FIRST PASSAGE TIMES FOR SOME CLASSES OF FRACTIONAL TIME-CHANGED DIFFUSIONS

Abstract. We consider some time-changed diffusion processes obtained by applying the Doob transformation rule to a time-changed Brownian motion. The time-change is obtained via the inverse of an α-stable subordinator. These processes are specified in terms of time-changed Gauss-Markov processes and fractional time-changed diffusions. A fractional pseudo-Fokker-Planck equation for such processes is given. We investigate their first passage time densities providing a generalized integral equation they satisfy and some transformation rules. First passage time densities for time-changed Brownian motion and Ornstein-Uhlenbeck processes are provided in several forms. Connections with closed form results and numerical evaluations through the level zero are given.

Keywords: First passage time; Fractional diffusion; Time-changed diffusion; Integral equation; Numerical evaluation

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1. Introduction

In last two decades an increasing attention has been given to time-changed processes, in order to construct new correlated processes and heavy-tailed processes preserving a sort of memory and also to be able to manipulate the time-scales of the consequent stochastic models ([3], [4], [6], [9], [28], [19], [21], [26]). The study of these new kind of processes required the large use of mathematical tools such as those of the fractional calculus, from which fractional processes and in particular fractional diffusions were introduced ([5], [27], [34], [40]).

The first passage time (FPT) problem for such processes is really interesting to be investigated because more realistic stochastic models can be constructed and many fields of applications can fruitfully use them. Indeed, the first passage time of fractional processes is actually the final goal of models embodying a sort of *memory*.

With the aim to investigate some time-changed diffusions and their FPT, we start with the time-changed Brownian motion and we consider those processes obtained by applying the Doob transformation rule ([23], [30]). Generally, the use of transformation rules between processes has a twofold purpose: from a mathematical point of view this is a strategy to understand how and how much can be enlarged a specific class of stochastic processes preserving known properties (see, for instance, [17] and reference therein), from an applicative point of view this is a way to construct ad hoc models satisfying specific requirements originated by phenomenological evidences ([16], [30]). Stimulated by these needs, we follow this strategy to construct the class of fractional time-changed diffusions on which we focus this paper.

Specifically, we start considering the classical Doob transform in order to identify the Gauss-Markov (GM) processes $\{X(t)\}$ such as

$$X(t) = m(t) + \eta(t)W(r(t)) \tag{1.1}$$

where $\{W(t), t \geq 0\}$ is the standard Brownian motion, and $m(t), \eta(t), r(t)$ suitable functions ([17]). Our idea is to consider the same Doob transform by substituting the standard Brownian motion with the time-changed Brownian motion $\{W_\alpha(t) = W(E_\alpha(t)), t \in I\}$ in such a way, for a parameter set $I \subseteq \mathbb{R}$, we construct the following time-changed process:

$$X_\alpha(t) = m(t) + \eta(t)W_\alpha(r(t)) \tag{1.2}$$

with $\eta(t)$, $m(t) \in C^1(I)$, $r(t)$ positive monotone increasing $C^1(I)-$functions (generally with $r(0) = 0$). Furthermore, $E_\alpha(t)$ is a non decreasing stochastic process, independent of $W(t)$; in particular we consider $E_\alpha(t) = \inf\{s > 0 : \sigma_\alpha(s) > t\}$ defined as the inverse of an $\alpha-$stable subordinator process σ_α, with $\alpha \in (0, 1)$. Substantially, starting from the time-changed process $W_\alpha(r(t)) = W(E_\alpha(r(t)))$, the process $X_\alpha(t) = m(t) + \eta(t)W_\alpha(r(t))$ is a time-changed Brownian motion, evaluated at a transformed time $r(t)$, transformed in space by $\eta(t)$, having mean $m(t)$. Then, we consider a second type of time-changed process, i.e. the following one:

$$\mathfrak{X}_\alpha(t) = m(E_\alpha(t)) + \eta(E_\alpha(t))W(r(E_\alpha(t))) \tag{1.3}$$

obtained as a time-changed GM process (1.1) by means of the same $E_\alpha(t)$. Note that in this last case we use the time-changed Brownian motion $W(r(E_\alpha(t)))$ different from the previous one $W(E_\alpha(r(t)))$ in (1.2). This kind of processes will be useful to obtain results also for the first kind ones.

The main difference between the two above processes $X_\alpha(t)$ and $\mathfrak{X}_\alpha(t)$ relies essentially on the order of application of the time-change and the space-time transformation. Indeed, for $X_\alpha(t)$, at first we apply the time-change to the Brownian motion $W(t)$ obtaining $W_\alpha(t) = W(E_\alpha(t))$, then the latter is evaluated in the transformed time $r(t)$, multiplied by $\eta(t)$ and then it is endowed with the mean $m(t)$. Instead, the process $\mathfrak{X}_\alpha(t)$ is obtained by considering at first the process $X(t)$ constructed by means of the Doob transform (1.1), that is a Brownian motion evaluated in the transformed time $r(t)$, multiplied by $\eta(t)$ and endowed with the mean $m(t)$, then the time-change is applied to $X(t)$ in such a way $\mathfrak{X}_\alpha(t) = X(E_\alpha(t))$.

We used to call the processes (1.2) the *first kind* time-changed GM processes and the processes in (1.3) the *second kind* of time-changed GM processes via the inverse of an $\alpha-$stable subordinator $E_\alpha(t)$ (even if they are non-Gaussian and non-Markov processes). We also use *subordinated* in place of *time-changed* with the same meaning.

In this paper we firstly provide an introduction on the time-change strategy and some basic facts related to time-changed processes. In Section 2 we define the time-changed processes by Doob transform as those in (1.2) and in (1.3), we study them by pointing differences and relationships between them. In Section 3 the pseudo Fokker-Planck equation is proved to hold for a class of fractional time-changed diffusions related to the above processes. In Section 4 the FPT topic is addressed: an integral equation is provided following the Volterra integral approach ([17], [35]) specialized for the specified time-changed processes. Due its key rule, we focus on FPT density of the time-changed Brownian motion. As example of application the

FPT density for a time-changed Ornstein-Uhlenbeck (OU) process is provided by means of several strategies and numerical evaluations are also provided.

1.1. **The time-change.** At first we recall some definitions and fundamental properties of the involved processes in the time-change.

Some basic facts: the α-stable subordinator and its inverse. Referring to [11] and [32], we recall some essential definitions. For $\alpha \in (0,1)$, we specifically consider an α-stable subordinator $\sigma_\alpha(t)$, i.e. a strictly increasing (pure jumps) positive Lévy process with the following Laplace transform, for $\lambda > 0, t > 0$,

$$\mathbb{E}[e^{-\lambda\sigma_\alpha(t)}] = e^{-t\lambda^\alpha},$$

with Laplace exponent λ^α. We also consider the inverse α-stable subordinator $E_\alpha(t)$, i.e.

$$E_\alpha(t) := \inf\{y > 0 : \ \sigma_\alpha(y) > t\}.$$

We recall that in [32], it was proved that $\sigma_\alpha(t)$ and $E_\alpha(t)$ are absolutely continuous random variables for any $t > 0$, but this feature can also be extended for $t < 0$ to suitable modifications of processes $\sigma_\alpha(t)$ and $E_\alpha(t)$. It was proved ([12], [14]) that the inverse stable subordinator $E_\alpha(t)$ has the following Laplace-Stieltjes transform

$$\mathbb{E}[e^{-sE_\alpha(t)}] = \sum_{n=0}^{\infty} \frac{(-st^\alpha)^n}{\Gamma(\alpha n+1)} = \mathcal{E}_\alpha(-st^\alpha) \tag{1.4}$$

where $\mathcal{E}_\alpha(-st^\alpha)$ is the Mittag-Leffler function. More specifically, we recall that he Mittag-Leffler function $\mathcal{E}_\alpha(z)$ is defined as

$$\mathcal{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, z, \alpha \in \mathcal{C}, \mathbb{R}(\alpha) > 0.$$

Moreover, keeping in mind that the notation $\stackrel{d}{=}$ means the equality of finite dimensional distributions (fdd), we recall the scaling property of $\sigma_\alpha(t)$:

$$\sigma_\alpha(t) \stackrel{d}{=} t^{1/\alpha}\sigma_\alpha(1).$$

Furthermore, the inverse of α-stable subordinator $E_\alpha(t)$ is a self-similar processes, indeed for $c > 0$

$$c^{-\alpha}E_\alpha(ct) \stackrel{d}{=} E_\alpha(t) \quad \forall t \geq 0. \tag{1.5}$$

Furthermore, the mean is

$$\mathbb{E}[E_\alpha(t)] = \frac{t^\alpha}{\Gamma(\alpha+1)} \tag{1.6}$$

and the covariance for $0 < s \leq t$ from [26]:

$$cov[E_\alpha(s), E_\alpha(t)] = \frac{[\alpha s^{2\alpha}B(\alpha,\alpha+1) + F(\alpha; s,t)]}{(\Gamma(\alpha+1))^2} \tag{1.7}$$

where $B(a,b)$ is the *Beta function*, and

$$F(\alpha; s,t) = \alpha t^{2\alpha}B(\alpha,\alpha+1; s/t) - (st)^\alpha$$

is the *hypergeometric function* (see, for instance, [20]) and $B(a, b; x)$, with $x \in [0, 1]$, is the *incomplete beta function*, i.e.

$$B(a, b; x) = \int_0^x u^{\alpha-1}(1-u)^{b-1} du,$$

with $B(a, b) = B(a, b; 1)$.

From (1.7) we also obtain the variance

$$var[E_\alpha(t)] = t^{2\alpha}\left[\frac{2}{\Gamma(2\alpha+1)} - \frac{1}{(\Gamma(\alpha+1))^2}\right]. \tag{1.8}$$

We recall that ([26])

$$cov[E_\alpha(s), E_\alpha(t)] \overset{t\to\infty}{\to} \frac{s^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{1.9}$$

Moreover, let us denote by $\gamma_\alpha(x)$ the probability density function (pdf) of $\sigma_\alpha(1)$ and by $\nu_\alpha(x, t)$ the pdf of $E_\alpha(t)$. It holds ([32])

$$\nu_\alpha(x, t) = \frac{t}{\alpha} x^{-1-\frac{1}{\alpha}} \gamma_\alpha(t x^{-\frac{1}{\alpha}}), \qquad x \geq 0,\ t > 0. \tag{1.10}$$

This density is zero for $x < 0$, whereas $E_\alpha(t)$ is positive for $t > 0$ with a discontinuity in $x = 0$. Additional details are provided in the Appendix.

In particular, we note that also E_α is an increasing, continuous process with constant values corresponding to the jumps of σ_α. Moreover, the Laplace transform of $\nu_\alpha(x, t)$ respect to t is

$$\mathcal{L}_{t\to\lambda}[\nu_\alpha(x, t)] = \lambda^{\alpha-1} e^{-x\lambda^\alpha}. \tag{1.11}$$

1.2. **Generalities on time-changed (or subordinated) processes.** A time-changed process is the composition of two independent processes: the outer process and the inverse of an α-stable subordinator. It is characterized by the continuous sample paths if the outer process has continuous paths. Here, the transformation rule (1.2) will involve the Brownian motion as the outer process and the inverse of an $\alpha-$stable subordinator processes for the time-change.

However, in general, if $f(x, s)$ is the probability density of the outer process, the time-changed process has the following pdf:

$$f_\alpha(x, t) = \int_0^{+\infty} f(x, s)\nu_\alpha(s, t) ds \quad \forall t \in I \subset \mathbb{R}. \tag{1.12}$$

We remark that there exist an alternative expression of $f_\alpha(x, t)$ by using (1.10) and the change of variable $ts^{-1/\alpha} = w$, i.e.

$$f_\alpha(x, t) = \int_0^{+\infty} f\left(x, \left(\frac{t}{w}\right)^\alpha\right) \gamma_\alpha(w) dw \quad \forall t \in I. \tag{1.13}$$

The Eq. (1.12) can also be interpreted as the application of a subordination operator in such a way $f_\alpha(x, t)$ is the subordinated density of $f(x, t)$ by means of $\nu_\alpha(s, t)$. The subordination operator is originally due to Bochner in 1955 [13]. Then, Bertoin [11] and Sato [38] studied Lévy subordinated processes. The fractional diffusions obtained by subordinators, further mathematical aspects and possible applications can be found in [26]-[30], [34], [40]. For more general subordinator, basically studying the Laplace exponent as Bernstein function, and involving also discrete subordinated processes see, for instance, Kochubei in [24].

2. The time-changed processes by Doob transform

About the first kind time-changed GM processes. In such a framework, we start by focussing on the first kind of time-changed processes as in (1.2). Indeed, by considering (1.12) and (1.13) for the case of $X_\alpha(t)$ of (1.2), we specifically have the following relation for its pdf:

$$f_{X_\alpha}(x,t) = \frac{1}{\eta(t)} \int_0^{+\infty} f_W\left(\frac{x-m(t)}{\eta(t)}, r(s)\right) \nu_\alpha(r(s), r(t)) dr(s), \; \forall t \in I \tag{2.1}$$

with $f_W(x,t)$ the pdf of the standard Brownian motion $W(t)$. After the change of variable $r(t)(r(s))^{-1/\alpha} = w$, we also have:

$$f_{X_\alpha}(x,t) = \frac{1}{\eta(t)} \int_0^{+\infty} f_W\left(\frac{x-m(t)}{\eta(t)}, \left(\frac{r(t)}{w}\right)^\alpha\right) \gamma_\alpha(w) dw \quad \forall t \in I. \tag{2.2}$$

We have to emphasize about the subordinated processes, i.e. for processes having the subordinated pdf as in (1.12) , that even for GM outer processes, the corresponding subordinated is not Gaussian. The time-changed process is not Gaussian anymore but we can give some specifications on its mean and covariance.

Indeed, the main moments of a such process are:

$$\mathbb{E}[X_\alpha(t)] = m(t), \qquad cov(X_\alpha(s), X_\alpha(t)) = \eta(t)\eta(s) cov(W_\alpha(r(s)), W_\alpha(r(t))). \tag{2.3}$$

Furthermore, we can also specify specific forms for the covariance in terms of the transforming functions $\eta(t)$ and $r(t)$.

Proposition 2.1. *By setting*

$$\zeta(s,t) = \eta(s) cov(W_\alpha(r(s)), W_\alpha(r(t))), \tag{2.4}$$

the following factorized form holds

$$cov(X_\alpha(s), X_\alpha(t)) = \eta(t)\zeta(s,t). \tag{2.5}$$

Then, if

$$r(t) = \left(\Gamma(\alpha+1)\frac{\zeta(t,t)}{\eta(t)}\right)^{1/\alpha}, \tag{2.6}$$

the $X_\alpha(t)$ covariance is, for $s \le t$,

$$cov(X_\alpha(s), X_\alpha(t)) = \frac{\eta(t)\eta(s)(r(s))^\alpha}{\Gamma(\alpha+1)}. \tag{2.7}$$

Proof. From (2.4) and the second of (2.3), Eq. (2.5) immediately follows. Furthermore, we have (cf. [22]):

$$cov(W_\alpha(r(s)), W_\alpha(r(t))) = \frac{(\min\{r(s), r(t)\})^\alpha}{\Gamma(\alpha+1)}$$

from which we derive, taking into account (2.4),

$$\zeta(s,t) = \eta(s)\frac{(\min\{r(s), r(t)\})^\alpha}{\Gamma(\alpha+1)}.$$

Then, for $s \leq t$, taking into account (2.6), we obtain that

$$cov(W_\alpha(r(s)), W_\alpha(r(t)) = \frac{(r(s))^\alpha}{\Gamma(\alpha+1)} = \frac{\zeta(s,s)}{\eta(s)} = \frac{\zeta(s,t)}{\eta(s)}.$$

Last equality implies

$$\zeta(s,t) = \eta(s)\frac{(r(s))^\alpha}{\Gamma(\alpha+1)}$$

that used in (2.5) leads to (2.7).

□

Examples of transforming functions $r(t)$ and $\eta(t)$ are $r(t) = e^{2\theta t}$ or $r(t) = (e^{2\theta t} - 1)/\theta$ and $\eta(t) = e^{-\theta t}$ with $\theta > 0$, that will be used in subsection 4.3.

We note that, even if the transformed process $X_\alpha(t)$ is neither Gaussian nor Markov, by adopting forms (2.7),(2.4) and (2.6), it preserves a factorized covariance (2.5).

Finally, we specify, from (2.2), the pdf of the process $X_\alpha(t) = m(t)+\eta(t)W_\alpha(r(t))$, i.e.

(2.8)
$$f_{X_\alpha}(x,t) = \frac{1}{\eta(t)\sqrt{2\pi(r(t))^\alpha}}\int_0^{+\infty} w^{\alpha/2}\exp\left\{-\frac{\left(\frac{x-m(t)}{\eta(t)}\right)^2}{2\left(\frac{r(t)}{w}\right)^\alpha}\right\}\gamma_\alpha(w)dw, \forall t \in I.$$

About the second kind time-changed GM processes. In order to highlight the difference between the second kind time-changed GM and the first kind one, we at first consider the subordinated pdf of processes (1.3) that, from (1.12), is the following one

$$f_{\mathfrak{X}_\alpha}(x,t) = \int_0^\infty \frac{1}{\eta(s)} f_W\left(\frac{x-m(s)}{\eta(s)}, r(s)\right)\nu_\alpha(s,t)ds \tag{2.9}$$

clearly different from (2.1). We remark that pdfs (2.1) and (2.9) are both derived from (1.12), but the integrand function $f(x,s)$ involved in (1.12) is different for each calculation due the different definition of the processes in (1.2) and (1.3).

Furthermore, about the mean of these kind of processes we have:

$$\mathbb{E}[\mathfrak{X}_\alpha(t)] = \mathbb{E}[m(E_\alpha(t))] + \mathbb{E}[\eta(E_\alpha(t))W(r(E_\alpha(t)))] = \mathbb{E}[m(E_\alpha(t))]. \tag{2.10}$$

It is easy to understand that, in case of the transforming function $m(t)$ in (1.2) is substituted by

$$m_\alpha(t) = \int_0^\infty m(s)\nu_\alpha(s,t)ds = \mathbb{E}[m(E_\alpha(t))] \tag{2.11}$$

in such a way we can consider

$$X_\alpha(t) = m_\alpha(t) + \eta(t)W_\alpha(r(t)),$$

the processes $X_\alpha(t)$ and $\mathfrak{X}_\alpha(t)$ have the same mean, i.e. $\mathbb{E}[X_\alpha(t)] = \mathbb{E}[\mathfrak{X}_\alpha(t)]$.

Then, the covariance of $\mathfrak{X}_\alpha(t)$ can be evaluated as follows:

$$\begin{aligned} cov(\mathfrak{X}_\alpha(s), \mathfrak{X}_\alpha(t)) &= cov(m(E_\alpha(s)), m(E_\alpha(t)) \\ &+ cov(\eta(E_\alpha(s))W(r(E_\alpha(s))), \eta(E_\alpha(t))W(r(E_\alpha(t)))) \end{aligned} \tag{2.12}$$

where

$$\begin{aligned} & cov(\eta(E_\alpha(s))W(r(E_\alpha(s))), \eta(E_\alpha(t))W(r(E_\alpha(t)))) \\ = \; & \mathbb{E}[\eta(E_\alpha(s))\eta(E_\alpha(t))]cov(W(r(E_\alpha(s))), W(r(E_\alpha(t)))). \end{aligned}$$

To specify completely this covariance the explicit forms of the functions $\eta(\cdot)$ and $r(\cdot)$ are finally required.

2.1. **Some comparisons.** In order to investigate relationships between special cases of first and second kind processes, we consider the transforming functions in (1.2) such that $m(t)$ as in (2.11) and

$$\eta_\alpha(t) = \int_0^\infty \eta(s)\nu_\alpha(s,t)ds$$

in such a way, we can consider

$$\bar{X}_\alpha(t) = m_\alpha(t) + \eta_\alpha(t)W_\alpha(r(t)) \tag{2.13}$$

that is a special case of (1.2). Furthermore, in order to do some comparisons, we can also consider

$$Y_\alpha(t) = m_\alpha(t) + \eta_\alpha(t)W(r(E_\alpha(t))).$$

Both processes $\bar{X}_\alpha(t)$ and $Y_\alpha(t)$ have the same mean of $\mathfrak{X}_\alpha(t)$, i.e.

$$\mathbb{E}[\bar{X}_\alpha(t)] = \mathbb{E}[Y_\alpha(t)] = \mathbb{E}[\mathfrak{X}_\alpha(t)], \quad \forall t \in I.$$

Instead, the pdf of $\bar{X}_\alpha(t)$ is

$$f_{\bar{X}_\alpha}(x,t) = \frac{1}{\eta_\alpha(t)}\int_0^{+\infty} f_W\left(\frac{x-m_\alpha(t)}{\eta_\alpha(t)}, z\right)\nu_\alpha(z,r(t))dz, \; \forall t \in I \tag{2.14}$$

obtained from (2.1) substituting z in place of $r(s)$, for r positive monotone increasing function with $r(0) = 0$ and $\lim_{t\to+\infty} r(t) = +\infty$, whereas the pdf of $Y_\alpha(t)$ is

$$f_{Y_\alpha}(x,t) = \frac{1}{\eta_\alpha(t)}\int_0^{+\infty} f_W\left(\frac{x-m_\alpha(t)}{\eta_\alpha(t)}, r(z)\right)\nu_\alpha(z,t)dz, \; \forall t \in I. \tag{2.15}$$

For $m(t) = 0$ and $\eta(t) = 1$, processes $\bar{X}_\alpha(t)$ and $Y_\alpha(t)$ reduce to $W_\alpha(r(t))$ and $W(r(E_\alpha(t)))$, respectively, and their pdf become

$$f_{W_\alpha}(x, r(t)) = \int_0^{+\infty} f_W\left(x, z\right)\nu_\alpha(z, r(t))dz \tag{2.16}$$

and

$$f_{W(r(E_\alpha))}(x,t) = \int_0^{+\infty} f_W\left(x, r(z)\right)\nu_\alpha(z,t)dz. \tag{2.17}$$

Hence, it appears clear that for $r(t) = t$, the two processes $W_\alpha(r(t))$ and $W(r(E_\alpha(t)))$ are the same process. Take in mind that, anyway, even if $r(t) = t$, the processes $X_\alpha(t)$ in (1.2) and $\mathfrak{X}_\alpha(t)$ in (1.3) differs. Obviously, for $\alpha = 1$ the processes $X_1(t)$ as in (1.2) and $\mathfrak{X}_1(t)$ as in (1.3) coincide.

Remark 2.1. *Relationships between $X_\alpha(t)$ in (1.2) and $\mathfrak{X}_\alpha(t)$ in (1.3) can be derived for specific choices of transforming functions $m(t)$, $\eta(t)$ and $r(t)$ in equations (1.2) and (1.3), respectively. Indeed, we can set differently functions in (1.2) and in (1.3), such as*

$$X_\alpha(t) = m_1(t) + \eta_1(t) W_\alpha(r_1(t)) \tag{2.18}$$

and

$$\mathfrak{X}_\alpha(t) = m_2(E_\alpha(t)) + \eta_2(E_\alpha(t)) W(r_2(E_\alpha(t))). \tag{2.19}$$

If the above transforming functions are chosen as $m_2(t) = 0$, $\eta_2(t) = 1$ and $r_2(t) = t$ in (2.19) in such a way $\mathfrak{X}_\alpha(t) = W(E_\alpha(t)) = W_\alpha(t)$, consequently we have that (2.18) becomes

$$X_\alpha(t) = m_1(t) + \eta_1(t)\mathfrak{X}_\alpha(r_1(t)). \tag{2.20}$$

Hence, in the special case $\mathfrak{X}_\alpha(t) = W_\alpha(t)$, from (2.20), we derive

$$f_{X_\alpha}(x,t) = \frac{1}{\eta_1(t)} f_{\mathfrak{X}_\alpha}\left(\frac{x - m_1(t)}{\eta_1(t)}, r_1(t)\right). \tag{2.21}$$

On the other hand, by setting the transforming functions $m_1(t) = m_2(t) = m$, $\eta_1(t) = \eta_2(t) = \eta$ and $r_2(t) = t$ in (2.18) and (2.19), one has

$$\mathfrak{X}_\alpha(r_1(t)) = X_\alpha(t) = m + \eta W_\alpha(r_1(t)). \tag{2.22}$$

Transformations between $X_\alpha(t)$ processes. Consider a GM process

$$X_D(t) = m_D(t) + \eta_D(t) W(r(t)) \tag{2.23}$$

and the transformed time-changed process obtained from it as follows

$$X_{D,\alpha}(t) = m_\alpha(t) + \eta_\alpha(t) \cdot W_\alpha(r(t)) \tag{2.24}$$

with

$$m_\alpha(t) = \int_0^\infty m_D(s)\nu_\alpha(s,t)ds, \quad \eta_\alpha(t) = \int_0^\infty \eta_D(s)\nu_\alpha(s,t)ds. \tag{2.25}$$

The process $X_{D,\alpha}(t)$ is a particular case of (1.2) when the transforming functions $m(t), \eta(t)$ in (1.2) are set in such a way $m(t) = m_\alpha(t)$ and $\eta(t) = \eta_\alpha(t)$, respectively. Then, it is also a particular case of $\bar{X}_\alpha(t)$ in (2.13) when functions $m_\alpha(t)$ and $\eta_\alpha(t)$ are defined for $m(t) = m_D(t)$ and $\eta(t) = \eta_D(t)$. Note that $X_{D,\alpha}(t)$ is different from $Y_\alpha(t)$, even if $\mathbb{E}[X_{D,\alpha}(t)] = \mathbb{E}[Y_\alpha(t)] = \mathbb{E}[\mathfrak{X}_\alpha(t).]$ We recall that for $\alpha = 1$ the two above processes coincide, i.e. $X_{D,1}(t) \equiv X_D(t)$. Now, we want to prove how it is possible to put in relation the two processes $X_\alpha(t)$ in (1.2), characterized by any transforming functions $m(t)$ and $\eta(t)$, and $X_{D,\alpha}(t)$ in (2.24), characterized by transforming functions $m_\alpha(t)$ and $\eta_\alpha(t)$ as in (2.25).

Proposition 2.2. *We have that there are functions $\tilde{m}(t)$ and $\tilde{\eta}(t)$ such that*

$$X_\alpha(t) = \tilde{m}(t) + \tilde{\eta}(t) X_{D,\alpha}(t). \tag{2.26}$$

Moreover, $X_\alpha(t) = X_{D,\alpha}(t)$ if and only if in (1.2) the transforming functions are specified as $m(t) = m_\alpha(t)$ and $\eta(t) = \eta_\alpha(t)$.

Proof. From (1.2) and (2.24), we can see that in both equations the time-changed standard Brownian motion $W_\alpha(r(t))$ is involved. From this we can write explicitly the relationships between processes $X_{D,\alpha}(t)$ and $X_\alpha(t)$, that is

$$X_{D,\alpha}(t) = m_\alpha(t) + \eta_\alpha(t)\left[\frac{X_\alpha(t) - m(t)}{\eta(t)}\right] \tag{2.27}$$

and *vice versa*

$$X_\alpha(t) = m(t) + \eta(t)\left[\frac{X_{D,\alpha}(t) - m_\alpha(t)}{\eta_\alpha(t)}\right]. \tag{2.28}$$

From the last one we recognize that:

$$X_\alpha(t) = \tilde{m}(t) + \tilde{\eta}(t) X_{D,\alpha}(t) \tag{2.29}$$

with

$$\tilde{m}(t) = m(t) - \frac{\eta(t)}{\eta_\alpha(t)} m_\alpha(t) \quad \text{and} \quad \tilde{\eta}(t) = \frac{\eta(t)}{\eta_\alpha(t)}.$$

Finally, if and only if $m(t) = m_\alpha(t)$ and $\eta(t) = \eta_\alpha(t)$, (2.29) implies that $X_\alpha(t) = X_{D,\alpha}(t)$. □

From the above proposition we put in evidence the link between the first kind time-change process with a specific GM process and we highlight how we can extend results valid for $X_\alpha(t)$ to the process $X_{D,\alpha}(t)$ by means of (2.27) for which the transformed functions are specified as above; on the other hand, we can also investigate specific properties of $X_{D,\alpha}(t)$ and by means of (2.28) to specialize these for the process $X_\alpha(t)$. Furthermore, even if in special cases, relationships between processes of $X_\alpha(t)$ of first kind and $\mathfrak{X}_\alpha(t)$ of second kind can be reciprocally exploited to investigate this classes of time-changed processes.

For such processes we aim to provide some specific results about a Fokker-Planck equation and first passage times just starting from some known results valid for GM processes. Indeed, for the general GM processes $X_D(t)$, under the assumption of differentiability of the involved transforming functions $m_D(t)$, $\eta_D(t)$ and $r(t)$, they are also diffusions ([16], [17]). Then, if random time is adopted, as the case of the inverse of a subordinator $\sigma_\alpha(t)$ is (with infinite mean [34]), the corresponding time-changed processes, among them $X_\alpha(t)$ and $\mathfrak{X}_\alpha(t)$, belong to the class of *anomalous diffusions*, also called *fractional diffusions* because their pdfs are solutions of special fractional differential equations.

Consequently, about the processes $\mathfrak{X}_\alpha(t)$, here constructed by means of the outer process GM process $X_D(t)$, i.e. $\mathfrak{X}_\alpha(t) = X_D(E_\alpha(t))$, we refer them as *$\alpha$-stable subordinated GM processes*. If all transforming functions are $C^1(I)$ these processes are included in the class of *subdiffusions*, for $\alpha \in (0,1)$. Here, we also refer to $\mathfrak{X}_\alpha(t)$ processes as *fractional diffusions*. Then, we will show that the results can be extended also to $X_\alpha(t)$.

In what follows we first recall the Fokker-Planck equation satisfied by the transition pdf of a GM process, and then we specialize it for $\mathfrak{X}_\alpha(t)$ processes.

3. The fractional pseudo-Fokker-Planck equation

The classical GM case. Referring to the GM process $X_D(t)$ as in (2.23), let $f_D(x,t|y,\tau)$ denote the normal transition pdf of X_D; more specifically,

(3.1)
$$f_D(x,t|y,\tau) = \frac{1}{\sqrt{2\pi\, var[X_D(t)|X_D(\tau)]}} \exp\left\{-\frac{(x-\mathbb{E}[X_D(t)|X_D(\tau)=y])^2}{2var[X_D(t)|X_D(\tau)]}\right\}$$

where

$$E[X_D(t)|X_D(\tau)=y] = m_D(t) + \frac{\eta_D(t)}{\eta_D(\tau)}[y - m_D(\tau)],$$
$$var[X_D(t)|X_D(\tau)] = \eta_D^2(t)[r(t)-r(\tau)].$$

The transition pdf of a GM process $X_D(t)$, i.e. $f_D(x,t|y,\tau)$, satisfies the following Fokker-Planck equation [17]:

$$\frac{\partial f_D(x,t|y,\tau)}{\partial t} = -\frac{\partial}{\partial x}\left[A_1(x,t) f_D(x,t|y,\tau)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(t) f_D(x,t|y,\tau)\right] \tag{3.2}$$

with the point source initial condition:

$$\lim_{t\to\tau} f_D(x,t|y,\tau) = \delta(x-y) \tag{3.3}$$

where $\delta(\cdot)$ is the delta function and

$$A_1(x,t) = m_D'(t) + [x - m_D(t)]\frac{\eta_D'(t)}{\eta_D(t)}, \qquad A_2(t) = (\eta_D(t))^2 r'(t) \tag{3.4}$$

are the infinitesimal moments.

About the time-changed Brownian motion $W_\alpha(t)$ and the Caputo derivative. Consider now the α-stable time changed Brownian motion $W_\alpha(t)$. We know ([21], [32], [40]) that its transition pdf $f_{W_\alpha}(x,t|y,\tau)$ satisfies the following fractional Fokker-Planck equation, with the initial condition (3.3),

$$ {}^C D_t^\alpha f_{W_\alpha}(x,t|y,\tau) = \frac{1}{2}\frac{\partial^2}{\partial x^2} f_{W_\alpha}(x,t|y,\tau) \tag{3.5}$$

where the operator ${}^C D_t^\alpha$ is the Caputo fractional derivative respect to t. The Caputo derivative of a function $f(x,t)$ can be defined by recalling the definition of the following fractional derivative:

$$\frac{\partial^\alpha f}{\partial t^\alpha} = \begin{cases} \frac{\partial f}{\partial t}(x,t), & \text{if } \alpha = 1 \\ {}^C D_t^\alpha f(x,t), & \text{if } \alpha \in (0,1) \end{cases}$$

where

$${}^C D_t^\alpha f(x,t) = \frac{1}{\Gamma(1-\alpha)}\left[\frac{\partial}{\partial t}\int_0^t f(x,\tau)(t-\tau)^{-\alpha} d\tau - \frac{f(x,0)}{t^\alpha}\right], \quad t>0,$$

is the regularized Riemann-Liouville fractional derivative, that for a C^1 function $f(t)$ is as in [32]:

$${}^C D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t f'(\tau)(t-\tau)^{-\alpha} d\tau.$$

Recall also that if f and $D^{\alpha}_{C,t}f(t)$ are Laplace transformable function, then it holds:

$$\mathcal{L}_{t\to\lambda}[{}^{C}D^{\alpha}_{t}f(t)] = \lambda^{\alpha}\mathcal{L}_{t\to\lambda}f(t) - \lambda^{\alpha-1}f(0). \tag{3.6}$$

Furthermore, the fractional Caputo derivative ${}^{C}D^{\alpha}_{t}$ can also be defined as the inverse Laplace transform of $\lambda^{\alpha}\mathcal{L}_{t\to\lambda}f(t) - \lambda^{\alpha-1}f(0)$ ([10]).

Then, considering the case of a subordinated probability density such as $f_{\alpha}(x,t) = \int_0^{\infty} f(x,s)\nu_{\alpha}(s,t)ds$, setting for short $\widehat{f}(x,\lambda) = \mathcal{L}_{t\to\lambda}f(x,t)$, we have

$$\begin{aligned}\widehat{f}_{\alpha}(x,\lambda) &= \mathcal{L}_{t\to\lambda}f_{\alpha}(x,t) = \int_0^{\infty} e^{-\lambda t}\int_0^{\infty} f(x,s)\nu_{\alpha}(s,t)dsdt \\ &= \int_0^{\infty} f(x,s)\int_0^{\infty} e^{-\lambda t}\nu_{\alpha}(s,t)dtds \\ &= \int_0^{\infty} f(x,s)\widehat{\nu}(s,\lambda)ds \\ &= \lambda^{\alpha-1}\int_0^{\infty} f(x,s)e^{-s\lambda^{\alpha}}ds = \lambda^{\alpha-1}\widehat{f}(x,\lambda^{\alpha}),\end{aligned} \tag{3.7}$$

where we used (1.11).

Some notes about the pdf of time-changed processes under a start conditioning. We consider the GM process $X_D(t)$ conditioned to start from y at initial time τ and we denote with $f_D(x,t|y,\tau)$ its pdf. Now, we specifically refer to $\mathfrak{X}_{\alpha}(t) = X_D(E_{\alpha}(t))$, for $t > \tau$, whose pdf is obtained from (1.12) as follows

$$f_{\mathfrak{X}_{\alpha}}(x,t|y,\tau) = \int_0^{\infty} f_D(x,s|y,\tau)\nu_{\alpha}(s,t)ds \tag{3.8}$$

with $f_D(x,s|y,\tau) = 0$ for $s < \tau$. We remark that the function $f_D(x,t|y,\tau)$ for Gauss-Diffusion processes ([16]) is specifically the transition Markov density. On the contrary, the $\mathfrak{X}_{\alpha}(t)$ processes are non Markov processes, (they can be semi-Markov, [7], [8]) nor (classical) diffusions. The following theorem is devoted to the above functions.

Furthermore, take into account that all involved functions in the following theorem have to be such that they belong not only to the domain of fractional Caputo derivative, i.e. $C^1(I)$ respect to the time variable t and $C^2(\mathbb{R})$ respect to space variable x, but also to the set of (time-)Laplace transformable functions and having finite inverse Laplace transforms.

Theorem 3.1. *Consider a time-changed GM process $X_D(t)$ as in (2.23) and the fractional diffusion $\mathfrak{X}_{\alpha}(t)$ for which (3.8) holds. The conditional density $f_{\mathfrak{X}_{\alpha}}(x,t|y,\tau)$ of the process $X_{\alpha}(t)$ satisfies the following fractional pseudo-Fokker-Planck equation:*

$${}^{C}D^{\alpha}_{t}f_{\alpha}(x,t|y,\tau) = \Phi_{\alpha}(f(x,t|y,\tau)) \tag{3.9}$$

where $f_{\alpha}(x,t|y,\tau)$ stands for $f_{\mathfrak{X}_{\alpha}}(x,t|y,\tau)$, $f(x,t|y,\tau)$ stands for $f_D(x,t|y,\tau)$, the operator Φ_{α} is such as:

$$\Phi_{\alpha}(f(x,t|y,\tau)) = \left[\left(-\frac{\partial}{\partial x}\mathcal{I}_{1,\alpha}f(x,t|y,\tau) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathcal{I}_{2,\alpha}f(x,t|y,\tau)\right)\right] \tag{3.10}$$

with

$$\mathcal{I}_{1,\alpha} f(x,t|y,\tau) = \int_0^\infty A_1(x,s) f(x,s|y,\tau)\nu_\alpha(s,t)ds \tag{3.11}$$

$$\mathcal{I}_{2,\alpha} f(x,t|y,\tau) = \int_0^\infty A_2(s) f(x,s|y,\tau)\nu_\alpha(s,t)ds \tag{3.12}$$

and the initial condition:

$$\lim_{t\to\tau} f_\alpha(x,t|y,\tau) = \delta(x-y). \tag{3.13}$$

Proof. Consider the Fokker-Planck equation (3.2) satisfied by the $f(x,t|y,\tau)$ and apply the Laplace transform (respect to t) to both sides of (3.2). We obtain:

$$\lambda \widehat{f}(x,\lambda|y,\tau) = \mathcal{L}_{t\to\lambda}\left(-\frac{\partial}{\partial x}\left[A_1(x,t)f(x,t|y,\tau)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(t)f(x,t|y,\tau)\right]\right). \tag{3.14}$$

Then, from (3.6) and by using the initial condition for $x \neq y$, one has

$$\mathcal{L}[{}^C D_t^\alpha f_\alpha(x,t|y,\tau)] = \lambda^\alpha \widehat{f}_\alpha(x,\lambda|y,\tau) = \lambda^{2\alpha-1}\widehat{f}(x,\lambda^\alpha|y,\tau).$$

Now, by using (3.14) in the above equation, we have:

$$\begin{aligned}
&\mathcal{L}[{}^C D_t^\alpha f_\alpha(x,t|y,\tau)] \\
&= \lambda^{\alpha-1}\mathcal{L}_{t\to\lambda^\alpha}\left(-\frac{\partial}{\partial x}\left[A_1(x,t)f(x,t|y,\tau)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[A_2(t)f(x,t|y,\tau)\right]\right) \\
&= \lambda^{\alpha-1}\left(-\frac{\partial}{\partial x}\mathcal{L}\left[A_1(x,t)f(x,t|y,\tau)\right](\lambda^\alpha) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathcal{L}\left[A_2(t)f(x,t|y,\tau)\right](\lambda^\alpha)\right) \\
&= \left(-\frac{\partial}{\partial x}\mathcal{L}\left[A_1(x,t)f(x,t|y,\tau)\right]_\alpha(\lambda) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathcal{L}\left[A_2(t)f(x,t|y,\tau)\right]_\alpha(\lambda)\right) \\
&= \left(-\frac{\partial}{\partial x}\mathcal{L}_{t\to\lambda}\mathcal{I}_{1,\alpha}f(x,t|y,\tau)(\lambda) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\mathcal{L}_{t\to\lambda}\mathcal{I}_{2,\alpha}f(x,t|y,\tau)(\lambda)\right).
\end{aligned}$$

Finally, by applying the inverse of Laplace transform to both sides, the thesis holds. □

Remark 3.1. *We remark that many authors dealt with fractional FP equations for time-changed processes (see, for instance,* [21], [22], [28], [34], [40] *and references therein). Specifically, FP-type equations was established in* [21] *for time-changed fractional Brownian motion, in* [22] *for time-changed Brownian motion with constant drift, in* [34] *and* [40] *for time-changed diffusions with non time-dependent infinitesimal moments. Note that the provided pseudo-Fokker-Planck equation* (3.9) *agrees with that in* [28] *established for an α-stable subordinated Brownian motion with a time-depending drift $F(t)$; in particular, the agreement is obtained in the specific case of the operator* (3.11) *is such that $\mathcal{I}_{1,\alpha}f(x,t|y,\tau) = F(t)f_\alpha(x,t|y,\tau)$.*

We can say that the pseudo-Fokker-Planck equation (3.9) *is a version of the fractional-type Fokker-Planck equations devoted to the case of more general time-dependent infinitesimal moments and, in particular, Theorem 3.1 shows how the infinitesimal moments of the outer GM process are involved in the operator of a such equation.*

In the following we specialize the result of Theorem 3.1 in specific cases.

Remark 3.2. *We remark that for the case in which $\mathfrak{X}_\alpha(t)$ (or $X_\alpha(t)$) is the α-stable subordinated Brownian motion $W_\alpha(t)$, Eq. (3.9) is the same of (3.5), i.e. the fractional pseudo-Fokker-Planck (FP) is exactly the fractional FP equation. Indeed, we specifically, have: $A_1(x,t)=0$, $A_2(t)=1$, $\mathcal{I}_{1,\alpha}\equiv 0$, $\mathcal{I}_{2,\alpha}f=f_\alpha$, such as*

$$\begin{aligned}\mathcal{L}\Phi_\alpha(f_W(x,t|y,\tau)) &= \left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}_{W_\alpha}(x,\lambda|y,\tau)\right)\\ &= \lambda^{\alpha-1}\left(\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}_W(x,\lambda^\alpha|y,\tau)\right)\\ &= \lambda^{2\alpha-1}\widehat{f}_W(x,\lambda^\alpha|y,\tau)=\lambda^\alpha\widehat{f}_{W_\alpha}(x,\lambda|y,\tau)\\ &= \mathcal{L}[{}^C D_t^\alpha f_{W_\alpha}(x,t|y,\tau)]\end{aligned}$$

where we also used (3.5) for $\alpha=1$, and its corresponding Laplace transformed equation, i.e.

$$\lambda\widehat{f}_W(x,t|y,\tau)(\lambda)=\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}_W(x,t|y,\tau)(\lambda).$$

For this specific case, i.e. when we refer to $\mathfrak{X}_\alpha(t)=W(E_\alpha(t))$, we denote by Φ_α^W the above corresponding operator.

Corollary 3.1. *(About the first kind time-changed processes) For the process $X_\alpha(t)$ as in Eq. (2.20) of Remark 2.1, i.e. $X_\alpha(t)=m_1(t)+\eta_1(t)W_\alpha(r_1(t))$, and from (2.21) and Theorem 3.1 we have that the conditional pdf $f_{X_\alpha}(x,t|y,\tau)$ satisfies the following fractional pseudo-Fokker-Planck equation:*

$$ {}^C D^\alpha_{r_1(t)}(f_{X_\alpha}(x,t|y,\tau))=\Phi_\alpha^W(f(x,t|y,\tau)) \tag{3.15}$$

where $f(x,t|y,\tau))$ is the conditioned pdf of the GM process $X_D(t)=m_1(t)+\eta_1(t)W(r_1(t))$, Φ_α^W is specified in Remark 3.2 and with the corresponding initial delta condition.

Proof. Under the assumptions for Eq. (2.20) in Remark 2.1, we have that $X_\alpha(t)=m_1(t)+\eta_1(t)\mathfrak{X}_\alpha(r_1(t))$ with $\mathfrak{X}_\alpha(t)=W(E_\alpha(t))$, hence

$$\begin{aligned}f_{X_\alpha}(x,t|y,\tau) &= \frac{1}{\eta_1(t)}f_{\mathfrak{X}_\alpha}\left(\frac{x-m_1(t)}{\eta_1(t)},r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)},r_1(\tau)\right)\\ &\frac{1}{\eta_1(t)}\int_0^{+\infty}f_W\left(\frac{x-m_1(t)}{\eta_1(t)},s\Big|\frac{y-m_1(t)}{\eta_1(t)},r_1(\tau)\right)\nu_\alpha(s,r_1(t))ds.\end{aligned}$$

From Theorem 3.1 we have that

$$\begin{aligned}&{}^C D^\alpha_{r_1(t)}f_{\mathfrak{X}_\alpha}\left(\frac{x-m_1(t)}{\eta_1(t)},r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)},r_1(\tau)\right)\\ &\qquad=\Phi_\alpha^W\left(f_W\left(\frac{x-m_1(t)}{\eta_1(t)},r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)},r_1(\tau)\right)\right).\end{aligned}\tag{3.16}$$

Furthermore, we recall that

$$f_W\left(\frac{x-m_1(t)}{\eta_1(t)},r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)},r_1(\tau)\right)=\eta_1(t)f(x,t|y,\tau)$$

where $f(x,t|y,\tau)$ is the conditioned pdf of $X_D(t) = m_1(t) + \eta_1(t)W(r_1(t))$. From all above identities, we obtain that

$$ {}^C D^{\alpha}_{r_1(t)} f_{\mathfrak{X}_\alpha}\left(\frac{x-m_1(t)}{\eta_1(t)}, r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)}, r_1(\tau)\right) $$
$$ = {}^C D^{\alpha}_{r_1(t)} \eta_1(t) f_{X_\alpha}(x,t|y,\tau) = \eta_1(t)\, {}^C D^{\alpha}_{r_1(t)} f_{X_\alpha}(x,t|y,\tau), $$

$$ \Phi^W_\alpha\left(f_W\left(\frac{x-m_1(t)}{\eta_1(t)}, r_1(t)\Big|\frac{y-m_1(t)}{\eta_1(t)}, r_1(\tau)\right)\right) = \eta_1(t)\Phi^W_\alpha\left(f(x,t|y,\tau)\right) $$

and the (3.15) follows. □

Corollary 3.2. *(Particular case of the previous corollary). For the process $X_\alpha(t)$ as in Eq. (2.22) of Remark 2.1, from Theorem 3.1 the conditional pdf $f_{X_\alpha}(x,t|y,\tau)$ satisfies the following fractional pseudo-Fokker-Planck equation:*

$$ {}^C D^\alpha_t f_{X_\alpha}(x,t|y,\tau) = \Phi^W_\alpha(f(x,t|y,\tau)) \tag{3.17} $$

where Φ_α is that in Remark 3.2 and with the initial delta condition.

Proof. It is sufficient to realize that, under the assumptions for Eq. (2.22) in Remark 2.1, $\mathfrak{X}_\alpha(t) = W(E_\alpha(t))$, then

$$ f_{X_\alpha}(x,t|y,\tau) = \int_0^{+\infty} \frac{1}{\eta} f_W\left(\frac{x-m}{\eta}, s\Big|\frac{y-m}{\eta}, r_1(\tau)\right)\nu_\alpha(s, r_1(t))ds $$
$$ = \frac{1}{\eta} f_{\mathfrak{X}_\alpha}\left(\frac{x-m}{\eta}, r_1(t)\Big|\frac{y-m}{\eta}, r_1(\tau)\right). $$

From Theorem 3.1 we have that

(3.18)

$$ {}^C D^{\alpha}_{r_1(t)} \frac{1}{\eta} f_{\mathfrak{X}_\alpha}\left(\frac{x-m}{\eta}, r_1(t)\Big|\frac{y-m}{\eta}, r_1(\tau)\right) = \Phi^W_\alpha\left(\frac{1}{\eta} f_W\left(\frac{x-m}{\eta}, r_1(t)\Big|\frac{y-m}{\eta}, r_1(\tau)\right)\right). $$

and, taking into account that

$$ \frac{1}{\eta} f_W\left(\frac{x-m}{\eta}, r_1(t)\Big|\frac{y-m}{\eta}, r_1(\tau)\right) = f(x,t|y,\tau) $$

where $f(x,t|y,\tau)$ is the conditioned pdf of $X_D(t) = m + \eta W(r_1(t))$, the (3.18) follows. □

Proposition 3.1. *(Time-depending drift case) For the case in which $X_D(t) = m_D(t) + W(t)$ and $\mathfrak{X}_\alpha(t) = m_D(E_\alpha(t)) + W_\alpha(t)$ with $m_D(t)$ (linear function of t), the (3.9) can be specialized in*

$$ D^\alpha_{C,t} f_\alpha(x,t|y,\tau)] = -m'_D(t)\frac{\partial}{\partial x} f_\alpha(x,t|y,\tau) + \frac{1}{2}\frac{\partial^2}{\partial x^2} f_\alpha(x,\lambda|y,\tau). \tag{3.19} $$

Proof. Starting from (3.9) with $A_1(x,t) = m'_D(t)$, $A_2(t) = 1$,

$$ \mathcal{I}_{1,\alpha} f(x,t|y,\tau) = \int_0^\infty m'_D(s) f(x,s|y,\tau)\nu_\alpha(s,t)ds $$
$$ \mathcal{I}_{2,\alpha} f(x,t|y,\tau) = \int_0^\infty f(x,s|y,\tau)\nu_\alpha(s,t)ds = f_\alpha(x,t|y,\tau). $$

we obtain

$$\begin{aligned}\mathcal{L}\Phi_\alpha(f(x,t|y,\tau)) &= \left(-\frac{\partial}{\partial x}\mathcal{L}_{t\to\lambda}\,\mathcal{I}_{1,\alpha}f(x,t|y,\tau)(\lambda)+\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}_\alpha(x,\lambda|y,\tau)\right)\\ &= \lambda^{\alpha-1}\left(-\frac{\partial}{\partial x}\mathcal{L}_{t\to\lambda^\alpha}(m'_D(t)f(x,t|y,\tau))(\lambda^\alpha)+\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}(x,\lambda^\alpha|y,\tau)\right)\\ &= \lambda^{2\alpha-1}\widehat{f}(x,\lambda^\alpha|y,\tau)=\lambda^\alpha\widehat{f}_\alpha(x,\lambda|y,\tau)\\ &= \mathcal{L}[{}^C D_t^\alpha f_\alpha(x,t|y,\tau)] \qquad (3.20)\end{aligned}$$

where we also used (3.2) valid for the GM process $X_D(t)$ and its corresponding Laplace transformed equation, i.e.

$$\lambda^\alpha\widehat{f}(x,\lambda^\alpha|y,\tau)=-\frac{\partial}{\partial x}\mathcal{L}_{t\to\lambda^\alpha}(m'_D(t)f(x,t|y,\tau))(\lambda^\alpha)+\frac{1}{2}\frac{\partial^2}{\partial x^2}\widehat{f}(x,\lambda^\alpha|y,\tau).$$

Finally, if $m_D(t)$ is a linear function of t, we can specify that (3.20) leads to the case of (3.19). □

4. First passage times

About the FPT of a GM process ([17]) $X_D(t)=m_D(t)+\eta_D(t)W(r(t))$, we recall the following formula:

$$g_{X_D}(S(t),t|x_0,t_0)=\frac{dr(t)}{dt}g_W(S^*(r(t)),r(t)|x_0^*,r(t_0)) \qquad (4.1)$$

where $g_{X_D}(S(t),t|x_0,t_0)$ is the pdf of the FPT

$$\mathcal{T}=\inf\{\theta>0: X_D(\theta)>S(\theta)\}, \qquad (4.2)$$

with $S(t)$ a $C^1(I)$-boundary, and $g_W(S^*(r(t)),r(t)|x_0^*,r(t_0))$ is the pdf of the FPT of the standard Brownian motion W, i.e. the FPT pdf of

$$\mathcal{T}_W=\inf\{\theta>0: W(r(\theta))>S^*(r(\theta))\}$$

with

$$S^*(r(t))=\frac{S(t)-m_D(t)}{\eta_D(t)},\qquad x_0^*=\frac{x_0-m_D(t_0)}{\eta_D(t_0)}. \qquad (4.3)$$

In order to investigate this problem, here, we first consider the *subordinated* FPT $\mathfrak{T}_\alpha$ defined as the random variable having with pdf $g_\alpha(S(t),t|x_0,t_0)$ such that

$$g_\alpha(S(t),t|x_0,t_0)=\int_0^\infty g(S(\theta),\theta|x_0,t_0)\nu_\alpha(\theta,t)d\theta \qquad (4.4)$$

where $g\equiv g_{X_D}$ is the FPT of X_D process. $\mathfrak{T}_\alpha$ is the subordinated FPT of the process $\mathfrak{X}_\alpha(t)=X_D(E_\alpha(t))$ to the boundary $S(t)$.

Note that, in (4.4), $g(S(\theta),\theta|x_0,t_0)$ is equal to zero for $\theta<t_0$, henceforth the evaluation of the integral at the RHS of (4.4) is on the domain $(t_0,+\infty)$, even if we leave it as in (4.4) due to $t_0\in(0,+\infty)$.

Hence, in order to investigate the first passage times of $X_\alpha(t)$ processes, we at first focus on main subordinated processes $W_\alpha(t)$ and $X_D(E_\alpha(t))$; we specialize some FPT results, such as those in [16], [17], and we recall those already known and then we give specific results about the subordinated first passage times.

4.1. **An integral equation for the subordinated first passage time.** We recall the following result holding for FPT of GM processes ([17]). Consider a GM process with the Doob representation, i.e. $X_D(t) = m_D(t) + \eta_D(t) W(r(t))$. For this process, we also consider a function $\zeta_D(t)$ such that that $\zeta_D(t) = r(t)\eta_D(t)$. Let $f[x, t|y, \tau]$ be its normal transition pdf as in (3.1) and

$$g[S(t), t|x_0, t_0] = \frac{d}{dt} P(\mathcal{T} \leq t)$$

be the pdf of the FPT defined in (4.2). Let $S(t), m_D(t), \eta_D(t), \zeta_D(t), r(t)$ be $C^1(I)$ functions. Then, $g[S(t), t|x_0, t_0]$ satisfies the following nonsingular second-kind Volterra integral equation

$$g[S(t), t|x_0, t_0] = -2\Psi[S(t), t|x_0, t_0] + 2\int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0]\, \Psi[S(t), t|S(\tau), \tau]\, d\tau \tag{4.5}$$

$$\big(x_0 < S(t_0)\big)$$

where

$$\Psi[S(t), t|y, \tau] = \left\{ \frac{S'(t) - m'_D(t)}{2} - \frac{S(t) - m_D(t)}{2} \, \frac{\zeta'_D(t)\eta_D(\tau) - \eta'_D(t)\zeta_D(\tau)}{\zeta_D(t)\eta_D(\tau) - \eta_D(t)\zeta_D(\tau)} \right.$$

$$\left. - \frac{y - m_D(\tau)}{2} \, \frac{\eta'_D(t)\zeta_D(t) - \eta_D(t)\zeta'_D(t)}{\zeta_D(t)\eta_D(\tau) - \eta_D(t)\zeta_D(\tau)} \right\} f[S(t), t|y, \tau]. \tag{4.6}$$

Theorem 4.1. *For the $\mathfrak{X}_\alpha(t)$ process, consider the α-stable subordinated pdf*

$$g_\alpha(S(t), t|x_0, t_0) = \int_0^\infty g(S(\theta), \theta|x_0, t_0)\nu_\alpha(\theta, t) d\theta \tag{4.7}$$

where $g[S(t), t|x_0, t_0]$ is the FPT pdf of a GM X_D process. Let $\mathfrak{T}_\alpha$ be the subordinated FPT with pdf $g_\alpha(S(t), t|x_0, t_0)$. Then, $g_\alpha(S(t), t|x_0, t_0)$ satisfies the following equation

$$g_\alpha[S(t), t|x_0, t_0] = -2\Psi_\alpha[S(t), t|x_0, t_0] + 2\mathfrak{I}_{\Psi_\alpha}^{t_0,t} g[S(t), t|x_0, t_0] \tag{4.8}$$

where the integral operator is defined as follows

$$\mathfrak{I}_{\Psi_\alpha}^{t_0,t} g[S(t), t|x_0, t_0] = \int_{t_0}^{t} g[S(\tau), \tau|x_0, t_0]\Psi_\alpha[S(t), t|S(\tau), \tau] d\tau \tag{4.9}$$

and

$$\Psi_\alpha[S(t), t|y, \tau] = \int_0^\infty \Psi[S(\theta), \theta|y, \tau]\nu_\alpha(\theta, t) d\theta. \tag{4.10}$$

Proof. By inserting in the right hand side of (4.7) the expression of g as at the right hand side of (4.5), we have

$$g_\alpha(S(t), t|x_0, t_0) = -2\Psi_\alpha[S(t), t|y, \tau] \tag{4.11}$$

$$+2\int_0^\infty \int_{t_0}^{\theta} g[S(\tau), \tau|x_0, t_0]\Psi[S(\theta), \theta|S(\tau), \tau] d\tau \nu_\alpha(\theta, t) d\theta \tag{4.12}$$

Under assumption that all involved functions are L^1 on their domains, Fubini theorem can be applied to the integral term at the right hand side, in such a way one has

$$\int_0^{\infty}\int_{t_0}^{\theta} g[S(\tau),\tau|x_0,t_0]\Psi[S(\theta),\theta|S(\tau),\tau]d\tau\nu_\alpha(\theta,t)d\theta \tag{4.13}$$

$$= \int_{t_0}^{t} g[S(\tau),\tau|x_0,t_0]\int_0^{\infty}\Psi[S(\theta),\theta|S(\tau),\tau]\nu_\alpha(\theta,t)d\theta d\tau \tag{4.14}$$

$$= \mathfrak{I}^{t_0,t}_{\Psi_\alpha} g[S(\tau),\tau|x_0,t_0] \tag{4.15}$$

where we used (4.10) and also that

$$\int_0^{\infty}\Psi[S(\theta),\theta|S(\tau),\tau]\nu_\alpha(\theta,t)d\theta = 0, \quad \tau > t,$$

□

Note that the equation (4.8) allows to obtain numerical approximations of g_α for general $C^1(0,+\infty)$ boundary $S(t)$, for which no closed form results are available. This can be done by using numerical procedure for coupled integral equations (4.5) and (4.8). Indeed, in cases in which g is unknown, the main advantage respect to the direct numerical quadrature of (4.7) is that generally both functions Ψ and Ψ_α involved in (4.5) and (4.8) can be analytically evaluated, and in addition the integration intervals for the numerical quadrature in (4.5) and (4.8) are limited.

Corollary 4.1. *Under the assumptions of Theorem 4.1, if the boundary* $S(t)$ *is such that*

$$S(t) = m_D(t) + a\zeta_D(t) + b\eta_D(t) \qquad \forall t \in I,\ a,b \in \mathbb{R}. \tag{4.16}$$

the subordinated FPT pdf $g_\alpha[S(t),t|x_0,t_0]$, *for* $x_0 < S(t_0)$, *can be written as follows*

$$g_\alpha[S(t),t|x_0,t_0] = \frac{S(t_0)-x_0}{\eta_D(t_0)}\int_0^{\infty}\frac{\eta_D(\theta)r'(\theta)}{r(\theta)-r(t_0)}f[S(\theta),\theta|x_0,t_0]\nu_\alpha(\theta,t)d\theta, \tag{4.17}$$

where $f[S(\theta),\theta|x_0,t_0]$ *is the normal transition pdf of the corresponding GM process* X_D *as given in* (3.1), *and* $r(t) = \zeta_D(t)/\eta_D(t)$.[1]

Proof. From Theorem 3.2 of [17], we recall that, specifically referring to $W(r(t))$,

$$\Psi_W[S(r(t)),r(t)|S(r(\tau)),r(\tau)] = 0, \quad \forall\tau,t\in I,\ \tau\le t$$

iff

$$S(r(t)) = ar(t) + b \quad \forall t \in I,\ a,b\in\mathbb{R}.$$

This result implies that

$$\Psi[S(t),t|S(\tau),\tau] = 0, \quad \forall\tau,t\in I,\ \tau\le t$$

iff

$$\begin{aligned} S(t) &= m_D(t) + \eta_D(t)S(r(t)) = m_D(t) + \eta_D(t)(ar(t)+b) \\ &= m_D(t) + \eta_D(t)\left(a\frac{\zeta_D(t)}{\eta_D(t)} + b\right) = m_D(t) + a\zeta_D(t) + b\eta_D(t). \end{aligned}$$

Hence, recalling (4.10), we have that if the boundary is (4.16), we have

$$\Psi_\alpha[S(t),t|S(\tau),\tau] = 0, \quad \forall\tau,t\in I.$$

[1] Note that $\zeta_D(t) \equiv \zeta(t,t)$ of Proposition 2.1.

Consequently, the operator $\mathfrak{I}_{\Psi_\alpha}^{t_0,t}$ (4.9), involved in (4.8), is identically zero. For the boundary (4.16), making use of Theorem 4.1, the integral equation (4.8) reduces to the following one

$$g_\alpha[S(t),t|x_0,t_0] = -2\,\Psi_\alpha[S(t),t|x_0,t_0] \tag{4.18}$$

where Ψ_α is defined in (4.10) with Ψ as in (4.6). Now, by substituting (4.16) in $\Psi[S(t),t|x_0,t_0]$, specialized from (4.6) with $y=x_0$, $\tau=t_0$, we obtain (cf. Corollary 3.1 of [17])

$$\Psi[S(t),t|x_0,t_0] = -\frac{1}{2}\frac{S(t_0)-x_0}{r(t)-r(t_0)}\frac{\eta_D(t)}{\eta_D(t_0)}r'(t)f[S(t),t|x_0,t_0],$$

that substituted in (4.10), written for $y=x_0$, $\tau=t_0$, leads to $\Psi_\alpha[S(t),t|x_0,t_0]$, and finally (4.17) follows from (4.18). □

An advanced investigation about other possible transformed closed forms such as those related with one-side and two-side Daniels-type boundaries ([17], [35]), but also additional asymptotic results ([7], [18]) will be the object of a future work.

4.2. **FPT density for time-changed Brownian motion.**

Due its central rule in this class of the fractional diffusions, now we point out some specific results about the FPT density of the time-changed Brownian motion $W_\alpha(t)=W(E_\alpha(t))$.

Proposition 4.1. *For the time-changed Brownian motion $W_\alpha(t)$ in presence of a linear boundary $S(t)=at+b$, with $b>x_0$, and $\forall t\geq t_0$, the FPT density is*

$$g_\alpha[at+b,t|x_0,t_0] = \\ = -af_\alpha[at+b,t|x_0,t_0] + \int_0^\infty\left(\frac{a\theta+b-x_0}{\theta-t_0}\right)f[a\theta+b,\theta|x_0,t_0]\nu_\alpha(\theta,t)d\theta \tag{4.19}$$

where $f_\alpha[at+b,t|x_0,t_0]$ is as in (1.12) *with*

$$f[a\theta+b,\theta|x_0,t_0] = \frac{1}{\sqrt{2\pi(\theta-t_0)}}\exp\left\{-\frac{(a\theta+b-x_0)^2}{2(\theta-t_0)}\right\}.$$

that is (3.1) *specialized for the standard Brownian motion $W(t)$.*

Proof. For the time-changed Brownian motion $W_\alpha(t)=W(E_\alpha(t))$ in presence of a linear boundary $S(t)=at+b$, with $b>x_0$, (4.17) can be re-written with $r(t)=t,\eta(t)=1$, to obtain

$$g_\alpha[at+b,t|x_0,t_0] = -2\,\Psi_\alpha[at+b,t|x_0,t_0] = \\ -2\int_0^\infty \Psi^W[a\theta+b,t|x_0,t_0]\nu_\alpha(\theta,t)d\theta \\ = -\int_0^\infty\left(a-\frac{a\theta+b-x_0}{\theta-t_0}\right)f[a\theta+b,\theta|x_0,t_0]\nu_\alpha(\theta,t)d\theta.$$

from which Eq. (4.19) follows.

□

Note that (4.19) for $\alpha=1$ coincides with the well-known formula due to Bachelier-Lévy (see, for instance, [1]), i.e. the Wald density

$$g[at+b,t|x_0,t_0] = \frac{|at_0+b-x_0|}{(t-t_0)^{3/2}}\phi\left(\frac{at+b-x_0}{\sqrt{t-t_0}}\right), \tag{4.20}$$

with $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$.

Remark 4.1. *We note that in the case of the linear boundary it is also possible to proceed as follows. At first, we remark that we can use* (4.20) *in* (4.7)*. Indeed, just for the FPT of the time-changed Brownian motion $W(E_\alpha(t))$ for the linear boundary $S(t) = at + b$, $(a, b \in \mathbb{R})$, we have that*

(4.21)
$$g_\alpha(S(t), t|x_0, t_0) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{|at_0 + b - x_0|}{(\theta - t_0)^{3/2}} \exp\left\{-\frac{(a\theta + b - x_0)^2}{2(\theta - t_0)}\right\} \nu_\alpha(\theta, t) d\theta$$

equivalent to (4.19)*.*

In particular, from (4.21), referring to the zero-boundary (i.e. for $0 = S(t) = at + b$ with $a = b = 0$), we can write the FPT pdf of the time-changed Brownian motion (for $x_0 > 0$, $t_0 = 0$), as the following

$$g_\alpha(0, t|x_0, 0) = \frac{x_0}{\sqrt{2\pi}} \int_0^\infty \frac{1}{\theta^{3/2}} \exp\left\{-\frac{x_0^2}{2\theta}\right\} \nu_\alpha(\theta, t) d\theta. \tag{4.22}$$

Moreover, by using (1.10) and the suitable change of variable as used in (1.13)

$$g_\alpha(0, t|x_0, 0) = \frac{x_0}{\sqrt{2\pi}} \int_0^\infty \left(\frac{w}{t}\right)^{\frac{3\alpha}{2}} \exp\left\{-\frac{x_0^2}{2}\left(\frac{w}{t}\right)^\alpha\right\} \gamma_\alpha(w) dw \tag{4.23}$$

with $\gamma_\alpha(w)$ the density of the α-stable subordinator.

For $0 < \alpha \leq 1$, the function $\gamma_\alpha(\cdot)$ in (4.23) can be numerically evaluated by means of R library routines. Specifically, this library allows to call the function *dstable* to evaluate the density $\gamma_\alpha(w)$ of a stable subordinator (see, [31]). For this case we implemented our R codes providing some numerical approximations plotted in Fig.1. Alternatively, it is possible to obtain further numerical approximations by using a series expansion for $\gamma_\alpha(w)$ such as (cf. [39])

$$\gamma_\alpha(w) = \sum_{i=1}^{+\infty} \frac{(-1)^{n-1}}{n!} \frac{\Gamma(\alpha n + 1)}{\Gamma(\alpha n)\Gamma(1 - \alpha n)} w^{-\alpha n - 1} \tag{4.24}$$

or by using its asymptotic behaviors that can be found, for instance, in [32]. In a preliminary numerical investigation, we can say that the use of the serie expansion (4.24) can be well exploited with the first $n = 100$ summands because this is a right balance between very short running times and order of accuracy of results. Furthermore, we remark that the numerical resolution of (4.8) constitutes a valid and general strategy to evaluate the FPT density for this process but also for the general case. A more detailed and comparative investigation about different approximation strategies will be done in a future work.

Finally, we stress the importance of the provided FPT results for time-changed Brownian process because this process is directly involved in the construction of fractional diffusions here constructed by the Doob transform (1.2). In addition, the above FPT results can be exploited for the class of time-changed processes X_α of the first kind as the following way.

Proposition 4.2. *The FPT density through a boundary $S(t)$ of the $X_\alpha(t) = m(t) + \eta(t) W_\alpha(r(t))$ process can be derived in the following way:*

$$g_{X_\alpha}(S(t), t|x_0, t_0) = \frac{dr(t)}{dt} g_{W_\alpha}(S^*(r(t)), r(t)|x_0^*, r(t_0)) \tag{4.25}$$

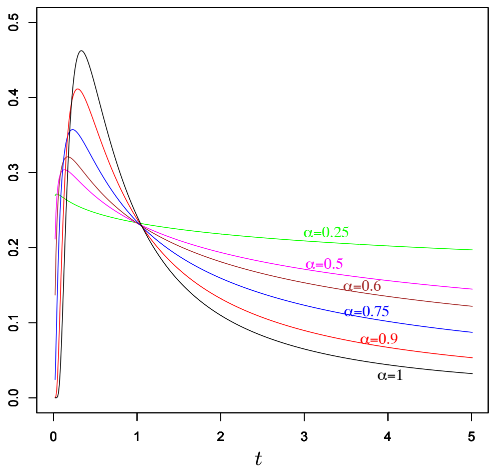

FIGURE 1. Numerical evaluations of FPT pdf $g_\alpha(0,t|x_0,0)$ as in (4.23) of the time-changed Brownian motion through zero-boundary for some values of α. For all plots we set $x_0=1$.

with

$$S^*(r(t)) = \frac{S(t)-m(t)}{\eta(t)}, \qquad x_0^* = \frac{x_0-m(t_0)}{\eta(t_0)} \tag{4.26}$$

with $g_{W_\alpha}(S^*(r(t)), r(t)|x_0^*, r(t_0))$ *is the FPT pdf of* $W_\alpha(t)$.

Proof. Following the strategy of derivation of (4.1) and (4.3) for the classical case (see [17]), also in this case (4.25) and (4.26) are obtained.

Note that the FPT pdf $g_{W_\alpha}(S^*(r(t)), r(t)|x_0^*, r(t_0))$, if it is not known in closed form, can be evaluated by means of numerical quadrature strategies applied to the integral equation (4.8) such as those in [16] or [17]. □

On the other hand, the already known results for the FPT of the time-changed Brownian motion through a specified boundary can be extended to the process X_α for a corresponding boundary as specified in the following proposition.

Proposition 4.3. *When the FPT pdf* $g_{W_\alpha}(S(\vartheta),\vartheta|x_0,\vartheta_0)$ *for* $W_\alpha(t)$ *through the boundary* $S(\vartheta)$ *is available, the following transformation formulae are useful to specify FPT pdf for the process* $X_\alpha(t)$*:*

$$g_{X_\alpha}(\tilde{S}(r^{-1}(\vartheta)), r^{-1}(\vartheta)|\tilde{x}_0, r^{-1}(\vartheta_0)) = \frac{g_{W_\alpha}(S(\vartheta),\vartheta|x_0,\vartheta_0)}{\frac{dr^{-1}(\vartheta)}{d\vartheta}} \tag{4.27}$$

with

$$\tilde{S}(r^{-1}(\vartheta)) = m(r^{-1}(\vartheta)) + \eta(r^{-1}(\vartheta))S(\vartheta), \;\; \tilde{x}_0 = m(r^{-1}(\vartheta_0)) + \eta(r^{-1}(\vartheta_0))x_0. \tag{4.28}$$

Proof. These formulae are derived by inverting (4.25) and (4.26). □

In the next subsection we give appication examples of (4.25)-(4.26) and (4.27)-(4.28).

4.3. **FPT density for time-changed OU-type processes.** At first, we specify a consequence of the previous FPT results on a time-changed OU process. Consider, at first, the stationary OU process such that can be written as the following GM process, for $t \in \mathbb{R}$,

$$U(t) = m(t) + e^{-\theta t} W(e^{2\theta t}) \tag{4.29}$$

and its of second kind time-changed version $\mathfrak{U}_\alpha(t) = U(E_\alpha(t))$.

Corollary 4.2. *For the stationary time-changed OU process* $U(E_\alpha(t))$ *the FPT pdf through the boundary* $S(t) = m(t) + ae^{\theta t} + be^{-\theta t}$ *is for* $t \geq t_0$

$$g_{\mathfrak{U}_\alpha}[S(t), t|u_0, t_0] = e^{\theta t_0}(S(t_0) - u_0) \int_0^\infty \frac{2\theta e^{\theta \xi}}{e^{2\theta \xi} - e^{2\theta t_0}} f[S(\xi), \xi|u_0, t_0] \nu_\alpha(\xi, t) d\xi \tag{4.30}$$

where

$$f[S(\xi), \xi|u_0, t_0] = \frac{1}{\sqrt{2\pi[1 - e^{-\theta(\xi - t_0)}]}} \times \exp\left\{ -\frac{[S(\xi) - m(\xi) - e^{-\theta(\xi - t_0)}(u_0 - m(t_0))]^2}{2[1 - e^{-\theta(\xi - t_0)}]} \right\}, \tag{4.31}$$

the latter obtained from (3.1) *adapted to the case of the considered OU process.*

Proof. Applying the Corollary 4.1 to the process $U(E_\alpha(t))$, we recall that it is the time-changed version of the GM process X_D as in (2.23) with transforming functions $r(t) = e^{2\theta t}$ and $\eta_D(t) = e^{-\theta t}$, and normal transition density $f[s, t|y, \tau]$ as in (3.1). Hence, the corresponding boundary is just the hyperbolic-type boundary $S(t) = m(t) + ae^{\theta t} + be^{-\theta t}$, being $\zeta_D(t) = e^{\theta t}$. Finally, from (4.17), we have (4.30). □

Furthermore, it is also possible to consider a non-stationary time-changed OU process $\{U_0(t), t \geq 0\}$ solution of the following stochastic differential equation, for $t \geq 0$, and $u_0 \in \mathbb{R}$,

$$dU_0(t) = -\theta U_0(t) dt + dW(t), \quad U_0(0) = u_0. \tag{4.32}$$

For $\theta > 0$, the case we consider, the process is recurrent and hence its FPT for any constant level $b > u_0$ is finite with probability one. It is interesting to recall that for this process a closed form of FPT pdf through the level $b = 0$ is available (see formula (2.8) in [2]), i.e.

$$g_{U_0}(0, t|u_0, 0) = \frac{|u_0|}{\sqrt{2\pi}} \left(\frac{2\theta}{e^{\theta t} - e^{-\theta t}} \right)^{3/2} \exp\left(-\frac{2\theta u_0^2}{e^{2\theta t} - 1} + \frac{\theta t}{2} \right) \tag{4.33}$$

due originally to Breiman [15], and Pitman and Yor [36].

In addition, $U_0(t)$ is the GM process with the following representation by the Brownian motion:

$$U_0(t) = m_{U_0}(t) + \eta_{U_0}(t) W(r(t) - r(0)) = u_0 e^{-\theta t} + e^{-\theta t} W\left(\frac{e^{2\theta t} - 1}{2\theta} \right) \tag{4.34}$$

with $r(t) = e^{2\theta t}/(2\theta)$.

We note that the formula (2.8) of [2] was determined starting from the well-known formula of FPT pdf of the Brownian motion $\{W(\tau), \tau \geq 0\}$ through a constant level $-u_0 > 0$ (see (4.20) with $a = 0$ and $b = -u_0$), taking into account (4.34), the inverse of transform (4.34), i.e.

$$W(\tau) = \sqrt{2\theta\tau + 1}\left[U_0\left(\frac{\log(2\theta\tau + 1)}{2\theta}\right)\right] - u_0, \tag{4.35}$$

and the analogous of relation (4.1), i.e.

$$g_{U_0}(S, t|u_0, 0) = \frac{d\rho(t)}{dt} g_W(S^*(\rho(t)), \rho(t)|x_0^*, \rho(0)) \tag{4.36}$$

with

$$S^*(\rho(t)) = \frac{S - m_{U_0}(t)}{\eta_{U_0}(t)}, \qquad x_0^* = \frac{x_0 - m_{U_0}(0)}{\eta_{U_0}(0)} \tag{4.37}$$

and $\rho(t) = r(t) - r(0)$. For the specific case of (4.33), in (4.37) with $S = 0, m_{U_0}(t) = u_0 e^{-\theta t}, \eta_{U_0}(t) = e^{-\theta t}$, we have

$$S^*(\rho(t)) = -u_0, \qquad x_0^* = 0 > u_0. \tag{4.38}$$

Note that a first kind time-changed version of such a process is

$$U_{0,\alpha}(t) = \qquad m_{U_0}(t) + \eta_{U_0}(t) W_\alpha\left(\frac{e^{2\theta t} - 1}{2\theta}\right). \tag{4.39}$$

It can obtain another version, possibly connected to the previous one according to Proposition 2.2, as follows

$$\begin{aligned} \bar{U}_{0,\alpha}(t) = &\qquad m_\alpha(t) + \eta_\alpha(t) W_\alpha\left(\frac{e^{2\theta t} - 1}{2\theta}\right) \\ = &\qquad u_0 \mathcal{E}(-\theta t^\alpha) + \mathcal{E}(-\theta t^\alpha) W(E_\alpha(\rho(t))) \end{aligned} \tag{4.40}$$

where $\mathcal{E}(\cdot)$ is the Mittag-Leffler function and $\rho(t) = \frac{e^{2\theta t}-1}{2\theta}$.

Regarding the second kind time-changed version $\mathfrak{U}_{0,\alpha}(t) = U_0(E_\alpha(t))$, the related FPT pdf $g_{\mathfrak{U}_{0,\alpha}}(S, t|u_0, 0)$, we know that it can be determined by its definition:

$$g_{\mathfrak{U}_{0,\alpha}}(S, t|u_0, 0) = \int_0^\infty g_{U_0}(S, \xi|u_0, 0) \nu_\alpha(\xi, t) d\xi. \tag{4.41}$$

Indeed, we can write the following formula for the case $S = 0$, by inserting (4.33) in (4.41), i.e.

(4.42)

$$g_{\mathfrak{U}_{0,\alpha}}(0, t|u_0, 0) = \int_0^\infty \frac{|u_0|}{\sqrt{2\pi}} \left(\frac{2\theta}{e^{\theta\xi} - e^{-\theta\xi}}\right)^{3/2} \exp\left(-\frac{2\theta u_0^2}{e^{2\theta\xi} - 1} + \frac{\theta\xi}{2}\right) \nu_\alpha(\xi, t) d\xi.$$

By using the form of $\nu_\alpha(\xi, t)$, as in (1.10), and the suitable change of variable $w = (t\xi^{-1/\alpha})$, as used in (1.13) , we finally have that (4.42) can also be alternatively written as follows:

(4.43)

$$g_{\mathfrak{U}_{0,\alpha}}(0, t|u_0, 0) = \int_0^\infty \frac{|u_0|}{\sqrt{2\pi}} \left(\frac{2\theta}{e^{\theta(\frac{t}{w})^\alpha} - e^{-\theta(\frac{t}{w})^\alpha}}\right)^{3/2} \exp\left(-\frac{2\theta u_0^2}{e^{2\theta(\frac{t}{w})^\alpha} - 1} + \frac{\theta(\frac{t}{w})^\alpha}{2}\right) \gamma_\alpha(w) dw$$

where $\gamma_\alpha(w)$ is the density of the α-stable subordinator.

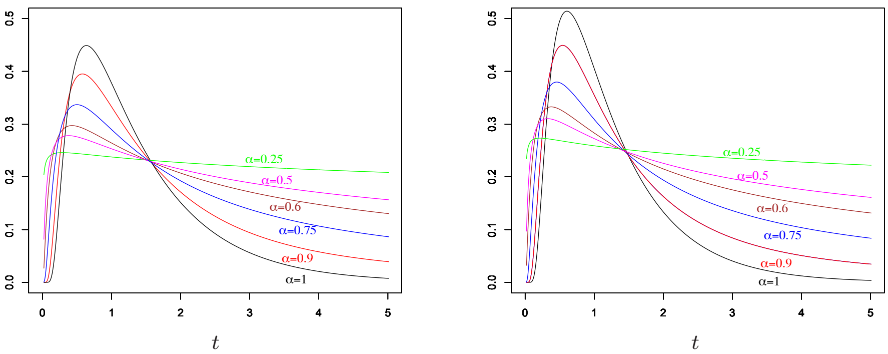

FIGURE 2. Numerical evaluations of FPT pdf $g_{\mathfrak{U}_0,\alpha}(0,t|u_0,0)$ (4.43) of the time-changed OU process $U_0(E_\alpha(t))$ through zero-boundary for some values of α. For all plots we set $u_0 = 1$ and $\theta = 1$ (on the left) and $\theta = 1.2$ (on the right).

In Fig. 2 we plot numerical evaluations of FPT pdf (4.43) of the time-changed OU process $U_0(E_\alpha(t))$ through zero-boundary for some values of α and θ. These evaluations are obtained by ad hoc R-codes we devised.

Before to give the following corollary, we remark that (4.42), due to (4.36), has been obtained by the following integration

$$g_{\mathfrak{U}_0,\alpha}(S,t|u_0,0) = \int_0^\infty d\rho(\xi) g_W(S^*(\rho(\xi)),\rho(\xi)|x_0^*,\rho(0))\nu_\alpha(\xi,t). \tag{4.44}$$

Instead, for the first kind time-changed process $U_{0,\alpha}(t)$, from Proposition 4.2, we have

$$g_{U_{0,\alpha}}(S,t|u_0,0) = \frac{d\rho(t)}{dt} g_{W_\alpha}(S^*(\rho(t)),\rho(t)|x_0^*,\rho(0)) \tag{4.45}$$

$$= \frac{d\rho(t)}{dt}\int_0^\infty g_W(S^*(\rho(t)),\xi|x_0^*,\rho(0))\nu_\alpha(\xi,\rho(t))d\xi. \tag{4.46}$$

In particular, for a constant boundary $S^*(\rho(t)) = S^*$ and $\rho(0) = 0$, one has

$$g_{U_{0,\alpha}}(S,t|u_0,0) = \frac{d\rho(t)}{dt}\int_0^\infty g_W(S^*,\xi|x_0^*,0)\nu_\alpha(\xi,\rho(t))d\xi. \tag{4.47}$$

Hence, for this specific case, we give the following corollary.

Corollary 4.3. *For the FPT pdf of $U_{0,\alpha}(t)$ process we have the following expression*

$$g_{U_{0,\alpha}}(0,t|u_0,0) = \frac{e^{2\theta t}}{\sqrt{2\pi}}\int_0^\infty \frac{|u_0|}{\xi^{3/2}}\exp\left\{-\frac{u_0^2}{2\xi}\right\}\nu_\alpha\left(\xi,\frac{e^{2\theta t}-1}{2\theta}\right)d\xi. \tag{4.48}$$

Proof. Indeed, by applying Proposition 4.2, if the FPT pdf for the time-changed Brownian motion through the corresponding boundary $S^*(\rho(t))$ in (4.37) was known, also the FPT pdf for the first kind time-changed OU process $U_0(E_\alpha(t))$ can be obtained by using (4.25). For this specific case, we have $S = 0$, $S^*(\rho(t)) = -u_0$, $x_0^* = 0$, $\rho(t) = (e^{2\theta t} - 1)/(2\theta)$ and $\rho(0) = 0$, in such a way

$$g_{U_0,\alpha}(0, t|u_0, 0) = e^{2\theta t} g_{W_\alpha}\left(-u_0, \frac{e^{2\theta t} - 1}{2\theta}\Big|0, 0\right) \tag{4.49}$$

where, from (4.21),

$$\begin{aligned} & g_{W_\alpha}\left(-u_0, \frac{e^{2\theta t} - 1}{2\theta}\Big|0, 0\right) \\ & = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{|u_0|}{\xi^{3/2}} \exp\left\{-\frac{u_0^2}{2\xi}\right\} \nu_\alpha\left(\xi, \frac{e^{2\theta t} - 1}{2\theta}\right) d\xi \end{aligned} \tag{4.50}$$

and finally (4.48) follows by inserting (4.50) in (4.49). □

Note that (4.48) is an example of application of the transforming formulae (4.25) and (4.26) of Proposition 4.2 between FPT densities. Finally, referring to (4.34), we can also give an example of application of the formulae (4.27) and (4.28) of Proposition 4.3 by which the FPT density of time changed OU process is derived from that of time-changed Brownian motion through zero-boundary as in (4.22); the FPT obtained is related to a different boundary. Specifically, we have:

$$g_{U_0,\alpha}\left(\frac{u_0}{\sqrt{1 + 2\theta t}}, \frac{\log(1 + 2\theta t)}{2\theta}\Big|u_0 + x_0, 0\right) = e^{2\theta t} g_{W_\alpha}\left(0, t\Big|x_0, 0\right). \tag{4.51}$$

Hence, by using $g_{W_\alpha}\left(0, t\Big|x_0, 0\right)$ as in (4.22) in (4.51), we immediately have the available FPT of the time-changed OU process through the boundary $\frac{u_0}{\sqrt{1+2\theta t}}$ in the logarithmic time $\frac{\log(1+2\theta t)}{2\theta}$.

An advanced study of the numerical approximations of all above equations, focussing on those derived from a numerical resolution of (4.8) and simulations will be the object of a future work.

Appendix

We give some additional details about the subordinator and its inverse process considered in the subsection 1.1. In particular, we can specify the density of $\sigma_\alpha(t)$ involved in (1.10) as follows (([37]))

$$\gamma_\alpha(x, t) = \frac{1}{\alpha t x^{1+\alpha}} M_\alpha\left(\frac{t}{x^\alpha}\right)$$

where the M-Wright function $M_\beta(z)$ is defined as

$$M_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\alpha k + (1 - \alpha))} = \frac{1}{\pi} \sum_{k=0}^{\infty} \frac{(-z)^{k-1}}{(k-1)!} \Gamma(\alpha k) \sin(\pi \alpha k), z \in \mathcal{C}, 0 < \alpha < 1.$$

Instead of (1.10), the density of $E_\alpha(t)$ can also be given in the following form

$$\nu_\alpha(x, t) = \frac{1}{t^\alpha} M_\alpha\left(\frac{x}{t^\alpha}\right).$$

Furthermore, we also note that there is a Mikusinski's representation [33] for density

$$\gamma_\alpha(x,1)=\gamma_\alpha(x)=\frac{\alpha}{1-\alpha}\cdot\frac{1}{\pi x}\int_0^\pi u(\varphi)\exp\{-u(\varphi)\}d\varphi,$$

where

$$u(\varphi)=\frac{\sin[(1-\alpha)\varphi]}{\sin\varphi}\left(\frac{\sin(\alpha\varphi)}{x\sin\varphi}\right)^{\frac{\alpha}{1-\alpha}}.$$

The following asymptotics are known

$$\gamma_\alpha(x)\sim C_2\frac{\exp\{-C_1x^{-\frac{\alpha}{1-\alpha}}\}}{x^{(2-\alpha)/(2-2\alpha)}}, x\to 0,$$

and

$$\gamma_\alpha(x)\sim\frac{C_3}{x^{1+\alpha}}, x\to\infty$$

where

$$C_1=(1-\alpha)\alpha^{\alpha/(1-\alpha)}, C_2=\frac{\alpha^{\frac{1}{2-2\alpha}}}{\sqrt{2\pi(1-\alpha)}}, C_3=\frac{\sin(\pi\alpha)}{\pi}\Gamma(1+\alpha).$$

Coming back to (1.10), for example, for $t=1$, we have

$$\nu_\alpha(x,1)\sim\frac{C_2}{\alpha}x^{(2\alpha-1)/(2-2\alpha)}\exp\{-C_1x^{1/(1-\alpha)}\},\ x\to\infty,$$

and

$$\nu_\alpha(x,1)\to\frac{\sin(\pi\alpha)}{\pi\alpha}\Gamma(1+\alpha),\ x\to 0.$$

Then, using self-similarity and setting $z=\frac{x}{t^\alpha}$, we have:

$$\nu_\alpha(z,t)=t^{-\alpha}\nu_\alpha(z,1)\sim t^{-\alpha}\frac{C_2}{\alpha}z^{(2\alpha-1)/(2-2\alpha)}\exp\{-C_1z^{1/(1-\alpha)}\},\ z\to\infty$$

or

$$\nu_\alpha(x,t)\sim C_2\cdot\frac{t^{-\alpha(1+\frac{2\alpha-1}{2-2\alpha})}}{\alpha}x^{\frac{2\alpha-1}{2-2\alpha}}\exp\{-\frac{C_1}{t^{\alpha/(1-\alpha)}}x^{1/(1-\alpha)}\},\ x\to\infty \tag{4.52}$$

for any fixed t, and

$$\nu_\alpha(t,x)\to t^{-\alpha}\frac{\sin(\pi\alpha)}{\pi\alpha}\Gamma(1+\alpha),\ x\to 0. \tag{4.53}$$

Finally, a further expression of $\nu_\alpha(x,t)$ can be found in [25], i.e. $\forall t\geq 0$

$$\nu_\alpha(x,t)=\frac{1}{\pi}\int_0^{+\infty}u^{\alpha-1}e^{-tu-xu^\alpha\cos(\alpha\pi)}\sin\left(\pi\alpha-xu^\alpha\sin(\pi\alpha)\right)du. \tag{4.54}$$

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