A PARTIALLY EXCLUSIVE RENT-SEEKING CONTEST

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12 April 2021

Abstract

This study presents a novel contest model in which interest groups compete for partially exclusive rents, and the number of winners is endogenous. Partial exclusivity can explain the low empirical estimates of rent dissipation that create the Tullock paradox. However, partial exclusivity also increases aggregate effort and social waste. The effort-maximising contest design follows a simple rule of limiting the expected number of winners to half the number of contestants.

JEL classification: C72, D72.

Keywords: rent seeking, interest groups, multiple-winner contests, rent dissipation, contest design.

1 Introduction

This study presents a novel class of contest models, in which the number of winners is stochastic and endogenous. This feature is due to the micro foundations of the winner-selection mechanism, which requires the winners to

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be within a certain fraction of the top performance. This fraction determines to what extent the prizes or rents are exclusive, and it can be considered as a key choice of the contest designer and an alternative to choosing the number of winners in advance. The potential applications of the contest model include promotion contests and recognition prizes, but, in particular, rent seeking as will be argued next.

Contrary to theoretical predictions, rent-seeking expenditures are typically small compared to the value of the rent. This empirical puzzle is known as the Tullock paradox (Tullock, 1980; Riley, 1999; Ansolabehere et al., 2003; Zingales, 2017). However, a large part of lobbying activities are conducted by or on behalf of interest groups that are not direct rivals, whereas the theoretical literature typically considers individuals or firms that compete for the same rent (see Hillman and Ursprung, 2016). The interest groups may seek completely different rents, for example, in the form of an import tariff on product A, a subsidy for industry B, or less stringent environmental regulation in sector C, which need not be mutually exclusive policies but are still limited in their eventual, enacted number. Thus, one theoretical contribution of this study is to present a multiple-winner Tullock contest model to account for this partial exclusivity and provide an explanation for the Tullock paradox. However, partial exclusivity not only leads to lower rent dissipation but also increases aggregate rent-seeking effort.

The tendency of theoretical models to predict too high levels of rent dissipation can be observed in the consistently higher indirect measures of rent-seeking expenditures as compared to their direct measures when available (Mueller, 2003; Del Rosal, 2011). This finding has led to various attempts to extend the standard model by considering limited entry and Stackelberg competition (Pérez-Castrillo and Verdier, 1992), asymmetric valuations or costs (Gradstein, 1995; Nti, 1999), risk or loss aversion (Van Long and Vousden, 1987; Cornes and Hartley, 2003), repeated games and collusion (Leininger and Yang, 1994), status quo bias (Polborn, 2006), and probability distortions (Baharad and Nitzan, 2008), and so on. Like these studies, my model, which incorporates partial exclusivity, results in lower rent dissipation. However, I also show that, with multiple winners, there is simultaneously more rent-seeking effort. Thus, rent dissipation is a misleading measure of social waste in the presence of multiple rents and interests.

From the point of view of contest theory, this study is connected to the broader literature on multiple-prize and multiple-winner contests (see Sisak, 2009). Among comparable models, the nested contest success function (CSF) of Clark and Riis (1996, 1998) has been the most popular choice in previous analyses. The micro foundations of the nested CSF have been studied by Fu and Lu (2012), who show it to be equivalent to a raking rule that considers the best shot of each contestant across multiple independent attempts. While this provides another perspective to the series of hypothetical sub-contests that underlie the nested CSF winner-selection mechanism, generating comparative statics results is nevertheless cumbersome.

In comparison, the proposed multiple-winner extension of the standard Tullock CSF with partially exclusive rents is a simple and intuitive mechanism for rewarding the top performers of the contest without fixing the number of winners in advance. In this model, the winners of the contest need to be within a specified range of the top performance. This requirement determines the exclusivity of the rents. The stochastic micro foundations of the CSF are similar to those of Jia (2012) with an endogenous probability of a draw. An important difference between this model and previous multiple-winner contests is that the number of winners in my model is endogenous, which is an intuitive feature of rent seeking, and extends to other contexts as well[1]. I show that the existence and uniqueness of the symmetric equilibrium are satisfied under fairly general conditions.

This study also contributes to the literature on contest design (e.g. Gradstein and Konrad, 1999). One central feature of the model is the relationship between partial exclusivity and the expected number of winners; these can be manipulated by the contest designer. In contrast to Fu and Lu (2009) and Chowdhury and Kim (2017), I show that the choice between a single grand contest and multiple symmetric sub-contests does not have any effect on

[1]Naturally, the proposed CSF is not applicable to contests in which the number of winners is predetermined, as is generally the case in sports and electoral contests. However, as suggested by an anonymous reviewer, the contest model could be applied to public good provision.

the aggregate effort under my CSF when the contestants have homogeneous valuations. As shown in the online appendix, with weakly heterogeneous valuations, it becomes optimal to sort the contestants into sub-contests that minimise variance.

The main result on contest design is that for a rent-seeking maximising designer, the optimal rule is to limit the expected number of winners to half the number of contestants. Furthermore, any sub-contest structure can then yield the maximal aggregate effort if the designer is free to choose the degree of exclusivity accordingly. I conclude the analysis by comparing the aggregate efforts between the different multiple-winner contests.

2 A multiple-winner Tullock contest

In many situations, rent seekers are not firms seeking the same rent but rather interest groups that represent the common interests of their members[2]. Furthermore, the groups are often interested in influencing different and separate policies[3]. The rents in these cases are transfers from consumers, potential or foreign firms, or the general public, who are unorganised or not represented in the process. This scenario implies lesser competition for rents than the standard winner-takes-all contest models. However, if a policymaker requires comparable efforts from the winners, then there will be some competition between groups that are not otherwise direct rivals.

Other contexts, where the number of winners can vary, the rents are in principle non-rival, but competition is created by the need to demonstrate comparable effort, include school admissions and internal labour market tournaments. In particular, recognition prizes, which are studied by Liu and Lu (2017) in an all-pay auction setup, share many of the same features as a partially exclusive rent-seeking contest[4].

[2]I do not model the efforts of individual group members in these contests. Please see Nitzan (1991) and Hillman and Ursprung (2016) for such models.

[3]In contrast, a common agency problem would be a different situation because the interest groups have divergent preferences regarding the same policy choice (Bernheim and Whinston, 1986; Grossman and Helpman, 1994).

[4]That is, recognition prizes to online sellers or artists, for example, can be considered

Since partial exclusivity provides a novel class of contest models that may be applied to various contexts other than rent seeking, I will use general labels by calling the players 'contestants' and a 'contest designer'. In this section, we first consider the stochastic foundations and properties of the multiple-winner CSF and then I derive the equilibrium efforts when the contestants have homogenous valuations. Heterogeneous valuations are studied in the online appendix.

2.1 Contest success function

In this study, I consider a rent-seeking contest between $n \geq 2$ risk-neutral contestants and the problem of a contest designer, whose objective is to maximise aggregate effort in the contest. Let $N = \{1, \ldots, n\}$ denote the set of the contestants. Contestant $i \in N$ attempts to secure a rent of value R, which is exclusive to this player. That is, there are n potential rents, one for each contestant. For tractability, I assume that the valuations are the same for all contestants but an extension considering heterogeneous valuations is presented in the online appendix.

The contestants are neither seeking the same rents nor trying to influence the same policy. Hence, the rents are not mutually exclusive. However, the contest designer awards rents based only on positive contest performance, which should not lag too far behind that of the other contestants. This requirement makes the rents partially exclusive. As such, the term 'partial exclusivity' refers to a situation in which more than one rent can be awarded and the number of winners, m, is determined in the equilibrium.

The stochastic foundations of the partially exclusive rent-seeking contest mirror those of Jia (2012), which considers draws between contestants. As such, the rent-seeking performance of contestant i is given by $y_i(x_i, \theta_i) = x_i^r \theta_i$, where $x_i \in [0, R]$ is the expended effort, $r > 0$ measures returns to scale, and $\theta_i \in \mathbb{R}_+$ is an i.i.d. random shock. Contestants do not observe θs when deciding on their effort.

homogenous in value and limited to one per winner. Since they have no cash value, the contest organiser can award them without costs or contraints, or (at least to some extent) diluting their value.

Contestant i wins a rent if and only if

$$x_i^r \theta_i > \max_{j \neq i} \alpha x_j^r \theta_j,$$

where $\alpha \in (0, 1]$ is the exclusivity parameter chosen by the designer (considered in Section 3). When $\alpha < 1$, there can be multiple winners as long as their actual performance is not too far below and more than the fraction α of the top performance among the contestants.

In the case of partial exclusivity, i.e. $\alpha < 1$, the top performer implicitly sets the standards and expectations regarding the minimally acceptable performance. Thus, the top lobbyist in a rent-seeking contest, for example, not only wins the rent but sets the bar for the others. A similar function is played by the top salesperson or student when the performance of others is scaled by the top performance.

Letting x_{-i} denote the vector of efforts of contestants other than i, the probability that i wins the rent is given by

$$p_i(x_i, x_{-i}, \alpha) = Prob(x_i^r \theta_i > \alpha x_j^r \theta_j, \forall j \neq i), \tag{1}$$

Following Jia (2012), I assume that the random component θ_i are drawn from a Fréchet distribution with a shape parameter $\lambda = 1$ and a scale parameter $\sigma > 0$, and show that this results in a modified Tullock CSF with stochastic and endogenous number of winners[5]. The distribution has the probability density function

$$h(z) = \sigma z^{-2} e^{-\sigma/z},$$

when $z > 0$ and 0 otherwise, and the cumulative probability function

$$H(z) = \int_0^z h(s)ds = e^{-\sigma/z}.$$

[5] As in Jia (2008), one can also derive the CSF by assuming that the performance is given by $y_i(x_i, \theta_i) = x_i \theta_i$, but that the distribution of θ_i has a general shape parameter $\lambda = r$. The only difference then is that we will have α^r term instead of α in the numerator of (2).

The following theorem shows that the corresponding CSF has a ratio form with α as its key variable. All proofs are in the Appendix.

Theorem 1 *If $\theta_i \sim Fr\acute{e}chet(1, \sigma)$ in (1), then the corresponding CSF has the ratio form*

$$p_i(x_i, x_{-i}, \alpha) = \frac{x_i^r}{\alpha \sum_{j \neq i} x_j^r + x_i^r}, \tag{2}$$

if $\sum x_i > 0$ and 0 otherwise.

Theorem 1 establishes the microeconomic foundations of the modified Tullock CSF, where $\alpha \in (0, 1]$ determines the degree of exclusivity and intensity of competition[6]. It provides a functional form for contests in which the number of winners is endogenous to effort x_i and the contest designer's choice of α. Note that $\alpha = 1$, which I refer to as 'perfect exclusivity' in this context, gives the standard Tullock CSF with a single winner. There is the usual discontinuity with respect to the effort vector $\mathbf{0}$ but the stochastic foundations imply that the number of winners and each contestant's probability of success should then be zero rather than a positive constant as is often assumed in the literature.

If $\sum x_i \neq 0$ and $\alpha < 1$, the number of winners is stochastic due to random shocks θ_i and distributed between 1 and l, where $l \in [1, n]$ is the number of contestants that exert positive effort. The expected number of winners (and awarded rents) is denoted by m and this will be endogenously determined by the sum of the success probabilities:

$$m \equiv \sum p_i(x_i, x_{-i}, \alpha) = \sum \frac{x_i^r}{\alpha \sum_{j \neq i} x_j^r + x_i^r}. \tag{3}$$

Proposition 1 *The expected number of winners is maximised, $m = k$, when the contestants choose identical efforts $x_i = x, \forall i \in N$. Furthermore, the*

[6] Godwin et al. (2006) incorporate a similar competition parameter into their two-player contest model.

relationship between α and k is given by

$$k = \frac{n}{\alpha(n-1)+1} \leftrightarrow \alpha = \frac{n-k}{k(n-1)}.$$

Since m is a sum of n probabilities (3), there exists a least upper bound $k \geq m$, which depends on α. As long as $\alpha > 0$, random shocks imply that not all contestants are expected to be winners. Furthermore, the greater the dispersion in the effort, the fewer the number of contestants that are expected to be within α of the top performance. In contrast, note that for contests with draws the case is the opposite and the expected number of winners is higher with asymmetric efforts as this decreases the probability of a draw (Jia, 2012; Deng et al., 2018; Vesperoni and Yildizparlak, 2019).

It is easy to see that when the contestants choose the same effort level, the condition $x_i^r \theta_i > \alpha x_j^r \theta_j, \forall j \neq i$ is likely to hold for a larger number of them. Thus, with unequal effort levels, the sum of the probabilities is less than k but with identical efforts, the constraint $m \leq k$ is binding. From this perspective, Proposition 1 is unsurprising. Furthermore, that any identical effort level leads to the same outcome is due to the homogeneity of degree zero of the CSF and, subsequently, m.

Rent-seeking is notoriously difficult to study empirically due to the lack of data (Laband and Sophocleus, 2019). However, rent-seeking effort is conceptually clear and, at least in principle, measurable. Rent-seeking performance is more problematic in this sense. One interpretation of performance in a rent-seeking contest is "political influence". While it is a subtle concept, it seems natural to think that rent-seeking effort buys political influence, but the process is fraught with uncertainties. Furthermore, political influence primarily matters in relation to that of others.

These considerations make it unlikely that either performance or α could be directly observed in rent-seeking contests. However, this may be different in other partially exclusive contests. For example, consider sales performance in internal labour market contests. As an alternative for rewarding a fixed number of salespeople, the employer can state that everyone within α of the top sales performance will receive a bonus, making the rule and its application observable to everyone. Similarly, there are other contests, such as school admissions and exams, where performance and how it is rewarded are readily observable but effort is not.

Given the relationship between α and k, we can cast the contest designer's choice equally well as that of first choosing the maximum number of expected winners, k, which then implies the corresponding

$$\alpha = \frac{n-k}{k(n-1)}.$$

It further follows from Proposition 1 that $\alpha = 0 \leftrightarrow k = n$ and $\alpha = 1 \leftrightarrow k = 1$. If $0 < \alpha < 1$ and $1 < k < n$, which is the focus of interest, the rents will be partially exclusive (i.e., neither mutually exclusive nor guaranteed to all). Although n and the realised number of winners are integers, m and k need not be and α can be considered a continuous variable.

The timeline of the model is such that, in Stage 1, the contest designer chooses k (and the corresponding α) and in Stage 2, the contestants choose their efforts x_i. I solve the game by backward induction. Hence, I will next determine the equilibrium efforts and return to contest design and endogenise α and k later in subsection 3.2.

2.2 Equilibrium efforts

Here, we focus on the symmetric equilibrium in a contest with homogeneous valuations, for which a closed-form solution can be derived. Contestant i chooses x_i to maximise the expected payoff

$$\pi_i = \frac{x_i^r}{\alpha \sum_{j \neq i} x_j^r + x_i^r} R - x_i. \tag{4}$$

The first- and second-order conditions of (4) are

$$\frac{r x_i^{r-1} \alpha \sum_{j\neq i} x_j^r}{\left(\alpha \sum_{j\neq i} x_j^r + x_i^r\right)^2} R - 1 = 0 \tag{5}$$

and

$$\frac{r x_i^{r-2} \alpha \sum_{j\neq i} x_j^r ((r-1)\alpha \sum_{j\neq i} x_j^r - (1+r) x_i^r)}{\left(\alpha \sum_{j\neq i} x_j^r + x_i^r\right)^3} R < 0. \tag{6}$$

Proposition 2 *If and only if* $r \leq (\alpha(n-1)+1)/(\alpha(n-1))$*, then there exists a unique symmetric pure strategy Nash equilibrium in which each contestant's effort is*

$$x = \frac{r\alpha(n-1)}{(\alpha(n-1)+1)^2} R. \tag{7}$$

While the objective function is not concave for $r > 1$, it is not necessary for the existence of the symmetric equilibrium (see also Cornes and Hartley, 2005). However, with increasing returns to scale it is still possible that there exist asymmetric equilibria, in particular with inactive contestants.

Proposition 3 *All active contestants choose the same equilibrium effort. Furthermore, there are no inactive contestants and the symmetric equilibrium is the unique equilibrium if i)* $n = 2$*, ii)* $r \leq 1$*, or iii)* $\alpha \leq [(n-1)(n-2)]^{-1/2}$.

This result shows that there are no additional asymmetric equilibria, where active contestants choose different effort levels[7]. However, when the returns to scale are increasing, $n > 2$, and the expected payoff in the symmetric equilibrium is close to zero, there can exist asymmetric equilibria, where $n-1$ contestants choose the symmetric effort and one remains inactive. While in general it depends on the assumptions on the contest technology and costs whether this possibility arises, apart from Pérez-Castrillo and

[7]While the first-order condition can be satisfied by two different effort levels in the case of increasing returns, the proof shows that then one needs to be a local minimum or saddle point. Note that the proof makes use of $\alpha \leq 1$, and hence the result does not directly extend to contests with draws.

Verdier (1992) and Cornes and Hartley (2005) it is rarely considered in the literature. Nevertheless, the uniqueness of the symmetric equilibrium in the partially exclusive contest still holds, if the rents are not too exclusive and the expected number of winners is high enough.

The symmetric effort (7) further implies that the probability of success of each contestant in the symmetric equilibrium is

$$p = \frac{1}{\alpha(n-1)+1},$$

which, after summing all i gives the expected number of winners as

$$m = k = \frac{n}{\alpha(n-1)+1}, \tag{8}$$

as already indicated by Proposition 1.

Given that the contestants are assumed to be seeking different rents, it is reasonable to consider the situation where the values of these rents, R_i, are also different. In the online appendix, I use a perturbation approach and approximate the deviations in the efforts of weakly heterogeneous contestants around the symmetric equilibrium. The analysis shows that the symmetric case provides a good approximation of the aggregate effort when the valuations are weakly heterogeneous and all contestants remain active.

3 Analysis

This section examines the effect of partial exclusivity on rent-seeking behaviour and its use in contest design. First, I show that two outcomes, which are seemingly in conflict, arise because of partial exclusivity. I conclude the section by considering the problem of an effort-maximising contest designer and endogenise the design with respect to partial exclusivity.

3.1 Partial exclusivity

I focus on symmetric equilibrium, since it has a closed-form solution and provides a good approximation of the aggregate equilibrium effort in the case of weakly heterogeneous valuations. That is, the comparative statistics are qualitatively the same in the latter case as long as all the contestants remain active.

In the case of endogenous number of winners and rents, we need to distinguish between two measures of rent dissipation:

Definition 1

(i) ***Expected rent dissipation*** *is the ratio of total efforts to the expected value of the awarded rents (i.e. $\sum x_i/mR$).*

(ii) ***Potential rent dissipation*** *is the ratio of total efforts to the total value of the rents (i.e. $\sum x_i/nR$).*

Expected rent dissipation is the measure of the amount of expected gains that will be wasted in the aggregate efforts[8]. Potential rent dissipation is the measure of the amount of sought gain the contestants are willing to spend in the aggregate[9]. In the case of a fixed number of winners, there is no difference between the two measures but here the distinction is crucial.

Theorem 2 *Partial exclusivity of rents leads to lower expected rent dissipation.*

This result is further illustrated by Figure 1, which shows how expected rent dissipation is increasing in exclusivity and maximised when $\alpha = 1$. The reason is that while the aggregate effort can be increasing or decreasing in α, the expected number of winners in the denominator, which is inversely related to α, is always decreasing and at a faster pace. Thus, partial exclusivity of rents leads to lower expected rent dissipation and can account for the Tullock

[8]The concept was first used by Vesperoni and Yildizparlak (2019) in the context of draws.

[9]Removing the scale effect of rents by normalising their value to $R = 1$ serves the same purpose in the symmetric equilibrium.

paradox. However, expected rent dissipation can be a misleading measure of social cost since it is not monotonically related to the aggregate rent-seeking effort.

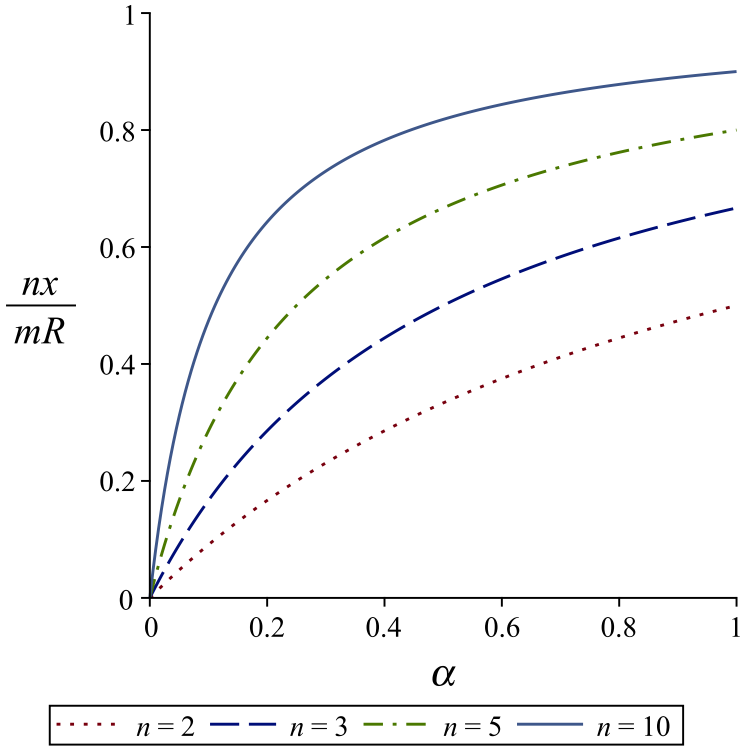

Figure 1: Expected rent dissipation and the degree of exclusivity (with $r = 1$ and for different values of n).

Theorem 3 *Partial exclusivity of rents leads to higher aggregate effort and potential rent dissipation if and only if* $\alpha > 1/(n-1)^2 \leftrightarrow k < n-1$. *Furthermore, the efforts and potential rent dissipation are maximised when* $\alpha = 1/(n-1) \leftrightarrow k = n/2$.

Theorem 3 shows that a partially exclusive contest yields the same aggregate effort as a perfectly exclusive single-winner contest when the maximum of the expected number of winners is $k = n-1$ and more effort when $k < n-1$. Thus, partial exclusivity leads to lower expected rent dissipation and more rent-seeking effort simultaneously.

We may further contrast the partially exclusive contest with multiple winners with the related contest with an endogenous probability of a draw by Jia (2012). If $\alpha > 1$ and the draw prize is zero, then Theorem 2 can be directly extended to draws such that the expected rent dissipation is increasing in α, which now stands for the required winning threshold. This is because the expected value of the prize decreases (as the draw becomes more probable) faster than the efforts. The overall negative effect on effort in the case of draws also implies that Theorem 3 looks different as then efforts and potential rent dissipation are maximised for $\alpha = 1$. However, this does not imply that the standard single-winner contest is generally superior in this sense as the relationship is non-monotonic for $\alpha \leq 1$, the case of multiple winners.

Figure 2 illustrates the expenditure-rent ratios or the potential rent dissipation for contests with different n. Apart from $n = 2$, we see how potential rent dissipation is first increasing and then decreasing in α. $n = 2$ is a special case where perfect exclusivity coincides with the effort-maximising degree of exclusivity.

The general, inverted-U shape in potential rent dissipation (and effort) is further illustrated in Figure 3. We can see here how $\alpha = 1/(n-1)^2$ yields the same amount of rent-seeking effort as perfect exclusivity, whereas $\alpha = 1/(n-1)$ maximises effort. At first, the strategic effect of increased competition induces more effort as it becomes harder to be a winner but when competition further increases and the expected number of winners becomes low, this tendency is reversed and the contestants start to cut back effort. That is, a little effort goes a long way in securing the rent when k is large, whereas for a small k, the marginal return to effort shrinks. The equilibrium effort is maximised when the chances to be a winner are average.

The effort-maximising $k = n/2$ is an interesting result for its simplicity and in comparison with the nested CSF, for which rent-seeking effort is maximised when $k \approx 0.632n$ (Clark and Riis, 1998)[10]. It is further verified by Proposition 3 that the symmetric equilibrium is also the unique equilibrium

[10]To be precise, this is the upperbound of effort-maximising k when $n \to \infty$. For small n it is less but can be shown to be always more than $0.576n$.

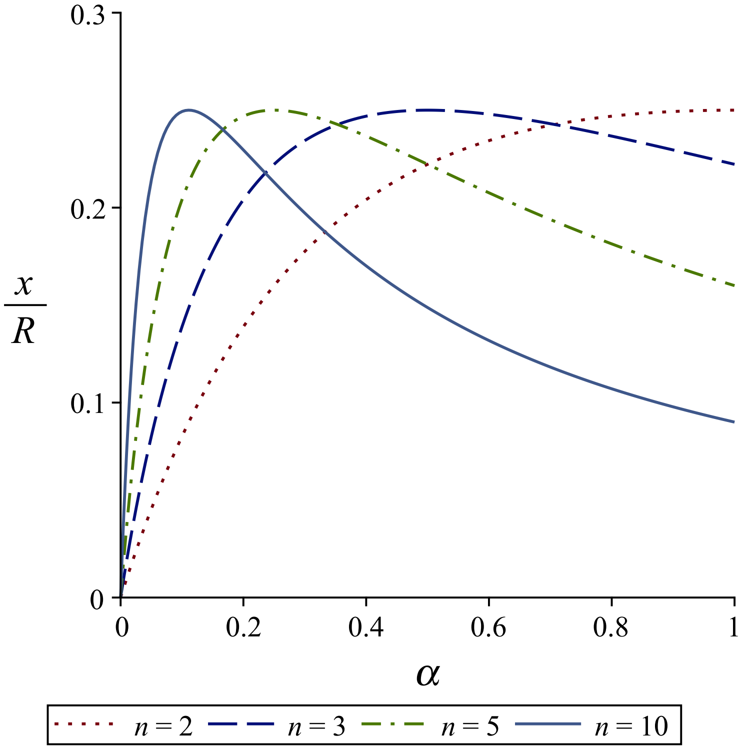

Figure 2: Potential rent dissipation and the degree of exclusivity (with $r = 1$ and for different values of n).

for the effort maximising α and k, which makes the analysis and comparison more robust.

The effort-maximising k further implies that the expected number of winners is also $m = n/2$ and that the probability of success of each contestant is $p = 1/2$. Furthermore, in this case the equilibrium effort is $x = rR/4$, the expected rent dissipation is $nx/mR = r/2$, and the potential rent dissipation is $nx/nR = r/4$, all of which are independent of n. Additionally, the expected payoff becomes $\pi = (2 - r)R/4$. Therefore, the participation constraint holds for all $r \leq 2$, which is the same as in the standard Tullock contest when $n = 2$ and is less restrictive for $n > 2$.

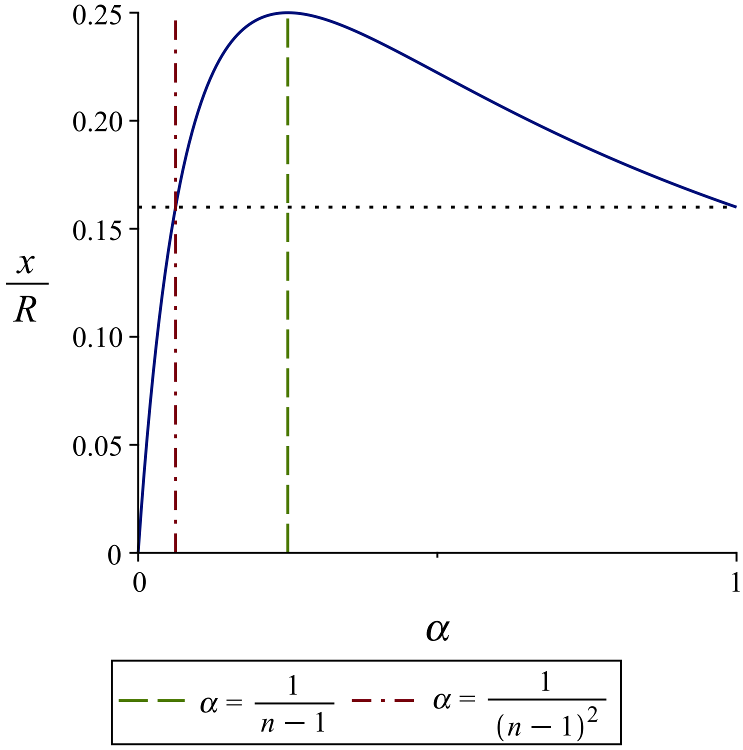

Figure 3: The general relationship between potential rent dissipation and the degree of exclusivity (with $r = 1$).

3.2 Contest design

Although the contest structure is exogenously given in some cases, it is reasonable to assume that the policymaker who allocates the rents could affect the design of this contest. A standard assumption in contest design literature is that the objective of a self-interested contest designer is to maximise aggregate rent-seeking efforts (e.g. Gradstein and Konrad, 1999). Since rents are transfers from consumers, potential or foreign firms, or the general public, the focus of the policymaker who designs the rent-seeking contest is in maximising the aggregate effort. Other contexts, where to designer's goal is typically to maximise effort, include promotion, innovation, and artistic contests. In particular, another context, where the designer can focus on maximising the effort without additional costs, budget constraints, or diluting the value of prizes, concerns recognition prizes (see Liu and Lu, 2017). I now consider the role of partial exclusivity in this setup.

Suppose the valuations are homogeneous or that the contestants' aggregate behaviour is reasonably well approximated by symmetric equilibrium. A key premise of the rent seeking theory is that the self-interested policymaker benefits from the interest groups' efforts to influence policy. Thus, let the designer's utility be an increasing function of the aggregate effort: $U(nx), U' > 0$. Maximising U with respect to α when x is given by (7) yields

$$\frac{dU}{d\alpha} = \frac{dU}{dnx}\frac{dnx}{d\alpha} = 0.$$

As $U' > 0$, it clearly follows that the designer chooses $\alpha = 1/(n-1)$ or, equivalently, $k = n/2$, as established by Theorem 3.

This outcome provides a simple rule, set $k = n/2$, for a policymaker who benefits from and wants to maximise rent-seeking effort. If the number of contestants increases (decreases), the designer can negate this change in the competition by making the rents less (more) exclusive. In essence, the designer induces maximum effort by virtually pairing the contestants for each prize, which eliminates the effect of n.

Fu and Lu (2009) and Chowdhury and Kim (2017) show that, for two other multiple-winner contests, the winner-selection mechanism is the determinant of whether a grand contest or multiple sub-contests will elicit a higher equilibrium effort. The outcome is different here because the allocation of contestants is irrelevant when the sub-contest structure is symmetric or the designer can choose the degree of exclusivity.

Theorem 4 *Consider an allocation of n contestants into s sub-contests such that the number of contestants is at least two in each sub-contest. Then, the aggregate effort is the same in a grand contest with n contestants and a collection of s identical sub-contests for any given total number of expected winners m. If, furthermore, m is not fixed and k can be chosen separately for each sub-contest, then the maximum aggregate effort is the same for any collection of sub-contests.*

This finding of the irrelevance of sub-contest structure provides an interesting contrast to the literature. However, the result is unsurprising in this case because the equilibrium effort depends only on the m/n ratio and not their levels, which is not the case in the winner selection or loser elimination models (Fu and Lu, 2009; Chowdhury and Kim, 2017). Furthermore, if the contest designer can choose the degree of exclusivity, then that will negate the competition effect arising from the number of contestants and elicit the maximum effort. Thus, there is no further benefit of splitting or combining sub-contests be they identical or not, which again is different in the other multiple-winner contests where the designer would prefer to control both m and n of each sub-contest.

As shown in the online appendix, the heterogeneous valuations paint a more nuanced picture of the rent-seeking contest. First, it becomes optimal to sort the weakly heterogeneous rent seekers into sub-contests that minimise the variance in the valuations. Thus, the choice between a grand contest and multiple sub-contests is no longer irrelevant. Second, with weakly heterogeneous valuations, the $k = n/2$ rule is no longer optimal. Nevertheless, the loss of efficiency is very small.

We could also consider variations of the designer's problem that include administrative costs or additional political gains that are increasing in m. If the net costs or gains are constant, z for each m, then the optimal k given the symmetric equilibrium effort becomes

$$\tilde{k} = \frac{(rR + z)n}{2rR},$$

where $z \gtrless 0$. While the additional costs and gains may be nonlinear in m or the valuations weakly heterogeneous, we may consider an intermediate k, where $k \approx fn$ and $f \in (0, 1)$ to be likely under various circumstances.

When k is a constant fraction of n, the symmetric equilibrium effort becomes $x = r(1 - f)fR$, which is independent of n. Therefore, a prediction of the model with endogenous contest design is that the aggregate effort nx is linear with the number of contestants. This analysis is not an exhaustive examination of the contest designer's problem but it serves to demonstrate that partially exclusive rents and the combination of low rent dissipation and high aggregate effort are natural outcomes under various circumstances.

As said earlier, a key difference in terms of winner selection is that the other multiple-winner contest models fix the number of winners from the outset, whereas in the partially exclusive contest it is both endogenous and stochastic. As Clark and Riis (1996) note, there is no unique method of how winners should be selected in multiple-winner contests. In some contexts, the contest designer may also be faced with a choice between these mechanisms. As such, I conclude the analysis by comparing the total effort in the symmetric equilibrium across the different multiple-winner contests for rents with uniform value R and given n, m and r.

The nested multiple-winner contest of Clark and Riis (1996, 1998) has m successive sub-contests, which each choose a single winner and remove the contestant from the subsequent stages. In contrast, the reverse nested contest of Fu et al. (2014) proceeds by sequentially choosing $n - m$ losers in its sub-contests. The loser elimination contest of Chowdhury and Kim (2014) is similar, but rather than directly choosing the loser in each stage, it chooses the group of survivors that continue to the next stage until all losers have been eliminated. Furthermore, this mechanism is equivalent to the very first, but simultaneous, multiple-winner mechanism suggested by Berry (1993) (see Chowdhury and Kim, 2014).

By making use of the digamma function and its relation to harmonic numbers,

$$\psi(n) = \sum_{k=1}^{n-1} \frac{1}{k} - \gamma,$$

with the Euler–Mascheroni constant γ, the comparable total symmetric equilibrium effort of Clark and Riis (1998) can be presented as

$$nx^C = rR \sum_{s=1}^{m} \left(1 - \sum_{h=0}^{s-1} \frac{1}{n-h} \right) = rR(n-m) \left(\psi(n+1) - \psi(n-m+1) \right). \tag{9}$$

Similarly, the total symmetric equilibrium effort of Fu et al. (2014) can be written as

$$nx^F = rR\sum_{s=1}^{m}\left(\sum_{h=0}^{n-s}\frac{1}{n-h}-1\right) = rRm\left(\psi(n+1)-\psi(m+1)\right). \qquad (10)$$

Finally, the total symmetric equilibrium effort in the contest models of Berry (1993) and Chowdhury and Kim (2014) is

$$nx^B = \frac{rR(n-m)}{n}. \qquad (11)$$

Theorem 5 *For given n, r and $m \in (1, n-1)$, the total symmetric equilibrium effort in the partially exclusive contest is less in the nested and reverse nested contests and more than in the loser elimination contest, i.e.*

$$nx^C, nx^F > nx > nx^B.$$

Figure 4 provides an example of how the different winner selection mechanisms rank for different number of winners with respect to potential rent dissipation. As was proven by Fu et al. (2014), we see that x^C and x^F are equal at $m = n/2$. x, on the other hand, is shadowed by the two and symmetric around the midpoint. x^B is markedly different from the other mechanisms as it monotonically decreasing in m.

Theorem 5 suggests that an effort-maximising contest designer may want to fix the number of winners and proceed by sequentially choosing the winners or losers rather than choose the level of exclusivity and then let the contest determine the number of winners. However, the analysis above assumes the symmetric equilibrium exists for the examined cases. As such, the partially exclusive contest may allow a higher r, for example. Indeed, this must be the case since it leads to full rent dissipation when $r = (\alpha(n-1)+1)/(\alpha(n-1))$ and it is not possible that the effort could be larger given another winner-selection mechanism. Thus, the choice of winner-selection mechanism, if indeed available, depends also on the existing constraints and what aspects of the design can be manipulated.

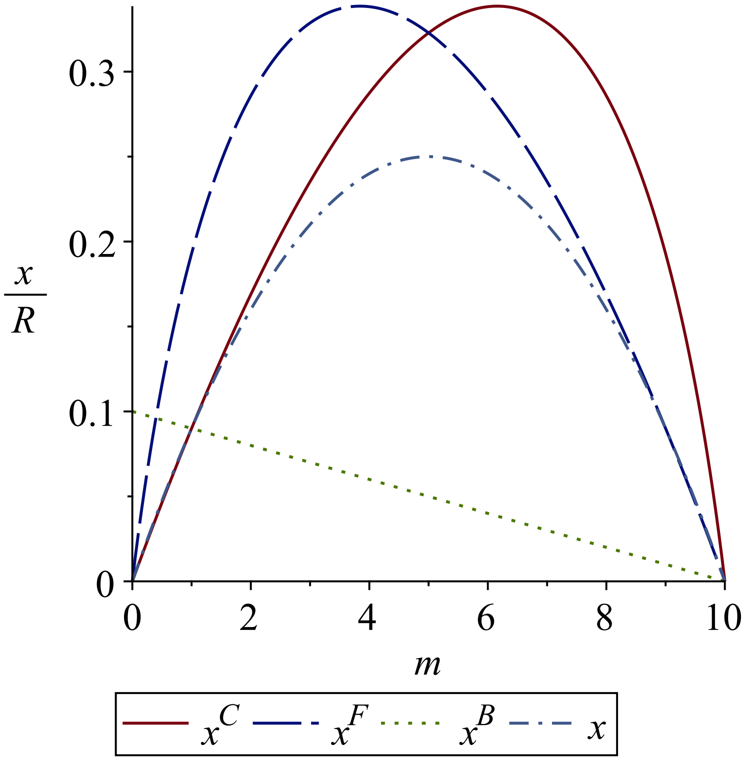

Figure 4: The (expected) number of winners and potential rent dissipation across the mechanisms (with $r = 1$ and $n = 10$).

4 Conclusion

Given the apparently high returns on rent seeking, the Tullock paradox raises the question: why is it not more prevalent in practice? The answer may be that rent seeking is more extensive than it appears. Rent seeking in the form of lobbying politicians is largely due to interest groups with separate goals, as opposed to firms fiercely competing for the same rent. Rents that are not mutually exclusive but still limited in their number, create imperfect competition for rents, leading to lower rent dissipation. However, low rent dissipation rates are deceptive in the context of partially exclusive rents as the aggregate rent-seeking effort can simultaneously be high. As such, the policymaker may appear to follow a relatively harmless policy towards lobbying interest groups while simultaneously inducing maximal rent-seeking efforts.

In this study, I propose a simple and intuitive multiple-winner extension to the Tullock contest success function. This extension provides a novel class of contest models, which could be applied to other contexts where the number of winners is endogenous. As shown in the online appendix, the symmetric equilibrium of the contest is a fairly accurate description of the aggregate behaviour even when the valuations are weakly heterogeneous. However, when this heterogeneity becomes more extensive, some contestants may become inactive. The analysis of effort-maximising contest design in such a case is left for future studies.

Acknowledgements

I am grateful to the associate editor and three anonymous reviewers for their valuable comments that greatly improved this paper. I would also like to thank Alberto Vesperoni, Anna Kochanova, Carmen Bevia, Ezgi Kaya, Helmuts Azacis, Iain Long, Indrajit Ray, James Strickland, John de Figueiredo, Joshy Easaw, Nigel Ashford, Pim Heijnen, Subhasish Chowdhury, Toomas Hinnosaar, Vassileos Logothetis, and the seminar audience at Cardiff, the SEA Annual Meetings in Tampa, Contests: Theory and Evidence conference in Norwich, Oligo Workshop in Nottingham, the SIOE Annual Meetings in Stockholm, and the European Meeting on Game Theory in Turku for their helpful comments on the earlier drafts of the paper. All errors remain my own.

Declaration of Competing Interest

Declarations of interest: none

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Appendix

Proof of Theorem 1. Note first that if $x_i = 0$, then it is not possible that $x_i^r\theta_i > \alpha x_j^r\theta_j$ for any $j \neq i$ and it is certain that contestant i will not be a winner. This also implies that if $\sum x_i = 0$, then there are no winners.

The remainder of the proof is adapted from Jia (2012). Suppose $x_i > 0$ and rewrite (1) as

$$Prob(\theta_j < \frac{x_i^r}{\alpha x_j^r}\theta_i, \forall j \neq i),$$

which, by the law of total probability, is equal to

$$\begin{aligned}
&\int_0^\infty Prob(\theta_j < \frac{x_i^r}{\alpha x_j^r} z, \forall j \neq i \mid \theta_i = z)h(z)dz \\
=&\int_0^\infty Prob(\theta_j < \frac{x_i^r}{\alpha x_j^r} z, \forall j \neq i \mid \theta_i = z)\sigma z^{-2}\exp\left(-\frac{\sigma}{z}\right)dz \\
=&\int_0^\infty \prod_{j\neq i} Prob(\theta_j < \frac{x_i^r}{\alpha x_j^r} z \mid \theta_i = z)\sigma z^{-2}\exp\left(-\frac{\sigma}{z}\right)dz \\
=&\int_0^\infty \prod_{j\neq i} \exp\left(-\frac{\sigma}{z}\frac{\alpha x_j^r}{x_i^r}\right)\sigma z^{-2}\exp\left(-\frac{\sigma}{z}\right)dz \\
=&\int_0^\infty \sigma z^{-2}\exp\left(-\frac{\sigma}{z}\frac{\alpha\sum\limits_{j\neq i} x_j^r + x_i^r}{x_i^r}\right)dz = \frac{x_i^r}{\alpha\sum\limits_{j\neq i} x_j^r + x_i^r},
\end{aligned}$$

which is the desired result. ■

Proof of Proposition 1. Note that if $\alpha = 1$, then $m = 1$ for any vector of efforts. Suppose that $\alpha < 1$. Differentiate m (3) with respect to x_i and x_j

to obtain

$$\frac{\partial m}{\partial x_i} = \frac{r x_i^{r-1} \alpha \sum\limits_{k \neq i} x_k^r}{\left(\alpha \sum\limits_{k \neq i} x_k^r + x_i^r\right)^2} - \sum_{k \neq i} \frac{x_k^r \alpha r x_i^{r-1}}{\left(\alpha \sum\limits_{l \neq k} x_l^r + x_k^r\right)^2} = 0 \tag{12}$$

and

$$\frac{\partial m}{\partial x_j} = \frac{r x_j^{r-1} \alpha \sum\limits_{k \neq j} x_k^r}{\left(\alpha \sum\limits_{k \neq j} x_k^r + x_j^r\right)^2} - \sum_{k \neq j} \frac{x_k^r \alpha r x_j^{r-1}}{\left(\alpha \sum\limits_{l \neq k} x_l^r + x_k^r\right)^2} = 0. \tag{13}$$

By dividing (12) by $r x_i^{r-1}$ and (13) by $r x_j^{r-1}$, and setting the two expressions equal to each other, we obtain

$$\frac{\alpha \sum x_i^r}{\left(\alpha \sum\limits_{k \neq i} x_k^r + x_i^r\right)^2} = \frac{\alpha \sum x_i^r}{\left(\alpha \sum\limits_{k \neq j} x_k^r + x_j^r\right)^2}$$

$$\leftrightarrow x_i^r(1-\alpha) = x_j^r(1-\alpha) \leftrightarrow x_i = x_j,$$

which implies that we have $x_i = x_j = x$ in any stationary point for any $i, j \in N$. What remains to be shown is that the maximum is obtained when $x_i = x$ for all $i \in N$ as compared to when some efforts are at the endpoints.

Suppose the efforts of n' contestants are equal to R, those of n'' are zero, and the efforts of the remaining $n-n'-n''$ contestants are equal to $\bar{x} \in (0, R)$. The expected number of winners is then given by

$$m'' = \frac{(n-n'-n'')\bar{x}^r}{\bar{x}^r + \alpha(n-n'-n''-1)\bar{x}^r + \alpha n' R^r} + \frac{n' R^r}{R^r + \alpha(n'-1)R^r + \alpha(n-n'-n'')\bar{x}^r}. \tag{14}$$

Next, we establish that m'' is decreasing in n'':

$$
\begin{aligned}
\frac{\partial m''}{\partial n''} = &\frac{-\bar{x}^r(\bar{x}^r + \alpha n' R^r)}{(\bar{x}^r + \alpha(n - n' - n'' - 1)\bar{x}^r + \alpha n' R^r)^2} \\
&+ \frac{\bar{x}^r \alpha n' R^r}{(R^r + \alpha(n' - 1)R^r + \alpha(n - n' - n'')\bar{x}^r)^2} \\
&< \frac{-\bar{x}^r \alpha n' R^r}{(\bar{x}^r + \alpha(n - n' - n'' - 1)\bar{x}^r + \alpha n' R^r)^2} \\
&+ \frac{\bar{x}^r \alpha n' R^r}{(R^r + \alpha(n' - 1)R^r + \alpha(n - n' - n'')\bar{x}^r)^2} < 0 \\
\leftrightarrow \bar{x}^r + \alpha(n - n' - n'' - 1)\bar{x}^r + \alpha n' R^r &< R^r + \alpha(n' - 1)R^r + \alpha(n - n' - n'')\bar{x}^r \\
&\leftrightarrow \bar{x}^r(1 - \alpha) < R^r(1 - \alpha).
\end{aligned}
$$

Hence, to maximise m, it is necessary that $n'' = 0$, which we substitute in (14) to obtain

$$
\begin{aligned}
m' = &\frac{(n - n')\bar{x}^r}{\bar{x}^r + \alpha(n - n' - 1)\bar{x}^r + \alpha n' R^r} \\
&+ \frac{n' R^r}{R^r + \alpha(n' - 1)R^r + \alpha(n - n')\bar{x}^r}. \qquad (15)
\end{aligned}
$$

Suppose there exists a $n' \in (0, n)$ that maximises (15). Then, we have

$$
\begin{aligned}
\frac{\partial m'}{\partial n'} = &\frac{-\bar{x}^r(\bar{x}^r(1 - \alpha) + \alpha n R^r)}{(\bar{x}^r + \alpha(n - n' - 1)\bar{x}^r + \alpha n' R^r)^2} \\
&+ \frac{R^r(R^r(1 - \alpha) + \alpha n \bar{x}^r)}{(R^r + \alpha(n' - 1)R^r + \alpha(n - n')\bar{x}^r)^2} = 0 \qquad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2 m'}{\partial (n')^2} = &\frac{2\bar{x}^r(\bar{x}^r(1 - \alpha) + \alpha n R^r)\alpha(R^r - \bar{x}^r)}{(\bar{x}^r + \alpha(n - n' - 1)\bar{x}^r + \alpha n' R^r)^3} \\
&- \frac{2R^r(R^r(1 - \alpha) + \alpha n \bar{x}^r)\alpha(R^r - \bar{x}^r)}{(R^r + \alpha(n' - 1)R^r + \alpha(n - n')\bar{x}^r)^3} < 0 \leftrightarrow \\
\frac{\bar{x}^r(\bar{x}^r(1 - \alpha) + \alpha n R^r)}{(\bar{x}^r + \alpha(n - n' - 1)\bar{x}^r + \alpha n' R^r)^3} &< \frac{R^r(R^r(1 - \alpha) + \alpha n \bar{x}^r)}{(R^r + \alpha(n' - 1)R^r + \alpha(n - n')\bar{x}^r)^3}. \qquad (17)
\end{aligned}
$$

Substitute (16) to (17) and obtain

$$R^r + \alpha(n'-1)R^r + \alpha(n-n')\bar{x}^r < \bar{x}^r + \alpha(n-n'-1)\bar{x}^r + \alpha n' R^r$$

$$\leftrightarrow R^r(1-\alpha) < \bar{x}^r(1-\alpha),$$

but this is a contradiction. Hence, n' that maximises (15) is either 0 or n, which implies that m is maximised when $x_i = x \in (0, R], \forall i \in N$. Substituting x in (3) for all x_i yields

$$m = \frac{n}{\alpha(n-1)+1} = k \leftrightarrow \alpha = \frac{n-k}{k(n-1)}.$$

■

Proof of Proposition 2. Let x denote the equilibrium effort exerted by each contestant in a symmetric Nash equilibrium. Note that $x = 0$ or $x = R$ cannot be an equilibrium, since a profitable deviation always exists, and hence any symmetric equilibrium must be interior. Substituting x for all x_i in (5), we obtain a unique solution

$$x = \frac{r\alpha(n-1)}{(\alpha(n-1)+1)^2}R.$$

(4) is strictly concave if condition (6) holds and

$$(r-1)\alpha \sum_{j \neq i} x_j^r - (1+r)x_i^r < 0, \tag{18}$$

for which a sufficient condition is $r \leq 1$.

Next, suppose $r > 1$. We see from (18) that (5) is increasing until

$$\hat{x}_i = \left(\frac{(r-1)\alpha \sum_{j \neq i} x_j^r}{1+r} \right)^{\frac{1}{r}}$$

and decreasing after. Hence, (4) has at most one interior maximum since there can be only one $x_i > \hat{x}_i$ such that (5) is satisfied.

Summing (18) over all i gives

$$(r-1)\alpha(n-1)\sum x_i^r - (1+r)\sum x_i^r < 0$$

and, thus, the symmetric effort (7) is a local maximum if and only if

$$(r-1)\alpha(n-1) - (1+r) < 0. \tag{19}$$

Finally, the symmetric effort (7) is also the global maximum if the expected payoff is no less than at the endpoints $\{0, R\}$, which requires that the participation constraint holds and $\pi \geq 0$. Substituting (7) into (4) gives

$$\pi_i = \frac{1}{\alpha(n-1)+1}R - \frac{r\alpha(n-1)}{(\alpha(n-1)+1)^2}R \geq 0,$$

which holds if and only if

$$(1-r)\alpha(n-1) + 1 \geq 0 \leftrightarrow r \leq \frac{\alpha(n-1)+1}{\alpha(n-1)}. \tag{20}$$

Collecting the results, the symmetric effort is a local maximum if and only if (19) holds, there can be no other local maxima, and the local maximum is also global if and only if (20) holds. I leave the confirmation that (20) also satisfies (19) to the reader. ■

Proof of Proposition 3. Let $Z \equiv \sum x_i^r$ denote the total equilibrium performance. Suppose that $x > 0$ and $y > 0$ are equilibrium efforts given the same Z and that $x \neq y$.

Substitute x and Z in (5) to obtain

$$f(x, Z) \equiv \frac{rx^{r-1}\alpha(Z - x^r)}{(\alpha Z + (1-\alpha)x^r)^2}R - 1 = 0. \tag{21}$$

It is necessary that both x and y satisfy (21).[11] Furthermore, to see whether

[11]One can show that this is not possible if $r \leq 1$, but non-increasing returns is not a

such x and y can both be feasible and optimal, differentiate (21) with respect to x to obtain the range of x for any given Z such that (21) is decreasing (or increasing) in the effort:

$$\frac{\partial f(x,Z)}{\partial x} = \frac{r(r-1)x^{r-1}\alpha(Z-x^r)}{x\left(\alpha Z+(1-\alpha)x^r\right)^2}R - \frac{r^2x^{r-1}\alpha x^r}{x\left(\alpha Z+(1-\alpha)x^r\right)^2}R$$

$$-\frac{2r^2x^{r-1}\alpha(Z-x^r)(1-\alpha)x^r}{x\left(\alpha Z+(1-\alpha)x^r\right)^3}R =$$

$$\frac{rx^{r-1}\alpha R}{x\left(\alpha Z+(1-\alpha)x^r\right)^3}\left((1-\alpha)x^{2r}-(1+r+\alpha(r-2))Zx^r+\alpha(r-1)Z^2\right) < 0$$

if and only if

$$(1-\alpha)X^2-(1+r+\alpha(r-2))ZX+\alpha(r-1)Z^2 < 0, \tag{22}$$

where $X \equiv x^r$. The roots of (22) separate the different intervals of X, each of which may potentially contain a unique optimal x satisfying (21).

Optimality requires non-negative expected payoff. Towards this end, (21) can be rewritten as

$$\frac{rX\alpha(Z-X)}{x_i\left(\alpha Z+(1-\alpha)X\right)^2}R-1=0$$

$$\leftrightarrow x_i = \frac{rX\alpha(Z-X)}{\left(\alpha Z+(1-\alpha)X\right)^2}R. \tag{23}$$

Substituting (23) in (4), we obtain

$$\pi_i = p_iR - x_i = \frac{X}{\alpha Z+(1-\alpha)X}\left(1-\frac{r\alpha(Z-X)}{\alpha Z+(1-\alpha)X}\right)R \geq 0$$

$$\leftrightarrow X \geq \bar{X} \equiv \frac{\alpha(r-1)}{1+\alpha(r-1)}Z.$$

The participation constraint and feasibility require that $X \in [\max\{0,\bar{X}\}, Z)$.

necessary requirement for the overall result.

If $\alpha = 1$, then (22) becomes

$$-(2r-1)XZ + (r-1)Z^2 < 0. \tag{24}$$

Furthermore, if $r > 1$ then solving (24) for X yields $X > (r-1)Z/(2r-1)$, which is always the case if $X \geq \bar{X} = (r-1)Z/r$. If $r \in [1/2, 1]$, then (24) holds for all $X, Z > 0$. If $r < 1/2$ then solving (24) for X yields $X < (1-r)Z/(1-2r)$, which is the case if $X < Z$. Hence, the inequality (24) holds for all feasible X that satisfy the participation constraint.

If $\alpha < 1$, (22) can be negative only in between its two roots:

$$X_L = \frac{-b - \sqrt{b^2 - 4ac}}{2a} Z \text{ and } X_H = \frac{-b + \sqrt{b^2 - 4ac}}{2a} Z,$$

where

$$a = 1 - \alpha, \quad b = -(1 + r + \alpha(r-2)), \quad c = \alpha(r-1).$$

Consider the discriminant, $b^2 - 4ac = (1+\alpha)^2 r^2 + (2 - 6\alpha)r + 1$, as a function $f(r)$. Since the discriminant of $f(r)$, in turn, is $32\alpha(\alpha - 1) < 0$, $f(r)$ is positive for all permissible α and r. Hence, the two roots of (22) are real.

For $X > X_H$ to be feasible it is required that

$$X_H < Z \leftrightarrow \frac{-b + \sqrt{b^2 - 4ac}}{2a} < 1 \leftrightarrow 2a + b > \sqrt{b^2 - 4ac}.$$

Since $2a + b > 0$ is necessary for the above inequality to hold, we can square both sides to obtain

$$(2a+b)^2 > b^2 - 4ac \leftrightarrow 4a(a+b+c) = -4(1-\alpha)r > 0,$$

which is an impossibility.

For $X < X_L$ to be feasible and optimal it is required that $X_L > \max\{0, \bar{X}\}$. Note that

$$X_L > 0 \leftrightarrow -b - \sqrt{b^2 - 4ac} > 0 \leftrightarrow -b > \sqrt{b^2 - 4ac}.$$

Since $-b > 0$ is necessary for the above inequality to hold, we can square both sides to obtain

$$b^2 > b^2 - 4ac \leftrightarrow 4ac > 0,$$

which is not the case if $r \leq 1$. Consider then $r > 1$ and

$$X_L > \bar{X} \leftrightarrow \frac{-b - \sqrt{b^2 - 4ac}}{2a} > \frac{c}{1+c} \leftrightarrow -b(1+c) - 2ac > (1+c)\sqrt{b^2 - 4ac}.$$

Since $-b(1+c) - 2ac > 0$ is necessary for the above inequality to hold, we can square both sides to obtain

$$(b(1+c) + 2ac)^2 > (1+c)^2(b^2 - 4ac) \leftrightarrow$$

$$4ac(b(1+c) + (1+c)^2 + ac) = -4(1-\alpha)\alpha(r-1)r(\alpha(r-2)+1) > 0,$$

but this is not possible when $r > 1$ and $\alpha \in (0, 1)$.

Hence, for any given Z only the single interval where $f(x, Z)$ is decreasing in x is feasible and optimal, and it contains a unique x such that $f(x, Z) = 0$. Conversely, if $f(x, Z) = f(y, Z) = 0$, $x \neq y$ and both are feasible, then one of them is a local minimum or saddle point and not an equilibrium choice.

Lastly, we consider asymmetric equilibria, in which some contestants expend zero effort. Consider first the case of $n = 2$. If contestant 1 chooses $x_1 = 0$, then it is optimal for contestant 2 to choose an arbitrarily small $x_2 = \epsilon$. But then contestant 1 can increase the expected payoff by participating with positive x_1, say 2ϵ. Hence, both contestants must be active in any equilibrium.

Consider now the case of $r \leq 1$. (23) can be rewritten as

$$\leftrightarrow x_i = rp_i(1 - p_i)R. \tag{25}$$

Substitute (25) in (4) to obtain

$$\pi_i = p_i R - x_i = p_i R(1 - r(1 - p_i)) > 0 \leftarrow r \leq 1.$$

That is, there is an incentive to be active as (5) will always yield positive expected payoff if $r \leq 1$.

Lastly, consider the case of $r > 1$, where (5) may yield negative expected payoff. Let $Z_{-i} = Z - x_i^r$ denote the opponents' total performance. For all $x_i > 0$, the expected payoff (4) is decreasing in Z_{-i}. In the symmetric equilibrium, where all n contestants are active, this is

$$Z_{-i}(n) = (n-1)\left(\frac{r\alpha(n-1)}{(\alpha(n-1)+1)^2}R\right)^r.$$

Conversely, the total performance of m active contestants faced by an inactive contestant is

$$Z(m) = m(x(m))^r = m\left(\frac{r\alpha(m-1)}{(\alpha(m-1)+1)^2}R\right)^r.$$

Note that

$$\frac{\partial Z(m)}{\partial m} = (x(m))^r\left(1+\frac{rm(1-\alpha(m-1))}{(m-1)(\alpha(m-1)+1)}\right) > 0 \leftarrow \alpha \leq \frac{1}{m-1}.$$

Suppose then that $\alpha > 1/(m-1)$. Since $r \leq (\alpha(m-1)+1)/(\alpha(m-1))$,

$$\frac{\partial Z(m)}{\partial m} \geq (x(m))^r\left(1+\frac{m}{(m-1)}\left(\frac{1}{\alpha(m-1)}-1\right)\right)$$

$$\geq (x(m))^r\left(1+\frac{m}{(m-1)}\left(\frac{1}{m-1}-1\right)\right) = (x(m))^r\frac{1}{(m-1)^2} > 0.$$

Therefore, the total performance faced by an inactive contestant is the greatest when $m = n-1$. Since the participant's expected payoff is non-negative in the symmetric equilibrium, it is also non-negative if $Z_{-i}(n) \geq Z(n-1)$. This holds when

$$(n-1)\left(\frac{r\alpha(n-1)}{(\alpha(n-1)+1)^2}R\right)^r \geq (n-1)\left(\frac{r\alpha(n-2)}{(\alpha(n-2)+1)^2}R\right)^r$$

$$\leftrightarrow (n-1)(\alpha(n-2)+1)^2 \geq (n-2)(\alpha(n-1)+1)^2$$

$$\leftrightarrow -\alpha^2 + \frac{1}{n-2} - \frac{1}{n-1} \geq 0 \leftrightarrow \alpha \leq [(n-1)(n-2)]^{-1/2},$$

which is sufficient to make participation always profitable ex ante. ■

Proof of Theorem 2. The expected rent dissipation in the symmetric equilibrium (7) is given by the ratio

$$\frac{nx}{mR} = \frac{r\alpha(n-1)}{\alpha(n-1)+1},$$

which is increasing in the degree of exclusivity, α, since

$$\partial\left(\frac{nx}{mR}\right)/\partial\alpha = \frac{r(n-1)}{(\alpha(n-1)+1)^2} > 0,$$

which completes the proof. ■

Proof of Theorem 3. Comparing the aggregate efforts, nx, under partial and perfect exclusivity of rents yields

$$\begin{aligned} nx(\alpha) = \frac{nr\alpha(n-1)}{(\alpha(n-1)+1)^2}R &> \frac{nr(n-1)}{n^2}R = nx(1) \\ \leftrightarrow -\alpha^2(n-1)^2 + \alpha(n^2 - 2(n-1)) - 1 &> 0 \\ \leftrightarrow (1-\alpha)(\alpha(n-1)^2 - 1) &> 0 \\ \leftrightarrow \alpha &> \frac{1}{(n-1)^2}. \end{aligned} \tag{26}$$

(26) together with (8) implies that

$$k < \frac{n}{\frac{1}{n-1}+1} = n-1.$$

The same comparison holds for potential rent dissipation, where both sides of the inequalities are divided by nR. This completes the proof of the first part of the theorem.

To solve for maximum effort, I differentiate nx with respect to α to obtain

$$\frac{\partial nx}{\partial\alpha} = \frac{nr(n-1)(1-\alpha(n-1))}{(\alpha(n-1)+1)^3}R. \tag{27}$$

Setting (27) equal to zero and solving for α yields

$$\alpha = \frac{1}{n-1}, \tag{28}$$

which is the argument of the maximum because (27) is decreasing in α. Clearly, the same argument (28) maximises potential rent dissipation, which is equal to $nx/nR = x/R$ in the symmetric equilibrium. Finally, from (8) it follows that (28) corresponds to

$$k = \frac{n}{2}.$$

This completes the proof of the second part of the theorem. ■

Proof of Theorem 4. Consider first the case of s symmetric sub-contests, each with n/s contestants and m/s expected winners. Let n_j denote the number of contestants and x_j their equilibrium efforts in sub-contest j. Using (7) and (8), the aggregate effort across all sub-contests is

$$\sum_{j=1}^{s} n_j x_j = \sum_{j=1}^{s} \frac{rR(n_j - m_j)m_j}{n_j} = s\frac{rR\left(\frac{n}{s} - \frac{m}{s}\right)\frac{m}{s}}{\frac{n}{s}} = \frac{rR(n-m)m}{n},$$

which is the same as in a single grand contest with $s = 1$.

Consider then the final statement of the theorem. We have $\sum_{j=1}^{s} n_j = n$. The effort-maximising k for sub-contest j is $k_j = n_j/2$ (as implied by Theorem 3), which yields the equilibrium effort $x = rR/4$. If each k_j is chosen similarly, the aggregate effort over s sub-contests is

$$\sum_{j=1}^{s} n_j \frac{rR}{4} = n\frac{rR}{4},$$

which is independent of the number of sub-contests s and the allocation of contestants n_j. ■

Proof of Theorem 5. The digamma function satisfies the recurrence relation

$$\psi(n+1) = \psi(n) + \frac{1}{n}.$$

Its derivative, the trigamma function, can be defined as

$$\psi'(n) = \sum_{k=0}^{\infty} \frac{1}{(n+k)^2}.$$

The trigamma function is decreasing in n and Elbert and Laforgia (2000) prove that it has a lower bound

$$\psi'(n) > \frac{1}{n} + \frac{1}{2n^2}.$$

The following steps make use of these properties.

Comparing (9) and nx given by (7) and (8) we see that

$$nx^C = rR(n-m)\left(\psi(n+1) - \psi(n-m+1)\right) \gtrless \frac{rR(n-m)m}{n} = nx$$

$$\leftrightarrow \psi(n+1) - \psi(n-m+1) \gtrless \frac{m}{n}. \tag{29}$$

If $m = 1$, both sides of (29) are equal to $1/n$ and $nx^C = nx$. The derivative of the LHS with respect to m is

$$\psi'(n-m+1) = \sum_{k=0}^{\infty} \frac{1}{(n-m+1+k)^2} \geq \psi'(n) > \frac{1}{n} + \frac{1}{2n^2},$$

which is greater than the derivative of the RHS, $1/n$. Thus, $nx^C > nx \leftrightarrow m > 1$.

Comparing (10) and nx we see that

$$nx^F = rRm\left(\psi(n+1) - \psi(m+1)\right) \gtrless \frac{rR(n-m)m}{n} = nx$$

$$\leftrightarrow \psi(n+1) - \psi(m+1) \gtrless \frac{n-m}{n}. \tag{30}$$

If $m = n - 1$, both sides of (30) are equal to $1/n$ and $nx^C = nx$. The derivative of the LHS with respect to m is

$$-\psi'(m+1) = -\sum_{k=0}^{\infty} \frac{1}{(m+1+k)^2} \leq -\psi'(n) < -\frac{1}{n} - \frac{1}{2n^2},$$

which is less than the derivative of the RHS, $-1/n$. Thus, $nx^F > nx \leftrightarrow m < n-1$.

Lastly, it is immediate from (11) that $nx^B = nx \leftrightarrow m = 1$ and $nx^B < nx$ otherwise. ∎