity $a_t = \log(A_t)$ follows an exogenous, stochastic AR(1) process represented by:

$$a_t = \rho_a a_{t-1} + u_{at} \tag{3.10}$$

Where $|\rho_a| < 1$ and $u_{at}$ is a white noise innovation of technology.

Firms may not have the chance to update their price in a given period, but they will always choose their labour input to minimize total cost each period. In other words, each intermediate goods firm chooses the amount of labour $h_t^d(j)$ to employ, taking the input price $W_t$ as given. This cost-minimization problem is

$$\min_{h_t^d(j)} W_t h_t^d(j)$$

*s.t*

$$y_t(j) = A_t h_t^d(j)$$

$$y_t(j) = (\frac{P_t(j)}{P_t})^{-\varepsilon_p} y_t$$

A Lagrange is

$$\mathcal{L} = -W_t h_t^d(j) + \varphi_t(j)(A_t h_t^d(j) - \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} y_t)$$

The first order condition with respect to $h_t^d(j)$ is

$$W_t = \varphi_t(j) A_t$$

Here, $\varphi_t(j)$ is the marginal cost for firm $j$. Since all intermediate goods firms face the same input prices and the same aggregate technology innovation, the subscript $j$ can be dropped. Re-arranging the above F.O.C for labour,

$$\varphi_t = \frac{W_t}{A_t}$$

This indicate that the marginal cost is the same for all intermediate goods firms. The real marginal cost can be written as $mc_t = \frac{\varphi_t}{P_t}$. Re-write the real marginal cost in terms of the real wage

$$mc_t = \frac{w_t}{A_t} \tag{3.11}$$

Here, $w_t = \frac{W_t}{P_t}$ is the real wage, which is the nominal wage divided by aggregate price index. Real marginal cost is also the same across firms.

Though the intermediate goods firms face the same wage and same cost minimization problem, they have the ability to set their own price, given that they face downward sloping demand curves as long as $\varepsilon_p$ is not infinite. It is assumed that the intermediate goods firms set their prices a lá Calvo (1983) and Yun (1996). In each period, a fixed fraction $\phi_p \in [0,1]$ of firms cannot adjust their prices and can only index their prices by the past inflation, which is governed by the indexation parameter $\xi_p \in [0,1]$. In other words, there is a fixed probability of $1 - \phi_p$ that a firm can adjust its price. When $\phi_p = 0$, the nominal prices are fully flexible. When $\xi_p = 0$, the nominal prices are not indexed at all to the inflation. When $\xi_p = 1$, the nominal prices are fully indexed to the inflation. Hence, the price which a typical intermediate goods firm can charge in period $t$ is

$$P_t(j) = \begin{cases} P_t{}^{\#}(j) \\ (1 + \pi_{t-1})^{\xi_p} P_{t-1}(j) \end{cases}$$

$P_t{}^{\#}(j)$ is the optimal reset price of the intermediate goods firm $j$. Then if a firm has not been given the opportunity to adjust its price again in period $t + s$, the price that this firm, who has updated its price in period $t$, will charge in period $t + s$ is

$$P_{t+s}(j) = (\frac{P_{t+s-1}}{P_{t-1}})^{\xi_p} P_t{}^{\#}(j)$$

The intermediate goods firm $j$, which is given the chance to adjust its price in period $t$, will maximize its profit by discounting its profit flows by the nominal stochastic discount factor, $\beta^s U'(c_t)$, as well as the probability that a price set today is still in effect in the future, ${\phi_p}^s$. $c_t$ is the aggregate consumption and $U'(c_t) > 0$ is the first derivative of household's utility function with respect to consumption.

$$\max_{P_t^{\#}(j)} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left[\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_t}^{\#}(j)^{1-\varepsilon_p} {P_{t+s}}^{\varepsilon_p - 1} y_{t+s} \right.$$
$$\left. - mc_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_t}^{\#}(j)^{-\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s}\right]$$

The first order condition with respect to ${P_t}^{\#}(j)$ is

$$\left(1-\varepsilon_p\right){P_t}^{\#}(j)^{-\varepsilon_p} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_{t+s}}^{\varepsilon_p - 1} y_{t+s}$$
$$+ \varepsilon_p {P_t}^{\#}(j)^{-\varepsilon_p - 1} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) mc_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s} = 0$$

Re-arranging this first order condition

$${P_t}^{\#}(j) = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) mc_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s}}{E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_{t+s}}^{\varepsilon_p - 1} y_{t+s}}$$

Since nothing on the right-hand side depends on $j$, all updating firms will adjust to the same reset price, ${P_t}^{\#}$. Then the price-setting condition can be written recursively as

$${P_t}^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}}{X_{2,t}}$$

Where

$$X_{1,t} = U'(c_t) mc_t {P_t}^{\varepsilon_p} y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} E_t X_{1,t+1}$$

$$X_{2,t} = U'(c_t) {P_t}^{\varepsilon_p - 1} y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} E_t X_{2,t+1}$$

Write these in real terms by defining $x_{1,t} = \frac{X_{1,t}}{P_t^{\varepsilon_p}}$, $x_{2,t} = \frac{X_{2,t}}{P_t^{\varepsilon_p - 1}}$, and the gross consumer price inflation $1 + \pi_t = \frac{P_t}{P_{t-1}}$,

$$x_{1,t} = U'(c_t) mc_t y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_p \varepsilon_p} \frac{E_t X_{1,t+1}}{P_t^{\varepsilon_p}}$$

$$= U'(c_t) mc_t y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_p \varepsilon_p} \frac{E_t X_{1,t+1}}{P_{t+1}^{\varepsilon_p}} \frac{P_{t+1}^{\varepsilon_p}}{P_t^{\varepsilon_p}}$$

$$= U'(c_t) mc_t y_t + \beta\phi_p (1+\pi_t)^{-\xi_p \varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1,t+1}$$

$$x_{2,t} = U'(c_t) y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{\xi_p (1-\varepsilon_p)} \frac{E_t X_{2,t+1}}{P_t^{\varepsilon_p - 1}}$$

$$= U'(c_t) y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{\xi_p (1-\varepsilon_p)} \frac{E_t X_{2,t+1}}{P_{t+1}^{\varepsilon_p - 1}} \frac{P_{t+1}^{\varepsilon_p - 1}}{P_t^{\varepsilon_p - 1}}$$

$$= U'(c_t) y_t + \beta\phi_p (1+\pi_t)^{\xi_p (1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2,t+1}$$

Then the reset price equation can be written as

$$P_t^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1,t}/P_t^{\varepsilon_p}}{X_{2,t}/P_t^{\varepsilon_p - 1}} \frac{P_t^{\varepsilon_p}}{P_t^{\varepsilon_p - 1}} = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{x_{1,t}}{x_{2,t}} P_t$$

$$x_{1,t} = c_t^{-\sigma} mc_t y_t + \beta\phi_p (1+\pi_t)^{-\xi_p \varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1,t+1}$$

$$x_{2,t} = c_t^{-\sigma} y_t + \beta\phi_p (1+\pi_t)^{\xi_p (1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2,t+1}$$

Define the gross reset price inflation as $1 + \pi_t^{\#} = \frac{P_t^{\#}}{P_{t-1}}$, the optimal price-setting conditions recursively to this problem can be written as

$$1 + \pi_t^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} (1+\pi_t) \frac{x_{1,t}}{x_{2,t}} \tag{3.12}$$

$$x_{1,t} = c_t^{-\sigma} mc_t y_t + \beta\phi_p (1+\pi_t)^{-\xi_p \varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1,t+1} \tag{3.13}$$

$$x_{2,t} = c_t^{-\sigma} y_t + \beta\phi_p (1+\pi_t)^{\xi_p (1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2,t+1} \tag{3.14}$$

### 3.2.3 Government

The consolidated government prints money and uses up all revenues coming from creating new money to make lump-sum transfers to households. The budget constraint of the government is

$$\bar{M}_t = \bar{M}_{t-1} + P_t s_t$$

Where $s_t$ is the lump-sum transfers or taxes and $\bar{M}_t$ is the money supply. It is assumed that $\bar{M}_t$ is set at time $t-1$, since under the timing convention here, $\bar{M}_{t-1}$ is predetermined with respect to time $t-1$. Hence, it means that the government chooses the optimal lump-sum transfers at the period $t-1$ before the technology innovation will be realized by households in period $t$. Let the real money balances $\bar{m}_t = \frac{\bar{M}_t}{P_t}$, then the budget constraint becomes

$$\bar{m}_t = \frac{\bar{m}_{t-1}}{1+\pi_t} + s_t \tag{3.15}$$

The monetary authority sets an exogenous process for the money supply which follows an AR(1) process in the growth rate. As money is the only asset in the whole economy, it is not uncommon to ignore the interest rate when setting monetary rules. Hence, it is assumed that money growth rate is targeted at the steady state inflation rate. This specification will generate positive trend inflation in nominal terms:

$$\ln \bar{M}_t - \ln \bar{M}_{t-1} = (1-\rho_m)\bar{\pi} + \rho_m(\ln \bar{M}_{t-1} - \ln \bar{M}_{t-2}) + u_{mt}$$

Here $\bar{\pi}$ is the steady state growth rate of the money supply. This will finally be equal to steady state inflation, which is assumed to be zero or inflation target.

Rewrite the above process in terms of real balance to make the money supply process become stationary, defining $\ln \bar{m}_t = \ln \bar{M}_t - \ln P_t$ and $\pi_t = \ln P_t - \ln P_{t-1}$

$$\Delta \ln \bar{m}_t = (1-\rho_m)\bar{\pi} - \ln \pi_t + \rho_m \Delta \ln \bar{m}_{t-1} + \rho_m \ln \pi_{t-1} + u_{mt} \tag{3.16}$$

Where $|\rho_m| < 1$ and $u_{mt}$ is a white noise innovation of money supply growth.

### 3.2.4 Household Problem

There is a continuum of differentiated households which is indexed by $i \in [0,1]$ in the economy. These households set their wages subject to a Calvo-style setting friction, which means they will charge heterogeneous wages, meaning they will supply differentiated labour capital, get different incomes and therefore have different consumptions and savings. Erceg, Henderson and Levin (2000) show that if there exist state-contingent securities that insure households against idiosyncratic wage risks, and if preferences are assumed to be separable between consumption and leisure, household will be identical among their choice of consumption, capital accumulation, capital utilization, and bond-holdings but will differ in the wage they charge and labour they supply. As the utility is separable between consumption and working hours or labour, households will be identical along all other margins but working hours and wages. This prohibits the heterogeneity in labour supply to spilling over into consumption heterogeneity. Also because of the existence of complete markets, households can fully ensure against the employment risks. Hence, the subscript $i$ is dropped for all variables except labour supply and wage. A typical household chooses her paths for consumption ($c_t$), working hours ($h_t(i)$) and the money holdings for next period consumption ($M_t$) to maximise the lifetime utility though money does not enter the utility function directly. As it will be illustrated later, money plays its role through entering both the budget constraint and the CIA constraint. The lifetime utility is described by the following function

$$E_t \sum_{t=0}^{\infty} \beta^t U(c_t, h_t(i))$$

Where $E_t$ denotes the rational expectations operator that is conditional on all information available at time $t$, $\beta \in (0,1)$ is the subjective discount factor and $U$ is assumed to be strictly increasing in its first argument, strictly decreasing in its second argument, strictly concave and take the following form

$$U\big(c_t, h_t(i)\big) = \frac{{c_t}^{1-\sigma}}{1-\sigma} - \frac{(h_t(i))^{1+\varphi}}{1+\varphi} \tag{3.17}$$

$\sigma, \varphi > 0$ are the Arrow-Pratt coefficients of relative risk aversion for consumption and working, respectively.

The consumption good is assumed to be a CES aggregator of the quantities of a continuum of differentiated goods, $c_t(j), j \in [0,1]$:

$$c_t = \left[\int_0^1 c_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} dj\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \tag{3.18}$$

Where the parameter $\varepsilon_p > 1$ denotes the intratemporal elasticity of substitution across different consumption goods. The optimal level of differentiated goods is solved by minimizing total expenditure, $\int_0^1 P_t(j)\, c_t(j) dj = P_t c_t$, subject to the aggregation constraint above

$$c_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} c_t \tag{3.19}$$

The identical cash-in-advance constraint is imposed to all households. The household must have enough money on hand to cover all her nominal purchase of consumption goods. In particular, the household finances all expenditure of consumption goods by money hold over from time $t-1$ ($M_{t-1}$) and lump-sum transfer given by the government at time $t$ ($P_t s_t$). Here, money transfer can be allowed for the consumption in current period $t$. Alternatively, the transfer could not be used for current consumption. According to Abel (1985), the dynamic behaviours are irrelevant to whether it can be used in period $t$ or not.

$$M_{t-1} + P_t s_t \geq P_t c_t$$

Rewrite in the real terms using $m_t = \frac{M_t}{P_t}$

$$\frac{m_{t-1}}{1+\pi_t} + s_t \geq c_t \tag{3.20}$$

In this economy, money is assumed to be only asset. Thus, it enters the budget constraint as the only store of value

$$P_t c_t + M_t = W_t(i) h_t(i) + M_{t-1} + \Pi_t^n + P_t s_t$$

$\Pi_t^n$ is the nominal profit the household received from monopolistic firms as all monopolistic firms are assumed to be owned by households and $W_t(i)$ is the nominal wage request by household $i$.

Define the real profit $\Pi_t = \frac{\Pi_t^n}{P_t}$, the budget constraint in real terms is

$$c_t + m_t = \frac{W_t(i) h_t(i)}{P_t} + \frac{m_{t-1}}{1+\pi_t} + \Pi_t + s_t \tag{3.21}$$

The Lagrange associated with this problem is

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \left( \frac{{c_t}^{1-\sigma}}{1-\sigma} - \frac{(h_t(i))^{1+\varphi}}{1+\varphi} \right) - \lambda_t (c_t + m_t - \frac{W_t(i) h_t(i)}{P_t} - \frac{m_{t-1}}{1+\pi_t} - \Pi_t - s_t) - \eta_t \left( c_t - \frac{m_{t-1}}{1+\pi_t} - s_t \right) \right\}$$

Household chooses $c_t$, $h_t(i)$, $\frac{W_t(i)}{P_t}$ and $m_t$; $\lambda_t$ represents the Lagrangian multiplier associated with the budget constraint, i.e. the shadow price of the budget constraint, and $\eta_t$ represents the Lagrangian multiplier associated with the CIA constraint, i.e. the shadow price of the CIA constraint.

And the problem yields the following first-order conditions with respect to $c_t$ and $m_t$ respectively, the first-order condition of differentiated labour is ignored here and will be illustrated later

$$c_t^{-\sigma} = (\lambda_t + \eta_t) \tag{3.22}$$

$$\lambda_t = \beta E_t \left\{ \left( \frac{\lambda_{t+1}}{1+\pi_{t+1}} + \frac{\eta_{t+1}}{1+\pi_{t+1}} \right) \right\} \tag{3.23}$$

$$\eta_t \geq 0 \tag{3.24}$$

$$\eta_t \left( \left( c_t - \frac{m_{t-1}}{1+\pi_t} - s_t \right) \right) = 0 \tag{3.25}$$

The first order conditions for $c_t$ and $m_t$ combine for the Euler equation, which describes the intertemporal substitution of consumption.

$$c_t^{-\sigma} - \eta_t = \beta E_t \left\{ \frac{{c_{t+1}}^{-\sigma}}{1+\pi_{t+1}} \right\} \tag{3.26}$$

When $\eta_t = 0$, equation above becomes the familiar Euler equation. The price of an extra unit of utility of consuming today is equal to tomorrow's expected utility of consumption discounted by time preference and expected inflation. Whether $\eta_t$ is zero or not has a key effect on the bind or slack of CIA constraints. Simply, when $\eta_t = 0$, the CIA constraint is nonbinding. When $\eta_t \neq 0$, then the CIA constraint binds. Note that the value $\eta_t$ can take is determined endogenously. The precise conditions when the CIA constraint is binding and when it is not are discussed in later Section.

When taking the first-order conditions with respect to labour and wages, it is assumed that the wage-setting process is Calvo-style as well. That is, in each period, a randomly selected fraction $1-\phi_w \in [0,1]$ of households are given the chance to optimally update their nominal wages. That is, a fraction $\phi_w$ of households cannot adjust their nominal wages. Instead, they partially index their nominal wage to lagged inflation at the rate $\xi_w \in [0,1]$. When $\phi_w = 0$, the nominal wages are fully flexible. When $\xi_w = 0$, the nominal wages are not indexed at all to inflation. When $\xi_w = 1$, the nominal wages are fully indexed to the nominal prices. This is the case when the real rigidity of wage is involved. Thus, the nominal wage of a household $i$ in period $t$ is

$$W_t(i) = \begin{cases} {W_t}^{\#}(i) \\ (1+\pi_{t-1})^{\xi_w} W_{t-1}(i) \end{cases}$$

Where ${W_t}^{\#}(i)$ is the optimal reset wage of household $i$.

Then the non-updated wage in period $t+s$ can be expressed as

$$W_{t+s}(i) = (\frac{P_{t+s-1}}{P_{t-1}})^{\xi_w} {W_t}^{\#}(i)$$

The Lagrange problem with respect to ${W_t}^{\#}(i)$ can be recreated as the household who has the opportunity to change wage in period $t$ will discount period $t+s$ by $(\beta\phi_w)^s$, which reflects the probability that a chosen wage in period $t$ is still in effect in $t+s$ is ${\phi_w}^s$

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \{-\frac{1}{1+\varphi} \left( \frac{\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w} {W_t}^{\#}(i)}{W_{t+s}} \right)^{-\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}$$
$$+ \frac{\lambda_{t+s}}{P_{t+s}} ((\frac{P_{t+s-1}}{P_{t-1}})^{\xi_w} {W_t}^{\#}(i))^{1-\varepsilon_w} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d\}$$

The first order condition with respect to ${W_t}^{\#}(i)$ is

$$\varepsilon_w {W_t}^{\#}(i)^{-\varepsilon_w(1+\varphi)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w \varepsilon_w (1+\varphi)} {W_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi} + (1-$$
$$\varepsilon_w) {W_t}^{\#}(i)^{-\varepsilon_w} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d = 0$$

Rearranging this first order condition

$$W_t^{\#}(i)^{1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {W_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}}{E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d}$$

Since nothing on the right-hand side depends on $i$, the dependence on $i$ can be eliminated. That is, all updating households will charge a common reset wage. Then the wage setting condition can be written as

$${W_t^{\#}}^{1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{N_{1,t}}{N_{2,t}}$$

$$N_{1,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {W_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}$$

$$N_{2,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d$$

Write $N_{1,t}$ and $N_{2,t}$ in real terms

$$N_{1,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {w_{t+s}}^{\varepsilon_w(1+\varphi)} {P_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}$$

$$N_{2,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \lambda_{t+s}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {w_{t+s}}^{\varepsilon_w} {P_{t+s}}^{\varepsilon_w - 1} h_{t+s}^d$$

Then write $N_{1,t}$ and $N_{2,t}$ recursively

$$N_{1,t} = {w_t}^{\varepsilon_w(1+\varphi)} {P_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi} + \beta\phi_w E_t \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} N_{1,t+1}$$

$$N_{2,t} = \lambda_t {w_t}^{\varepsilon_w} {P_t}^{\varepsilon_w - 1} h_t^d + \beta\phi_w E_t \left(\frac{P_t}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} N_{2,t+1}$$

Define $n_{1,t} = \frac{N_{1,t}}{{P_t}^{\varepsilon_w(1+\varphi)}}$ and $n_{2,t} = \frac{N_{2,t}}{{P_t}^{\varepsilon_w - 1}}$,

$$n_{1,t} = {w_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi} + \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t \frac{N_{1,t+1}}{{P_t}^{\varepsilon_w(1+\varphi)}}$$

$$= {w_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t \frac{N_{1,t+1}}{{P_{t+1}}^{\varepsilon_w(1+\varphi)}} \frac{{P_{t+1}}^{\varepsilon_w(1+\varphi)}}{{P_t}^{\varepsilon_w(1+\varphi)}}$$

$$= {w_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t (1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} n_{1,t+1}$$

$$n_{2,t} = \lambda_t {w_t}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \frac{N_{2,t+1}}{{P_t}^{\varepsilon_w-1}}$$

$$= \lambda_t {w_t}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \frac{N_{2,t+1}}{{P_{t+1}}^{\varepsilon_w-1}} \frac{{P_{t+1}}^{\varepsilon_w-1}}{{P_t}^{\varepsilon_w-1}}$$

$$= \lambda_t {w_t}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t (1+\pi_{t+1})^{\varepsilon_w-1} n_{2,t+1}$$

Hence, the reset wage can be expressed as

$$ {W_t^{\#}}^{1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{\frac{N_{1,t}}{{P_t}^{\varepsilon_w(1+\varphi)}}}{\frac{N_{2,t}}{{P_t}^{\varepsilon_w-1}}} \frac{{P_t}^{\varepsilon_w(1+\varphi)}}{{P_t}^{\varepsilon_w-1}} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{n_{1,t}}{n_{2,t}} {P_t}^{1+\varepsilon_w\varphi}$$

Define the real reset wage as ${w_t}^{\#} = \frac{{W_t}^{\#}}{P_t}$

$${w_t^{\#}}^{1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{n_{1,t}}{n_{2,t}}$$

$$n_{1,t} = {w_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi} + \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t (1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} n_{1,t+1}$$

$$n_{2,t} = \lambda_t {w_t}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t (1+\pi_{t+1})^{\varepsilon_w-1} n_{2,t+1}$$

Equations (44)-(46) describe the optimal wage-setting process. In order to calculate steady states, it is computationally easier to write these conditions in terms of relative reset wage by defining $\hat{n}_{1,t} = \frac{n_{1,t}}{{w_t^{\#}}^{\varepsilon_w(1+\varphi)}}$ and $\hat{n}_{2,t} = \frac{n_{2,t}}{{w_t^{\#}}^{\varepsilon_w}}$.

$$\hat{n}_{1,t} = (\frac{w_t}{w_t^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d^{1+\varphi}} + \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t (1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} \frac{n_{1,t+1}}{w_t^{\#^{\varepsilon_w(1+\varphi)}}}$$

$$= (\frac{w_t}{w_t^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d^{1+\varphi}}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t(1$$

$$+ \pi_{t+1})^{\varepsilon_w(1+\varphi)} \frac{n_{1,t+1}}{w_{t+1}^{\#^{\varepsilon_w(1+\varphi)}}} \frac{w_{t+1}^{\#^{\varepsilon_w(1+\varphi)}}}{w_t^{\#^{\varepsilon_w(1+\varphi)}}}$$

$$= (\frac{w_t}{w_t^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d^{1+\varphi}}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t[(1$$

$$+ \pi_{t+1})^{\varepsilon_w(1+\varphi)} \left(\frac{w_{t+1}^{\#}}{w_t^{\#}}\right)^{\varepsilon_w(1+\varphi)} \hat{n}_{1,t+1}]$$

$$\hat{n}_{2,t} = \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} \frac{n_{2,t+1}}{w_t^{\#^{\varepsilon_w}}}$$

$$= \lambda_t \left(\frac{w_t}{w_t^{\#}}\right)^{\varepsilon_w} h_t^d$$

$$+ \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} \frac{n_{2,t+1}}{w_{t+1}^{\#^{\varepsilon_w}}} \frac{w_{t+1}^{\#^{\varepsilon_w}}}{w_t^{\#^{\varepsilon_w}}}$$

$$= \lambda_t (\frac{w_t}{w_t^{\#}})^{\varepsilon_w} h_t^d$$

$$+ \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \left[(1+\pi_{t+1})^{\varepsilon_w-1} (\frac{w_{t+1}^{\#}}{w_t^{\#}})^{\varepsilon_w} \hat{n}_{2,t+1}\right]$$

Then the wage-setting equation can be written as

$$w_t^{\#^{1+\varepsilon_w\varphi}} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\frac{n_{1,t}}{w_t^{\#^{\varepsilon_w(1+\varphi)}}}}{\frac{n_{2,t}}{w_t^{\#^{\varepsilon_w}}}} \frac{w_t^{\#^{\varepsilon_w(1+\varphi)}}}{w_t^{\#^{\varepsilon_w}}} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}} w_t^{\#^{\varepsilon_w\varphi}}$$

More compactly

$$w_t^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}}$$

Finally, the optimal wage-setting conditions recursively to this problem are

$$w_t^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}} \tag{3.27}$$

$$\hat{n}_{1,t} = \left(\frac{w_t}{{w_t}^{\#}}\right)^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi}$$

$$+\beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)}E_t\left[(1+\pi_{t+1})^{\varepsilon_w(1+\varphi)}\left(\frac{{w_{t+1}}^{\#}}{{w_t}^{\#}}\right)^{\varepsilon_w(1+\varphi)}\hat{n}_{1,t+1}\right] \quad (3.28)$$

$$\hat{n}_{2,t} = \lambda_t\left(\frac{w_t}{{w_t}^{\#}}\right)^{\varepsilon_w} h_t^d$$

$$+\beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)}E_t\left[(1+\pi_{t+1})^{\varepsilon_w-1}\left(\frac{{w_{t+1}}^{\#}}{{w_t}^{\#}}\right)^{\varepsilon_w}\hat{n}_{2,t+1}\right] \quad (3.29)$$

### 3.2.5 Equilibrium and Aggregation

Both goods market clearing and money market clearing are required to close the model. In quantity terms, the supply of final goods should be equal to the demand for consumption goods.

$$y_t = c_t$$

In value terms, money demand is equated to money supply. The money demand side is represented by the money holdings of the household for next period's consumption and is described by the CIA constraint. Money is supplied obviously by the government.

$$M_t = \bar{M}_t$$

Write it in real terms

$$m_t = \bar{m}_t$$

Integrating over demand for intermediate goods of all firms and equating this to the intermediate goods production function

$$\int_0^1 A_t h_t^d(j)dj = \int_0^1 (\frac{P_t(j)}{P_t})^{-\varepsilon_p} y_t dj$$

Distributing the integral, Equation above becomes

$$A_t\int_0^1 h_t^d(j)dj = y_t\int_0^1 (\frac{P_t(j)}{P_t})^{-\varepsilon_p} dj$$

The labour market clearing requires that total labour supply by the labour packer must equal the sum of demand of labour from firms, that is,

$$\int_0^1 h_t^d(j)dj = h_t^d$$

Then the aggregate production function can be written as

$$y_t = \frac{A_t h_t^d}{v_t^p}$$

Where $v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} dj$ is the price dispersion. Considering the Calvo price setting assumption that $1-\phi_p$ of firms reset their price to the same price, while $\phi_p$ of firms can only partially index their price to what they charged in the last period, the integral $v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} dj$ can be split up as

$$\begin{aligned}
v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} dj &= \int_0^{1-\phi_p} (\frac{P_t^\#}{P_t})^{-\varepsilon_p} dj + \int_{1-\phi_p}^1 (\frac{(1+\pi_{t-1})^{\xi_p} P_{t-1}(j)}{P_t})^{-\varepsilon_p} dj \\
&= (1-\phi_p)(\frac{P_t^\#}{P_t})^{-\varepsilon_p} \\
&+ \int_{1-\phi_p}^1 (1+\pi_{t-1})^{-\xi_p\varepsilon_p} (\frac{P_{t-1}}{P_t})^{-\varepsilon_p} (\frac{P_{t-1}(j)}{P_{t-1}})^{-\varepsilon_p} dj \\
&= (1-\phi_p)(\frac{P_t^\#}{P_t})^{-\varepsilon_p} \\
&+ (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p} \int_{1-\phi_p}^1 (\frac{P_{t-1}(j)}{P_{t-1}})^{-\varepsilon_p} dj \\
&= (1-\phi_p)(\frac{P_t^\#}{P_{t-1}})^{-\varepsilon_p}(\frac{P_{t-1}}{P_t})^{-\varepsilon_p} + (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p}\phi_p v_{t-1}^p \\
&= (1-\phi_p)(\frac{P_t^\#}{P_{t-1}})^{-\varepsilon_p}\left(\frac{1}{1+\pi_t}\right)^{-\varepsilon_p} \\
&+ (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p}\phi_p v_{t-1}^p
\end{aligned}$$

Define $1+{\pi_t}^\# = \frac{P_t^\#}{P_{t-1}}$, the price dispersion term is

$$v_t^p = (1-\phi_p)\left(\frac{1+{\pi_t}^\#}{1+\pi_t}\right)^{-\varepsilon_p} + (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p}\phi_p v_{t-1}^p$$

The evolution of aggregate price can be described as

$$P_t{}^{1-\varepsilon_p} = (1-\phi_p)P_t^{\#\,1-\varepsilon_p} + \int_{1-\phi_p}^{1} (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} P_{t-1}(j)^{1-\varepsilon_p} dj$$

$$= (1-\phi_p)P_t^{\#\,1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} \int_{1-\phi_p}^{1} P_{t-1}(j)^{1-\varepsilon_p} dj$$

$$= (1-\phi_p)P_t^{\#\,1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} \phi_p P_{t-1}{}^{1-\varepsilon_p}$$

Divide both sides by $P_{t-1}{}^{1-\varepsilon_p}$, the aggregate inflation index can be written in terms of inflation rates

$$(1+\pi_t)^{1-\varepsilon_p} = (1-\phi_p)(1+\pi_t{}^{\#})^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)}\phi_p$$

The labour market clearing condition is integrating over demand function for labour of all households. Recall that

$$W_t{}^{1-\varepsilon_w} = \int_0^1 W_t(i)^{1-\varepsilon_w} di$$

Use properties of Calvo wage setting assumption to break the right-hand side

$$W_t{}^{1-\varepsilon_w} = \int_0^{1-\phi_w} W_t{}^{\#\,1-\varepsilon_w} di + \int_{1-\phi_w}^{1} (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} W_{t-1}(i)^{1-\varepsilon_w} di$$

$$= (1-\phi_w)W_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \int_{1-\phi_w}^{1} W_{t-1}(i)^{1-\varepsilon_w} di$$

$$= (1-\phi_w)W_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w W_{t-1}{}^{1-\varepsilon_w}$$

Rewrite in terms of real wage by dividing both sides by $P_t{}^{1-\varepsilon_w}$ and define $w_t{}^{\#} = \frac{W_t{}^{\#}}{P_t}$

$$w_t{}^{1-\varepsilon_w} = (1-\phi_w)w_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)}\phi_w(\frac{W_{t-1}}{P_t})^{1-\varepsilon_w}$$

$$= (1-\phi_w)w_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)}\phi_w(\frac{W_{t-1}}{P_{t-1}})^{1-\varepsilon_w}(\frac{P_{t-1}}{P_t})^{1-\varepsilon_w}$$

$$= (1-\phi_w)w_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)}\phi_w(w_{t-1})^{1-\varepsilon_w}(\frac{P_{t-1}}{P_t})^{1-\varepsilon_w}$$

Finally, the real wage is then be expressed as

$$w_t{}^{1-\varepsilon_w} = (1-\phi_w)w_t{}^{\#\,1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)}\phi_w(1+\pi_t)^{\varepsilon_w-1}w_{t-1}{}^{1-\varepsilon_w}$$

The system of optimality conditions and constraints is given below. All endogenous and exogenous variables are expressed in their levels.

$$\begin{cases} c_t = \dfrac{m_{t-1}}{1+\pi_t} + s_t, & \text{binding CIA constraint} \\ \eta_t = 0, & \text{nonbinding CIA constraint} \end{cases} \tag{3.30}$$

$$c_t{}^{-\sigma} = (\lambda_t + \eta_t) \tag{3.31}$$

$$\lambda_t = \beta E_t \left\{ \left( \frac{\lambda_{t+1}}{1+\pi_{t+1}} + \frac{\eta_{t+1}}{1+\pi_{t+1}} \right) \right\} \tag{3.32}$$

$$\begin{gathered} w_t{}^{1-\varepsilon_w} = (1-\phi_w) w_t{}^{\#1-\varepsilon_w} \\ +(1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w (1+\pi_t)^{\varepsilon_w - 1} w_{t-1}{}^{1-\varepsilon_w} \end{gathered} \tag{3.33}$$

$$w_t{}^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1t}}{\hat{n}_{2t}} \tag{3.34}$$

$$\begin{gathered} \hat{n}_{1t} = \left( \frac{w_t}{w_t{}^{\#}} \right)^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi} \\ +\beta \phi_w (1+\pi_t)^{-\xi_w \varepsilon_w (1+\varphi)} E_t \left[ (1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} \left( \frac{w_{t+1}{}^{\#}}{w_t{}^{\#}} \right)^{\varepsilon_w(1+\varphi)} \hat{n}_{1t+1} \right] \end{gathered} \tag{3.35}$$

$$\begin{gathered} \hat{n}_{2t} = \lambda_t \left( \frac{w_t}{w_t{}^{\#}} \right)^{\varepsilon_w} h_t^d \\ +\beta \phi_w (1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \left[ (1+\pi_{t+1})^{\varepsilon_w - 1} \left( \frac{w_{t+1}{}^{\#}}{w_t{}^{\#}} \right)^{\varepsilon_w} \hat{n}_{2t+1} \right] \end{gathered} \tag{3.36}$$

$$1 + \pi_t^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} (1+\pi_t) \frac{x_{1t}}{x_{2t}} \tag{3.37}$$

$$x_{1t} = c_t{}^{-\sigma} mc_t y_t + \beta \phi_p (1+\pi_t)^{-\xi_p \varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1t+1} \tag{3.38}$$

$$x_{2t} = c_t{}^{-\sigma} y_t + \beta \phi_p (1+\pi_t)^{\xi_p(1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2t+1} \tag{3.39}$$

$$mc_t = \frac{w_t}{A_t} \tag{3.40}$$

$$y_t = \frac{A_t h_t^d}{v_t^p} \tag{3.41}$$

$$(1+\pi_t)^{1-\varepsilon_p} = \left(1-\phi_p\right) (1+\pi_t{}^{\#})^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} \phi_p \tag{3.42}$$

$$v_t^p = (1-\phi_p)\left(\frac{1+\pi_t{}^{\#}}{1+\pi_t}\right)^{-\varepsilon_p} + (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p}\phi_p v_{t-1}^p \tag{3.43}$$

$$\overline{m}_t = \frac{\overline{m}_{t-1}}{1+\pi_t} + s_t \tag{3.44}$$

$$c_t = y_t \tag{3.45}$$

$$m_t = \overline{m}_t \tag{3.46}$$

$$\Delta \ln \overline{m}_t = \ln \overline{m}_t - \ln \overline{m}_{t-1} \tag{3.47}$$

$$\Delta \ln \overline{m}_t = (1-\rho_m)\overline{\pi} - \ln \pi_t + \rho_m \Delta \ln \overline{m}_{t-1} + \rho_m \ln \pi_{t-1} + u_{mt} \tag{3.48}$$

$$\ln A_t = \rho_a \ln A_{t-1} + u_{at} \tag{3.49}$$

The last two equations describe the uncertainties in this model: money supply growth and technology innovation. Note that the Lagrange multiplier of the CIA constraint has not been eliminated because the CIA constraint is assumed to be occasionally binding rather than always binding or always nonbinding.

The equilibrium sequence is characterized by two different states: one where the CIA constraint is binding, and one where it is not. How these two states are divided depends on the sequences of productivity and monetary shocks with two extreme cases that the CIA constraint is always binding and is never binding. The logic for the following analysis of dynamics is that first consider these two extremes separately and then combine them together by allowing the CIA constraint to move freely.

Generally, the steady states can be illustrated as follows. There is a unique steady state when the CIA constraint is always binding. Due to the neutrality of money in the condition when the CIA constraint never binds, there could be hard to find a definite steady states as the economy is at its efficiency and output is produced according to the equilibrium of demand and supply given the market prices and wages if the same variables are includes as the binding CIA model.

Above equations list is about the scenario when price and wage are all sticky. This is the most comprehensive case. The other three scenarios can be derived from these equations by setting $\phi_p$ or/and $\phi_w$ to be zero. When $\phi_p$ are equal to zero, the probability that a firm can adjust its price is 1, which means the price is fully flexible. When $\phi_w$ are equal to zero, the probability that a household can adjust his/her wage is 1, which means the nominal wage is fully flexible. Thus, the flexible price and flexible wage case can be seen as $\phi_p$ and $\phi_w$ are both equal to be zero. The sticky price but flexible wage case is when $\phi_w$ is zero and $\phi_p$ is not. The sticky wage but flexible price case is then $\phi_p$ is zero while $\phi_w$ is not.

### 3.2.6 Occasionally Binding CIA Constraints

$$\eta_t \left( \left( c_t - \frac{m_{t-1}}{1+\pi_t} - s_t \right) \right) = 0 \tag{3.50}$$

Above Karush-Kuhn-Tucker condition shows that there are two kind of cases for the value $\eta_t$ to take. One case is that $\eta_t$ is equal to zero. According to the Karush-Kuhn-Tucker condition, the CIA constraint could be nonbinding or weakly binding in this case. The other one is $\eta_t > 0$. However, in this case, the Karush-Kuhn-Tucker condition implies that the CIA constraint has to be strictly binding. These two different conditions shifting endogenously with different values for $\eta_t$ is the basic idea for occasionally binding CIA constraint.

Next consider the intertemporal condition

$$c_t^{-\sigma} - \eta_t = \beta E_t \left\{ \frac{c_{t+1}{}^{-\sigma}}{1+\pi_{t+1}} \right\} \tag{3.51}$$

When the CIA constraint is nonbinding or weakly binding, it is the usual intertemporal condition or so-called Euler equation describing the link between consumption today and tomorrow that marginal utility of consumption today should equal to the discounted and deflated marginal utility of consumption tomorrow. However, when the CIA constraint strictly binds, the marginal utility of consumption becomes larger than the discounted and deflated marginal utility of consumption tomorrow, which means that households tend to consume less due to binding CIA constraint.

It is straightforward to get money demand condition from this intertemporal condition.

$$E_t \left\{ \beta \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{1}{1+\pi_{t+1}} \right\} \begin{cases} < 1, & \text{binding CIA constraint} \\ = 1, & \text{nonbinding CIA constraint} \end{cases} \tag{3.52}$$

The term $E_t\{\frac{1}{1+\pi_{t+1}}\}$ is the expected gross return of money. The expected net return of money is then $E_t\{\frac{1}{1+\pi_{t+1}}-1\}$ . If the expected inflation is always non-negative or the inflation is expected to increase, then the expected net return on money is always less or equal to zero, $E_t\{\frac{1}{1+\pi_{t+1}}-1\} \leq 0$, that is, non-positive or decreasing. Money is used as a transaction tool instead of a real asset to generate returns. This is when the households tend to use up all their money holdings which triggers a binding CIA constraint. On the other hand, if the expected inflation can be negative or the inflation is expected to decrease, then the expected net return on money can be positive or increasing now, $E_t\left\{\frac{1}{1+\pi_{t+1}}-1\right\} > 0$. This time the households earn positive return by holding more money. Money can be seen as a real asset that have returns. Under this situation, the CIA constraint can be nonbinding. The term $\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$ is the intertemporal rate of substitution between consumption tomorrow and today. Thus, the whole term on the left-hand side can be explained as the expected return of money in terms of the expected utility of consumption tomorrow proportional to the utility of consumption today. This is the expected relative value of money (Dixon and Pourpourides, 2016). When households expect that this relative value of money will decrease next period, they will rush to spend all their money holdings this period which makes the CIA constraint bind. If their expectations of the relative value of money to be the same, they will keep extra money for next period's consumption and the CIA constraint is nonbinding. Since the expected relative value of money does not change, the consumers are indifferent between spending money today and holding it to next period, rather than spend all money today.

Next consider the intra-temporal condition that states the relationship between the marginal rate of substitution between work and consumption and the real wage. To keep it simple, ignore the sticky wage for a while. Then the intra-temporal condition for flexible wage is

$$\frac{{h_t^d}^{\varphi}}{{c_t}^{-\sigma}} = \frac{\lambda_t}{(\lambda_t+\eta_t)} w_t \qquad (3.53)$$

The left-hand side of the equation is the marginal rate of substitution between consumption and working hours, which is equal to the price ratio of labour, $w_t$, only when $\eta_t = 0$.

When the CIA constraint is nonbinding or weakly binding, i.e. $\eta_t = 0$, this is the familiar intra-temporal condition describing the equal relationship and is consistent with the RBC literatures, implying households can supply and demand to the extent they like. When the CIA constraint is strictly binding, i.e. $\eta_t > 0$, the marginal rate of substitution becomes lower than the real wage, which means that the labour supply will be lower given the level of consumption. This is called the substitution effect. There is also an income effect. The income is lower, which will, in turn, increase the labour supply. Since that the real wage remains unchanged for a given technology level, the combination of both substitution effect and income effect will lead to a decreasing labour supply. Hence, a binding CIA constraint in fact acts as a tax on consumption as the household substitutes from consumption which is constrained by the CIA constraint to leisure.

To sum up, whether the CIA constraint is binding or not depends on the expected relative value of money by consumers and the expectations on the relative value of money relies mainly on the expected inflation rate which is closely related to the growth rate of money supply through money supply rule in this model.

## 3.3 Baseline Dynamics

A starting calibration and a set of dynamics for the baseline models are presented to illustrate the logic of endogenously binding CIA constraints. Occasionally binding CIA constraints can be separated into two regimes, binding regime and nonbinding regime, and shocks make contributions to the switch between these two regimes. Parameters of both regimes take the same value to let the shocks be the only cause for switch. IRFs for binding regime or reference regime produce dynamics consistent with the existing RBC literatures. There is no IRF for nonbinding regime or alternative regime due to the infinite steady states.

### 3.3.1 Baseline Calibration

The calibration procedure involves the choices of functional forms for the utility function and production function and making value assignments to the parameters of the model based on either micro-evidence or long-run macroeconomic facts. Given the simple structure of this model, there is a range of values for the parameters to take. Here, a baseline calibration is outlined to confirm that the CIA constraint can be nonbinding endogenously for some periods. The model calibrated and estimated using the US data is illustrated in next Section.

A unit of period corresponds to a quarter. The subjective discount factor, $\beta$, is set at 0.99 to be consistent with a steady state real interest rate of 4 percent per year. Mehra and Prescott (1985) analyse the micro-evidence for the coefficient of relative risk aversion and find that its value fall in the range between 1 and 2. The coefficient of relative risk aversion of consumption, $\sigma$, is set to equal 1, which is commonly used in the RBC literatures. The baseline value for the elasticity of wages with respect to hours, $\varphi$, is set at 0.2. This is consistent with Galí et al. (2004). This also follows Rotemberg and Woodford's (1997, 1999) calibration of the elasticity of output with respect to hours of $\frac{2}{3}$ and the elasticity of wages in terms of output of 0.3. $\varepsilon_p$, the elasticity of substitution among different intermediate goods is set to 11 so that the gross markup in goods market, $\frac{\varepsilon_p}{\varepsilon_p-1}$, is equal to 1.1, following Christiano et al. (2005). Similarly, $\varepsilon_w$, the elasticity of substitution among different types of labour takes the value 11 consistent with that the gross markup in labour market, $\frac{\varepsilon_w}{\varepsilon_w-1}$, is equal to 1.1. For baseline simulations, the full indexation for both price and wage are assumed because the data shows some indexation in price and wage and the indexation parameters will be estimated in the next Section. The probability of sticky price and sticky wage are set to be 0.5 and 0.2, respectively. This setting is different from the literatures that wages are always stickier than prices and the sticky wage parameter is 0.73 suggest by Smets and Wouters (2007). But this smaller number of the wage stickiness is highly related to the smaller labour supply elasticity of 0.2 and this is consistent with the Smets and Wouters (2007)'s robustness result that the reduced degree of wage stickiness leads to a decrease in the labour supply elasticity. This calibration is also supported by Beraja et al. (2016). Their benchmark estimate of sticky wage parameter is equal to 0.24 by using the regional data of US. This result is very close to Grigsby's (2018) micro estimates of annual base wage adjustments using administrative payroll data. According to Galí et al. (2017), the U.S. optimal annual inflation target is close to 2% and takes the range between 1.85% and 2.21%. For simplicity, the quarterly steady state inflation, $\bar{\pi}$, is set to be 0.5%.

The autoregressive parameter for the technology shock, $\rho_a$, and its standard deviation, $\sigma_a$, are calibrated at 0.9720 and 0.01 respectively, after Gomme et al. (2017). For money supply shock, the value that the autoregressive coefficient, $\rho_m$, takes is 0.45 with its standard deviation $\sigma_m = 0.01$. Here, autoregressive coefficient of money growth is set to be 0.45 to adjust the sensitivity of inflation to a lower level.

The baseline calibration is summarised for model with sticky price and sticky wage in Table 3-1 in Appendix 3.A. For other three models, all parameters remain the same except for the sticky price and sticky wage parameters. More precisely, $\phi_w = 0$ for sticky price model; $\phi_p = 0$ for sticky wage model; and $\phi_w = \phi_p = 0$ for flexible price and flexible wage model.

## 3.3.2 Dynamics for Binding CIA Constraints

Though the focus here is to analyse endogenously binding cash-in-advance constraints, it is helpful to firstly analyse the dynamics when the CIA constraint is always binding and when it is always nonbinding separately. And it is not inappropriate to start from the flexible price and flexible wage model.

Since it is assumed by most of literatures that the CIA constraint always binds, the impulse response functions (IRFs) of the model analysed here are consistent with the literatures when $\eta_t > 0$. The IRFs to a technology shock are shown in Figure 3-1.

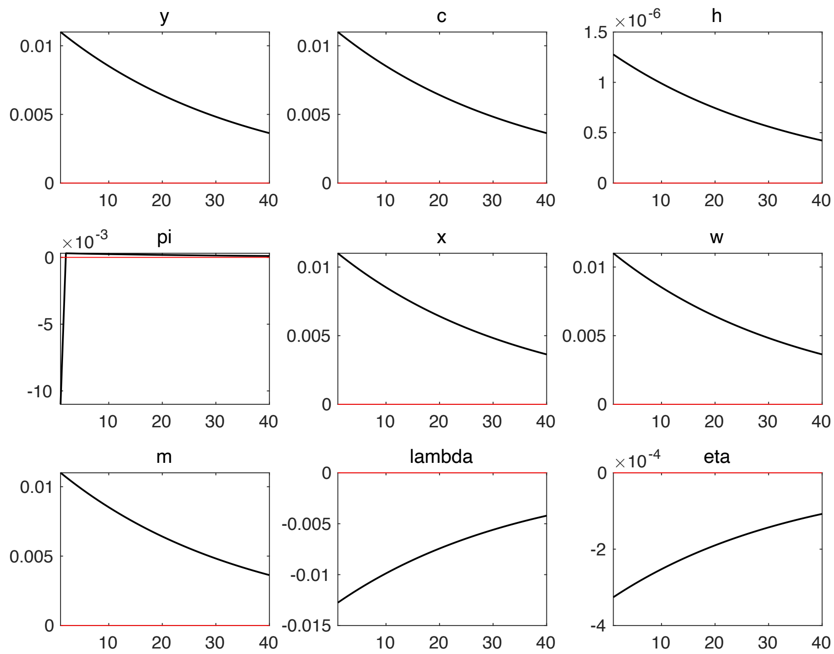


*Figure 3-1 IRFs following a technology shock*

The responses of the real variables are identical as they would be in an RBC model without money explicitly included. A persistent positive technology shock lowers the price level reflecting by the immediate jumping down of inflation and this leads to an increase in real money balances. An increase in technology has an increasing effect on the real wage, which is appeal for households to supply more labour. When the only shock in the economy is technology innovation, the CIA constraint plays no role. More labour supply increases the output and consumption. If there is only one technology shock, money is actually neutral.

However, money could be non-neutral if a money supply shock is included instead of a technology shock. The IRFs to a money supply shock are shown in Figure 3-2. Here, money has real effects. A temporary increase in the growth rate of money, which is a permanent change in the level of the nominal money supply, lowers output, labour supply and consumption. Why can this happen? An increase in the growth rate of money supply causes higher inflation which is essentially a tax on householders who hold money. If there is no cash-in-advance constraint that requires them to hold money, then people will give up all their money holdings. However, in this case, they have to hold money because of the cash-in-advance constraint, while higher inflation gives consumers an incentive to get rid of money. Since money is required only for consumption, they cannot substitute from money to consumption. Instead, they substitute from money to leisure. Thus, there is a reduction in both consumption and labour supply, which leads to a decline in output as well.

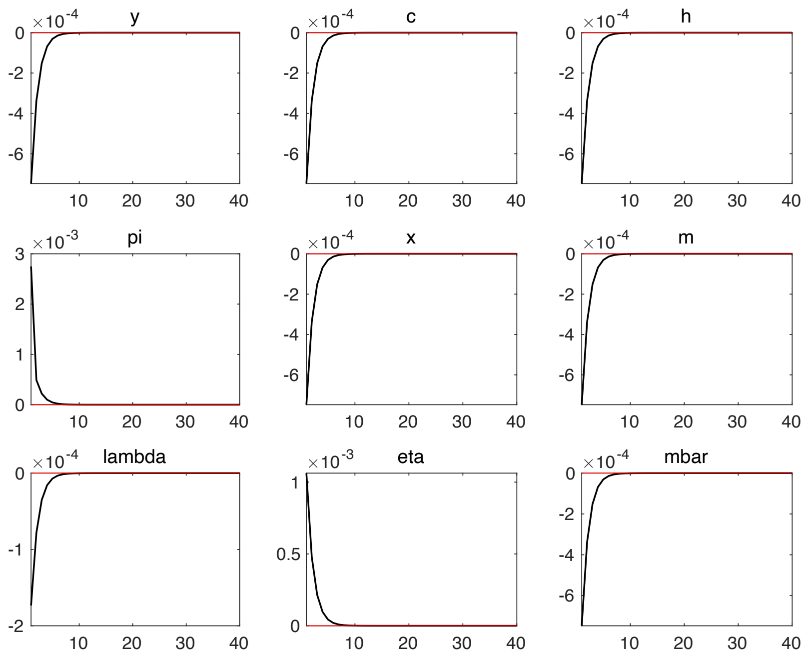


*Figure 3-2 IRFs following a money supply shock*

The addition of sticky price and sticky wage will not change the dynamics of shocks for an always binding CIA constraint but will show some sluggishness, compared to the model with both price and wage flexible. More specific IRFs for models with sticky price or/and sticky wage will be explained in next Section when analysing the occasionally binding CIA constraints.

### 3.3.3 Dynamics for Nonbinding CIA Constraints

The original intuition pointed out by Friedman (1956) is that money is a good thing in the sense when it can reduce transaction frictions and increase welfare. It happens only when there is no cost of producing money. The inflation rate being positive imposes a tax on those who holds money and this tax distorts welfare. From this perspective, a negative inflation is what desires by the government. Having a negative inflation rate in CIA model means that there could be a positive return on money, and then households tend to hold extra money, that is, $m_t \to \infty$. Eventually, this will lead to a nonbinding CIA constraint. If the CIA constraint is always nonbinding, then the whole model is back to a basic RBC case. Money is neutral in this case. The demand and supply by households are totally determined by the market prices and wages. They can adjust their decisions immediately when something unexpected happens when there is no nominal or real rigidities. This means the market is efficient and, intuitively, households must be better off in this case though it might be a weakly one. Hence, it is hard to find a definite steady state for the nonbinding CIA case due to the neutrality of money if the same parameters and variables are included as the binding CIA case. This is why no IRFs to shocks can be plot when CIA constraint does not bind.

## 3.3.4 Dynamics for Occasionally Binding CIA Constraints

### 3.3.4.1 Algorithm for Occasionally Binding CIA Constraints

Whether CIA constraints bind or not is assumed to be determined endogenously here. Inspired by the toolkit proposed by Guerrieri and Iacoviello (2015) to solve dynamic models with occasionally binding constraints, the CIA constraint is allowed to change between two regimes, namely, binding regime and nonbinding regime, of the same RBC model. The same RBC model here means that the same set of variables should be included in these two different regimes. The binding CIA regime is defined as the reference regime while the nonbinding regime is seen as the alternative regime. The two important requirements for the toolkit are as follows.

1. A rational expectations equilibrium must exist at the reference regime. This requires a definite set of steady states of the reference regime.
2. Before the economy is shocked, the model stays in the reference regime. If the shocks happen, the model is moved away from the reference regime to the alternative regime for some finite time and then must return to the reference regime by the assumption that agents expect that there will be no future shocks.

The reference regime and the alternative regime only differ because of the Kuhn-Tucker condition. More specifically, the set of conditions under the reference regime encompasses $\eta_t > 0$, which is corresponding to $c_t = \frac{m_{t-1}}{1+\pi_t} + s_t$, implying a binding CIA constraint that households have no incentive to hold extra money as the relative value of money is expected to decrease. The set of conditions under the alternative regimen encompasses $\eta_t = 0$, which is corresponding to $c_t < \frac{m_{t-1}}{1+\pi_t} + s_t$, implying a slack CIA constraint that the relative value of money is expected to be the same or increase and households can hold money as they like. The solution algorithm proposed by Guerrieri and Iacoviello (2015) employs a guess-and-verify approach. First, the date when the model will return to the reference regime is guessed. Second, this initial guess is verified and, if necessary, updated. This algorithm is implemented using the MATLAB programming language and the routines are designed as an add-on to Dynare. The model at the reference regime and the model at the alternative regime with the same variable set are specified into two Dynare model files respectively. The guess-and-verify process is specified in another MATLAB file which allows the switch between two regime directories.

Specifically, the economy features two regimes: a regime when the CIA constraints bind, and a regime when they do not but are expected to bind in the future. When the CIA constraints bind, the linearized system of necessary conditions for an equilibrium can be written as

$$A_1 E_t X_{t+1} + A_0 X_t + A_{-1} X_{t-1} + B u_t = 0$$

Where $A_1$, $A_0$, and $A_{-1}$ are matrices of coefficients, the vector $X$ collects the endogenous variables excluding shocks that are deviating from the steady state for the regime when constraints are binding, and $u$ is a vector of all shock processes. The linearized system for the regime with nonbinding constraints can be similarly expressed as

$$A_1{}^* E_t X_{t+1} + A_0{}^* X_t + A_{-1}{}^* X_{t-1} + B^* u_t + C^* = 0$$

Where $A_1{}^*$, $A_0{}^*$, $A_{-1}{}^*$, and $B^*$ are coefficient matrices and $C^*$ is a vector of constants capturing the differences in deviations from steady states. When the constraints bind, the decision rule of the model is

$$X_t = G X_{t-1} + K u_t$$

Where $G$ and $K$ are nonlinear functions of the matrices $A_1$, $A_0$, and $A_{-1}$.

The guess-and-verify method is used when the CIA constraints are nonbinding. For example, if the constraints are nonbinding in period $t$, but are expected to be binding next period, the decision rule for period $t$ using the decision rule for the next period $t+1$ that $E_t X_{t+1} = G X_t$ can be written as

$$X_t = -({A_1}^* G + {A_0}^*)^{-1}({A_{-1}}^* X_{t-1} + B^* u_t + C^*)$$

As shown in this decision rule, the dynamics for the model with nonbinding CIA constraints rely not only on the current regime implied by the matrices, ${A_1}^*$, ${A_0}^*$, ${A_{-1}}^*$, $B^*$, and $C^*$ but also on the expectations of future regimes when the CIA constraints are binding again through matrix $G$. The same logic can be used to compute the case when the CIA constraints are guessed to be nonbinding for more than one periods or when they are expected to be nonbinding in later periods.

The occasionally binding CIA constraint makes first-order perturbation method inapplicable. The model is solved using the piecewise linear method described in Guerrier and Iacoviello (2015), which links the first-order approximation of the model around the same points under each regime. Specifically, the points used for first-order approximation are the steady states of the binding CIA regime. It might be helpful to have the steady states of both regimes to be as close as possible. However, if money is included in both regimes, it is impossible to find the steady states of the nonbinding CIA regime as money has no role in this case. Taking this into account, the differences between these two regimes can be seen as a function of the discount factor $\beta$ and the steady state inflation, $\bar{\pi}$. This may lead to large jumps for simulation results as illustrated below. Despite of this drawbacks, this algorithm provides an important contribution to solve nonlinear models.

As is mentioned above, the endogeneity when the CIA constraint binds or not is related to the order of the occurrences of technology innovation and money supply shock. Either the technology shock or the money supply shock itself can lead to a nonbinding CIA constraint if the effects of shocks are large enough to cause a persistent negative inflation or a decrease in inflation. In this model with money only, the interaction between these two shocks is possible to trigger the nonbinding constraints even the effects of shocks are small. If the technology shock happens before money supply shock, then the CIA constraint is always binding. When the economy is shocked by the money growth first and then a technology shock, the CIA constraint could be nonbinding for some certain periods.

The following dynamics are plot by adapting the toolkit of Guerrieri and Iacoviello (2015), referring to their paper for more details, and four different scenarios are shown separately.

All the figures below have the same property: the blue lines are the dynamics for always binding CIA constraints while the red lines are the dynamics for endogenously binding CIA constraints. The word occbinds is referred to the case when the CIA constraint occasionally binds.

### 3.3.4.2 Model with Flexible Price and Flexible Wage

Figure 3-3 and Figure 3-4 confirms the conclusion that the CIA constraints are always binding when there is only technology shock or monetary shock.

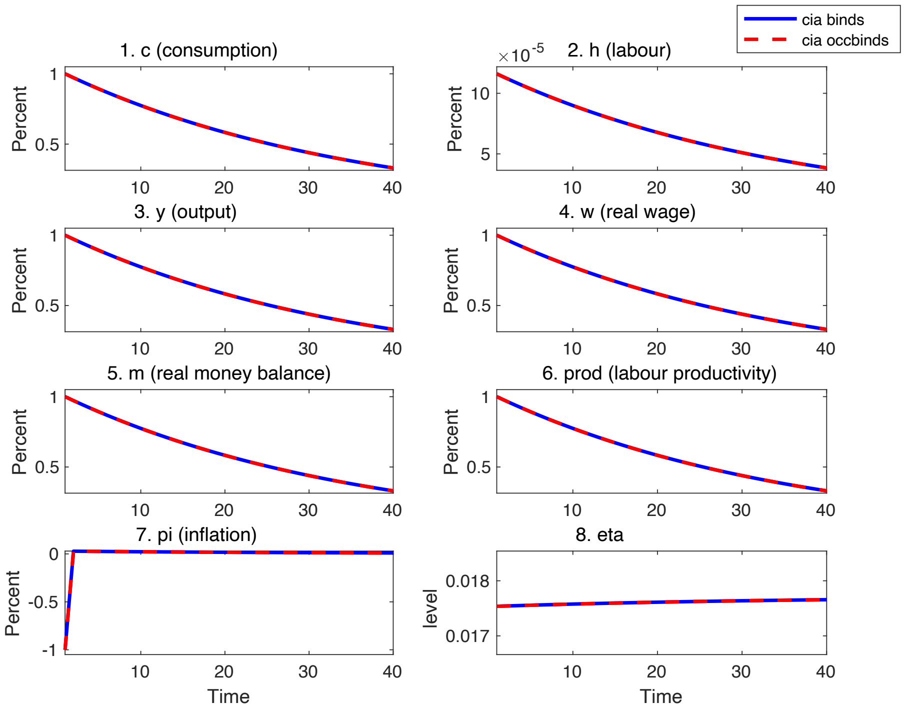


*Figure 3-3 Dynamics following a 1% technology shock*

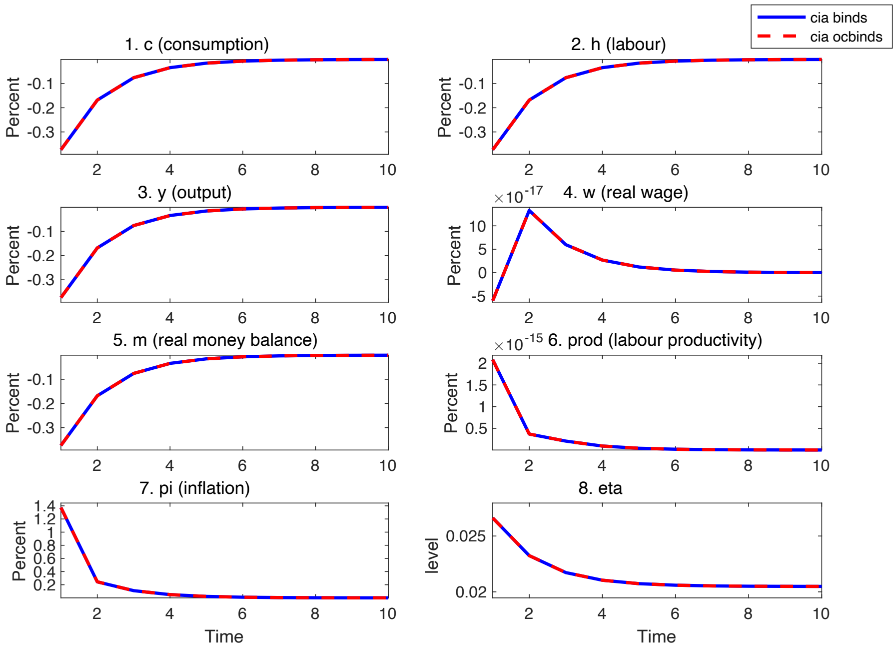


*Figure 3-4 Dynamics following a 1% monetary shock*

Figure 3-5 shows the IRFs of the occasionally binding CIA constraints for the flexible price and flexible wage model when the monetary shock hits before the technology shock. This is consistent with the assumption that the growth rate of money supply and money transfer to households are determined by the monetary authority before the technology innovation is realized by the agents. If the transfer of money is chosen by the monetary authority before the realization of technology innovation, then this transfer may be welfare improving by making CIA constraints deviate from binding to nonbinding for certain periods as output is higher when the CIA constraint does not bind. The positive monetary shock increases the inflation, which makes CIA constraint even more binding. In period 4, the money growth shock has not faded away and the money supply continues to increase. However, a technology innovation hits the economy, which causes a decrease in inflation. The households will hold the excess money supply in the pocket instead of spending as they expect the inflation will decrease, which gives a space for nonbinding CIA constraint.

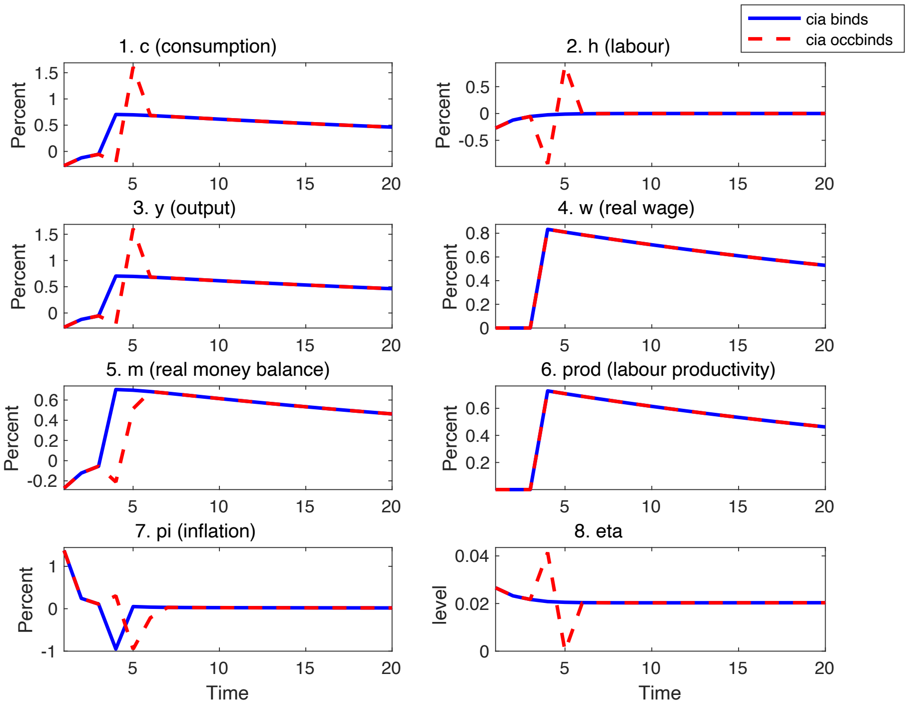


*Figure 3-5 Dynamics following 1% money supply shock and 1% technology shock*

The blue line is the dynamics for the always binding CIA cases when there is one 1% positive money supply shock, i.e. 1 standard deviation money supply shock, at period 1 and a 1% positive technology shock, i.e. 1 standard deviation technology shock, at period 4. Consumers expect that the relative value of money will decrease due to a positive money supply growth and they highly prefer to spend all their money in current period, which finally leads to a binding CIA constraint. Although the decrease in inflation can cause an increase in the relative value of money in period 4, the always binding CIA constraints do not allow the households to hold extra money.

The red line is when there is 1% positive money supply shock at period 1 and 1% positive technology shock (1 standard deviation technology shock) at period 4 as the CIA constraint is allowed to be nonbinding for some periods instead. The money supply shock happens at period 1 and positive growth in money supply reduces the consumption, labour supply and output but increases inflation as usual. At period 4, the technology innovates unexpectedly. Then dynamics of the economy now is contributed to the combined effects of both money supply shock and technology shock. This combined effect shows some sluggishness in inflation. In period 4, inflation increases for a little bit instead of decreasing immediately due to the persistence of money supply shock. The real wage increases significantly and creates an income effect on labour supply, which leads to a decreasing in labour supply. This decrease in labour supply then causes a decrease in both output and consumption. After period 4, inflation starts to decrease as a consequence that the technology shock dominates money supply shock. Decreasing inflation results in an increase in the relative value of money and households tend to hold extra money as an asset, which triggers a nonbinding CIA constraint. In period 5, the technology shock is highly persistent, and the CIA constraint continues to be nonbinding as the consumers continue to expect an increase in the relative value of money. However, consumption and output turn to increase in this case. This is because the real wage continues to increase, and the substitution effect dominates the income effect instead. Hence, technology innovation attracts more labour supply by increasing the real wage, and therefore increases the output and consumption level sharply by around 2%. This huge jump up in consumption, output and labour is a result that the market works at its efficiency when the CIA constraint is nonbinding as money is neutral, which will be explained more precisely later. As the only input in production is labour and the only uncertainty in production is technology, with a technology shock, the productivity increases by 1%, which inherits from the 1% technology shock.

The interaction between monetary shock and technology shock results in the CIA constraint to be nonbinding for only one period or one quarter, i.e. period 5. That is, the probability of a binding CIA constraint in the model with flexible price and flexible wage is 95% for a 20-period or 5-year dimension. When the inflation goes back to its target which is around 2% annually after period 6, the relative value of money decreases again and consumers have no incentive to hold extra money, which makes the CIA constraint to bind again.

Consumption, output, labour supply and real wage are all larger when the CIA constraint is nonbinding than when it does bind as shown by the huge jump in period 5. This is because the piecewise linear solution used here links the first-order approximation of the model around the same point under each regime, which is the steady states of the binding CIA regime. When the CIA constraint is always binding, for a given level of consumption, the marginal rate of substitution between consumption and leisure is less than the real wage, leading to a lower labour supply. Though there is also an income effect, this small income effect is strictly dominated by the substitution effect. In other words, Inflation serving as a tax on money holdings disappoints the consumers who hold money. But they are constrained by the CIA constraint and they cannot simply substitute from money to consumption. Instead, they have higher preference to substitute from money to leisure, which causes a reduction in labour supply. Compared to the case when the CIA constraint is always nonbinding, consumption, output and labour are lower when the CIA constraint always binds. If the first-order approximations of these two regimes are both around the steady states of the binding CIA regime, it may result in a larger deviation when the CIA constraint does not bind. When money supply shock dies out in period 6, technology still innovates, productivity and real wage have not reached its steady state and continue to decrease. Higher real wage cannot attract more labour supply as the employment, which is less volatile, is at its steady state but can increase the total income of households, which results in higher consumption as well as higher output. This is consistent with Galí (1999) that a significant decline in hours after a technology shock raises labour productivity. Hence, when working hours return back to its steady state after period 6, the consumption and output still decrease gradually to their steady state as the labour productivity remains higher.

In short words, one standard deviation money growth shock followed by one standard deviation technology shock tends to amplify the effect of both monetary shock and technology shock as the equilibrium in the binding regime is always welfare inferior to that in the nonbinding regime. In this case, decreases in inflation generated by technology innovation is the reason to cause nonbinding CIA constraints. Allowing the CIA constraint to be endogenously nonbinding generates some persistence in inflation without including any nominal or real rigidities.

#### 3.3.4.3 Model with Sticky Price but Flexible Wage

Here, price is assumed to be sticky and follows a Calvo-style. Sticky price introduces more sluggishness. It is the same as before that the CIA constraints are always binding when there is only technology shock or money supply shock shown in Figure 3-6 and Figure 3-7, but it can be nonbinding for some periods when there is a positive technology shock following a positive money supply shock. The IRFs for occasionally binding CIA constraints are shown in Figure 3-8 where there is a positive one standard deviation money supply shock in period 0 followed by a positive one standard deviation technology shock in period 4.

As shown in Figure 3-6, sticky price generates some inflation persistence, which in turn leads to a hump shape in consumption and output as a result of a positive technology shock. With respect to a positive money supply shock shown in Figure 3-7, sticky price leads to a gradual increase in inflation with the biggest effect in period 2, which is a cause of gradual decrease in consumption and output.

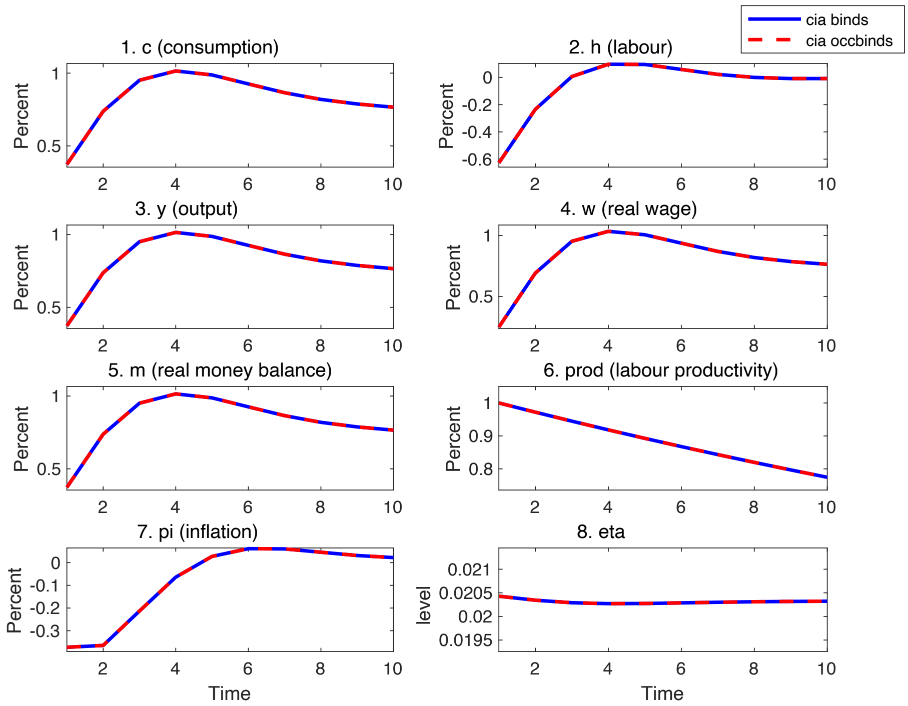


*Figure 3-6 Dynamics following a 1% technology shock*

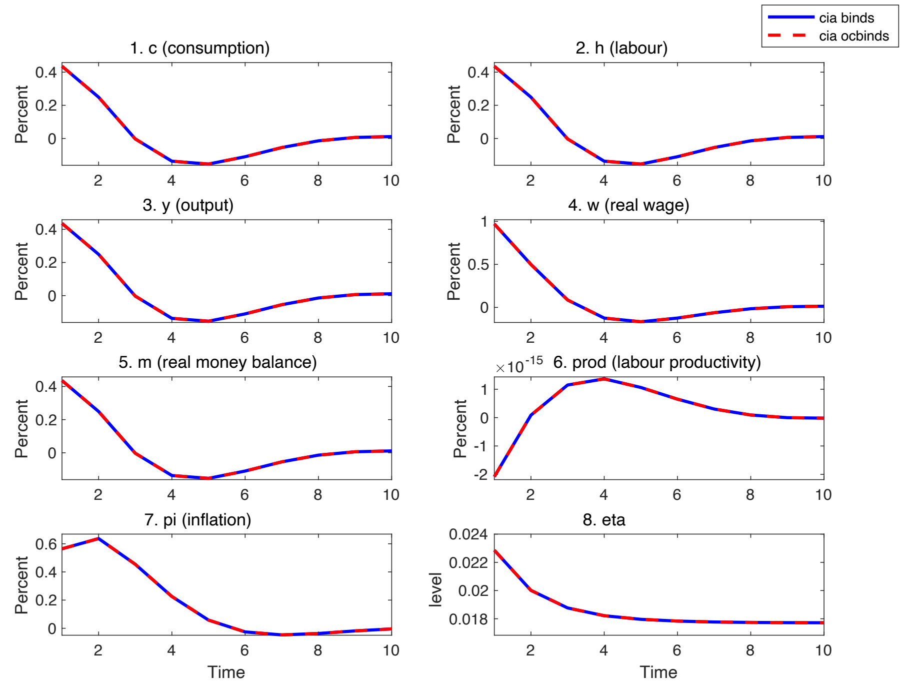


*Figure 3-7 Dynamics following a 1% monetary shock*

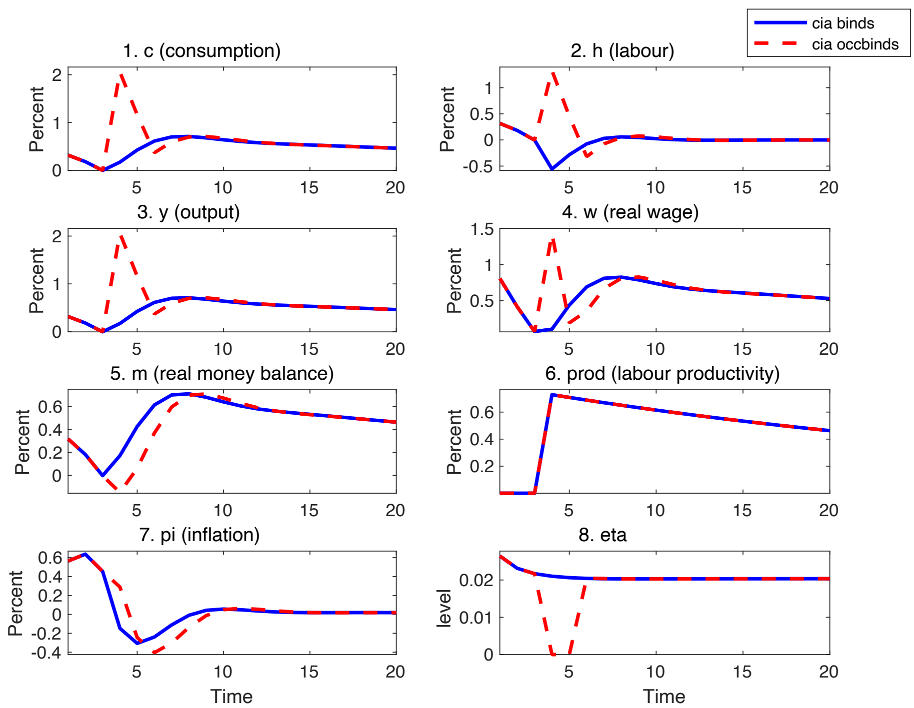


*Figure 3-8 Dynamics following 1% money supply shock and 1% technology shock*

Here, the blue line is when there is one positive money supply shock in period 1 and one positive technology shock in period 4 for the case when the CIA constraint is always binding. The red line is when there is one positive money supply shock in period 1 followed by a positive technology shock in period 4 for the case when the CIA constraint is occasionally binding. The inflation shows the biggest effect in 2 periods after a positive one standard deviation monetary shock. This lagged effect on inflation leads to a lagged decrease in consumption as well as output and the biggest effect on these two variables is in period 5 as shown in Figure 3-7. However, in period 4, a positive one standard deviation technology shock hits the economy which increases the labour productivity immediately. The inflation also increases by a little bit as a result of combined effect of both money supply shock and technology shock. After period 4, the inflation starts to decrease leading to an increase in the relative value in money, which gives a chance for nonbinding CIA constraints. By allowing the CIA constraints to be occasionally binding, the inflation shows more persistence with the biggest negative effect in period 6 compared to the biggest effect in period 5 in the case when the CIA constraint is always binding. This sluggishly negative inflation, which indicates positive relative value of money, gives the chance for the CIA constraints to be nonbinding for longer periods, i.e. 2 periods or 2 quarters in this case. Hence, the probability of a binding CIA constraint for the model with sticky price but flexible wage is 90% for a 20-period or 5-year dimension. As this probability is highly related to the sluggishness in the economy, the stickier the price is, the higher probability of a nonbinding CIA constraint.

The positive technology shock in period 4 results in a significant increase in the real wage. This significant increase leads to an increase in labour supply as the substitution effect of real wage dominates the income effect instead of creating a significant income effect on labour supply at the very beginning as shown in the case of flexible price and flexible wage. The increase in labour supply then causes the increase in both output and consumption.

The rigidities introduced here leads to an even larger amplified effect of both monetary shock and technology shock compared to the flexible price and flexible wage model. This is the reason why consumption and output jump even higher by around 3% after the technology innovation. The effect on labour is smaller than that on output as a result of lower volatility in employment.

When the CIA constraint becomes binding again in period 6 as the inflation turns to increase, labour, output and consumption still decrease to a lower level as the technology shock is persistent and the income effect of real wage dominates in this case. If the CIA constraint is allowed to be endogenously binding, a positive money supply shock followed by a positive technology shock will result in a consumption boom followed by a consumption decline due to the nominal or real rigidities.

### 3.3.4.4 Model with Sticky Wage but Flexible Price

Next the nominal wage is sticky while the nominal price is flexible. As before, Figure 3-9 and Figure 3-10 show that only technology shock or monetary shock cannot cause the CIA constraint to be nonbinding. Similar to the sticky price model, sticky wage also generates some sluggishness in inflation, output and consumption but less compared to that generated by the sticky price model as wage is assumed to be more flexible in this setup. As nominal wage is fully indexed to lagged inflation, real wage does not show a hump shape to a positive technology shock.

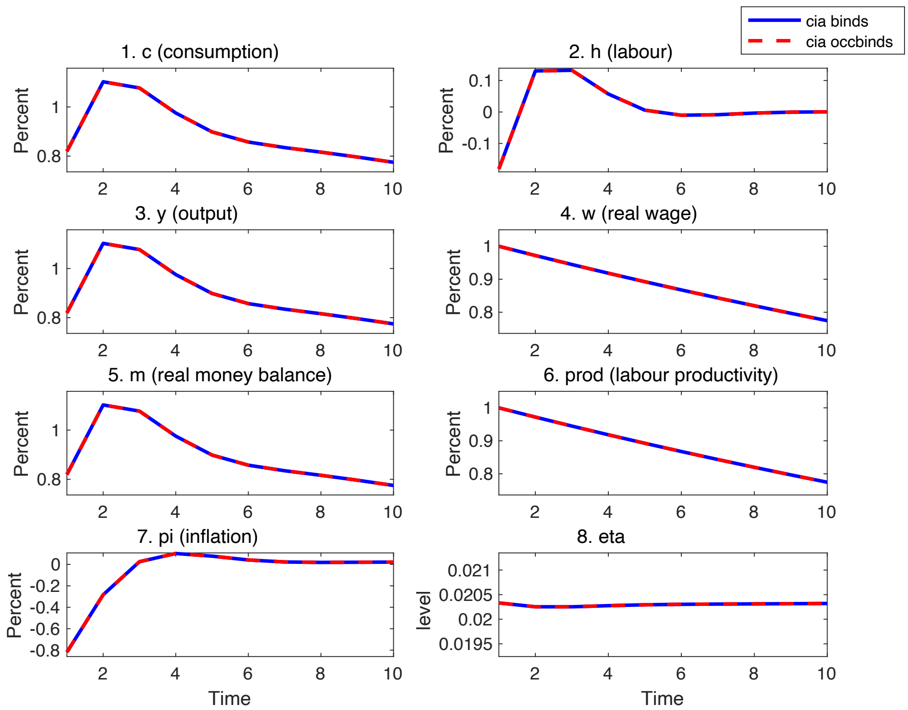


*Figure 3-9 Dynamics following a 1% monetary shock*

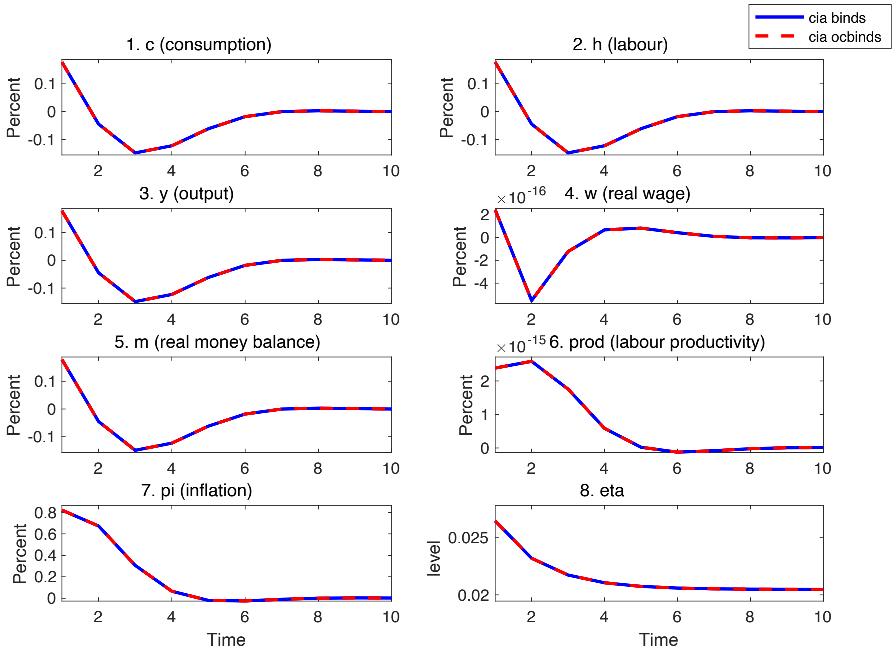


*Figure 3-10 Dynamics following a 1% monetary shock*

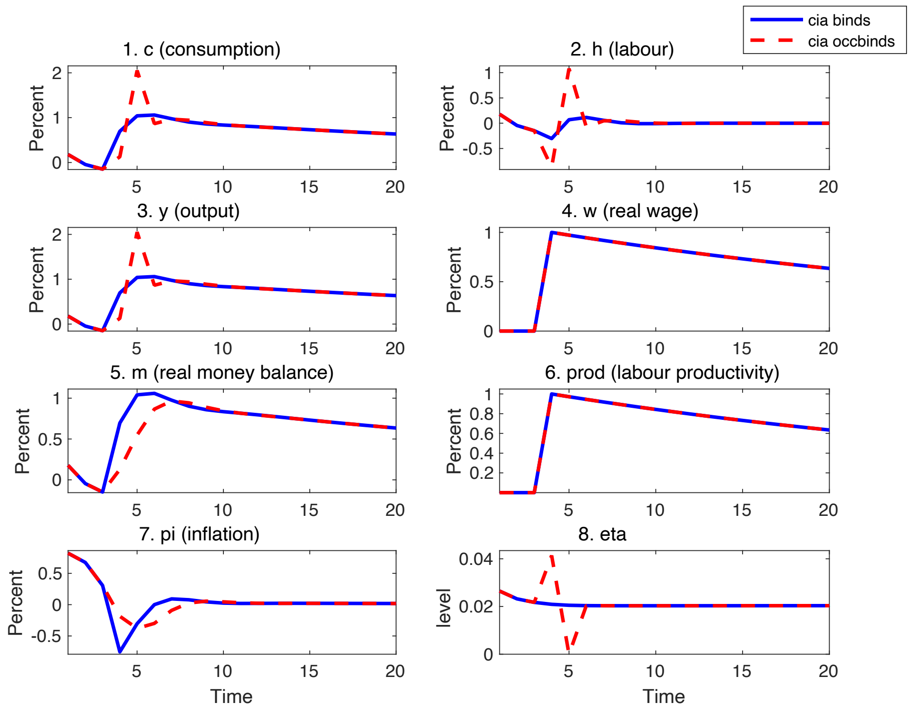


*Figure 3-11 Dynamics following 1% money supply shock and 1% technology shock*

The dynamics of one positive standard deviation money supply shock in period

1 followed by a positive standard deviation technology shock in period 4 are displayed in Figure 3-11. Same as specified before, the blue line is when there is one positive money supply shock in period 1 and one positive technology shock in period 4 for the case with always binding CIA constraints. The red line is for the case when there is one positive money supply shock in period 1 followed by a positive technology shock in period 4, in which the CIA constraints can be nonbinding for some periods. Sticky wage does show some sluggishness, but this sluggishness is not big enough for more nonbinding CIA constraints because of the calibrated sticky wage parameter is small. When the technology innovates in period 4, the productivity increases instantly leading to a decrease in labour as the real wage increases significantly and creates an income effect on labour supply. This decrease in labour supply then results in decreases in both consumption and output. The inflation continues to decrease in period 5 before it starts to increase, which causes the CIA constraint to be nonbinding for only one period. Thus, the probability of a binding CIA constraint in this case for a 20-period or 5-year dimension is 95%. When the CIA constraint continues to be nonbinding, the substitution effect of real wage dominates the income effect, which leads to an increase in labour supply. Since the labour supply increases, output increases as well as consumption. When the CIA constraint binds in period 6, the income effect of real wage on labour supply dominates, which leads to a decrease in labour supply and thus a decrease in output and consumption. That is, there is a consumption boom followed by a consumption decline when the CIA constraint can be occasionally binding.

#### 3.3.4.5 Model with Sticky Price and Sticky Wage

The nominal price and nominal wage are both assumed to be sticky in this model with the sticky parameters of nominal price and nominal wage are 0.5 and 0.2. Neither technology shock nor money supply shock itself can trigger a nonbinding CIA constraint as shown in Figure 3-12 and Figure 3-13 but more sluggishness is generated in this case.

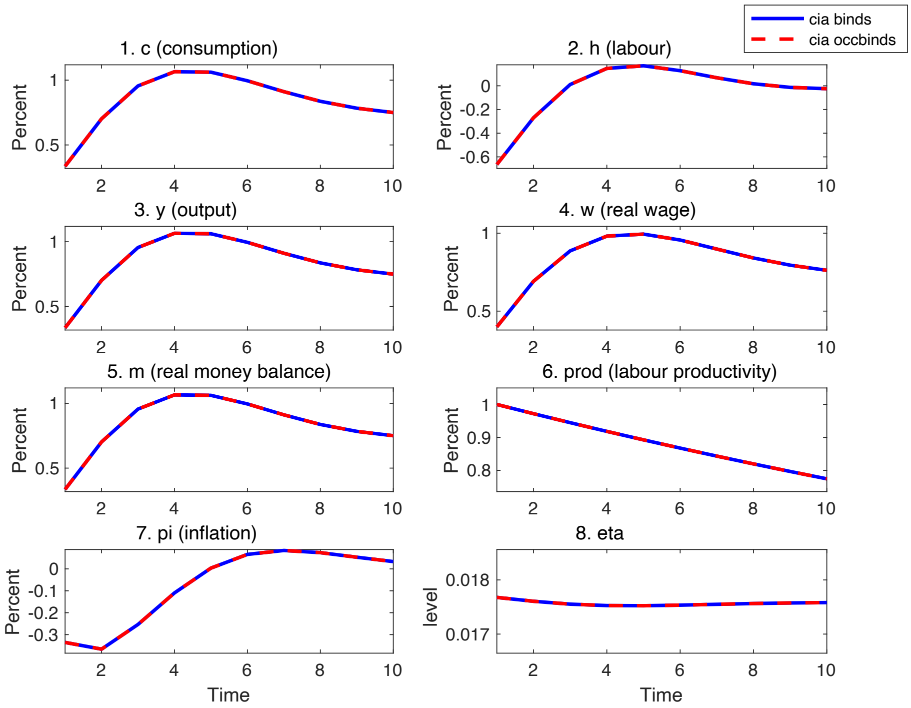


*Figure 3-12 Dynamics following a 1% technology shock*

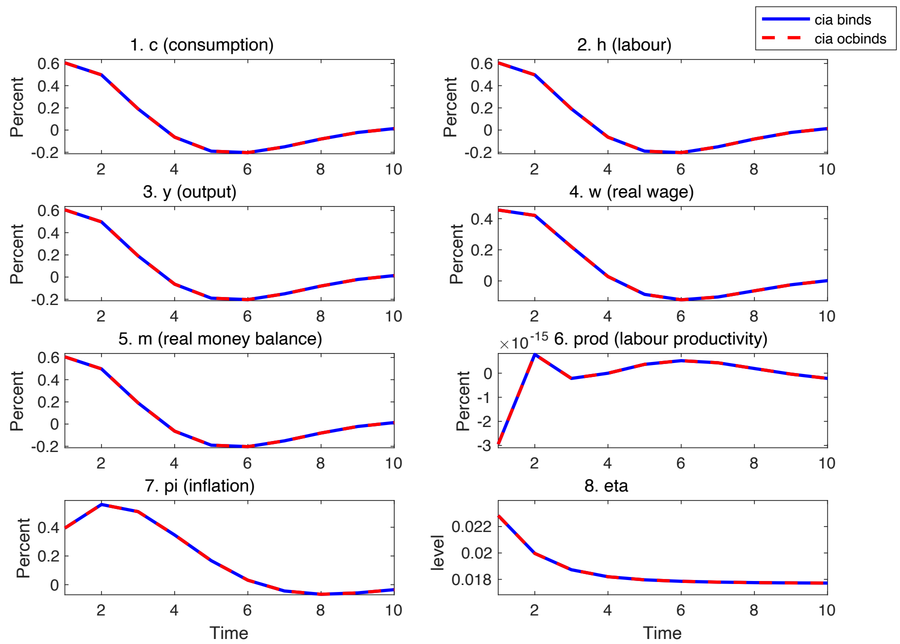


*Figure 3-13 Dynamics following a 1% monetary shock*

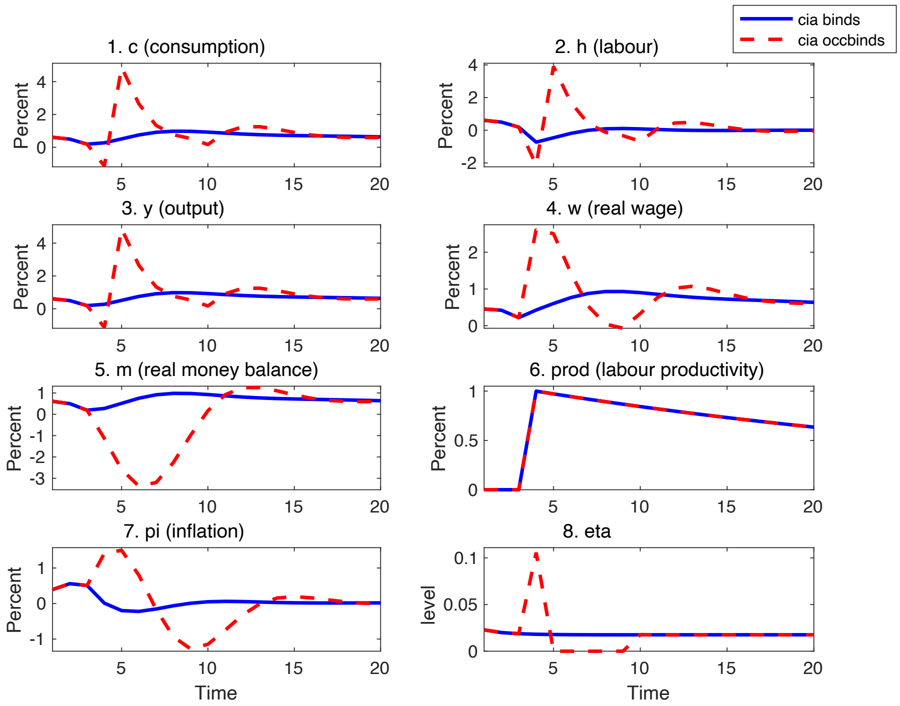


*Figure 3-14 Dynamics following 1% money supply shock and 1% technology shock*

Figure 3-14 is the IRFs for occasionally binding CIA constraints, where the blue line is when there is one positive money supply shock in period 1 and one positive technology shock in period 4 with the CIA constraint binding all the time while the red line is when there is one positive money supply shock in period 1 followed by a positive technology shock in period 4 with the CIA constraint to be nonbinding for some periods. The stickiness in both nominal price and nominal wage results in an even higher probability for the CIA constraints to be nonbinding. The constraint is nonbinding during the periods from period 5 to period 9, that is, 5 periods or 1.25 years in total. Hence, the probability of a binding CIA constraint in this model is 75% for a 20-period or 5-year dimension.

Inflation shows more sluggishness in this case. It reaches its highest level at period 5 and decreases gradually to its lowest level in period 9 as inflation lagged reacts to a positive technology shock. When the technology shock hits in period 4, the labour productivity increases immediately, which, in turn, decreases the labour demand and output as the inflation continues to increase. In period 5, when inflation is highest and starts to decline, the CIA constraint becomes nonbinding, the economy falling in a classical one where the households can supply and demand at a level they desire. This causes a huge jump up in consumption, output and working hours as the binding CIA case is always welfare inferior to the nonbinding case. Since the CIA constraint is nonbinding in this case, the substitution effect of real wage can dominate the income effect. Thus, when the real wage continues to increase, the labour supply increases as the substitution effect dominates, which leads to increases in output and consumption. The interaction between sticky price and sticky wage results in an even larger amplified effects compared to models with only sticky price or sticky wage. After inflation is lowest in period 9 and turns to increase, the CIA constraint binds again. In this case, the substitution effect cannot dominate as the CIA constraint binds and the income effect dominates, which leads to a decrease in labour and thus decreases in output and consumption. As a result of two rigidities here, the consumption boom is followed by a longer consumption decline.

## 3.4 Estimation

### 3.4.1 Calibration and Priors

The adjusted Bayesian estimation method follows Guerrieri and Iacoviello (2017) to estimate the deep structural parameters of the model with occasionally binding CIA constraints while the rest parameters remain calibrated based on the information of estimation sample. Each estimated parameter must be assigned with a type of distribution (beta, normal, gamma, inverse gamma) along with the mean and standard deviation of the distribution. The standard errors of the distribution measure how strongly the prior mean is believed. The smaller is the prior standard deviation, the bigger penalty is assigned to the estimation procedure which picks a value of the parameter far from the prior mean. Turning to the type of distribution chosen, if the parameter is restricted to lie between 0 and 1, the beta distribution is more suitable. The inverse gamma distribution is commonly used for the standard deviations of shocks. For other parameters, it is optional to choose either a gamma or a normal distribution. The main difference between these two distributions is that the gamma distribution is skewed but the normal distribution is not.

As both price and wages show some stickiness in the data, the model estimated here is the model with sticky price and sticky wage. The calibrated parameters used in estimation are labour elasticity, $\varphi = 0.2$, the elasticity of substitution among intermediate goods, $\varepsilon_p = 11$, the elasticity of substitution among different type of labour, $\varepsilon_w = 11$. Instead, the discount factor, $\beta$, is set to equal 0.9999. This setting is reasonable in that it is important to make sure the steady state for two different regimes in a switching model to be as close as possible. The steady state inflation, $\bar{\pi}$, however, is set to 0, which corresponds to a zero growth in steady state money supply growth implied in the data. All other parameters are estimated by Bayesian estimation. Deviating from above simulation calibration, the price indexation $\xi_p$ and the wage indexation $\xi_w$ are not fully indexed but partially indexed that will be estimated. The sluggishness in the model has an important impact on the fluctuation of inflation which triggers nonbinding CIA constraints. Hence, the sticky parameters, $\phi_p$ and $\phi_w$, will be estimated. In addition, the persistence parameters for technology shock and money growth shock, $\rho_a$ and $\rho_m$, and their standard deviations, $\sigma_a$ and $\sigma_m$, are estimated.

All the priors are selected from what are commonly used in the literatures. The Calvo probabilities for both prices and wages are assumed to be around 0.5, implying the average length of price and wage contracts are half a year[18]. The degree of indexation to past inflation follows a beta distribution with mean 0.5 and standard deviation 0.2 in both goods and labour markets. The priors for shocks are harmonized. The standard deviations of both shocks are inverse gamma distributed with mean 0.01 and 1 degree of freedom. The persistence of the AR(1) processes are assumed to be different. The technology innovation is assumed to be more persistent. The persistence parameter for technology shock is beta distributed with mean 0.75 and standard deviation 0.1. The persistence parameter for money growth follows beta distribution with mean 0.5 and standard deviation 0.2. The prior distributions are reported in Table 3-2 in Appendix 3.A.

### 3.4.2 Data and Likelihood

The estimation model is based on observations for two series summarised in Table 3-3: price inflation (CPI growth) and GDP growth calculated on GDP per capita[19]. These observations cover the period from 1985Q1 to 2017Q4. The model features two observations and two shocks (i.e. technology shock and

[18] This is consistent with the findings of Bils and Klenow (2004).
[19] Following Gurrieri and Iacoviello (2017), a one-side HP filter is used to construct the data prior to estimation.

money supply shock), which saves the model from identification problems.

Table 3-3 Data sources for estimation

| Data Sources for Estimation | |
|---|---|
| Price inflation | Quarterly change in CPI deflator from Bureau of Economic Analysis |
| GDP growth | GDP from Bureau of Economic Analysis, log transformed and detrended with one-side HP filter |

The dynamics in each regime highly rely on how long one expects to be in that regime, which in turn depends on the state vector. The model solution takes the following reduced form as suggested by Guerrieri and Iacoviello (2017)

$$X_t = P(X_{t-1}, \epsilon_t)X_{t-1} + D(X_{t-1}, \epsilon_t) + Q(X_{t-1}, \epsilon_t)\epsilon_t$$

$X_t$ is a vector that includes all the variables in the model except the exogenous shock processes. The innovations to the shocks are collected in the vector $\epsilon_t$. $P$ is a matrix of reduced-form coefficients and is state-dependent. The same are for vector $D$ and matrix $Q$. These vectors and matrices are functions of the lagged state vector as well as of the current shock processes. However, as the shock processes can have the impact on the changes in the reduced-form coefficients, $X_t$ is still locally linear in $\epsilon_t$.

To cope with the observed variables, the solution above can be represented by multiplying the state vector $X_t$ by the matrix $H_t$, which is a matrix of the observed variables. Then, the vector of observed series $Y_t$ can be expressed by $Y_t = H_t X_t$. As the reduced-form coefficients are a function of shock processes, the Kalman filter is not suitable to get the estimates of the innovations in shocks. An inversion filter can be used instead under the condition that the number of observed variables and the number of innovations plus the number of measurement errors are the same, which is an approach outlined by Fair and Taylor (1983) to estimating nonlinear DSGE models. Following Gurrieri and Iacoviello (2017) that a medium-scale model with occasionally binding constraints and no measurement error is solved, $\epsilon_t$ can be recursively solved given $X_{t-1}$ and current observation of $Y_t$, the nonlinear equations can be written as

$$Y_t = H_t P(X_{t-1}, \epsilon_t)X_{t-1} + H_t D(X_{t-1}, \epsilon_t) + H_t Q(X_{t-1}, \epsilon_t)\epsilon_t$$

The unobserved components are contained in the vector $X_t$, in order to use this inversion filter, an initialization is required to filtering the system. The same initialization is used as Gurrieri and Iacoviello (2017) that the initial $X_0$ coincides with the model's steady state and the filter is trained by the first 20 observations[20].

The innovations process vector $\epsilon_t$ is assumed to follow a multivariate normal distribution with mean zero and a covariance matrix $\Sigma$. Under this assumption, the logarithmic transformation of the likelihood function $f$ for the observed data $Y^T$ implied by a change in variables argument can be written as

$$\log(f(Y^T)) = -\frac{T}{2}\log(\det(\Sigma)) - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t{}'(\Sigma^{-1})\epsilon_t + \sum_{t=1}^{T}\log(\left|det\frac{d\epsilon_t}{dY_t}\right|)$$

The formation of Jacobian matrix $\frac{d\epsilon_t}{dY_t}$ requires the inverse transformation from the shocks to the observations only being given implicitly by

$$H_tQ(X_{t-1},\epsilon_t)\epsilon_t - (Y_t - H_tP(X_{t-1},\epsilon_t)X_{t-1} - H_tD(X_{t-1},\epsilon_t)\,) = 0$$

The determinant of $H_tQ(X_{t-1},\epsilon_t)$ is verified to be nonzero for further differentiation and the implicit transformation is locally invertible accordingly. The Jacobian of the inverse transformation, which depends on the local linearity between $X_t$ and $\epsilon_t$, is

$$\frac{d\epsilon_t}{dY_t} = (H_tQ(X_{t-1},\epsilon_t))^{-1}$$

This Jacobian of the inverse transformation for the change in variables is already known from the model's solution and no extra derivative calculations are needed by using this piece-wise solution, which shows a superior advantage in calculating time compared to the general approach proposed by Fair and Taylor (1983).

Given that $\left|\det\left(\left(H_tQ(X_{t-1},\epsilon_t)\right)^{-1}\right)\right| = \frac{1}{|\det(H_tQ(X_{t-1},\epsilon_t))|}$ as well as the Jacobian result, the logarithmic transformation of the likelihood can be rewritten as

[20] They find that the estimation results are insensitive to the assumption that $X_0$ is equal to the model' s steady state value.

$$\log(f(Y^T)) = -\frac{T}{2}\log(\det(\Sigma)) - \frac{1}{2}\sum_{t=1}^{T}\epsilon_t'(\Sigma^{-1})\epsilon_t - \sum_{t=1}^{T}\log(|\det(H_t Q(X_{t-1}, \epsilon_t))|)$$

## 3.4.3 Estimation Results

The calibrated IRFs of occasionally binding CIA constraints show that there is a consumption boom followed by a smaller consumption collapse with the only asset money and some rigidities. The estimated model with both sticky price and sticky wage confirms that the CIA constraints are not always binding, especially during the period of the Great Recession. As a positive money supply shock followed by a positive technology shock results in a nonbinding CIA constraint for some periods, unconventional monetary rules, such as Quantitative Easing, play a key role for the CIA constraint to be nonbinding.

### 3.4.3.1 Estimated Parameters

The posterior is construct and maximized as a function to evaluate the likelihood function with prior distributions about the parameters, given the observed data. A standard random walk Metropolis-Hastings algorithm with a chain of 50000 draws are then used to construct the posterior density of the estimated parameters.

The posterior modes and other statistics of the estimated parameters are reported in Table 3-2 in Appendix 3.A. In terms of the parameters that govern the frequency of price and wage adjustment, the posterior modes for the Calvo parameters of nominal rigidities are both equal to 0.4950, implying the price and wage are reset every half year. This slightly low degree of price and wage rigidities are likely to be compensated by the real rigidities, that is, partial indexation of prices and wage. The estimated indexation parameters of price and wage are both 0.4950. These estimated parameters related to rigidities are not in line with the estimation results in most literatures. This may be the result that money is the only asset in the model and other real assets, such as capital, are totally excluded. The estimated technology innovation function and money supply function are consistent with literatures that technology shock is always more persistent than money supply shock. The estimated persistence parameters of technology shock and monetary shock are 0.7810 and 0.4995 respectively, with the standard deviation of both shocks are equal to 0.0047.

Given the estimated parameters, the model captures some key empirical properties in the data. First, the key variable of the model is the CIA multiplier, which is shown as a function of the household's expectation of inflation. The estimated model shows that the correlation between the CIA multiplier and inflation is 0.6. Second, the estimated model line up well with the data with respect to key moments. The standard deviation of output growth is 0.0121 in model, compared to 0.0075 in the data. The correlation between the output growth and inflation in the model, 0.28, is higher than that in the data, 0.13. Though consumption is not used as an observable when estimating the model, the standard deviation of estimated consumption is 0.025, which is close to 0.029 in data.

Finally, the probability of a binding CIA constraint is 73.48% and the estimated CIA multiplier $\eta_t$ is plotted in Figure 3-15. It shows that the CIA constraint becomes more nonbinding during the period of the Great Recession and after, that is, the period from 2008 till now. After the financial crisis in 2008, the US central bank used the unconventional monetary policy instruments, forward guidance and quantitative easing (QE), to stimulate the economy and help the economy recover from the recession. The dynamics of the CIA multiplier in Figure 3-15 are consistent with different stages of QE. A more detailed explanation is in the next Section.

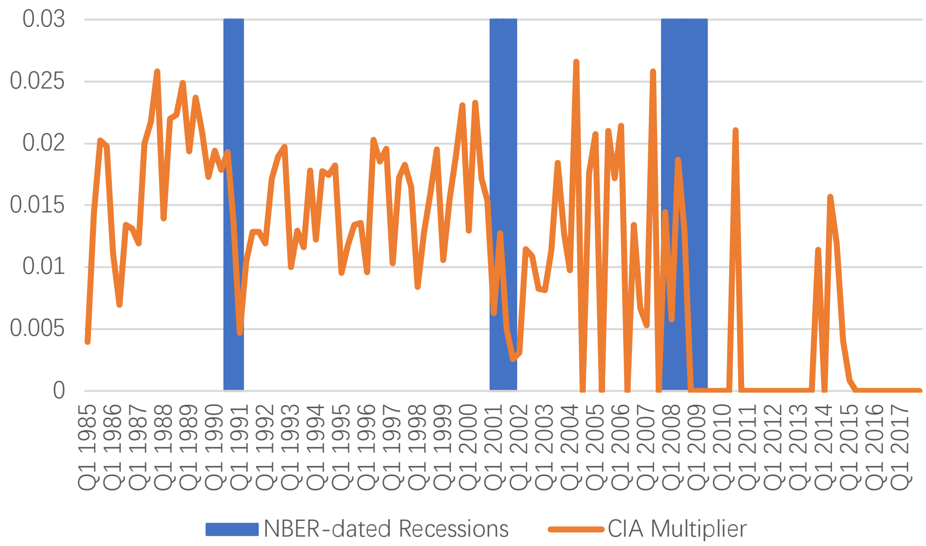


*Figure 3-15 Estimated CIA multiplier*

### 3.4.3.2 Unconventional Monetary Policies and the CIA Constraints

The global financial crisis started in July 2007 and led to a bank collapse in later 2008, especially in major economies, such as the US. The US Federal Reserve has tried different tools to help the economy recover from the Great Recession, which is the period from 2008 to 2009. Given that the conventional monetary policy that altered the short-term interest rate was ineffective and the short-term interest rate has been pushed to near their lower limit zero (or ZLB), the Fed seek to implement unconventional monetary policy tools, namely, forward guidance and quantitative easing[21].

Forward guidance refers to a way that attempts to communicate the guidance of the expected interest rate path to different agents in the economy on the purpose to influence their financial decisions. This monetary policy instrument lines up with the baseline assumption of the occasionally binding CIA constraint model that the money transfer is announced before the technology innovation. When the monetary shock hits the economy before the technology shock, the households' expectation of the relative value of money will be changed. This, in turn, will influence their decisions on the substitution between consumption, leisure and money.

A second unconventional monetary policy tool that has been adopted is the quantitative easing or QE. QE refers that the central banks purchase a large scale of long-term bonds and other financial assets, leading to a large increase in their balance sheet. The central bank purchases of both long-term government debt and private assets, such as corporate debt or asset-backed securities, is also called open market operations with the aim to manipulate the supply of base money in the economy.

The Bank of Japan started the process of QE back in 2001 and maintained the policy till 2006. Under this policy, the main instrument used by the BoJ to reach its operating target of current account balances of financial institutions at the BoJ was purchases of Japanese Government Bonds (JGBs). Since the global financial crisis, the central banks adopted a number of unconventional tools to promote the financial stability. Instead of focusing on the liability side of central

[21] See Rudebusch (2018) for more details.

banks' balance sheet like the standard QE implemented by the BoJ, the Federal Reserve's credit easing policy entails an expansion of the asset side of the Federal Reserve's balance sheet by allowing purchases not of government securities but of private assets, such as corporate bonds and mortgage-based securities (MBS). In response to the Great Recession, the Federal Reserve engaged in credit easing by purchasing large sums of Treasuries and mortgage-backed securities. The Bank of England reduced the interest rate to 0.5% and took on a QE from 2009 with purchases of approximately £200 billion or 14% of GDP to combat a recession. Most of the assets purchased were UK government securities (gilts). The Bank also purchased smaller quantities of high-quality private-sector assets. The BoE did it again in 2016 to keep interest rate low and help with issues on Brexit. The European Central Bank adopted QE for the first time from 2015 to 2018 as a complementary of austerity measures by buying €60 billion worth of bonds.

Typically, the Fed buys debts and mortgage-based securities (MBS) from commercial banks in exchange for money, which expands the Fed's balance sheet by the amount of asset purchases. This activity will then indirectly control the money supply through the electronically money. This is evident by the increase in the US monetary base, which is the total amount of currency that circulates in the public or hold as commercial bank deposits in central bank reserves. There were 3 separate waves of purchases, namely QE1, QE2, and QE3, by the Federal Reserve Board over the period from late 2008 to 2014. Figure 3-16 shows the changes in the US base money following different stages of QE since the global financial crisis.

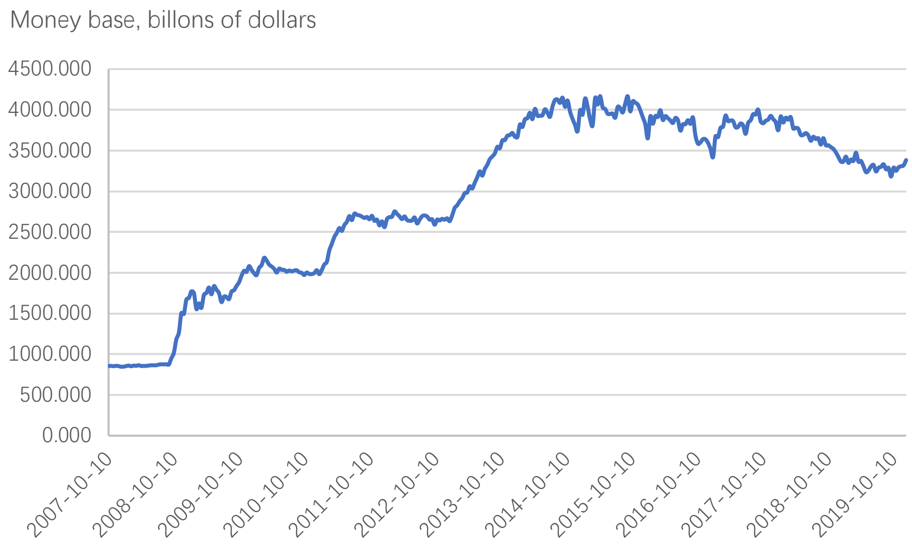


*Figure 3-16 Money base in the US*

*Sources: Federal Reserve Bank of St. Louis*

The first QE program, QE1, was announced by the Fed in November 2008 to purchase a large scale of long-term assets with the intention to buy $600 billion bonds and MBS. Further in March 2009, the asset purchase was expanded to up to $1.75 trillion in agency securities with $1.25 trillion in MBS included. In general, total QE1 government purchases were equal to 12 percent of GDP. Although this unconventional policy was served as if it were a one-shot policy, the whole purchases were completed until March 2010.

This initial announcement provided a signal of target policy rate in order to affect the future expectations of agents, which is a form of forward guidance. Figure 3-15 shows that the CIA constraints are nonbinding during the period of QE1 from the end of 2008 to the beginning of 2010. When the Fed firstly announced purchases of agency securities and mortgage-backed securities, a way equivalent to increase the money supply unexpectedly, and consumers assumed that the relative value of money would be decrease, which caused the CIA constraints to bind and the CIA multiplier increased. Given the rising money supply, the total factor of productivity innovated, which gave a chance for a larger decrease in inflation and the consumers' expectations of future value of money turned increase. The CIA constraints became nonbinding and the economy was stimulated as GDP growth and employment were higher.

Although the QE1 helped the US recovery, it was not sufficient to generate a sustainable economic recovery. When the asset purchases finished in March 2010, the CIA constraints started to bind again, which is consistent with the spike in the CIA multiplier in 2010 Q3. The Federal Open Market Committee then implemented a series of extra asset purchase programs. From November 2010 to June 2011, the Fed completed a purchase of $600 billions of long-maturity Treasury securities, which was known as QE2. In September 2009, the Fed announced another program called the Maturity Extension Program (MEP), which included a swap of $400 billion in Treasury securities with maturities of six years with an equal amount of securities with maturities of three years or less. This program was extended to June 2012 with an overall swap of more than $600 billion. This MEP can be seen as a transition from QE2 to QE3. Different from earlier programs, QE3 was a flow-based program, which specified a monthly purchase instead of an overall purchase. In September 2009, the Fed started to purchase $40 billions of MBS per month. Later in December 2012, the Fed expanded to purchase $45 billion of long-term Treasury securities per month. Then the total purchases were $85 billion per month. The QE3 was officially ended in October 2014. There was a spike in the CIA multiplier in 2014 Q2 in anticipation of this. The earlier spike in 2013 Q4 was during the “taper tantrum” that followed Ben Bernanke’s announcement that QE was going to end. The periods when the CIA constraint does not bind is consistent with the periods of different waves of QE. Since August 2010, the reinvestment policy that the Fed replaces mature securities to maintain a constant nominal size of its balance sheet when there is no QE program, is the main reason why the MBS purchases are non-zero even after QE3 was officially finished. This can be seen as a slight increase in money base from the end of 2016 to 2017 in Figure 3-16 and zero CIA multipliers during the same period in Figure 3-15.

To sum up, this simple DSGE model with occasionally binding CIA constraints can mimic different waves of the QE. The economy was stimulated by the implementation of QE programs during the period of liquidity trap when the nominal interest rate was at its zero bound. Since agents’ expectations of inflation are raised by QE, the economy is able to have a short escape from liquidity traps.

## 3.5 Conclusion

Though the model is quite simple with money as the only asset, there are some policy implications. Nominal interest rate has hit it zero lower bound and the Fed has tried to use the unconventional monetary policies, QE and the forward guidance, to generate some recovery. The private marginal cost of holding money is nominal interest rate, while the public marginal cost of producing money is almost zero. Thus, having a positive nominal interest rate can be seen as imposing a tax on money holders and this actually distorts welfare. According to Fisher equation that nominal interest rate should be equal to real interest rate plus inflation rate, setting zero nominal interest rate is equal to negative inflation rate. Nominal interest rate at its zero lower bound is consistent with the case when the CIA constraint is nonbinding that households are indifferent among holding money and consuming. During this period, the economy is stimulated, and households are kind of better off by increasing consumption, output and employment. This is consistent with the zero CIA multipliers shown in Figure 3-16 after 2009. However, the probability of nonbinding CIA was too low, and the recovery was not satisfied. After the nominal interest rate was stuck at its lower limit, the central banks seek the help from unconventional monetary policies to generate longer recovery periods. The occasionally CIA constraint DSGE model confirms the contributions of unconventional monetary policies. The monetary authority should not avoid increase money supply as endogenously nonbinding CIA constraint can somehow improve the welfare of households, though a larger consumption boom is followed by a smaller consumption decline. As the nominal interest rate is zero when the CIA constraint is nonbinding, the unconventional monetary policies may not be helpful to escape the liquidity trap quickly. However, from the above calibrated IRFs that the CIA constraint will be binding eventually, it is not trivial to believe that the economy will get out of liquidity trap one day, but it takes longer time. The periods of liquidity trap are highly contributed to the size or persistence of shocks in this case. There could be other shocks dampen the escape but is out of the scope of this Chapter. If the technology shock is more persistent than money supply shock, consumption and output continue to be stimulated when the economy escapes the liquidity trap. From this perspective, unconventional monetary policies can still have a positive influence after the economy recovers from financial crisis.

## Appendix 3.A Tables

Table 3-1 Parameter values

| Parameter Assignment | | |
|---|---|---|
| $\beta$ | 0.99 | Discount factor |
| $\sigma$ | 1 | Coefficient of relative risk aversion |
| $\varphi$ | 0.2 | The elasticity of wage with respect to hours |
| $\varepsilon_p$ | 11 | The elasticity of substitution among intermediate goods |
| $\varepsilon_w$ | 11 | The elasticity of substitution among different type of labour |
| $\bar{\pi}$ | 0.005 | Steady state inflation |
| $\phi_p$ | 0.5 | The Calvo probability that a firm does not change its price |
| $\phi_w$ | 0.2 | The Calvo probability that a household does not change its wage |
| $\xi_p$ | 1 | Nominal price indexation to lagged inflation |
| $\xi_w$ | 1 | Nominal wage indexation to lagged inflation |
| $\rho_a$ | 0.972 | Autoregressive parameter for technology shock |
| $\rho_m$ | 0.45 | Autoregressive parameter for money growth |
| $\sigma_a$ | 0.01 | Standard deviation for technology shock |
| $\sigma_m$ | 0.01 | Standard deviation for money supply shock |

Table 3-2 Estimated parameters for the model with sticky price and sticky wage

| Estimated Parameters | | Priors Type [mean, std.] | Posteriors | | | | |
|---|---|---|---|---|---|---|---|
| | | | Mode | Mean | 5% | Median | 95% |
| $\phi_p$ | Calvo parameter, prices | BETA [0.5, 0.075] | 0.4950 | 0.5668 | 0.4805 | 0.5666 | 0.6629 |
| $\phi_w$ | Calvo parameter, wages | BETA [0.5, 0.075] | 0.4950 | 0.4336 | 0.3182 | 0.4382 | 0.5373 |
| $\xi_p$ | Price indexation | BETA [0.5, 0.2] | 0.4950 | 0.6067 | 0.4185 | 0.5802 | 0.8394 |
| $\xi_w$ | Wage indexation | BETA [0.5, 0.2] | 0.4950 | 0.5501 | 0.2966 | 0.5726 | 0.7788 |
| $\rho_a$ | AR(1) technology shock | BETA [0.75, 0.1] | 0.7810 | 0.7809 | 0.6176 | 0.7887 | 0.8897 |
| $\rho_m$ | AR(1) money growth shock | BETA [0.5, 0.1] | 0.4995 | 0.5236 | 0.4306 | 0.5101 | 0.6777 |
| $\sigma_a$ | std. technology shock | INV.GAMMA [0.01, 1] | 0.0047 | 0.0080 | 0.0034 | 0.0071 | 0.0164 |
| $\sigma_m$ | std. money growth shock | INV.GAMMA [0.01, 1] | 0.0047 | 0.0071 | 0.0029 | 0.0058 | 0.0156 |

# Chapter 4 A closed economy model with bonds and capital

## 4.1 Introduction

A closed economy DSGE model without bonds and capital provides an empirical proof for the effectiveness of open market operations. An increase in money can trigger nonbinding cash-in-advance (CIA) constraints which can contribute to the stimulation of the economy when money is the only asset, especially for the period during the quantitative easing (QE) programs. However, the model fails to explain why there is a persistent liquidity trap and why there are disappointing levels of investment and of output growth after the Great Recession. Can increased money holdings crowd out physical investment and depress output growth? When other assets, such as bonds and capital, are introduced, money can serve as a safer store of value with positive returns when the CIA constraint is nonbinding and nominal interest rate hit its zero bound. Excess money holdings are saved rather than spent since the other two assets cannot provide liquidity premium. As long as the central banks stick to the zero or nearly zero policy rate, the agents will continue expecting lower inflation or even deflation regardless of the QE programs. In this case, the economic growth is limited, and the economy tends to be stuck in the liquidity trap. A closed economy model with bonds and capital is discussed in this chapter.

Capital is assumed to partially or fully subject to the CIA constraint to bring more frictions into the model. If the CIA constraint is only assumed on consumption and is always binding, monetary shocks have no persistent effects on output as investment can be very volatile by working as a buffer for consumption. A binding CIA constraint distorts the labour-leisure choice as in Chapter 3 but has no distortion on the investment decisions. If the CIA constraint is imposed on aggregate spending (consumption plus investment), the consumers are forced to intertemporally smooth aggregate spending via real money balance accumulation over time, which can generate a hump-shaped output persistence (see Wang and Wen, 2005; Auray and Blas, 2007; Akay, 2010). Nominal interest rate is introduced via the one-period nominal bonds, which gives standard Taylor rule and ZLB constraint a chance to play their roles. Svensson (1985) extends Lucas (1978, 1982)'s general equilibrium asset-pricing model and specifies that the nominal interest rate is the expected utility of the liquidity services (or the shadow price of the real balances) over the expected utility of wealth. This is the formal equation that sums up the relationship between the nominal interest rate and slackness of the CIA constraint. As is derived and illustrated in later Sections, the expectation of nominal interest rate is a function of the Lagrange multiplier of the CIA constraint. Whether the CIA constraint is binding or not depends on whether the nominal interest rate is expected to be zero or not, meanwhile the relaxing of the CIA constraints has an influence on the expectation of nominal interest rate.

It is argued in Chapter 3 that nonbinding CIA constraints can stimulate the economy when inflation is expected to be low. This is true when money is the only asset. However, as other assets, such as bonds and capital, which can serve as a store of value when the CIA constraint is binding, are included, nonbinding CIA constraints have limited ability to stimulate the economy. Increases in output and its components are depressed, especially when the economy is already in the liquidity traps. When the shadow price of real balances is zero and the nominal interest rate is extremely low or zero, money becomes perfect substitute to other assets and can also provide a return. Money now serves as a safe store of value that stops real interest rates from falling, which depresses investment.

Two pillars of monetary rules are compared: a money growth rule (benchmark model) and an interest rate feedback rule with zero-lower bound (ZLB) constraint (alternative model). When the model is closed with a money growth rule, the main driving force for the nonbinding CIA constraint is still the expected inflation. When there is an expected deflation, money return becomes positive, and the households tend to hold more money. The CIA constraint is nonbinding, and the nominal interest rate can be or nearly be at its ZLB. When the model is closed with an interest rate feedback rule (Taylor rule, for example) with ZLB constraint, policy rate, rather than inflation, makes significant contribution to the nonbinding CIA constraint. In contrast to the results of Bhattacharjee and Thoenissen (2007) that the CIA model closed by a money growth rule comes closest to the data, the model with occasionally binding CIA constraint and ZLB constraint closed by an interest rate feedback rule fits the data better, especially for the period after the global financial crisis.

The model with both nominal and real rigidities is described in Section 4.2. Section 4.3 discusses its calibration and estimation results for both benchmark model and alternative model. Section 4.4 provides some policy implications.

## 4.2 The model

The model in this Chapter lays out and analyses a so-called medium scale New Keynesian DSGE model following Smets and Wouters (2007). This model takes a real business cycle model as its backbone and adds nominal rigidities. It furthers the model from Chapter 3 by including both bonds and capital to look at how the inclusion of real assets in the model affects the response of the economy. More importantly, whether the inclusion of alternative assets will have impact on occasionally binding cash-in-advance constraint will give better intuition for the central banks to conduct monetary policies.

The features of the model can be summarised as follows. There is a labour packer who combines heterogenous labour inputs into a homogeneous labour capital used in firm's productions. The households consume, supply labour, invest in physical capital, make capital utilization decisions, lease capital to firms, set wages following the downward-sloping demand function and accumulate bonds. A continuum of intermediate goods firms uses both capital services and labour to produce differentiated goods. A representative final good firm then bundles these heterogeneous goods of intermediate goods firms into a final good that can be consumed by the households. The monetary policies are conduct by a central bank according to a money growth rule or a Taylor rule. The government chooses its money supply and finances this with a mix of lump sum transfers and debt.

### 4.2.1 Labour Packer

The nominal rigidities in wages are introduced by the same way in Chapter 3. The main assumption is that there is a continuum of households, indexed by $i \in [0,1]$. Each household supplies differentiated labour into labour packing firm (or a labour packer), who then bundles the differentiated labour capital into a homogeneous labour capital available for production, $h_t^d$. The labour bundling technology can be, thus, be expressed as

$$h_t^d = \left(\int_0^1 \left(h_t(i)\right)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} di\right)^{\frac{\varepsilon_w}{\varepsilon_w - 1}}, \varepsilon_w > 1 \tag{4.1}$$

Where $\varepsilon_w$ measures the elasticity of substitution among different types of labour. This elasticity of substitution is assumed to be greater than one, meaning that different types of labour are substitutes. The profit maximization problem is as follows

$$\max_{h_t(i)} W_t(\int_0^1 \left(h_t(i)\right)^{\frac{\varepsilon_w - 1}{\varepsilon_w}} di)^{\frac{\varepsilon_w}{\varepsilon_w - 1}} - \int_0^1 W_t(i) h_t(i) di$$

Here $W_t$ is the aggregate nominal wage, while $W_t(i)$ denotes the nominal wage of labour variety $i$. A downward-sloping demand for each variety of labour $i$ is obtained by solving this profit maximization problem

$$h_t(i) = \left(\frac{W_t(i)}{W_t}\right)^{-\varepsilon_w} h_t^d \tag{4.2}$$

Labour demand for each individual variety is a function of aggregate demand for labour and the relative wage of the variety. Since demand for labour of variety $i$ depends on the relative wage, this specification is the same in terms of the real wage or the nominal wage ratio.

If the aggregate wage index can be defined as

$$W_t h_t^d = \int_0^1 W_t(i) h_t(i) di \tag{4.3}$$

Given the above demand function, the aggregate wage index can be simplified as

$$W_t^{1-\varepsilon_w} = \int_0^1 (W_t(i))^{1-\varepsilon_w} di \tag{4.4}$$

Define the aggregate labour supply to be equal to the sum of labour by variety

$$h_t = h_t^d v_t^w \tag{4.5}$$

Here, $v_t^w = \int_0^1 \left(\frac{W_t(i)}{W_t}\right)^{-\varepsilon_w} di$ measures the wage dispersion across different varieties of labour. It is assumed to be greater than one so that the aggregate labour used in production is always smaller than the aggregate labour supplied by individuals.

## 4.2.2 Firm Problem

The production is made of two sectors, one is intermediate goods sector and the other one is the final goods sector. The final goods sector is assumed to be perfectly competitive while the intermediate goods sector is assumed to be imperfectly competitive. The final goods sector can be seen as a representative firm, which only bundles intermediate goods into final goods without using any inputs for production. The intermediate goods sector consists of a continuum of firms indexed by $j \in [0,1]$ producing differentiated goods $y_t(j)$ at price $P_t(j)$. These intermediate goods firms hire labour $h_t^d(j)$ and use capital services $\hat{k}_t(j)$ for production. All differentiated goods also populate the unit interval as in Chapter 3.

### 4.2.2.1 Final Goods Firm

The differentiated intermediate goods are bundled into a final output via a production function

$$y_t = \left[\int_0^1 y_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} dj\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \tag{4.6}$$

Where $\varepsilon_p > 1$ is the elasticity of substitution among intermediates. The profit maximization problem can then be written as

$$\max_{y_t(i)} \Pi_t = P_t \, [\int_0^1 y_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} dj]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} - \int_0^1 P_t(j) \, y_t(j) dj$$

The relative demand curve for each intermediate good is derived from the first order condition of the profit maximization problem

$$y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} y_t \tag{4.7}$$

Since the final goods market is perfectly competitive, the economic profit of the firm is always zero. The aggregate price index is given by

$$P_t = \left[\int_0^1 P_t(j)^{1-\varepsilon_p} dj\right]^{\frac{1}{1-\varepsilon_p}} \tag{4.8}$$

### 4.2.2.2 Intermediate Goods Firms

The production function of intermediate goods is given by

$$y_t(j) = A_t \hat{k}_t(j)^\alpha h_t^d(j)^{1-\alpha} - F \tag{4.9}$$

$F \geq 0$ is a fixed cost, which is set to ensure that profits are zero at steady state so that free entry is ruled out. It is assumed to be zero here to exclude the effect of fixed cost on the equilibrium. $\hat{k}_t(j)$ is capital service, which is the product of utilization and physical capital leased from households:

$$\hat{k}_t(j) = u_t k_{t-1}(j) \tag{4.10}$$

$A_t$ is aggregate technology and is assumed to be common for all intermediate goods firms. The log productivity is represented by an exogenous, stochastic AR(1) process:

$$a_t = \rho_a a_{t-1} + u_{at} \tag{4.11}$$

Here, $|\rho_a| < 1$ and $u_{at}$ is a white noise technical innovation.

Intermediate firms may not have chance to update their price in a given period, but they will always choose their inputs to minimize the total cost each period. The cost-minimization problem can be summarised as

$$\min_{\hat{k}_t(j), h_t^d(j)} W_t h_d^t(j) + r_t^n \hat{k}_t(j)$$

*s.t.*

$$A_t \hat{k}_t(j)^\alpha h_t^d(j)^{1-\alpha} - F \geq y_t(j)$$

$$y_t(j) = (\frac{P_t(j)}{P_t})^{-\varepsilon_p} y_t$$

The firms' optimization is

$$\mathcal{L} = -W_t h_d^t(j) - r_t^n \hat{k}_t(j) + \gamma_t(j)(A_t \hat{k}_t(j)^\alpha h_t^d(j)^{1-\alpha} - F - \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} y_t)$$

The first order conditions with respect to $h_d^t(j)$ and $\hat{k}_t(j)$ are

$$W_t = (1-\alpha)\gamma_t(j) A_t (\frac{\hat{k}_t(j)}{h_t^d(j)})^\alpha$$

$$r_t^n = \alpha \gamma_t(j) A_t (\frac{\hat{k}_t(j)}{h_t^d(j)})^{\alpha-1}$$

Here, $\gamma_t(j)$ is the marginal cost for firm $j$.

Divide these two conditions

$$\frac{W_t}{r_t^n} = \frac{1-\alpha}{\alpha} (\frac{\hat{k}_t(j)}{h_t^d(j)})$$

Since firms face the same factor prices, $W_t$ and $r_t^n$, the above equation shows that all firms must have the same capital/labour ratio, which will in turn equal to the aggregate ratio of these two factors. The factor price ratio equation can be rewritten in real terms and the subscripts $j$ can be dropped:

$$\frac{w_t}{r_t} = \frac{1-\alpha}{\alpha} \left(\frac{\hat{k}_t}{h_t^d}\right) \tag{4.12}$$

Here, $w_t = \frac{W_t}{P_t}$ is the real wage and $r_t = \frac{r_t^n}{P_t}$ is the real return on capital. $\hat{k}_t = \int_0^1 \hat{k}_t(j) dj$ is the aggregate capital services used and $h_t^d = \int_0^1 h_t^d(j)\, dj$ is the aggregate labour inputs since all intermediate goods produced populate the unit interval.

If the factor prices faced by firms are the same, capital and labour are hired in the same ratio, and the productivity shock is the same among all firms, all intermediate firms will have the same marginal cost. The marginal cost can be expressed in terms of labour demand condition as:

$$mc_t = \frac{w_t}{(1-\alpha)A_t\left(\frac{\hat{k}_t}{h_t^d}\right)^{\alpha}} \tag{4.13}$$

$mc_t = \frac{\gamma_t}{P_t}$ is real marginal cost which is the same across firms.

The nominal profits of firm $j$ are

$$\Pi_t^n(j) = P_t(j)y_t(j) - W_t h_t^d(j) - r_t^n \hat{k}_t(j)$$

From the above first order conditions

$$W_t h_t^d(j) = (1-\alpha)\gamma_t A_t \hat{k}_t(j)^{\alpha} h_t^d(j)^{1-\alpha}$$

$$r_t^n \hat{k}_t(j) = \alpha\gamma_t A_t \hat{k}_t(j)^{\alpha} h_t^d(j)^{1-\alpha}$$

Here, all intermediate firms having the same nominal marginal cost $\gamma_t$ is assumed. The sum of these is then

$$W_t h_t^d(j) + r_t^n \hat{k}_t(j) = \gamma_t A_t \hat{k}_t(j)^{\alpha} h_t^d(j)^{1-\alpha}$$

Now substitute the production function in, the sum of firm's cost can be written as:

$$W_t h_t^d(j) + r_t^n \hat{k}_t(j) = \gamma_t y_t(j) + \gamma_t F$$

In this sense, the profit function is rewritten as:

$$\Pi_t^n(j) = P_t(j)y_t(j) - \gamma_t y_t(j) - \gamma_t F$$

Firms are also subject to Calvo price-setting as in Chapter 3. there is a fixed probability of $1-\phi_p$ that a firm can update its price. If the firm does not have the chance to change the price, it can partially index its price to lagged aggregate inflation at $\xi_p \in [0,1]$. Thus, the price a firm can charge in period $t$ is

$$P_t(j) = \begin{cases} P_t^{\#}(j) \\ (1+\pi_{t-1})^{\xi_p} P_{t-1}(j) \end{cases}$$

$P_t^{\#}(j)$ is the optimal price that the intermediate goods firm $j$ resets in period $t$. If a firm cannot update its price again in period $t+s$ but has reset its price in period $t$, the price it will charge in period $t+s$ is a product of gross inflation rates:

$$P_{t+s}(j) = (\frac{P_{t+s-1}}{P_{t-1}})^{\xi_p} {P_t}^{\#}(j)$$

The profit maximization problem of an intermediate firm which is given the opportunity to adjust its price in period $t$ is dynamic because the price it sets in period $t$ will be relevant to future periods in expectations. The firm will discount its profit flows by a nominal stochastic discount factor, equal to $\beta^s U'(c_t)$, as well as the probability that a price set today is still in effect later, ${\phi_p}^s$

$$\max_{{P_t}^{\#}(j)} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left[ \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_t}^{\#}(j)^{1-\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s} - \gamma_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_t}^{\#}(j)^{-\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s} \right]$$

The first order condition in terms of ${P_t}^{\#}(j)$ is

$$(\varepsilon_p - 1){P_t}^{\#}(j)^{-\varepsilon_p} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_{t+s}}^{\varepsilon_p} y_{t+s}$$

$$= \varepsilon_p {P_t}^{\#}(j)^{-\varepsilon_p - 1} E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \gamma_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s}$$

It can be re-arranged as

$${P_t}^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) mc_{t+s} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} {P_{t+s}}^{\varepsilon_p} y_{t+s}}{E_t \sum_{s=0}^{\infty} \left(\beta\phi_p\right)^s U'(c_{t+s}) \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} {P_{t+s}}^{\varepsilon_p - 1} y_{t+s}}$$

Then the price-setting condition can be expressed recursively as

$${P_t}^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} \frac{X_{1t}}{X_{2t}}$$

Where

$$X_{1t} = U'(c_t) mc_t {P_t}^{\varepsilon_p} y_t + \beta\phi_p \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_p\varepsilon_p} E_t X_{1t+1}$$

$$X_{2t} = U'(c_t){P_t}^{\varepsilon_p - 1}y_t + \beta\phi_p\left(\frac{P_t}{P_{t-1}}\right)^{\xi_p(1-\varepsilon_p)} E_t X_{2t+1}$$

Define $x_{1,t} = \frac{X_{1,t}}{{P_t}^{\varepsilon_p}}$, $x_{2,t} = \frac{X_{2,t}}{{P_t}^{\varepsilon_p - 1}}$, and the gross consumer price inflation $1 + \pi_t = \frac{P_t}{P_{t-1}}$, the above conditions can be written in real terms as

$$x_{1t} = \frac{X_{1t}}{{P_t}^{\varepsilon_p}} = U'(c_t)mc_t y_t + \beta\phi_p(1+\pi_t)^{-\xi_p\varepsilon_p} E_t \frac{X_{1t+1}}{{P_t}^{\varepsilon_p}}$$

$$= U'(c_t)mc_t y_t + \beta\phi_p(1+\pi_t)^{-\xi_p\varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1t+1}$$

$$x_{2t} = \frac{X_{2t}}{{P_t}^{\varepsilon_p - 1}} = U'(c_t)y_t + \beta\phi_p(1+\pi_t)^{\xi_p(1-\varepsilon_p)} E_t \frac{X_{2t+1}}{{P_t}^{\varepsilon_p - 1}}$$

$$= U'(c_t)y_t + \beta\phi_p(1+\pi_t)^{\xi_p(1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2t+1}$$

The reset price can be represented as

$${P_t}^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1}\frac{X_{1,t}/{P_t}^{\varepsilon_p}}{X_{2,t}/{P_t}^{\varepsilon_p - 1}}\frac{{P_t}^{\varepsilon_p}}{{P_t}^{\varepsilon_p - 1}} = \frac{\varepsilon_p}{\varepsilon_p - 1}\frac{x_{1,t}}{x_{2,t}}P_t$$

$$x_{1,t} = {c_t}^{-\sigma} mc_t y_t + \beta\phi_p(1+\pi_t)^{-\xi_p\varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1,t+1}$$

$$x_{2,t} = {c_t}^{-\sigma} y_t + \beta\phi_p(1+\pi_t)^{\xi_p(1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2,t+1}$$

If the reset price inflation is defined as $1 + {\pi_t}^{\#} = \frac{{P_t}^{\#}}{P_{t-1}}$, the optimal price-setting conditions can be summarised as

$$1 + \pi_t^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1}(1+\pi_t)\frac{x_{1t}}{x_{2t}} \tag{4.14}$$

$$x_{1t} = {c_t}^{-\sigma} mc_t y_t + \beta\phi_p(1+\pi_t)^{-\xi_p\varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1t+1} \tag{4.15}$$

$$x_{2t} = {c_t}^{-\sigma} y_t + \beta\phi_p(1+\pi_t)^{\xi_p(1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2t+1} \tag{4.16}$$

### 4.2.3 Household Problem

There is a continuum of households indexed by $i \in [0,1]$ which are subject to a Calvo-style wage-setting friction. This means these households will have heterogeneous wages and supply heterogeneous labour inputs and hence income. The existence of state-contingent securities is assumed to insure households against idiosyncratic wage risks. If preferences are assumed to be separable between consumption and leisure, the households will then be identical among their decisions on consumption, capital accumulation, capital utilization, and bond-holdings but will be different in wages charged and labour inputs supplied (Erceg, Henderson and Levin, 2000). Since utility is separable between consumption and labour, all households will be identical among all margins except labour inputs and wages. Here, the dependence on $i$ is suppressed with the exception of those two margins. In this model, the assumption that money is the only asset is relaxed. It is assumed that the capital stock is owned by households. They can choose how intensively to utilize their capital stock, and then lease capital services $\hat{k}_t$ which is the product of physical capital $k_{t-1}$ and utilization $u_t$ to firms. The capital accumulation equation can be written as

$$k_t = Z_t\left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right)I_t + \left(1 - \delta(u_t)\right)k_{t-1} \tag{4.17}$$

$$\delta(u_t) = \delta_0 + \delta_1(u_t - 1) + \frac{\delta_2}{2}(u_t - 1)^2 \tag{4.18}$$

Here, $\tau$ is a measure of the investment adjustment cost. $\delta(u_t)$ is a function that maps the utilization of capital stock into depreciation rate, with parameters $\delta_0$, $\delta_1$ and $\delta_2$. $I_t$ is investment. $Z_t$ is an exogenous marginal efficiency of investment shock, with its log $z_t$ also following an AR(1) process:

$$z_t = \rho_z z_{t-1} + u_{zt} \tag{4.19}$$

The household can also choose how many nominal government bonds they are going to accumulate. To sum up, an individual household makes decision on consumption, labour inputs, investment on capital stocks, money and nominal bond holdings to maximise his/her lifetime utility with money playing its role via the CIA constraint. The household utility function is described as

$$E_t \sum_{t=0}^{\infty} \beta^t U(c_t, h_t(i))$$

Rational expectation operator $E_t$ is conditional on all information available at time $t$. $\beta \in (0,1)$ is the discount factor. $U$ is a strictly concave function and takes the following form

$$U\big(c_t, h_t(i)\big) = \frac{{c_t}^{1-\sigma}}{1-\sigma} - \frac{(h_t(i))^{1+\varphi}}{1+\varphi} \tag{4.20}$$

$\sigma > 0$ is the coefficients of relative risk aversion for consumption and $\varphi > 0$ is the inverse Frisch elasticity of labour supply. The consumption good is a CES aggregator of a continuum of goods produced by differentiated intermediated goods firms, $c_t(j), j \in [0,1]$:

$$c_t = \left[\int_0^1 c_t(j)^{\frac{\varepsilon_p - 1}{\varepsilon_p}} di\right]^{\frac{\varepsilon_p}{\varepsilon_p - 1}} \tag{4.21}$$

$\varepsilon_p > 1$ is a parameter for the intratemporal elasticity of substitution across differentiated goods. Thus, the optimal level of differentiated goods is solved by minimizing total expenditure, $\int_0^1 P_t(j)\, c_t(j) dj = P_t c_t$, subject to the above CES aggregator:

$$c_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} c_t \tag{4.22}$$

The timing is important for a cash-in-advance economy. Here, time is discrete and indexed by $t = 1, 2, 3, \ldots$. The timing of production and consumption follows that of the cash-in-advance economy set up by Svensson (1985). Households must choose their cash holdings before they know the current state of the economy. They then purchase consumption goods. After the goods market closes, they receive a lump sum transfer and make arrangements of their portfolio. This timing mechanism ensures money demand to embody three motives, that is, the transactions motives, precautionary motives and store-of-value functions. The timing of transactions within period $t$ in this model can be described as follows. The consumer enters period $t$ with $M_{t-1}$ units of currency and $B_{t-1}$ one-period nominal bonds with a promised return. The asset market opens, on which the consumer can exchange money, nominal bonds, and make their investment decisions. Then the consumer learns the current period shocks and receives a cash transfer from the government. The asset market closes and the labour market where the consumer supplies labour to firms opens. The goods market opens, and consumers purchase consumption goods with their cash holdings. Finally, the goods market closes and consumers receive their labour earnings in cash. Hence, the budget constraint of a typical household can be expressed as

$$P_t c_t + M_t + P_t I_t + B_t$$

$$= W_t(i)h_t(i) + M_{t-1} + (1 + i_{t-1})B_{t-1} + r_t^n u_t k_{t-1} + \Pi_t^n + P_t s_t$$

$P_t$ is the nominal price of goods. $r_t^n$ is the nominal rental rate on capital services. $\Pi_t^n$ is nominal profit distributed from monopolistic firms. $s_t$ is a real lump sum transfer from the government or a lump sum tax if $s_t < 0$. This transfer is allowed for the current consumption. Abel (1985) finds that whether the transfer can be used for consumption in current period or not has no impact on the dynamics. $W_t(i)$ is the nominal wage charged by household $i$. $i_t$ is the nominal interest rate and $B_t$ is the stock of one-period nominal government bonds with which a household enters period $t + 1$. In this economy, bonds are assumed in zero net supply. That is, $B_t = 0$ in equilibrium.

Define the real money balances $m_t = \frac{M_t}{P_t}$, the real bond holdings $b_t = \frac{B_t}{P_t}$, the real return rate on capital services $r_t = \frac{r_t^n}{P_t}$, and the real profits $\Pi_t = \frac{\Pi_t^n}{P_t}$, given the gross inflation $1 + \pi_t = \frac{P_t}{P_{t-1}}$, the above budget constraint in real terms is

$$c_t + m_t + I_t + b_t$$
$$= \frac{W_t(i)h_t(i)}{P_t} + \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + r_t u_t k_{t-1} + \Pi_t + s_t \qquad (4.23)$$

The households are also subject to a cash-in-advance constraint. This is the constraint that the consumer must finance her consumption purchases, nominal bonds purchases and capital investment from the asset stocks that she enters the period with

$$P_t c_t + \phi_I P_t I_t + B_t \leq M_{t-1} + (1 + i_{t-1}) B_{t-1} + P_t s_t$$

This constraint can also be rewritten in real terms as

$$c_t + \phi_I I_t + b_t \leq \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t \qquad (4.24)$$

$\phi_I \in [0, 1]$ controls the fraction of investment which is subject to the cash-in-advance constraint[22]. The consideration of both consumption and investment as cash goods is more consistent with the estimation of the function of aggregate money demand[23]. When $\phi_I = 0$, consumption is the only cash goods, then the model is reduced to that of Yun (1996) and Ellison and Scott (2000)[24]. When $\phi_I = 1$, investment is fully financed with cash[25]. This fraction may be more substantial in developing economies which may not have a well-functioning credit market[26]. Here, investment is not assumed to be fully

---

[22] Christiano, Eichenbaum, and Evans (2005) show that a sticky-price and sticky-wage model with investment in the CIA constraint can generate enough persistence in output and inflation without price indexation. Wang and Wen (2006) find that the inclusion of investment as a cash good in a sticky-price model is crucial to generate enough persistence in output that is otherwise missing in a standard sticky price model. This result is also confirmed by Auray and Blas (2013).

[23] Mulligan and Sala-I-Martin (1992), for example, find that aggregate income is better than aggregate consumption as a scale variable in estimating money demand.

[24] Hansen (1996) suggests that when all investment is subject to the CIA constraint, the real effects of higher money growth can be quite significant. An alternative of the model would be imposing a CIA constraint on investment of individual firms instead of on the aggregate investment. Chu and Cozzi (2014) consider a Schumpeterian growth model with CIA constraints on R&D investment, consumption and manufacturing.

[25] See Wang and Wen (2006) and Auray and Blas (2007) for example.

[26] Worthington (1995) find that the elasticity of investment to cash flow is between 0.2 and 0.65. Wang and Wen (2006) find that output shows a hump-shaped response pattern when only 30% of aggregate investment is subject to a CIA constraint. For the US and most OECD countries, $\phi_I$ is probably close to zero. Thus, in a deterministic world, sustained inflation

financed by cash, which shows some illiquidity of investment. When the CIA constraint works as a tax on consumption, household cannot fully substitute from consumption to investment. To keep it simple, $\phi_I$ is abstract from the fact that itself may be endogenous[27].

The Lagrange of household problem is

$$\begin{aligned}\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \Bigg\{ & \left( \frac{{c_t}^{1-\sigma}}{1-\sigma} - \frac{(h_t(i))^{1+\varphi}}{1+\varphi} \right) \\ & - \lambda_t \left( c_t + m_t + I_t + b_t - \frac{W_t(i) h_t(i)}{P_t} - \frac{m_{t-1}}{1+\pi_t} - \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} \right. \\ & \left. - r_t u_t k_{t-1} - \Pi_t - s_t \right) - \mu_t \left[ k_t - Z_t \left( 1 - \frac{\tau}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 \right) I_t \right. \\ & \left. - \left( 1 - \delta_0 - \delta_1 (u_t - 1) - \frac{\delta_2}{2} (u_t - 1)^2 \right) k_{t-1} \right] \\ & - \eta_t \left( c_t + \phi_I I_t + b_t - \frac{m_{t-1}}{1+\pi_t} - \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} - s_t \right) \Bigg\}\end{aligned}$$

The household maximizes her utility subject to the budget constraint and the CIA constraint by choosing the paths of $c_t$, $h_t(i)$, $\frac{W_t(i)}{P_t}$, $m_t$, $b_t$, $I_t$, $u_t$ and $k_t$. $\lambda_t$ is the Lagrange multiplier associated with the budget constraint meaning the shadow price of the budget constraint. $\mu_t$ is the Lagrange multiplier associated with the capital accumulation condition. $\eta_t$ represents the Lagrange multiplier associated with the CIA constraint, i.e. the shadow price of the CIA constraint. The first order conditions with respect to $c_t$, $m_t$, $b_t$, $I_t$, $u_t$ and $k_t$ are summarised as follows. The first order condition of differentiated labour will be illustrated in sticky wage problem later.

$$c_t^{-\sigma} = (\lambda_t + \eta_t) \tag{4.25}$$

$$\lambda_t = \beta E_t \left\{ \left( \frac{\lambda_{t+1}}{1+\pi_{t+1}} + \frac{\eta_{t+1}}{1+\pi_{t+1}} \right) \right\} \tag{4.26}$$

---

should not be expected to generate significant growth. The results in Chari et al. (1996) are underpinned by this relationship.

[27] The degree of financial sophistication may be contingent on inflation. See Bencivenga and Smith (1992), Ireland (1994), and Boyd and Smith (1998) for example.

$$\lambda_t + \eta_t = \beta E_t \left\{ \lambda_{t+1} \frac{(1+i_t)}{1+\pi_{t+1}} + \eta_{t+1} \frac{(1+i_t)}{1+\pi_{t+1}} \right\} \tag{4.27}$$

$$\lambda_t + \eta_t \phi_I = \mu_t Z_t \left( 1 - \frac{\tau}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 - \tau \left( \frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right)$$
$$+ \beta E_t \left\{ \mu_{t+1} Z_{t+1} \tau \left( \frac{I_{t+1}}{I_t} - 1 \right) \left( \frac{I_{t+1}}{I_t} \right)^2 \right\} \tag{4.28}$$

$$\lambda_t r_t = \mu_t \left( \delta_1 + \delta_2 (u_t - 1) \right) \tag{4.29}$$

$$\mu_t = \beta E_t \left\{ \begin{matrix} \lambda_{t+1} r_{t+1} u_{t+1} + \\ \mu_{t+1} \left( 1 - \delta_0 - \delta_1 (u_{t+1} - 1) - \frac{\delta_2}{2} (u_{t+1} - 1)^2 \right) \end{matrix} \right\} \tag{4.30}$$

$$\eta_t \geq 0 \tag{4.31}$$

$$\eta_t \left( \left( c_t + \phi_I I_t + b_t - \frac{m_{t-1}}{1+\pi_t} - \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} - s_t \right) \right) = 0 \tag{4.32}$$

The first order conditions for $c_t$ and $m_t$ give the Euler equation for money

$$c_t^{-\sigma} - \eta_t = \beta E_t \left\{ \frac{c_{t+1}^{-\sigma}}{1+\pi_{t+1}} \right\} \tag{4.33}$$

The first order conditions for $c_t$ and $b_t$ combine for the Euler equation for bonds

$$\beta E_t \left\{ \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{1+i_t}{1+\pi_{t+1}} \right\} = 1 \tag{4.34}$$

As before, whether $\eta_t$ is zero or not is crucial for endogenous occasionally binding CIA constraints. When $\eta_t = 0$, the expected nominal interest rate $i_t$ is also equal to zero, and then the CIA constraint is nonbinding. When $\eta_t > 0$, the expected nominal interest rate $i_t$ is greater than zero, and then the CIA constraint binds. The interaction of these two Euler equations are discussed in later Section[28].

[28] Monnet and Weber (2001) summarise two contradictory views of the relationship between money supply and changes in interest rate. One view is called the liquidity effect view, which are used to analyse the effect of unexpected changes in money growth. According to this view, money demand is a decreasing function of the nominal interest rate as the interest rate is the opportunity cost of holding money. Another view, following from the Fisher equation, is that money demand is positively related to the interest rate.

Next, consider the Lagrange related to Calvo-style wage-setting. Each period, a fraction, $1-\phi_w \in [0,1]$, of households is randomly selected and is given the chance to update their wages. This means that the other fraction, $\phi_w$, of households cannot update their wages. It is also assumed that the part of households who have no chance to adjust their wages can index their nominal wages at a rate $\xi_w \in [0,1]$ to lagged inflation. Hence, the nominal wage of a household $i$ in period $t$ can be described as

$$W_t(i) = \begin{cases} W_t{}^{\#}(i) \\ (1+\pi_{t-1})^{\xi_w} W_{t-1}(i) \end{cases}$$

$W_t{}^{\#}(i)$ is the optimal wage reset by household $i$ during period $t$. If the household cannot adjust her wage again in period $t+s$ but has reset her wage in period $t$, the wage she will charge in period $t+s$ is

$$W_{t+s}(i) = (\frac{P_{t+s-1}}{P_{t-1}})^{\xi_w} W_t{}^{\#}(i)$$

The Lagrange problem related to wage-setting for a household who is given the chance to adjust her wage can be recreated by discounting period $t+s$ by $(\beta\phi_w)^s$. This means that the probability that a wage set in period $t$ is still in effect in $t+s$ is $\phi_w{}^s$.

$$\mathcal{L} = E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \{-\frac{1}{1+\varphi}\left(\frac{\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w} W_t{}^{\#}(i)}{W_{t+s}}\right)^{-\varepsilon_w(1+\varphi)} h_{t+s}^{d}{}^{1+\varphi}$$
$$+\frac{\lambda_{t+s}}{P_{t+s}}((\frac{P_{t+s-1}}{P_{t-1}})^{\xi_w} W_t{}^{\#}(i))^{1-\varepsilon_w} W_{t+s}{}^{\varepsilon_w} h_{t+s}^{d}\}$$

The first order condition with respect to $W_t{}^{\#}(i)$ is

$$\varepsilon_w W_t{}^{\#}(i)^{-\varepsilon_w(1+\varphi)-1} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} W_{t+s}{}^{\varepsilon_w(1+\varphi)}\, h_{t+s}^{d}{}^{1+\varphi}$$

$$+(1-\varepsilon_w) W_t{}^{\#}(i)^{-\varepsilon_w} E_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}} \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} W_{t+s}{}^{\varepsilon_w} h_{t+s}^{d} = 0$$

Simplifying further,

$$W_t^{\#}(i)^{1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {W_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}}{E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d}$$

Since nothing on the right-hand side depends on the subscript $i$, all households who have the opportunity to update wage will charge a common reset wage. This wage setting condition can be written as

$${W_t^{\#}}^{\varepsilon_w\varphi+1} = \frac{\varepsilon_w}{\varepsilon_w - 1}\frac{N_{1,t}}{N_{2,t}}$$

$$N_{1,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {W_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}$$

$$N_{2,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \frac{\lambda_{t+s}}{P_{t+s}}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {W_{t+s}}^{\varepsilon_w} h_{t+s}^d$$

Define $w_t = \frac{W_t}{P_t}$, $N_{1,t}$ and $N_{2,t}$ can be written in real terms as

$$N_{1,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} {w_{t+s}}^{\varepsilon_w(1+\varphi)} {P_{t+s}}^{\varepsilon_w(1+\varphi)} {h_{t+s}^d}^{1+\varphi}$$

$$N_{2,t} = E_t \sum_{s=0}^{\infty}(\beta\phi_w)^s \lambda_{t+s}\left(\frac{P_{t+s-1}}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} {w_{t+s}}^{\varepsilon_w} {P_{t+s}}^{\varepsilon_w-1} h_{t+s}^d$$

Then write $N_{1,t}$ and $N_{2,t}$ recursively

$$N_{1,t} = {w_t}^{\varepsilon_w(1+\varphi)} {P_t}^{\varepsilon_w(1+\varphi)} {h_t^d}^{1+\varphi} + \beta\phi_w E_t \left(\frac{P_t}{P_{t-1}}\right)^{-\xi_w\varepsilon_w(1+\varphi)} N_{1,t+1}$$

$$N_{2,t} = \lambda_t {w_t}^{\varepsilon_w} {P_t}^{\varepsilon_w-1} h_t^d + \beta\phi_w E_t \left(\frac{P_t}{P_{t-1}}\right)^{\xi_w(1-\varepsilon_w)} N_{2,t+1}$$

Define $n_{1,t} = \frac{N_{1,t}}{{P_t}^{\varepsilon_w(1+\varphi)}}$ and $n_{2,t} = \frac{N_{2,t}}{{P_t}^{\varepsilon_w-1}}$,

$$n_{1,t} = w_t{}^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi} + \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t \frac{N_{1,t+1}}{P_t{}^{\varepsilon_w(1+\varphi)}}$$

$$= w_t{}^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t \frac{N_{1,t+1}}{P_{t+1}{}^{\varepsilon_w(1+\varphi)}} \frac{P_{t+1}{}^{\varepsilon_w(1+\varphi)}}{P_t{}^{\varepsilon_w(1+\varphi)}}$$

$$= w_t{}^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t(1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} n_{1,t+1}$$

$$n_{2,t} = \lambda_t w_t{}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \frac{N_{2,t+1}}{P_t{}^{\varepsilon_w-1}}$$

$$= \lambda_t w_t{}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \frac{N_{2,t+1}}{P_{t+1}{}^{\varepsilon_w-1}} \frac{P_{t+1}{}^{\varepsilon_w-1}}{P_t{}^{\varepsilon_w-1}}$$

$$= \lambda_t w_t{}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} n_{2,t+1}$$

Hence, the reset wage condition is

$$W_t{}^{\#\,1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{\frac{N_{1,t}}{P_t{}^{\varepsilon_w(1+\varphi)}}}{\frac{N_{2,t}}{P_t{}^{\varepsilon_w-1}}} \frac{P_t{}^{\varepsilon_w(1+\varphi)}}{P_t{}^{\varepsilon_w-1}} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{n_{1,t}}{n_{2,t}} P_t{}^{1+\varepsilon_w\varphi}$$

Define the real reset wage as $w_t{}^{\#} = \frac{W_t{}^{\#}}{P_t}$, this condition can then be written

$$w_t{}^{\#\,1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w-1} \frac{n_{1,t}}{n_{2,t}}$$

$$n_{1,t} = w_t{}^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi} + \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t(1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} n_{1,t+1}$$

$$n_{2,t} = \lambda_t w_t{}^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} n_{2,t+1}$$

For the purpose of steady state calculation, these conditions can be rewritten in terms of relative reset wage. Define $\hat{n}_{1,t} = \frac{n_{1,t}}{w_t{}^{\#\,\varepsilon_w(1+\varphi)}}$ and $\hat{n}_{2,t} = \frac{n_{2,t}}{w_t{}^{\#\,\varepsilon_w}}$

$$\hat{n}_{1,t} = (\frac{w_t}{w_t{}^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t(1+\pi_{t+1})^{\varepsilon_w(1+\varphi)} \frac{n_{1,t+1}}{w_t{}^{\#\varepsilon_w(1+\varphi)}}$$

$$= (\frac{w_t}{w_t{}^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t(1$$

$$+ \pi_{t+1})^{\varepsilon_w(1+\varphi)} \frac{n_{1,t+1}}{w_{t+1}{}^{\#\varepsilon_w(1+\varphi)}} \frac{w_{t+1}{}^{\#\varepsilon_w(1+\varphi)}}{w_t{}^{\#\varepsilon_w(1+\varphi)}}$$

$$= (\frac{w_t}{w_t{}^{\#}})^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+ \beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)} E_t[(1$$

$$+ \pi_{t+1})^{\varepsilon_w(1+\varphi)} \left(\frac{w_{t+1}{}^{\#}}{w_t{}^{\#}}\right)^{\varepsilon_w(1+\varphi)} \hat{n}_{1,t+1}$$

$$\hat{n}_{2,t} = \lambda_t \left(\frac{w_t}{w_t{}^{\#}}\right)^{\varepsilon_w} h_t^d + \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} \frac{n_{2,t+1}}{w_t{}^{\#\varepsilon_w}}$$

$$= \lambda_t \left(\frac{w_t}{w_t{}^{\#}}\right)^{\varepsilon_w} h_t^d$$

$$+ \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t(1+\pi_{t+1})^{\varepsilon_w-1} \frac{n_{2,t+1}}{w_{t+1}{}^{\#\varepsilon_w}} \frac{w_{t+1}{}^{\#\varepsilon_w}}{w_t{}^{\#\varepsilon_w}}$$

$$= \lambda_t (\frac{w_t}{w_t{}^{\#}})^{\varepsilon_w} h_t^d$$

$$+ \beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)} E_t \left[(1+\pi_{t+1})^{\varepsilon_w-1} (\frac{w_{t+1}{}^{\#}}{w_t{}^{\#}})^{\varepsilon_w} \hat{n}_{2,t+1}\right]$$

Then the wage-setting expression can be written as

$$w_t{}^{\#1+\varepsilon_w\varphi} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\frac{n_{1,t}}{w_t{}^{\#\varepsilon_w(1+\varphi)}}}{\frac{n_{2,t}}{w_t{}^{\#\varepsilon_w}}} \frac{w_t{}^{\#\varepsilon_w(1+\varphi)}}{w_t{}^{\#\varepsilon_w}} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}} w_t{}^{\#\varepsilon_w\varphi}$$

More compactly,

$$w_t{}^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}}$$

Finally, the optimal wage-setting conditions recursively to this problem are

$$w_t{}^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1,t}}{\hat{n}_{2,t}} \tag{4.35}$$

$$\hat{n}_{1,t} = \left(\frac{w_t}{w_t{}^{\#}}\right)^{\varepsilon_w(1+\varphi)} h_t^{d\,1+\varphi}$$

$$+\beta\phi_w(1+\pi_t)^{-\xi_w\varepsilon_w(1+\varphi)}E_t\left[(1+\pi_{t+1})^{\varepsilon_w(1+\varphi)}\left(\frac{w_{t+1}{}^{\#}}{w_t{}^{\#}}\right)^{\varepsilon_w(1+\varphi)}\hat{n}_{1,t+1}\right] \quad (4.36)$$

$$\hat{n}_{2,t} = \lambda_t\left(\frac{w_t}{w_t{}^{\#}}\right)^{\varepsilon_w} h_t^d$$

$$+\beta\phi_w(1+\pi_t)^{\xi_w(1-\varepsilon_w)}E_t\left[(1+\pi_{t+1})^{\varepsilon_w-1}\left(\frac{w_{t+1}{}^{\#}}{w_t{}^{\#}}\right)^{\varepsilon_w}\hat{n}_{2,t+1}\right] \quad (4.37)$$

### 4.2.4 Government

The consolidated government raises revenue via printing money or issuing new debts to finance its nominal expenditures which are lump-sum transfers to households. The budget constraint of this consolidated government is

$$\bar{M}_t + B_t = \bar{M}_{t-1} + P_t s_t + (1+i_{t-1})B_{t-1}$$

$B_{t-1}$ is the stock of nominal bonds with which the government enters the period. The government also owes interest at a nominal rate $i_{t-1}$ on that. $\bar{M}_t$ is the money supply and $s_t$ is the lump-sum transfer/tax. Hence, the government can issue new nominal bonds, $B_t - B_{t-1}$, and increase money supply, $\bar{M}_t - \bar{M}_{t-1}$ to pay its interests on debt and make lump-sum transfer to households if $s_t > 0$.

Define the real money balance $\bar{m}_t = \frac{\bar{M}_t}{P_t}$, the budget constraint in real terms is

$$\bar{m}_t + b_t = \frac{\bar{m}_{t-1}}{1+\pi_t} + s_t + \frac{1+i_{t-1}}{1+\pi_t}b_{t-1} \quad (4.38)$$

The monetary authority can choose between two types of monetary policies. As a benchmark case, the monetary authority controls the growth of money by setting an exogenous process for the money supply. Here, an AR(1) process is assumed for the rate of money growth and this growth rate is targeted at the steady state inflation.

$$\ln \bar{M}_t - \ln \bar{M}_{t-1} = (1-\rho_m)\bar{\pi} + \rho_m(\ln \bar{M}_{t-1} - \ln \bar{M}_{t-2}) + u_{mt}$$

$\bar{\pi}$ is the steady state of inflation or the steady state growth rate of money supply. Since this process is non-stationary in nominal terms, it can become stationary by rewriting in real terms

$$\Delta \ln \bar{m}_t = (1-\rho_m)\bar{\pi} - \ln \pi_t + \rho_m \Delta \ln \bar{m}_{t-1} + \rho_m \ln \pi_{t-1} + u_{mt} \qquad (4.39)$$

Where $\ln \bar{m}_t = \ln \bar{M}_t - \ln P_t$ and $\pi_t = \ln P_t - \ln P_{t-1}$. $|\rho_m| < 1$ and $u_{mt}$ is a white noise innovation of money supply growth.

Alternatively, the monetary authority can set monetary policy according to a partial adjustment Taylor rule[29]. It allows for interest rate smoothing and responds to a year-on-year inflation, output in deviation from steady state and output growth. This monetary policy is also subject to the zero-lower bound (ZLB) constraint[30]:

$$1 + i_t = \max\{1, 1 + \bar{\imath}_t\} \qquad (4.40)$$

$$1 + \bar{\imath}_t = (1 + i_{t-1})^{\rho_i}(1+\bar{\imath})^{1-\rho_i}[\left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\rho_\pi}\left(\frac{y_t}{\bar{y}}\right)^{\rho_y}]^{1-\rho_i}(\frac{y_t}{y_{t-1}})^{\rho_{\Delta y}}\exp(u_{mt})$$

Where $0 \le \rho_i < 1$ captures the degree of interest rate smoothing. $\bar{\imath}$, $\bar{\pi}$ and $\bar{y}$ are the steady state interest rate, inflation rate and output, respectively. The coefficients $\rho_\pi$, $\rho_y$, and $\rho_{\Delta y}$ are positive. $u_{mt}$ is a white noise innovation of interest rate shock. When $i_t = 0$, money and bonds are both used as a store of value, and the economy is said to be inside the liquidity trap.

### 4.2.5 Equilibrium and Aggregation

Total profits in the economy is the integral of profits across intermediate

[29] Taylor (2012) argued that the Fed followed the Taylor rule closely around 2003.

[30] According to Christiano, Eichenbaum and Rebelo (2011), the presence of ZLB creates an additional nonlinearity.

goods firm:

$$\Pi_t^n = \int_0^1 \Pi_t^n(j)dj = \int_0^1 \left[P_t(j)y_t(j) - W_t h_t^d(j) - r_t^n \hat{k}_t(j)\right]dj$$

Break up the integral,

$$\Pi_t^n = \int_0^1 P_t(j)y_t(j)dj - W_t \int_0^1 h_t^d(j)dj - r_t^n \int_0^1 \hat{k}_t(j)dj$$

Market-clearing for labour market and capital market requires $h_t^d = \int_0^1 h_t^d(j)dj,\ u_t k_{t-1} = \int_0^1 \hat{k}_t(j)dj$. Hence,

$$\Pi_t^n = \int_0^1 P_t(j)y_t(j)dj - W_t h_t^d - r_t^n u_t k_{t-1}$$

Since the demand function for goods, $y_t(j) = (\frac{P_t(j)}{P_t})^{-\varepsilon_p} y_t$, and aggregate price ${P_t}^{1-\varepsilon_p} = \int_0^1 P_t(j)^{1-\varepsilon_p} dj$, the total profit function becomes

$$\Pi_t^n = P_t y_t - W_t h_t^d - r_t^n u_t k_{t-1}$$

Integrate over household constraint,

$$\begin{aligned} P_t c_t + M_t + P_t I_t + B_t \\ &= \int_0^1 W_t(i)h_t(i)di + M_{t-1} + (1 + i_{t-1})B_{t-1} + r_t^n u_t k_{t-1} + \Pi_t^n \\ &+ P_t s_t \end{aligned}$$

Using the demand curve for labour, $h_t(i) = (\frac{W_t(i)}{W_t})^{-\varepsilon_w} h_t^d$, and aggregate wage index ${W_t}^{1-\varepsilon_w} = \int_0^1 (W_t(i))^{1-\varepsilon_w} di$, the integrated budget constraint is

$$P_t c_t + M_t + P_t I_t + B_t = W_t h_t^d + M_{t-1} + (1 + i_{t-1})B_{t-1} + r_t^n u_t k_{t-1} + \Pi_t^n + P_t s_t$$

Then plug the expression for total profits into budget constraint,

$$P_t c_t + M_t + P_t I_t + B_t = P_t y_t + M_{t-1} + (1 + i_{t-1})B_{t-1} + P_t s_t$$

Money market is cleared by equalling the money demand to the money supply. The money holdings of households represent the demand side for money while the money stock supplied by the government represents the supply side.

$$M_t = \overline{M}_t$$

Or in real terms

$$m_t = \overline{m}_t$$

Since the government bonds is issued by the government and is held by households, the aggregate accounting identity is given by plugging the budget constraint of government into integrated household budget constraint:

$$P_t c_t + P_t I_t = P_t y_t$$

Or in real terms

$$c_t + I_t = y_t$$

Integrate over production function of the intermediate goods, equating this to the integration over demand for intermediate goods

$$\int_0^1 [A_t \hat{k}_t(j)^{\alpha} h_t^d(j)^{1-\alpha} - F] dj = \int_0^1 (\frac{P_t(j)}{P_t})^{-\varepsilon_p} y_t dj$$

Since all firms hire capital and labour in the same ratio, the integral can be broken up as

$$A_t (\frac{\hat{k}_t}{h_t^d})^{\alpha} \int_0^1 h_t^d(j) dj - F = y_t \int_0^1 (\frac{P_t(j)}{P_t})^{-\varepsilon_p} dj$$

Using the labour market clearing condition, $h_t^d = \int_0^1 h_t^d(j) dj$, and defining price dispersion $v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} dj$,

$$A_t {\hat{k}_t}^{\alpha} {h_t^d}^{1-\alpha} - F = y_t v_t^p$$

Recall the Calvo assumption that $1 - \phi_p$ of firms adjust to the same optimal reset price while $\phi_p$ of firms can only partially index their price to the one they charged in the previous period. Then the price dispersion can be split up as

$$v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\varepsilon_p} dj = \int_0^{1-\phi_p} (\frac{P_t^\#}{P_t})^{-\varepsilon_p} dj + \int_{1-\phi_p}^1 (\frac{(1+\pi_{t-1})^{\xi_p} P_{t-1}(j)}{P_t})^{-\varepsilon_p} dj$$

$$= (1-\phi_p)(\frac{P_t^\#}{P_t})^{-\varepsilon_p}$$

$$+ \int_{1-\phi_p}^1 (1+\pi_{t-1})^{-\xi_p\varepsilon_p} (\frac{P_{t-1}}{P_t})^{-\varepsilon_p} (\frac{P_{t-1}(j)}{P_{t-1}})^{-\varepsilon_p} dj$$

$$= (1-\phi_p)(\frac{P_t^\#}{P_t})^{-\varepsilon_p}$$

$$+ (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p} \int_{1-\phi_p}^1 (\frac{P_{t-1}(j)}{P_{t-1}})^{-\varepsilon_p} dj$$

$$= (1-\phi_p)(\frac{P_t^\#}{P_{t-1}})^{-\varepsilon_p} (\frac{P_{t-1}}{P_t})^{-\varepsilon_p}$$

$$+ (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p} \phi_p v_{t-1}^p$$

$$= (1-\phi_p)(\frac{P_t^\#}{P_{t-1}})^{-\varepsilon_p} \left(\frac{1}{1+\pi_t}\right)^{-\varepsilon_p}$$

$$+ (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p} \phi_p v_{t-1}^p$$

Since $1+{\pi_t}^\# = \frac{P_t^\#}{P_{t-1}}$, the price dispersion can be written as

$$v_t^p = (1-\phi_p)\left(\frac{1+{\pi_t}^\#}{1+\pi_t}\right)^{-\varepsilon_p} + (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p} \phi_p v_{t-1}^p$$

The aggregate price index can be described as

$${P_t}^{1-\varepsilon_p} = (1-\phi_p){P_t^\#}^{1-\varepsilon_p} + \int_{1-\phi_p}^1 (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} P_{t-1}(j)^{1-\varepsilon_p} dj$$

$$= (1-\phi_p){P_t^\#}^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\varepsilon_p} dj$$

$$= (1-\phi_p){P_t^\#}^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)} \phi_p {P_{t-1}}^{1-\varepsilon_p}$$

Dividing both side by ${P_{t-1}}^{1-\varepsilon_p}$, the evolution of aggregate inflation index is

$$(1+\pi_t)^{1-\varepsilon_p} = (1-\phi_p)(1+{\pi_t}^\#)^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)}\phi_p$$

Recall ${W_t}^{1-\varepsilon_w} = \int_0^1 W_t(i)^{1-\varepsilon_w} di$ and the Calvo wage setting assumption,

$$W_t{}^{1-\varepsilon_w} = \int_0^{1-\phi_w} W_t{}^{\#1-\varepsilon_w} di + \int_{1-\phi_w}^1 (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} W_{t-1}(i)^{1-\varepsilon_w} di$$

$$= (1-\phi_w) W_t{}^{\#1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \int_{1-\phi_w}^1 W_{t-1}(i)^{1-\varepsilon_w} di$$

$$= (1-\phi_w) W_t{}^{\#1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w W_{t-1}{}^{1-\varepsilon_w}$$

Dividing both side by $P_t{}^{1-\varepsilon_w}$ and defining $w_t{}^{\#} = \frac{W_t{}^{\#}}{P_t}$, the real wage expression is

$$w_t{}^{1-\varepsilon_w} = (1-\phi_w) w_t{}^{\#1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w (\frac{W_{t-1}}{P_t})^{1-\varepsilon_w}$$

$$= (1-\phi_w) w_t{}^{\#1-\varepsilon_w}$$

$$+ (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w (\frac{W_{t-1}}{P_{t-1}})^{1-\varepsilon_w} (\frac{P_{t-1}}{P_t})^{1-\varepsilon_w}$$

$$= (1-\phi_w) w_t{}^{\#1-\varepsilon_w}$$

$$+ (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w (w_{t-1})^{1-\varepsilon_w} (\frac{P_{t-1}}{P_t})^{1-\varepsilon_w}$$

As $1+\pi_{t-1} = \frac{P_{t-1}}{P_t}$, the real wage is finally be expressed as

$$w_t{}^{1-\varepsilon_w} = (1-\phi_w) w_t{}^{\#1-\varepsilon_w} + (1+\pi_{t-1})^{\xi_w(1-\varepsilon_w)} \phi_w (1+\pi_t)^{\varepsilon_w - 1} w_{t-1}{}^{1-\varepsilon_w}$$

The system of optimality conditions and constraints is given below. All endogenous and exogenous variables are in levels.

$$\begin{cases} c_t + \phi_I I_t + b_t = \dfrac{m_{t-1}}{1+\pi_t} + \dfrac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t, \text{binding CIA constraint} \\ \eta_t = 0, \qquad \text{nonbinding CIA constraint} \end{cases} \tag{4.41}$$

$$k_t = Z_t \left(1 - \frac{\tau}{2}\left(\frac{I_t}{I_{t-1}} - 1\right)^2\right) I_t$$
$$+ \left(1 - \delta_0 - \delta_1(u_t - 1) - \frac{\delta_2}{2}(u_t - 1)^2\right) k_{t-1} \tag{4.42}$$

$$c_t{}^{-\sigma} = (\lambda_t + \eta_t) \tag{4.43}$$

$$\lambda_t = \beta E_t \left\{\left(\frac{\lambda_{t+1}}{1+\pi_{t+1}} + \frac{\eta_{t+1}}{1+\pi_{t+1}}\right)\right\} \tag{4.44}$$

$$\lambda_t + \eta_t = \beta E_t \left\{ \lambda_{t+1} \frac{(1+i_t)}{1+\pi_{t+1}} + \eta_{t+1} \frac{(1+i_t)}{1+\pi_{t+1}} \right\} \tag{4.45}$$

$$\begin{aligned} \lambda_t + \eta_t \phi_I = \mu_t Z_t \left( 1 - \frac{\tau}{2} \left( \frac{I_t}{I_{t-1}} - 1 \right)^2 - \tau \left( \frac{I_t}{I_{t-1}} - 1 \right) \frac{I_t}{I_{t-1}} \right) \\ + \beta E_t \left\{ \mu_{t+1} Z_{t+1} \tau \left( \frac{I_{t+1}}{I_t} - 1 \right) \left( \frac{I_{t+1}}{I_t} \right)^2 \right\} \end{aligned} \tag{4.46}$$

$$\lambda_t r_t = \mu_t \big( \delta_1 + \delta_2 (u_t - 1) \big) \tag{4.47}$$

$$\mu_t = \beta E_t \left\{ \lambda_{t+1} r_{t+1} u_{t+1} + \mu_{t+1} \left( 1 - \delta_0 - \delta_1 (u_t - 1) - \frac{\delta_2}{2} (u_t - 1)^2 \right) \right\} \tag{4.48}$$

$$w_t{}^{\#} = \frac{\varepsilon_w}{\varepsilon_w - 1} \frac{\hat{n}_{1t}}{\hat{n}_{2t}} \tag{4.49}$$

$$\begin{aligned} \hat{n}_{1t} = \left( \frac{w_t}{w_t{}^{\#}} \right)^{\varepsilon_w (1+\varphi)} h_t^{d\,1+\varphi} \\ + \beta \phi_w (1+\pi_t)^{-\xi_w \varepsilon_w (1+\varphi)} E_t \left[ (1+\pi_{t+1})^{\varepsilon_w (1+\varphi)} \left( \frac{w_{t+1}{}^{\#}}{w_t{}^{\#}} \right)^{\varepsilon_w (1+\varphi)} \hat{n}_{1t+1} \right] \end{aligned} \tag{4.50}$$

$$\begin{aligned} \hat{n}_{2t} = \lambda_t \left( \frac{w_t}{w_t{}^{\#}} \right)^{\varepsilon_w} h_t^d \\ + \beta \phi_w (1+\pi_t)^{\xi_w (1-\varepsilon_w)} E_t \left[ (1+\pi_{t+1})^{\varepsilon_w - 1} \left( \frac{w_{t+1}{}^{\#}}{w_t{}^{\#}} \right)^{\varepsilon_w} \hat{n}_{2t+1} \right] \end{aligned} \tag{4.51}$$

$$\begin{aligned} w_t{}^{1-\varepsilon_w} = (1-\phi_w) w_t{}^{\#\,1-\varepsilon_w} \\ + (1+\pi_{t-1})^{\xi_w (1-\varepsilon_w)} \phi_w (1+\pi_t)^{\varepsilon_w - 1} w_{t-1}{}^{1-\varepsilon_w} \end{aligned} \tag{4.52}$$

$$\hat{k}_t = u_t k_{t-1} \tag{4.53}$$

$$\frac{w_t}{r_t} = \frac{1-\alpha}{\alpha} \left( \frac{\hat{k}_t}{h_t^d} \right) \tag{4.54}$$

$$mc_t = \frac{w_t}{(1-\alpha) A_t \left( \frac{\hat{k}_t}{h_t^d} \right)^{\alpha}} \tag{4.55}$$

$$1 + \pi_t^{\#} = \frac{\varepsilon_p}{\varepsilon_p - 1} (1+\pi_t) \frac{x_{1t}}{x_{2t}} \tag{4.56}$$

$$x_{1t} = c_t{}^{-\sigma} mc_t y_t + \beta \phi_p (1+\pi_t)^{-\xi_p \varepsilon_p} E_t (1+\pi_{t+1})^{\varepsilon_p} x_{1t+1} \tag{4.57}$$

$$x_{2t} = c_t{}^{-\sigma} y_t + \beta \phi_p (1+\pi_t)^{\xi_p (1-\varepsilon_p)} E_t (1+\pi_{t+1})^{\varepsilon_p - 1} x_{2t+1} \tag{4.58}$$

$$\bar{m}_t + b_t = \frac{\bar{m}_{t-1}}{1+\pi_t} + s_t + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} \tag{4.59}$$

$$y_t = \frac{A_t \hat{k}_t{}^{\alpha} h_t^{d\,1-\alpha}}{v_t^p} \tag{4.60}$$

$$(1+\pi_t)^{1-\varepsilon_p} = \left(1-\phi_p\right)(1+\pi_t{}^{\#})^{1-\varepsilon_p} + (1+\pi_{t-1})^{\xi_p(1-\varepsilon_p)}\phi_p \tag{4.61}$$

$$v_t^p = \left(1-\phi_p\right)\left(\frac{1+\pi_t{}^{\#}}{1+\pi_t}\right)^{-\varepsilon_p} + (1+\pi_{t-1})^{-\xi_p\varepsilon_p}(1+\pi_t)^{\varepsilon_p}\phi_p v_{t-1}^p \tag{4.62}$$

$$c_t + I_t = y_t \tag{4.63}$$

$$m_t = \overline{m}_t \tag{4.64}$$

$$b_t = 0 \tag{4.65}$$

$$z_t = \rho_z z_{t-1} + u_{zt} \tag{4.66}$$

$$a_t = \rho_a a_{t-1} + u_{at} \tag{4.67}$$

The last two equations describe the uncertainties in productivity $A_t$, and marginal efficiency of investment $Z_t$. All the AR parameters are assumed to be lie between 0 and 1. All shocks are drawn from standard normal distributions.

The differences between two type of monetary policies provide important implications for monetary authority. In benchmark model, the monetary authority adopts an exogenous money growth rule.

$$\Delta \ln \overline{m}_t = (1-\rho_m)\overline{\pi} - \ln \pi_t + \rho_m \Delta \ln \overline{m}_{t-1} + \rho_m \ln \pi_{t-1} + u_{mt} \tag{4.68}$$

$$\Delta \ln \overline{m}_t = \ln \overline{m}_t - \ln \overline{m}_{t-1} \tag{4.69}$$

The equilibrium sequence in this model is characterized by two different states as before: one where the CIA constraint is binding, and one where it is not. There is a unique steady state when the CIA constraint is always binding. The non-stochastic steady states are solved in Appendix 4.A. When the CIA constraint never binds, money is neutral. There is no definite steady state in this case.

In alternative model, the monetary policy follows a modified Taylor rule, which is also subject to a ZLB constraint.

$$1 + i_t = \max\{1, 1 + \bar{\iota}_t\} \tag{4.70}$$

$$1 + \bar{\iota}_t = (1 + i_{t-1})^{\rho_i}(1 + \bar{\iota})^{1-\rho_i}[\left(\frac{1 + \pi_t}{1 + \bar{\pi}}\right)^{\rho_\pi} \left(\frac{y_t}{\bar{y}}\right)^{\rho_y}]^{1-\rho_i}(\frac{y_t}{y_{t-1}})^{\rho_{\Delta y}} \exp(u_{mt})$$

Since the ZLB constraint is taken into account in this model, another nonlinearity has been introduced. The case when the CIA constraint is always binding and when the interest rate is always positive is commonly discussed in literatures. The unique steady state is the same as the one when the money growth rule is applied and when the CIA constraint is always binding. Three more cases need to be considered: When the CIA constraint is always binding while the interest rate is zero; When the CIA constraint does not bind while the interest rate is positive; When the CIA constraint is nonbinding while the interest rate is zero. The relationship between CIA constraint and ZLB constraint is discussed in the next Section.

### 4.2.6 Occasionally Binding CIA Constraints

Recall Karush-Kuhn-Tucker condition related to the CIA constraint,

$$\eta_t\left(\left(c_t + \phi_I I_t + b_t - \frac{m_{t-1}}{1 + \pi_t} - \frac{1 + i_{t-1}}{1 + \pi_t} b_{t-1} - s_t\right)\right) = 0$$

There are two cases of values for Lagrange multiplier $\eta_t$ to take. The first case is the usual case when $\eta_t$ is strictly positive, meaning that the CIA constraint should be strictly binding for the Karush-Kuhn-Tucker condition to hold. In this case, credit strongly dominates money as a store of value and households will not hold extra money. The second case is the unusual case when $\eta_t$ is equal to zero. In this case, the CIA constraint can be nonbinding, and money and credit are both used as a store of value as the households can allocate some savings in money.

Then consider the Euler equations for three important assets in this model[31]. Firstly, consider the Euler equation for money holdings[32],

$$c_t^{-\sigma} - \eta_t = \beta E_t \left\{ \frac{c_{t+1}^{-\sigma}}{1 + \pi_{t+1}} \right\} \tag{4.71}$$

Nonbinding CIA constraints ($\eta_t = 0$) suggest that the marginal utility of consumption today is equal to the discounted and deflated marginal utility of consumption tomorrow, while binding CIA constraints ($\eta_t > 0$) show that the marginal utility of consumption today is larger than that of consumption tomorrow. This relationship can be expressed as

$$\beta E_t \left\{ \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{1}{1 + \pi_{t+1}} \right\} \begin{cases} < 1, & \text{binding CIA constraint} \\ = 1, & \text{nonbinding CIA constraint} \end{cases} \tag{4.72}$$

This is also the condition for money demand. The term $E_t\{\frac{1}{1+\pi_{t+1}}\}$ represents the expected gross return of money. The expected net return on money is in turn $E_t\{\frac{1}{1+\pi_{t+1}} - 1\}$. This expected net return is always less or equal to zero as long as the expected inflation is non-negative. Since $E_t\{\frac{1}{1+\pi_{t+1}} - 1\} \leq 0$, the motive for money demand is transaction need. Hence, the consumers rush to use up all their money holdings and then the CIA constraint binds. However, when the expected inflation is negative or a deflation is expected, the expected net return can be positive. Consumers' motives for money demand include both transaction need of money and its function as a store of value. in this case, households may hold money as real asset. Actually, they do not care how much money they hold as long as their money holdings at least can cover their consumption needs. Thus, the CIA constraint has the chance to be nonbinding. The whole term $\beta E_t \left\{ \left( \frac{c_{t+1}}{c_t} \right)^{-\sigma} \frac{1}{1+\pi_{t+1}} \right\}$ is defined as

[31] A Euler equation is an intertemporal first order condition that characterizes the optimal choice by equating the expected marginal costs to the expected marginal benefits.

[32] The intuitions behind this condition has been discussed thoroughly in Chapter 3.

the expected relative value of money[33]. When it is expected to decrease next period, consumers tend to spend all their money holdings and the CIA constraint is binding. When it is expected to be the same next period, consumers are indifferent between spending money current period and holding it for one period, and the CIA constraint can be nonbinding.

Secondly, consider the Euler equation for bond holdings[34],

$$\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\frac{1+i_t}{1+\pi_{t+1}}\right\}=1 \tag{4.73}$$

This is also the condition for bond demand. It is worth mentioning that $i_t$ is the rate at which the households will receive interests in period $t+1$ but is decided prior to the opening of the goods market. When the expected nominal interest rate is strictly positive, that is $1+i_t>1$, the net return on bonds is larger than zero and thus consumers strictly prefer to buy bonds. It also suggests that

$$\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\frac{1}{1+\pi_{t+1}}\right\}<1 \tag{4.74}$$

This is a condition for binding CIA constraint as well. Since the net return on money is non-positive, consumers always hold the amount of money to only cover their consumption as long as the nominal interest rate is positive. When the nominal interest rate can be zero, that is, the nominal interest rate is at its ZLB, nominal bonds have no positive return in this case. Hence, nominal bonds and money can be seen as perfect substitutes and bonds show their liquid properties. The economy is inside the so-called liquidity trap. The Euler equation becomes

$$\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\frac{1}{1+\pi_{t+1}}\right\}=1 \tag{4.75}$$

[33] See Dixon and Pourpourides (2016)

[34] Without loss of generality, the nominal government bond is assumed to be of zero supply ultimately. Whether bonds are subject to the CIA constraint will not change the results.

This is in line with the case when the CIA constraint is nonbinding[35]. Since nominal bonds can no longer provide any positive return, consumers are indifferent between holding bonds and holding money as long as they have enough money for future consumption.

Combine the Euler equation for money holdings with that for bond holdings, the relationship between Lagrange multiplier $\eta_t$ and the expected nominal interest rate $i_t$ is

$$i_t = \frac{\eta_t}{\lambda_t} \tag{4.76}$$

Hence, the utility of one unit of money must be equal to the expected utility of the interest on one unit of money invested as a nominal bond[36]. When the consumers expect that the nominal interest rate is subject to a zero lower bound or the economy is inside the liquidity trap, the Lagrange multiplier is zero and the CIA constraint does not bind.

Thirdly, consider the Euler equation for capital, which is relatively complicated here. For explanation simplicity, investment adjustment costs, capital utilization and marginal efficiency of investment shock are ignored as they just bring in more frictions. Simplified capital accumulation process are as follows.

$$k_t = I_t + (1-\delta)k_{t-1} \tag{4.77}$$

Recall the CIA constraint with investment

$$c_t + \phi_I I_t + b_t \leq \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t$$

New first order condition for investment and capital stock becomes

$$\mu_t = \lambda_t + \phi_I \eta_t \tag{4.78}$$

[35] The Friedman rule obtains, which implies that the nominal interest rate on a bond that sells before the goods market opens is zero (Svensson, 1985).

[36] $\eta_t$ is also called the utility of liquidity services of real balances by Svensson (1985).

$$\mu_t = \beta E_t\{\lambda_{t+1} r_{t+1} + \mu_{t+1}(1-\delta)\} \tag{4.79}$$

Here, $\lambda_t$ is the shadow price of the budget constraint or wealth, $\mu_t$ is the shadow prices of investment, and $\eta_t$ the shadow price of the CIA constraint or the real balance. $\phi_I$ is the fraction of investment which should be financed with cash.

Consider two corner cases for $\phi_I$, i.e. $\phi_I = 0$ and $\phi_I = 1$, and ignore the nonbinding CIA constraint for a while.

When $\phi_I = 0$,

$$\mu_t = \lambda_t \tag{4.80}$$

This equation indicates that wealth and the investment have the same shadow price[37]. An increase in $\phi_I$ raises the shadow price of investment as long as the CIA constraint is always binding.

The first order condition for capital is reduced to

$$\lambda_t = \beta E_t\{\lambda_{t+1}(r_{t+1} + 1 - \delta)\} \tag{4.81}$$

Or

$$E_t\{(r_{t+1} + 1 - \delta)\} = \frac{\lambda_t}{\beta E_t\{\lambda_{t+1}\}} \tag{4.82}$$

$\lambda_t$ is the marginal utility of wealth. The real return on capital thus is defined as the present marginal utility of wealth over the discounted expected marginal utility of wealth in the next period.

Combine with the first order conditions for consumption and money

$$\beta E_t\left\{\left(\frac{c_{t+2}}{c_{t+1}}\right)^{-\sigma} \frac{1+\pi_{t+1}}{1+\pi_{t+2}}(r_{t+1} + 1 - \delta)\right\} = 1 \tag{4.83}$$

The left-hand side is the expected real return on capital in terms of the deflated utility of consumption in period $t+2$ proportional to the deflated

[37] Stockman (1981) finds that capital can be freely obtained by bartering as long as the shadow price ratio of capital to real balance is unchanged.

utility of consumption in period $t+1$, which can be defined as the expected relative value of capital. Since wealth cannot buy consumption instantaneously, the relative value of capital is irrelevant to the current state and is expected to be the same across the periods.

Using the first order conditions for money demand and bonds demand,

$$E_t\left\{\frac{1+i_{t+1}}{1+\pi_{t+1}}\right\} = E_t\{r_{t+1} + 1 - \delta\} \tag{4.84}$$

Since $\phi_I = 0$, meaning investment is not financed by cash, money growth plays its role via the CIA constraint, which is similar to that of a consumption tax only. The left-hand side $E_t\left\{\frac{1+i_{t+1}}{1+\pi_{t+1}}\right\}$ is the expected real return on nominal bonds. When the investment is not subject to the CIA constraint, the expected real return on capital equals the expected real return on nominal bonds, meaning that the consumers cannot earn a premium on investing in capital over buying bonds. Hence, a binding CIA constraint which acts as a tax only on consumption does not distort the allocation between bonds and capital. As long as the consumers can earn positive returns on bonds and capital, they will not choose to hold extra money but increase their bond holdings and investment.

Money is non-neutral as the CIA constraint distorts the labour-leisure choice. An increase in money growth or an increase in inflation gives the households an incentive to get rid of money. However, since consumption needs to be financed by cash, the households cannot substitute from money to consumption. They, instead, substitutes from money to leisure.[38] There will be a reduction in labour supply and consumption, which leads to a decline in output.

[38] Ignore Calvo-style of wage setting, the intra-temporal condition is $\frac{h_t^{d^{\varphi}}}{c_t^{-\sigma}} = \frac{\lambda_t}{(\lambda_t+\eta_t)} w_t$. The marginal rate of substitution between leisure and consumption equals real wage only when the CIA constraint is nonbinding $\eta_t = 0$.

The second case is when $\phi_I = 1$, that is, the investment is fully financed by cash. In this case, money can have impact on capital via the CIA constraint. The first order condition for investment is

$$\mu_t = \lambda_t + \eta_t \tag{4.85}$$

As long as $\eta_t$ is positive or the CIA constraint is always binding, the shadow price of the investment is larger than that of the real balance as the CIA constraint make capital holdings more expensive[39].

The first order condition for capital is

$$\lambda_t + \eta_t = \beta E_t\{\lambda_{t+1}(r_{t+1} + 1 - \delta) + \eta_{t+1}(1 - \delta)\} \tag{4.86}$$

Substitute the first order condition for consumption in

$$c_t^{-\sigma} = \beta E_t\{c_{t+1}{}^{-\sigma}(r_{t+1} + 1 - \delta) - \eta_{t+1}r_{t+1}\} \tag{4.87}$$

Or

$$\beta E_t\left\{\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}(r_{t+1} + 1 - \delta)\right\}\begin{cases} > 1, & \text{binding CIA constraint} \\ = 1, & \text{nonbinding CIA constraint} \end{cases} \tag{4.88}$$

The left-hand side can be explained as the expected real return on capital in terms of the expected utility of consumption tomorrow proportional to the utility of consumption today. It can also be seen as the expected relative value of capital. When households expect that the relative value of capital will increase next period, they will rush to invest their money holdings to capital and the CIA constraint is binding. When the relative value of capital is expected to remain the same next period, there is no incentive for households use up all their money holdings, and instead they are indifferent between holding money for one period and spending it.

Combine with the Euler equation for bonds,

[39] An increase in $\phi_I$ also raises the price of investment.

$$E_t\{(r_{t+1}+1-\delta)\}\begin{cases} > E_t\left\{\dfrac{1+i_t}{1+\pi_{t+1}}\right\}, & \text{binding CIA constraint} \\ = E_t\left\{\dfrac{1+i_t}{1+\pi_{t+1}}\right\}, & \text{nonbinding CIA constraint} \end{cases} \tag{4.89}$$

When the expected return on capital is strictly higher than both the expected real return on bonds and the return on money holdings or the consumers can earn a higher liquidity premium on investment, there is no extra demand for money as money is strictly dominated by these alternative assets, and the CIA constraint binds. In this case, capital can be seen as the only asset which works as a store of value. The CIA constraint not only distorts the labour-leisure (as when the investment is not subject to the CIA constraint) but also distorts the capital accumulation decision because it works as a tax on investment as well. Since there is a tax on investment, although the households have a chance to earn a premium on investment, they need enough money holdings to cover the investment. Hence, the households will substitute from both consumption and investment to leisure and both consumption and investment are lower, which in turn decreases the output.

What if the CIA constraint is nonbinding? This is when the Lagrange multiplier for the CIA constraint is zero, $\eta_t = 0$, and the expected interest rate is zero, $i_t = 0$. There is thus no inflation tax on consumption or/and investment. The above condition becomes

$$E_t\left\{\frac{1}{1+\pi_{t+1}}\right\} = E_t\{r_{t+1}+1-\delta\} \tag{4.90}$$

Buying nominal bonds and investing in capital cannot generate extra return now and capital can easily be attained from bartering as money is fully pledged. Since expected real return on money, bonds and capital are the same, capital, bonds and money are perfect substitutes, and it is costless for consumers to hold money[40]. If the consumers hold money beyond their

[40] As long as the nominal interest rate is zero or the Lagrange multiplier for the CIA constraint is zero, $\phi_I$ has no role in determining the level of investment.

transaction and investment needs when both consumption and investment are subject to the CIA constraint, money, which providing a safe store of value, here acts like a risk-sharing asset. The investment cannot provide a positive premium, which makes capital even less attractive. In this case, liquidity traps (when the nominal interest rate is zero) with low inflation or deflation depresses investment.

Recall the firm's problem

$$r_t = \alpha mc_t A_t (\frac{\hat{k}_t}{h_t^d})^{\alpha-1}$$

$$w_t = (1-\alpha) mc_t A_t (\frac{\hat{k}_t}{h_t^d})^{\alpha}$$

If there is a liquidity trap with deflation, for a given technology, the marginal product of capital $r_t$, which is inversely related to expected inflation, will be higher and hence the capital labour ratio will be lower. Lower capital labour ratio will in turn decreases the real wage. The substitution effect of real wage, which dominates its income effect, will decrease the labour supply, which will further decrease the output and consumption.

Compared with the results in Chapter 3 where money is the only asset and labour is the only input for production, a nonbinding CIA constraint may lose its ability to boost the economy. Instead, a nonbinding CIA constraint may further depress the economy as zero lower bound prevents interest rate from falling[41].

## 4.3 Calibrated and Estimated Dynamics

It is mentioned above that the monetary policy is conduct by the government and has two pillars: the money supply rule through an exogenous process

41 Di Tella (2017) shows that investment is too high during booms and too low during liquidity traps and suggests that an optimal allocation is possible by implementing a tax or subsidy on capital and the Friedman rule.

(benchmark model) and the interest rate rule which is determined through a Taylor rule and takes zero lower bound (ZLB) into account (alternative model). The basic idea for the CIA constraint to be endogenously binding is the same as in the Chapter 3. The whole benchmark model system is separate into two regimes, binding CIA constraint regime (reference regime) and nonbinding CIA constraint regime (alternative regime). Parameters of both regimes take the same value to ensure that shocks are the only contributions to the switching between these two regimes. However, for the alternative model, another nonlinearity has been introduced by ZLB. As there are two nonlinearities in this model, four regimes need to be considered. Regime 1 is when the CIA constraint is binding and the ZLB on interest rate does not bind. Regime 2 is when the CIA constraint is binding and the ZLB on interest rate binds. Regime 3 is both the CIA constraint and the ZLB on interest rate are nonbinding. Regime 4 is when the CIA constraint is nonbinding while the ZLB on interest rate is binding. Although the number of nonlinearities increases, it is argued that a positive money supply shock or a negative interest rate shock followed by a positive technology shock give a chance for occasionally binding constraints. Outline for the section is as follows. In Section 4.3.1, calibration results for the baseline model is listed. Section 4.3.2 shows the estimation results for the benchmark model. Section 4.3.3 and Section 4.3.4 illustrates the calibration results and estimation results for the alternative model. A comparison between the benchmark and alternative models is discussed in Section 4.3.5[42].

### 4.3.1 Calibration for Benchmark Model

Recall that the benchmark model is the one where the monetary policy

[42] Value assignments are kept being the same to all parameters except for the parameters associated with different monetary rules in both models.

follows an exogenous money growth rule that is targeted at the steady state inflation

$$\Delta \ln \overline{m}_t = (1-\rho_m)\overline{\pi} - \ln \pi_t + \rho_m \Delta \ln \overline{m}_{t-1} + \rho_m \ln \pi_{t-1} + u_{mt}$$

$$\Delta \ln \overline{m}_t = \ln \overline{m}_t - \ln \overline{m}_{t-1}$$

#### 4.3.1.1 Calibration Parameters

Table 4-1 in Appendix 4.B contains the calibrated parameters for benchmark model. The choice of parameters is important as it must represent economic features and is also able to ensure the stability of the system. New Keynesian parameters are mostly chosen as in Galí (2008). A unit of period corresponds to a quarter. The subjective discount factor $\beta$ is set at 0.995, which implies an annualized risk-free real interest rate of about 2 percent. Mehra and Prescott (1985) point that the value for the coefficient of relative risk aversion fall in the range between 1 and 2 based on the micro-evidence. The coefficient of relative risk aversion of consumption, $\sigma$, following most literatures, is set to be 1. The coefficient of labour disutility, $\varphi$, is set at 1.2. The parameters associated with function $\delta(u_t)$ which maps the capital utilization into depreciation rate, $\delta_0$, $\delta_1$ and $\delta_2$ are set as follows. $\delta_0$ is set at 0.025, which implies an average annual rate of depreciation on capital is 10 percent. The parameter $\delta_1$ is pinned down to ensure that the normalised steady state utilization is 1[43]. $\delta_2$ is difficult to estimate and is assumed to be 0.01[44]. The measure of the investment adjustment cost, $\tau$, is assumed to be small as well to ensure the endogenously binding constraints more pronounced and the investment will be very responsive to shocks. The fixed cost of production, $F$, is set to be consistent with zero steady state profit. The capital share in production $\alpha$ is fixed at 1/3, which is a value commonly

[43] The calculation of $\delta_1 = \frac{1}{\beta} - (1-\delta_0)$ can be found in Appendix 4.A.

[44] Most literatures follow Christiano, Eichenbaum, and Evans (2005) and set it to a low value.

used in literatures. The elasticity of substitution among differentiated intermediate goods, $\varepsilon_p$, is set at 6, which implies that the gross mark-up in goods market, $\frac{\varepsilon_p}{\varepsilon_p-1}$, is 1.2. This is also roughly consistent with the steady state labour hours in neighbourhood of one-third. Similarly, the elasticity of substitution among heterogenous labour, $\varepsilon_w$, is given the value 6 as well, which implies a gross mark-up in labour market, $\frac{\varepsilon_w}{\varepsilon_w-1}$, is 1.2. The quarterly trend inflation is assumed to be 0.5% as in Chapter 3[45]. The probability of sticky wage and sticky wage are calibrated roughly following Smets and Wouters (2007), that is, $\phi_p$ is 0.7 and $\phi_w$ is 0.6, respectively. Since the data show some indexation in wages and prices, the indexation parameters, $\xi_p$ and $\xi_w$, are assumed to be 0.5. Investment is assumed to be partially subject to the CIA constraint and the fraction of aggregate investment that should be financed by cash $\phi_I$ is set at 30%[46].

There are three shocks in the baseline model, i.e. technology shock, investment shock and money supply shock. The autoregressive parameter for the technology shock, $\rho_a$, and its standard deviation, $\sigma_a$, are calibrated at 0.9 and 0.01. The autoregressive parameter for the investment shock, $\rho_z$, and its standard deviation, $\sigma_z$, are also set at 0.9 and 0.01. For money supply shock, the autoregressive coefficient, $\rho_m$, is assumed to be 0.6 with its standard deviation $\sigma_m$ also set at 0.01.

#### 4.3.1.2 Dynamics for Benchmark Model

It is argued in Chapter 3 that an expansionary monetary policy shock followed by a technology shock is possible to generate an endogenously

[45] The optimal annual inflation takes the range between 1.85% and 2.21%.
[46] Wang and Wen (2005) find that output can generate a hump-shaped pattern even when only as little as 30% of investment is subject to the CIA constraint.

nonbinding CIA constraint[47]. An increase in money supply always leads to an increase in inflation. If the inflation is expected to increase, the gross return on money, which is captured by $E_t\{\frac{1}{1+\pi_{t+1}}\}$, is expected to be strictly less than 1. Since holding money cannot generate positive return, the consumers will give up all their money holdings if there is no CIA constraint. However, they have to hold money for their consumption needs. Hence, they choose to hold the amount of money just to cover their consumption, which triggers a binding CIA constraint. A positive technology shock lowers the price level, causing a decrease in inflation or even a deflation. If this effect of technology innovation on the price is large enough to drive down the inflation to negative, the gross return on money can be greater than one or the net return on money can be positive. Investment is assumed to partially subject to the CIA constraint and there is almost no adjustment cost. A nonbinding CIA constraint, which prevents the expected interest rate from going down, depresses the investment. The magnitude of investment depression is related to the magnitude of adjustment cost. If there is a high cost for investment to adjust to shocks, nonbinding CIA constraints have quite small effect on investment distortion because it costs a lot for consumers to make responses to changing states. If there is almost no investment adjust cost, nonbinding CIA constraints can lead to a larger decrease in investment.

The algorithm for occasionally binding CIA constraints follows the toolkit of Guerrieri and Iacoviello (2015), where occasionally binding constraints are allowed to switch between two regimes, reference and alternative regimes[48]. The model set and parameter values are kept the same in both regimes. Here, the reference regime refers to the binding CIA model and the

---

[47] Either money supply shock or technology shock can generate nonbinding CIA constraints. When money is the only asset in the economy, the order of shocks matters.

[48] A summary for the explanation of the algorithm by Guerrieri and Iacoviello (2015) are provided in Chapter 3.

alternative regime is the nonbinding CIA model. The condition under the reference regime is $\eta_t > 0$, which is consistent with $c_t + \phi_I I_t + b_t = \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t$. The condition under the alternative regime is $\eta_t = 0$, which is corresponding to $c_t + \phi_I I_t + b_t < \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t$ . Two requirements for the toolkit to work need to be emphasized: (1) there must be a rational expectation equilibrium in the reference regime; (2) before there is a disturbance, the economy stays in the reference regime. When the economy is hit by the disturbance, it switches from the reference regime to the alternative regime for a finite time period and must return back to the reference regime when there is no disturbance expected.

Figure 4-1 shows the impulse functions when there is a 10% positive money supply shock, or the government conducts a monetary easing. The blue line is when the CIA constraint binds all the time and the red line is when the CIA constraint can be nonbinding for some periods.

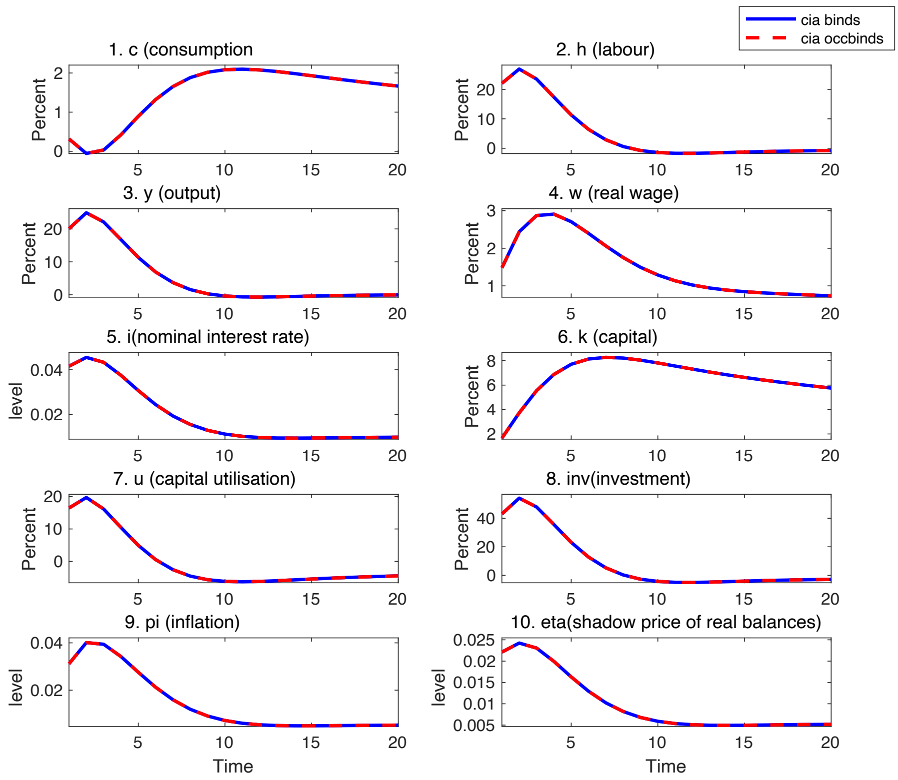


*Figure 4-1 IRFs following a 10% money supply shock*

An increase in the growth rate of money supply can generate hump shaped responses to output, consumption, investment and employment. An initial rise in the aggregate demand is limited by the current rise in money. Thus, consumers and firms have to wait for future injections of money to fully adjust to the money supply shock. The dynamics of models with a money growth rule mainly relies on the impact of money growth on inflation. A money injection results in an increase in the expectation of inflation, which will decrease the return on money. Thus, the households rush to use up all their money holdings and the CIA constraint is always binding. The nominal interest rate must also increase for a rise in inflation. It is a function of the shadow price of real balances. Since the shadow price of real balance (the Lagrange multiplier) increases, the expected nominal interest rate should increase. Hence, the model fails to generate a liquidity effect. There is no chance for the CIA constraint to be nonbinding for some periods when the inflation is strictly positive.

What will happen if there is a negative money growth shock that leads to a negative inflation? Figure 4-2 shows the IRFs when there is a 10% negative money supply shock, or the government implements a contractionary monetary policy. The blue line is when the CIA constraint is always binding, and the red line is when the CIA constraint is allowed to be nonbinding for some periods.

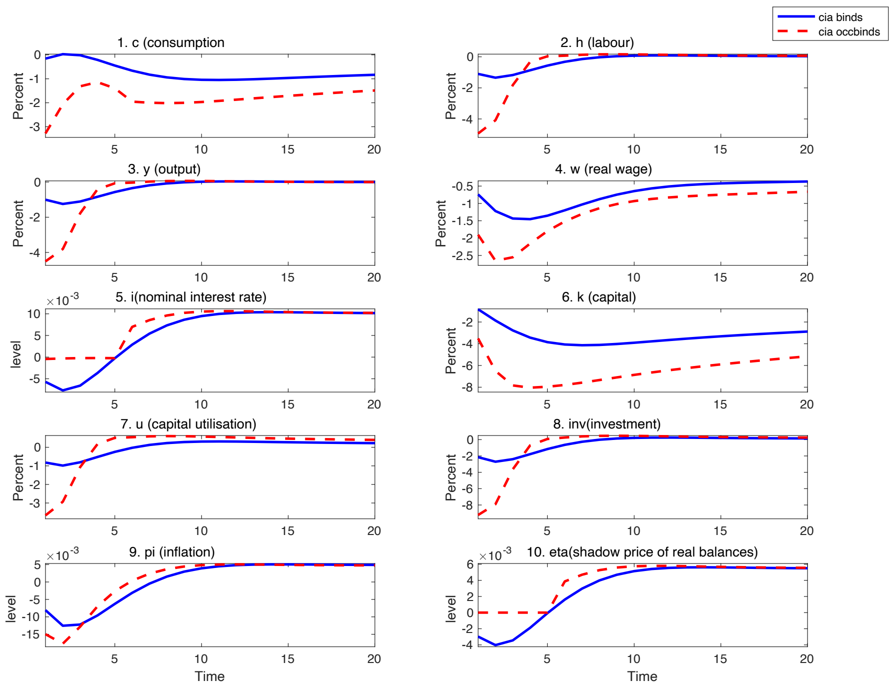


*Figure 4-2 IRFs following a negative 10% money supply shock*

Literatures that assume always binding CIA constraints have shown that money can have effects on real economy, but the effects are indeed very small. However, when the nonbinding CIA constraints are taken into account, the effects of money are magnified. A contractionary monetary policy is consistent with a decrease in money supply. The households expect that money is more expensive to get next period and are more willing to hold more money for their future consumption and investment requirements because cash is needed to cover these expenditures when the economy is hit by a negative money supply shock. The incentive for households to increase their money holding decisions will lead to an excess demand for money, which will in turn increase the value of money. The excess demand of money will result in a decrease in the marginal utility of real balance. The nominal interest rate decreases as a result of negative money supply shock as well[49]. A decrease in the interest rate is consistent with a decrease in the marginal utility of real balances as the expected interest rate is directly related to the marginal utility of real balance[50]. If there is no constraint on the marginal utility of real balances, it can be negative. However, the marginal utility of real balance cannot go below zero because there is no marginal cost of producing money, which in turn will result in a zero expected nominal interest rate. Hence, the expectation of nominal interest rate is directly related to whether the CIA constraint is binding or not in the current period. When the CIA constraint is nonbinding, the expected nominal interest rate is zero, and the rate of deflation will be equal to the real interest rate according to the Fisher relationship. The return on capital, which is the same as the return on money, is less. The households invest less into capital. The real interest rate is higher when nominal interest rate hits its zero bound and thus the cost of borrowing is higher. Firms will decrease their borrowing of capital from households. Since money can provide positive return and is the perfect substitute of capital and bonds, households tend to hold more money in pocket for the precautionary motivations. Faced by the decreasing demand of their goods, the firms not only decrease their borrowing of capital but also

[49] If money transfers can be used for purchases on the goods market as in Lucas' s (1982) model, temporary monetary disturbance have no effect on the real interest rate.

[50] Recall $i_t = \frac{\eta_t}{\lambda_t}$ and $i_t$ is the expected nominal interest rate at which the government will pay at the beginning of period $t+1$.

decrease the capital utilisation, which lead to a huge fall in output. Technically, the huge depression of output and investment can be explained by the assumption that the model is approximated around the same points under each regime. The points used are the steady states of the binding regime. The difference can be seen as a function of the discount factor and the steady state inflation. When the CIA constraint is nonbinding, $\beta = 1 + \bar{\pi}$. When the CIA constraint is binding, $\beta < 1 + \bar{\pi}$. This may also result in a huge deviations of variables.

The inflation is lowest in period 2 because it takes time for firms to update their price and starts to return back to its steady state. In period 4, the inflation of nonbinding CIA regime (alternative regime) is going to be higher than that of binding CIA regime (reference regime). This is because the expected inflation is independent of monetary contraction when the CIA constraint does not bind. Less deflation is consistent with lower real interest rate, which increases the investment, employment and output. However, a nonbinding CIA constraint fails to generate an increase in consumption as the households tend to invest more instead of consuming. When the CIA constraint binds after 6 periods, it takes longer for consumption to return back to its steady state because consumption is fully subject to the CIA constraint and households need time to accumulate real balances. Therefore, in a 20-period dimension, a negative money supply shock can contribute to a nonbinding CIA constraint for 6 periods. In other words, the probability of a binding CIA constraint in 20 periods is 70% after a 10% negative money supply shock.

The expectation of a negative inflation or a deflation can be seen as a source of nonbinding CIA constraint. A decrease in the supply of money is one kind of cause of deflation that is under the control of the monetary authority. A deflation can also happen naturally when technology innovates the productivity of the economy. Figure 4-3 shows the IRFs when there is a 10% technology shock. The blue line is when the CIA constraint is assumed to bind all the time, and the red line is when the CIA constraint can be occasionally binding.

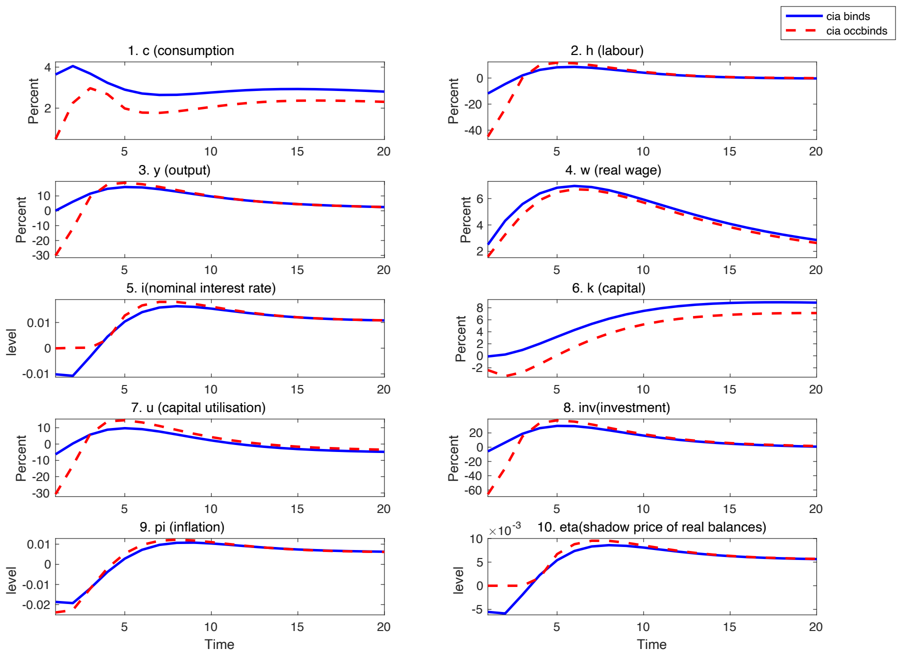


*Figure 4-3 IRFs following a 10% technology shock*

The dynamics of the real variables are the same as they would be in a real business cycle model without money when the CIA constraint is always binding. The direct impact of a technology innovation is an increase in productivity and thus a decrease in the price level. The hump-shape of increase in consumption and investment can be explained as the households need to accumulate real money balance for the need of consumption and investment[51]. The nominal interest rate decreases on impact and increases above its steady state along the adjustment path. Employment declines initially as a result of the increase in productivity and rises thereafter following

[51] Since the investment is subject to almost no adjustment cost, it appears more volatile.

the increase in real wage. However, when the nonbinding CIA constraint is considered, the positive technology shock can generate endogenously nonbinding CIA constraints if the technology innovation is large enough and real assets other than money are also included. A persistent positive 10% technology shock lowers the price level which is reflected by the decrease in inflation. As the inflation is jumping down to negative, the real return on money can be positive. This will give the households an incentive to increase their money holdings, which will trigger a nonbinding CIA constraint. Since the expected interest rate is a function of the shadow price of real balances, the nominal interest rate is expected to be zero when the shadow price of real balance is zero. As a consequence, agents are indifferent among money holdings, bonds and capital investment, which distorts the consumption and investment on impact. When the inflation rises above zero after 4 periods, the return on money becomes zero or negative again and the consumers have no incentive to save in money holdings. In this case, the CIA constraint is binding again. The consumers' needs to use up all their money holdings increase the price level and thus inflation. The nominal interest rate also increases, and the agents invest more in capital. As a result, output rises. However, the consumption does not increase above the level when the CIA constraint is always binding because the households tend to make more investment and it takes time for them to accumulate real balance[52]. Therefore, a 10% positive technology shock can generate a nonbinding CIA constraint for 4 period in a 20-period time dimension. It can be concluded that the probability of a binding CIA constraint is 80% in 20 periods after a 10% technology shock.

[52] There is an increase in investment because investment is only partially subject to the CIA constraint.

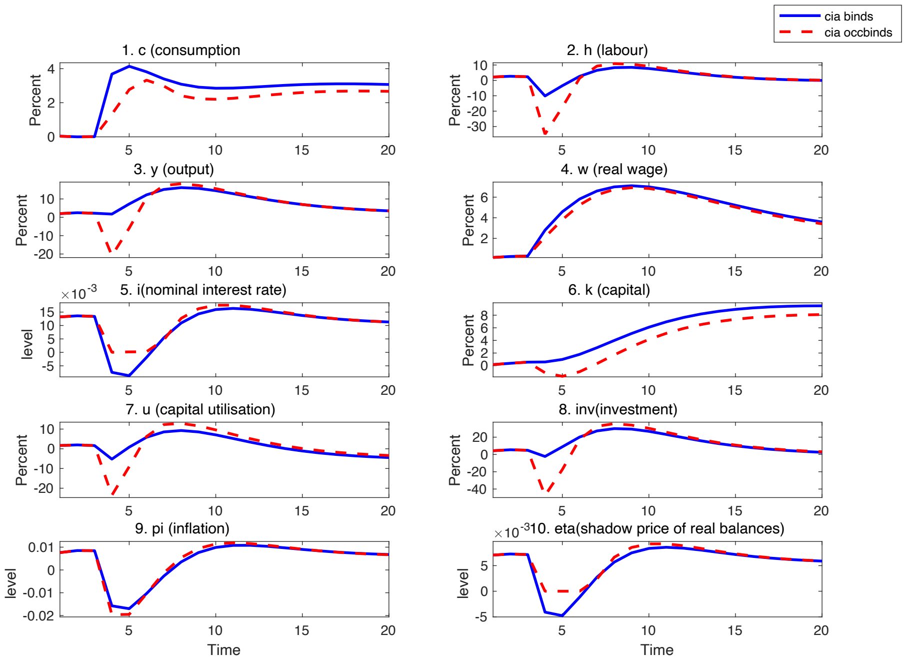


*Figure 4-4 IRFs following a 1% money supply shock and a 10% technology shock*

Figure 4-4 shows the dynamics of occasionally binding CIA constraints when there is a 1% money growth shock in period 1 followed by a 10% technology shock in period 4. Although positive money growth shock cannot generate liquidity effect, a significant positive technology shock can lead to nonbinding CIA constraint naturally. The significant technology innovation depresses the economy by reducing the output, investment and employment on impact and the CIA constraint is nonbinding as a result of negative inflation. The households are indifferent among money holdings, bond holdings and investment when the CIA constraint does not bind. When the CIA constraint is nonbinding, the marginal utility of consumption is equal to the marginal utility of wealth. Although there is an incentive for consumers to consume more but the amount of goods they can purchase is constrained by the depression in output. This is in turn the reason why the consumption cannot increase above the level when the CIA constraint always binds. However, it helps stimulus the economy when the CIA constraint binds again as the output and investment increase by more than those when the CIA constraint is always binding, though the effect is quite small. This is because the nominal interest rates increase by more following the increase in inflation and more investment is made. But the consumption is not stimulated as the consumption is fully subject to the CIA constraint while the investment is only partially subject to the CIA constraint. The dynamics of nonbinding CIA constraint is mainly contributed to the technology innovation. Hence, when there is a 1% positive money supply shock in period 1 followed by a 10% technology shock in period 4, the CIA constraint does not bind for 4 periods in a 20-period dimension, i.e. the probability of a binding CIA constraint is 80% for a 20-period dimension. Compared to the model with money as the only asset, model with the introduction of other assets, i.e. bonds and capital, fail to stimulate the consumption and the nonbinding CIA constraints or liquidity traps lead to further depression in investment and output.

## 4.3.2 Estimation for Benchmark Model

### 4.3.2.1 Calibration and Priors

The Bayesian estimation method follows Guerrieri and Iacoviello (2017) as in Chapter 3. The deep structural parameters of the benchmark model are estimated while the rest are still calibrated.

The prior distributions are reported in Table 4-2 in Appendix 4.B. The standard deviations of the shocks are assumed to follow an inverse-gamma distribution with a mean of 0.01 and 1 degree of freedom. The persistence of the AR(1) processes for technology and investment innovations are beta distributed with mean 0.75 and standard deviation 0.2. The persistence of money growth shock is assumed to be less and follows a beta distribution with mean 0.5 and standard deviation 0.2. These are quite standard calibrations. The fraction of investment that is subject to the CIA constraint is also estimated. It is assumed to follow a normal distribution with mean of 0.5 and standard deviation of 0.05. The priors describing the price and wage setting refer to Smets and Wouters (2007). The average length of price and wage contracts are assumed to be half a year. Thus, the Calvo parameters for prices and wages are assumed to be beta distributed with mean of 0.5 and standard deviation 0.1. The parameters measuring the degree of indexation to past inflation in goods and labour market are set to follow a beta distribution with mean 0.5 and standard deviation 0.15.

#### 4.3.2.2 Data and Estimation Results

Observations for three series shown in Table 4-3 are used to estimate the model: price inflation (CPI growth), GDP growth calculated on GDP per capita and personal consumption growth. These observations are based on the period from 1985Q1 to 2017Q4. The three series of data are consistent with three shocks (i.e. technology shock, money supply shock and investment shock) so that the model is exactly identified.

Table 4-3 Data sources for estimation

| Data Sources for Estimation | |
|---|---|
| Price inflation | Quarterly change in CPI deflator from Bureau of Economic Analysis, minus 0.5 percent |
| GDP growth | GDP per capital from Bureau of Economic Analysis, log transformed and detrended with one-sided HP filter |
| Consumption growth | Real personal consumption expenditures from Bureau of Economic Analysis, log transformed and detrended with one-sided HP filter |

The estimated model confirms the findings of calibrated model that the CIA constraint does not bind all the time, especially during the period after the global crisis. Table 4.2 in Appendix 4.B summarises the mode, the mean, and the 5 and 95 percentiles of the posterior distribution of the parameters which are obtained by the Metropolis-Hastings algorithm[53]. The productivity and investment processes are estimated to be more persistent, with an AR(1) coefficient of 0.9990 and 0.9962, respectively. The mean of the standard error of the productivity shock and investment shock are 0.0048 and 0.0049. The high persistence of productivity and investment shock indicates that the explanation of most of the forecast error variances of the real variables are contributed to those two shocks. In contrast, the persistence of the monetary shock is relatively low, with an AR(1) coefficient of 0.4806. It turns out that the mean of the posterior distribution is relatively close to the mean of the prior assumption for the estimates of the main behavioural parameters. The

[53] A standard Metropolis-Hastings algorithm with a chain of 50000 draws are used, see Guerrieri and Iacoviello (2017) for more details.

degree of price stickiness is estimated to be a bit higher than 0.5 while the Calvo parameter for wage is estimated to be slightly less than 0.5. The average duration of wage contracts is less than half a year; whereas the average duration of price contracts is about half a year. The mean of the degree of price and wage indexation are estimated to be a bit less than 0.5. The estimated mean fraction of investment that is subject to the CIA constraint is also close to the mean of the prior assumption 0.5. Wang and Wen (2005) show that, in consistent with the U.S. data, the peak of output response is not reached until three quarters after the monetary shock when the fraction of investment financed by cash is assumed to be 0.6. The estimated mean about 0.5 is not a loss of generality.

When the government conducts the monetary policy via a money growth rule, the shadow price of real balances (the CIA multiplier) is a function of household’s expected inflation and the expected nominal interest rate is a function of the shadow price of real balances. The correlation between the CIA multiplier and inflation is high (0.8997) as shown by the estimated model. Turning to the correlation between the inflation and nominal interest rate, it turns out the nominal interest rate is extremely highly correlated to inflation with the estimated correlation of 0.9097. This is in line with findings from calibrated model that inflation is the main driven force for nonbinding CIA constraints and liquidity traps.

The dynamics of the CIA multiplier for the benchmark model is displayed in Figure 4-5. It indicates that the probability of a binding CIA constraint is 86.36%. This dynamic is consistent with the results in Chapter 3 that the CIA constraint is more nonbinding during the period of the Great Recession. The periods when the CIA constraint is nonbinding also mimic the different stages of QE as argued in Chapter 3. There were three phases of QE by the Federal Reserve Board for the period from late 2008 to 2014. The CIA constraint does not bind for the first time during the period of Q1 (from the end of 2008 to the beginning of 2010) and the period of Q2 (from the end of 2010 to the middle of 2011). The economy escapes the liquidity traps for a while before the QE3 from the end of 2012 to the late 2014 which leads to the second and third binding CIA constraint periods. After the QE3, the economy slightly recovers from the Great Recession. It can be referred to a binding CIA constraint after 2016.

However, the estimated model failed to match the fact that the CIA constraint always binds before the Great Recession. One possible explanation for the nonbinding CIA constraint during 2000s is dot-com bubbles. After the dot-com bubbles burst after 2001, there was a huge collapse in stock market. This may motivate the consumers to hold more money and the CIA constraint is nonbinding.

As suggested by the calibration IRFs, when money is not the only asset in the economy, unconventional monetary policies like QE may lose their abilities to stimulate output and its components via nonbinding CIA constraints. The nonbinding CIA constraints also drives the nominal interest rates down to an extremely low level or zero. Money then becomes a perfect substitute to bonds and capital and serves as a safe store of value, which depresses the investment in capital and in turn depresses output.

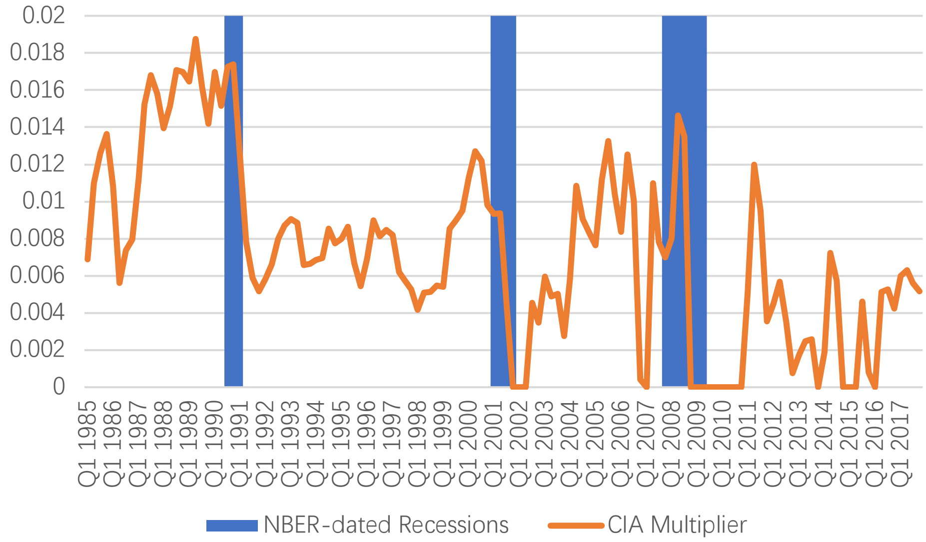


*Figure 4-5 Estimated CIA multiplier for the benchmark model*

### 4.3.3 Calibration for Alternative Model

Recall the alternative model is the one where the monetary policy follows a modified Taylor rule, which is also subject to a ZLB constraint.

$$1 + i_t = \max\{1, 1 + \bar{\iota}_t\}$$

$$1 + \bar{\iota}_t = (1 + i_{t-1})^{\rho_i}(1 + \bar{\iota})^{1-\rho_i}[\left(\frac{1 + \pi_t}{1 + \bar{\pi}}\right)^{\rho_\pi}\left(\frac{y_t}{\bar{y}}\right)^{\rho_y}]^{1-\rho_i}(\frac{y_t}{y_{t-1}})^{\rho_{\Delta y}}\exp(u_{mt})$$

#### 4.3.3.1 Calibration Parameters

Table 4-4 in Appendix 4.B contains the calibrated parameters for alternative model. The parameter choices are the same as those in the benchmark model except that the adjustment cost of investment is introduced in the alternative model. $\tau$ is the measure of the investment adjust costs by introducing more frictions into the model[54]. It is set at 6 according to the

[54] According to Galí (1999), Francis and Ramey (2005), and Galí and Rabanal (2004) argued that positive productivity shocks cause an immediate fall in hours worked due to nominal price rigidities, habit formation, and adjustment costs to investment.

estimation of Smets and Wouters (2007) that the mean of adjustment parameter for investment is 5.74. The parameters associated with the Taylor rule are also set following the estimation results of Smets and Wouters (2007). $\rho_\pi = 2$ is the long-run response to inflation, which is assumed to be always larger than 1. The output gap is defined as the difference between actual output and natural output. Here, the natural output is the output under the assumption that both price and wage are flexible. $\rho_y = 0.08$ is the long-run response to the output gap and $\rho_{\Delta y} = 0.2$ is the short-run response to the output gap.

There are also three shocks in the alternative model, i.e. technology shock, investment shock and interest rate shock. The autoregressive parameters and their standard deviations for both technology shock and investment shock are the same as in the baseline model. For interest rate shock, the autoregressive coefficient parameter, $\rho_i = 0.7$ is assumed to be significantly smoothing and its standard deviation $\sigma_m$ is also set at 0.01.

#### 4.3.3.2 Dynamics for Alternative Model

The alternative model differs from the benchmark model in that there are two constraints in the alternative model rather than only one constraint. The first constraint is the CIA constraint that gives the households an incentive to hold money. The second constraint is a constraint on nominal interest rate that it cannot go below zero. According to the algorithm of Guerrieri and Iacoviello (2015), the alternative model can be separated into four different regimes; the model sets and parameter assignments are kept the same in all these four regimes. Here, regime 1 refers to the usual case when the CIA constraint is binding and the ZLB constraint is nonbinding. The conditions under this regime are $\eta_t > 0$ or $c_t + \phi_I I_t + b_t = \frac{m_{t-1}}{1+\pi_t} + \frac{1+i_{t-1}}{1+\pi_t} b_{t-1} + s_t$ and $i_t = (1 +$

$i_{t-1})^{\rho_i}(1+\bar{\imath})^{1-\rho_i}[\left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\rho_\pi}\left(\frac{y_t}{\bar{y}}\right)^{\rho_y}]^{1-\rho_i}(\frac{y_t}{y_{t-1}})^{\rho_{\Delta y}}\exp(u_{mt})-1$. Regime 2 is the case when both the CIA constraint and the ZLB constraint are nonbinding with the conditions $\eta_t=0$ and $i_t=(1+i_{t-1})^{\rho_i}(1+\bar{\imath})^{1-\rho_i}[\left(\frac{1+\pi_t}{1+\bar{\pi}}\right)^{\rho_\pi}\left(\frac{y_t}{\bar{y}}\right)^{\rho_y}]^{1-\rho_i}(\frac{y_t}{y_{t-1}})^{\rho_{\Delta y}}\exp(u_{mt})-1$. Regime 3 refers to the case when both the CIA constraint and the ZLB constraint are binding with the conditions $c_t+\phi_I I_t+b_t=\frac{m_{t-1}}{1+\pi_t}+\frac{1+i_{t-1}}{1+\pi_t}b_{t-1}+s_t$ and $i_t=0$. The last regime, regime 4, is when the CIA constraint is nonbinding while the ZLB constraint is binding with the conditions $\eta_t=0$ and $i_t=0$. The alternative model also satisfied the two requirements for the toolkit to work. Since the expected nominal interest rate is a function of the shadow price of real balances, whether the CIA constraint is binding or not and whether the ZLB constraint is binding or not are positively linked[55]. More specifically, when the CIA constraint is binding, then the ZLB constraint is expected to be nonbinding. When the CIA constraint is nonbinding, then the ZLB constraint is expected to be binding. Hence, it can still be summarised into two groups that when the CIA constraint is binding and when the CIA constraint is nonbinding, while the nonlinearity of ZLB will bring more volatilities into the alternative model[56].

Figure 4-6 shows the IRFs when there is a 2% negative interest rate shock. The blue line is when the CIA constraint is always binding, and the red line is when the CIA constraint is allowed to be nonbinding for some periods.

[55] When the nominal interest rate is zero, the ZLB constraint is binding. When the nominal interest rate is positive, the ZLB constraint is nonbinding.

[56] Nakata (2017) finds that consumption, inflation and output are reduced by a larger amount when the ZLB constraint is binding than when it is not.

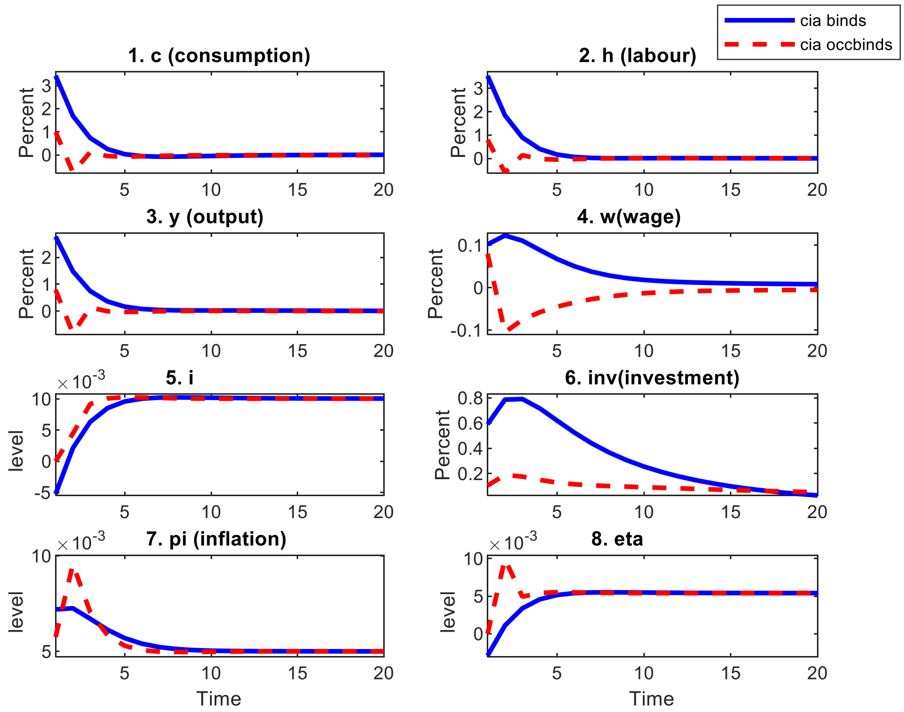


*Figure 4-6 IRFs following a negative 2% interest rate shock*

A cut in the policy rate is consistent to a monetary easing. Here, monetary shocks affect the economy through an interest rate feedback rule instead of simply through inflation. Hence, whether the CIA constraint binds or not relies on whether the ZLB constraint is expected to bind or not. The blue line shows when the CIA constraint is always binding and the ZLB constraint never binds. In response to an unexpected cut in the policy rate, output, consumption and investment increase. Inflation also rises in response to the monetary easing. Since prices and nominal wages are sticky, there are hump shapes in the dynamics of inflation and real wage. Employment, which is determined by demand side, increases along with the increase in output. The nominal interest rate reacts to the interest rate shock endogenously. The government has the incentive to raise the interest rates when the shock hits the economy in response to the rise in inflation and a positive output gap. But the increase in interest rate is not enough to offset the decrease in policy rate. Thus, the policy rate still falls relative to its equilibrium value. The red line is when the CIA constraint can be endogenously nonbinding. The nonbinding CIA constraint is a result of expected binding ZLB constraint when there is a negative interest rate shock. A negative 2% interest rate shock drives down the interest rate to negative. However, the nominal interest rate is constrained by ZLB, i.e. the interest rate cannot fall below zero. Since the interest rate are stopped from falling, the inflation increases by less on impact. An expected binding ZLB constraint implies a nonbinding CIA constraint. As a result, the households tend to hold more money as a store of value because bonds and investment cannot give extra return. In this situation of liquidity trap, money works as a safe asset which prevents the real interest rate from falling and depresses the investment. Consumption and output are also depressed on impact. When the nominal interest rate becomes positive and the CIA constraint binds again in period 2, the greatest effect of monetary easing on inflation occurs. Because prices adjust more frequently than nominal wages, the real wage falls. After the CIA constraint binds, consumption, output and employment return back to the equilibrium level. However, it takes longer for the investment to return back because of high investment adjustment cost. A small cut in the policy rate only generate the binding ZLB constraint and nonbinding CIA constraint for one period in a 20-period dimension. When both ZLB constraint and CIA constraint are considered endogenously binding, nonbinding CIA constraints fail to stimulate the economy. It takes even longer time for the investment to return back after the nominal interest rate is positive and the CIA constraint binds.

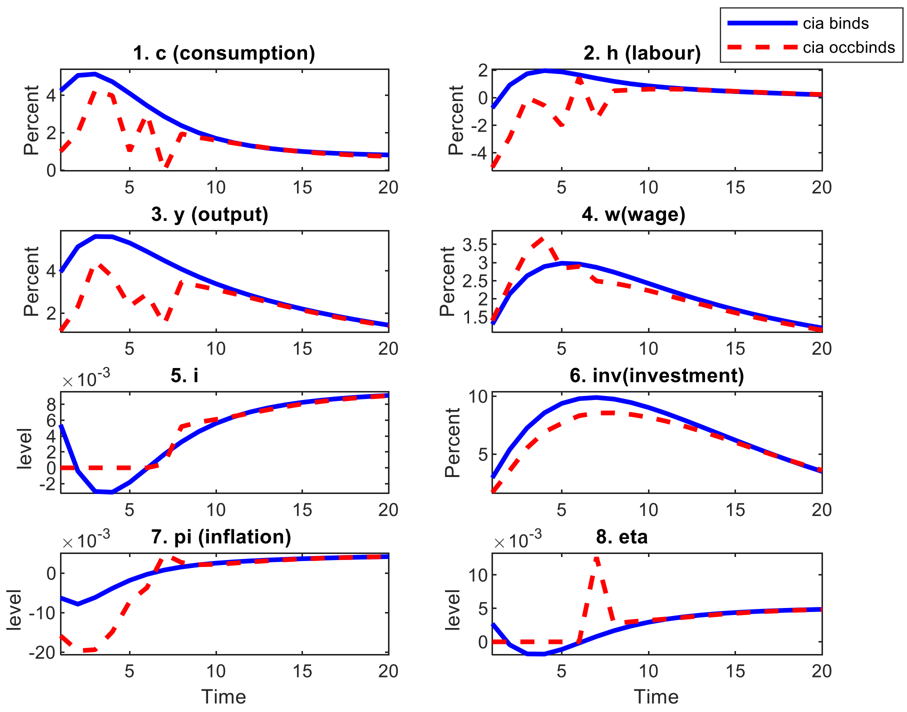


*Figure 4-7 IRFs following a 5% technology shock*

Figure 4-7 shows the dynamics when there is a 5% technology shock. A 5% technology shock is large enough to generate the occasionally binding CIA constraint for a model with an interest rate feedback rule. Unlike interest rate shocks, technology shocks can naturally lead to a negative inflation, which is the main contribution to a nonbinding CIA constraint and thus a binding ZLB constraint. The IRFs following a technology innovation for the model with an interest rate feedback rule (the alternative model) are similar to those for the model with a money growth rule (the benchmark model). A technology shock lowers the price level and thus inflation. The monetary authority responds to the fall in inflation by lowering the nominal interest rate. If the CIA constraint is assumed to be binding all the time, money is neutral when there is only technology shock. However, money can have real effect when the CIA constraint and ZLB constraint can be endogenously binding. On the one hand, inflation is reduced to negative on impact, which implies money can have return. The households expect that there will be positive return on money and tend to hold more money. The excess holding of money leads to a decrease in the marginal utility of money. When the marginal utility of money is reduced to zero, the CIA constraint is nonbinding. As a result, the nominal interest rate is zero, implying the ZLB constraint is binding. On the other hand, the monetary authority makes response to the decrease in inflation by lowering the policy rate. Since there is a limit for the policy rate that it cannot be negative, the policy rate can only be lowered to zero, which is consistent with the binding ZLB constraint. A zero-interest rate indicates that the shadow price of real balance should be zero, which results in a nonbinding CIA constraint. Money is considered as a safe store of value compared to bonds and capital when the ZLB constraint binds and the CIA constraint does not bind. The economy falls in a liquidity trap. Consumption, output and investment are all depressed as a result of excess money holdings. Employment decreases on impact because of the depression in output and the rise in real wage. The IRFs for the alternative model differs from those for the benchmark model in that output and consumption appears to be more volatile. This may due to the interactions between two nonlinearities introduced in the alternative model and the construction of the interest rate feedback rule. The expected nominal interest rate is a function of the current shadow price of real balances. When the shadow price of real balances becomes positive, the expected nominal interest rate is still zero. There should be a large increase in the shadow price of real balances to save the economy from liquidity traps. As a result, the CIA constraint is nonbinding for 6 periods while the ZLB is binding for one period more, 7 periods. In conclusion, the probability of a nonbinding CIA constraint is 70% and the probability of a binding ZLB constraint is 65%.

Next, Figure 4-8 shows the simulations which is closer to what happened after the global financial crisis. There is a negative interest rate that triggers a non-persistent liquidity trap in period 1, and then a technology innovation that leads to a more persistent liquidity trap. As before, the blue line is the case when the CIA constraint is always binding while the red line is the case when the CIA constraint is occasionally binding. Although a negative interest rate shock cannot generate a persistent liquidity trap, the economy is under the depression if the technology innovation happens before the effect of negative interest rate fades away. Figure 4-8 shows that the economy will return back to its equilibrium after 6 periods except for the investment which takes longer. If the economy is hit by a technology innovation in period 4, consumption, output and investment are still in the depression. This technology innovation, instead of stimulating the economy, triggers a longer liquidity trap. Consumption, output and investment, though, rise but are still depressed compared to those when the CIA constraint is always binding and the ZLB constraint is always nonbinding. To sum up, the economy is in the liquidity trap (i.e. the ZLB constraint binds) for 8 periods: 1 period when the negative interest rate shock hit the economy and 7 periods when the technology innovation happens. The CIA constraint is nonbinding for 7 periods: 1 period for negative interest rate shock and 6 periods for the technology shock. Hence, the probability of a nonbinding CIA constraint is 65% and the probability of a liquidity trap is 60%.

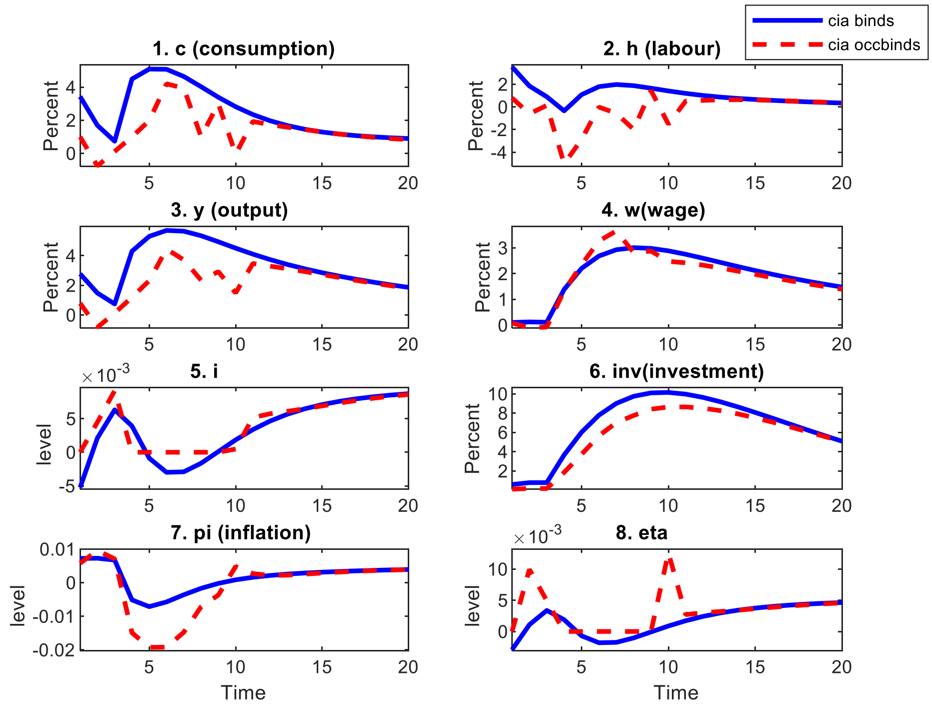


*Figure 4-8 IRFs following a negative 2% interest rate shock and a 5% technology shock*

### 4.3.4 Estimation for Alternative Model

#### 4.3.4.1 Calibration and Priors

Similar to the benchmark model, the deep structural parameters are estimated while the rest remains to be calibrated. The prior distributions are reported in Table 4-5 in Appendix 4.B. All parameters are assigned the same prior distributions as in the benchmark model except for the parameters describing the monetary policy rule which are based on a standard Taylor rule. The prior distributions for parameters of the Taylor rule are borrowed from Smets and Wouters (2007): the long-run reaction on inflation are described by a normal distribution with mean 1.5 and standard deviation 0.25. The long-run reaction on the output gap and the short-run reaction on the change in the output gap are assumed to be restricted to lie between 0 and 1. Thus, a beta distribution is assigned to these two parameters. Specifically, both the long-run reaction on the output gap and the short-run reaction on the change in the output gap follow a beta distribution with mean 0.125 (0.5 divided by 4) and standard deviation 0.05. The persistent of the interest rate shock is determined by the coefficient on the lagged interest rate. This coefficient is assumed to be beta distributed around a mean of 0.75 with a standard deviation of 0.1. The estimated posterior mean for the fraction of investment that is subject to the CIA constraint is very close to its prior mean. Hence, for the alternative model, this parameter is, instead, calibrated at 0.5.

#### 4.3.4.2 Data and Estimation Results

Since there are three shocks in the model, for the model to be exactly identified observations for three series need to be used in estimation. The nonlinearity of ZLB has been included in the alternative model. Nominal interest rate, instead, are an essential data series for the estimation as well as price inflation. Another data series used is the GDP growth as in the benchmark model. Data sources used for estimation are summarised in Table 4-6. The data period is also from 1985Q1 to 2017Q4.

Table 4-6 Data sources for estimation

| Data Sources for Estimation | |
| --- | --- |
| Price inflation | Quarterly change in CPI deflator from Bureau of Economic Analysis, minus 0.5 percent |
| GDP growth | GDP per capital from Bureau of Economic Analysis, log transformed and detrended with one-sided HP filter |
| Interest rate | Effective Federal fund rate, annualized percent, divided by 400 to convert into quarterly units |

The mode, the mean, and the 5 and 95 percentiles of the posterior distribution of deep structural parameters are summarised in Table 4-5 in Appendix 4.B and a Metropolis-Hastings algorithm with a chain of 50000 draws are implemented. Firstly, the observations regarding the estimated process for the exogenous disturbances seem to be informative. The productivity and investment processes are estimated to be slightly more persistent, with an AR(1) coefficient of 0.7917 and 0.7960, respectively. The mean of standard deviations of these two shocks are quite similar (0.0050 and 0.0047, respectively). The mean of standard error of interest rate feedback shock (0.0047) is also similar to those of technology and investment innovations. This suggests that the combination of these three disturbances will make contributions to explain the forecast error variance of the real variables at a long horizon. Secondly, the estimates of the main behaviour parameters appear to differ from those of the benchmark model. The estimated degree of price stickiness is a bit higher than 0.5 while that of wage stickiness are very close to the mean of the prior assumption. The average duration of wage contracts remains around half a year; whereas the average duration of price contracts is about three quarters[57]. The degree of indexation to past inflation in both goods market and labour market is estimated to much closer to the prior mean. The mean of the degree of price indexation (0.5275) is estimated to be much larger than that of Smets and Wouters (2007). The mean of the posterior degree of wage indexation (0.5622), on the other hand, is consistent with the estimation of Smets and Wouters (2007). Thirdly, turning to what is new in the alternative model (the monetary policy reaction function parameters), the estimated mean of the long-run reaction coefficient to inflation is estimated to be relatively high (2.0037). This may be due to the reason that the interaction between the ZLB constraint and the CIA constraint is a phenomenon of the monetary effects on policy rate as well as inflation. There is a considerable degree of interest rate smoothing. The mean of the coefficient on the lagged interest rate is estimated to be 0.7814, which is slightly less than that estimated by Smets and Wouters (2007). It seems that policy does not react very strongly to both the level of output gap in the long run (0.0539) and the changes in the output gap in the short run (0.0961).

When two nonlinearities, i.e. the ZLB constraint on policy rates and the CIA constraint, are needed to account for, the estimation results show that the period when the CIA constraint is nonbinding is consistent with the period when the ZLB constraint is binding. The ZLB is binding for one more period after the CIA constraint is already binding as the nominal interest rate is a function of the shadow price of real balances (the CIA multiplier). Hence, the dynamics of nonbinding CIA constraint, instead of a simple result of inflation, follows the joint dynamics of both the nominal interest rate and inflation. The estimated correlation between the CIA multiplier and the policy rate is

[57] The estimated degree of price stickiness is close to that of Smets and Wouters (2007). The average duration of wage contracts is, however, estimated shorter than that of Smets and Wouters (2007).

extremely high (0.9992); whereas the estimated correlation between the CIA multiplier and inflation (0.4456) is much lower. This suggests that the nominal interest rate is the main driven force for nonbinding CIA constraints in the alternative model when the ZLB constraint is taken into account, as argued in calibrated IRFs, rather than the inflation in the benchmark model. A liquidity trap is obviously a result of zero or extreme low policy rate.

Figure 4-9 plots the dynamics of the CIA multiplier for the alternative model. It indicates that the probability of a binding CIA constraint is 82.58%, which is close to the estimated probability of a binding CIA constraint in the benchmark model. The estimated alternative model succeeds in matching the fact that the CIA constraint is always binding before the Great Recession. When the monetary policy is conducted via an interest rate feedback rule which is also subject to a ZLB constraint, different stages of QE can lead to nonbinding CIA constraints but are not the main driven force for the nonbinding constraints. As suggested by the estimation results, the correlation between the CIA multiplier and policy rate is extremely high, the periods when the CIA constraint is nonbinding mimics the periods when the policy rate is set to be nearly zero (the ZLB constraint is binding).

In December 2008, the Federal Reserve cut its target for the Federal Funds Rate to between 0% and 0.25% and the interest rates stayed at 0% until the end of 2009. The Federal Reserve finally raised its Federal Funds Rate in December 2015, which can be marked as the end of the ZLB on short-run nominal interest rate for the US. This can be confirmed by the dynamics of the CIA multiplier: between December 2008 and 2015, the CIA multiplier is zero or nearly zero, which is corresponding to the period when the Federal Reserve set its target policy rate to almost zero; after December 2015, the CIA multiplier obviously increases above zero as a result that the target policy rate is raised. Since the CIA multiplier is already at zero and the economy is in a liquidity trap, increase in the money supply is likely to be ineffective in stimulating the economy, or even further depresses the economy. This is because the households are generally indifferent to an increase in the money base when there is a liquidity trap. Instead of spending the extra money, they tend to hold the extra money as a safe asset.

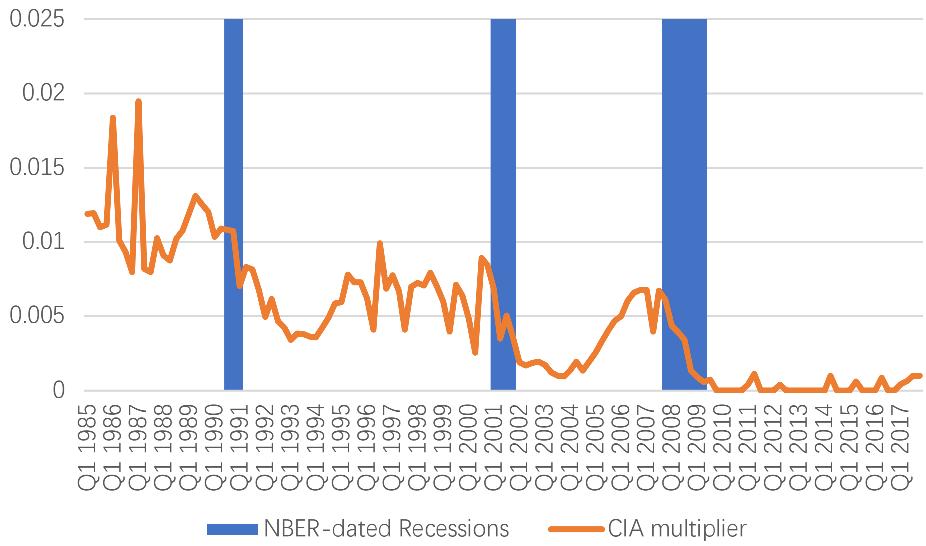


*Figure 4-9 Estimated CIA multiplier for the alternative model*

### 4.3.5 Comparison Between Benchmark and Alternative Models

Section 4.3.1 and Section 4.3.3 analyse the calibration properties for the benchmark and alternative models while Section 4.3.2 and Section 4.3.4 illustrates the estimation results for those two models. It can be concluded that whether the CIA constraint is binding or not mainly depends on whether the inflation is negative or not in the benchmark model; whereas whether the CIA constraint is binding or not mainly depends on whether the ZLB constraint is binding or not (or whether the nominal interest rate is expected to be zero or not) in the alternative model. When money is not the only asset, and nominal bonds and capital are included, nonbinding CIA constraint fails to stimulate output and its components. If the economy is already in a liquidity trap, that is the nominal interest rate cannot fall any further, the depression in output and its components is even larger. This is because when there is a liquidity trap or the ZLB constraint is binding, money becomes perfect substitute for bonds and capital. The households, instead of using up their cash, hold money as a safe store of value. Money, in this case, prevents the reduction of the real interest rate.

Figure 4-10 compares consumption growth, output growth and inflation in the data (blue line) with their estimated model counterparts (red line) for the benchmark model. It is obvious that occasionally binding CIA constraints magnify the effects of nonbinding CIA constraints on output and consumption growth. As is argued that the expected nominal interest rate is a function of the CIA multiplier, Figure 4-11 shows that the dynamics of the interest rate follows the dynamics of the CIA multiplier, which is consistent with the high correlation between the CIA multiplier and interest rate, but fails to match the real data. This maybe the result that the interest rate is generated from the CIA multiplier rather than follows the policy rate set by the monetary authority. Under this circumstance, the increase in money base still helps the economy to escape the liquidity traps, though the effects are quite limited.

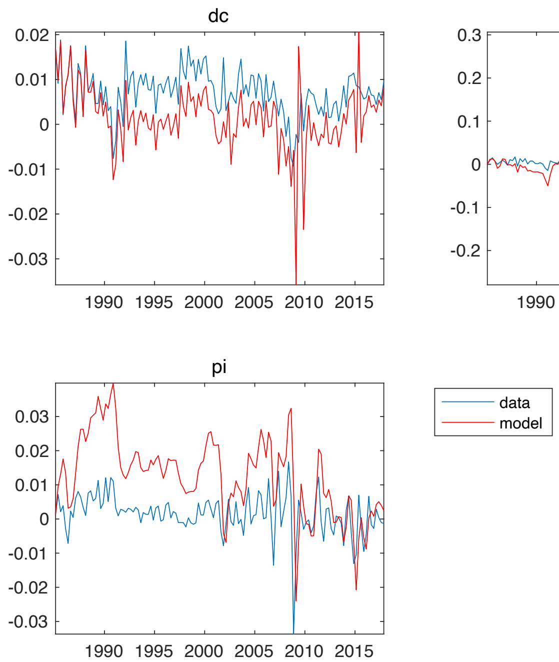


*Figure 4-10 Comparison between data and estimated benchmark model*

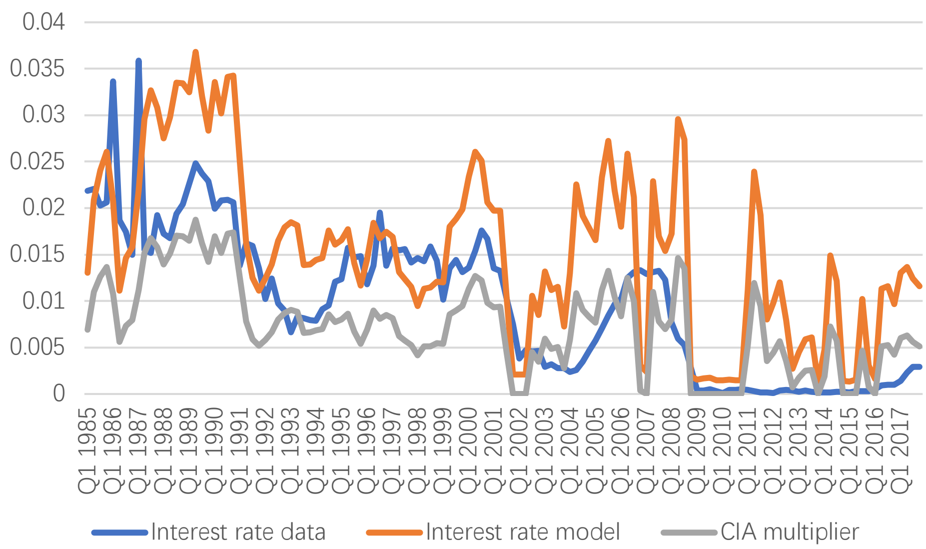


*Figure 4-11 Comparison between data and estimated benchmark model for interest rate*

The Federal Reserve adhered to the Taylor rule since 1995. Taylor (2012) also argued that the Fed followed the Taylor rule closely until around 2003.

Hence, it is understandable that the monetary policy that follows an exogenous money growth rule in the benchmark model cannot match the data. Figure 4-12 proves that the alternative model with endogenously binding CIA constraint and ZLB constraint can match the data very well, especially for the period of the Great Recession. The alternative model helps to solve the critiques against standard linearized DSGE model to fit the data after the global crisis. The first critique is that the linearized DSGE models severely underestimate the actual fall in GDP. The second one is that these models predict that inflation should decrease much more than it actually did. The alternative successfully predicts the output growth but slightly underestimates the decrease in inflation. Figure 4-13 confirms that the main driven force for the nonbinding CIA constraint in the alternative model is the dynamics of the nominal interest rate. Under this circumstance, the increase in money stock fails to generate an escape of liquidity trap unless the monetary authority raises its policy rate.

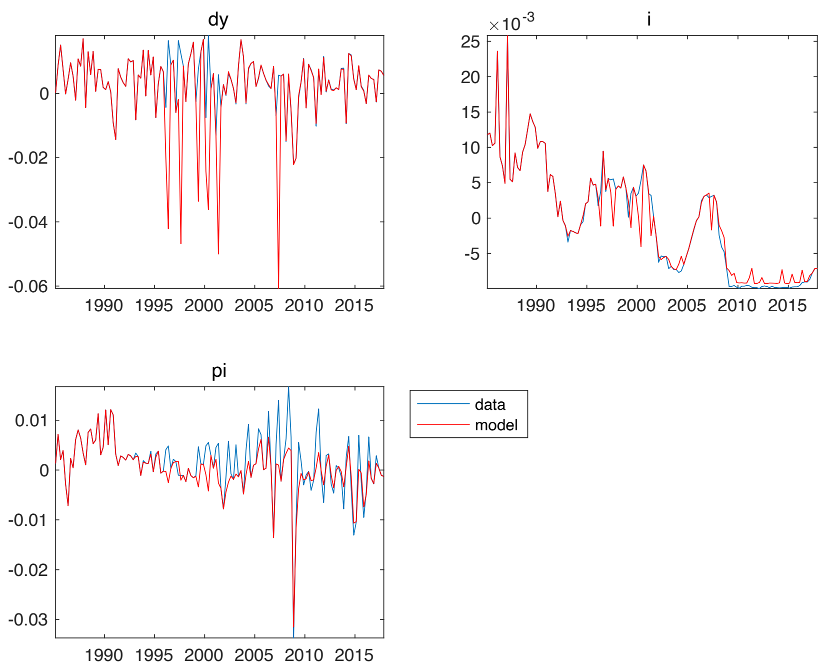


*Figure 4-12 Comparison between data and estimated alternative model*

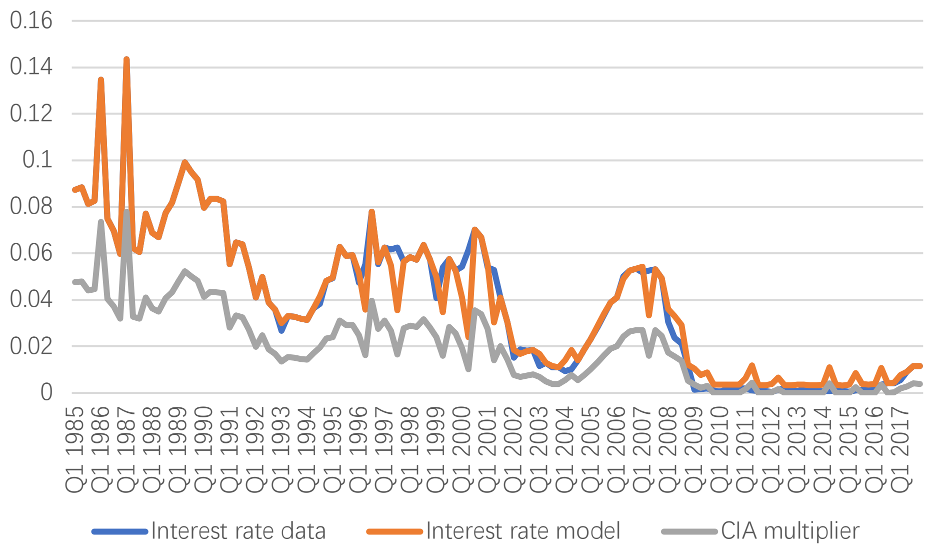


*Figure 4-13 Comparison between data and estimated alternative model for interest rate*

## 4.4 Conclusion

This Chapter is an extension of Chapter 3 by introducing other real assets into the model. When nominal bonds and capital are included and the CIA constraint is assumed to be on consumption as well as investment, nonbinding CIA constraint loses its ability to boost the economy as money becomes a safe asset that prevents the interest rate from falling. Things are even worse when the monetary policy follows an interest rate feedback rule, such as Taylor rule. As long as the nominal interest rate is stuck at its ZLB, money depresses the investment, and thus output and consumption compared to the case when the CIA constraint always binds and the ZLB never binds. Unconventional monetary policy that increases the monetary base will not necessarily have significant effect on inflation in a liquidity trap. The increase in the money supply tends to be saved rather than consumed. The economy still suffers deflation and deep recession, which results in a substantial spare capacity. The interest feedback rule, such as Taylor rule, may response to this recession by suggesting high negative interest rates, which will lead to a more serious ZLB. From this perspective, liquidity traps are more persistent[58]. Unconventional monetary policies may not help the economy to escape the liquidity trap, and it may generate slower recovery by improving the deflation through a large increase in monetary base or bonds.

[58] Bacchetta, Benhima and Kalantzis (2019) argued that quantitative easing causes a deeper liquidity trap.

## Appendix 4.A Non-stochastic Steady States

From the Euler equation for nominal bonds, given the condition for steady state inflation $\pi = \bar{\pi}$, the steady state nominal interest rate is solved as

$$i = \frac{1+\bar{\pi}}{\beta} - 1 \qquad (4.A.1)$$

Recall the Euler equation for real balance,

$$\eta = \frac{1+\bar{\pi}-\beta}{\beta}\lambda \qquad (4.A.2)$$

Go to the first-order condition for investment and note that $Z = 1$,

$$\lambda + \phi_I \eta = \mu \qquad (4.A.3)$$

As long as $\eta > 0$ in steady state, the steady state Tobin's q is larger than 1. Equation 4.A.3 can be reduced to

$$\mu = \left(1 + \frac{1+\bar{\pi}-\beta}{\beta}\phi_I\right)\lambda \qquad (4.A.4)$$

Impose $u = 1$ in the steady state, then steady state depreciation is just $\delta_0$. Consider the Euler equation for capital,

$$\mu = \beta[\lambda r + \mu(1-\delta_0)] \qquad (4.A.5)$$

Hence,

$$r = \frac{[1-\beta(1-\delta_0)]\left(1 + \frac{1+\bar{\pi}-\beta}{\beta}\phi_I\right)}{\beta} \qquad (4.A.6)$$

From the first-order condition for utilization, the value of $\delta_1$ can be solved as

$$\delta_1 = \frac{r}{1 + \frac{1 + \bar{\pi} - \beta}{\beta}\phi_I} \quad (4.A.7)$$

The steady state reset price inflation can be solved from the evolution of inflation condition given the exogenous steady state inflation rate

$$\pi^{\#} = \left[\frac{(1+\bar{\pi})^{1-\varepsilon_p} - \phi_p(1+\bar{\pi})^{\xi_p(1-\varepsilon_p)}}{1-\phi_p}\right]^{\frac{1}{1-\varepsilon_p}} - 1 \quad (4.A.8)$$

Only is $\bar{\pi} = 0$ or $\xi_p = 1$, $\pi^{\#} = \bar{\pi}$. Then, an expression for steady state dispersion is

$$v^p = \frac{(1-\phi_p)\left(\frac{1+\pi^{\#}}{1+\bar{\pi}}\right)^{-\varepsilon_p}}{1-\phi_p(1+\bar{\pi})^{(1-\xi_p)\varepsilon_p}} \quad (4.A.9)$$

Only if $\bar{\pi} = 0$ or $\xi_p = 1$ will $v^p = 1$.

The steady states of $x_1$ and $x_2$ are expressed as

$$x_1 = \frac{c^{-\sigma}mcy}{1-\beta\phi_p(1+\bar{\pi})^{(1-\xi_p)\varepsilon_p}} \quad (4.A.10)$$

$$x_2 = \frac{c^{-\sigma}y}{1-\beta\phi_p(1+\bar{\pi})^{(1-\xi_p)(\varepsilon_p-1)}} \quad (4.A.11)$$

The ratio of these two expressions is

$$\frac{x_1}{x_2} = mc\frac{1-\beta\phi_p(1+\bar{\pi})^{(1-\xi_p)(\varepsilon_p-1)}}{1-\beta\phi_p(1+\bar{\pi})^{(1-\xi_p)\varepsilon_p}} \quad (4.A.12)$$

If $\bar{\pi} = 0$ or $\xi_p = 1$, this ration will be equal to $mc$. In conjunction with the reset inflation condition, the steady state marginal cost is obtained

$$mc = \frac{\varepsilon_p - 1}{\varepsilon_p} \frac{1 + \pi^{\#}}{1 + \bar{\pi}} \frac{1 - \beta\phi_p (1 + \bar{\pi})^{(1-\xi_p)\varepsilon_p}}{1 - \beta\phi_p (1 + \bar{\pi})^{(1-\xi_p)(\varepsilon_p - 1)}} \quad (4.A.13)$$

Only if $\bar{\pi} = 0$ or $\xi_p = 1$ will $mc = \frac{\varepsilon_p - 1}{\varepsilon_p}$.

Then the steady state capital-labour ratio is

$$\frac{k}{h^d} = \left(\frac{\alpha A mc}{r}\right)^{\frac{1}{1-\alpha}} \quad (4.A.14)$$

Note that if $u = 1$, there is no difference between steady state capital and steady state capital services $k = \hat{k}$.

Once knowing the capital-labour ratio, the steady state wage is

$$w = (1 - \alpha)\left(\frac{k}{h^d}\right)^{\alpha} mc \quad (4.A.15)$$

Note that $A = 1$ in steady state. The steady state reset wage is given from the wage evolution equation

$$w^{\#} = \left[\frac{1 - \phi_w (1 + \bar{\pi})^{(1-\xi_w)(\varepsilon_w - 1)}}{1 - \phi_w}\right]^{\frac{1}{1-\varepsilon_w}} w \quad (4.A.16)$$

Only if $\bar{\pi} = 0$ or $\xi_w = 1$ will $w^{\#} = w$. Now, the steady state $\hat{n}_1$ and $\hat{n}_2$ can be expressed as

$$\hat{n}_1 = \frac{\left(\frac{w}{w^{\#}}\right)^{\varepsilon_w (1+\varphi)} {h^d}^{1+\varphi}}{1 - \beta\phi_w (1 + \bar{\pi})^{(1-\xi_w)\varepsilon_w (1+\varphi)}} \quad (4.A.17)$$

$$\hat{n}_2 = \frac{\lambda \left(\frac{w}{w^{\#}}\right)^{\varepsilon_w} h^d}{1 - \beta\phi_w (1 + \bar{\pi})^{(1-\xi_w)(\varepsilon_w - 1)}} \quad (4.A.18)$$

The ratio of these is

$$\frac{\hat{n}_1}{\hat{n}_2} = \lambda^{-1}\left(\frac{w}{w^{\#}}\right)^{\varepsilon_w\varphi} h^{d^{\varphi}}\,\frac{1-\beta\phi_w(1+\bar{\pi})^{(1-\xi_w)(\varepsilon_w-1)}}{1-\beta\phi_w(1+\bar{\pi})^{(1-\xi_w)\varepsilon_w(1+\varphi)}} \qquad (4.A.19)$$

From the reset wage condition, this should equal

$$w^{\#} = \frac{\varepsilon_w}{\varepsilon_w-1}\lambda^{-1}\left(\frac{w}{w^{\#}}\right)^{\varepsilon_w\varphi} h^{d^{\varphi}}\,\frac{1-\beta\phi_w(1+\bar{\pi})^{(1-\xi_w)(\varepsilon_w-1)}}{1-\beta\phi_w(1+\bar{\pi})^{(1-\xi_w)\varepsilon_w(1+\varphi)}} \qquad (4.A.20)$$

Isolate $h^d$,

$$h^{d^{\varphi}} = \frac{\varepsilon_w-1}{\varepsilon_w}\lambda w^{\#}\left(\frac{w}{w^{\#}}\right)^{-\varepsilon_w\varphi}\frac{1-\beta\phi_w(1+\pi)^{(1-\xi_w)\varepsilon_w(1+\varphi)}}{1-\beta\phi_w(1+\pi)^{(1-\xi_w)(\varepsilon_w-1)}} \qquad (4.A.21)$$

From the first-order condition for consumption

$$\lambda = \frac{\beta}{1+\bar{\pi}}c^{-\sigma} \qquad (4.A.22)$$

Plug equation $4.A.21$ into the condition $4.A.22$

$$h^{d^{\varphi}} = \frac{\varepsilon_w-1}{\varepsilon_w}\frac{\beta}{1+\bar{\pi}}c^{-\sigma}w^{\#}\left(\frac{w}{w^{\#}}\right)^{-\varepsilon_w\varphi}\frac{1-\beta\phi_w(1+\pi)^{(1-\xi_w)\varepsilon_w(1+\varphi)}}{1-\beta\phi_w(1+\pi)^{(1-\xi_w)(\varepsilon_w-1)}} \qquad (4.A.23)$$

Consider the aggregate resource constraint for a while,

$$y = c + I \qquad (4.A.24)$$

From the capital accumulation equation,

$$I = \delta_0 k \qquad (4.A.25)$$

Then,

$$y = c + \delta_0 k \qquad (4.A.26)$$

Divide both sides by $h^d$,

$$\frac{y}{h^d} = \frac{c}{h^d} + \delta_0 \frac{k}{h^d} \qquad (4.A.27)$$

$\frac{k}{h^d}$ is given by equation $4.A.14$, then

$$\frac{y}{h^d} v^p = \left(\frac{k}{h^d}\right)^{\alpha} - \frac{F}{h^d} \qquad (4.A.28)$$

It is assumed that there is no fixed cost, $F = 0$. The $\frac{c}{h^d}$ is solved as

$$\frac{c}{h^d} = \frac{\left(\frac{k}{h^d}\right)^{\alpha}}{v^p} - \delta_0 \frac{k}{h^d} \qquad (4.A.29)$$

Then $h^d$ can be solved as

$$h^d = \left[ w^{\#} \left(\frac{w}{w^{\#}}\right)^{-\varepsilon_w \varphi} \frac{\varepsilon_w - 1}{\varepsilon_w} \frac{\beta}{1+\bar{\pi}} \left(\frac{c}{h^d}\right)^{-\sigma} \frac{1 - \beta\phi_w (1+\bar{\pi})^{(1-\xi_w)\varepsilon_w(1+\varphi)}}{1 - \beta\phi_w (1+\bar{\pi})^{(1-\xi_w)(\varepsilon_w - 1)}} \right]^{\frac{1}{\varphi+\sigma}} \qquad (4.A.30)$$

Once the steady state labour supply has been solved, everything else can be obtained straightforward:

$$k = \frac{k}{h^d} h^d \qquad (4.A.31)$$

$$c = \frac{c}{h^d} h^d \qquad (4.A.32)$$

$$y = \frac{\left(\frac{k}{h^d}\right)^{\alpha} h^d}{v^p} \qquad (4.A.33)$$

$$m = c + \phi_I I \qquad (4.A.34)$$

$$s = m - \frac{m}{1 + \bar{\pi}} \qquad (4.A.35)$$

## Appendix 4.B Tables

Table 4-1 Parameter values for benchmark model

| Parameter Assignment (Benchmark Model) | | |
|---|---|---|
| $\beta$ | 0.995 | Discount factor |
| $\sigma$ | 1 | Coefficient of relative risk aversion |
| $\varphi$ | 1.2 | The elasticity of wage with respect to hours |
| $F$ | 0 | The fixed cost of production |
| $\alpha$ | 1/3 | The capital share in production |
| $\delta_0$ | 0.025 | Capital depreciation parameter |
| $\delta_2$ | 0.01 | The utilization adjustment cost parameter |
| $\tau$ | 0.01 | The magnitude of capital adjustment costs |
| $\phi_I$ | 0.3 | The fraction of investment that must be financed by cash |
| $\varepsilon_p$ | 6 | The elasticity of substitution among intermediate goods |
| $\varepsilon_w$ | 6 | The elasticity of substitution among different type of labour |
| $\bar{\pi}$ | 0.005 | Steady state inflation |
| $\phi_p$ | 0.7 | The Calvo probability that a firm does not change its price |
| $\phi_w$ | 0.6 | The Calvo probability that a household does not change its wage |
| $\xi_p$ | 0.5 | Nominal price indexation to lagged inflation |
| $\xi_w$ | 0.5 | Nominal wage indexation to lagged inflation |

| | | |
|---|---|---|
| $\rho_a$ | 0.9 | Autoregressive parameter for technology shock |
| $\rho_z$ | 0.9 | Autoregressive parameter for investment shock |
| $\rho_m$ | 0.6 | Autoregressive parameter for money growth |
| $\sigma_a$ | 0.01 | Standard deviation for technology shock |
| $\sigma_z$ | 0.01 | Standard deviation for investment shock |
| $\sigma_m$ | 0.01 | Standard deviation for money supply shock |

Table 4-2 Estimated parameters for benchmark model

| Estimated Parameters | | Priors Type [mean, std.] | Posteriors | | | | |
|---|---|---|---|---|---|---|---|
| | | | Mode | Mean | 5% | Median | 95% |
| $\phi_I$ | Fraction of investment should be financed by cash | NORMAL [0.5, 0.05] | 0.5052 | 0.4937 | 0.4213 | 0.5052 | 0.5424 |
| $\phi_p$ | Calvo parameter, prices | BETA [0.5, 0.1] | 0.5081 | 0.5858 | 0.5081 | 0.5162 | 0.7386 |
| $\phi_w$ | Calvo parameter, wages | BETA [0.5, 0.1] | 0.4707 | 0.4663 | 0.4286 | 0.4707 | 0.6686 |
| $\xi_p$ | Price indexation | BETA [0.5, 0.15] | 0.4800 | 0.5224 | 0.4800 | 0.5802 | 0.8394 |
| $\xi_w$ | Wage indexation | BETA [0.5, 0.15] | 0.4600 | 0.4592 | 0.3864 | 0.4600 | 0.5843 |
| $\rho_a$ | AR(1) technology shock | BETA [0.75, 0.2] | 0.9990 | 0.9911 | 0.9691 | 0.9981 | 0.9990 |
| $\rho_m$ | AR(1) money growth shock | BETA [0.5, 0.2] | 0.4806 | 0.5077 | 0.4755 | 0.4806 | 0.5913 |
| $\rho_z$ | AR(1) investment shock | BETA [0.75, 0.2] | 0.9962 | 0.9873 | 0.9638 | 0.9962 | 0.9962 |
| $\sigma_a$ | std. technology shock | INV.GAMMA [0.01, 1] | 0.0048 | 0.0061 | 0.0048 | 0.0048 | 0.0130 |
| $\sigma_m$ | std. money growth shock | INV.GAMMA [0.01, 1] | 0.0050 | 0.0054 | 0.0041 | 0.0050 | 0.0076 |
| $\sigma_z$ | std. investment shock | INV.GAMMA [0.01, 1] | 0.0049 | 0.0059 | 0.0049 | 0.0049 | 0.0101 |

Table 4-4 Parameter values for alternative model

| Parameter Assignment (Alternative Model) | | |
|---|---|---|
| $\beta$ | 0.995 | Discount factor |
| $\sigma$ | 1 | Coefficient of relative risk aversion |
| $\varphi$ | 1.2 | The elasticity of wage with respect to hours |
| $F$ | 0 | The fixed cost of production |
| $\alpha$ | 1/3 | The capital share in production |
| $\delta_0$ | 0.025 | Capital depreciation parameter |
| $\delta_2$ | 0.01 | The utilization adjustment cost parameter |
| $\tau$ | 6 | The magnitude of capital adjustment costs |
| $\phi_I$ | 0.3 | The fraction of investment that must be financed by cash |
| $\varepsilon_p$ | 6 | The elasticity of substitution among intermediate goods |
| $\varepsilon_w$ | 6 | The elasticity of substitution among different type of labour |
| $\bar{\pi}$ | 0.005 | Steady state inflation |
| $\phi_p$ | 0.7 | The Calvo probability that a firm does not change its price |
| $\phi_w$ | 0.6 | The Calvo probability that a household does not change its wage |
| $\xi_p$ | 0.5 | Nominal price indexation to lagged inflation |
| $\xi_w$ | 0.5 | Nominal wage indexation to lagged inflation |

| | | |
|---|---|---|
| $\rho_a$ | 0.9 | Autoregressive parameter for technology shock |
| $\rho_z$ | 0.9 | Autoregressive parameter for investment shock |
| $\rho_i$ | 0.7 | Autoregressive parameter for Taylor rule |
| $\rho_\pi$ | 2 | Inflation response |
| $\rho_y$ | 0.08 | Long-run response to the output gap |
| $\rho_{\Delta y}$ | 0.2 | Short-run response to the change in the output gap |
| $\sigma_a$ | 0.01 | Standard deviation for technology shock |
| $\sigma_z$ | 0.01 | Standard deviation for investment shock |
| $\sigma_m$ | 0.01 | Standard deviation for money supply shock |

Table 4-5 Estimated parameters for alternative model

| Estimated Parameters | | Priors Type [mean, std.] | Posteriors | | | | |
|---|---|---|---|---|---|---|---|
| | | | Mode | Mean | 5% | Median | 95% |
| $\phi_p$ | Calvo parameter, prices | BETA [0.5, 0.1] | 0.7245 | 0.7094 | 0.6565 | 0.7126 | 0.7588 |
| $\phi_w$ | Calvo parameter, wages | BETA [0.5, 0.1] | 0.5045 | 0.5541 | 0.4552 | 0.5618 | 0.6433 |
| $\xi_p$ | Price indexation | BETA [0.5, 0.15] | 0.5270 | 0.5416 | 0.4397 | 0.5444 | 0.7107 |
| $\xi_w$ | Wage indexation | BETA [0.5, 0.15] | 0.5622 | 0.5410 | 0.4639 | 0.5510 | 0.6543 |
| $\rho_a$ | AR(1) technology shock | BETA [0.75, 0.2] | 0.7917 | 0.7855 | 0.6731 | 0.7917 | 0.8779 |
| $\rho_i$ | Interest rate smoothing parameter | BETA [0.75, 0.1] | 0.7814 | 0.7941 | 0.7254 | 0.7834 | 0.8599 |
| $\rho_\pi$ | The long-run reaction on inflation | NORMAL [1.5, 0.25] | 2.0037 | 1.8994 | 1.3956 | 1.8410 | 2.4670 |
| $\rho_y$ | The long-run reaction on the output gap | NORMAL [0.125, 0.025] | 0.0539 | 0.0640 | 0.0403 | 0.0662 | 0.0968 |
| $\rho_{\Delta y}$ | The short-run reaction to the change in the output gap | NORMAL [0.125, 0.025] | 0.0961 | 0.1060 | 0.0669 | 0.1049 | 0.1426 |
| $\rho_z$ | AR(1) investment shock | BETA [0.75, 0.2] | 0.7960 | 0.7635 | 0.6998 | 0.7697 | 0.8106 |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $\sigma_a$ | std. technology shock | INV.GAMMA [0.01, 1] | 0.0050 | 0.0062 | 0.0039 | 0.0059 | 0.0097 |
| $\sigma_m$ | std. money growth shock | INV.GAMMA [0.01, 1] | 0.0047 | 0.0096 | 0.0035 | 0.0106 | 0.0172 |
| $\sigma_z$ | std. investment shock | INV.GAMMA [0.01, 1] | 0.0047 | 0.0076 | 0.0040 | 0.0062 | 0.0200 |

# Chapter 5 Conclusion

This thesis is motivated to investigate why the economy tends to be stuck in a persistent liquidity trap and how probably to escape it. The more recent COVID-19 pandemic in 2020 has urgently prompted a reconsideration of how effective the monetary policies are. DSGE models are believed to not simply have a sound micro-foundation but also emerge as the major tools for quantitative macroeconomics policy analysis. Why there is a need of money and how money plays its role to the economic growth are always the central questions for monetarists. Money is superior to other asset in that it can provide liquidity services. However, it appears to be too much liquidity in the economy since the 2008 global crisis, leading to an even persistent liquidity trap. My research provides a new scope of understanding liquidity traps via a liquidity constraint, the cash-in-advance constraint.

Money is treated as an asset similar to other assets. The only difference is that money has the value by providing liquidity services while the other assets earn value from giving dividends. The value of liquidity services is determined endogenously via the slackness of the CIA constraint, which relies on the effects of shocks on the expected interest rate and inflation.

If money is the only asset as in Chapter 3 and the monetary policy follows a money growth rule, an increase in money supply followed by a technology innovation tends to stimulate the economy via a nonbinding CIA constraint. This is because the disturbances affect the economy through the only channel of inflation expectations. Open market operations, which trigger nonbinding CIA constraint for a consumption boom, also guides an increase in household's expectations of inflation that help the economy to escape liquidity traps. This provides a theoretical and empirical rationale for quantitative easing programs.

If money as well as bonds and capital are included as in Chapter 4, nonbinding CIA constraints are no longer able to stimulate the economy. Money is held in pockets since the other two assets cannot provide liquidity premium when the CIA constraint is nonbinding and nominal interest rate hits its zero bound. Money, compared with these two assets, is much safer and depresses output and its components. As a result, the economic growth is limited. Different monetary policies can have different effects. When the central banks adopt money growth rules, inflation is still the only channel for shocks to relax the CIA constraint. Thus, the quantitative easing programs still help the economy escape liquidity traps. However, when the interest rate feedback rule which takes zero-lower bound of nominal interest rate into account is adhered, interest rate, besides inflation, can also influences the ability of shocks to relax the CIA constraint. As long as the central banks stick to the zero or nearly zero policy rate, the households will continue expecting lower inflation or even deflation regardless of the quantitative easing programs, and the economy tends to have no chance to escape the liquidity trap. Endogenously binding CIA constraints, in this case, explains why there is a persistent liquidity trap after the Great Recession even though there are huge increases in the money supply.

Inflation is always a monetary phenomenon. Contrary to the past, the monetary policies contribute too little to help the central banks reach their inflation targets. This thesis provides a possible explanation. When the central banks embarked on their near-zero interest rate target policies, the CIA constraint is slack in response to an expected binding ZLB constraint. A nonbinding CIA constraint is also associated with the expectations of households on the relative value of money, which is ultimately reflected by the expectations on inflation rate. As long as the households believe the inflation is going to be low with a tendency to deflation, the CIA constraint will continue to be relaxing, and thus the nominal interest rate will remain stuck at its zero-lower bound. Therefore, the economy is suffering from a persistent liquidity trap.

Friedman rule that it is optimal to have a negative inflation is not violated as long as the central banks themselves do not target at a zero-policy rate. The policies that involve large-scale asset purchase programmes to avoid a downwards trend in inflation succeed in escaping liquidity traps, but their effectiveness is limited. If the increase in money or bonds are not large enough and the technology innovates faster, the household's expectations on inflation remain low. The tendency of the public to hold money in pockets will crowd out investment, which results in a low productivity that surprises the world after the Great Recession.

It is the time for the governments to have a deep think of how to raise the public's expectations on inflation rate or how to stimulate aggregate demand. A straight way to promote a stimulus is to punish the one who hoards money. If the policy authority decides to maintain their zero-policy rate target, a large increase in the money base should be substituted by the policies that subside specific sectors. For example, a subside on capital may raise the motivations of investment, which will contribute to an increase in aggregate demand and thus output growth. A cheap loan to medium-sized, small and micro businesses may also be helpful. At the zero-lower bound, inflation can be managed only through the real interest rate. It comes to a problem of the government credibility. Future uncertainty is the main cause for the consumers to stock up money. To solve this problem, a responsible and transparent government should be established to build up the confidence of the public. Instead of concerning more with the short-run economic prosperity for the election procedure, the policy authority should focus more on the investment in fundamental sectors or frontline infrastructure services. For example, learning from the unexpected health crisis of COVID-19, a more comprehensive public health care system is urgently to be improved.

This thesis gives a first glance of how endogenously binding CIA constraint works in the policy making process. However, several challenges and extensions are still worth investigating. Firstly, this simple framework concentrates only on two fundamental shocks, technology shock and monetary shock. More frictions, such as financial frictions caused by financial accelerator mechanism or financial intermediaries, could be included. It is popular to add an endogenously binding collateral constraint on housing to analyse the cause of the Great Recession and predict future financial crisis. Three nonlinearities may even complicate the model structure, but it could provide a clearer mechanism to study the interactions between major actors of the economy, namely, consumers, investors and governors.

The second novelty could be having the negative interest rates in this framework. Denmark had a negative deposit rate in 2012. There was also a short period when Swiss banks charged foreign savers. Bank of England is seeking a new deposit rate to encourage loans to small businesses. This is a new promising direction pursued by the central banks who suffer most from the stagnation recently. However, it is quite challenging to model the negative interest rate. It may also be questioned whether a negative interest rate would distort the market and lead to an even worse financial crisis.

Another natural extension could be a consideration of open economy framework. In an open economy, private households run into the bank frequently to exchange domestic currency for foreign reverses when the CIA constraint is always binding. The central bank then tends to lose a large amount of their foreign reserves stock. A balance of payment crisis is possible, and the central bank has to devalue the domestic currency, which usually causing a recession.

However, when the CIA constraint is nonbinding, money becomes a safe asset and there is less likely that a balance of payment crisis will occur. An open economy structure also provides some spaces for analysing monetary policies in a cooperative setting. Galí and Monacelli (2005), Benigno (2009), De Paoli (2009), and Engel (2014) make good contributions to cooperative setting in a small-country context.

In general, it is commonly believed that supply-side policies are unable to solve the fundamental problem of the shortage in aggregate demand and it takes longer time for the supply-side policies to have real effects. A nonbinding CIA constraint can stimulate the aggregate demand when money is the only asset and labour is the only input for production but fail to play its role when the economy is already stuck in a liquidity trap.

# Bibliography


[1] Abel, A. B. 1985. Dynamic behavior of capital accumulation in a cash-in-advance model. *Journal of Monetary Economics* 16(1), pp. 55-71.

[2] Adam, K. and Billi, R. M. 2007. Discretionary monetary policy and the zero lower bound on nominal interest rates. *Journal of Monetary Economics* 54(3), pp. 728-752.

[3] Adão, B. and Correia, I. and Teles, P. 2011. Unique monetary equilibria with interest rate rules. *Review of Economic Dynamics* 14(3), pp. 432-442.

[4] Aiyagari, S. R. 1994. Uninsured idiosyncratic risk and aggregate saving. *The Quarterly Journal of Economics* 109(3), pp. 659-684.

[5] Akay, K. 2010. Indeterminacy in cash-in-advance models and the role of frictions. *Macroeconomic Dynamics* 14(1), p. 119.

[6] Akerlof, G. A. and Yellen, J. L. 1985. A near-rational model of the business cycle, with wage and price inertia. *The Quarterly Journal of Economics* 100(Supplement), pp. 823-838.

[7] Alvarez, F. and Atkeson, A. and Edmond, C. 2009. Sluggish responses of prices and inflation to monetary shocks in an inventory model of money demand. *The Quarterly Journal of Economics* 124(3), pp. 911-967.

[8] Asriyan, V. and Fornaro, L. and Martin, A. and Ventura, J. 2016. *Monetary policy for a bubbly world*.

[9] Auray, S. and de Blas, B. 2009. On Stickiness, Cash in Advance, and Persistence. *Cahier de recherche/Working Paper* 9, p. 19.

[10] Auray, S. and de Blas, B. 2013. Investment, matching and persistence in a modified cash-in-advance economy. *Journal of Economic Dynamics and Control* 37(3), pp. 591-610.

[11] Bacchetta, P. and Benhima, K. and Kalantzis, Y. 2019. Money and capital in a persistent liquidity trap. *Journal of Monetary Economics*.

[12] Ball, L. and Mankiw, N. G. 1995. Relative-price changes as aggregate supply shocks. *The Quarterly Journal of Economics* 110(1), pp. 161-193.

[13] Barnes, M. and Boyd, J. H. and Smith, B. D. 1999. Inflation and asset returns. *European Economic Review* 43(4-6), pp. 737-754.

[14] Basu, S. and Fernald, J. G. 1997. Returns to scale in US production: Estimates and implications. *Journal of political Economy* 105(2), pp. 249-283.

[15] Baumol, W. J. 1952. The transactions demand for cash: An inventory theoretic approach. *The Quarterly Journal of Economics*, pp. 545-556.

[16] Benassy, J.-P. 1975. Neo-Keynesian disequilibrium theory in a monetary economy. *The review of economic studies* 42(4), pp. 503-523.

[17] Benassy, J.-P. 1996. Taste for variety and optimum production patterns in monopolistic competition. *Economics Letters* 52(1), pp. 41-47.

[18] Bencivenga, V. R. and Smith, B. D. 1992. Deficits, inflation, and the banking system in developing countries: The optimal degree of financial repression. *Oxford Economic Papers* 44(4), pp. 767-790.

[19] Bernanke, B. S. and Gertler, M. 1995. Inside the black box: the credit channel of monetary policy transmission. *Journal of Economic Perspectives* 9(4), pp. 27-48.

[20] Bewley, T. 1983. A difficulty with the optimum quantity of money. *Econometrica: Journal of the Econometric Society*, pp. 1485-1504.

[21] Bhattacharjee, A. and Thoenissen, C. 2007. Money and monetary policy in dynamic stochastic general equilibrium models. *The Manchester School* 75, pp. 88-122.

[22] Brunner, K. 1951. Inconsistency and indeterminacy in classical economics. *Econometrica, Journal of the Econometric Society*, pp. 152-173.

[23] Brunnermeier, M. K. and Sannikov, Y. 2016. On the optimal inflation rate. *American economic review* 106(5), pp. 484-489.

[24] Brycz, M. 2012. Keynesian and Monetary Approach to the Liquidity Trap–looking for cointegration evidence from 2008–Crisis in the United States. *Journal of International Studies* 5(2).

[25] Buera, F. and Nicolini, J. P. 2017. *Liquidity Traps and Monetary Policy: Managing a Credit Crunch Online Appendix*.

[26] Buera, F. J. and Moll, B. 2015. Aggregate implications of a credit crunch: The importance of heterogeneity. *American Economic Journal: Macroeconomics* 7(3), pp. 1-42.

[27] Caballero, R. J. and Farhi, E. and Gourinchas, P.-O. 2017. The safe assets shortage conundrum. *Journal of Economic Perspectives* 31(3), pp. 29-46.

[28] Calvo, G. A. 1983. Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics* 12(3), pp. 383-398.

[29] Campbell, J. R. and Fisher, J. D. and Justiniano, A. and Melosi, L. 2017. Forward guidance and macroeconomic outcomes since the financial crisis. *NBER macroeconomics annual* 31(1), pp. 283-357.

[30] Carré, J.-J. and Dubois, P. and Malinvaud, E. 1975. *French economic growth*. Stanford University Press.

[31] Chamley, C. and Polemarchakis, H. 1984. Assets, general equilibrium and the neutrality of money. *The review of economic studies* 51(1), pp. 129-138.

[32] Chari, V. V. and Christiano, L. J. and Eichenbaum, M. 1998. Expectation traps and discretion. *Journal of Economic Theory* 81(2), pp. 462-492.

[33] Chari, V. V. and Christiano, L. J. and Kehoe, P. J. 1996. Optimality of the Friedman rule in economies with distorting taxes. *Journal of Monetary Economics* 37(2), pp. 203-223.

[34] Christiano, L. and Eichenbaum, M. and Rebelo, S. 2011. When is the government spending multiplier large? *Journal of political Economy* 119(1), pp. 78-121.

[35] Christiano, L. J. and Eichenbaum, M. and Evans, C. L. 2005. Nominal rigidities and the dynamic effects of a shock to monetary policy. *Journal of political Economy* 113(1), pp. 1-45.

[36] Chu, A. C. and Cozzi, G. 2014. R&D and economic growth in a cash-in-advance economy. *International Economic Review* 55(2), pp. 507-524.

[37] Chung, H. and Laforte, J.-P. and Reifschneider, D. and Williams, J. C. 2011. Estimating the macroeconomic effects of the Fed’s asset purchases. *FRBSF Economic Letter* 3, pp. 1-5.

[38] Chung, H. and Laforte, J. P. and Reifschneider, D. and Williams, J. C. 2012. Have we underestimated the likelihood and severity of zero lower bound events? *Journal of Money, Credit and Banking* 44, pp. 47-82.

[39] Clarida, R. and Gali, J. and Gertler, M. 1999. The science of monetary policy: a new Keynesian perspective. *Journal of economic literature* 37(4), pp. 1661-1707.

[40] Clower, R. 1967. A reconsideration of the microfoundations of monetary theory. *Economic Inquiry* 6(1), pp. 1-8.

[41] Clower, R. W. 1965. A Keynesian Counter Revolution: A Theoretical Appraisal. *The Theory of Interests*,

[42] Cogley, T. and Sbordone, A. M. 2008. Trend inflation, indexation, and inflation persistence in the New Keynesian Phillips curve. *American economic review* 98(5), pp. 2101-2126.

[43] Cooley, T. F. and Hansen, G. D. 1989. The inflation tax in a real business cycle model. *The American Economic Review*, pp. 733-748.

[44] Debortoli, D. and Galí, J. 2017. Monetary policy with heterogeneous agents: Insights from TANK models. *Manuscript, September*.

[45] Debortoli, D. and Galí, J. and Gambetti, L. 2020. On the empirical (ir) relevance of the zero lower bound constraint. *NBER macroeconomics annual* 34(1), pp. 141-170.

[46] Del Negro, M. and Eggertsson, G. and Ferrero, A. and Kiyotaki, N. 2017. The great escape? A quantitative evaluation of the Fed's liquidity facilities. *American economic review* 107(3), pp. 824-857.

[47] Demary, M. and Hüther, M. 2015. When low interest rates cause low inflation. *Intereconomics* 50(6), pp. 350-355.

[48] Devereux, M. B. and Siu, H. E. 2007. State dependent pricing and business cycle asymmetries. *International Economic Review* 48(1), pp. 281-310.

[49] Diamond, P. A. 1965. National debt in a neoclassical growth model. *The American Economic Review* 55(5), pp. 1126-1150.

[50] Díaz-Giménez, J. and Giovannetti, G. and Marimon, R. and Teles, P. 2008. Nominal debt as a burden on monetary policy. *Review of Economic Dynamics* 11(3), pp. 493-514.

[51] Dıaz-Giménez, J. and Kirkby, R. 2013. Illustrating the quantity theory of money in the United States and in three model economies.

[52] Dixit, A. K. and Stiglitz, J. E. 1977. Monopolistic competition and optimum product diversity. *The American Economic Review* 67(3), pp. 297-308.

[53] Dixon, H. and Kara, E. 2010. Can we explain inflation persistence in a way that is consistent with the microevidence on nominal rigidity? *Journal of Money, Credit and Banking* 42(1), pp. 151-170.

[54] Dixon, H. and Kara, E. 2011. Contract length heterogeneity and the persistence of monetary shocks in a dynamic generalized Taylor economy. *European Economic Review* 55(2), pp. 280-292.

[55] Dixon, H. and Le Bihan, H. 2011. DE TRAVAIL N 324.

[56] Dixon, H. and Pourpourides, P. M. 2016. On imperfect competition with occasionally binding cash-in-advance constraints. *Journal of Macroeconomics* 50, pp. 72-85.

[57] Dotsey, M. and King, R. G. and Wolman, A. L. 1999. State-dependent pricing and the general equilibrium dynamics of money and output. *The Quarterly Journal of Economics* 114(2), pp. 655-690.

[58] Dotsey, M. and Sarte, P. D. 2000. Inflation uncertainty and growth in a cash-in-advance economy. *Journal of Monetary Economics* 45(3), pp. 631-655.

[59] Eggertsson, G. B. 2008. Great Expectations and the End of the Depression. *American economic review* 98(4), pp. 1476-1516.

[60] Eggertsson, G. B. and Woodford, M. 2003. *Optimal monetary policy in a liquidity trap*.

[61] Ellison, M. and Scott, A. 2000. Sticky prices and volatile output. *Journal of Monetary Economics* 46(3), pp. 621-632.

[62] Erceg, C. J. and Henderson, D. W. and Levin, A. T. 2000. Optimal

monetary policy with staggered wage and price contracts. *Journal of Monetary Economics* 46(2), pp. 281-313.

[63] Evans, G. W. and Honkapohja, S. M. and Marimon, R. 2007. Stable sunspot equilibria in a cash-in-advance economy. *The BE Journal of Macroeconomics* 7(1).

[64] Feenstra, R. C. 1986. Functional equivalence between liquidity costs and the utility of money. *Journal of Monetary Economics* 17(2), pp. 271-291.

[65] Feenstra, R. C. 2003. A homothetic utility function for monopolistic competition models, without constant price elasticity. *Economics Letters* 78(1), pp. 79-86.

[66] Fehr, E. and Goette, L. 2005. Robustness and real consequences of nominal wage rigidity. *Journal of Monetary Economics* 52(4), pp. 779-804.

[67] Francis, N. and Ramey, V. A. 2005. Is the technology-driven real business cycle hypothesis dead? Shocks and aggregate fluctuations revisited. *Journal of Monetary Economics* 52(8), pp. 1379-1399.

[68] Friedman, M. 1956a. The quantity theory of money: a restatement. *Studies in the quantity theory of money* 5, pp. 3-31.

[69] Friedman, M. 1956b. *Studies in the quantity theory of money*. University of Chicago Press.

[70] Friedman, M. 1969. *The optimum quantity of money, and other essays*.

[71] Gali, J. 1999. Technology, employment, and the business cycle: do technology shocks explain aggregate fluctuations? *American economic review* 89(1), pp. 249-271.

[72] Galí, J. and López-Salido, D. and Vallés, J. 2004. Understanding the effects of government spending on consumption.

[73] Gali, J. and Monacelli, T. 2008. Optimal monetary and fiscal policy in a currency union. *Journal of international economics* 76(1), pp. 116-132.

[74] Galí, J. and Rabanal, P. 2004. Technology shocks and aggregate fluctuations: How well does the real business cycle model fit postwar US data? *NBER macroeconomics annual* 19, pp. 225-288.

[75] Gambacorta, L. and Hofmann, B. and Peersman, G. 2014. The effectiveness of unconventional monetary policy at the zero lower bound: A cross-country analysis. *Journal of Money, Credit and Banking* 46(4), pp. 615-642.

[76] Giraud, G. and Tsomocos, D. P. 2010. Nominal uniqueness and money non-neutrality in the limit-price exchange process. *Economic Theory* 45(1-2), pp. 303-348.

[77] Gomme, P. and Ravikumar, B. and Rupert, P. 2017. Return to Capital in a Real Business Cycle Model.

[78] Grandmont, J.-M. and Laroque, G. 1976. The liquidity trap. *Econometrica: Journal of the Econometric Society*, pp. 129-135.

[79] Grandmont, J.-M. and Younes, Y. 1972. On the Role of Money and the Existence of a Monetary Equilibrium. *The review of economic studies* 39(3), pp. 355-372.

[80] Guerrieri, L. and Iacoviello, M. 2015. OccBin: A toolkit for solving dynamic models with occasionally binding constraints easily. *Journal of Monetary Economics* 70, pp. 22-38.

[81] Guerrieri, L. and Iacoviello, M. 2017. Collateral constraints and macroeconomic asymmetries. *Journal of Monetary Economics* 90, pp. 28-49.

[82] Hall, B. H. 1986. *The relationship between firm size and firm growth in the US manufacturing sector*.

[83] Hall, B. H. and Lerner, J. 2010. The financing of R&D and innovation.*Handbook of the Economics of Innovation*. Vol. 1. Elsevier, pp. 609-639.

[84] Heer, B. and Maußner, A. 2015. The cash-in-advance constraint in monetary growth models with labor market search. *Macroeconomic Dynamics* 19(1), pp. 144-166.

[85] Hicks, J. R. 1937. Mr. Keynes and the" classics"; a suggested interpretation. *Econometrica: Journal of the Econometric Society*, pp. 147-159.

[86] Hornstein, A. 1993. Monopolistic competition, increasing returns to scale, and the importance of productivity shocks. *Journal of Monetary Economics*

31(3), pp. 299-316.

[87] Howitt, P. 1992. Cash-in-advance economy. *The New Palgrave Dictionary of Money and Finance* 1, pp. 318-319.

[88] Hromcová, J. 2008. Learning-or-doing in a cash-in-advance economy with costly credit. *Journal of Economic Dynamics and Control* 32(9), pp. 2826-2853.

[89] Huang, C. and Chang, J.J. and Ji, L. 2013. *Cash-in-advance constraint on R&D in a Schumpeterian growth model with endogenous market structure*.

[90] Ireland, P. N. 1994. Money and growth: an alternative approach. *The American Economic Review*, pp. 47-65.

[91] J Caballero, R. and Farhi, E. 2018. The safety trap. *The review of economic studies* 85(1), pp. 223-274.

[92] Kacperczyk, M. and Schnabl, P. 2010. When safe proved risky: Commercial paper during the financial crisis of 2007-2009. *Journal of Economic Perspectives* 24(1), pp. 29-50.

[93] Keynes, J. M. 1936. The general theory of interest, employment and money. london: MacMillan.

[94] Krugman, P. R. and Dominquez, K. M. and Rogoff, K. 1998. It's baaack: Japan's slump and the return of the liquidity trap. *Brookings Papers on Economic Activity* 1998(2), pp. 137-205.

[95] Lagos, R. and Rocheteau, G. 2005. Inflation, output, and welfare. *International Economic Review* 46(2), pp. 495-522.

[96] Lagos, R. and Wright, R. 2003. Dynamics, cycles, and sunspot equilibria in ‘genuinely dynamic, fundamentally disaggregative’models of money. *Journal of Economic Theory* 109(2), pp. 156-171.

[97] Leijonhufvud, A. 1968. the Economics of Keynes: a study in monetary theory. *New York*.

[98] Lhuissier, S. and Mojon, B. and Rubio-Ramirez, J. F. 2020. Does the liquidity trap exist? .

[99] Lucas Jr, R. E. 1988. On the mechanics of economic development.

*Journal of Monetary Economics* 22(1), pp. 3-42.

[100] Lucas, R. E. 1980a. Equilibrium in a pure currency economy. *Economic Inquiry* 18(2), p. 203.

[101] Lucas, R. E. 1980b. Methods and problems in business cycle theory. *Journal of Money, Credit and Banking* 12(4), pp. 696-715.

[102] Lucas, R. E. 1980c. Two illustrations of the quantity theory of money. *The American Economic Review* 70(5), pp. 1005-1014.

[103] Matsuyama, K. and Ushchev, P. 2017. Beyond CES: Three Alternative Cases of Flexible Homothetic Demand Systems. *Global Poverty Research Lab Working Paper* (17-109).

[104] McCallum, B. T. and Goodfriend, M. S. 1987. *Money: Theoretical analysis of the demand for money*.

[105] Mehra, R. and Prescott, E. C. 1985. The equity premium: A puzzle. *Journal of Monetary Economics* 15(2), pp. 145-161.

[106] Michel, P. and Wigniolle, B. 2005. Cash-in-advance constraints, bubbles and monetary policy.

[107] Monnet, C. and Weber, W. E. 2001. Money and interest rates. *Federal Reserve Bank of Minneapolis Quarterly Review* 25(4), pp. 2-13.

[108] Mouabbi, S. and Sahuc, J. G. 2019. Evaluating the macroeconomic effects of the ECB's unconventional monetary policies. *Journal of Money, Credit and Banking* 51(4), pp. 831-858.

[109] Mulligan, C. B. and Sala-i-Martin, X. and Mankiw, N. G. and Rotemberg, J. J. 1992. US money demand: surprising cross-sectional estimates. *Brookings Papers on Economic Activity*, pp. 285-343.

[110] Nakata, T. 2017. Uncertainty at the zero lower bound. *American Economic Journal: Macroeconomics* 9(3), pp. 186-221.

[111] Palivos, T. and Wang, P. and Zhang, J. 1993. Velocity of money in a modified cash-in-advance economy: theory and evidence. *Journal of Macroeconomics* 15(2), pp. 225-248.

[112] Panizza, U. and Wyplosz, C. 2018. The folk theorem of decreasing

effectiveness of monetary policy: What do the data say? *Russian Journal of Money and Finance* 77(1), pp. 71-107.

[113] Patinkin, D. 1965. *Money, interest, and prices; an integration of monetary and value theory*.

[114] Pissarides, C. A. 2011. Equilibrium in the labor market with search frictions. *American economic review* 101(4), pp. 1092-1105.

[115] Ragot, X. 2014. The case for a financial approach to money demand. *Journal of Monetary Economics* 62, pp. 94-107.

[116] Robert, L. and Stokey, N. 1987. Money and interest in a cash-in-advance economy. *Econometrica* 55(3), pp. 491-513.

[117] Rotemberg, J. J. and Woodford, M. 1999. Interest rate rules in an estimated sticky price model.*Monetary policy rules*. University of Chicago Press, pp. 57-126.

[118] Rudebusch, G. 2018. A Review of the Fed's Unconventional Monetary Policy. *FRBSF Economic Letter* 27, pp. 1-5.

[119] Samuelson, P. A. 1958. An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of political Economy* 66(6), pp. 467-482.

[120] Shapiro, C. and Stiglitz, J. E. 1984. Equilibrium unemployment as a worker discipline device. *The American Economic Review* 74(3), pp. 433-444.

[121] Smets, F. and Wouters, R. 2007. Shocks and frictions in US business cycles: A Bayesian DSGE approach. *American economic review* 97(3), pp. 586-606.

[122] Stockman, A. C. 1981. Anticipated inflation and the capital stock in a cash-in-advanced economy.

[123] Svensson, L. 2001. The foolproof way of escaping from a liquidity trap: is it really and can it help Japan? *Graham Lecture, Princeton University*.

[124] Svensson, L. E. 1985. Money and asset prices in a cash-in-advance economy. *Journal of political Economy* 93(5), pp. 919-944.

[125] Taylor, J. B. 1979. Staggered wage setting in a macro model. *The*

*American Economic Review* 69(2), pp. 108-113.

[126] Taylor, J. B. 2012. Monetary policy rules work and discretion doesn’t: A tale of two eras. *Journal of Money, Credit and Banking* 44(6), pp. 1017-1032.

[127] Tella, S. D. 2018. *A neoclassical theory of liquidity traps*.

[128] Telyukova, I. A. 2013. Household need for liquidity and the credit card debt puzzle. *Review of Economic Studies* 80(3), pp. 1148-1177.

[129] Telyukova, I. A. and Visschers, L. 2011. Precautionary demand for money in a monetary business cycle model.

[130] Tirole, J. 1985. Asset bubbles and overlapping generations. *Econometrica: Journal of the Econometric Society*, pp. 1499-1528.

[131] Tobin, J. 1956. The interest-elasticity of transactions demand for cash. *The review of Economics and Statistics*, pp. 241-247.

[132] Wang, P.f. and Wen, Y. 2006. Another look at sticky prices and output persistence. *Journal of Economic Dynamics and Control* 30(12), pp. 2533-2552.

[133] Williamson, S. D. 2012a. Liquidity, monetary policy, and the financial crisis: A new monetarist approach. *American economic review* 102(6), pp. 2570-2605.

[134] Williamson, S. D. 2012b. New monetarist economics: Understanding unconventional monetary policy. *Economic Record* 88, pp. 10-21.

[135] Williamson, S. D. and Wright, R. 2010. New monetarist economics: Methods.

[136] Wolman, A. L. 1999. Sticky prices, marginal cost, and the behavior of inflation. *FRB Richmond Economic Quarterly* 85(4), pp. 29-48.

[137] Worthington, P. R. 1995. Investment, cash flow, and sunk costs. *The journal of industrial economics*, pp. 49-61.

[138] Wu, J. C. and Xia, F. D. 2016. Measuring the macroeconomic impact of monetary policy at the zero lower bound. *Journal of Money, Credit and Banking* 48(2-3), pp. 253-291.

[139] Wu, J. C. and Xia, F. D. 2017. Time-varying lower bound of interest rates in Europe. *Chicago Booth Research Paper* (17-06).

[140] Yun, T. 1996. Nominal price rigidity, money supply endogeneity, and business cycles. *Journal of Monetary Economics* 37(2), pp. 345-370.