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INVERSE SPECTRAL AND SCATTERING THEORY FOR THE HALF-LINE LEFT-DEFINITE STURM–LIOUVILLE PROBLEM*

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Abstract. The problem of integrating the Camassa–Holm equation leads to the scattering and inverse scattering problem for the Sturm–Liouville equation $-u'' + \frac{1}{4}u = \lambda wu$, where w is a weight function which may change sign but where the left-hand side gives rise to a positive quadratic form so that one is led to a left-definite spectral problem. In this paper the spectral theory and a generalized Fourier transform associated with the equation $-u'' + \frac{1}{4}u = \lambda wu$ posed on a half-line are investigated. An inverse spectral theorem and an inverse scattering theorem are established. A crucial ingredient of the proofs of these results is a theorem of Paley–Wiener type which is shown to hold true. Additionally, the accumulation properties of eigenvalues are investigated.

Key words. inverse scattering problems, inverse spectral problems, left-definite problems, Sturm–Liouville, Camassa–Holm equation

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1. Introduction.

Standard Sturm–Liouville theory deals with the eigenvalue problem

$$-(pu')' + qu = \lambda wu, \tag{1.1}$$

together with appropriate boundary conditions, in the space L^2_w of functions square integrable with respect to the weight w, *i.e.*, the norm-square of the space is $\|u\|^2 = \int |u|^2 w$. A basic assumption for this to be possible is that $w \geq 0$. In some situations of interest this is not the case, but instead one has $p > 0$, $q \geq 0$. One may then use as a norm-square the integral $\int(p|u'|^2 + q|u|^2)$, and a problem of this type is usually called *left-definite*. A left-definite problem of current interest is the spectral problem associated with the Camassa–Holm equation, which is of the form

$$-u'' + \tfrac{1}{4}u = \lambda wu. \tag{1.2}$$

The Camassa–Holm equation is an integrable system in a similar sense as the Korteweg–de Vries (KdV) equation. It was first derived as an abstract bi-Hamiltonian system by Fuchssteiner and Fokas [22]. Subsequently, it was shown by Camassa and Holm [11] that it may serve as an integrable model for shallow water waves. In that paper Camassa and Holm also showed that the solitons are peaked and called them peakons (see also Fokas and Liu [21] and Johnson [23]). In contrast to the KdV equation the Camassa–Holm equation may model breaking waves, *i.e.*, smooth

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initial data may develop singularities in finite time; cf. Constantin and Escher [15] and Constantin [13] (see also Bressan and Constantin [10] for a way to resolve the singularities due to wave breaking). This, however, happens only when w changes sign and it is this fact which motivates us to consider (1.2) without the assumption that w is positive. The well developed theory of scattering and inverse scattering for the Schrödinger equation is of crucial importance to the theory of the KdV equation. In the same way scattering/inverse scattering theory for (1.2) is important for dealing with the Camassa–Holm equation. Unfortunately, no such theory is available unless $w \geq 0$, and even then current theory requires more smoothness of w than is convenient to assume, in view of the lack of smoothness for the corresponding peakons.

The problem of inverse scattering for (1.2) is considerably more difficult than for the Schrödinger equation, which may be viewed as a rather mild perturbation of the equation $-u'' = \lambda u$. In case of (1.2) the perturbation is of the equation $-u''+\frac{1}{4}u = \lambda u$, and thus changes the coefficient containing the eigenvalue parameter λ. It appears that the methods used so far for dealing with the Schrödinger equation are no longer applicable.

In this paper we will prove some uniqueness results for inverse spectral theory and inverse scattering for the left-definite case which apply to (1.2) posed on a half-line. One would also like to have results for the full-line, but this appears to be more difficult. One exception is the case of odd initial data for the Camassa–Holm equation on the full-line because the problem can be reduced to one on a half-line. We mention here that the half-line case was also investigated by Boutet de Monvel and Shepelsky [8], [9], who employ Riemann–Hilbert techniques but assume that w is positive. Our approach is via the inverse spectral theory for the left-definite problem, which also is not very well developed. Even the spectral theory for left-definite problems is not widely known (but see for example [1]), in the level of detail necessary for dealing with the inverse problem. We will therefore start by presenting a reasonably comprehensive spectral theory, then prove some uniqueness theorems for the inverse spectral problem, and finally a uniqueness theorem for inverse half-line scattering.

Spectral theory for left-definite Sturm–Liouville problems seems to have been initiated by Weyl [28], who called such problems *polar*. Later many authors have dealt with more or less general left-definite problems. In particular we mention a series of papers by Niessen, Schneider, and their collaborators on singular left-definite so-called S-hermitian systems; see, *e.g.*, [26]. See also [1] and the references cited there. For a more recent contribution, see Kong, Wu, and Zettl [24]. However, papers in inverse spectral theory for left-definite problems are much more scarce; one example is Binding, Browne, and Watson [7].

Because of the connection with the Camassa–Holm equation the inverse scattering problem for (1.2) has attracted some attention. From the physical point of view the full-line case where w decays at infinity and the periodic case are most interesting. The former was treated by Fokas [20] and Constantin and various co-authors, for example in [14], [16], and [17]. The latter was addressed by Constantin and McKean [18], Constantin [12], and Vaninsky [27]. The full-line case with odd initial data reduces to a half-line case, but the half-line case is also of interest independently.

It will be convenient to deal only with the equation

$$-u'' + qu = \lambda wu. \tag{1.3}$$

There is no loss of generality in doing this, since the change of variable $t = \int_0^x 1/p$ will, as is readily seen, turn (1.1) into an equation of this form.

The plan of the paper is as follows. In section 2 we give a general spectral theory for left-definite problems on intervals with at least one regular endpoint, modelled on standard Titchmarsh–Weyl theory. One may extend this to intervals with two singular endpoints, in the same way as one can extend the right-definite theory, but since we will have no use of it here we have abstained from this.

In section 3 we deal with the generalized Fourier transform associated with a left-definite problem. To simplify the discussion we have restricted ourselves to one case, when so-called finite functions are dense in the Hilbert space associated with the equation. There are no fundamental difficulties involved in dealing with the general situation, but again we have no need of it in the applications we are thinking of.

Section 4 discusses uniqueness of the inverse spectral problem. Unfortunately we have neither a characterization nor a reconstruction algorithm, but the fundamental uniqueness theorem is quite general.

In section 5 we prove a theorem of Paley–Wiener type which is crucial for our approach to the inverse spectral theory, and section 6 deals with the uniqueness theorem for the half-line inverse scattering of a left-definite problem. Section 7 is devoted to some results about the number of eigenvalues for a left-definite problem under scattering conditions. Some elementary, but rather lengthy, calculations needed in section 4 have been relegated to the appendix.

2. Spectral theory. We shall consider (1.3) on an interval $[0,b)$ and assume that q and w are real-valued and integrable on compact subsets of $[0,b)$, that $q \geq 0$, and that neither q nor w vanish a.e. Let $\mathcal{H}_1$ be the set of locally absolutely continuous functions u defined in $[0,b)$ such that $u' \in L^2(0,b)$ and $q|u|^2 \in L^1(0,b)$. As we shall see presently $\mathcal{H}_1$ is a Hilbert space with scalar product

$$\langle u, v\rangle = \int_0^b (u'\overline{v'} + qu\overline{v})$$

and norm $\|u\| = \sqrt{\langle u,u\rangle}$. In order to show completeness of $\mathcal{H}_1$ and discuss how to find self-adjoint realizations corresponding to (1.3) we first note the following simple result.

LEMMA 2.1. *For any $a \in [0,b)$ there exists a constant C_a such that*

$$|u(x)| \leq C_a\|u\| \tag{2.1}$$

for any $x \in [0,a]$ and any $u \in \mathcal{H}_1$.

Proof. By the fundamental theorem of calculus and the Cauchy–Schwarz inequality $|u(x)| \leq |u(y)| + |y-x|^{1/2}\big(\int_0^b |u'|^2\big)^{1/2}$. If $c \in [a,b)$ is such that $\int_0^c q > 0$, multiplication by $q(y)$ and integrating with respect to y gives

$$|u(x)|\int_0^c q \leq \int_0^c q|u| + c^{1/2}\int_0^c q\left(\int_0^b |u'|^2\right)^{1/2}.$$

Using Cauchy–Schwarz again we obtain (2.1) with $C_a = \big(c + 1/\int_0^c q\big)^{1/2}$. ∎

PROPOSITION 2.2. *The space $\mathcal{H}_1$ is complete.*

Proof. By (2.1) a Cauchy sequence $u_1, u_2, \ldots$ in $\mathcal{H}_1$ converges locally uniformly to a continuous function u. Furthermore, $\sqrt{q}u_j$ and u_j' converge in $L^2[0,b)$ to $\sqrt{q}u$ and, say, v, respectively. Now

$$u_j(x) - u_j(0) = \int_0^x u_j'.$$

Letting $j \to \infty$ we obtain $u(x) = u(0) + \int_0^x v$. Thus u is absolutely continuous with derivative v and u_j converges to u in $\mathcal{H}_1$. ∎

Denote the set of integrable functions with compact support in $(0, b)$ by L_0. Then, if $u \in \mathcal{H}_1$ and $v \in L_0$, it follows that $|\int u\overline{v}| \le C_a \int |v| \, \|u\|$ if $\operatorname{supp} v \subset [0, a]$, so that the linear form $\mathcal{H}_1 \ni u \mapsto \int u\overline{v}$ is bounded. By Riesz's representation theorem we may therefore find a unique $v^* \in \mathcal{H}_1$ so that $\int u\overline{v} = \langle u, v^* \rangle$. Clearly v^* depends linearly on v, so we obtain a (bounded) operator $G_0 : L_0 \to \mathcal{H}_1$ such that

$$\langle u, G_0 v \rangle = \int_0^b u\overline{v} \quad \text{for} \quad u \in \mathcal{H}_1, \ v \in L_0.$$

The operator G_0 is central for the left-definite spectral theory of (1.3).

PROPOSITION 2.3. *The operator G_0 is an integral operator $G_0 u(x) = \int u \, g_0(x, \cdot)$, it is injective, and its restriction to $L_0 \cap \mathcal{H}_1$ is symmetric with range dense in $\mathcal{H}_1$.*

Proof. By (2.1) the map $\mathcal{H}_1 \ni u \mapsto u(x)$ is for each fixed $x \in [0, b)$ a bounded linear form, so there exists an element $g_0(x, \cdot) \in \mathcal{H}_1$ so that $u(x) = \langle u, \overline{g_0(x, \cdot)} \rangle$ for $u \in \mathcal{H}_1$, and therefore $G_0 v(x) = \langle G_0 v, \overline{g_0(x, \cdot)} \rangle = \int_0^b v \, g_0(x, \cdot)$ for any $v \in L_0$. Thus G_0 is an integral operator with kernel $g_0(x, y)$ (actually, as we shall see in Proposition 2.7, g_0 is real-valued). If u and $v \in L_0 \cap \mathcal{H}_1$, then

$$\langle G_0 u, v \rangle = \overline{\langle v, G_0 u \rangle} = \int_0^b u\overline{v} = \langle u, G_0 v \rangle,$$

so the restriction of G_0 to $L_0 \cap \mathcal{H}_1$ is symmetric.

Let $[c, d] \subset (0, b)$ and $u_j(x) = \min(1, j(x - c), j(d - x))$ for $x \in [c, d]$ and $u_j(x) = 0$ otherwise. Then $u_j \in L_0 \cap \mathcal{H}_1$ and tends boundedly to the characteristic function of $[c, d]$ as $j \to \infty$, so if $G_0 v = 0$, it follows from $0 = \langle G_0 v, u_j \rangle = \int v\overline{u_j}$ that $\int_c^d v = 0$ for all $[c, d] \subset (0, b)$. Thus $v = 0$ a.e. so that G_0 is injective. On the other hand, if $u \in \mathcal{H}_1$ is orthogonal to $G_0 v$ for all $v \in L_0 \cap \mathcal{H}_1$, we may put $v = u_j$, so that $0 = \langle u, G_0 u_j \rangle \to \int_c^d u$. It follows that $u = 0$ so the range of G_0 restricted to $L_0 \cap \mathcal{H}_1$ is dense and the proof is complete. ∎

We shall have to briefly use the theory of symmetric relations as presented in [1, section 1], and define maximal and minimal relations corresponding to (1.3). We start by setting

$$T_c = \{(G_0(wv), v) \mid v \in L_0 \cap \mathcal{H}_1\}.$$

Then, since w is real-valued, T_c is a symmetric relation in $\mathcal{H}_1$ for

$$\langle G_0(wu), v \rangle = \overline{\langle v, G_0(wu) \rangle} = \int_0^b wu\overline{v} = \langle u, G_0(wv) \rangle.$$

Proposition 2.3 implies that T_c is the graph of a densely defined symmetric operator in $\mathcal{H}_1$ if $\operatorname{supp} w = [0, b)$, but at this point we do not want to exclude the possibility of w vanishing on an open set. We define the *minimal relation* T_0 as the closure (in $\mathcal{H}_1 \oplus \mathcal{H}_1$) of T_c, and the *maximal relation* T_1 as the adjoint of this, *i.e.*,

$$T_1 = \{(u, f) \in \mathcal{H}_1 \oplus \mathcal{H}_1 \mid \langle u, v \rangle = \langle f, G_0(wv) \rangle \text{ for all } v \in L_0 \cap \mathcal{H}_1\}.$$

We must show that T_1 is a differential relation.

PROPOSITION 2.4. *We have $(u, f) \in T_1$ if and only if u and $f \in \mathcal{H}_1$, u' is locally absolutely continuous, and $-u'' + qu = wf$.*

Proof. First note that if u and $f \in \mathcal{H}_1$, then the definition of G_0 shows that

$$\langle u, v\rangle - \langle f, G_0(wv)\rangle = \int_0^b (u'\overline{v'} + qu\overline{v} - wf\overline{v}) \tag{2.2}$$

for any $v \in L_0 \cap \mathcal{H}_1$. If in addition u' is locally absolutely continuous and satisfies $-u'' + qu = wf$, integrating by parts gives

$$\langle u, v\rangle - \langle f, G_0(wv)\rangle = \int_0^b (-u'' + qu - wf)\overline{v} = 0.$$

This proves one direction of the proposition.

In proving the other direction the assumption is that the quantity (2.2) is zero. But since $C_0^\infty(0,b) \subset \mathcal{H}_1$ this means that the distributional derivative of u' is $qu - wf$ so that u' is locally absolutely continuous and u satisfies the differential equation.

To give a proof without the use of distribution theory we prove a variant of the classical du Bois-Reymond lemma. If $v \in L_0 \cap \mathcal{H}_1$, integration by parts in (2.2) gives

$$\int_0^b \left\{ u' - \int_0^x (qu - wf) - C \right\} \overline{v'} = 0 \tag{2.3}$$

for any constant C. Now let $[c,d] \subset (0,b)$ and choose $C = \frac{1}{d-c}\int_c^d \{u' - \int_0^x (qu - wf)\}$. Put $v(y) = 0$ for $y \notin [c,d]$ and

$$v(y) = \int_c^y \left\{ u'(x) - \int_0^x (qu - wf) - C \right\} dx$$

for $y \in [c,d]$. Then $v \in L_0 \cap \mathcal{H}_1$ and (2.3) gives

$$\int_c^d \left| u' - \int_0^x (qu - wf) - C \right|^2 = 0$$

so that $u' - \int_0^x (qu - wf)$ is constant in $[c,d]$. Thus u' is locally absolutely continuous, and differentiation gives $-u'' + qu = wf$. $\square$

Let $\mathcal{D}_\lambda = \{(u, \lambda u) \in T_1\}$ and let D_λ be the projection of $\mathcal{D}_\lambda$ onto its first components, *i.e.*, $u \in D_\lambda$ means that $u \in \mathcal{H}_1$ and u satisfies $-u'' + qu = \lambda w u$. We then have

$$T_1 = T_0 \dot{+} \mathcal{D}_\lambda \dot{+} \mathcal{D}_{\overline{\lambda}}$$

as a direct sum for any nonreal λ. Here $\dim \mathcal{D}_\lambda = \dim D_\lambda$ is constant in each of the upper and lower half-planes, and these dimensions will be called the *deficiency indices* of T_1. See [1, Theorem 1.4] for this simple generalization of the von Neumann formula for symmetric operators and its consequences. It is clear that $\dim D_\lambda \le 2$, and that $\dim D_{\overline{\lambda}} = \dim D_\lambda$, since $\overline{u} \in D_{\overline{\lambda}}$ if and only if $u \in D_\lambda$. Thus deficiency indices are always equal, and there are always self-adjoint extensions of T_0, which will at the same time be restrictions of T_1, and therefore realizations of (1.3). It is of course of interest to have criteria in terms of the coefficients q and w for different values of the deficiency indices $\dim D_\lambda$. In surprising contrast to the right-definite case, we have the following simple and explicit criteria.

THEOREM 2.5. *Suppose* $\operatorname{Im}\lambda \ne 0$ *and let* W *be an antiderivative of* w. *Then* $\dim D_\lambda = 2$ *if* $b < \infty$ *and* $q + W^2 \in L^1[0,b)$. *Otherwise* $\dim D_\lambda = 1$ *for* $\operatorname{Im}\lambda \ne 0$.

The theorem is a special case of [2, Theorem 2.3]. See also [5]. In the right-definite case a simple variation of constants argument shows that if $\dim D_\lambda = 2$ for one real or nonreal value of λ, then this holds for all $\lambda \in \mathbb{C}$. A similar argument shows that this remains true in the left-definite case, with the exception that it is possible that $\dim D_0 = 2$ even if $\dim D_\lambda < 2$ for all $\lambda \neq 0$. This is to be expected, since D_0 does not depend on the choice of w. We characterize $\dim D_0$ completely in the following theorem, which also brings out the significance of the space D_0. We use the expression *finite function* in $\mathcal{H}_1$ to denote a function which vanishes near b.

THEOREM 2.6.

(1) *The set D_0 is the orthogonal complement in $\mathcal{H}_1$ of $L_0 \cap \mathcal{H}_1$ and has dimension* 1 *or* 2*.*
(2) $\dim D_0 = 2$ *if and only if* $b < \infty$ *and* $q \in L^1[0, b)$*.*
(3) *If $b < \infty$ and $q \in L^1[0, b)$, then v and v' have finite limits at b for all $v \in D_0$, and these limits uniquely determine v.*
(4) *If $b < \infty$ and $q \in L^1[0, b)$, then every $u \in \mathcal{H}_1$ has a limit at b which is a bounded linear form on $\mathcal{H}_1$.*
(5) *If $\dim D_0 = 1$ and $D_0 \ni v \not\equiv 0$, then $v(0)\overline{v'(0)} < 0$ and $u(x)\overline{v'(x)} \to 0$ as $x \to b$ for any $u \in \mathcal{H}_1$.*
(6) *Finite functions are dense in $\mathcal{H}_1$ if and only if* $\dim D_0 = 1$*.*

Most of this is also a special case of the results of [2] and [5], but we give a simple proof, an elaboration of which can also prove Theorem 2.5.

Proof. We have $u \in D_0$ precisely if $(u, 0) \in T_1$, which holds precisely if $\langle u, v\rangle = \langle u, v\rangle - \langle 0, G_0(wv)\rangle = 0$ for all $v \in L_0 \cap \mathcal{H}_1$, proving the first claim. Since there are elements $v \in \mathcal{H}_1$ with $v(0) \neq 0$, and since $u(0) = 0$ for every $u \in L_0 \cap \mathcal{H}_1$, it follows from (2.1) for $x = 0$ that $\dim D_0 \geq 1$ and we have proved (1).

If b is finite and q integrable, standard existence and uniqueness theorems show that all solutions of $-v'' + qv = 0$ are continuously differentiable with absolutely continuous derivative in $[0, b]$, and thus in $\mathcal{H}_1$, and that they are uniquely determined by the values of v and v' at b. In this case the proof of Lemma 2.1 clearly also works for $a = b$, so we have proved (3), (4), and one direction of (2).

Now let $u \in \mathcal{H}_1$ and $v \in D_0$. Integration by parts gives

$$\int_0^x (u'\overline{v'} + qu\overline{v}) + u(0)\overline{v'(0)} = u(x)\overline{v'(x)}. \tag{2.4}$$

Thus $u(x)\overline{v'(x)}$ has a limit at b. If this is not 0, then $(u(x)\overline{v'(x)})^{-1}$ is bounded close to b. Therefore $u'/u = u'\overline{v'}/(u\overline{v'})$ is integrable near b, so that u has a nonzero limit at b. Since $q|u|^2$ is integrable it follows that $q \in L^1(0, b)$. Similarly, $v''/v' = qv/v' = qv\overline{u}/(v'\overline{u})$ is integrable near b, so v' has a nonzero limit at b. Since $|v'|^2$ is integrable it follows that b is finite.

Now, setting $u = v \not\equiv 0$ in (2.4) the integral is increasing, ≥ 0, and not constant, so if $v(0)\overline{v'(0)} \geq 0$, then $v(x)\overline{v'(x)}$ cannot tend to 0 at b. However, if $\dim D_0 = 2$, we may choose $v \in D_0$ with $v'(0) = 0$, so it follows that $q \in L^1(0, b)$ and b finite, completing the proof of (2).

On the other hand, if $\dim D_0 = 1$, then $u(x)\overline{v'(x)}$ must tend to zero for any $u \in \mathcal{H}_1$. In particular, for $u = v$ one therefore has $v(0)\overline{v'(0)} < 0$ for any nonzero $v \in D_0$ which proves (5).

Finally, if $u \in \mathcal{H}_1$ is finite and $v \in D_0$, integration by parts shows that $\langle u, v\rangle = -u(0)\overline{v'(0)}$, so the orthogonal complement of the finite functions consists of those

$v \in D_0$ for which $v'(0) = 0$. According to (5) this implies $v = 0$ if $\dim D_0 = 1$ and the proof is complete. $\square$

It is now possible to give a detailed description of the kernel g_0.

PROPOSITION 2.7. *The kernel $g_0(x, y)$ is real-valued and symmetric in x, y. As a function of y it satisfies* (1.3) *with $\lambda = 0$ for $y \neq x$, and there are real-valued functions ψ_0 and φ_0 which solve* (1.3) *with $\lambda = 0$, such that if $u \in \mathcal{H}_1$, then*

(1) *$\psi_0 \in \mathcal{H}_1$, $\psi_0'(0) = 1$ and $\psi_0'(x)u(x) \to 0$ as $x \to b$,*

(2) *$\varphi_0(0) = -1$, $\varphi_0'(0) = 0$,*

(3) *$g_0(x, y) = \varphi_0(\min(x, y))\psi_0(\max(x, y))$.*

Proof. The existence of the solution φ_0 is not in question, and if a solution with the properties of ψ_0 exists, it is easy to verify that the kernel $\varphi_0(\min(x, y))\psi_0(\max(x, y))$ has the properties required of $g_0(x, y)$.

The existence of ψ_0 follows from Theorem 2.6. Indeed, if $\dim D_0 = 2$, the element $v \in D_0$ with $v(b) = 1$, $v'(b) = 0$ is real-valued and must have $v(0)v'(0) < 0$ by (2.4), so $v'(0) \neq 0$, and an appropriate multiple will have the properties required of ψ_0.

On the other hand, if $\dim D_0 = 1$, any nonzero $v \in D_0$ satisfies $v(0)\overline{v'(0)} < 0$ so $v'(0) \neq 0$, and an appropriate multiple will satisfy the requirements for ψ_0. Note that this solution is real-valued, since its real and imaginary parts also are in D_0, and are thus proportional, and the initial condition guarantees that the imaginary part vanishes. $\square$

Now let T be a self-adjoint restriction of T_1 and assume that (u, f) and $(v, g) \in T$. Integrating by parts we then obtain

$$\int_0^x (u'\overline{g'} + qu\overline{g}) - \int_0^x (f'\overline{v'} + qf\overline{v}) = (u'\overline{g} - f\overline{v'})\big|_0^x. \tag{2.5}$$

As $x \to b$ this vanishes, since the left-hand side tends to $\langle u, g\rangle - \langle f, v\rangle$. Thus the condition for symmetry is that

$$(u'\overline{g} - f\overline{v'})\big|_0^b = 0.$$

Comparing this with $(u'\overline{v} - u\overline{v'})\big|_0^b = 0$, which is the similar condition in the right-definite case, we see that only exceptionally would self-adjoint boundary conditions in the left-definite case also be self-adjoint boundary conditions in the right-definite case.

Separated boundary conditions are those that make $u'\overline{g} - f\overline{v'}$ vanish at each endpoint separately, and are thus at 0 of the form

$$f(0)\cos\alpha + u'(0)\sin\alpha = 0, \tag{2.6}$$

for some $\alpha \in [0, \pi)$. Again comparing with the right-definite case, where the condition is $u(0)\cos\alpha + u'(0)\sin\alpha = 0$, the conditions coincide only in the case $\alpha = \pi/2$, the Neumann boundary condition. However, for eigenfunctions, where $f = \lambda u$, it is clear that also $\alpha = 0$, the Dirichlet boundary condition, gives the same spectra outside of $\lambda = 0$.

We shall not need a detailed description of self-adjoint boundary conditions at a singular endpoint. However, one may always impose the condition (2.6) at 0. It is easy to see that the corresponding restriction of T_1 has a symmetric adjoint, which is a strict extension of T_0. If the deficiency indices of T_0 equal 1, this is sufficient to obtain a self-adjoint restriction T of T_1, and all self-adjoint realizations are of this form. Otherwise, a condition needs to be imposed also at b. From (2.5) it follows immediately that every $(u, f) \in T_1$ satisfying such a condition at b must satisfy $\operatorname{Im}(u'(x)\overline{f(x)}) \to 0$ as $x \to b$.

Assuming now that we have a self-adjoint relation T, the spectral theorem looks as follows (see [1, Theorem 1.15]). Consider the set $\mathcal{H}_\infty = \{u \in \mathcal{H}_1 \mid (0, u) \in T\}$. Then $\mathcal{H}_\infty$ is a subspace of $\mathcal{H}_1$, and setting $\mathcal{H} = \mathcal{H}_1 \ominus \mathcal{H}_\infty$ the *domain* D_T of T (*i.e.*, the set of first components of T) is a dense subset of $\mathcal{H}$, and $T \cap \mathcal{H} \oplus \mathcal{H}$ is the graph of a self-adjoint operator in $\mathcal{H}$. We will denote this operator by T as well, and may now apply the usual spectral theorem to T. If the resolution of the identity for the operator T is $\{E_t\}_{t\in\mathbb{R}}$, we extend the domain of the projection E_t to all of $\mathcal{H}_1$ by setting $E_t\mathcal{H}_\infty = 0$. Clearly one may view $\mathcal{H}_\infty$ as an eigenspace for the relation T belonging to the eigenvalue ∞, so adjoining the orthogonal projection onto $\mathcal{H}_\infty$ to $\{E_t\}_{t\in\mathbb{R}}$ gives a resolution of the identity in $\mathcal{H}_1$ for the relation T. In the present case one may give a rather complete description of $\mathcal{H}_\infty$.

PROPOSITION 2.8. *The space $\mathcal{H}_\infty$ consists of those elements $g \in \mathcal{H}_1$ for which* $wg = 0$ a.e., *and for which $(0, g)$ satisfies the boundary conditions that define T. In particular, if $wg = 0$ a.e. and $g \in L_0 \cap \mathcal{H}_1$, then $g \in \mathcal{H}_\infty$.*

Proof. Now $g \in \mathcal{H}_\infty$ means that $(0, g) \in T$, which therefore satisfies the boundary conditions defining T. In particular, $0 = \langle g, G_0(wf)\rangle - \langle 0, f\rangle = \langle g, G_0(wf)\rangle = \int g\overline{f}w$ for any $f \in L_0 \cap \mathcal{H}_1$. It follows, as in the proof of Proposition 2.3, that $wg = 0$ a.e.

Conversely, if $(0, g)$ satisfies the boundary conditions and $gw = 0$ a.e., then if $(u, f) \in T$, an integration by parts gives

$$\langle u, g\rangle - \langle f, 0\rangle = \lim_{x\to b}(u'\overline{g} - f \cdot 0)\big|_0^x = 0,$$

i.e., $(0, g) \in T$, so the proof is complete. ∎

We remark that if an endpoint is regular, then the boundary condition implied by $u \in \mathcal{H}_\infty$ is in most cases the vanishing of u in that endpoint. For separated boundary conditions an exception occurs when the boundary condition is of Neumann type (*i.e.*, when $\alpha = \pi/2$ in (2.6)). If we have Neumann conditions at both ends, or at one end when deficiency indices equal 1, there are *no* boundary conditions for elements of $\mathcal{H}_\infty$.

We will base our derivation of the expansion theorem for the operator T on a detailed description of the resolvent $R_\lambda = (T - \lambda)^{-1}$. Thus R_λ is defined on $\mathcal{H}$, but we extend its domain to $\mathcal{H}_1$ by setting $R_\lambda\mathcal{H}_\infty = 0$. The range of R_λ is of course D_T, which is a dense set in $\mathcal{H}$. Using the kernel g_0 for the evaluation operator on $\mathcal{H}_1$ introduced in the proof of Proposition 2.3, we have $R_\lambda u(x) = \langle R_\lambda u, \overline{g_0(x,\cdot)}\rangle = \langle u, \overline{R_{\overline{\lambda}} g_0(x,\cdot)}\rangle$, since the adjoint of R_λ is $R_{\overline{\lambda}}$. Thus we may view $G(x,\cdot,\lambda) = \overline{R_{\overline{\lambda}} g_0(x,\cdot)}$ as Green's function for our operator; note, however, that G is not the kernel of a standard integral operator. It will turn out to be convenient to introduce the kernel $g(x, y, \lambda) = G(x, y, \lambda) + g_0(x, y)/\lambda$, so that we obtain

$$R_\lambda u(x) = \langle u, \overline{g(x,\cdot,\lambda)}\rangle - u(x)/\lambda. \tag{2.7}$$

Note that $G(x,\cdot,\lambda) \in \mathcal{H}$ but this is not true of $g(x,\cdot,\lambda)$ unless $\mathcal{H}_\infty = \{0\}$. We shall need a precise description of $g(x, y, \lambda)$. To do this we must introduce solutions of (1.3) satisfying initial conditions at 0, so let $\varphi(x, \lambda)$, $\theta(x, \lambda)$ be solutions of (1.3) for $\lambda \neq 0$ satisfying

$$\begin{cases} \lambda\varphi(0,\lambda) = -\sin\alpha \\ \varphi'(0,\lambda) = \cos\alpha \end{cases}, \quad \begin{cases} \lambda\theta(0,\lambda) = \cos\alpha \\ \theta'(0,\lambda) = \sin\alpha \end{cases}. \tag{2.8}$$

This means that φ satisfies the boundary condition (2.6) and θ another similar boundary condition at 0. We have the following theorem.

THEOREM 2.9. *Suppose T is a self-adjoint realization of* (1.3) *given by* (2.6) *and, if needed, an appropriate condition at b. Then there exists a function $m(\lambda)$ defined for* $\operatorname{Im}\lambda \neq 0$, *the* Titchmarsh–Weyl m-function for T, *depending only on λ and such that $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$, called the* Weyl solution *for T, is in $\mathcal{H}_1$ and satisfies the boundary condition at b, if any. Furthermore*

$$g(x,y,\lambda) = \varphi(\min(x,y),\lambda)\psi(\max(x,y),\lambda).$$

Proof. For nonreal λ neither φ nor θ can be in $\mathcal{H}_1$ and satisfy the boundary condition at b, since that would make λ a nonreal eigenvalue for a self-adjoint problem. Thus there is a solution $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ in $\mathcal{H}_1$ which also satisfies the boundary condition at b, since if $\dim D(\lambda) = 2$, one linear, homogeneous condition still leaves a one-dimensional space, whereas if $\dim D(\lambda) = 1$, no boundary condition is imposed at b.

Define, for fixed x and $\lambda \notin \mathbb{R}$, the function

$$F(y) = \varphi(\min(x,y),\lambda)\psi(\max(x,y),\lambda) - \lambda^{-1}g_0(x,y).$$

Since $\psi(\cdot,\lambda)$ and ψ_0 are in $\mathcal{H}_1$ so is F. We claim that $F \in D_T$. In fact, one easily checks that F' is locally absolutely continuous and that F satisfies $-F'' + qF = \lambda wF + wg_0(x,\cdot)$. It is also easy to check that F satisfies the boundary condition (2.6).

Finally, for $y > x$ the function F is a linear combination of $\psi(\cdot,\lambda)$ and ψ_0. The former satisfies the boundary condition at b by construction, and ψ_0 satisfies the boundary condition at b by Theorem 2.6(5), since if $(u,f) \in T$, then $\psi_0'\overline{f} - 0\overline{u'} = \psi_0'\overline{f} \to 0$ at b. All this means that $F = R_\lambda g_0(x,\cdot) = \overline{R_{\overline{\lambda}} g_0(x,\cdot)} = G(x,\cdot,\lambda)$ so that $g(x,y,\lambda)$ is as claimed. ∎

THEOREM 2.10. *The function m is analytic outside* $\mathbb{R}$, *it maps the upper half plane into itself, and it satisfies $\overline{m(\lambda)} = m(\overline{\lambda})$.*

Proof. Since R_λ is analytic outside $\mathbb{R}$ in the strong operator topology $R_\lambda u(x)$ is, by (2.1), pointwise analytic. It follows that $g(x,\cdot,\lambda)$ is weakly analytic for each x, and thus, again by (2.1), $g(x,y,\lambda)$ is analytic outside $\mathbb{R}$ for each x and y. Since $\varphi(x,\lambda)$ and $\theta(x,\lambda)$ also are analytic and since an integration by parts shows that they are nonzero for $x > 0$ and $\lambda \notin \mathbb{R}$, it follows that $m(\lambda)$ is analytic in $\mathbb{C} \setminus \mathbb{R}$.

If (v,g) defines a boundary condition at b, then so does either its real part or its imaginary part, which is easily seen. Therefore, since $\psi(x,\lambda)$ satisfies (1.3) and the boundary condition at b, so does $\overline{\psi(x,\overline{\lambda})}$, and is thus a multiple of $\psi(x,\lambda)$. Now $\overline{\varphi(x,\overline{\lambda})} = \varphi(x,\lambda)$, $\overline{\theta(x,\overline{\lambda})} = \theta(x,\lambda)$ and $\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\varphi(x,\lambda)$ so it follows that $m(\overline{\lambda}) = \overline{m(\lambda)}$.

Integrating by parts we have

$$\operatorname{Im}\lambda \int_0^x (|\psi'(\cdot,\lambda)|^2 + q|\psi(\cdot,\lambda|^2) = \operatorname{Im}(\overline{\psi'(\cdot,\lambda)}\lambda\psi(\cdot,\lambda))\big|_0^x.$$

Since ψ satisfies a boundary condition at b, the integrated term vanishes as $x \to b$. At 0 the integrated term evaluates to $-\operatorname{Im} m(\lambda)$, so we obtain

$$\|\psi(\cdot,\lambda)\|^2 = \operatorname{Im} m(\lambda)/\operatorname{Im}\lambda. \tag{2.9}$$

Thus m maps the upper and lower half-planes into themselves. ∎

A function with the properties of m is a so-called Nevanlinna or Herglotz function, and has a unique representation

$$m(\lambda) = A + B\lambda + \int_{\mathbb{R}} \left(\frac{1}{t-\lambda} - \frac{t}{t^2+1} \right) d\rho, \tag{2.10}$$

where $A \in \mathbb{R}$, $B \geq 0$, and $d\rho$ is a positive measure with $\int_{\mathbb{R}} \frac{d\rho(t)}{1+t^2} < \infty$. We will call the measure $d\rho$ the *spectral measure* for T, for reasons that will become clear presently.

We finally note the following proposition.

PROPOSITION 2.11. *Unless $\alpha = \pi/2$ and $0 \notin \operatorname{supp} w$ the functions ψ_0 and $\psi(\cdot, \lambda)$ are in $\mathcal{H}$.*

Proof. Suppose $g \in \mathcal{H}_\infty$. An integration by parts then gives

$$\langle g, \psi \rangle = -g(0)\overline{\psi'(0)},$$

where $\psi = \psi_0$ or $\psi(\cdot, \lambda)$. The boundary condition at 0 requires $g(0) = 0$ unless $\alpha = \pi/2$, and even then $g(0) = 0$ unless $w = 0$ in a neighbourhood of 0. ∎

3. The Fourier transform. We shall call functions that vanish near b *finite* and from now on make the following simplifying assumption.

Assumption 3.1. Assume that finite functions are dense in $\mathcal{H}_1$.

According to Theorem 2.6 this means exactly that either $q \notin L^1(0, b)$ or else $b = \infty$. Note that, according to Theorem 2.5, the assumption implies that the deficiency indices of T_1 equal 1.

The spectral measure introduced in the previous section gives rise to a Hilbert space L^2_ρ with scalar product $\langle \hat{u}, \hat{v} \rangle_\rho = \int_{-\infty}^{\infty} \hat{u}\overline{\hat{v}}\, d\rho$. We shall define a generalized Fourier transform $\mathcal{F} : \mathcal{H}_1 \to L^2_\rho$ with the following properties.

THEOREM 3.2.

(1) *The map $u \mapsto \int_0^b (u'\varphi'(\cdot, t) + qu\varphi(\cdot, t))$, defined for finite $u \in \mathcal{H}_1$, extends by continuity to a map $\mathcal{F} : \mathcal{H}_1 \to L^2_\rho$ called the* generalized Fourier transform. *The image of $u \in \mathcal{H}_1$ is denoted by $\mathcal{F}(u)$ or $\hat{u}$. We write this as $\hat{u}(t) = \langle u, \varphi(\cdot, t) \rangle$ although the integral in general does not converge pointwise.*

(2) *The mapping $\mathcal{F} : \mathcal{H}_1 \to L^2_\rho$ has kernel $\mathcal{H}_\infty$ and is unitary between $\mathcal{H}$ and L^2_ρ so that* Parseval's formula *$\langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle_\rho$ holds if at least one of u and v is in $\mathcal{H}$.*

(3) *If $u \in D_T$, then $\mathcal{F}(Tu)(t) = t\hat{u}(t)$. Conversely, if $\hat{u}$ and $t\hat{u}(t)$ are in L^2_ρ, then $\mathcal{F}^{-1}(\hat{u}) \in D_T$.*

(4) *Suppose $\alpha \neq 0$ in (2.6). Then $\varphi(x, \cdot) \in L^2_\rho$ for each x and $\int_{-\infty}^{\infty} \hat{u}\varphi(x, \cdot)\, d\rho = \langle \hat{u}, \varphi(x, \cdot) \rangle_\rho$ converges in $\mathcal{H}$, and hence locally uniformly in x, for $\hat{u} \in L^2_\rho$. This is the adjoint of $\mathcal{F} : \mathcal{H}_1 \to L^2_\rho$ and thus the inverse of $\mathcal{F}$ restricted to $\mathcal{H}$. If M is a Borel set in $\mathbb{R}$, then*

$$E_M u(x) = \int_M \hat{u}\varphi(x, \cdot)\, d\rho. \tag{3.1}$$

If $\alpha = 0$, the same is true, except that we must replace $\varphi(\cdot, t)$ for $t = 0$ by the function ψ_0 of Proposition 2.7. Note that ψ_0 is the eigenfunction for the eigenvalue 0 in this case.

We first consider the Fourier transform for finite functions $u \in \mathcal{H}_1$, for every $\lambda \in \mathbb{C}$ setting

$$\hat{u}(\lambda) = \langle u, \varphi(\cdot, \overline{\lambda}) \rangle.$$

It is clear that $\hat{u}$ is an entire function, since integration by parts shows that

$$\hat{u}(\lambda) = \langle u, \varphi(\cdot, \overline{\lambda})\rangle = \int_0^b u\lambda\varphi(\cdot,\lambda)w - u(0)\cos\alpha,$$

and by (2.8) $\lambda\varphi(x,\lambda)$ is an entire function of λ, locally uniformly in x.

LEMMA 3.3. *For finite u and $v \in \mathcal{H}_1$ we have $\hat{u}$ and $\hat{v} \in L^2_\rho$. If E_Δ is the spectral projection for T associated with an interval Δ, then $\langle E_\Delta u, v\rangle = \int_\Delta \hat{u}\overline{\hat{v}}\, d\rho$.*

Proof. We have $\langle R_\lambda u, v\rangle = \hat{u}(\lambda)\overline{\hat{v}(\overline{\lambda})}m(\lambda) + g(\lambda)$, where g is entire, as is easily verified by direct calculation. Integrating around a rectangle γ with corners at $c \pm i$ and $d \pm i$ we therefore have $\int_\gamma \langle R_\lambda u, v\rangle\, d\lambda = \int_\gamma \hat{u}(\lambda)\overline{\hat{v}(\overline{\lambda})}m(\lambda)\, d\lambda$ whenever one of the integrals exists. By the spectral theorem the first integral equals $\int_\gamma\int_{\mathbb{R}} \frac{d\langle E_t u, v\rangle}{t-\lambda}\, d\lambda$, so if the integral is absolutely convergent, changing the order of integration gives $-2\pi i\langle E_{(c,d)}u, v\rangle$ if c and d are points of continuity for $\langle E_t u, v\rangle$.

Similarly, using the Nevanlinna representation (2.10), the other integral equals $-2\pi i\int_c^d \hat{u}(t)\overline{\hat{v}(t)}\, d\rho(t)$ if it is absolutely convergent and c, d are points of continuity for ρ.

The absolute convergence of the double integrals is ensured if $\langle E_t u, v\rangle$ and ρ are differentiable at c and d as is easily seen. For more details of the identical calculation carried out for the right-definite case, see [6, Lemmas 14.3, 14.4].

As functions of bounded variation $\langle E_t u, v\rangle$ and ρ are both differentiable a.e., so the second claim of the lemma is true if the endpoints of Δ belong to this dense set of points, and so in general by continuity. In particular, letting $c \to -\infty$, $d \to \infty$ through such points it follows that $\langle E_{\mathbb{R}} u, u\rangle = \langle \hat{u}, \hat{u}\rangle_\rho$, so that $\hat{u}, \hat{v} \in L^2_\rho$. ∎

Since finite functions are dense in $\mathcal{H}_1$, and since $E_{\mathbb{R}}$ has kernel $\mathcal{H}_\infty$, we now obtain Theorem 3.2(1) by continuity and also (2) except for the surjectivity of $\mathcal{F}$. To prove this we need the following lemmas.

LEMMA 3.4. *The transform of $R_\lambda u$ is $\hat{u}(t)/(t-\lambda)$.*

Proof. According to the spectral theorem we have $\langle R_\lambda u, v\rangle = \int_{\mathbb{R}} \frac{d\langle E_t u, v\rangle}{t-\lambda}$ and by Lemma 3.3 we have $\langle E_t u, v\rangle = \int_{-\infty}^t \hat{u}\overline{\hat{v}}\, d\rho$ so that

$$\langle R_\lambda u, v\rangle = \int_{\mathbb{R}} \frac{\hat{u}(t)}{t-\lambda}\,\overline{\hat{v}(t)}\, d\rho(t).$$

We also have $R_\lambda - R_{\overline{\lambda}} = (\lambda - \overline{\lambda})R_{\overline{\lambda}}R_\lambda$ and $\langle R_\lambda u, R_\lambda u\rangle = \langle R_{\overline{\lambda}}R_\lambda u, u\rangle$ so

$$\langle R_\lambda u, R_\lambda u\rangle = \frac{1}{\lambda - \overline{\lambda}}(\langle R_\lambda u, u\rangle - \langle R_{\overline{\lambda}} u, u\rangle) = \left\|\frac{\hat{u}(t)}{t-\lambda}\right\|_\rho^2.$$

Expanding $\|\frac{\hat{u}(t)}{t-\lambda} - \mathcal{F}(R_\lambda u)\|_\rho^2$ and using Parseval's formula and the above yields 0, thus proving the lemma. ∎

LEMMA 3.5. *The operator T has eigenvalue 0 if and only if $\alpha = 0$, in which case the eigenfunction is ψ_0, and for any $u \in \mathcal{H}_1$ we then have $\hat{u}(0) = -u(0)$.*

Furthermore, the measure $d\rho$ has mass at 0 ($\{0\}$ is not a nullset with respect to $d\rho$) precisely if $\alpha = 0$. In this case $\hat{\psi}_0 = \chi_{\{0\}}/\rho\{0\}$, where $\chi_{\{0\}}$ is the characteristic function of the singleton $\{0\}$ and $\rho\{0\}$ the spectral measure of this set.

Proof. According to Theorem 2.6 the only nontrivial solutions of (1.3) for $\lambda = 0$ in $\mathcal{H}_1$ are multiples of a solution u for which $u'(0)\overline{u(0)} < 0$, so that $u'(0) \neq 0$. These solutions satisfy the boundary condition (2.6) precisely if $\alpha = 0$, which proves the first claim. If u is any finite function, integrating by parts gives $\hat{u}(0) = \langle u, \varphi(\cdot, 0)\rangle = -u(0)$. This holds in general by continuity, $u(0)$ being a bounded linear form on $\mathcal{H}_1$ by (2.1), and $\hat{u}(0)$ on L^2_ρ since $d\rho$ has mass at 0, as we shall see presently.

Now $u \in D_T$ and $Tu = 0$ precisely if $u + \lambda R_\lambda u = 0$, and the Fourier transform of $u + \lambda R_\lambda u$ is $(1 + \frac{\lambda}{t-\lambda})\hat{u}(t) = \frac{t\hat{u}(t)}{t-\lambda}$. If this is 0, then $\hat{u} = 0$ a.e. with respect to $d\rho$ except possibly at $t = 0$. Thus, if $\alpha = 0$, then $\{0\}$ cannot be a nullset with respect to $d\rho$. It also follows that $\hat{\psi}_0$ is a multiple of the characteristic function of the set $\{0\}$. On the other hand, since $\dim D_\lambda = 1$, Weyl solutions for different α are proportional so it immediately follows that

$$m_0(\lambda) = \frac{\psi'(0,\lambda)}{\lambda\psi(0,\lambda)} = \frac{\sin\alpha + m_\alpha(\lambda)\cos\alpha}{\cos\alpha - m_\alpha(\lambda)\sin\alpha}, \tag{3.2}$$

where m_α denotes the m-function associated with the boundary condition parameter α. Now $m_0(i\nu) \to \infty$ as $\nu \downarrow 0$, as a consequence of the mass at 0, so that $m_\alpha(i\nu) \to \cot\alpha$ for $\alpha \neq 0$. For $\alpha \neq 0$ the spectral measure therefore has no mass at 0.

It only remains to prove the formula for $\hat{\psi}_0$. By Parseval's formula (note that $\psi_0 \in \mathcal{H}$ by Proposition 2.11) we have $\hat{\psi}_0(0) = -\psi_0(0) = \|\psi_0\|^2 = \|\hat{\psi}_0\|^2_\rho = |\hat{\psi}_0(0)|^2\rho\{0\}$. Hence $-\psi_0(0) = \hat{\psi}_0(0) = 1/\rho\{0\}$. ∎

It is now easy to prove that $\mathcal{F}$ is surjective.

LEMMA 3.6. *The Fourier transform $\mathcal{H} \to L^2_\rho$ is surjective.*

Proof. Suppose that $\hat{u} \in L^2_\rho$ is orthogonal to all Fourier transforms $\hat{v}$. Since $\hat{v}(t)/(t-\lambda)$ is also a transform, for any nonreal λ, we have $\int \frac{1}{t-\lambda}\hat{u}(t)\overline{\hat{v}(t)}\,d\rho(t) = 0$ for all nonreal λ. Thus the Stieltjes transform of the measure $\hat{u}\overline{\hat{v}}\,d\rho$ is 0, so by the uniqueness of the Stieltjes transform it follows that this measure is the zero measure.

Now, if $\hat{v}$ is the transform of a finite function in $\mathcal{H}_1$, then it is an entire function, so to prove that t is outside the support of $\hat{u}\,d\rho$ it is enough to show that there is such a $\hat{v}$ for which $\hat{v}(t) \neq 0$. If $t \neq 0$ and $\hat{v}(t) = 0$ for all compactly supported $v \in \mathcal{H}_1$, then as in the proof of Proposition 2.4 it follows that $\varphi(\cdot, t)$ satisfies (1.3) both for $\lambda = 0$ and $\lambda = t$, so that $\varphi(\cdot, t)w = 0$ a.e., which is not possible since it implies that $\varphi(\cdot, t) = 0$ in a set of positive Lebesgue measure. It therefore follows that $\hat{u}\,d\rho$ vanishes outside 0. But according to Lemma 3.5 this proves that the measure is zero, unless $\alpha = 0$. However, also in this case $\hat{u} = 0$ since otherwise $\hat{u}$ would be the transform of an eigenfunction. ∎

We next turn to Theorem 3.2(3).

LEMMA 3.7. *If $u \in D_T$, then $\mathcal{F}(Tu)(t) = t\hat{u}(t)$. Conversely, if $\hat{u}$ and $t\hat{u}(t)$ are in L^2_ρ, then $\mathcal{F}^{-1}(\hat{u}) \in D_T$.*

Proof. We have $u \in D_T$ if and only if for some $v \in \mathcal{H}_1$ we have $u = R_\lambda(v - \lambda u)$, *i.e.*, if and only if $\hat{u}(t) = (\hat{v}(t) - \lambda\hat{u}(t))/(t-\lambda)$ or $t\hat{u}(t) = \hat{v}(t)$ for some $\hat{v} \in L^2_\rho$. ∎

We obtain the following corollary which will be useful later on.

COROLLARY 3.8. *If $u \in D_T$, then $\hat{u}$ is integrable with respect to $d\rho$.*

Proof. The functions $t\hat{u}(t)$, $\hat{u}$, and $1/(t-i)$ are all in L^2_ρ, so that $\hat{u}(t) = (t\hat{u}(t) - i\hat{u}(t))/(t-i)$ is integrable with respect to $d\rho$. ∎

To finish the proof of Theorem 3.2 it only remains to consider the inverse transform.

LEMMA 3.9. *If $\alpha \neq 0$, the integral $\langle\hat{u}, \varphi(x, \cdot)\rangle_\rho$ converges in $\mathcal{H}$ and locally uniformly for every $\hat{u} \in L^2_\rho$. If $\hat{u} = \mathcal{F}(u)$ for some $u \in \mathcal{H}_1$, then the integral is the orthogonal projection of u onto $\mathcal{H}$.*

If $\alpha = 0$, the same statement is true if one replaces $\varphi(\cdot, 0)$ by ψ_0 in the integral.

Remark 3.10. A simple integration by parts shows that every finite function is orthogonal to φ_0. Now suppose $\alpha = 0$ and let $\theta_0 = \varphi(\cdot, 0)$ so that θ_0 solves (1.3) for $\lambda = 0$ with initial data $\theta_0(0) = 0$, $\theta_0'(0) = 1$. Then, when calculating the Fourier transform at 0 we may replace $\varphi(\cdot, 0)$ by any function $\theta_0 + A\varphi_0$ for A constant, with no change to the Fourier transform.

In particular we may choose $A = -\psi_0(0) = 1/\rho\{0\}$, according to Lemma 3.5, so that $\theta_0 + A\varphi_0 = \psi_0$. This might seem a more natural choice of kernel for the Fourier transform, in view of the fact that it must be used for the inverse transform, and that ψ_0 is an eigenfunction to the eigenvalue 0, but would thus not actually change the Fourier transform.

Proof of Lemma 3.9. We have $u(x) = \langle u, g_0(x, \cdot)\rangle = \langle \hat{u}, e(x, \cdot)\rangle_\rho$ for $u \in \mathcal{H}$, where $e(x, t) = \mathcal{F}(g_0(x, \cdot))(t)$. If $u \in \mathcal{H}_1$, we instead get the projection of u onto $\mathcal{H}$, so that the integral operator $\hat{u} \mapsto \langle \hat{u}, e(x, \cdot)\rangle_\rho$ is the adjoint of $\mathcal{F}$. We must prove that $e(x, t) = \varphi(x, t)$, so suppose $\hat{u}$ has compact support and consider $\tilde{u}(x) = \langle \hat{u}, \varphi(x, \cdot)\rangle_\rho$ which satisfies the equation $-\tilde{u}'' + q\tilde{u} = w(x)\langle \hat{u}, t\varphi(x, \cdot)\rangle_\rho$, differentiating under the integral sign. Since $\hat{u}$ has compact support $u \in D_T$, so that $-u'' + qu = w(x)\langle t\hat{u}(t), e(x, t)\rangle_\rho$. Thus $u_1 = u - \tilde{u}$ satisfies $-u_1'' + qu_1 = w(x)\langle t\hat{u}(t), e(x, t) - \varphi(x, t)\rangle_\rho$.

Now, if v is finite, then

$$\langle \tilde{u}, v\rangle = \iint \hat{u}(t)(\varphi'(\cdot, t)\overline{v'} + q\varphi(\cdot, t)\overline{v})\, d\rho(t) = \langle \hat{u}, \hat{v}\rangle_\rho = \langle u, v\rangle,$$

since the double integral is absolutely convergent. Hence u_1 is orthogonal to all finite v so it satisfies $-u_1'' + qu_1 = 0$. It follows that $w(x)\langle t\hat{u}(t), e(x, t) - \varphi(x, t)\rangle_\rho = 0$ a.e., so that $\langle t\hat{u}(t), e(x, t) - \varphi(x, t)\rangle_\rho = 0$ on a set of positive measure. But this function also satisfies (1.3) for $\lambda = 0$, as is seen by replacing $\hat{u}$ by $t\hat{u}(t)$ in the previous calculations. It follows that $t(e(x, t) - \varphi(x, t)) = 0$ for a.a. t with respect to $d\rho$, so that $e(x, t) = \varphi(x, t)$ except possibly if $t = 0$ and $\alpha = 0$.

However, 0 is an eigenvalue for $\alpha = 0$ and the eigenfunction ψ_0 has transform $\chi_{\{0\}}/\rho\{0\}$ according to Lemma 3.5, so we must choose $e(x, 0) = \psi_0(x)$. ∎

The proof of Theorem 3.2 is now complete if we note that from $\langle E_t u, v\rangle = \int_{-\infty}^t \hat{u}\overline{\hat{v}}$ follows that the transform of $E_t u$ is $\hat{u}$ multiplied by the characteristic function of $(-\infty, t]$. The formula $E_M u(x) = \int_M \hat{u}\varphi(x, \cdot)\, d\rho$ therefore follows from the inversion formula.

In Lemma 3.5 we calculated the Fourier transform of ψ_0 in the case $\alpha = 0$. We shall need to find a few more Fourier transforms.

LEMMA 3.11. *If $\lambda \notin \mathbb{R}$, the Fourier transform of $\psi(\cdot, \lambda)$ is $\hat{\psi}(t, \lambda) = 1/(t - \lambda)$. Furthermore, the Fourier transform of ψ_0 equals $\hat{\psi}_0(t) = \sin\alpha/t$ for $\alpha \neq 0$ and $1/\rho\{0\}$ times the characteristic function of the set $\{0\}$ for $\alpha = 0$.*

Proof. We have already calculated $\hat{\psi}_0$ for $\alpha = 0$ in Lemma 3.5. If $\alpha \neq 0$, we note that $\psi_0(x) = -g_0(0, x)$ so its Fourier transform is $-e(0, t) = -\varphi(0, t) = \sin\alpha/t$.

According to (2.7), Theorem 2.9, and Lemma 3.9, for $u \in \mathcal{H}$ we have

$$\begin{aligned} -\sin\alpha\langle \hat{u}, \hat{\psi}(\cdot, \lambda)\rangle_\rho &= \overline{\lambda}\varphi(0, \overline{\lambda})\langle u, \psi(\cdot, \lambda)\rangle \\ &= \overline{\lambda}R_{\overline{\lambda}}u(0) + u(0) = \left\langle \left(\frac{\overline{\lambda}}{t - \overline{\lambda}} + 1\right)\hat{u}(t), e(0, t)\right\rangle_\rho \\ &= \left\langle \hat{u}(t), \frac{t\, e(0, t)}{t - \lambda}\right\rangle_\rho = -\sin\alpha\left\langle \hat{u}(t), \frac{1}{t - \lambda}\right\rangle_\rho \end{aligned}$$

so that we have $\hat\psi(t,\lambda) = 1/(t-\lambda)$ if $\alpha \neq 0$. If $\alpha = 0$, we assume $\hat u$ has compact support so that we may differentiate $u(x) = \langle \hat u, e(x,\cdot)\rangle_\rho$ under the integral sign to obtain

$$\begin{aligned}\langle \hat u, \hat\psi(\cdot,\lambda)\rangle_\rho &= \varphi'(0,\overline{\lambda})\langle u, \psi(\cdot,\lambda)\rangle = (R_{\overline{\lambda}}u)'(0)\\ &= \left\langle \frac{\hat u(t)}{t-\overline{\lambda}}, e_x'(0,t)\right\rangle_\rho = \left\langle \hat u(t), \frac{e_x'(0,t)}{t-\lambda}\right\rangle_\rho = \left\langle \hat u(t), \frac{1}{t-\lambda}\right\rangle_\rho.\end{aligned}$$

Thus, also in this case we obtain $\hat\psi(t,\lambda) = 1/(t-\lambda)$. ∎

COROLLARY 3.12. *Suppose $u \in \mathcal{H}$. Then $\langle u, \psi(\cdot, t\lambda)\rangle \to 0$ as $t \to \infty$, locally uniformly for $\lambda \notin \mathbb{R}$. By* (2.1) *this means that $\psi(x, t\lambda) \to 0$ as $t \to \infty$, locally uniformly in x and $\lambda \notin \mathbb{R}$.*

In fact, unless $0 \notin \operatorname{supp} w$ and $\alpha = \pi/2$ we have $\psi(\cdot, t\lambda) \to 0$ in $\mathcal{H}$, locally uniformly in $\lambda \notin \mathbb{R}$ as $t \to \infty$.

Proof. We have $\langle u, \psi(\cdot,\lambda)\rangle = \langle \hat u, \hat\psi(\cdot,\lambda\rangle_\rho$. With the extra assumptions Proposition 2.11 shows that $\psi(\cdot,\lambda) \in \mathcal{H}$ so that $\|\psi(\cdot,\lambda)\| = \|\hat\psi(\cdot,\lambda)\|_\rho$.

It follows immediately by dominated convergence from Lemma 3.11 that the claims are true. ∎

Remark 3.13. All of the theory of sections 2 and 3 extends with no essential change to the case when w is just a measure, or even an element of $H^{-1}_{\mathrm{loc}}(0, b)$.

4. Uniqueness of the inverse problem. We shall here deal with the following question: *To what extent is the operator T,* i.e.*, the interval $[0, b)$, the coefficients q and w, and the boundary condition parameter α, determined by the spectral measure $d\rho$?* To answer this question we introduce the concept of a *Liouville transform* as a map $v \mapsto u$ given by $u(x) = f(x)v(g(x))$, where f and g are fixed functions. We suppose that g is strictly increasing and continuous, and that f is never 0. It is then easy to see that the inverse of a Liouville transform is also a Liouville transform, as is the composition of two Liouville transforms.

Now consider another relation $\breve T$ of the same type as T, with Hilbert space $\breve{\mathcal{H}}_1$, interval $[0, \breve b)$, boundary condition parameter $\breve\alpha$, and coefficients $\breve q$ and $\breve w$. We will assume, as we do for $\mathcal{H}_1$, that finite functions are dense in $\breve{\mathcal{H}}_1$.

THEOREM 4.1. *Suppose that $\alpha = \breve\alpha$, or $0 < \alpha = \pi/2 - \breve\alpha < \pi/2$, or $\pi/2 < \alpha = 3\pi/2 - \breve\alpha < \pi$ and that there is a continuously differentiable bijection g from $[0, b)$ to $[0, \breve b)$ with the following properties: g, g', and g'' are locally absolutely continuous, $g' > 0$, $g(0) = g''(0) = 0$, $g'(0) = (\sin\breve\alpha/\sin\alpha)^2$ if $\alpha \neq 0 \neq \breve\alpha$, $g'(0) = 1$ if $\alpha = \breve\alpha = 0$, and the coefficients of T and $\breve T$ satisfy $\breve q(g(x)) = (-f(x)f''(x) + q(x)f(x)^2)/g'(x)$ and $\breve w(g(x)) = w(x)/g'(x)^2$, where $f(x) = g'(x)^{-1/2}$.*

Then the spectral measures associated with T and $\breve T$ are identical.

Proof. The functions g and f give rise to Liouville transform $\mathcal{L}$ from functions defined on $[0, \breve b)$ to functions defined on $[0, b)$, in particular to a transform from $\breve{\mathcal{H}}_1$ to $\mathcal{H}_1$. We will first show that this latter transform is unitary. To that end assume that $\breve u$ and $\breve v$ are in $\breve{\mathcal{H}}_1$ and that at least one of them is a finite function. Obviously $\mathcal{L}\breve u$ and $\mathcal{L}\breve v$ are locally absolutely continuous. Furthermore we obtain after a partial integration

$$\begin{aligned}\langle \mathcal{L}\breve u, \mathcal{L}\breve v\rangle_{\mathcal{H}_1} &= \int_0^b (g'(\breve u'\overline{\breve v'})\circ g + (-ff'' + qf^2)(\breve u\overline{\breve v})\circ g)\\ &= \int_0^{\breve b} (\breve u'\overline{\breve v'} + \breve q\breve u\overline{\breve v}) = \langle \breve u, \breve v\rangle_{\breve{\mathcal{H}}_1}.\end{aligned}$$

This proves first that $\mathcal{L}\breve{u} \in \mathcal{H}_1$ whenever $\breve{u}$ is a finite function in $\breve{\mathcal{H}}_1$ and second that $\mathcal{L}$ is an isometry from the finite functions in $\breve{\mathcal{H}}_1$ onto the finite functions in $\mathcal{H}_1$. As an isometry $\mathcal{L}$ can be extended to a unitary operator from $\breve{\mathcal{H}}_1$ to $\mathcal{H}_1$.

Next, a straightforward computation, using that $2f'g' + fg'' = 0$, shows that $-u'' + qu = wr$ if $u = \mathcal{L}\breve{u}$, $r = \mathcal{L}\breve{r}$, and $-\breve{u}'' + \breve{q}\breve{u} = \breve{w}\breve{r}$. In particular, $(\breve{u}, \breve{r}) \in \breve{T}$ implies that $(\mathcal{L}\breve{u}, \mathcal{L}\breve{r}) \in T$ and $\mathcal{L}\breve{\psi}(\cdot, \lambda)$ must be a multiple of $\psi(\cdot, \lambda)$.

Also, since $\breve{\varphi}(\cdot, \lambda)$ satisfies the differential equation $-\breve{u}'' + \breve{q}\breve{u} = \lambda\breve{w}\breve{u}$ the function $\mathcal{L}\breve{\varphi}(\cdot, \lambda)$ satisfies $-u'' + qu = \lambda wu$. Our assumptions on α, $\breve{\alpha}$, $g'(0)$, and $g''(0)$ imply that $f(0) = \sin\alpha/\sin\breve{\alpha} = \cos\breve{\alpha}/\cos\alpha$ and that $f'(0) = 0$. Therefore we find $\lambda(\mathcal{L}\breve{\varphi}(\cdot, \lambda))(0) = \lambda f(0)\breve{\varphi}(0, \lambda) = -\sin\alpha$ and $(\mathcal{L}\breve{\varphi}(\cdot, \lambda))'(0) = \breve{\varphi}'(0, \lambda)/f(0) = \cos\alpha$ which shows that $\varphi(\cdot, \lambda) = \mathcal{L}\breve{\varphi}(\cdot, \lambda)$. The situation is a little more complicated for the relationship between θ and $\breve{\theta}$ where one finds that

$$\mathcal{L}\breve{\theta}(\cdot, \lambda) = \theta(\cdot, \lambda) + (\tan\breve{\alpha} - \tan\alpha)\varphi(\cdot, \lambda).$$

By the linearity of $\mathcal{L}$ we have

$$\mathcal{L}\breve{\psi}(\cdot, \lambda) = \theta(\cdot, \lambda) + (\tan\breve{\alpha} - \tan\alpha + \breve{m})\varphi(\cdot, \lambda) = \psi(\cdot, \lambda).$$

This proves that $\breve{m} + \tan\breve{\alpha} = m + \tan\alpha$ and hence that $\breve{\rho} = \rho$. ∎

In the rest of this section we will make the following additional assumption about (1.3).

Assumption 4.2. The coefficients w and $\breve{w}$ satisfy $\operatorname{supp} w = [0, b)$, $\operatorname{supp} \breve{w} = [0, \breve{b})$.

Note that this does not mean that $w \neq 0$ a.e.; w could vanish on a nowhere dense set of strictly positive measure. However, it does mean that $\mathcal{H}_\infty = \{0\}$, $\mathcal{H} = \mathcal{H}_1$.

Remark 4.3. One may also allow w to be an arbitrary measure. However, then in the definition of the function h below, and in the statement of Lemma 5.1, w should be replaced by the density of the absolutely continuous part of the measure w, and Assumption 4.2 will have to be made on this density. If this is done, the results in the rest of the paper are still true, *mutatis mutandis*, with essentially the same proofs.

Now define the functions $h(x) = \int_0^x \sqrt{|w|}$ on $[0, b)$ and $\breve{h}(x) = \int_0^x \sqrt{|\breve{w}|}$ on $[0, \breve{b})$, respectively. By Assumption 4.2 these are strictly increasing, locally absolutely continuous functions.

Our main theorem is the following.

THEOREM 4.4. *Suppose that T and $\breve{T}$ have the same spectral measure $d\rho$. Then there is a unitary Liouville transform $\mathcal{U}$ taking $\breve{T}$ into T, in the sense that $\mathcal{H} \ni u \mapsto \mathcal{U}u \in \breve{\mathcal{H}}$ through $u(x) = f(x)\mathcal{U}u(g(x))$ and $\mathcal{U}T = \breve{T}\mathcal{U}$. Here $g(x) = \breve{h}^{-1} \circ h(x)$ and $f(x) = (g'(x))^{-1/2}$.*

The functions f and g are continuously differentiable, f is strictly positive, and f' is locally absolutely continuous with $f'(0) = 0$. Also $\alpha = \breve{\alpha}$, in which case $f(0) = 1$, or else $0 < \alpha = \pi/2 - \breve{\alpha} < \pi/2$ or $\pi/2 < \alpha = 3\pi/2 - \breve{\alpha} < \pi$, in which case $f(0) = |\tan\alpha|$.

The relations between the coefficients are $\breve{w}(g(x)) = w(x)/(g'(x))^2$ and $\breve{q}(g(x)) = (-f''(x) + q(x)f(x))/(f(x)(g'(x))^2)$.

It is clear from Theorem 4.1 that Theorem 4.4 is optimal in the sense that it is not possible to deduce more about the relation between T and $\breve{T}$ from the equality of their spectral measures than is done in Theorem 4.4. Sufficient additional information, however, will imply that T and $\breve{T}$ are identical. We give two corollaries of this type.

COROLLARY 4.5. *Suppose T and $\breve{T}$ have the same spectral measure and that $|w| = |\breve{w}|$ in $[0, \min(b, \breve{b}))$. Then $T = \breve{T}$,* i.e., *$b = \breve{b}$, $\alpha = \breve{\alpha}$, $q = \breve{q}$, and $w = \breve{w}$.*

Proof. The assumptions together with Theorem 4.4 show that $g(x) = x$ so that $b = \breve{b}$, and that $f(x) = 1$, so that T and $\breve{T}$ are identical. ∎

Note that only the absolute value of w need be known, so that all information about sign changes in w is encoded in the spectral measure. Also note that if $|w| = |\breve{w}|$ only in $[0, a)$, where $0 < a < \min(b, \breve{b})$, we still have $\alpha = \breve{\alpha}$ and $q = \breve{q}$, $w = \breve{w}$ in $[0, a)$.

COROLLARY 4.6. *Suppose T and $\breve{T}$ have the same spectral measure, that $q = \breve{q}$ on $[0, \min(b, \breve{b}))$, and that either $b = \breve{b}$ or $\alpha = \breve{\alpha}$. Then $T = \breve{T}$,* i.e., *$b = \breve{b}$, $\alpha = \breve{\alpha}$, $q = \breve{q}$, and $w = \breve{w}$.*

We will postpone the proof and first prove Theorem 4.4. To do this we will use a theorem of Paley–Wiener type. For its statement it will be convenient to introduce a special class of entire functions.

DEFINITION 4.7. *Let $\mathcal{A}$ be the set of entire functions $\hat{u}$ of order $\leq 1/2$ which satisfy*

$$\limsup_{t\to\infty} t^{-1} \ln|\hat{u}(t^2\lambda)| \leq \int_0^a \operatorname{Re}\sqrt{-\lambda w} \tag{4.1}$$

for some $a \in (0, b)$ and all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Here the branch of the square root is that with a positive real part.

THEOREM 4.8. *Let $\hat{u}$ be the generalized Fourier transform of $u \in \mathcal{H}$. Then $\hat{u}$ has at most one entire continuation in $\mathcal{A}$, and if* $\sup\operatorname{supp} u = a < b$*, such a continuation is given by*

$$\hat{u}(\lambda) = \int_0^a (u'\varphi'(\cdot, \lambda) + qu\varphi(\cdot, \lambda))$$

in which case (4.1) *holds with equality for all $\lambda \in \mathbb{C}$.*

Conversely, if $\hat{u}$ has an entire continuation of order $\leq 1/2$ satisfying (4.1) *for λ on at least two different rays from the origin, then* $\operatorname{supp} u \subset [0, a]$.

We will postpone the proof of Theorem 4.8 to the next section and instead turn to the proof of Theorem 4.4.

LEMMA 4.9. *Let $g : [0, b) \to [0, \breve{b})$ be increasing and $g(0) = 0$. Suppose $\mathcal{U} : \mathcal{H}_1 \to \breve{\mathcal{H}}_1$ is linear with the properties that $(\mathcal{U}u)(0) = 0$ if $u(0) = 0$, that* $\operatorname{supp}\mathcal{U}u \subset [0, g(x)]$ *if* $\operatorname{supp} u \subset [0, x]$*, and that* $\operatorname{supp}\mathcal{U}u \subset [g(x), \breve{b})$ *if* $\operatorname{supp} u \subset [x, b)$*. Then there exists a function f such that $(\mathcal{U}u)(g(x)) = f(x)u(x)$ for all $u \in \mathcal{H}_1$.*

Proof. Fix $x \in [0, b)$. Suppose u, $v \in \mathcal{H}_1$ and that $u(x) = v(x)$. We will first show that $(\mathcal{U}(u - v)(g(x)) = 0$. If $x = 0$, this is by assumption.

For $x > 0$ we define[1] $u_- = \chi_{[0,x]}(u-v)$ and $u_+ = \chi_{[x,b)}(u-v)$. These are elements of $\mathcal{H}$. Thus $\operatorname{supp}\mathcal{U}u_- \subset [0, g(x)]$ and $\operatorname{supp}\mathcal{U}u_+ \subset [g(x), b)$ so that the functions $\mathcal{U}u_\pm$ vanish in $g(x)$. Adding them gives $\mathcal{U}(u - v)(g(x)) = 0$ as desired.

It follows that the value of $\mathcal{U}u$ at $g(x)$ only depends on the value of u at x. Thus, for each fixed $x \in [0, b)$, the map $u(x) \mapsto \mathcal{U}u(g(x))$ is well-defined and linear on $\mathbb{C}$, so we may find $f(x)$ so that $\mathcal{U}u(g(x)) = f(x)u(x)$. ∎

We will also need the following lemma.

LEMMA 4.10. *Put $m(x, \lambda) = \psi'(x, \lambda)/(\lambda\psi(x, \lambda))$. Then $m(x, \lambda) \to 0$ and $\lambda m(x, \lambda) \to \infty$ for every $x \in [0, b)$ as $\lambda \to \infty$ along any nonreal ray starting from the origin.*

Proof. First note that $m(x, \lambda)$ is the m-function for (1.3) on the interval $[x, b)$, with the Dirichlet boundary condition ($\alpha = 0$) at x. The first claim is then an immediate consequence of [3, Theorem 3.6].

[1] χ_I denotes the characteristic function of an interval I.

To prove the second claim, first assume that q does not have compact support, so that it does not vanish identically on $[x, b)$. Now note that, according to (3.2), $\tilde{m}(\lambda) = -1/m(x, \lambda)$ is the m-function for the Neumann boundary condition ($\alpha = \pi/2$) at x, so we need to show this to be $o(|\lambda|)$. Now, in the Nevanlinna representation (2.10) it is easy to see that the integral is always $o(|\lambda|)$, so we simply need to prove that $B = 0$ in the representation of $\tilde{m}$. Denote the corresponding Weyl solution by $\tilde{\psi}$ and the spectral measure by $d\tilde{\rho}$. Using (2.9) and Lemma 3.11 we obtain

$$\|\tilde{\psi}(\cdot, \lambda)\|^2_{[x,b)} = \frac{\operatorname{Im} \tilde{m}(\lambda)}{\operatorname{Im} \lambda} = B + \int_{-\infty}^{\infty} \frac{d\tilde{\rho}(t)}{|t - \lambda|^2} = B + \|\hat{\tilde{\psi}}(\cdot, \lambda)\|^2_{\tilde{\rho}}.$$

However, by Proposition 2.11, Parseval's formula is correct for $\tilde{\psi}$, so that $B = 0$ and we are done in the case when q does not have compact support.

Now suppose q vanishes identically in $[x, b)$. Consider an auxiliary equation for which q does not have compact support, but which has the same coefficients as (1.3) up to some point c, $x < c < b$. For this equation the above proof of the lemma is valid. Moreover, let $\tilde{\theta}$ and $\tilde{\varphi}$ denote functions analogous to θ and φ for $\alpha = 0$, but with initial data given in the point x. In view of (2.9) both the original $m(x, \lambda)$ and the corresponding function for the auxiliary equation are in the "Weyl disk" defined by

$$\int_x^c |\tilde{\theta}' + m\tilde{\varphi}'|^2 \le \frac{\operatorname{Im} m}{\operatorname{Im} \lambda},$$

so their distance is bounded by the diameter of the disk, which is exponentially small as λ becomes large (see [3, Theorem 6.3] for this result). Since $m(x, \lambda)$ is a nontrivial Nevanlinna function it cannot tend to 0 faster than a multiple of $1/|\lambda|$ for large $|\lambda|$, so that asymptotically $m(x, \lambda)$ is the same as the corresponding function for the auxiliary equation. Thus the lemma is actually valid in all cases. ∎

Proof of Theorem 4.4. Note first that by Lemma 3.5 we must have either $\alpha = \breve{\alpha} = 0$ or else $\alpha \neq 0 \neq \breve{\alpha}$.

Let $\mathcal{H}$ and $\breve{\mathcal{H}}$ denote the Hilbert spaces and $\mathcal{F}$ and $\breve{\mathcal{F}}$ the generalized Fourier transforms associated with the two equations, and put $\mathcal{U} = \breve{\mathcal{F}}^{-1} \circ \mathcal{F} : \mathcal{H} \to \breve{\mathcal{H}}$, which is unitary since the target space is L^2_ρ for both $\mathcal{F}$ and $\breve{\mathcal{F}}$. By Lemma 3.11 we have $\mathcal{U}\psi_0 = \breve{\psi}_0$ if $\alpha = \breve{\alpha}$, and if $\alpha \neq 0 \neq \breve{\alpha}$, we have $\mathcal{U}\psi_0 = \frac{\sin \alpha}{\sin \breve{\alpha}} \breve{\psi}_0$. Since $\langle u, \psi_0 \rangle = -u(0)$ it follows that

$$u(0) = -\langle u, \psi_0 \rangle = -\langle \mathcal{U}u, \mathcal{U}\psi_0 \rangle = \frac{\sin \alpha}{\sin \breve{\alpha}} \mathcal{U}u(0), \tag{4.2}$$

where the quotient of the sines is to be read as 1 for $\alpha = \breve{\alpha} = 0$. In particular, $\mathcal{U}u(0) = 0$ if and only if $u(0) = 0$.

Now, applying Theorem 4.8 for the rays generated by $\pm i$, it is clear that if $\breve{a} \in (0, \breve{b})$ and $u \in \mathcal{H}$, then $\sup \operatorname{supp} u = a$ if $\sup \operatorname{supp} \mathcal{U}u = \breve{a}$, where $h(a) = \breve{h}(\breve{a})$, provided there is such an $a \in (0, b)$[2] (see [4, p. 29] for more details). This will certainly be the case if $\breve{a}$ is sufficiently close to 0. Suppose for some $\breve{a} \in (0, \breve{b})$ we have $h(b) \le \breve{h}(\breve{a})$. Then, since compactly supported functions are dense in $\mathcal{H}$, the range of $\mathcal{U}$ would be orthogonal to all elements of $\breve{\mathcal{H}}$ with supports in $(\breve{a}, \breve{b})$, contradicting the fact that $\mathcal{U}$ is unitary.

A similar reasoning applied to $\mathcal{U}^{-1}$ shows that the mapping

$$g : [0, b) \ni a \mapsto \breve{a} \in [0, \breve{b})$$

[2]Note that $\operatorname{Re} \sqrt{\pm iw} = \sqrt{|w|/2}$.

is bijective, and that $\sup\operatorname{supp}\mathcal{U}u = \breve{a}$ if $\sup\operatorname{supp} u = a$. It follows that $\sup\operatorname{supp} u = a$ if and only if $\sup\operatorname{supp}\mathcal{U}u = g(a)$.

We also have $\inf\operatorname{supp} u = a$ if and only if $\inf\operatorname{supp}\mathcal{U}u = g(a)$. To see this, note that what we have already proved implies that if $\inf\operatorname{supp} u = a > 0$, then $\mathcal{U}u$ is orthogonal to all elements of $\breve{\mathcal{H}}$ with support in $[0, g(a)]$. This means that in this interval $\mathcal{U}u$ is a multiple of $\breve{\varphi}_0$. However, since $u(0) = 0$ we also have $\mathcal{U}u(0) = 0$, so that the multiple is 0, and thus $\inf\operatorname{supp}\mathcal{U}u \geq g(a)$. A similar reasoning applied to $\mathcal{U}^{-1}$ proves the other direction.

We have now verified that $\mathcal{U}$ and $\mathcal{U}^{-1}$ both have the properties required in Lemma 4.9. This implies that there is a nonvanishing function f so that

$$u(x) = f(x)\mathcal{U}u(g(x)). \tag{4.3}$$

We must have f real-valued since $\mathcal{F}$ and $\breve{\mathcal{F}}^{-1}$, and thus $\mathcal{U}$, map real-valued functions to real-valued functions. We note that (4.2) implies that $f(0) = 1$ if $\alpha = \breve{\alpha} = 0$ and $f(0) = \frac{\sin\alpha}{\sin\breve{\alpha}} > 0$ if $\alpha \neq 0 \neq \breve{\alpha}$. Now choose $\mathcal{U}u = 1$ in a neighborhood of $g(x)$. We then have $u = f$ in a neighborhood of x. Since $u \in \mathcal{H}$ is locally absolutely continuous, so is f. This also implies that f is strictly positive, since it cannot change sign and $f(0) > 0$. Similarly, choosing $\mathcal{U}u$ linear in a neighborhood of $g(x)$ it follows that also g is locally absolutely continuous.

According to Lemma 3.11 $\mathcal{U}\psi(\cdot,\lambda) = \breve{\psi}(\cdot,\lambda)$, so we have $\psi(x,\lambda) = f(x)\breve{\psi}(g(x),\lambda)$. Taking the logarithmic derivative we obtain

$$\frac{\psi'(x,\lambda)}{\psi(x,\lambda)} = \frac{f'(x)}{f(x)} + g'(x)\frac{\breve{\psi}'(g(x),\lambda)}{\breve{\psi}(g(x),\lambda)}.$$

Here the left member and the coefficient for $g'(x)$ are locally absolutely continuous, and the coefficient for $g'(x)$ is not independent of λ by Lemma 4.10. It follows that g' and f' are locally absolutely continuous, and differentiating, using the differential equations, we obtain

$$-\frac{f''}{f} + q - (g')^2\breve{q}\circ g - \lambda(w - (g')^2\breve{w}\circ g) = \frac{(f^2g')'}{f^2}\,\frac{\breve{\psi}'(g(\cdot),\lambda)}{\breve{\psi}(g(\cdot),\lambda)}.$$

Here the right member is $o(|\lambda|)$ according to Lemma 4.10 so the coefficient of λ to the left vanishes. On the other hand, the right member is not independent of λ unless $(f^2g')' = 0$, so that we obtain

$$\breve{q}\circ g = \frac{1}{f(g')^2}(-f'' + qf),$$

$$\breve{w}\circ g = (g')^{-2}w,$$

$$f^2g' = C$$

for some constant C. Evaluating (4.3) and its derivative at 0 for $u = \psi(\cdot,\lambda)$ elementary calculations now shows[3] that $C = 1$ and $f'(0) = 0$. One also deduces that either $\alpha = \breve{\alpha}$ or else $0 < \alpha = \pi/2 - \breve{\alpha} < \pi/2$ or $\pi/2 < \alpha = 3\pi/2 - \breve{\alpha} < \pi$. In these calculations one uses that $\breve{m}$ is not a Möbius transform, which is clear since this would give

[3] See the appendix.

a transform space of dimension 1. This can only happen if w, and $d\rho$, is a point mass. $\square$

Finally we have to prove Corollary 4.6.

Proof of Corollary 4.6. The function $\tilde{f} = -\varphi_0$ solves $-\tilde{f}'' + q\tilde{f} = 0$ with initial data $\tilde{f}(0) = 1$, $\tilde{f}'(0) = 0$. Since $q \geq 0$ this solution is strictly positive on $[0, b)$, so we may put $\tilde{g}(x) = \int_0^x 1/\tilde{f}^2$. The pair of functions $\tilde{f}$, $\tilde{g}$ gives us a Liouville transform F_0 mapping $[0, b)$ onto some interval $[0, c)$ and $[0, \breve{b})$ onto $[0, \breve{c})$, and transforming the equations into $-u_0'' = \lambda w_0 u_0$ and $-\breve{u}_0'' = \lambda \breve{w}_0 \breve{u}_0$, respectively. Thus $F_0 F F_0^{-1}$, where F is the Liouville transform of Theorem 4.4, transforms one of these equations into the other.

Being a composition of Liouville transforms this is itself a Liouville transform given, say, by $u_0(x) = f_1(x)\breve{u}_0(g_1(x))$. By construction we obtain $f_1(0) = f(0)$, $f_1'(0) = 0$, and $f_1^2 g_1' \equiv 1$. Since both potentials are identically 0 it follows that $f_1'' = 0$. This means that $f_1 \equiv f(0)$ and $g_1(x) = x/(f(0))^2$.

If $\alpha = \breve{\alpha}$, then by Theorem 4.4 $f(0) = 1$ so that $F_0 F F_0^{-1}$ is the identity, implying that also F is the identity. Similarly, if $b = \breve{b}$, then $c = \breve{c}$ so that $f(0) = 1$, unless $c = \breve{c} = \infty$. We will show that c is always finite, and then it again follows that F is the identity.

Now $c = \int_0^b 1/\tilde{f}^2$, so we need to show that this integral is finite. Put $H = \tilde{f}'\tilde{f}$ which will be strictly positive sufficiently close to b by (2.4).

Differentiating $H' = (\tilde{f}')^2 + \tilde{f}''\tilde{f} = (\tilde{f}')^2 + q\tilde{f}^2 \geq (\tilde{f}')^2$. Thus $1/\tilde{f}^2 = (\tilde{f}')^2/H^2 \leq H'/H^2$ so that $\int_d^b 1/\tilde{f}^2 \leq 1/H(d) < \infty$ if d is sufficiently close to b. This completes the proof. $\square$

5. The Paley–Wiener theorem. The proof of Theorem 4.8 relies on the following lemma, which is taken from [3, Theorem 6.1, Corollary 6.2].

LEMMA 5.1. *The following asymptotic formulas hold, locally uniformly for $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $x > 0$. The square root refers to the branch with positive real part:*

$$\lim_{t\to\infty} t^{-1} \ln \varphi(x, t^2\lambda) = \int_0^x \sqrt{-\lambda w},$$

$$\lim_{t\to\infty} t^{-1} \ln \psi(x, t^2\lambda) = -\int_0^x \sqrt{-\lambda w}.$$

The next lemma implies the simple direction of Theorem 4.8.

LEMMA 5.2. *Suppose $u \in \mathcal{H}$ and $\operatorname{supp} u \subset [0, a]$. Then $\hat{u}(\lambda)$ is entire of order $\leq 1/2$ and $\hat{u}(\lambda) = o(|\lambda\varphi(a + \varepsilon, \lambda)|)$ for every $\varepsilon > 0$ as $\lambda \to \infty$ along any nonreal ray originating at the origin.*

Proof. For finite u we have $\langle u, \varphi(\cdot, \overline{\lambda})\rangle = -u(0)\cos\alpha + \int_0^b u\lambda\varphi(\cdot, \lambda)w$. Now write

$$\hat{u}(\lambda) = -u(0)\cos\alpha + \lambda\varphi(a + \varepsilon, \lambda)\int_0^a u\varphi(\cdot, \lambda)w/\varphi(a + \varepsilon, \lambda).$$

The function $\varphi(x, \lambda)/\varphi(a + \varepsilon, \lambda)$ tends to zero uniformly for $x \in [0, a]$ and $\lambda\varphi(a + \varepsilon, \lambda) \to \infty$ according to Lemma 5.1 as $\lambda \to \infty$ along a nonreal ray. The lemma follows. $\square$

The hard direction of Theorem 4.8 follows from the next lemma.

LEMMA 5.3. *Suppose $u \in \mathcal{H}$, that $\hat{u}$ has an entire continuation of order $\leq 1/2$, and that $\hat{u}(\lambda) = \mathcal{O}(1/|\psi(a, \lambda)|)$ as $\lambda \to \infty$ along two different nonreal rays originating at the origin. Then $\operatorname{supp} u \subset [0, a]$ and $\hat{u}(\lambda) = \langle u, \overline{\varphi(\cdot, \lambda)}\rangle$.*

Proof. Let $\varepsilon > 0$ and consider $F(\lambda) = \langle R_\lambda u, v\rangle - \hat{u}(\lambda)\langle \psi(\cdot,\lambda), v\rangle$, where $v = G_0(wf)$ and $f \in \mathcal{H}$ has compact support in $(a+\varepsilon, b)$. In particular $v \in D_T$. We shall show that F has an entire continuation of order $\leq 1/2$ which tends to 0 along the given rays. By the Phragmén–Lindelöf principle it follows that F is bounded everywhere and is therefore constant by Liouville's theorem, thus actually identically 0.

Now $F(\lambda) = \int_0^b (R_\lambda u - \hat{u}(\lambda)\psi(\cdot,\lambda))\overline{f}w$ so, arguing like in the proof of Proposition 2.3, it follows that $R_\lambda u - \hat{u}(\lambda)\psi(\cdot,\lambda)$ has support in $[0, a+\varepsilon]$. Applying the differential equation it follows that also u has support in $[0, a+\varepsilon]$. Since $\varepsilon > 0$ is arbitrary, in fact u has support in $[0, a]$. For $x > a$ the formula (2.7) gives $R_\lambda u(x) = \psi(x,\lambda)\langle u, \overline{\varphi(\cdot,\lambda)}\rangle$ so that $\psi(x,\lambda)(\hat{u}(\lambda) - \langle u, \overline{\varphi(\cdot,\lambda)}\rangle) = 0$. The lemma follows from this.

To prove that F is entire, Parseval's formula and Lemma 3.11 show that

$$F(\lambda) = \int_{-\infty}^{\infty} \frac{\hat{u}(t) - \hat{u}(\lambda)}{t-\lambda}\,\overline{\hat{v}(t)}\,d\rho(t).$$

It is obvious that this is an entire function, at least if we can bound the integrand properly. To do this and see that the order is at most $1/2$, note that for $|t-\lambda| \leq 1$ we may estimate the integrand by $\sup_{|z|\leq 1} |\hat{u}'(\lambda+z)||\hat{v}(t)|$. For $|t-\lambda| > 1$ we may estimate the integrand by $|\hat{u}(t)\hat{v}(t)| + |\hat{u}(\lambda)||\hat{v}(t)|$. Hence we have locally uniformly dominated convergence of the integral and

$$|F(\lambda)| \leq \|u\|\|v\| + \big(\sup_{|z|\leq 1} |\hat{u}'(\lambda+z)| + |\hat{u}(\lambda)|\big)\int_{-\infty}^{\infty} |\hat{v}|d\rho,$$

which is the required estimate, the integral being finite by Corollary 3.8 and $\hat{u}$ and therefore $\hat{u}'$ being of order $\leq 1/2$.

Finally, to show that F tends to 0 along the rays, we first note that $\psi(x,\lambda)/\psi(a,\lambda)$ converges to 0 uniformly for $x \in [a+\varepsilon, b)$, according to Lemma 5.1. Assuming f has compact support in $[a+\varepsilon, b)$ we obtain $\int_0^b \psi(\cdot,\lambda)\overline{f}w = o(|\psi(a,\lambda)|)$. Since $R_\lambda \to 0$ strongly as $\operatorname{Im}\lambda \to \infty$, it follows that F tends to 0 along the given rays. This finishes the proof. ∎

Theorem 4.8 is a simple consequence of these lemmas.

Proof of Theorem 4.8. If $\operatorname{supp} u \subset [0,a]$, it follows from Lemmas 5.2 and 5.1 that $\hat{u}(\lambda) = \langle u, \varphi(\cdot,\overline{\lambda})\rangle$ is an entire continuation of $\hat{u}$ of order $\leq 1/2$ such that

$$\limsup_{t\to\infty} t^{-1}\ln|\hat{u}(t^2\lambda)| \leq \lim_{t\to\infty} t^{-1}\ln|\varphi(a+\varepsilon, t^2\lambda)| = \int_0^{a+\varepsilon} \operatorname{Re}\sqrt{-\lambda w}$$

for nonreal λ and all $\varepsilon > 0$.

On the other hand, suppose there is an entire continuation of $\hat{u}$ of order $\leq 1/2$ and such that

$$\limsup_{t\to\infty} t^{-1}\ln|\hat{u}(t^2\lambda)| \leq \int_0^{a} \operatorname{Re}\sqrt{-\lambda w}$$

for λ on two different rays from the origin. If one or both of these are real, an immediate application of the Phragmén–Lindelöf principle shows this to be true for all other rays as well, so we may assume them nonreal. By Lemma 5.1 this implies that $\hat{u}(\lambda) = \mathcal{O}(|\psi(a+\varepsilon,\lambda)|^{-1})$ for large λ on these rays if $0 < \varepsilon < b-a$. Lemma 5.3 now shows that $\operatorname{supp} u \subset [0, a+\varepsilon]$ for small $\varepsilon > 0$ and thus for $\varepsilon = 0$. The uniqueness of the continuation also follows from Lemma 5.3. If we have strict inequality on one ray, a simple argument using the Phragmén–Lindelöf principle (see [4, Lemma 3.6]) shows this to hold on all nearby rays as well, so that in fact $\sup\operatorname{supp} u < a$. The proof is now complete. ∎

6. Inverse scattering on the half-line. In this section we will show that scattering data for the half-line problem determines the coefficient w if q is known. We will of course have to assume that our equation is sufficiently close to a model equation, which, as usual, has constant coefficients.

Thus we consider (1.3) on $[0,\infty)$ with the following additional assumption, which will be in force throughout this section.

Assumption 6.1. There is a constant $q_0 \geq 0$ such that $q(x) - q_0$ and $w(x) - 1$ are both in $L^1(0,\infty)$.

Note that according to Theorems 2.5 and 2.6 finite functions are dense in $\mathcal{H}_1$ and, given the boundary condition (2.6), there is a unique self-adjoint realization T of (1.3) in $\mathcal{H}_1$.

We will need the following standard result.

PROPOSITION 6.2. *For* $\operatorname{Im} k \geq 0$, $k \neq 0$, *there exists a solution* $f(\cdot,k)$ *of* (1.3) *with* $\lambda = k^2+q_0$ *having the following properties:* (1) $f(x,\cdot)$ *and* $f'(x,\cdot)$ *are analytic for* $\operatorname{Im} k > 0$ *and continuous for* $\operatorname{Im} k \geq 0$, $k \neq 0$; (2) $f(x,k) \sim e^{ikx}$ *and* $f'(x,k) \sim ike^{ikx}$ *as* $x \to \infty$.

This is standard. It is easily proved by first writing the equation for $g(x,k) = f(x,k)e^{-ikx}$ as $g''+2ikg' = (q-q_0-(k^2+q_0)(w-1))g$ and then solving this equation by successive approximations from its desired initial values $g(\infty) = 1$, $g'(\infty) = 0$ at ∞ using the estimate $|e^{2ik(t-x)} - 1| \leq 2$. See, for instance, Deift and Trubowitz [19].

If $\operatorname{Im} k > 0$, then $f(\cdot,k) \in \mathcal{H}_1$. Thus, if $\lambda \notin \mathbb{R}$ (*i.e.*, also $\operatorname{Re} k \neq 0$), then

$$f(x,k) = F(k)\psi(x,\lambda)$$

for some function F defined in $\operatorname{Im} k > 0$, $\operatorname{Re} k \neq 0$.

Let $[u,v] = u'v - uv'$ denote the Wronskian of the functions u and v and recall that Wronskians of solutions to (1.3) are independent of x. Since

$$[\lambda\varphi(\cdot,\lambda), f(\cdot,k)] = F(k)[\lambda\varphi(\cdot,\lambda), \psi(\cdot,\lambda)] = F(k) \tag{6.1}$$

is analytic for $\operatorname{Im}(k) > 0$ we find that F is analytic and can be extended analytically to the positive imaginary axis. Moreover, since $[\lambda\varphi(\cdot,\lambda), f(\cdot,k)]$ is continuous in $\operatorname{Im}(k) \geq 0$, $k \neq 0$, the function F extends continuously to the positive and negative real line. The zeros of F are located exactly where φ and f are linearly dependent, *i.e.*, when $\lambda = q_0 + k^2$ is an eigenvalue.

Equation (6.1) gives also that $F(-k) = \overline{F(k)}$ for real $k \neq 0$ and that F has no zeros on either the positive or the negative real line since $\varphi(\cdot,\lambda)$ is real for real λ and the real and imaginary parts of $f(x,k) \sim e^{ikx}$ are linearly independent.

For $k > 0$ and thus $\lambda = k^2 + q_0 > q_0$ define

$$\psi_\pm(\cdot,\lambda) = \lim_{\epsilon\to 0} \psi(\cdot,(\pm k + i\epsilon)^2 + q_0)$$

and

$$m_\pm(\lambda) = \lim_{\epsilon\to 0} m((\pm k + i\epsilon)^2 + q_0).$$

Since $\overline{m(\lambda)} = m(\overline{\lambda})$ when λ is not real we find that $m_+(\lambda) = \overline{m_-(\lambda)}$ when λ is real. Therefore

$$\frac{2ik\lambda}{|F(k)|^2} = \lambda[\psi_+(\cdot,\lambda), \psi_-(\cdot,\lambda)] = m_+(\lambda) - m_-(\lambda) = 2i \operatorname{Im} m_+(\lambda)$$

when $k > 0$ so that $\lambda > q_0$. This in turn implies

$$\pi\rho'(\lambda) = \operatorname{Im} m(\lambda + i0) = \frac{k\lambda}{|F(k)|^2}$$

for $\lambda > q_0$. Thus the restriction of F to the positive real line determines the spectral measure on the interval (q_0, ∞). It follows from this that the spectrum of T is absolutely continuous[4] in (q_0, ∞).

In the interval $(-\infty, q_0)$, where λ corresponds to the positive half of the imaginary axis for k, the spectrum is discrete since F is analytic there. There might also be an eigenvalue for $k = 0$, $\lambda = q_0$. Suppose $\lambda \neq 0$ is an eigenvalue. Then $\varphi(\cdot, \lambda)$ is a corresponding eigenfunction, and its Fourier transform $\hat{\varphi}(\lambda)$ is a multiple of the characteristic function of the set $\{\lambda\}$. The inversion formula (3.1) gives $\varphi(x,\lambda) = \hat{\varphi}(\lambda)\varphi(x,\lambda)\rho\{\lambda\}$, where $\rho\{\lambda\}$ is the spectral measure of the set $\{\lambda\}$. Thus $\hat{\varphi}(\lambda) = 1/\rho\{\lambda\}$. Parseval's formula gives $\|\varphi(\cdot,\lambda)\|^2 = |\hat{\varphi}(\lambda)|^2\rho\{\lambda\} = 1/\rho\{\lambda\}$. On the interval $(-\infty, q_0]$ we therefore know the spectral measure if we know all eigenvalues λ and the corresponding *normalization constants* $\|\varphi(\cdot,\lambda)\|^2$. Similarly, if $\alpha = 0$, then by Lemma 3.5 also $\lambda = 0$ is an eigenvalue, and $1/\rho\{0\}$ is the normalization constant for the eigenfunction ψ_0. We obtain the following theorem.

THEOREM 6.3. *Given the absolute value of the coefficient $F(k)$ for positive k, all eigenvalues, the corresponding normalization constants, and either q or $|w|$, the coefficients q and w and the boundary value parameter α are uniquely determined.*

Proof. We have already seen that the given data determine the spectral measure, and may now apply Corollaries 4.5 and 4.6 to draw the desired conclusion. ∎

7. Eigenvalues. This section is devoted to the proof of the following theorem. Part of the proof is an adaptation of Marchenko [25].

THEOREM 7.1. *Assume that q and w satisfy Assumption* 6.1. *Then we have the following:*

(1) *The eigenvalues of T are isolated and can accumulate only at q_0 or negative infinity.*
(2) *There will be infinitely many negative eigenvalues if and only if w is negative on a set of positive measure.*

If in addition we have $\int_0^\infty t|q(t) - q_0 w(t)|dt < \infty$, we also have the following:

(3) *Eigenvalues will not accumulate at q_0.*
(4) *q_0 is not an eigenvalue unless $q_0 = 0$ and $\alpha = 0$.*

To prove this we need the following strengthening of Proposition 6.2.

PROPOSITION 7.2. *Suppose q and w satisfy Assumption* 6.1 *and the integral $\int_0^\infty t|q(t) - q_0 w(t)|dt$ is finite. Then, for every $x \in [0,\infty)$, the function $f(x,\cdot)$ and its x-derivative, which were previously defined for* $\operatorname{Im}(k) \geq 0$, $k \neq 0$, *extend continuously to $k = 0$.*

The additional assumption and the improved estimate

$$|e^{2ik(t-x)} - 1| \leq \min(2|k|t, 2)$$

[4]For $q_0 < s < t$ we have $\int_s^t \operatorname{Im} m(\mu + i\varepsilon)\, d\mu \to \pi(\rho(t) - \rho(s))$ as $\varepsilon \downarrow 0$. But the left-hand side converges to $\int_s^t \operatorname{Im} m(\mu + i0)\, d\mu$ so ρ is absolutely continuous.

allow us to perform the successive approximations also near $k = 0$. The proposition follows from this.

Proof of Theorem 7.1. If $\mu = k^2 + q_0 < q_0$ is an eigenvalue of T, then, since F is analytic in the upper half-plane, eigenvalues are isolated and hence cannot accumulate at any point in $(-\infty, q_0)$. This proves (1).

To prove the second statement we make first the assumption that $q_0 > 0$ and $\alpha \neq 0$. By Lemma 3.5 zero is then not in the spectrum of T so that the range of T is $\mathcal{H}$ and we may define a bilinear form Q on $\mathcal{H}$ by setting

$$Q(u, v) = \int_{\mathbb{R}} \frac{1}{t} \hat{u}(t)\overline{\hat{v}(t)} d\rho(t).$$

Note that $Q(u, v) = 0$ if the supports of $\hat{u}$ and $\hat{v}$ do not intersect, which happens, for instance, if u and v are eigenvectors for different eigenvalues. Furthermore, by Lemma 3.7 $Q(u, Tv) = \int_{\mathbb{R}} \hat{u}(t)\overline{\hat{v}(t)} d\rho(t) = \langle u, v\rangle$. An integration by parts gives

$$\int_0^x (u'\overline{v'} + qu\overline{v}) = u(x)\overline{v'(x)} - u(0)\overline{v'(0)} + \int_0^x wu\overline{Tv}$$

for $u \in \mathcal{H}$ and $v \in D_T$. Hence if v is in the range of T and u is finite, or if u and v are exponentially decaying eigenfunctions, then we obtain

$$Q(u, v) = \int_0^\infty wu\overline{v} + \cot(\alpha)u(0)\overline{v(0)} \tag{7.1}$$

taking into account the boundary condition satisfied by $(T^{-1}v, v)$.

Now assume that $w \geq 0$. If $\cot(\alpha) \geq 0$, there can be no negative eigenvalue since $Tv = \lambda v$, $\lambda < 0$, $\|v\| \neq 0$ would imply that

$$0 \leq \int_0^\infty w|v|^2 + \cot\alpha|v(0)|^2 = \frac{1}{\lambda} Q(v, Tv) = \frac{1}{\lambda}\|v\|^2 < 0,$$

giving a contradiction. If $\cot\alpha < 0$, there can be at most one negative eigenvalue as we shall show now. If there were two distinct negative eigenvalues λ_1 and λ_2 with associated eigenvectors v_1 and v_2, we could assume that $v_1(0) = v_2(0)$. This would entail that

$$0 \leq \int_0^\infty w|v_1 - v_2|^2 = Q(v_1 - v_2, v_1 - v_2) = Q(v_1, v_1) + Q(v_2, v_2) < 0$$

since eigenfunctions decay exponentially so that we are allowed to employ (7.1).

Next assume $w < 0$ on a set of positive Lebesgue measure. We shall show that there are infinitely many negative eigenvalues. For any integer n one can choose elements $u_1, \ldots, u_n$ in $\mathcal{H}$, compactly supported in $(0, \infty)$, such that $Q(u_j, u_j) < 0$ and $Q(u_j, u_k) = 0$ if $j \neq k$. To achieve this one may for instance choose first bounded sets $A_1, \ldots, A_n$ of positive measure and positive distances from zero and each other on which w is negative. Then one lets u_j be a suitable mollification of the characteristic function of A_j. Equation (7.1) now guarantees that they have the desired properties.

Thus $Q(u, u) < 0$ whenever u is in the linear span B of $u_1, \ldots, u_n$. Let P be the orthogonal projection of B into the negative spectral subspace of $\mathcal{H}$, *i.e.*, $Pu = \mathcal{F}^{-1}(u\chi)$, where χ is the characteristic function of $(-\infty, 0)$. Suppose now that n is larger than the number of negative eigenvalues. Then the kernel of P cannot be trivial so that there is a nontrivial $u \in B$ such that $\hat{u}$ is supported in $[0, \infty)$. Hence

$$0 > Q(u,u) = \int_{\mathbb{R}} \frac{1}{t} |\hat{u}(t)|^2 d\rho(t) \geq 0.$$

Since this is impossible the number of negative eigenvalues must be infinite.

If we only have $q_0 \geq 0$, but still $\alpha \neq 0$, then Q remains defined for functions u, v with Fourier transforms bounded near 0, since in this case $1/t \in L^2_\rho$ by Lemma 3.11. But the Fourier transforms of eigenfunctions to nonzero eigenvalues are supported away from 0, and the Fourier transform of a finite function is entire and thus locally bounded. Also, u_j is in the range of T. To see this, solve $-y'' + qy = wu_j$ with 0 initial data at a point to the right of $\operatorname{supp} u_j$ which yields a finite function y. Adding an appropriate multiple of ψ_0 (Proposition 2.7) gives a function in D_T. Thus the proof applies also in this case.

Allowing also $\alpha = 0$ the form Q is still defined if $\hat{u}(t)\overline{\hat{v}(t)}/t$ is continuous at 0. This is the case if u and v are eigenfunctions to negative eigenvalues. Also, if u is a finite function orthogonal to the eigenfunction ψ_0, then $\hat{u}(0) = 0$; so Q is defined for such functions. This last condition is just one linear condition on the space B, so the remainder can still have arbitrarily large dimension. All of the u_j are in the range of T, since the boundary condition now reads $u_j(0) = 0$. Thus the proof applies also in this case, and the proof of (2) is finished.

Now assume that $\int_0^\infty t|q(t) - q_0 w(t)| dt$ is finite, and that, contrary to our claim, there is a sequence $\mu_n = k_n^2 + q_0 < q_0$ of eigenvalues converging to q_0. Since eigenfunctions are orthogonal and satisfy the boundary condition an integration by parts shows

$$\int_0^\infty w f(\cdot, k_n) \overline{f(\cdot, k_m)} = -f(0, k_n) \overline{f(0, k_m)} \cot \alpha \tag{7.2}$$

if $n \neq m$. If $\alpha = 0$, the right-hand side has to be replaced by zero.

Since $\int_0^\infty t|q(t) - q_0 w(t)| dt < \infty$, our construction of f shows that $f(x, k) \sim e^{ikx}$ as $x \to \infty$, uniformly for $k \in i[0, 1]$. This shows first that (7.2) is bounded as n and m tend to infinity, and second that we may find a positive c such that $|f(x, k) - e^{ikx}| \leq e^{-|k|x}/4$ if $x \geq c$, $k \in i[0, 1]$. Simple estimates then show that

$$\frac{7}{16} e^{-(|k_n|+|k_m|)x} \leq \operatorname{Re}(f(x, k_n)\overline{f(x, k_m)}) \leq \frac{25}{16} e^{-(|k_n|+|k_m|)x}$$

if n and m are large. Since $w - 1$ is integrable this shows that the integral

$$\int_c^\infty \operatorname{Re}(f(x, k_n)\overline{f(x, k_m)}) w \to +\infty$$

as n, m tend to infinity. Now, since $f(x, k)$ is uniformly continuous on $[0, c] \times i[0, 1]$ it follows that the integral over $[0, c]$ is bounded, so the integral over $[0, \infty)$ tends to infinity, contradicting the previously established boundedness and proving (3).

Finally, if $q_0 = 0$, we already know q_0 is an eigenvalue if and only if $\alpha = 0$. On the other hand, if $q_0 > 0$, then $f(\cdot, q_0)$ is asymptotic to 1, and any other solution to (1.3) is asymptotically linear, as is easily seen from the well-known reduction of order method. Thus no such solution is in $\mathcal{H}$ and there is no eigenfunction with eigenvalue q_0. This proves (4). $\square$

Remark 7.3. If we allow w to be a general measure, then the negative part of w could be a finite sum of Dirac measures. In this case one may in the same way show that the number of negative eigenvalues is equal to the number of these Dirac measures if $\alpha \neq 0$, $\cot\alpha \geq 0$, and $q_0 > 0$, with suitable modifications in the other cases.

8. Appendix. Here we present some calculations which were omitted from the proof of Theorem 4.4.

For $x = 0$ the relation $\psi(x,\lambda) = f(x)\breve{\psi}(g(x),\lambda)$ gives

$$\cos\alpha - m(\lambda)\sin\alpha = f(0)\{\cos\breve{\alpha} - \breve{m}(\lambda)\sin\breve{\alpha}\}, \tag{8.1}$$

while $\psi'(x,\lambda) = f'(x)\breve{\psi}(g(x),\lambda) + f(x)g'(x)\breve{\psi}'(g(x),\lambda)$ for $x = 0$ gives

$$\sin\alpha + m(\lambda)\cos\alpha = \frac{f'(0)}{\lambda}\{\cos\breve{\alpha} - \breve{m}(\lambda)\sin\breve{\alpha}\} + \frac{C}{f(0)}\{\sin\breve{\alpha} + \breve{m}(\lambda)\cos\breve{\alpha}\}. \tag{8.2}$$

From (8.1), (8.2) we obtain

$$1 = \left\{f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha\right\}\{\cos\breve{\alpha} - \breve{m}(\lambda)\sin\breve{\alpha}\} + \frac{C\sin\alpha}{f(0)}\{\sin\breve{\alpha} + \breve{m}(\lambda)\cos\breve{\alpha}\}$$

and

$$m(\lambda) = \left\{-f(0)\sin\alpha + \frac{f'(0)}{\lambda}\cos\alpha\right\}\{\cos\breve{\alpha} - \breve{m}(\lambda)\sin\breve{\alpha}\} + \frac{C\cos\alpha}{f(0)}\{\sin\breve{\alpha} + \breve{m}(\lambda)\cos\breve{\alpha}\},$$

which after rearranging gives

$$\begin{aligned} &1 - \left(f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha\right)\cos\breve{\alpha} - \frac{C\sin\alpha\sin\breve{\alpha}}{f(0)} \\ &\quad = \breve{m}(\lambda)\left\{-\left(f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha\right)\sin\breve{\alpha} + \frac{C\sin\alpha\cos\breve{\alpha}}{f(0)}\right\} \end{aligned} \tag{8.3}$$

and

$$\begin{aligned} &\left(f(0)\sin\alpha - \frac{f'(0)}{\lambda}\cos\alpha\right)\cos\breve{\alpha} - \frac{C\cos\alpha\sin\breve{\alpha}}{f(0)} \\ &\quad = \breve{m}(\lambda)\left\{\left(f(0)\sin\alpha - \frac{f'(0)}{\lambda}\cos\alpha\right)\sin\breve{\alpha} + \frac{C\cos\alpha\cos\breve{\alpha}}{f(0)}\right\} - m(\lambda). \end{aligned} \tag{8.4}$$

In (8.3) the left member and the coefficient of $\breve{m}$ are linear in $1/\lambda$, while $\breve{m}(\lambda)$ is not constant or a Möbius transform (this would give a one-dimensional transform space). From (8.3) we therefore obtain

$$\left(f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha\right)\cos\breve{\alpha} = 1 - \frac{C\sin\alpha\sin\breve{\alpha}}{f(0)},$$

$$\left(f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha\right)\sin\breve{\alpha} = \frac{C\sin\alpha\cos\breve{\alpha}}{f(0)},$$

which gives

$$f(0)\cos\alpha + \frac{f'(0)}{\lambda}\sin\alpha = \cos\breve{\alpha},$$

$$\frac{C\sin\alpha}{f(0)} = \sin\breve{\alpha}.$$

From this it is (again) clear that $\sin\alpha = 0$ if and only if $\sin\breve{\alpha} = 0$, so that we have two cases.

- $\alpha = \breve{\alpha} = 0$. We obtain $f(0) = 1$, and insertion in (8.4) shows that $\frac{f'(0)}{\lambda} = m(\lambda) - C\breve{m}(\lambda)$. The right member is $(1-C)m(\lambda)$ since $m(i\nu)$ and $\breve{m}(i\nu) \to 0$ as $\nu \to +\infty$ by Lemma 4.10, and m, $\breve{m}$ have the same spectral measure. Again by Lemma 4.10 it follows that $C = 1$, and thus $f'(0) = 0$.
- $\alpha \neq 0 \neq \breve{\alpha}$. We obtain $f'(0) = 0$, $f(0) = C\sin\alpha/\sin\breve{\alpha}$, and $C\sin(2\alpha) = \sin(2\breve{\alpha})$. But we know that $f(0) = \sin\alpha/\sin\breve{\alpha}$ so that $C = 1$. Insertion in (8.4) gives $m(\lambda) - \breve{m}(\lambda) = \cot\alpha - \cot\breve{\alpha}$.
  Since $\sin(2\alpha) = \sin(2\breve{\alpha})$ we have either $\alpha = \breve{\alpha}$ or $0 < \alpha = \pi/2 - \breve{\alpha} < \pi/2$ or $\pi/2 < \alpha = 3\pi/2 - \breve{\alpha} < \pi$. If $\alpha = \breve{\alpha}$, we obtain $f(0) = 1$ and $m(\lambda) = \breve{m}(\lambda)$. In the other cases we obtain $f(0) = |\tan\alpha|$ and $m(\lambda) - \breve{m}(\lambda) = 2\cot(2\alpha)$.

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