Modelling Concept Interpolation in Description Logics using Abstract Betweenness Relations

Steven Schockaert, Yazmín Ibáñez-García, and Víctor Gutiérrez-Basulto

School of Computer Science & Informatics, Cardiff University, UK
{SchockaertS1,IbanezGarciaY,GutierrezBasultoV}@cardiff.ac.uk

Abstract. Interpolation is a strategy for deriving plausible conclusions based on background knowledge about a particular kind of conceptual relatedness. Specifically, we say that a concept B is between the concepts $A_1, ..., A_n$ if natural properties that hold for each of the concepts $A_1, ..., A_n$ are likely to hold for the concept B as well. In the context of description logics, such conceptual betweenness relations allow us to infer plausible concept inclusions. In previous work, two semantics have been proposed for characterising this interpolation mechanism: a feature-based semantics inspired by formal concept analysis and a geometric semantics inspired by conceptual spaces. While interpolation is sound under both semantics, their motivation has to some extent been ad hoc. Taking a different approach, in this paper we start from ternary betweenness relations, defined on triples of individuals, and we impose certain desirable properties on such relations. As our main result, we show a close correspondence between the feature based semantics and the proposed semantics based on betweenness relations.

Keywords: Description logics · Plausible Reasoning · Concept Interpolation · Betweenness.

1 Introduction

Description logics are used to characterise concepts in terms of their logical relationships to other concepts. Despite having many advantages, such logic-based formalisations lack some of the flexibility of vector representations, especially with respect to supporting inductive generalisation. For instance, suppose we know that *banana*, *apple* and *kiwi* are types of fruit, and suppose we are given vector representation of these entities, as well as vector representations of other entities such as *orange*. By observing that the representation of *orange* is located in the same region of the vector space as *banana*, *apple* and *kiwi*, we can then infer that oranges are likely to be fruit as well. This view of inductive generalization in terms of vector space similarity has been extensively studied in cognitive science [9]. From a practical point of view, such strategies have also been found effective for modelling concepts in vector space embeddings of individuals [3, 4].

The motivation of our work is to make a similar inductive generalisation mechanism available for flexible reasoning with description logic ontologies. The key idea is to rely on a type of conceptual relationship which we call conceptual *betweenness*: we say that A is between the concepts B_1 and B_2, written $A \sqsubseteq B_1 \bowtie B_2$, if properties that are true for both B_1 and B_2 can be expected to be true for A as well. We are concerned with defining a suitable semantics for $\bowtie$, such that from $A \sqsubseteq B_1 \bowtie B_2$, $B_1 \sqsubseteq C$ and $B_2 \sqsubseteq C$, we can derive $A \sqsubseteq C$, provided that C is *natural* in some sense. We refer to this inference pattern as *interpolation*[1]. Note that the notion of naturalness is common in theories of induction [11, 19, 8]. It is easy to see that some kind of condition to limit inductive generalisations is indeed required; e.g. for $C = B_1 \sqcup B_2$ the inference pattern is obviously not valid. For example, from $\{\mathsf{Orange} \sqsubseteq \mathsf{Apple} \bowtie \mathsf{Kiwi}, \mathsf{Apple} \sqsubseteq \mathsf{Apple} \sqcup \mathsf{Kiwi}, \mathsf{Kiwi} \sqsubseteq \mathsf{Apple} \sqcup \mathsf{Kiwi}\}$ there is no reason to infer $\mathsf{Orange} \sqsubseteq \mathsf{Apple} \sqcup \mathsf{Kiwi}$.

In [12], we introduced two semantics for betweenness and naturalness, both of which support interpolation but differ in how betweenness interacts with intersection, among others. In both cases, rather strong assumptions are made about how concepts are represented and how *natural concepts* are defined. In this paper, we take a different approach and start from an abstract ternary betweenness relation over individuals, where we write $bet(a, b, c)$ to denote that b is between a and c. We then say that $A \sqsubseteq B_1 \bowtie B_2$ is satisfied in an interpretation $\mathcal{I}$ if every individual in $A^{\mathcal{I}}$ is between some individual from $B_1^{\mathcal{I}}$ and some individual from $B_2^{\mathcal{I}}$. The two semantics from [12] can be seen as special cases of the approach we introduce here, where the betweenness relation *bet* is defined in a particular way. The interest in starting from an abstract betweenness relation is that we can be specific about the properties that we want to impose on this relation. Our main contributions in this paper are as follows:

1. We introduce a semantics for interpolation based on abstract ternary betweenness relations, and we discuss a number of natural properties that such relations should ideally satisfy.
2. We show that this semantics coincides with a generalization of the feature-enriched semantics from [12], provided that the ternary betweenness relation is required to satisfy a number of particular conditions.

The paper is structured as followed. In the next section, we recall the logic $\mathcal{EL}^{\bowtie}$ from [12], which extends $\mathcal{EL}$ with in-between concepts and an associated interpolation mechanism. Section 3 subsequently introduces a generalisation of the feature-enriched semantics from [12], introducing the notion of *abstract feature-enriched interpretations*. This generalised semantics allows us to consider the logic $\mathcal{EL}_{\perp}^{\bowtie}$, which extends $\mathcal{EL}^{\bowtie}$ with the ability to express disjointness. In Section 4, we then introduce a new semantics for $\mathcal{EL}_{\perp}^{\bowtie}$, based on ternary betweenness relations. Finally, we study how this new semantics can be related to the (abstract) feature-enriched semantics. In particular, Section 5 shows how an abstract feature-enriched interpretation can be constructed from a given betweenness relation in a satisfiability preserving way, while 6 considers the opposite direction.

[1] This is not to be confused with the notions of interpolation that are used to relate logical theories [6, 16].

2 Background

In this section, we recall the logic $\mathcal{EL}^{\bowtie}$ from [12], which extends the logic $\mathcal{EL}$ with the aim of supporting interpolation.

Syntax. The logic $\mathcal{EL}^{\bowtie}$ extends the standard description logic $\mathcal{EL}$ with *in-between concepts* of the form $C \bowtie D$, describing the set of objects that are between the concepts C and D. Further, $\mathcal{EL}^{\bowtie}$ includes countably infinite but disjoint sets of *concept names* $\mathsf{N_C}$ and *role names* $\mathsf{N_R}$, where $\mathsf{N_C}$ contains a distinguished infinite set of *natural concept names* $\mathsf{N_C^{Nat}}$. The syntax of $\mathcal{EL}^{\bowtie}$ *concepts* C, D is defined by the following grammar, where $A \in \mathsf{N_C}$, $A' \in \mathsf{N_C^{Nat}}$ and $r \in \mathsf{N_R}$:

$$C, D := \top \mid A \mid C \sqcap D \mid \exists r.C \mid N \tag{1}$$

$$N, N' := A' \mid N \sqcap N' \mid N \bowtie N' \tag{2}$$

Concepts of the form N, N' are called *natural concepts*. An $\mathcal{EL}^{\bowtie}$ *TBox* is a finite set of concept inclusions $C \sqsubseteq D$, where C, D are $\mathcal{EL}^{\bowtie}$ concepts.

Feature-Enriched Semantics The semantics of $\mathcal{EL}^{\bowtie}$ can be defined in terms of feature-enriched interpretations, which extend standard first-order interpretations by also specifying a mapping π from individuals to sets of *features* $\mathcal{F}$. The intuition is that these features characterise concepts at a sufficiently fine-grained level to capture similarity in a way that is sufficient for modelling inductive generalisation. Note that this is a common approach for representing concepts in cognitive science [25]. It is important to emphasise that these features may not correspond to properties that can be encoded in the syntax.

Formally, a *feature-enriched* interpretation is a tuple $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ in which $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a classical DL interpretation, $\mathcal{F}$ is a non-empty finite set of features and π is a mapping assigning to every $d \in \Delta^{\mathcal{I}}$ a proper subset of $\mathcal{F}$ such that the following hold:

1. For each $d \in \Delta^{\mathcal{I}}$ it holds that $\pi(d) \subset \mathcal{F}$;
2. for each $F \subset \mathcal{F}$ there exists some individual $d \in \Delta^{\mathcal{I}}$ such that $\pi(d) = F$.

For a standard $\mathcal{EL}$ concept C, we define $C^{\mathfrak{I}}$ as $C^{\mathcal{I}}$, where $C^{\mathcal{I}}$ is defined as usual [1]. To define the semantics of in-between concepts, with each concept C we associate a corresponding set of features $\varphi^{\mathfrak{I}}(C)$ as follows:

$$\varphi^{\mathfrak{I}}(C) = \bigcap \{\pi(d) \mid d \in C^{\mathcal{I}}\}.$$

We then define:

$$(N \bowtie N')^{\mathfrak{I}} = \{d \in \Delta^{\mathcal{I}} \mid \varphi^{\mathfrak{I}}(N) \cap \varphi^{\mathfrak{I}}(N') \subseteq \pi(d)\}.$$

Intuitively, $(N \bowtie N')^{\mathfrak{I}}$ contains those elements from $\Delta^{\mathcal{I}}$ that have all the features that N and N' have in common. Note that for any individual d we have required $\pi(d) \neq \mathcal{F}$. This is useful because it implies that $\varphi^{\mathfrak{I}}(C) = \mathcal{F}$ iff $C^{\mathfrak{I}} = \emptyset$. A feature-enriched interpretation $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ *satisfies* a concept inclusion $C \sqsubseteq D$ if $C^{\mathfrak{I}} \subseteq D^{\mathfrak{I}}$. $\mathfrak{I}$ is a *model* of an $\mathcal{EL}^{\bowtie}$ TBox $\mathcal{T}$ if it satisfies all CIs in $\mathcal{T}$ and for every natural concept N in $\mathcal{T}$, it holds that

$$N^{\mathfrak{I}} = \{d \in \Delta^{\mathcal{I}} \mid \varphi^{\mathfrak{I}}(N) \subseteq \pi(d)\} \tag{3}$$

i.e. N is fully specified by its features. If (3) is satisfied, we say that N is natural in $\mathfrak{I}$. It is easy to verify that (3) is satisfied for a complex natural concept, as soon as it is satisfied for its constituent natural concept names. Note that for natural concepts C and D we have that $C \sqsubseteq D$ is satisfied iff $\varphi^{\mathfrak{I}}(D) \subseteq \varphi^{\mathfrak{I}}(C)$.

3 Abstract Feature-Enriched Semantics

We now consider the logic $\mathcal{EL}_{\bot}^{\bowtie}$, which extends $\mathcal{EL}_{\bot}$ in the same way that $\mathcal{EL}^{\bowtie}$ extends $\mathcal{EL}$. In the feature-enriched semantics, all proper subsets $F \subset \mathcal{F}$ are witnessed, in the sense that there is some d such that $\pi(d) = F$. As shown below, it turns out that this assumption is too restrictive when $\bot$ is added to the language. First, we define satisfiability: a concept C is *satisfiable* w.r.t. a TBox $\mathcal{T}$ if there is a model $\mathfrak{I}$ of $\mathcal{T}$ such that $C^{\mathfrak{I}} \neq \emptyset$.

Example 1. The concept B cannot be satisfied w.r.t. $\{B \sqsubseteq A \bowtie C, A \sqcap B \sqsubseteq \bot, B \sqcap C \sqsubseteq \bot\}$ using a feature-enriched interpretation. Indeed, from $A \sqcap B \sqsubseteq \bot$ and $B \sqcap C \sqsubseteq \bot$, we find $\varphi^{\mathfrak{I}}(A) \cup \varphi^{\mathfrak{I}}(B) = \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(B) = \mathcal{F}$ and thus $(\varphi^{\mathfrak{I}}(A) \cap \varphi^{\mathfrak{I}}(C)) \cup \varphi^{\mathfrak{I}}(B) = \mathcal{F}$. However, from $B \sqsubseteq A \bowtie C$ we find $\varphi^{\mathfrak{I}}(A) \cap \varphi^{\mathfrak{I}}(C) \subseteq \varphi^{\mathfrak{I}}(B)$. Together we thus find $\varphi^{\mathfrak{I}}(B) = \mathcal{F}$ or equivalently $B^{\mathcal{I}} = \emptyset$.

This example shows that under the current semantics it is not possible for a concept B to be between the concepts A and C if all these concepts are disjoint. To address this limitation, we introduce abstract feature-enriched interpretations as follows.

Definition 1. *An abstract feature-enriched interpretation is a tuple $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ s.t. $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a classical DL interpretation, $\mathcal{F}$ is a finite set of features, and $\pi : \Delta^{\mathcal{I}} \to 2^{\mathcal{F}}$ such that $\pi(d) \subset \mathcal{F}$ for all $d \in \Delta^{\mathcal{I}}$.*

Abstract feature-enriched interpretations thus generalise feature-enriched interpretations by no longer requiring that all subsets X of $\mathcal{F}$ are witnessed, in the sense that there is some individual x such that $\pi(x) = X$. The abstract feature-enriched semantics is then defined as before, where abstract feature-enriched interpretations are used instead of feature-enriched interpretations. Notably, the following properties of the feature-enriched semantics, which are required for making some plausible inferences, remain satisfied for abstract feature-enriched interpretations.

Proposition 1 (Interpolation). *Let $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ be an abstract feature-enriched interpretation, satisfying $C \sqsubseteq X$ and $D \sqsubseteq Y$. Then $\mathfrak{I}$ also satisfies $C \bowtie D \sqsubseteq X \bowtie Y$.*

Proposition 2. *For any abstract feature-enriched interpretation $\mathfrak{I}$ and any concepts C and D it holds that*

$$\varphi^{\mathfrak{I}}(C \bowtie D) = \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$$

However, there are some properties from the feature-enriched semantics that are no longer satisfied for abstract feature-enriched interpretations. First, we no longer have that $\varphi^{\mathfrak{I}}(C \sqcap D) = \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$ in general, even when C and D are natural concepts, as the following counterexample illustrates.

Example 2. Let the abstract feature-enriched interpretation $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ be defined as $\Delta^{\mathcal{I}} = \{x_1, x_2, x_3, x_4\}$, $\mathcal{F} = \{f_1, f_2, f_3, f_4, f_5\}$ and

$$\pi(x_1) = \{f_1, f_2\} \qquad \pi(x_2) = \{f_2, f_3, f_4\} \qquad \pi(x_3) = \{f_3, f_5\}$$
$$C^{\mathfrak{I}} = \{x_1, x_2\} \qquad D^{\mathfrak{I}} = \{x_2, x_3\}$$

Then we have $\varphi^{\mathfrak{I}}(C) = \{f_2\}$, $\varphi^{\mathfrak{I}}(D) = \{f_3\}$ and $\varphi^{\mathfrak{I}}(C \sqcap D) = \{f_2, f_3, f_4\}$.

The fact that $C \sqcap D$ may have features beyond those of C and D intuitively makes sense. From a practical point of view, however, the fact that $\varphi^{\mathfrak{I}}(C \sqcap D)$ cannot be determined from $\varphi^{\mathfrak{I}}(C)$ and $\varphi^{\mathfrak{I}}(D)$ limits the kinds of plausible inferences we can make. For instance, this means that we can no longer infer $B \sqcap X \sqsubseteq Y$ from $A \sqcap X \sqsubseteq Y$, $C \sqcap X \sqsubseteq Y$ and $B \sqsubseteq A \bowtie C$, with all concepts assumed to be natural. This means in particular that a notion of non-interference, restricting how X and $A \bowtie C$ interact, would need to be added to the language, similar to what was done for the geometric semantics in [12].

However, we still have that $C \sqcap D$ is natural in $\mathfrak{I}$ whenever C and D are natural. In particular, we have the following result.

Proposition 3. *Let $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ be an abstract feature-enriched interpretation. If N is a natural concept, as defined by* (2)*, then it holds that N is natural in $\mathfrak{I}$, in the sense of* (3)*.*

Let us now consider how the semantics is affected if we impose conditions on which subsets of $\mathcal{F}$ are witnessed. First, let us consider the following condition, which intuitively states that for all individuals x and y there must be some individual that is in-between.

Definition 2 (Downward closure). *An abstract feature-enriched interpretation $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ satisfies downward closure if for all $x, y \in \Delta^{\mathcal{I}}$ there exists an individual $z \in \Delta^{\mathcal{I}}$ such that $\pi(z) = \pi(x) \cap \pi(y)$.*

Second, we also consider the following dual condition.

Definition 3 (Upward closure). *An abstract feature-enriched interpretation $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ satisfies upward closure if for all $x, y \in \Delta^{\mathcal{I}}$ there exists an individual $z \in \Delta^{\mathcal{I}}$ such that $\pi(z) = \pi(x) \cup \pi(y)$.*

This second condition is closely related to how intersections are modelled. In particular, requiring upward closure restores the equality between $\varphi^{\mathfrak{I}}(C \sqcap D)$ and $\varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$.

Proposition 4. *Suppose that $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ satisfies upward closure. Then for natural concepts C and D it holds that*

$$\varphi^{\mathfrak{I}}(C \sqcap D) = \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$$

However, upward closure also implies that the conjunction of any two concepts is satisfiable.[2] For this reason, upward closure does not seem to be a desirable property. Downward closure, on the other hand, will play an important role in this paper. The main consequence of imposing downward closure is stated in the following proposition, which essentially says that each non-empty natural concept C has a prototype when downward closure is satisfied.

Proposition 5. *Suppose that $\mathfrak{I} = (\mathcal{I}, \mathcal{F}, \pi)$ satisfies downward closure. If concept C is natural in $\mathfrak{I}$ and $C^{\mathfrak{I}} \neq \emptyset$, then there exists some $x \in C^{\mathfrak{I}}$ such that $\pi(x) = \varphi^{\mathfrak{I}}(C)$.*

4 Abstract Betweenness Semantics

The intuition of in-between concepts is that $C \bowtie D$ contains all individuals that are between instances of C and instances of D. However, the feature-enriched semantics only captures this intuition indirectly, and it is unclear which unintended consequences this semantics might have (beyond the issue already identified in Example 1). For this reason, we now introduce a semantics for $\mathcal{EL}_{\perp}^{\bowtie}$ that is directly built from a ternary betweenness relation over the set of individuals. Formally, we define an abstract betweenness interpretation as follows.

Definition 4. *An abstract betweenness interpretation is a tuple $\mathfrak{I} = (\mathcal{I}, bet)$ such that $(\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ is a classical DL interpretation and $bet \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$.*

Similar to abstract feature-enriched interpretations, we refer to the interpretations from Definition 4 as "abstract" interpretations, to highlight that we will need to impose some further conditions, in this case on the relation *bet*, to ensure that the semantics behaves in an intuitive way. The semantics of in-between concepts is now defined as follows:

$$(C \bowtie D)^{\mathfrak{I}} = C^{\mathfrak{I}} \cup D^{\mathfrak{I}} \cup \{y \in \Delta^{\mathcal{I}} \mid \exists x \in C^{\mathfrak{I}}, z \in D^{\mathfrak{I}} \,.\, bet(x, y, z)\} \tag{4}$$

A concept C is natural in $\mathfrak{I}$ if the following equality is satisfied:

$$C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}} \tag{5}$$

This definition is inspired by the theory of conceptual spaces [8], where natural concepts are those which are represented by convex regions. The definition in

[2] It also follows that there is some feature f such that $f \notin \pi(x)$ for all $x \in \Delta^{\mathcal{I}}$.

(5) can indeed be seen as capturing the idea of convexity: any individual which is between individuals from $C^{\mathfrak{I}}$ must itself also belong to $C^{\mathfrak{I}}$. Satisfaction is defined as before. The following result follows trivially from the definition of the abstract betweenness semantics, without requiring any additional conditions.

Proposition 6 (Interpolation). *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation satisfying $\mathcal{T} = \{C_1 \sqsubseteq D_1, C_2 \sqsubseteq D_2\}$. Then $\mathfrak{I}$ also satisfies $(C_1 \bowtie C_2) \sqsubseteq (D_1 \bowtie D_2)$.*

We also have the following result.

Proposition 7. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation. If C and D are natural in $\mathfrak{I}$ then $C \sqcap D$ is natural in $\mathfrak{I}$ as well.*

4.1 Conditions on Betweenness Relations

We now consider a number of additional conditions that we may impose on *bet*. A useful starting point is the notion of betweenness space.

Definition 5. *The pair $(\Delta^{\mathcal{I}}, bet)$ is called a betweenness space if the following conditions are satisfied [17]:*

Acyclicity $\forall x, y, z \in \Delta^{\mathcal{I}} \,.\, bet(x, y, x) \Rightarrow (x = y)$.
Left-reflexivity $\forall x, y \in \Delta^{\mathcal{I}} \,.\, bet(x, x, y)$.
Symmetry $\forall x, y, z \in \Delta^{\mathcal{I}} \,.\, bet(x, y, z) \Leftrightarrow bet(z, y, x)$.
Transitivity1 $\forall x, y, z, u \in \Delta^{\mathcal{I}} \,.\, bet(x, y, z) \wedge bet(x, z, u) \Rightarrow bet(x, y, u)$.
Transitivity2 $\forall x, y, z, u \in \Delta^{\mathcal{I}} \,.\, bet(x, y, z) \wedge bet(x, z, u) \Rightarrow bet(y, z, u)$.

Ternary relations satisfying the conditions from Definition 5 are called metrizable betweenness relations in [17], as they are satisfied whenever *bet* can be defined as $bet(a, b, c) = \{(a, b, c) \mid d(a, c) = d(a, b) + d(b, c)\}$ for some metric d on $\Delta^{\mathcal{I}}$.

Clearly, the conditions satisfied by *bet* have a direct impact on the semantics of in-between concepts. For instance, the following result follows trivially.

Proposition 8. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation. If bet satisfies symmetry, then for any concepts C and D, it holds that:*

$$(C \bowtie D)^{\mathfrak{I}} = (D \bowtie C)^{\mathfrak{I}}$$

The following condition on *bet* is needed to ensure that $A \bowtie B$ is natural whenever A and B are natural.

Continuity $\forall a_1, a_3, b_1, b_2, b_3, c_1, c_3 \in \Delta^{\mathcal{I}} \,.\, bet(a_1, b_1, c_1) \wedge bet(a_3, b_3, c_3)$
$\wedge bet(b_1, b_2, b_3) \Rightarrow \exists a_2, c_2 \in \Delta^{\mathcal{I}} \,.\, bet(a_1, a_2, a_3) \wedge bet(c_1, c_2, c_3) \wedge bet(a_2, b_2, c_2)$.

In particular, we have the following result.

Proposition 9. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation such that bet satisfies continuity. If C and D are natural in $\mathfrak{I}$, it holds that $C \bowtie D$ is natural in $\mathfrak{I}$ as well.*

Combining Propositions 7 and 9, we obtain the following corollary.

Corollary 1. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation such that bet satisfies continuity. If N is a natural concept, as defined by* (2)*, it holds that N is natural in $\mathfrak{I}$, in the sense of* (5)*.*

As another notable consequence of continuity, we find that the in-between connective $\bowtie$ satisfies associativity.

Proposition 10. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation such that bet satisfies left-reflexivity, symmetry and continuity. For all natural concepts A, B, C it holds that:*

$$((A \bowtie B) \bowtie C)^{\mathfrak{I}} = (A \bowtie (B \bowtie C))^{\mathfrak{I}}$$

Note that from the above proposition, we immediately find the following counterpart to the transitivity1 condition.

Corollary 2. *Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation such that bet satisfies left-reflexivity, symmetry and continuity. For all natural concepts A, B, C, D it holds that:*

$$\big(\mathfrak{I} \models \{B \sqsubseteq A \bowtie C, C \sqsubseteq A \bowtie D\}\big) \quad \Rightarrow \quad \big(\mathfrak{I} \models B \sqsubseteq A \bowtie D\big)$$

Finally, we will also consider the following notion of non-triviality:

Non-triviality $\forall x \in \Delta^{\mathcal{I}} \,.\, \exists y \in \Delta^{\mathcal{I}} \,.\, \neg bet(y, x, y)$.

Note that acyclicity implies non-triviality, provided that $|\Delta^{\mathcal{I}}| \geq 2$.

5 From Betweenness Relations to Features

Let $\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation. In Section 5.1, we first introduce a construction for deriving an abstract feature-enriched interpretation $\mathfrak{K} = (\mathcal{I}, \mathcal{F}, \pi)$ from $\mathfrak{I}$. In 5.2 we then discuss under what conditions the interpretations $\mathfrak{I}$ and $\mathfrak{K}$ are equivalent, in the sense that $C^{\mathfrak{I}} = C^{\mathfrak{K}}$ for every concept C. Throughout the section, we assume that $\Delta^{\mathcal{I}}$ is finite.

5.1 Construction

Definition 6. *We call a set of individuals $A \subseteq \Delta^{\mathcal{I}}$ convex (w.r.t. the relation bet) if*

$$\forall x, z \in A \,.\, bet(x, y, z) \Rightarrow y \in A$$

It is easy to see that for every set $A \subseteq \Delta^{\mathcal{I}}$, there must exist a smallest convex set which contains A, i.e. the least fixpoint of the following sequence, where $A_0 = A$:

$$A_{i+1} = A_i \cup \{y \mid \exists x, z \in A_i \,.\, bet(x, y, z)\} \tag{6}$$

We will call this least fixpoint the *convex hull of* A and will denote it by $CH(A)$. We say that A is convex if $A = CH(A)$. Let $\mathcal{C}$ be the set of all convex subsets of $\Delta^{\mathcal{I}}$. We associate with each convex set $A \in \mathcal{C}$ a feature f_A and we define:

$$\mathcal{F} = \{f_A \mid A \in \mathcal{C}\} \qquad \pi(x) = \{f_A \mid x \in A\} \tag{7}$$

The following result shows that $\mathfrak{K} = (\mathcal{I}, \mathcal{F}, \pi)$ is an abstract feature enriched interpretation, provided that *bet* is non-trivial.

Proposition 11. *$\mathfrak{I} = (\mathcal{I}, bet)$ be an abstract betweenness interpretation such that bet satisfies non-triviality and let π be defined as in* (7)*. For each $x \in \Delta^{\mathcal{I}}$ there exists a feature $f \in \mathcal{F}$ such that $f \notin \pi(x)$.*

5.2 Equivalence

Let us fix an abstract betweenness interpretation $\mathfrak{I} = (\mathcal{I}, bet)$ and let $\mathfrak{K} = (\mathcal{I}, \mathcal{F}, \pi)$, with $\mathcal{F}$ and π defined as in (7). We now analyse what conditions we need to impose on *bet* such that $C^{\mathfrak{I}} = C^{\mathfrak{K}}$ for every concept C. If C is a standard $\mathcal{EL}$ concept, then we trivially have $C^{\mathfrak{I}} = C^{\mathfrak{K}} = C^{\mathcal{I}}$, hence the main question is about the interpretation of in-between concepts. Before studying when $(C \bowtie D)^{\mathfrak{I}} = (C \bowtie D)^{\mathfrak{K}}$, we first show that the natural concepts in $\mathfrak{I}$ are also natural in $\mathfrak{K}$.

Lemma 1. *It holds that $C^{\mathfrak{I}}$ is convex iff C is natural in $\mathfrak{I}$.*

Lemma 2. *Let A be a concept name and suppose that bet satisfies non-triviality. If A is natural in $\mathfrak{I}$ then A is natural in $\mathfrak{K}$.*

We now analyse under what conditions it holds that $(C \bowtie D)^{\mathfrak{I}} = (C \bowtie D)^{\mathfrak{K}}$.

Lemma 3. *Suppose that bet satisfies continuity, symmetry and left-reflexivity, and let A and B be convex sets. Then it holds that*

$$CH(A \cup B) = A \cup B \cup \{y \mid \exists x \in A, z \in B \,.\, bet(x, y, z)\}$$

Lemma 4. *Suppose that bet satisfies continuity, symmetry and left-reflexivity, and let C and D be concepts that are natural in $\mathfrak{K}$. If $C^{\mathfrak{I}} = C^{\mathfrak{K}}$ and $D^{\mathfrak{I}} = D^{\mathfrak{K}}$, it holds that $(C \bowtie D)^{\mathfrak{I}} = (C \bowtie D)^{\mathfrak{K}}$.*

Proposition 12. *Suppose that bet satisfies continuity, symmetry, left-reflexivity and non-triviality. It holds that $C^{\mathfrak{I}} = C^{\mathfrak{K}}$ for every $\mathcal{EL}^{\bowtie}$ concept C.*

6 From Features to Betweenness Relations

In this section, we start from an abstract feature-enriched interpretation $\mathfrak{K} = (\mathcal{I}, \mathcal{F}, \pi)$, from which we derive an abstract betweenness interpretation $\mathfrak{I} = (\mathcal{I}, bet)$ such that $C^{\mathfrak{K}} = C^{\mathfrak{I}}$ for all concepts C. We again assume that $\Delta^{\mathcal{I}}$ is finite.

6.1 Construction

To define $\mathfrak{I} = (\mathcal{I}, bet)$, we only need to specify the relation *bet*. This relation is defined in terms of π as follows:

$$bet(x, y, z) \equiv \pi(y) \supseteq \pi(x) \cap \pi(z) \tag{8}$$

It is trivial to verify that the betweenness relation *bet* defined in (8) satisfies left-reflexivity and symmetry. Moreover, this relation also satisfies transitivity1, since $bet(x, y, z)$ and $bet(x, z, u)$ mean that $\pi(y) \supseteq \pi(x)\cap\pi(z) \supseteq \pi(x)\cap(\pi(x)\cap\pi(u)) = \pi(x) \cap \pi(u)$, and thus $bet(x, y, u)$. On the other hand, acyclicity is clearly not satisfied. As the following counter example shows, transitivity2 is not satisfied either.

Example 3. Let π be defined as follows:

$$\pi(x) = \pi(z) = \{f\} \qquad\qquad \pi(y) = \pi(u) = \{f, g\}$$

Then we have $bet(x, y, z)$ and $bet(x, z, u)$ but $bet(y, z, u)$

We also have the following result

Lemma 5. *If $\mathfrak{K}$ satisfies downwards closure, then the betweenness relation defined by* (8) *satisfies continuity.*

Non-triviality is not satisfied in general, but could among others be obtained by imposing that $\mathcal{F} = \bigcup_{x\in\Delta^{\mathcal{I}}} \pi(x)$, i.e. by assuming that all of the features in $\mathcal{F}$ are actually used in some way.

6.2 Equivalence

Let $\mathfrak{K} = (\mathcal{I}, \mathcal{F}, \pi)$ be an abstract feature-enriched interpretation, and let $\mathfrak{I} = (\mathcal{I}, bet)$ be the corresponding abstract betweenness interpretation, with *bet* defined as in (8). We find that $C^{\mathfrak{K}} = C^{\mathfrak{I}}$ for all concepts C, provided that $\mathfrak{K}$ satisfies downward closure. In particular, we can show the following results.

Lemma 6. *If C is natural in $\mathfrak{K}$ then C is natural in $\mathfrak{I}$.*

Lemma 7. *Assume that $\mathfrak{K}$ satisfies downwards closure and suppose that C and D are natural in $\mathfrak{K}$. If $C^{\mathfrak{K}} = C^{\mathfrak{I}}$ and $D^{\mathfrak{K}} = D^{\mathfrak{I}}$ then we also have that $(C \bowtie D)^{\mathfrak{K}} = (C \bowtie D)^{\mathfrak{I}}$.*

Proposition 13. *Suppose that $\mathfrak{K}$ satisfies downward closure. It holds that $C^{\mathfrak{I}} = C^{\mathfrak{K}}$ for every $\mathcal{EL}^{\bowtie}$ concept C.*

7 Related Work

One can think of comparative similarity and conceptual betweenness as two complementary approaches for reasoning about similarity in a qualitative way. The problem of formally combining logics and similarity is addressed in [21, 23], where an operator is introduced to express that a concept A is more similar to some concept B than to some concept C. Extensions of description logics based on rough sets [14, 20, 18] rely on the notion of indistinguishability, which is also closely related to qualitative similarity. Beyond qualitative approaches and in the context of description logics, fuzzy description logics [24, 2, 13] directly model degrees of similarity.

Plausible inferences in description logics has also been addressed by incorporating some form of defeasible reasoning. For example, Giordano et al. [10, 5], proposed preferencial semantics of concept inclusion to reason about typicality, and Britz et al. [5] introduced a semantic framework for plausible subsumption in description logics.

Within a broader context, [15] is also motivated by the idea of combining description logics with ideas from cognitive science, although their focus is on modelling typicality effects and compositionality, e.g. inferring the meaning of *pet fish* from the meanings of *pet* and *fish*, which is a well-known challenge for cognitive systems since typical pet fish are neither typical pets nor typical fish.

8 Conclusions and Future Work

We have provided a new semantics of in-between concepts, in terms of an abstract ternary betweenness relation, and we have shown how this semantics is closely related to the feature-enriched semantics from [12]. The overall aim of our work is to develop better mechanisms for adding inductive capabilities to description logic reasoners, by exploiting vector representations of concepts that can be learned from large text collections (among others). Our work is thus related to previous efforts for adding aspects of similarity-based reasoning to description logics [22, 7]. The notion of betweenness can be linked to vector spaces in different ways, however. The fact that A is between concepts $B_1, ..., B_n$ merely means that natural properties which are satisfied for $B_1, ..., B_n$ can be expected to hold for A as well. One important area for future work is thus to study specific ways of deriving betweenness relations from vector spaces. Another important issue is the notion of non-interference. In general, if B is between A and C, we do not necessarily have that $B \sqcap X$ is between $A \sqcap X$ and $C \sqcap X$, which is problematic as it drastically limits the kinds of inferences that can be made. The solution proposed in [12] is to introduce a mechanism for asserting that X does not "interfere" with the conceptual relationship between A, B and C. However, it remains poorly understood how such non-interference knowledge could be learned from data.

References

1. Baader, F., Horrocks, I., Lutz, C., Sattler, U.: An Introduction to Description Logic. Cambridge University Press (2017)
2. Bobillo, F., Cerami, M., Esteva, F., García-Cerdaña, À., Peñaloza, R., Straccia, U.: Fuzzy description logic. In: Handbook of Mathematical Fuzzy Logic Volume 3, Studies in Logic, vol. 58. College Publications (2015)
3. Boleda, G., Gupta, A., Padó, S.: Instances and concepts in distributional space. In: Proceedings EACL. pp. 79–85 (2017)
4. Bouraoui, Z., Schockaert, S.: Learning conceptual space representations of interrelated concepts. In: Proceedings IJCAI. pp. 1760–1766 (2018)
5. Britz, K., Heidema, J., Meyer, T.A.: Semantic preferential subsumption. In: KR. pp. 476–484. AAAI Press (2008)
6. Craig, W.: Three uses of the herbrand-gentzen theorem in relating model theory and proof theory. J. Symb. Log. **22**(3), 269–285 (1957). https://doi.org/10.2307/2963594, https://doi.org/10.2307/2963594
7. d'Amato, C., Fanizzi, N., Fazzinga, B., Gottlob, G., Lukasiewicz, T.: Ontology-based semantic search on the web and its combination with the power of inductive reasoning. Ann. Math. Artif. Intell. **65**(2-3), 83–121 (2012)
8. Gärdenfors, P.: Conceptual spaces: The geometry of thought. MIT press (2000)
9. Gärdenfors, P., Makinson, D.: Nonmonotonic inference based on expectations. Artif. Intell. **65**(2), 197–245 (1994)
10. Giordano, L., Gliozzi, V., Olivetti, N., Pozzato, G.L.: Preferential description logics. In: LPAR. Lecture Notes in Computer Science, vol. 4790, pp. 257–272. Springer (2007)
11. Goodman, N.: Fact, fiction, and forecast. Harvard University Press (1955)
12. Ibáñez-García, Y., Gutiérrez-Basulto, V., Schockaert, S.: Plausible reasoning about EL-ontologies using concept interpolation. In: Proceedings KR. pp. 506–516 (2020)
13. Jiang, Y., Tang, Y., Wang, J., Tang, S.: Reasoning within intuitionistic fuzzy rough description logics. Inf. Sci. **179**(14), 2362–2378 (2009)
14. Klein, M.C.A., Mika, P., Schlobach, S.: Rough description logics for modeling uncertainty in instance unification. In: Proc. of URSW (2007)
15. Lieto, A., Pozzato, G.L.: A description logic of typicality for conceptual combination. In: Proceedings of the 24th International Symposium on the Foundations of Intelligent Systems. pp. 189–199 (2018)
16. Lutz, C., Wolter, F.: Foundations for uniform interpolation and forgetting in expressive description logics. In: Proceedings of the 22nd International Joint Conference on Artificial Intelligence (IJCAI-11) (2011)
17. Mendris, R., Zlatoš, P.: Axiomatization and undecidability results for metrizable betweenness relations. Proceedings of the American Mathematical Society **123**(3), 873–882 (1995)
18. Peñaloza, R., Zou, T.: Roughening the envelope. In: Proc. of FroCos (2013)
19. Quine, W.V.: Natural kinds. In: Essays in honor of Carl G. Hempel, pp. 5–23. Springer (1969)
20. Schlobach, S., Klein, M.C.A., Peelen, L.: Description logics with approximate definitions - precise modeling of vague concepts. In: Proc. of IJCAI (2007)
21. Sheremet, M., Tishkovsky, D., Wolter, F., Zakharyaschev, M.: Comparative similarity, tree automata, and diophantine equations. In: Proc. of LPAR-2005 (2005)
22. Sheremet, M., Tishkovsky, D., Wolter, F., Zakharyaschev, M.: A logic for concepts and similarity. J. Log. Comput. **17**(3), 415–452 (2007)

23. Sheremet, M., Wolter, F., Zakharyaschev, M.: A modal logic framework for reasoning about comparative distances and topology. Ann. Pure Appl. Logic **161**(4), 534–559 (2010)
24. Straccia, U.: Reasoning within fuzzy description logics. Journal of Artificial Intelligence Research **14**, 137–166 (2001)
25. Tversky, A.: Features of similarity. Psychological review **84**(4), 327 (1977)

A Proofs

Proof of Proposition 1

Let $x \in (C \bowtie D)^{\mathfrak{I}}$. Then we have $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$. From the fact that $C \sqsubseteq X$ and $D \sqsubseteq Y$ are satisfied, we know that $\varphi^{\mathfrak{I}}(C) \supseteq \varphi^{\mathfrak{I}}(X)$ and $\varphi^{\mathfrak{I}}(D) \supseteq \varphi^{\mathfrak{I}}(Y)$. Thus we find $\pi(x) \supseteq \varphi^{\mathfrak{I}}(X) \cap \varphi^{\mathfrak{I}}(Y)$, and $x \in (X \bowtie Y)^{\mathfrak{I}}$.

Proof of Proposition 2

By definition we have $x \in (C \bowtie D)^{\mathfrak{I}}$ iff $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$ hence we immediately find $\varphi^{\mathfrak{I}}(C \bowtie D) \supseteq \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$. To show the converse direction, suppose $f \in \varphi^{\mathfrak{I}}(C \bowtie D)$. Then it must be the case that every x such that $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$ is also such that $f \in \pi(x)$. That means in particular that $f \in \pi(x)$ for any $x \in C^{\mathfrak{I}}$ and any $x \in D^{\mathfrak{I}}$, and thus that $f \in \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$.

Proof of Proposition 3

We need to show that whenever C and D are natural in $\mathfrak{I}$ it holds that $C \bowtie D$ and $C \sqcap D$ are also natural in $\mathfrak{I}$. The result for $C \bowtie D$ follows immediately from Proposition 2; we now show the result for $C \sqcap D$. Assume that C and D are natural in $\mathfrak{I}$. We have that

$$x \in (C \sqcap D)^{\mathfrak{I}} \quad \text{iff} \quad \pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$$

whereas we need to show that

$$x \in (C \sqcap D)^{\mathfrak{I}} \quad \text{iff} \quad \pi(x) \supseteq \varphi^{\mathfrak{I}}(C \sqcap D)$$

Clearly $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C \sqcap D)$ implies $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$, since $\varphi^{\mathfrak{I}}(C \sqcap D) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$. To complete the proof, we show that $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$ also implies $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C \sqcap D)$. This follows because $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$ implies $x \in (C \sqcap D)^{\mathfrak{I}}$, which in turn implies $\pi(x) \supseteq \varphi^{\mathfrak{I}}(C \sqcap D)$.

Proof of Proposition 4

Clearly we have that $\varphi^{\mathfrak{I}}(C \sqcap D) \supseteq \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$. We now show the other direction. Let $f \notin \varphi^{\mathfrak{I}}(C) \cup \varphi^{\mathfrak{I}}(D)$. Note that this implies $C^{\mathfrak{I}} \neq \emptyset$ and $D^{\mathfrak{I}} \neq \emptyset$, and moreover that there is some $x \in C^{\mathfrak{I}}$ such that $f \notin \pi(x)$ and some $y \in D^{\mathfrak{I}}$ such that $f \notin \pi(y)$. Since $\mathfrak{I}$ satisfies upward closure, there must be some z such that $\pi(z) = \pi(x) \cup \pi(y)$. We then have $z \in (C \sqcap D)^{\mathfrak{I}}$ and $f \notin \pi(z)$, from which we find $f \notin \varphi^{\mathfrak{I}}(C \sqcap D)$.

Proof of Proposition 5

Clearly, given that the number of features is finite, $\varphi^{\mathfrak{I}}(C)$ is the intersection of a finite number of sets $\pi(y_1), ..., \pi(y_n)$. Repeatedly applying the downward closure property, we find that there must be some individual x such that $\pi(x) = \pi(y_1) \cap ... \cap \pi(y_n) = \varphi^{\mathfrak{I}}(C)$. Since C is natural, it must also be the case that $x \in C^{\mathfrak{I}}$.

Proof of Proposition 6

Let $y \in (C_1 \bowtie C_2)^{\mathfrak{I}}$. Then there exist $x \in C_1^{\mathfrak{I}}$ and $z \in C_2^{\mathfrak{I}}$ such that $bet(x, y, z)$ holds. But then we also have $x \in D_1^{\mathfrak{I}}$ and $z \in D_2^{\mathfrak{I}}$, and thus $y \in (D_1 \bowtie D_2)^{\mathfrak{I}}$.

Proof of Proposition 7

We trivially have that $(C \sqcap D)^{\mathfrak{I}} \subseteq ((C \sqcap D) \bowtie (C \sqcap D))^{\mathfrak{I}}$ holds. Conversely, due to the monotonicity of $\bowtie$ w.r.t. subset inclusion, we have

$$((C \sqcap D) \bowtie (C \sqcap D))^{\mathfrak{I}} \subseteq (C \bowtie C)^{\mathfrak{I}} = C^{\mathfrak{I}}$$

where the last equality holds because C was assumed to be natural. In the same way, we find

$$((C \sqcap D) \bowtie (C \sqcap D))^{\mathfrak{I}} \subseteq D^{\mathfrak{I}}$$

and thus we also have that:

$$((C \sqcap D) \bowtie (C \sqcap D))^{\mathfrak{I}} \subseteq C^{\mathfrak{I}} \cap D^{\mathfrak{I}} = (C \sqcap D)^{\mathfrak{I}}$$

Proof of Proposition 9

Given $C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}}$ and $D^{\mathfrak{I}} = (D \bowtie D)^{\mathfrak{I}}$, we have to show that $(C \bowtie D)^{\mathfrak{I}} = ((C \bowtie D) \bowtie (C \bowtie D))^{\mathfrak{I}}$.

If $y \in (C \bowtie D)^{\mathfrak{I}}$ then there must exist $x \in C^{\mathfrak{I}}$ and $z \in D^{\mathfrak{I}}$ such that $bet(x, y, z)$. Then we also have $x \in (C \bowtie D)^{\mathfrak{I}}$ and $z \in (C \bowtie D)^{\mathfrak{I}}$. We thus find $y \in ((C \bowtie D) \bowtie (C \bowtie D))^{\mathfrak{I}}$.

Now, conversely, assume that $y \in ((C \bowtie D) \bowtie (C \bowtie D))^{\mathfrak{I}}$. Then there are $y_1, y_3 \in (C \bowtie D)^{\mathfrak{I}}$ such that $bet(y_1, y, y_3)$ holds, which in turn means that there are $x_1, x_3 \in C^{\mathfrak{I}}$ and z_1, z_3 in $D^{\mathfrak{I}}$ such that $bet(x_1, y_1, z_1)$ and $bet(x_3, y_3, z_3)$ hold. Since bet satisfies continuity, we know that there must exist individuals x_2, z_2 such that $bet(x_1, x_2, x_3)$, $bet(z_1, z_2, z_3)$, and $bet(x_2, y, z_2)$ hold. Since C is natural and $x_1, x_3 \in C^{\mathfrak{I}}$, it follows from $bet(x_1, x_2, x_3)$ that $x_2 \in C^{\mathfrak{I}}$. Similarly we find $z_2 \in D^{\mathfrak{I}}$. From $bet(x_2, y, z_2)$ we can thus conclude $y \in (C \bowtie D)^{\mathfrak{I}}$.

Proof of Proposition 10

If $x \in ((A \bowtie B) \bowtie C)^{\mathfrak{I}}$ there exist $a \in A^{\mathfrak{I}}$, $b \in B^{\mathfrak{I}}$, $c \in C^{\mathfrak{I}}$ and $y \in \Delta^{\mathcal{I}}$ such that $bet(y, x, c)$ and $bet(a, y, b)$. Using left-reflexivity and symmetry, we also have that $bet(a, c, c)$ holds. From continuity, it then follows that there must be some a' and z such that $bet(a, a', a)$, $bet(b, z, c)$ and $bet(a', x, z)$. Note that from $a \in A^{\mathcal{I}}$, $bet(a, a', a)$ and the fact that A is natural, it follows that $a' \in A^{\mathfrak{I}}$. From $bet(b, z, c)$, we furthermore find $z \in (B \bowtie C)^{\mathfrak{I}}$. Finally, from $bet(a', x, z)$ we find $x \in (A \bowtie (B \bowtie C))^{\mathfrak{I}}$. The converse direction follows entirely in the same way.

Proof of Proposition 11

Let $x \in \Delta^{\mathcal{I}}$. Since bet satisfies non-triviality, there must be some y such that $\neg bet(y, x, y)$. We then have $x \notin CH(y)$, and in particular that $f_{CH(y)} \notin \pi(x)$.

Proof of Lemma 1

Assume that $C^{\mathfrak{I}}$ is convex. We need to show that $C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}}$. Let C_i be defined as in (6), with $C_0 = C^{\mathfrak{I}}$. If $y \in (C \bowtie C)^{\mathfrak{I}}$ then by definition there are $x, z \in A^{\mathfrak{I}}$ such that $bet(x, y, z)$ holds. This means $y \in A_1$ and thus $y \in CH(A^{\mathfrak{I}}) = A$. In other words, we already have $C^{\mathfrak{I}} \supseteq (C \bowtie C)^{\mathfrak{I}}$. We trivially have that $C^{\mathfrak{I}} \subseteq (C \bowtie C)^{\mathfrak{I}}$ holds as well.

Now, conversely, suppose that $C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}}$; we show that $C^{\mathfrak{I}}$ is convex. To this end, we need to prove that $C_1 = C^{\mathfrak{I}}$. We trivially have $C_1 \supseteq C^{\mathfrak{I}}$, hence it suffices to show $C_1 \subseteq C^{\mathfrak{I}}$. Suppose $y \in C_1 \setminus C^{\mathfrak{I}}$. Then by definition of C_1 (i.e. by definition of the fixpoint procedure), there must exist $x, z \in C^{\mathfrak{I}}$ such that $bet(x, y, z)$ holds. This means $y \in (C \bowtie C)^{\mathfrak{I}}$, which means $y \in C^{\mathfrak{I}}$ since we assumed $C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}}$, a contradiction.

Proof of Lemma 2

If $A^{\mathcal{I}} = \emptyset$ then we have $\varphi^{\mathfrak{I}}(A) = \mathcal{F}$. Given Proposition 11, we know that $\{x \mid \pi(x) \supseteq \mathcal{F}\} = \emptyset$ and thus that A is natural in $\mathfrak{K}$.

Now suppose $A^{\mathcal{I}} \neq \emptyset$. Since A is natural in $\mathfrak{I}$, we know from Lemma 1 that A is convex, and thus we have that $f_A \in \mathcal{F}$. By definition of π, we have $f_A \in \pi(x)$ iff $x \in A^{\mathfrak{I}}$. Since $A^{\mathcal{I}} \neq \emptyset$ we thus have $f_A \in \varphi^{\mathfrak{I}}(A)$. From $\pi(x) \supseteq \varphi^{\mathfrak{I}}(A)$ we can thus infer $x \in A^{\mathfrak{I}} = A^{\mathcal{I}}$. In other words, we have

$$A^{\mathfrak{K}} = A^{\mathcal{I}} \supseteq \{x \mid \pi(x) \supseteq \varphi^{\mathfrak{I}}(A)\}$$

By definition of $\varphi^{\mathfrak{I}}$, we also trivially have

$$A^{\mathfrak{K}} = A^{\mathcal{I}} \subseteq \{x \mid \pi(x) \supseteq \varphi^{\mathfrak{I}}(A)\}$$

Proof of Lemma 3

Let us write $X = A \cup B$ and let X_i be the sequence of sets in the iterative process (6) for computing the convex hull of X. Then we have Note that

$$\begin{aligned} X_1 =& A \cup B \cup \{y \mid \exists x, z \in A \cup B \,.\, bet(x, y, z)\} \\ &= A \cup B \cup \{y \mid \exists x, z \in A \,.\, bet(x, y, z)\} \cup \{y \mid \exists x, z \in B \,.\, bet(x, y, z)\} \\ &\quad \cup \{y \mid \exists x \in A, z \in B \,.\, bet(x, y, z)\} \\ &= A \cup B \cup \{y \mid \exists x \in A, z \in B \,.\, bet(x, y, z)\} \end{aligned}$$

where the last step follows from the assumption that A and B are convex sets, hence $\{y \mid \exists x, z \in A \,.\, bet(x, y, z)\} \subseteq A$ and $\{y \mid \exists x, z \in B \,.\, bet(x, y, z)\} \subseteq B$. Note that we do not need to consider the case where $x \in B$ and $z \in A$ because we assumed that bet satisfies symmetry.

To conclude the proof, we show that $X_2 = X_1$. Suppose there were some $y \in X_2 \setminus X_1$. Then there must be $x, z \in X_1$ such that $bet(x, y, z)$ holds.

- We cannot have $x \in A \cup B$ and $z \in A \cup B$ since otherwise we clearly have $y \in X_1$.
- Suppose there exist $a_1, a_3 \in A$ and $b_1, b_3 \in B$ such that $bet(a_1, x, b_1)$ and $bet(a_3, z, b_3)$ hold. Using continuity, it then follows that there must exist a_2, b_2 such that $bet(a_1, a_2, a_3)$, $bet(b_1, b_2, b_3)$ and $bet(a_2, y, b_2)$ hold. Moreover, since A and B are assumed to be convex, from $bet(a_1, a_2, a_3)$ and $bet(b_1, b_2, b_3)$ it follows that $a_2 \in A$ and $b_2 \in B$. We thus again find $y \in X_1$.
- Suppose $x \in A$ and we have $a_3 \in A$ and $b_3 \in B$ such that $bet(a_3, z, b_3)$ holds. Because bet satisfies left-reflexivity, we also have that $bet(x, x, b_3)$ holds. Due to continuity there must exist some a_2, b_2 such that $bet(x, a_2, a_3)$, $bet(b_3, b_2, b_3)$ and $bet(a_2, y, b_2)$ hold. As in the previous case, we find that $a_2 \in A$ and $b_2 \in B$, and thus $y \in X_1$.
- Finally the case where $z \in B$ and we have $a_1 \in A$ and $b_1 \in B$ such that $bet(a_1, x, b_1)$ is entirely analogous to the previous case.

Proof of Lemma 4

Suppose $y \in (C \bowtie D)^{\mathfrak{I}}$. Then either we have (i) $y \in C^{\mathfrak{I}} \cup D^{\mathfrak{I}}$ or (ii) there exist $x \in C^{\mathfrak{I}}$ and $z \in D^{\mathfrak{I}}$ such that $bet(x, y, z)$ holds. In the former case, we have $y \in C^{\mathfrak{K}} \cup D^{\mathfrak{K}}$ and thus also $y \in (C \bowtie D)^{\mathfrak{K}}$. In the latter case, we have $x \in C^{\mathfrak{K}}$ and $z \in D^{\mathfrak{K}}$. Now suppose there is some feature $f_A \in \mathcal{F}$ such that $f_A \in \pi(x)$ and $f_A \in \pi(z)$. Since $bet(x, y, z)$ holds and A is convex, we must also have $y \in A$ and thus $f_A \in \pi(y)$. Thus, $\pi(y) \supseteq \pi(x) \cap \pi(z) \supseteq \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$, meaning $y \in (C \bowtie D)^{\mathfrak{K}}$ since we assumed that C and D were natural in $\mathfrak{K}$, which means that $C \bowtie D$ is also natural in $\mathfrak{K}$ by Proposition 3.

Now, conversely, assume that $y \in (C \bowtie D)^{\mathfrak{K}}$. Let $A = CH(C^{\mathfrak{I}} \cup D^{\mathfrak{I}})$. Then we have $f_A \in \varphi^{\mathfrak{I}}(C) \cap \varphi^{\mathfrak{I}}(D)$ and thus $f_A \in \pi(y)$. This means $y \in A$. By Lemma 3, we must then either have (i) $y \in C^{\mathfrak{I}} \cup D^{\mathfrak{I}}$ or (ii) there exist $x \in C^{\mathfrak{I}}$ and $z \in D^{\mathfrak{I}}$ such that $bet(x, y, z)$ holds. In both cases, we find $y \in (C \bowtie D)^{\mathfrak{I}}$.

Proof of Proposition 12

We show this result by structural induction. If C is a concept name or $C = \top$, we trivially have $C^{\mathfrak{I}} = C^{\mathfrak{K}} = C^{\mathcal{I}}$. If C is of the form $C_1 \sqcap C_2$, we have $(C_1 \sqcap C_2)^{\mathfrak{I}} = C_1^{\mathfrak{I}} \cap C_2^{\mathfrak{I}} = C_1^{\mathfrak{K}} \cap C_2^{\mathfrak{K}} = (C_1 \sqcap C_2)^{\mathfrak{K}}$. If C is of the form $\exists r.C$, we have

$$\begin{aligned}
(\exists r.C)^{\mathfrak{I}} &= \{d \in \Delta^{\mathcal{I}} \mid \exists d' \in C^{\mathfrak{I}}, (d, d') \in r^{\mathcal{I}}\} \\
&= \{d \in \Delta^{\mathcal{I}} \mid \exists d' \in C^{\mathfrak{K}}, (d, d') \in r^{\mathcal{I}}\} \\
&= (\exists r.C)^{\mathfrak{K}}
\end{aligned}$$

Finally, we consider the case where C is of the form $C_1 \bowtie C_2$. In this case, C_1 and C_2 must be natural concepts, and thus we have that all the concept names appearing in C_1 and C_2 must be natural in $\mathfrak{I}$. From Lemma 2 we know that all the concept names appearing in C_1 and C_2 must then also be natural in $\mathfrak{K}$. It follows from Proposition 3 that C_1 and C_2 are natural in $\mathfrak{K}$. From Lemma 4 we can thus conclude that $(C_1 \bowtie C_2)^{\mathfrak{I}} = (C_1 \bowtie C_2)^{\mathfrak{K}}$.

Proof of Lemma 5

Suppose that $bet(a_1, b_1, c_1)$, $bet(a_3, b_3, c_3)$ and $bet(b_1, b_2, b_3)$ hold. We choose a_2 and c_2 as individuals with the following features: $\pi(a_2) = \pi(a_1) \cap \pi(a_3)$ and $\pi(c_2) = \pi(c_1) \cap \pi(c_3)$. Note that these individuals have to exist, since $\mathfrak{K}$ was assumed to satisfy downward closure. From the assumptions $bet(a_1, b_1, c_1)$, $bet(a_3, b_3, c_3)$ and $bet(b_1, b_2, b_3)$, we know that $\pi(b_1) \supseteq \pi(a_1) \cap \pi(c_1)$, $\pi(b_3) \supseteq \pi(a_3) \cap \pi(c_3)$ and $\pi(b_2) \supseteq \pi(b_1) \cap \pi(b_3)$. Together this implies $\pi(b_2) \supseteq \pi(a_1) \cap \pi(c_1) \cap \pi(a_3) \cap \pi(c_3)$, which implies $\pi(b_2) \supseteq \pi(a_2) \cap \pi(c_2)$, which is equivalent with $bet(a_1, b_2, c_2)$.

Proof of Lemma 6

Assume that C is natural in $\mathfrak{K}$. We need to show that $C^{\mathfrak{I}} = (C \bowtie C)^{\mathfrak{I}}$. In particular, let $x, z \in C^{\mathfrak{I}}$ and assume that $bet(x, y, z)$ holds. We need to show that $y \in C^{\mathfrak{I}}$. This follows because $\pi(y) \supseteq \pi(x) \cap \pi(z)$, given $bet(x, y, z)$, and $\pi(x) \cap \pi(z) \supseteq \varphi^{\mathfrak{I}}(C)$, given that $x, z \in C^{\mathfrak{I}}$.

Proof of Lemma 7

If $y \in (C \bowtie D)^{\mathfrak{K}}$ then by definition we have $\pi(y) \supseteq \varphi^{\mathfrak{K}}(C) \cap \varphi^{\mathfrak{K}}(D)$. From Proposition 5, we know that there exist $x \in C^{\mathfrak{K}}$ and $z \in d^{\mathfrak{K}}$ such that $\pi(x) = \varphi^{\mathfrak{K}}(C)$ and $\pi(z) = \varphi^{\mathfrak{K}}(D)$. This means $\pi(y) \supseteq \pi(x) \cap \pi(z)$ and thus $bet(x, y, z)$, which means $y \in (C \bowtie D)^{\mathfrak{I}}$.

Conversely, if $y \in (C \bowtie D)^{\mathfrak{I}}$, there must be $x \in C^{\mathfrak{I}}$ (and thus $x \in C^{\mathfrak{K}}$) and $z \in D^{\mathfrak{I}}$ (and thus $x \in D^{\mathfrak{K}}$) such that $bet(x, y, z)$, i.e. $\pi(y) \supseteq \pi(x) \cap \pi(y)$. But then we must also have $\pi(y) \supseteq \varphi^{\mathfrak{K}}(C) \cap \varphi^{\mathfrak{K}}(D)$, which means $y \in (C \bowtie D)^{\mathfrak{K}}$.

Proof of Proposition 13

We show this result by structural induction. For concept names and concepts of the form $C_1 \sqcap C_2$ or $\exists r.C$, we find the result in the same way as in Proposition 12. Now consider the case where C is of the form $C_1 \bowtie C_2$. In this case, C_1 and C_2 must be natural concepts. From Lemma 7 we can thus conclude that $(C_1 \bowtie C_2)^{\mathfrak{I}} = (C_1 \bowtie C_2)^{\mathfrak{K}}$.