Financial Development and Economic Growth in a Microfounded Small Open Economy Model

Bo Zhang [1], Peng Zhou [2,*]

[1] School of Economics and Management, Beijing University of Chemical Technology, 15 North Third Ring Road, Beijing, 100029, China; zhangbo2001_ren@hotmail.com
[2] Cardiff Business School, Cardiff University, D47 Aberconway Building, Colum Drive, Cardiff, CF10 3EU, United Kingdom.
* Correspondence: zhoup1@cardiff.ac.uk; Tel.: +44-2920688778.

Abstract: The global financial crisis since 2008 revived the debate on whether or not and to what extent financial development contributes to economic growth. This paper reviews different theoretical schools of thought and empirical findings on this nexus, building on which we aim to develop a unified, microfounded model in a small open economy setting to accommodate various theoretical possibilities and empirical observations. The model is then calibrated to match some well-documented stylized facts. Numerical simulations show that, in the long run, the welfare-maximizing level of financial development is lower than the growth-maximizing level. In the short run, the price channel (through world interest rate) dominates the quantity-channel (through financial productivity), suggesting a vital role of international cooperation in tackling systemic risk of the global financial system.

Keywords: economic growth; financial development; open economy; DSGE

1. Introduction

There has been a long history of debate over whether financial development (FD) contributes to economic growth (Schumpeter, 1912; Keynes, 1930; Levine et al., 2000). The 2008 global financial crisis attracted much attention in the literature since then (Aghion et al., 2009; Luintel et al., 2016; Osei and Kim, 2020). Most argue that FD can sustainably promote growth through advancing productivity and accumulating financial capital (Mahi et al, 2020). As a result, models with different financial mechanisms are elaborated, including both indirect financing (banking and insurance, e.g. Ehrlich and Becker, 1972; Diamond and Dybvig, 1983; Bongini et al., 2017) and direct financing (FDI, stock markets, and bond markets, e.g. Saint-Paul, 1992; ; Levine, 2005; Mallick and Moore, 2008). A common feature of these models is that, FD provides a transfer from the traditional non-growth sectors to the modern sectors where entrepreneurial responses are promoted. Such school of thought is termed as **supply-leading hypothesis** (Patrick, 1966). However, not everyone shares the same opinion. Levine (2001) points out that the supply-leading hypothesis reverses the causality. They contend that it is the augmented economic growth of an economy that creates a demand for financial services—where enterprise leads, finance follows. In this view, FD is not a leading cause, but a following result, of economic growth. We can call this school of thought the **demand-following hypothesis** (Lucas, 1988).

In contrast, some believe that economic growth is entirely independent of FD (Stern, 1989). In the spirit of Modigliani-Miller Theorem (Modigliani and Miller, 1958), the value of a firm (and hence the entire economy) does not depend on how the firm seeks its finances if the efficient market hypothesis holds. Some researchers even find that too much finance has negative effects on economic growth due to induced credit crunches (Arcand et al., 2011).

Out of the theoretical debate grows a vast empirical literature on whether and to what extent finance affects growth. The answer varies across countries and over time. Law and Demetriades (2005) show that FD has no effect on growth in countries with poor institutions. Rousseau and Wachtel (1998) find similar conclusions for countries with extremely high inflation. Levine (2001) finds that as FD rises, its effect on growth diminishes. Li et al (2015) estimate an inverted U-shaped relationship between FD and growth based on 102 countries. Nevertheless, these empirical studies are mainly econometric models rather than structurally derived, microfounded models which have been the mainstream macroeconomic modeling paradigm since 2000s.

The purpose of this paper is to provide a theoretical, unified explanation of the well-documented inverted U-shaped relationship between FD and growth using a microfounded model. We are not the first attempt in literature trying to develop a theoretical model to accommodate divergencies in empirical evidence (Beck et al., 2008). For example, Gertler et al (Gertler et al, 2020) develop an edge-cutting Dynamic Stochastic General Equilibrium (DSGE) model to explain how financial systems can be both growth-inducing and growth-induced but they focus more on the efficiency of financial services. This paper extends their modeling approach to an open economy setting, arguing that FD can affect economic growth in two ways and in a nonlinear fashion.

Seeing that the global financial crisis is highly contagious cross borders, our DSGE model is developed in the context of a small open economy (SOE), where two fundamental channels enable FD to affect economic growth nonlinearly and dynamically. One is the *quantity* channel—a higher financial sector productivity (due to FD) can immediately raise the financial output which in turn boosts the growth in productivity and output. The other is the *price* channel—a lower interest rate spread (due to FD) leads to a higher demand for financial capital and a higher growth. Under the assumption of SOE, the former is mainly a country-specific financial sector shock, while the latter is a world-wide financial sector shock (SOE implies price taking). A significant contribution of this paper to the literature is to *quantitatively* evaluate the importance of these two channels by which FD affect economic growth. In an estimated DSGE model, our simulation shows that the price channel can explain 11.4% of the dynamics of economic growth, while the quantity channel has a trivial role (0.4%). Technological progress is still the main driver of economic growth (76%).

2. Data and Stylized Facts

A popular conventional measure of FD is financial depth, F_t, which quantifies the relative abundance of financial services for economic activities. The Penn World data in Figure 1 show a clear inverted U-shaped or quadratic relationship[1] between financial depth and long-run growth rate. This is a stylized fact observed in number of empirical studies (Li et al., 2015; Gambocorta et al., 2014), so we will build this feature into our DSGE model. As will be explained in detail in the model section, this inverse U-shaped feature between financial depth and growth can be derived from the corporate finance literature on capital structure, in which the optimal financial leverage ratio of a firm depends on the trade-off between the benefits and costs. A country's financial depth is simply an aggregate of this trade-off. In other words, we establish a microfoundation of the macroeconomic model in the light of the theory of corporate finance. Additionally, this is also a necessary feature to avoid corner solution under a monotonic relationship—a linear relationship between FD and economic growth would lead to an extreme resource allocation to the financial sector.

The data on labor input to the financial sector is not well available for all countries, but we can use the US as an example to inform how financial capital is *dynamically* produced and maintained (Levine et al., 2000) as shown in the simple time-series model (Figure 2). Financial depth is a stock measure, so it has a high persistence over time. Thus, we model financial depth as equation (F3) in the next section.

[1] Strictly speaking, a simple model as such is subject to endogeneity problem due to reverse causality, i.e. whether the financial development causes the economic growth, or the economic growth leads to a higher financial development. However, the purpose of this graph is simply to show some informative patterns between the two key variables. The DSGE model in the next section will fully explore the complicated dynamics and causalities.

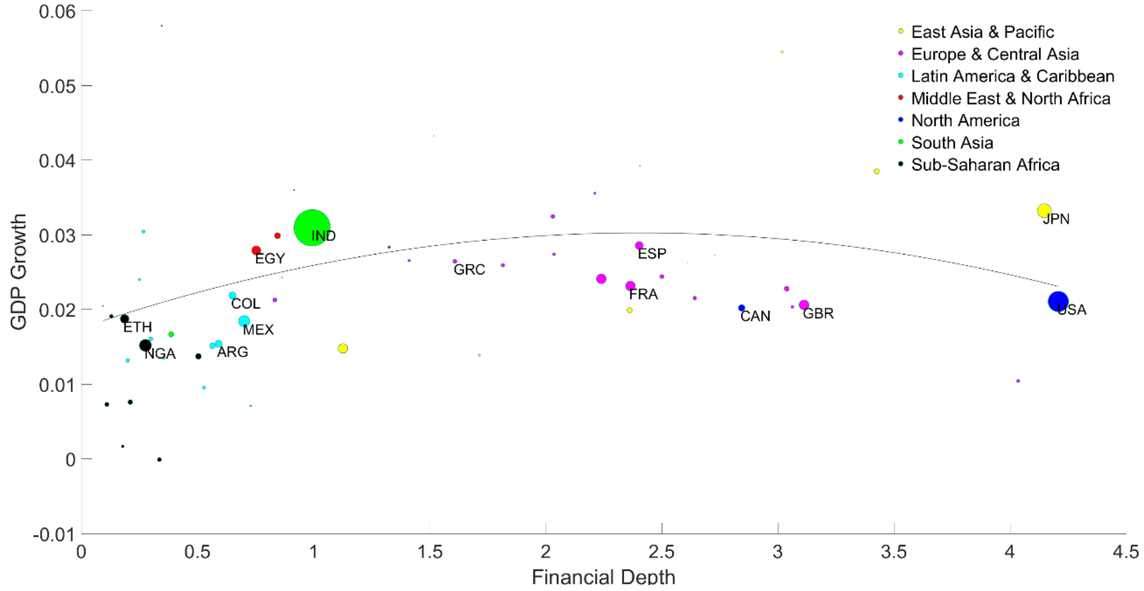

Figure 1 Relationship between Financial Capital Growth and Economic Growth. Notes: Penn World Table 9.0. Data points are annual average growth rates and financial depth for 52 countries of 7 regions for 52 years (1960-2011). Financial depth $\tilde{F}$ is defined as the value of the total financial capital (including bank credit, private bond, public bond, stock market and foreign debt) relative to GDP. The scatter plot and fitted curve is based on the cross-sectional average over the sample periods, weighted by population size. The between-effects estimator is: $g_Y = \underset{(0.0026)}{0.0175} + \underset{(0.0033)}{0.0106}\tilde{F} + \underset{(0.0007)}{-0.0022}\tilde{F}^2$ (standard errors in the brackets).

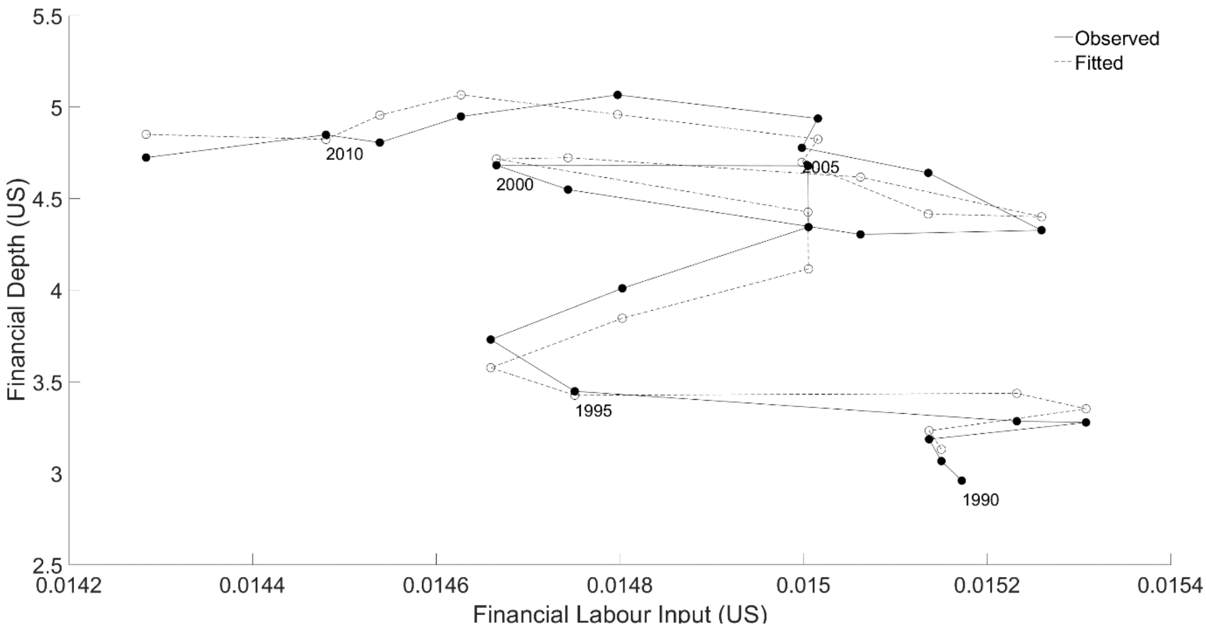

Figure 2 Relationship between Labor Input and Financial Depth. Notes: The relationship between financial capital to GDP ratio and labor input ratio of the US data (Source: Bureau of Labor Statistics, 1990-2018). The fitted path (dash) is estimated based on the model equation (F3).

3. The Model

To capture the stylized features summarized in the previous subsection, we develop a DSGE model following the real business cycle paradigm (Mendoza, 1991; Canova and Ubide, 1998). Throughout this paper, variables with superscript d are the domestic demand for domestic products/bonds (e.g. C^d, B^d), and those with superscript f are the domestic demand for foreign products/bonds (e.g. C^f, B^f). All variables with a star $*$ indicate the foreign counterparts (e.g. $C_t^*, i_t^*, s^*, \gamma^*$). The timing convention is such that the subscript t means the variable is determined at t or during period t (between $t-1$ and t), but it can take effect in period $t+1$ as a state variable (e.g. K_t).

[Consumer] The representative consumer maximizes her expected lifetime utility:

$$\max_{\left\{C_t, C_t^d, C_t^f, N_t, L_t, Z_t, B_t, B_t^f\right\}_{t=0}^{\infty}} \mathbf{E}_t \sum_{t=0}^{\infty} \beta^t U(C_t, L_t).$$

We assume a time-separable isoelastic utility function similar to McCallum and Nelson (2000), where θ is the relative utility weight of leisure (L_t) and ϵ_t^L is an exogenous preference shock with respect to leisure ($\epsilon_t^L > 0$ means leisure is more desirable).

$$U(C_t, L_t) = \ln C_t + \theta e^{\epsilon_t^L} \ln L_t \tag{C1}$$

Furthermore, to introduce open economy, the composite consumption C_t is an aggregator between domestic and foreign products similar to Armington (1969), where γ is the relative utility weight for imported foreign product $C_t^f \equiv M_t$ and ϵ_t^M is the exogenous preference shock with respect to C_t^f ($\epsilon_t^M > 0$ means imported foreign product is more desirable than usual), and s is the elasticity of substitution. Note that in steady state ($\epsilon_t^M = 0$), the utility weights of the domestic and foreign products are respectively equal to $\frac{1}{1+\gamma}$ and $\frac{\gamma}{1+\gamma}$.

$$C_t \equiv \left[\left(\frac{1}{1+\gamma e^{\epsilon_t^M}}\right)^{\frac{1}{s}} \left(C_t^d\right)^{\frac{s-1}{s}} + \left(\frac{\gamma e^{\epsilon_t^M}}{1+\gamma e^{\epsilon_t^M}}\right)^{\frac{1}{s}} \left(C_t^f\right)^{\frac{s-1}{s}}\right]^{\frac{s}{s-1}} \tag{C2}$$

There are three constraints restricting the optimization process. Firstly, time endowment is split between labor (N_t), financial activities (Z_t) such as working in the banking sector, and leisure (L_t).

$$N_t + Z_t + L_t = 1 \tag{C3}$$

Secondly, after a lump-sum tax T_t (net of any transfer payment) the dispensable income (including labor income at the rate of w_t, financial income at the rate of ω_t, and the dividend income per capita Π_t) is spent on consumption and financial investment (on both domestic[2] and foreign bonds). Here, the relative prices of financial assets are normalized to 1, so we need to interpret B_t^d and B_t^f as quantities (net holdings) respectively denominated by domestic and foreign output units.

[2] The domestic bonds here are government bonds only because the private bonds either cancel out among domestic households and firms or absorbed in foreign bond net holdings B_t^f.

Moreover, the real exchange rate χ_t is needed to account for the value difference between domestic and foreign output units (1 unit of foreign output is equal to χ_t units of domestic output). The world interest rate i_t^* is exogenous.

$$C_t^d + \chi_t C_t^f + B_t^d - (1+i_{t-1})B_{t-1}^d + \chi_t B_t^f - \chi_t(1+i_{t-1}^*)B_{t-1}^f = w_t N_t + \omega_t Z_t + \Pi_t - T_t \quad \text{(C4)}$$

[Firm] The representative firm maximizes the sum of discounted future profit flows. The discount rate is equal to the market (real) interest rate $r_t = i_t$ and the discount factor $D_t = \frac{1}{(1+i_0)\ldots(1+i_{t-1})}$ for $t \geq 1$ and $D_0 = 1$. The output sector and the financial sector are consolidated to one composite firm, but this setup is equivalent to the decentralized two-sector model according to the First Fundamental Theorem of Welfare Economics.

$$\max_{\{Y_t, K_t, I_t, N_t, Z_t\}_{t=0}^{\infty}} \mathbf{E}_t \sum_{t=0}^{\infty} D_t (Y_t - I_t - w_t N_t - \omega_t Z_t \equiv \Pi_t).$$

There are four constraints. The first is the production function of the aggregate output Y_t, with A_t being the Harrod neutral (or labor augmenting) technology and α being the income share of labor. The advantage of this specification is well documented that the growth rate of output is equal to the growth rate of technology in the balanced growth path (Uzawa, 1965).

$$Y_t = (A_t N_t)^{\alpha} K_{t-1}^{1-\alpha} \quad \text{(F1)}$$

The second is to endogenize the technological progress by financial depth in a similar way to other endogenous growth models (e.g. human capital, Lucas, 1988; knowledge capital, Romer, 1990). This feature makes the firm's optimization problem dynamic because the decision today (on Z_t) affects both the present and the future. A quadratic feature of the relationship between technological growth (g_A) and financial depth ($\tilde{F}_t$) is supported by the empirical evidence (Figure 1) and corporate finance theory (explained in the market clearing subsection). Given that our model follows a neoclassical paradigm, the contribution of capital (including financial capital) is diminishing. This is the fundamental reason for this observed inverted-U relationship in data. $\tilde{F}_t \equiv F_t / Y_t$, a standard measure of financial depth, is defined as the ratio between the total financial capital and GDP, delineating the relative abundance of financial instruments to facilitate the real economy. ϵ_t^A is an exogenous productivity shock.

$$g_{At} \equiv \frac{A_t}{A_{t-1}} - 1 = a_0 + a_1 \tilde{F}_t + a_2 \tilde{F}_t^2 + \epsilon_t^A \quad \text{(F2)}$$

The third constraint describes the financial capital production function. In the light of the evidence shown in Figure 2, the financial depth is determined by both financial labor input and the previous relative financial capital in a similar fashion to the aggregate output production function (F1). The difference is that we do not restrict it to constant returns to scale, so $\phi_1 + \phi_2$ can be greater than 1[3]. ϵ_t^F is an exogenous productivity shock specific to the financial sector.

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[3] The estimated (F3) based on the US data is: $\ln \tilde{F}_t = \underset{(2.1344)}{0.8162} + \underset{(0.0543)}{0.9061} \ln \tilde{F}_{t-1} + \underset{(0.5165)}{0.1571} \ln Z_t$. We will use the estimates to calibrate ϕ_1 and ϕ_2. It turns out that the financial capital production function is very close to constant returns to scale ($\hat{\phi}_1 + \hat{\phi}_2 = 1.0632$).

$$\tilde{F}_t = \Phi_0 \tilde{F}_{t-1}^{\phi_1} Z_t^{\phi_2} e^{\epsilon_t^F} \Leftrightarrow \ln \tilde{F}_t \approx \phi_0 + \phi_1 \ln \tilde{F}_{t-1} + \phi_2 \ln Z_t + \epsilon_t^F \tag{F3}$$

Finally, the law of motion for physical capital is specified in (F4). We grant the ownership of capital is to firms rather than household following Lucas (1967), but according to the Coase theorem, whoever owns the capital does not make any difference if there is no transaction cost.

$$K_t - (1-\delta)K_{t-1} = I_t \tag{F4}$$

[Government] The government finances its expenditure G_t by a lump-sum tax T_t and government bond B_t (G1), while the expenditure is a fraction (ζ) of GDP disturbed by a fiscal policy shock ϵ_t^G (G2).

$$G_t = T_t + B_t^d - (1 + i_{t-1})B_{t-1}^d \tag{G1}$$

$$G_t = (\zeta Y_t)e^{\epsilon_t^G} \tag{G2}$$

[Rest of the World] The equations below describe the balance of payment, i.e. current account surplus (trade balance + factor income) is equal to capital account deficit, with import and export being derived from consumer's marginal conditions by symmetry:

$$\left[(X_t - \chi_t M_t) + \chi_t i_{t-1}^* B_{t-1}^f\right] = \chi_t\left(B_t^f - B_{t-1}^f\right) \tag{R1}$$

$$M_t \equiv C_t^f = \left(\frac{\gamma e^{\epsilon_t^M}}{1+\gamma e^{\epsilon_t^M}}\right)^{-\frac{1}{s-1}} \left(1 + \frac{\chi_t^{s-1}}{\gamma e^{\epsilon_t^M}}\right)^{-\frac{s}{s-1}} C_t \tag{R2}$$

$$X_t = \left(\frac{\gamma^* e^{\epsilon_t^X}}{1+\gamma^* e^{\epsilon_t^X}}\right)^{-\frac{1}{s^*-1}} \left(1 + \frac{(1/\chi_t)^{s^*-1}}{\gamma^* e^{\epsilon_t^X}}\right)^{-\frac{s^*}{s^*-1}} C_t^* \tag{R3}$$

Note that the first order conditions for C_t^f and C_t^d are used to obtain equation (R2) as derived in the online appendix. Also, χ_t is inversed in equation (R3) is because the real exchange rate facing the rest of the world is the reciprocal of that facing the domestic consumers. Moreover, ϵ_t^X is the exogenous preference shock with respect to domestic output in the world market ($\epsilon_t^X > 0$ means the exported domestic output is more desirable). It is a common modeling choice to include a preference shock as such in the literature of international business cycles to match the persistence observed in the data (Kollmann, 2016; Rothert, 2020). C_t^* is the exogenous world consumption per capita, which is the counterpart of C_t.

[Market Clearing] The clearing conditions hold for output markets, labor markets, capital markets and financial markets, both domestically and internationally. The two domestic labor markets (in the financial sector and nonfinancial sector) are competitive and the two wages are equalized. Since capital is owned by firms, the cost of investment is internalized and there is no explicit capital market. Note that the consumer's budget constraint (C4), the definition of the firm's profit, the government's budget constraint (G1) and the balance of payment (R1) imply the domestic output market clearing condition $Y_t = C_t^d + I_t + G_t + X_t$. Moreover, under the small open economy assumption, the international output and financial markets are exogenous so the demand and supply can always meet. The only relevant market clearing condition is therefore the domestic financial market:

$$F_t = \left(St_t + B_t^{pr} + Cr_t\right) + B_t^d \tag{MC}$$

The left-hand-side F_t is the total domestic financial capital produced and maintained by the financial sector, and the right-hand-side includes the external finance demanded by the firm (St_t: stock market capitalisation; B_t^{pr}: private bond; Cr_t: bank credit/loan) and the public bond demanded by the government (B_t^d). According to the corporate finance literature (e.g. trade-off theory and pecking order theory), the demand for external finance $\left(St_t + B_t^{pr} + Cr_t\right)$ is a result of optimization leading to a ratio of the total capital (K_t). Panel data evidence (Rajan and Zingales, 1995) shows that this optimal ratio ranges from 20% in the US, 30% in Canada, 36% in the UK to 50% in Japan. In our data, this ratio (κ) of external finance is derived to be 25.88%. The financial market clearing condition can therefore be rewritten as:

$$F_t = \kappa K_t + B_t^d \tag{MC'}$$

A summary of how to derive and stationarize the dynamic stochastic system of model equations can be found in the Appendix. The bottom line is that this system consists of $N_n = 21$ endogenous variables, $N_x = 8$ exogenous variables (i.e. stochastic shocks) and the same number of innovations. Mathematically, there is no technical difference between endogenous and exogenous variables, so let's group them together into a 29-by-1 vector $\mathbf{x}_t$. The 8 innovations are grouped into an 8-by-1 vector $\boldsymbol{\eta}_t$. The structural form of the equation system can be summarized as:

$$\mathbf{E}_t[f(\mathbf{x}_t, \mathbf{x}_{t-1}, \mathbf{x}_{t+1}, \boldsymbol{\eta}_t | \boldsymbol{\theta})] = 0, \text{ where } \boldsymbol{\theta} \in \Theta.$$

This dynamic stochastic equation system can be solved and simulated using the perturbation method. The online supplementary material details how the model equations are derived and stationarized/detrended, and the next subsection describes the method of solving for the steady state and calibrating the parameters consistent to both the data and the model.

4. Solution and Calibration

The stationarized model can be solved by the perturbation method, of which the first step is to obtain the steady state of the 29 variables in $\mathbf{x}_t$. In the steady state, all the $N_x = 8$ exogenous variables are equal to zero or growing at the balanced growth path, so effectively there are only 21 equations for the $N_n = 21$ endogenous variables to be solved.

$$\mathbf{E}_t[f(\mathbf{x}_t, \mathbf{x}_{t-1}, \mathbf{x}_{t+1}, \boldsymbol{\eta}_t | \boldsymbol{\theta})] = 0 \to f(\bar{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{x}}, \mathbf{0} | \boldsymbol{\theta}) = 0.$$

If we know all the parameters ($\boldsymbol{\theta}$), then it is straightforward to solve for the steady state of the 21 endogenous variables from the 21 equations using numerical methods (e.g. Newton algorithm). However, some combinations of the parameter values can easily stumble into cases of no steady state solution. Let alone the uncertainty of parameter values. To resolve the problems, we pre-assign steady state solutions to some of the endogenous variables and some parameters based on data and theory, and then solve for the other endogenous variables and parameters consistent with the equation system. In this way, we blur the distinction between the parameters and the steady state solutions to the endogenous variables. The identification condition requires that the total number of the "unknowns" (either unknown parameters or the unknown steady states) must be equal to the number of equations. The key of this solution/calibration procedure is to partition the steady states and parameters into known and unknown blocks.

In our case, we use regressions and weighted average to calibrate some of the known parameters $\boldsymbol{\theta}^{(0)} = \phi_1, \phi_2, \rho s, \sigma s$ and steady states $\bar{\mathbf{x}}^{(0)} = \bar{Y}, \bar{C}, \bar{I}, \bar{M}, \bar{N}, \bar{Z}, \bar{F}, \bar{B}^d, \bar{\iota}^*, \bar{C}^*, \bar{\Pi}, \bar{G}, \bar{B}^f$, based on which the unknown parameters $\boldsymbol{\theta}^{(1)}$ and steady states $\bar{\mathbf{x}}^{(1)}$ are derived recursively from the equations: $f\left(\bar{\mathbf{x}}^{(1)}, \boldsymbol{\theta}^{(1)} \middle| \bar{\mathbf{x}}^{(0)}, \boldsymbol{\theta}^{(0)}\right) = 0$. Note that the identification condition requires $\dim\left(\boldsymbol{\theta}^{(1)}\right) + \dim\left(\bar{\mathbf{x}}^{(1)}\right) = N$, or equivalently, $\dim\left(\boldsymbol{\theta}^{(1)}\right) = \dim\left(\bar{\mathbf{x}}^{(0)}\right)$.

There are two advantages of this partition method. On the one hand, it makes full use of the "prior" information on both data and parameters to ensure the internal consistency in the calibration process. On the other hand, it greatly improves the efficiency of the perturbation method by providing an analytical solution to the steady state. It also avoids obtaining implausible solutions due to the drawbacks of numerical algorithms (imprecision, starting point, multiple solution, nonconvergence). To some extent, this way of calibrating the parameters resembles the simulated method of moments in the sense that it effectively matches some of the data moments and regression coefficients.

One limitation of this method, however, is that it does not utilise the dynamic features of the data, because the pre-assignment of the steady state solutions only uses the long-run averages. Therefore, the shock structures (like ρs and σs) cannot be systematically estimated.

Another limitation is that some parameters are simply not identifiable, because they cancel out in the steady state relationships. This is the case of "under-identification" defined by Canova and Sala (2009). One example in our model is the parameter s (elasticity of substitution between domestic and foreign goods), which disappears in the simplified form of the steady state equations. To fully estimate these parameters, either full-information methods (such as Maximum Likelihood and Bayesian) or limited-information methods (such as Indirect Inference) must be applied to make use of the dynamic information of the data. However, the feasibility of these methods is dependent on the existence of the steady state solution. Without a steady state solution, the estimation procedure cannot proceed. With this said, our solution/calibration method is very useful to provide a valid starting point even if the other estimation methods are to be used. Given that the primary purpose of this paper is theoretical, we will leave a more sophisticated empirical adventure to future studies.

5. Results and Discussions

The model that is estimated and calibrated to match the data features is then solved and simulated using the perturbation method. Our analysis of the model will focus on the relationship between FD and growth rate.

The *long-run* properties of the model can be analyzed by its steady state. Table 2 summarizes implied steady states of the endogenous and exogenous variables. A particularly interesting implication is that the long-run steady state of FD level ($\bar{F} = 1.596$) is lower than the turning point of the quadratic equation (F2), i.e. $-\frac{a_1}{2a_2} = 2.4$. In other words, the social welfare maximizing solution is different from the growth maximizing solution, under which the maximum growth rate is 6.75%, greater than the steady state growth rate 2.62%. To understand this difference, remember that the equilibrium FD level is derived to maximize the consumer's utility, rather than to maximize the growth rate. Therefore, to maximize a different objective function, the FD level needs to deviate from the welfare-maximizing level. An economy can grow faster if the financial depth is distorted to a higher level at the cost of the social welfare. One example is China, which experienced an unprecedented financial development and high economic growth over the last three decades. In 2016, the financial sector in China already accounts for over 10% of its GDP, much higher than 6.5% in the US. This is obviously a distortion of the economy, leading to higher risks of financial bubbles and a loss of welfare (Sun et al., 2020; Jung & Vijverberg, 2019).

The *short-run* properties of the estimated model are summarized by the forecast error variance decomposition (Table 1) and impulse response functions (Figure 3).

Table 1 Forecast Error Variance Decomposition of the Key Variables

		productivity shock η_t^A	financial shock η_t^F	leisure preference shock η_t^L	export preference shock η_t^X	import preference shock η_t^M	expenditure shock η_t^G	world interest rate shock η_t^{i*}	world consumption shock η_t^{C*}
1 year	g_Y	73.68	0.40	11.43	1.23	1.23	0.00	11.91	0.11
	g_A	99.12	0.66	0.13	0.01	0.01	0.00	0.08	0.00
	$\tilde{F}$	58.44	31.02	5.92	0.38	0.38	0.00	3.82	0.03
	Z	82.16	3.03	8.32	0.53	0.54	0.00	5.36	0.05
5 years	g_Y	76.13	0.43	9.54	1.19	1.20	0.01	11.40	0.10
	g_A	98.58	0.69	0.13	0.05	0.05	0.00	0.49	0.00
	$\tilde{F}$	81.26	9.13	1.74	0.65	0.65	0.00	6.52	0.06
	Z	82.13	4.31	6.17	0.61	0.61	0.00	6.12	0.05
infinity	g_Y	76.14	0.43	9.51	1.19	1.20	0.01	11.42	0.10
	g_A	96.73	0.67	0.13	0.20	0.21	0.00	2.04	0.02
	$\tilde{F}$	75.55	5.03	0.96	1.53	1.54	0.00	15.26	0.13
	Z	81.50	4.41	6.08	0.66	0.66	0.00	6.63	0.06

In Table 1, like any RBC-type models, the productivity shock (η^A) dominates the fluctuations of the four key endogenous variables, especially the economic growth (g_Y). Other supply-side shocks, such as leisure preference shock, also play an important role. The demand-side shocks in export, import and world consumption are trivial to output growth, but important for other variables like foreign bonds and trade balance. The financial shocks, including financial productivity shock η^F and world interest rate shock η^{i*}, need to be discussed separately.

As mentioned in the introduction, we can distinguish between two channels through which FD can affect the economic growth: the quantity channel and the price channel. The former originates in domestic shocks, and the latter comes from the rest of the world. In the quantity channel, a higher (domestic) financial productivity ($\eta^F > 0$) attracts more resources (labor and capital) to the financial sector, resulting in a higher financial output and economic growth. However, this effect is weak (around 0.4%) because the output sector and the financial sector are competing for the same resources (labor). In contrast, in the price channel, a higher (world-wide) financial productivity is usually associated with a negative world interest rate shock ($\eta^{i*} < 0$), which will stimulate the firm's investment (see the firm's optimality condition with respect to capital) and accordingly raise the demand for financial capital. Meanwhile, a lower price of financial capital will drive labor input from the financial sector to the output sector, leading to a lower supply of financial capital. Overall, the price channel is the second largest source of fluctuations in economic growth (11.9% in the short run and 11.4% in the medium and long run).

A considerable proportion of fluctuations in financial depth ($\tilde{F}$) is accounted for by the financial shock in the short run (31%), but its role vanishes quickly to 9% after 5 years and to 5% in the long run. In contrast, the role interest rate shock rises from 3.8% to 15.3% in explaining the fluctuations in

$\tilde{F}$. Like the reason for the small effect of η^F to economic growth, a domestic shock on quantity is less important than a world-wide shock on price, both because of the competition across sectors and because of the small open economy assumption.

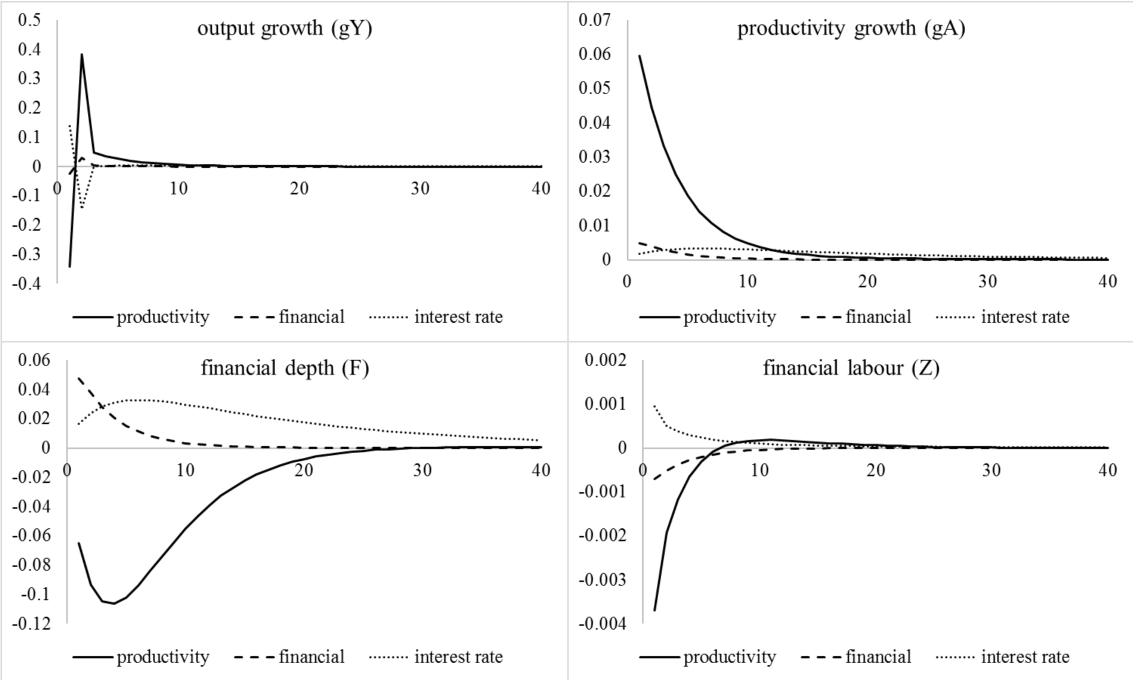

Figure 3 Impulse Response Functions of the Key Variables. Notes: The impulses are one standard deviations of productivity shock (η^A), financial shock (η^F) and world interest rate shock (η^{i*}).

Figure 3 shows how the four key variables (output growth, productivity growth, financial depth, and financial labor) respond to the three key shocks (productivity shock, financial shock, and interest rate shock). The economic growth rate (g_Y) will generally be higher after a positive productivity shock and financial shock (supply-side shocks). Initial negative responses of g_Y are due to the unbalancedness of labor allocation between the two sectors. For example, assume there is a positive productivity shock. Upon the arrival of the shock, the agent observes that working in the nonfinancial sector is more productive (and higher wage), so Z (as shown in the lower right panel) and $\tilde{F}$ (as shown in the lower left panel) will both drop. The net benefit of a higher productivity g_A (as shown in the upper right panel) is not enough to offset the effect of the resulting shortage in financial capital, so the overall effect on the output growth is initially negative. In addition, a surprisingly higher interest rate means a greater demand for domestic financial assets (bonds and stocks), resulting in a higher financial depth and financial labor. It is also interesting to observe a hump-shaped responses of growth rate after different shocks, suggesting an overshooting behavior.

Magnitude wise, the price channel (the interest rate shock) generates a greater impact to the fluctuations in growth rate compared to the quantity channel (the financial shock). However, the largest contributor remains to be the total factor productivity shock, like in any RBC-type models.

Table 2 The Calibrated Parameters and Steady States of the Model

Parameter	Meaning	Calibration	NB
θ	relative utility weight of leisure	4.492	D
β	subjective discount factor	0.989	D
γ	relative weight of foreign goods	0.279	D
s	elasticity of substitution	1.200	F
α	income share of labor	0.737	D
a_0	constant term	-0.298	D
a_1	linear term	0.304	D
a_2	quadratic term	-0.063	D
ϕ_0	constant term	0.712	D
ϕ_1	income share of $\tilde{F}_{t-1}$	0.906	E
ϕ_2	income share of Z	0.157	E
δ	capital depreciation rate	0.019	D
ζ	share of G	0.134	D
κ	optimal external finance rate	0.259	D
ρ_A	AR(1) coefficient	0.815	E
ρ_F	AR(1) coefficient	0.063	E
ρ_L	AR(1) coefficient	0.063	F
ρ_X	AR(1) coefficient	0.063	F
ρ_M	AR(1) coefficient	0.063	F
ρ_G	AR(1) coefficient	0.885	F
ρ_{i^*}	AR(1) coefficient	0.683	E
i^*	steady state i^*	0.038	E
ρ_{C^*}	AR(1) coefficient	0.441	E
g_{C^*}	steady state growth rate of C^*	0.026	E
σ_A	standard deviation	0.066	E

Variable	Meaning	Steady State	NB
$\bar{C}$	aggregate consumption	0.656	E
$\bar{C}^d$	domestic goods	0.513	D
$\bar{N}$	nonfinancial labor	0.197	E
$\bar{Z}$	financial labor	0.014	E
$\bar{B}^d$	domestic bonds	0.360	E
$\bar{B}^f$	foreign bonds	0.000	F
$\bar{K}$	physical capital stock	4.779	D
$\bar{I}$	investment	0.210	E
$\bar{Y}$	output	1.026	E
$\bar{g}_Y$	growth rate of output	0.026	D
$\bar{A}$	labor-augmented productivity	2.929	D
$\bar{g}_A$	growth rate of A	0.026	D
$\bar{F}$	financial depth	1.596	E
$\bar{\Pi}$	economic profit	0.000	F
$\bar{T}$	tax revenue	0.138	D
$\bar{G}$	government expenditure	0.134	D
$\bar{X}$	export	0.143	D
$\bar{M}$	import	0.143	E
$\bar{w}$	real wage	3.737	D
$\bar{\imath}$	interest rate	0.038	D
$\bar{\chi}$	real exchange rate	1.000	F
$\bar{C}^*$	world consumption	0.656	F
$\bar{\imath}^*$	world interest rate	0.038	E
$\bar{\epsilon}_A$	productivity shock	0.000	F
$\bar{\epsilon}_F$	financial shock	0.000	F

σ_F	standard deviation	0.038	E	$\bar{\epsilon}_L$	leisure preference shock	0.000	F
σ_L	standard deviation	0.100	F	$\bar{\epsilon}_X$	export preference shock	0.000	F
σ_X	standard deviation	0.090	E	$\bar{\epsilon}_M$	import preference shock	0.000	F
σ_M	standard deviation	0.090	E	$\bar{\epsilon}_G$	government expenditure shock	0.000	F
σ_G	standard deviation	0.024	E				
σ_{i^*}	standard deviation	0.023	E				
σ_{C^*}	standard deviation	0.016	E				

Notes: In the "NB" columns, "E" stands for estimated from data by GMM regression (with lagged endogenous variables as instruments), "F" stands for fixed based on the empirical DSGE literature (Smets & Wouters, 2007), and "D" stands for derived from the equation system listed in the Appendix. Note that the number of derived parameters is equal to the number of equations. The steady states are for the stationarized model.

5. Conclusions

This paper empirically summarizes and theoretically models the relationship between financial development and economic growth in a small open economy setting. We use the panel data (52 countries over 52 periods) to estimate an inverted U-shaped long-run feature and use the time-series data (the US over 22 years) to estimate a dynamic short-run feature. These two stylized facts are well documented in the literature, but we develop a theoretically microfounded DSGE model based on the empirical observations. This is an extension and a consolidation of existing theoretical models discussing the nexus between FD and economic growth in the following three important aspects.

First, the model is consistently derived from the microeconomic principles of optimization and equilibrium, in contrast to the ad hoc econometric models mostly adopted in empirical research. We do incorporate the two stylized facts estimated in econometric models, so our model has the merits of being both theoretically sound and empirically relevant.

Second, the model is extended from closed economy to open economy. Most theoretical models in the literature discussing financial development assume either a closed economy or an exogenous international market. The small open economy assumption provides a better microfoundation to model the relationship between FD and growth, given the globalization in the financial market in the modern economy.

Third, we accommodate the dynamic causality between FD and economic growth in our model, so they can affect each other over time, rather than sticking to either supply-leading hypothesis or demand-following hypothesis.

Based on empirical observations and optimal calibration/estimation, we numerically simulate the DSGE model to analyze the how FD affects growth. In the long run, we find that the growth rate of a small open economy can achieve 6.75% at the growth-maximizing level of financial development. However, the welfare-maximizing level of financial development is much lower, arriving at a lower long-run growth rate of 2.62%. This finding suggests that if people care more than just economic growth, the extent of financial development should be re-considered because the talents and resources used in the financial sector cannot be used in the goods sector. In other words, it is a trade-off between speed of growth and sustainability of growth. In the short run, we find that the price channel (the interest rate shock) dominates the quantity channel (the financial productivity shock) in determining the fluctuations of growth. Given that the interest rate is determined by the rest of the world in a small open economy, fluctuations in growth rate are basically a global phenomenon and a systemic risk. It implies that no country can be an island and it is vital to fight against financial instability with international cooperation.

This paper provides an elegant, neoclassical style theoretical framework for analyzing the non-monotonic relationship between FD and economic growth, and this prototype has a great potential to be extended to additional factors like education, institutions, and environment related to long-run growth of an economy. Moreover, future empirical work can be done to formally estimate the model using full-information methods like maximum likelihood or Bayesian inference.

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Supplementary Materials: Derivation and Stationarisation of the FOCs

For the consumer's optimisation problem, we substitute out U_t, C_t, L_t, C_t^d by (C1)-(C4) respectively. Then take partial derivatives with respect to the remaining control variables:

$$\begin{aligned}
&\partial C_t^f: C_{C_t^d}(-\chi_t) + C_{C_t^f} = 0 \\
&\partial N_t: U_{C_t} C_{C_t^d} w_t + U_{L_t}(-1) = 0 \\
&\partial Z_t: U_{C_t} C_{C_t^d} \omega_t + U_{L_t}(-1) = 0 \\
&\partial B_t^d: U_{C_t} C_{C_t^d}(-1) + \beta \mathbf{E}_t \left[U_{C_{t+1}} C_{C_{t+1}^d}(1 + i_t) \right] = 0 \\
&\partial B_t^f: U_{C_t} C_{C_t^d}(-\chi_t) + \beta \mathbf{E}_t \left[U_{C_{t+1}} C_{C_{t+1}^d}(1 + i_t^*) \right] = 0
\end{aligned}$$

Note that the first three are intratemporal conditions and the last two are intertempral conditions. In particular, the first marginal condition with respect to C_t^f is used to derive the export and import equations (R2)-(R3). To see this:

$$\begin{aligned}
&\partial C_t^f: C_{C_t^d}(-\chi_t) + C_{C_t^f} = 0 \\
&\to \left(\frac{\gamma e^{\epsilon_t^M}}{1 + \gamma e^{\epsilon_t^M}} \right)^{\frac{1}{s}} \left(C_t^f \right)^{\frac{s-1}{s} - 1} = \left(\frac{1}{1 + \gamma e^{\epsilon_t^M}} \right)^{\frac{1}{s}} \left(C_t^d \right)^{\frac{s-1}{s} - 1} \chi_t \\
&\to C_t^d = \frac{\chi_t^s}{\gamma e^{\epsilon_t^M}} C_t^f
\end{aligned}$$

Substitute this relationship back to (C2) to get (R2)[1]. By symmetry, we can obtain (R3) but the real exchange rate needs to be inversed. Moreover, the conditions ∂N_t and ∂Z_t implies labour market price equalisation $w_t = \omega_t$, and the conditions ∂B_t^d and ∂B_t^f implies financial market price equalisation (uncovered interest parity) $1 + i_t = \frac{E_t[\chi_{t+1}]}{\chi_t}(1 + i_t^*)$.

For the firm's optimisation problem, we replace ω_t by w_t using the conclusion above, and substitute out $Y_t, A_t, g_{At}, \tilde{F}_t, I_t$ by (F1)-(F4):

$$\mathbf{E}_t \sum_{t=0}^{\infty} \frac{1}{(1 + i_0) \dots (1 + i_{t-1})} \left[(A_t N_t)^\alpha K_{t-1}^{1-\alpha} - K_t + (1 - \delta) K_{t-1} - w_t N_t - w_t Z_t \right]$$

Take partial derivative with respect to the remaining control variables:

$$\partial N_t: \frac{\partial Y_t}{\partial N_t} = w_t$$

[1] Using the marginal condition ∂C_t^f to derive (R2) will kick this original marginal condition out of the model's equilibrium condition system, because it is redundant once (R2) and the definition of C_t is included.

$$\partial Z_t: \frac{\partial Y_t}{\partial A_t}\frac{\partial A_t}{\partial g_{At}}\frac{\partial g_{At}}{\partial \tilde{F}_t}\frac{\partial \tilde{F}_t}{\partial Z_t} + \mathbf{E}_t\left[\frac{1}{1+i_t}\frac{\partial Y_{t+1}}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial A_t}\frac{\partial A_t}{\partial g_{At}}\frac{\partial g_{At}}{\partial \tilde{F}_t}\frac{\partial \tilde{F}_t}{\partial Z_t}\right] + \mathbf{E}_t\left[\frac{1}{1+i_t}\frac{\partial Y_{t+1}}{\partial A_{t+1}}\frac{\partial A_{t+1}}{\partial g_{At+1}}\frac{\partial g_{At+1}}{\partial \tilde{F}_{t+1}}\frac{\partial \tilde{F}_{t+1}}{\partial \tilde{F}_t}\frac{\partial \tilde{F}_t}{\partial Z_t}\right] = w_t$$

$$\partial K_t: -1 + \mathbf{E}_t\left[\frac{1}{1+i_t}\left(\frac{\partial Y_{t+1}}{\partial K_t} + (1-\delta)\right)\right] = 0$$

To complete the system of equilibrium conditions, drop the redundant variables ($\omega_t = w_t$, $C_t^f \equiv M_t$), redefine some of the endogenous variables $C_t, C_t^d, I_t, Y_t, g_{Yt}, A_t, g_{At}, \tilde{F}_t, \Pi_t$, and combine with the government's budget constraints (G1)-(G2), the balance of payment (R1)-(R3), and the market clearing condition. Finally, there are 21 independent equilibrium conditions for the $N_n = 21$ endogenous variables:

$$[C_t; C_t^d; N_t; Z_t; B_t^d; B_t^f; I_t; K_t; Y_t; g_{Yt}; A_t; g_{A_t}; \tilde{F}_t; \Pi_t; T_t; G_t; X_t; M_t; w_t; i_t; \chi_t].$$

There are $N_x = 8$ exogenous shocks/variables, which can be further decomposed into 8 orthogonal Gaussian innovations:

$[\epsilon_t^A; \epsilon_t^F; \epsilon_t^L; \epsilon_t^X; \epsilon_t^M; \epsilon_t^G; i_t^*; C_t^*] \to \eta_t \equiv [\eta_t^A; \eta_t^F; \eta_t^L; \eta_t^X; \eta_t^M; \eta_t^G; \eta_t^{i*}; \eta_t^{C*}]$, where:

$\epsilon_t^j = \rho_j \epsilon_{t-1}^j + \eta_t^j$, where $\eta_t^j \sim N(0, \sigma_j^2)$ and $j = A, F, L, X, M, G$;

$i_t^* = (1-\rho_{i*})\bar{\imath}^* + \rho_{i*} i_{t-1}^* + \eta_t^{i*}$, where $\eta_t^{i*} \sim N(0, \sigma_{i*}^2)$;

$\ln\frac{C_t^*}{C_{t-1}^*} = (1-\rho_{C*})\bar{g}_{C*} + \rho_{C*}\ln\frac{C_{t-1}^*}{C_{t-2}^*} + \eta_t^{C*}$, where $\eta_t^{C*} \sim N(0, \sigma_{C*}^2)$.

Based on the data, we can estimate the last two error processes by weighted OLS:

$$\rho_{i*} = 0.68, \bar{\imath}^* = 3.76\%, \sigma_{i*} = 0.02; \rho_{C*} = 0.44, \bar{g}_{C^*} = 2.59\%, \sigma_{C*} = 0.02.$$

To summarise, we list all the model equations ($N_n + N_x = 29$) here:

[Consumer Block]

$$\frac{1}{\tilde{C}_t}\left(\frac{1}{1+\gamma e^{\epsilon_t^M}}\frac{\tilde{C}_t}{\tilde{C}_t^d}\right)^{\frac{1}{s}}\tilde{w}_t = \frac{\theta e^{\epsilon_t^L}}{1-N_t-Z_t}$$

$$\frac{1}{\tilde{C}_t}\left(\frac{1}{1+\gamma e^{\epsilon_t^M}}\frac{\tilde{C}_t}{\tilde{C}_t^d}\right)^{\frac{1}{s}} = \beta\mathbf{E}_t\left[\frac{1}{\tilde{C}_{t+1}\tilde{Y}_{t+1}}\left(\frac{1}{1+\gamma e^{\epsilon_{t+1}^M}}\frac{\tilde{C}_{t+1}}{\tilde{C}_{t+1}^d}\right)^{\frac{1}{s}}(1+i_t)\right]$$

$$1 + i_t = \frac{E_t[\chi_{t+1}]}{\chi_t}(1 + i_t^*)$$

$$\tilde{C}_t = \left[\left(\frac{1}{1 + \gamma e^{\epsilon_t^M}}\right)^{\frac{1}{s}} \left(\tilde{C}_t^d\right)^{\frac{s-1}{s}} + \left(\frac{\gamma e^{\epsilon_t^M}}{1 + \gamma e^{\epsilon_t^M}}\right)^{\frac{1}{s}} \left(\widetilde{M}_t\right)^{\frac{s-1}{s}}\right]^{\frac{s}{s-1}}$$

$$\left[\tilde{C}_t^d + \chi_t \widetilde{M}_t\right] + \begin{bmatrix} \tilde{B}_t^d - (1 + i_{t-1})\tilde{B}_{t-1}^d / \tilde{Y}_t \\ +\chi_t \tilde{B}_t^f - \chi_t (1 + i_{t-1}^*)\tilde{B}_{t-1}^f / \tilde{Y}_t \end{bmatrix} = \widetilde{w}_t N_t + \widetilde{w}_t Z_t + \widetilde{\Pi}_t - \tilde{T}_t$$

[Firm Block]

$$\frac{\alpha}{N_t} = \widetilde{w}_t$$

$$\frac{\alpha}{\tilde{A}_t}\frac{\tilde{A}_{t-1}}{\tilde{Y}_t}\left(a_1 + 2a_2\tilde{F}_t\right)\left(\frac{\phi_2 \tilde{F}_t}{Z_t}\right) + \mathbf{E}_t\left[\frac{1}{1 + i_t}\frac{\alpha}{\tilde{A}_{t+1}}(1 + g_{At+1})\right]\frac{\tilde{A}_{t-1}}{\tilde{Y}_t}\left(a_1 + 2a_2\tilde{F}_t\right)\left(\frac{\phi_2 \tilde{F}_t}{Z_t}\right) + \mathbf{E}_t\left[\frac{1}{1 + i_t}\frac{\alpha}{\tilde{A}_{t+1}}\tilde{A}_t\left(a_1 + 2a_2\tilde{F}_{t+1}\right)\left(\frac{\phi_1 \tilde{F}_{t+1}}{\tilde{F}_t}\right)\left(\frac{\phi_2 \tilde{F}_t}{Z_t}\right)\right] = \widetilde{w}_t$$

$$\mathbf{E}_t\left[\frac{1}{1 + i_t}\left(\frac{(1 - \alpha)\tilde{Y}_{t+1}}{\widetilde{K}_t} + (1 - \delta)\right)\right] = 1$$

$$\tilde{I}_t = \widetilde{K}_t - \frac{(1 - \delta)\widetilde{K}_{t-1}}{\tilde{Y}_t}$$

$$\tilde{Y}_t^{1-\alpha} = \left(\tilde{A}_t N_t\right)^{\alpha} \widetilde{K}_{t-1}^{1-\alpha}$$

$$g_{Yt} = \tilde{Y}_t - 1$$

$$\tilde{A}_t = \frac{\tilde{A}_{t-1}}{\tilde{Y}_t}(1 + g_{At})$$

$$g_{At} = a_0 + a_1\tilde{F}_t + a_2\tilde{F}_t^2 + \epsilon_t^A$$

$$\tilde{F}_t = \Phi_0 \tilde{F}_{t-1}^{\phi_1} Z_t^{\phi_2} e^{\epsilon_t^F}$$

$$\widetilde{\Pi}_t = 1 - \tilde{I}_t - \widetilde{w}_t N_t - \widetilde{w}_t Z_t$$

[Government, ROW and MC Block]

$$\tilde{G}_t = \tilde{T}_t + \tilde{B}_t^d - (1 + i_{t-1})\tilde{B}_{t-1}^d / \tilde{Y}_t$$

$$\tilde{G}_t = \zeta e^{\epsilon_t^G}$$

$$\left[\left(\tilde{X}_t - \chi_t \widetilde{M}_t\right) + \chi_t i_{t-1}^* \tilde{B}_{t-1}^f / \tilde{Y}_t\right] = \chi_t\left(\tilde{B}_t^f - \tilde{B}_{t-1}^f / \tilde{Y}_t\right)$$

$$\widetilde{M}_t = \left(\frac{\gamma e^{\epsilon_t^M}}{1+\gamma e^{\epsilon_t^M}}\right)^{-\frac{1}{s-1}} \left(1+\frac{\chi_t^{s-1}}{\gamma e^{\epsilon_t^M}}\right)^{-\frac{s}{s-1}} \tilde{C}_t$$

$$\tilde{X}_t = \left(\frac{\gamma^* e^{\epsilon_t^X}}{1+\gamma^* e^{\epsilon_t^X}}\right)^{-\frac{1}{s^*-1}} \left(1+\frac{(1/\chi_t)^{s^*-1}}{\gamma^* e^{\epsilon_t^X}}\right)^{-\frac{s^*}{s^*-1}} \tilde{C}_t^*$$

$$\tilde{F}_t = \kappa \widetilde{K}_t + \tilde{B}_t^d$$

[Exogenous Processes]

$\epsilon_t^j = \rho_j \epsilon_{t-1}^j + \eta_t^j$, where $\eta_t^j \sim N(0, \sigma_j^2)$ and $j = A, F, L, X, M, G$

$i_t^* = (1-\rho_{i*})\bar{\imath}^* + \rho_{i*} i_{t-1}^* + \eta_t^{i*}$, where $\eta_t^{i*} \sim N(0, \sigma_{i*}^2)$

$\ln \frac{\tilde{C}_t^* \tilde{Y}_t}{\tilde{C}_{t-1}^*} = (1-\rho_{C*})\bar{g}_{C*} + \rho_{C*} \ln \frac{\tilde{C}_{t-1}^* \tilde{Y}_{t-1}^*}{\tilde{C}_{t-2}^*} + \eta_t^{C*}$, where $\eta_t^{C*} \sim N(0, \sigma_{C*}^2)$

Note that this dynamic stochastic system does not have a steady state due to its endogenous growth structure. Apart from $[N_t; Z_t; g_{Yt}; g_{A_t}; g_{F_t}; i_t; \chi_t]$ which are already stationary by definition, we need to stationarise the nonstationary variables before applying the perturbation method. $[C_t; C_t^d; B_t^d; B_t^f; I_t; K_t; A_t; \Pi_t; T_t; G_t; X_t; M_t; w_t]$ can be stationarised by dividing the co-trending variable Y_t. Let's name all the stationarised variables $\tilde{\blacksquare}$, e.g. $\tilde{C}_t \equiv C_t/Y_t$ (interpreted as proportion of output). Y_t itself can be stationarised by Y_{t-1}, i.e. $\tilde{Y}_t \equiv Y_t/Y_{t-1} \equiv 1 + g_{Yt}$. For example, to stationarise the production function, divide both sides by Y_t:

$$Y_t = (A_t N_t)^\alpha K_{t-1}^\alpha \rightarrow 1 = \left(\frac{A_t}{Y_t} N_t\right)^\alpha \left(\frac{K_{t-1}}{Y_t}\right)^{1-\alpha};$$

$$1 = \left(\frac{A_t}{Y_t} N_t\right)^\alpha \left(\frac{K_{t-1}}{Y_{t-1}} \frac{Y_{t-1}}{Y_t}\right)^{1-\alpha} \rightarrow 1 = \left(\tilde{A}_t N_t\right)^\alpha \left(\frac{\widetilde{K}_{t-1}}{\tilde{Y}_t}\right)^{1-\alpha} \text{ or } \tilde{Y}_t^{1-\alpha} = \left(\tilde{A}_t N_t\right)^\alpha \left(\widetilde{K}_{t-1}\right)^{1-\alpha}.$$

As for the exogenous variables, only C_t^* needs to be stationarised. It can be done by dividing Y_t as well, but $\tilde{C}_t^* \equiv C_t^*/Y_t$ will be different across countries because Y_t is different.