Perverse Schobers and the McKay Correspondence

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February 2021

A thesis submitted in partial fulfilment of
the requirements for the degree of
Doctor of Philosophy

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Summary

In [BKS18], Bondal, Kapranov and Schechtman gave the definition of a conjectural categorical analogue of perverse sheaves, known as *perverse schobers*. More accurately, due to the difficulties involved in categorifying the definition of perverse sheaves directly, they take a description of the category of perverse sheaves on a linear hyperplane arrangement $\mathcal{H} \subseteq \mathbb{R}^n$ in terms of a quiver representation due to Kapranov and Schechtman [KS16], and categorify this description. They call this notion an $\mathcal{H}$-schober.

In Chapter 1, we provide the background material for this thesis. In particular, we give the aforementioned quiver description of the category of perverse sheaves and the definition of an $\mathcal{H}$-schober.

In Chapter 2, we introduce the notion of geometric invariant theory quotients, which depend on a choice of stability parameter $\mathcal{L}$; studying how this quotient changes as we vary the stability parameter is known as variation of geometric invariant theory (VGIT). For a given choice of stability parameter, we recount an iterative process for stratifying the unstable locus into a disjoint union of pieces, known as Kempf-Ness strata. An analysis of these KN strata leads to a method of constructing wall-crossing equivalences in VGIT via window subcategories.

In Chapter 3, we describe a VGIT problem arising from the McKay correspondence. This naturally produces a hyperplane arrangement $\mathcal{H}$ and, for each cell in this hyperplane arrangement, the derived category of a quotient stack. In the remainder of this chapter we investigate the geometry of these quotient stacks for the particular case of $G = \mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$.

In Chapter 4, we build an $\mathcal{H}$-schober from the McKay correspondence as indicated. In particular, we are able to verify most of the schober conditions. The remaining conditions will be treated in future work.

Acknowledgements

First and foremost, I'd like to thank my supervisor, Timothy Logvinenko, for all the advice, guidance and knowledge he has given me over the past few years. Without him, this thesis would simply never have been written.

I'd also like to give a special thank-you to my academic brother, Lorenzo De Biase, for the fun years we spent doing maths together.

Thanks are also due to Emma Aspland, Joe Bishop, Martina Cracco, Nikoleta Glynatsi, Raffaele Grande, Danny Groves, Hassan Izanloo, Jack Noonan, Emily O'Riordan, Geraint Palmer, Michalis Panayides, Sam Richardson, Álvaro Torras Cassas, Henry Wilde, Emily Williams and Lizzie Williams, as well as everyone else who made the department such a welcoming place.

In the school office, I'd like to thank Caroline Frame-Morgan and Pauline Hutchings for all their help over the years with things too numerous and varied to list here.

In addition, I'd like to thank Tracey Lintern, Kirstin Strokorb and Iskander Aliev for their kind words and practical advice when Mum was ill.

Outside of the School of Mathematics, I'd like to thank Ali Gray, Fergus Nimmo, Jack Taylor, Cody Tennant, Aaron English, Owen Escott-Allen, Hubie Griffiths and Paul Revill; once the current Covid restrictions are lifted I hope to actually see some of you again!

I'd also like to thank Charles Brookman, whose lessons played a large role in first setting me on this journey.

I'd like to express my gratitude to EPSRC and Cardiff University School of Mathematics, whose funding made this project possible.

Lastly, I'd like to thank my family, in particular my mum Claire and my dad David, as well as my fiancée Becky, who have all supported me so much over the past few years. This last year in particular has had some unexpected twists and turns, but you've all been consistently wonderful throughout.

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Introduction and Background Material

The bounded derived category of coherent sheaves of $\mathcal{O}_X$-modules, $D^b(X) := D^b(\mathrm{Coh}(X))$, of a given variety, scheme or stack X can be thought of as the ultimate cohomological invariant of X. They were introduced by Grothendieck and Verdier in the 1960s, together with the notion of a *triangulated* category which axiomatises them. Originally considered as formally constructed objects without too much geometric flavour, these derived categories have turned out to be a very reasonable invariant to consider. For example, a result of Bondal and Orlov shows that, if X is a smooth projective variety with ample (anti-)canonical bundle, the derived category is a strong invariant. That is to say, for two such varieties X and Y, $D^b(X) \simeq D^b(Y)$ if and only if $X \simeq Y$. Of course, if this was always the case, then studying these derived categories would perhaps not be terribly interesting. An early result of Mukai, however, exhibits a derived equivalence between an abelian variety A and its dual $\hat{A}$, which in general are not isomorphic. He did this by considering the Poincaré bundle on the product space $A \times \hat{A}$ and used this to construct an analogue of a Fourier transform (here the Poincaré bundle plays the role of the integral kernel in the Fourier transform). Today, functors of this type are known as Fourier-Mukai transforms. In fact, these transforms form a far more general class of functors than they appear to at first sight; a famous result of Orlov states that all fully faithful functors (satisfying some mild extra conditions) between the derived categories of two smooth projective varieties can be written as the Fourier-Mukai transform for some kernel. As all equivalences are necessarily fully faithful, studying these transforms is a powerful method for constructing derived equivalences.

An additional motivation for studying these derived categories comes from mathematical physics, where derived categories are closely linked to string theory.

This inspired Kontsevich's 1994 Mirror Symmetry Conjecture [Kon94], which conjectures a 'mirror' between algebraic geometry and symplectic geometry. In particular, it states that for each 'mirror pair' of certain projective varieties X and X', the derived category of coherent sheaves on X should be equivalent to some other category, known as the Fukaya category, associated to the symplectic geometry of X', and vice versa.

In this introductory chapter we give some background material and motivation for the remainder of the thesis. In Section 1.1 we give a brief reminder of what the derived category of an arbitrary abelian category is and why the autoequivalence group of the derived category of coherent sheaves might reasonably be a useful thing to study; we also remind the reader of the technology of Fourier-Mukai transforms. In Section 1.2 we discuss the McKay correspondence in two dimensions, and in 1.3 we define and discuss spherical functors. In Section 1.4 we discuss a proposed categorification of the notion of a perverse sheaf, known as a *perverse schober*, and note that we have already met an example of a schober in the form of spherical functors. Section 1.5 serves as a primer on stacks, and in Section 1.6 we extend the technology of Fourier-Mukai transforms to the equivariant setting of quotient stacks. In Section 1.7 we examine quivers and quiver representations, and in Section 1.8 we fix some notation and conventions for the subsequent chapters.

1.1 Derived categories and their autoequivalences

Given an abelian category $\mathcal{A}$, define the category $C(\mathcal{A})$ to be the category whose objects are complexes of objects of $\mathcal{A}$, i.e. everything of the form

$$\dots \longrightarrow A_{i-1} \xrightarrow{d_{i-1}} A_i \xrightarrow{d_i} A_{i+1} \xrightarrow{d_{i+1}} \dots$$

where $A_k \in \mathcal{A}$ and $d_{k+1} \circ d_k = 0$ for all $k \in \mathbb{Z}$. The morphisms in this category are maps of complexes. We define an intermediate category $K(\mathcal{A})$ whose objects are the same as those of $C(\mathcal{A})$, but we consider two morphisms of complexes to be equal if they are homotopically equivalent. The derived category $D(\mathcal{A})$ is then the category whose objects are again chain complexes, and the morphisms are equivalence classes of diagrams of the form

$$F \xleftarrow{s} H \longrightarrow G$$

where s is a quasi-isomorphism. A quasi-isomorphism is, by definition, a morphism of complexes such that the induced morphism at the level of cohomology is an isomorphism. The derived category can therefore be thought of as the category in which we formally invert quasi-isomorphisms. The full subcategory of $D(\mathcal{A})$ in which all the objects are cohomologically bounded is denoted $D^b(\mathcal{A})$. The derived category we will most frequently work with in this thesis is the bounded derived category of the abelian category of coherent sheaves of $\mathcal{O}_X$-modules on X, which we denote $D^b(X) := D^b(\text{Coh}(X))$. If we instead consider the derived category of quasi-coherent sheaves, we indicate this with a subscript. For a conceptual introduction to derived categories we suggest [Tho01]. For technical details there are many excellent references, such as [Huy06] or [Har66].

Definition 1.1.0.1. *Let X and Y be two separated schemes of finite type over $\mathbb{C}$, and let $D_{QCoh}(-)$ denote the unbounded derived category of quasi-coherent sheaves. The Fourier-Mukai transform $\Phi_K : D_{QCoh}(X) \to D_{QCoh}(Y)$ is defined as*

$$\Phi_K(\mathcal{F}) = q_*(p^*\mathcal{F} \otimes K)$$

where p and q are the projections $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$, and $K \in D^b(X \times Y)$ is said to be the kernel *of the transform.*

Remark 1.1.0.2. The functors q_*, p^* and $\otimes$ are a priori taken to be the derived versions. Note however that p is a flat map, and so p^* is the pullback in the usual sense. If K is a complex of locally free sheaves, then the derived tensor product is also the usual tensor product of complexes of sheaves.

We can also compose Fourier-Mukai transforms as follows. Let X, Y and Z be separated schemes of finite type and consider $P \in D_{\text{QCoh}}(X \times Y)$ and $Q \in D_{\text{QCoh}}(Y \times Z)$.

$$\begin{array}{ccccc} & & X \times Y \times Z & & \\ & \swarrow^{\pi_{12}} & \downarrow{\scriptstyle \pi_{13}} & \searrow^{\pi_{23}} & \\ X \times Y & & X \times Z & & Y \times Z \end{array}$$

Define an object in $D_{\text{QCoh}}(X \times Z)$ by $R := \pi_{13*}(\pi_{12}^* P \otimes \pi_{23}^* Q)$.

Proposition 1.1.0.3. *The composition*

$$D_{QCoh}(X) \xrightarrow{\Phi_P} D_{QCoh}(Y) \xrightarrow{\Phi_Q} D_{QCoh}(Z)$$

is isomorphic to the Fourier-Mukai transform $\Phi_R : D_{QCoh}(X) \to D_{QCoh}(Z)$.

Proof. See [Huy06] or [Muk81]. □

Initially, these Fourier-Mukai transforms might appear to define a very limited class of maps between derived categories, but the following result gives a powerful motivation for why this class is worthy of study. In certain situations, such as for smooth projective varieties, Fourier-Mukai transforms descend nicely to functors between the bounded derived categories of coherent sheaves, $D^b(-)$.

Theorem 1.1.0.4 ([Orl97]). *Let X and Y be smooth projective varieties, and let $F : D^b(X) \to D^b(Y)$ be a fully faithful exact functor. If F admits left and right adjoints, then there exists an object $K \in D^b(X \times Y)$, unique up to isomorphism, such that $F \simeq \Phi_K$.*

In particular, all equivalences between the derived categories of two smooth projective varieties can therefore be written as Fourier-Mukai transforms.
Historically, one of the most important ways of studying a given derived category was to consider its autoequivalence group, the group of equivalences $\mathrm{Aut}(D^b(X)) := \{D^b(X) \xrightarrow{\sim} D^b(X)\}$. These can be highly non-trivial in general. Letting X be a smooth projective variety, a much celebrated result of Bondal & Orlov [BO01] tells us that if the canonical bundle ω_X is ample or anti-ample (i.e. if X is of general type or Fano), then the autoequivalence group has a particularly nice structure. More precisely, it is generated by three classes of autoequivalences, sometimes referred to as the *standard autoequivalences*. These are:

i) tensoring with an invertible sheaf $\mathcal{L}$, with inverse given by tensoring with $\mathcal{L}^{-1}$,

ii) the shift functor $[n]$, which shifts chain complexes n places to the left in $D^b(X)$, with inverse $[-n]$, and

iii) the pushforward functor f_*, with inverse the pullback functor f^*, where f is an automorphism of X.

If we define the diagonal morphism $\Delta : X \to X \times X$ and the graph $(\mathrm{id}, f) : X \to X \times X$ of an automorphism $f : X \to X$, these three classes of autoequivalences correspond to the Fourier-Mukai kernels $\Delta_*\mathcal{L}$, $\Delta_*\mathcal{O}_X[n]$ and $(\mathrm{id}, f)_*\mathcal{O}_X$, respectively. A special case of all three of these classes is the identity functor, whose kernel is given by the structure sheaf of the diagonal, $\mathcal{O}_\Delta := \Delta_*\mathcal{O}_X$:

$$\begin{aligned}
\Phi_{\mathcal{O}_\Delta}(\mathcal{F}) &= q_*(p^*\mathcal{F} \otimes \Delta_*\mathcal{O}_X) \\
&\simeq q_*\Delta_*(\Delta^*p^*\mathcal{F} \otimes \mathcal{O}_X) && \text{by the projection formula} \\
&\simeq (q \circ \Delta)_*(p \circ \Delta)^*\mathcal{F} && \text{by functoriality of pullback/pushforward} \\
&\simeq \mathcal{F} && \text{as } q \circ \Delta = p \circ \Delta = \mathrm{id}_X.
\end{aligned}$$

When $\omega_X \simeq \mathcal{O}_X$ (i.e. X is Calabi-Yau in the weak sense) we still have these three classes, but there can be considerably more autoequivalences than just these. A class of new autoequivalences inspired by mirror symmetry were constructed by Seidel & Thomas [ST01] and are known as *spherical twists*, named after the spherical objects which induce them. Under mirror symmetry, these spherical twists correspond to Dehn twists around Lagrangian spheres on the symplectic side of the mirror. To avoid confusion with the more general concept of a twist around a spherical functor which we will encounter later, we shall refer to these as *geometric* spherical twists.

Definition 1.1.0.5 ([ST01])**.** *Let X be a smooth projective variety over $\mathbb{C}$. An object $\mathcal{E} \in D^b(X)$ is said to be spherical if*

$$i) \qquad Hom_{D^b(X)}(\mathcal{E}, \mathcal{E}[n]) \simeq \begin{cases} \mathbb{C} & \text{if } n = 0 \text{ or } dim(X), \\ 0 & \text{else} \end{cases}$$

$$ii) \qquad \mathcal{E} \otimes \omega_X \simeq \mathcal{E},$$

where ω_X denotes the canonical bundle of X.

Note that, if ω_X is ample or anti-ample, then this second condition ensures that the support of $\mathcal{E}$ must be zero-dimensional.

Definition 1.1.0.6. *Given a spherical object $\mathcal{E} \in D^b(X)$, define*

$$\mathcal{P} := Cone(\eta : \pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E} \to \mathcal{O}_\Delta) \in D^b(X \times X)$$

where $\pi_i : X \times X \to X$ denotes the projection onto the i^{th} component and the map η is the natural pairing given by the composition of the restriction to the diagonal followed by the diagonal pushforward of the evaluation map $ev : \mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$

$$\eta : \pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E} \to \Delta_*\Delta^*(\pi_1^*\mathcal{E}^\vee \otimes \pi_2^*\mathcal{E}) \simeq \Delta_*(\mathcal{E}^\vee \otimes \mathcal{E}) \xrightarrow{\Delta_* ev} \mathcal{O}_\Delta.$$

We define the geometric spherical twist *$T_\mathcal{E}$ as the Fourier-Mukai transform $\Phi_\mathcal{P} : D^b(X) \to D^b(X)$.*

Proposition 1.1.0.7. *Geometric spherical twists $T_\mathcal{E}$ are autoequivalences of $D^b(X)$.*

Proof. See the original paper [ST01], or [Plo05] for an alternative proof. □

Definition 1.1.0.8. *A set of m spherical objects $\mathcal{E}_1, \dots \mathcal{E}_m \in D^b(X)$ is said to form an A_m-configuration if*

$$\bigoplus_{n \in \mathbb{Z}} Hom_{D^b(X)}(\mathcal{E}_i, \mathcal{E}_j[n]) \simeq \begin{cases} \mathbb{C} & \text{if } |i-j| = 1 \\ 0 & \text{if } |i-j| \geq 2. \end{cases}$$

Proposition 1.1.0.9 ([ST01]). *Given an A_m-configuration of spherical objects $\mathcal{E}_i \in D^b(X)$, the corresponding geometric spherical twists satisfy the braid relations, up to graded natural isomorphism, i.e.*

$$T_{\mathcal{E}_i}T_{\mathcal{E}_{i+1}}T_{\mathcal{E}_i} \simeq T_{\mathcal{E}_{i+1}}T_{\mathcal{E}_i}T_{\mathcal{E}_{i+1}} \qquad \text{for } i \in \{1, \dots, m-1\}$$
$$T_{\mathcal{E}_i}T_{\mathcal{E}_j} \simeq T_{\mathcal{E}_j}T_{\mathcal{E}_i} \qquad \text{for } |i-j| \geq 2.$$

Define the categorical group action ρ : Br$_{m+1} \to$ Aut$(D^b(X))$ by sending the i^{th} standard generator of the braid group to $T_{\mathcal{E}_i}$. If dim$(X) \geq 2$, ρ is injective.

Example 1.1.0.10 (Example 3.5, [ST01]). Let X be a surface. Then any smooth rational curve C with self-intersection $C \cdot C = -2$ is a spherical object. A collection of such curves $(C_1, \dots, C_m)$ such that $C_i \cap C_j = \emptyset$ for $|i-j| \geq 2$ and $C_i \cdot C_{i+1} = 1$ for $i = 1, \dots, m-1$ is an A_m-configuration.

1.2 The McKay correspondence

Let us now give a quick introduction to the McKay Correspondence, before returning to study it in more detail in Chapter 3. Far more comprehensive surveys exist than what appears in this thesis, see e.g. [Rei97], [Rei99] or the introduction to [BKR01] for the point of view of derived categories; the original article is [McK80]. Let $G \subset SL(2, \mathbb{C})$ be a finite subgroup with an action on $\mathbb{C}^2 = $ Spec$(\mathbb{C}[x, y])$. With this information, we can naively[1] define the quotient variety as the prime spectrum of the invariant functions, $\mathbb{C}^2/G :=$ Spec$(\mathbb{C}[x, y]^G)$, which has an isolated singularity at the origin. These singularities are known by various different names, for example *Kleinian*, *Du Val* or *simple surface* singularities, or *rational double points*. The rings of invariant polynomials were classified by Klein in 1884:

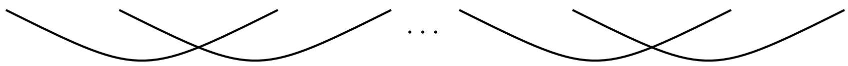
Theorem 1.2.0.1 ([Kle84]). *Let $G \subset SL(2, \mathbb{C})$ be a finite subgroup with an action on $\mathbb{C}^2$. Then $\mathbb{C}[x, y]^G \simeq \mathbb{C}[X, Y, Z]/(f)$, where f belongs to one of the following cases:*

$$A_n : f = Z^2 + X^2 + Y^{n+1}$$
$$D_n : f = Z^2 + X(X^{n-2} + Y^2)$$
$$E_6 : f = Z^2 + X^3 + Y^4$$
$$E_7 : f = Z^2 + X(X^2 + Y^3)$$
$$E_8 : f = Z^2 + X^3 + Y^5.$$

[1]There are certainly different ways of taking quotients by group actions; this is the central point of geometric invariant theory, which we shall meet later.

In the two-dimensional case, it is well-known that we can resolve the quotient singularity $\mathbb{C}^2/G$ by a finite sequence of blow-ups at the origin, and through this we obtain the minimal resolution $\pi : Y_{\text{min}} \to \mathbb{C}^2/G$.

Example 1.2.0.2. Consider the action of $G = \mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ on $\mathbb{C}^2$ given by $1 \cdot (x, y) = (\zeta x, \zeta^{n-1} y)$ for ζ a principal root of unity. This has the ring of invariant functions $\mathbb{C}[x^n, xy, y^n]$ which is of type A_{n-1}. Then $\mathbb{C}^2/G$ has a single isolated singularity at the origin, of type often denoted $\frac{1}{n}(1, n-1)$ in reference to the action which induces it, and we can resolve this singularity by blowing it up at the singular point to get the minimal resolution $\pi : Y_{\text{min}} \to \mathbb{C}^2/G$. Geometrically, π is a bijection away from the origin, and a chain of $n-1$ exceptional $\mathbb{P}^1$ curves over the singular point at the origin, intersecting as follows:



The intersection graph[2] of this chain of exceptional rational curves is exactly the Dynkin diagram of type A_{n-1}. This is a particular instance of part of John McKay's famous 1980 observation, which we now state.

Theorem 1.2.0.3 (McKay's observation, [McK80]). *If we forget the edge directions and multiplicities in the McKay quiver[3] of the group G, we obtain the affine Dynkin diagram of some semi-simple Lie algebra $\mathfrak{g}$ of ADE type. In addition, the intersection graph of the irreducible components of the exceptional divisor $\pi^{-1}(0)$ is the Dynkin diagram of $\mathfrak{g}$.*

Roughly, the idea of this observation is that the non-trivial irreducible representations of G are in bijection with the irreducible components of the exceptional divisor of the minimal resolution. The set-up of this provides a fruitful source of spherical twists as follows.

Example 1.2.0.4. For E_i an irreducible component of the exceptional divisor of the minimal resolution of a Kleinian singularity, E_i satisfies the self-intersection condition of Example 1.1.0.10 and $\mathcal{O}_{E_i} \in D^b(Y_{\text{min}})$ is therefore a spherical object. If f is of A_n type, the irreducible components form an A_n-configuration in the sense of Definition 1.1.0.8, and so induces a faithful categorical action of the braid group on $D^b(Y_{\text{min}})$.

[2]This is the (undirected) graph whose vertices i correspond to irreducible components E_i of the exceptional locus $\pi^{-1}(0)$, with an edge connecting vertices i and j if and only if $E_i \cap E_j \neq \emptyset$.

[3]See Definition 3.1.0.3 for the formal definition.

1.3 Spherical functors

The following remarkable concept, first introduced by Anno and Logvinenko [AL17], provides a natural direct generalisation of the concept of a geometric spherical twist. Unlike geometric spherical twists, these spherical functors are not autoequivalences themselves, but rather induce autoequivalences on both their source and target category. We first give the definition, then explain in what sense these functors generalise geometric spherical twists.

Definition 1.3.0.1 (Spherical Functors, [AL17])**.** *Let A and B be two enhanced triangulated categories and let $s : A \to B$ be an enhanceable functor with enhanceable adjoints $l \dashv s \dashv r$. Use the following distinguished triangles to define the twist t, the dual twist t', the cotwist f and the dual cotwist f':*

$$sr \longrightarrow id_B \longrightarrow t \qquad\qquad t' \longrightarrow id_B \longrightarrow sl$$

$$f \longrightarrow id_A \longrightarrow rs \qquad\qquad ls \longrightarrow id_A \longrightarrow f'$$

The functor s is spherical if all of the following hold:

i) t and t' are quasi-inverse autoequivalences of B.

ii) f and f' are quasi-inverse autoequivalences of A.

iii) The composition $lt[-1] \to lsr \to r$ is an isomorphism of functors.

iv) The composition $r \to rsl \to fl[1]$ is an isomorphism of functors.

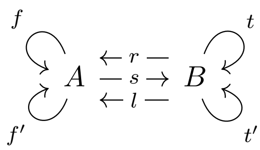


Figure 1.1: Diagrammatic representation of a spherical functor.

In fact it suffices to check any two of these requirements, as the following proposition shows.

Proposition 1.3.0.2. *Any two of the conditions in the definition imply the other two.*

Proof. See [AL17]. □

Remark 1.3.0.3. A spherical object $\mathcal{E} \in D^b(X)$ in the sense of Seidel & Thomas [ST01] is a spherical functor $D^b(\text{Spec } \mathbb{C}) \to D^b(X)$ which sends $\mathcal{O}_{\text{Spec}\mathbb{C}} \mapsto \mathcal{E}$, with the twist t around this functor then being the corresponding geometric spherical twist. This is Example 3.5 of [AL10]. In this sense, spherical functors $D^b(Z) \to D^b(X)$ are a natural direct generalisation of geometric spherical twists where we replace the point with a scheme Z.

Remark 1.3.0.4. A result of Segal [Seg18] tells us that *all* autoequivalences of B are induced by some spherical functor. That is, given any autoequivalence of B, we can always find some A and a spherical functor $s : A \to B$ such that the twist t is the given autoequivalence.

1.4 Perverse schobers

The concept of a perverse sheaf[4] was introduced in [BBD82] and has come to play an important role in both representation theory and algebraic geometry. As outlined in [KS15], it has since become clear that there should be some sort of categorical analogue of the concept of a perverse sheaf, and the name *perverse schober* (or simply *schober*) has become attached to these conjectured analogues. Due to technical difficulties with constructing such schobers, coupled with the fact that the definition is still conjectural, there aren't yet many known examples of these schobers. Allowing for some flexibility in precisely what is meant by a 'schober', examples can be found in [BKS18; ŠVdB19; KS15; Don17; Don19].

The main reason that this definition is not yet nailed-down is that the standard description of perverse sheaves does not lend itself well to categorification. This is because, in general, it's not completely clear what should take the role of "complexes of triangulated categories". However, in some situations the category of perverse sheaves can be described as representations of certain quivers (directed graphs) obeying certain commutativity relations. This description sometimes suggests a natural categorical analogue, such as the following example which we recall from [KS15]. Let D be the unit disc in $\mathbb{C}$ and let $\text{Perv}(D, 0)$ be the category of perverse sheaves on D with the only possible singularity at the origin. Then the category $\text{Perv}(D, 0)$ is equivalent [Bei87] [GGM85] to the category of quadruples (V, W, f, g), where V and W are k-vector spaces and f, g are linear maps

[4] Note that a perverse sheaf is not actually a sheaf, rather it is an object in some abelian subcategory of the derived category of constructible complexes of sheaves of vector spaces. This subcategory appears naturally as the heart of a t-structure, known as the perverse t-structure.

$$V \underset{g}{\overset{f}{\rightleftarrows}} W$$

Figure 1.2: The quiver description of Perv$(D, 0)$.

such that the linear map

$$T_W := \mathrm{id}_W - fg \tag{1.1}$$

is an isomorphism. In [KS15], the authors propose considering spherical functors as the schobers corresponding to this description. The intuition behind this is that the data of a spherical functor provides two diagrams

$$A \underset{l}{\overset{s}{\rightleftarrows}} B \qquad B \underset{r}{\overset{s}{\leftrightarrows}} A$$

and this data should be considered the categorical analogue of Figure 1.2, with the equivalent condition to (1.1) being an isomorphism being that the twist and cotwist of s are equivalences. In this situation, taking the cone to define the (co)twist plays the role of the subtraction in (1.1).

In the cases where we have a quiver description of the category of perverse sheaves and any relations in the quiver are *monomial*[5] in nature, we immediately have a natural categorification where we replace vector spaces by triangulated categories and linear maps by exact functors. The monomial condition on the relations of the linear maps then becomes important, as it suggests a natural commutativity relation on the functors. Taking one of these schobers and applying some functor from the category of triangulated categories to the category of k-vector spaces (e.g. taking Grothendieck groups $V_i := K_0(\mathcal{V}) \otimes k$) would then produce a perverse sheaf. In [BKS18], Bondal, Kapranov and Schechtman take such a quiver description of the category of perverse sheaves on the complexification of a linear hyperplane arrangement $\mathcal{H} \subset \mathbb{R}^n$. This quiver description does admit a natural categorical analogue, and they term categorical representations satisfying these conditions *$\mathcal{H}$-schobers*. We now recall both this quiver description of the category of perverse sheaves, and the conjectured categorical analogue. We first state what we mean by a *stratification* of a variety.

Definition 1.4.0.1. *A* stratification *of a variety X is a partition $X = \bigsqcup_{\lambda\in\Lambda} X_\lambda$ of X into smooth, locally closed, connected subvarieties parametrised by a poset Λ, such that $\overline{X_\lambda} = \bigsqcup_{\mu\leq\lambda} X_\mu$ for all $\lambda \in \Lambda$, where $\overline{X_\lambda}$ denotes the topological closure of X_λ.*

[5] i.e. the relations do not involve adding linear maps together or scaling them by some element of the field k, as it's not always clear what the version of this for functors should be.

Now, let $\mathcal{H}$ be a collection of linear hyperplanes in $\mathbb{R}^n$. This produces a finite disjoint collection of cells C_i such that $\mathbb{R}^n = \bigsqcup_i C_i$, and we endow these cells with a partial ordering by setting $C_i \leq C_j$ if and only if $C_i \subseteq \overline{C_j}$, where we take the closure in the normal Euclidean topology. Denote by $\mathcal{C} = \{C_i\}$ this poset. The complexification of this hyperplane arrangement, $\mathcal{H}_{\mathbb{C}}$, provides a stratification of $\mathbb{C}^n$. As an example, consider Figure 1.3 which shows the stratification induced by the unique linear "hyperplane" in $\mathbb{R}^1$.

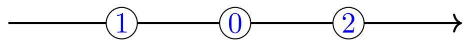


Figure 1.3: The unique linear hyperplane arrangement in $\mathbb{R}^1$. The "hyperplane" C_0 is the single point at the origin and C_1 (resp. C_2) is the set of strictly negative (resp. strictly positive) points in $\mathbb{R}$. In the partial ordering, $C_0 \leq C_1$, C_2 as their closures contain the point at the origin; C_1 and C_2 are incomparable.

With the structure of this linear hyperplane arrangement in $\mathbb{R}^n$, the following theorem gives a description of the perverse sheaves on the complexified space $\mathbb{C}^n$. Fix some base field k and let Perv$(\mathbb{C}^n, \mathcal{H})$ denote the category of perverse sheaves of k-vector spaces on $\mathbb{C}^n$, smooth with respect to $\mathcal{H}$. In [KS16], Kapranov and Schechtman give the following quiver representation description of perverse sheaves on the *complex* space based on the cells in the *real* space. We say an ordered triple of three cells C_i, C_j and C_k are *collinear* if there exists a triple of points $a \in C_i$, $b \in C_j$ and $c \in C_k$, such that b lies on the straight line segment $[a, c] \subseteq \mathbb{R}^n$.

Theorem 1.4.0.2 (Theorem 8.1, [KS16]). *Perv$(\mathbb{C}^n, \mathcal{H})$ is equivalent to the category of diagrams formed by a finite-dimensional k-vector space E_i for each $C_i \in \mathcal{C}$ and, for all $C_i \leq C_j$, linear maps*

$$E_i \underset{\delta_{ji}}{\overset{\gamma_{ij}}{\rightleftarrows}} E_j$$

satisfying the following relations:

i) *Transitivity: if $C_i \leq C_j \leq C_k$, then $\gamma_{ik} = \gamma_{jk}\gamma_{ij}$ and $\delta_{ki} = \delta_{ji}\delta_{kj}$.*

ii) *Idempotency: if $C_i \leq C_j$, then $\gamma_{ij}\delta_{ji} = \mathrm{id}_{E_j}$. For any $C_i, C_j, C_k \in \mathcal{C}$ with $C_i \leq C_j, C_k$, this allows us to define $\varphi_{jk} := \gamma_{ik}\delta_{ji} : E_j \to E_k$ without ambiguity.*

iii) *Collinear transitivity: if three cells C_i, C_j, C_k are collinear, then $\varphi_{ik} = \varphi_{jk}\varphi_{ij}$.*

iv) *Invertibility: if C_j, C_k are cells of dimension d, lie in a linear subspace of $\mathbb{R}^n$ of dimension d, and are separated by a cell $C_i \leq C_j, C_k$ of dimension $d-1$, then φ_{jk} is an isomorphism.*

We now simply *define* $\mathcal{H}$-schobers to be the immediate categorical analogue of the result of this theorem. With a slight abuse of notation by now taking γ_{ij} and δ_{ji} to be functors, we state the following definition which was first given by Bondal, Kapranov and Schechtman [BKS18].

Definition 1.4.0.3 (Definition 3.6, [BKS18]). *An $\mathcal{H}$-schober is a collection of triangulated categories $\mathcal{E}_{C_i}$ and adjoint pairs of exact functors $\gamma_{ij} \dashv \delta_{ji}$, where $\gamma_{ij} : \mathcal{E}_{C_i} \to \mathcal{E}_{C_j}$ and $\delta_{ji} : \mathcal{E}_{C_j} \to \mathcal{E}_{C_i}$ for every $C_i \leq C_j$, such that the following conditions hold:*

i) *Transitivity: If $C_i \leq C_j \leq C_k$, then $\gamma_{ik} \simeq \gamma_{jk}\gamma_{ij}$ and $\delta_{ki} \simeq \delta_{ji}\delta_{kj}$.*

ii) *Idempotency: the counit of adjunction $\gamma_{ij}\delta_{ji} \to id_{\mathcal{E}_{C_j}}$ for all $C_i \leq C_j$ is an isomorphism, and so $\varphi_{ij} := \gamma_{kj}\delta_{ik}$ is well-defined for $C_k \leq C_i, C_j$. We refer to the φ_{ij} as* flopping functors.

iii) *Collinear transitivity: for 3 collinear cells C_i, C_j, C_k, we have a natural isomorphism of functors $\varphi_{jk}\varphi_{ij} \tilde{\to} \varphi_{ik}$.*

iv) *Invertibility: for all C_i and C_j of dimension d which lie in the same d-dimensional subspace of $\mathbb{R}^n$, and which are separated by a cell of dimension $d-1$, φ_{ij} is an equivalence. We refer to such φ_{ij} as wall-crossing functors.*

Remark 1.4.0.4. In the original paper [BKS18], they take $\gamma_{ij} \dashv \delta_{ji}$, whereas in [ŠVdB19] the authors take $\delta_{ji} \dashv \gamma_{ij}$, with the idempotency condition then being phrased in terms of the *unit* of adjunction rather than the *counit*.

Often, the invertibility condition in this definition is the hardest to verify in practice. In [BKS18], the authors consider the hyperplane arrangement given by the root system of $\mathfrak{sl}_3$ and construct a candidate for an $\mathcal{H}$-schober on this. In their paper they verify all the schober conditions with the exception of the wall crossing corresponding to going from a one-dimensional cell to its opposite via the cell at the origin.

The first fully verified example of an $\mathcal{H}$-schober was given in the remarkable paper [ŠVdB19]. In this paper, the authors construct the triangulated categories in vast generality as derived categories of GIT quotient stacks for a reductive group (see Definition/Proposition 1.6.0.5) acting on a quasi-symmetric representation[6]. This built on earlier work of the authors [ŠVdB17], as well as work of Halpern-Leistner and Sam [HS20].

1.5 A quick introduction to stacks

In the same way as schemes were constructed in order to generalise algebraic varieties, stacks should be viewed as some generalisation of the notion of a scheme. In this thesis, the only stacks we will encounter are global quotient stacks $[X/G]$ formed from a reductive group G acting on a variety X, so we refrain from discussing the general theory in too much detail. For a global quotient stack, the points of the stack should be thought of as a pair consisting of a G-orbit in X together with the data of an automorphism group as follows. For a point in $[X/G]$, choose a point p in the corresponding G-orbit in X. The automorphism group is the stabiliser subgroup of elements of G which fix p. When the stabilisers are all finite the stack is Deligne-Mumford. When the action is free (i.e. the stabiliser of all points is trivial), the quotient stack is a scheme. Roughly, this is because there is no additional information kept by retaining knowledge of the stabilisers. We will informally refer to points in the quotient stack with non-trivial stabilisers as "stacky" points. For a very readable introduction to algebraic stacks, see [Góm01].

We denote by (Sch/S) the category of schemes over a given scheme S, i.e. the category whose objects are pairs (X, s) where X is a scheme and $s : X \to S$ is a morphism of schemes, and whose morphisms $(X, s) \to (X', s')$ are morphisms of schemes $f : X \to X'$ such that $s = s' \circ f$. Recall that a *groupoid* is a (small) category in which every morphism is invertible. This gives us diagrams

$$i \circlearrowright R \underset{t}{\overset{s}{\rightrightarrows}} U \xrightarrow{\varepsilon} R, \qquad R \times_{s,U,t} R \xrightarrow{c} R$$

where

i) U is the set of objects, R the set of morphisms,

ii) s (resp. t) is the map which takes a morphism to the object at its source (resp. target),

[6]Quasi-symmetric representations were introduced in [ŠVdB17]. Roughly, let G be a reductive group acting on a representation V, and let a maximal torus $G \supseteq T \simeq \mathbb{G}_m^n$ act with weights $\{\beta_i\} \in \operatorname{Hom}(T, \mathbb{G}_m) \simeq \mathbb{Z}^n \subseteq \mathbb{R}^n$. The representation V is quasi-symmetric if and only if $\sum_{\beta_i \in l} \beta_i = 0$ for all lines $l \subseteq \mathbb{R}^n$ passing through the origin (recall that the empty sum is equal to zero by convention).

iii) i inverts the morphism,

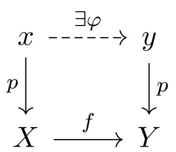
iv) ε takes an object to its identity morphism,

v) c composes morphisms.

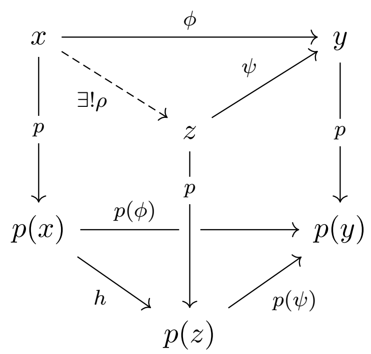
Definition 1.5.0.1. *A category over* (Sch/S) *is a pair* $(\mathcal{C}, p)$ *consisting of a category* $\mathcal{C}$ *and a (covariant) functor* $p : \mathcal{C} \to (Sch/S)$*. We say that* $C \in \mathcal{C}$ *(resp.* $f : C \to D$*) lies over* $A \in (Sch/S)$ *(resp. lies over* $g : A \to B$*) if* $p(C) = A$ *(resp.* $p(f) = g$*).*

Definition 1.5.0.2. *A category* $\mathcal{C}$ *over* (Sch/S) *is called a category fibred in groupoids if*

i) for every morphism $f : X \to Y$ *in* (Sch/S) *such that* $p(y) = Y$ *for some* $y \in \mathcal{C}$*, there exists* $x \in \mathcal{C}$ *and* $\varphi : x \to y$ *such that* $p(x) = X$ *and* $p(\varphi) = f$*.*



ii) if $\phi : x \to y$ *and* $\psi : z \to y$ *are morphisms in* $\mathcal{C}$ *and there exists* $h : p(x) \to p(z)$ *in* (Sch/S) *such that* $p(\psi) \circ h = p(\phi)$*, then there exists a unique morphism* $\rho : x \to z$ *such that* $\psi \circ \rho = \phi$ *and* $p(\rho) = h$*.*



Let $\mathcal{C}$ be a category fibred in groupoids and let $B \in (Sch/S)$. We define the fibre over B to be the subcategory of $\mathcal{C}$ consisting of objects lying over B and morphisms lying over id_B. This is a groupoid, which justifies the name of the previous definition. The formal definition of a stack is as follows.

Definition 1.5.0.3. *A stack is a category fibred in groupoids for which descent data is effective, and for which a condition known as the prestack condition holds. A morphism* *of stacks* $(\mathcal{X}_1, p_1 : \mathcal{X}_1 \to (Sch/S)) \to (\mathcal{X}_2, p_2 : \mathcal{X}_2 \to (Sch/S))$ *is a functor* $F : \mathcal{X}_1 \to \mathcal{X}_2$ *such that* $p_2 \circ F \simeq p_1$.

We refrain from explaining either of these conditions, and instead direct the reader to the above references. Motivated by the above description of a groupoid, we define the following.

Definition 1.5.0.4. *An algebraic groupoid over* S *is a tuple* $(R, U, s, t, \varepsilon, i, c)$*, where* R, U *are objects of* (Sch/S) *and* s, t, ε, i, c *are morphisms of* S*-schemes satisfying the obvious compatibilities. Where the maps are clear we shall denote an algebraic groupoid by* $R \rightrightarrows U$.

Example 1.5.0.5. A prime source of such algebraic groupoids comes from an algebraic group G acting on a variety X. In this case we can take $R = G \times X$ and $U = X$. Then the maps are as follows:

$$\begin{aligned} s &: (g, x) \mapsto x \\ t &: (g, x) \mapsto g \cdot x \\ i &: (g, x) \mapsto (g^{-1}, x) \\ \varepsilon &: x \mapsto (1_G, x) \end{aligned}$$

where $g \cdot x$ is the group action. In this situation we shall refer to $G \times X \rightrightarrows X$ as an *action groupoid*.

Definition 1.5.0.6. *A quasi-coherent sheaf on an algebraic groupoid* $R \rightrightarrows U$ *is a pair* $(\mathcal{F}, \alpha)$*, where* $\mathcal{F}$ *is a quasi-coherent sheaf on* U*, and* $\alpha : s^*\mathcal{F} \xrightarrow{\sim} t^*\mathcal{F}$ *is a choice of isomorphism satisfying a cocycle condition.*

For $R = G \times X$ and $U = X$ as considered above, this definition coincides with that of a *G-equivariant* quasi-coherent sheaf, as we shall define in Definition 1.6.0.6.

Proposition 1.5.0.7. *The category of quasi-coherent sheaves on the quotient stack* $[X/G]$ *is equivalent to the category of quasi-coherent sheaves on the algebraic groupoid* $G \times X \rightrightarrows X$*, where the two maps are given by the projection onto the second factor and the group action.*

Proof. See e.g. Remark 2.1.2 in [BFK19] or the contained reference to [Vis89]. □

This proposition tells us that the quasi-coherent sheaves on $[X/G]$ are precisely the G-equivariant quasi-coherent sheaves on X. In fact, the same statement is true with "quasi-coherent" replaced by e.g. *coherent* or *locally free* [BFK19].

1.6 Fourier-Mukai transforms for quotient stacks

We now give the construction of the equivariant analogue of Fourier-Mukai transforms. This is essentially nothing but Definition 1.1.0.1 where we must keep track of the equivariant structure. We fix an algebraically closed field k, with no assumptions on characteristic (at least initially). By an algebraic variety we mean a reduced scheme of finite type over k.

Definition 1.6.0.1. *An algebraic group G is a group with the structure of an algebraic variety, where the multiplication map $\mu : G \times G \to G$ and the inverse map $\nu : G \to G$ are morphisms of algebraic varieties.*

Example 1.6.0.2. $GL(n, k)$ is an algebraic group for all $n \in \mathbb{N}$.

Definition 1.6.0.3. *A linear algebraic group G is an algebraic group which is isomorphic to an algebraic subgroup of $GL(n, k)$ for some $n \in \mathbb{N}$.*

Definition 1.6.0.4. *A linear algebraic group G is a linearly reductive group if, for any finite-dimensional representation V of G, and any non-zero $v_0 \in V^G$, there exists a non-zero linear map $f : V \to k$ such that $f(v_0) \neq 0$.*

Definition/Proposition 1.6.0.5. *If k is now assumed to be of characteristic zero, G is a reductive group if and only if it is a linearly reductive group.*

From now on, we fix $k = \mathbb{C}$.

Definition 1.6.0.6 ([MFK94]). *Let X be a scheme with a (left) action of an algebraic group G, and let $\pi_2, act : G \times X \to X$ denote the projection onto the second factor and the action map, respectively. A G-equivariant quasi-coherent sheaf is a pair $(\mathcal{E}, \alpha)$, where $\mathcal{E}$ is a quasi-coherent sheaf on X and $\alpha : act^*\mathcal{E} \xrightarrow{\sim} \pi_2^*\mathcal{E}$ is a choice of isomorphism obeying the cocycle condition, which is the commutativity of*

$$\begin{array}{ccc}
(act \circ (id_G \times act))^*\mathcal{E} & \xrightarrow{(id_G \times act)^*\alpha} & (\pi_2 \circ (id_G \times act))^*\mathcal{E} \\
\| & & \| \\
 & & (act \circ p_{23})^*\mathcal{E} \\
 & & \downarrow{\scriptstyle p_{23}^*\alpha} \\
 & & (\pi_2 \circ p_{23})^*\mathcal{E} \\
\| & & \| \\
(act \circ (\mu \times id_X))^*\mathcal{E} & \xrightarrow[(\mu \times id_X)^*\alpha]{} & (\pi_2 \circ (\mu \times id_X))^*\mathcal{E}
\end{array} \tag{CC}$$

where $(id_G \times act), p_{23}$ and $(\mu \times id_X)$ are the obvious maps $G \times G \times X \to G \times X$.

Remark 1.6.0.7. The maps marked as such in (CC) are genuine equalities, rather than merely isomorphisms, as $\mathrm{act} \circ (\mathrm{id}_G \times \mathrm{act}) = \mathrm{act} \circ (\mu \times \mathrm{id}_X)$, $\pi_2 \circ (\mathrm{id}_G \times \mathrm{act}) = \mathrm{act} \circ p_{23}$ and $\pi_2 \circ p_{23} = \pi_2 \circ (\mu \times \mathrm{id}_X)$.

Remark 1.6.0.8. For a G-equivariant line bundle $\mathcal{L}$, the choice of isomorphism α is often referred to in the literature as a *linearisation* of $\mathcal{L}$.

Definition 1.6.0.9. *The category of G-equivariant quasi-coherent sheaves on a scheme X is the category whose objects are G-equivariant quasi-coherent sheaves, and whose morphisms are morphisms of sheaves compatible with the maps α, i.e. those morphisms $f : \mathcal{E} \to \mathcal{F}$ for which the following diagram commutes*

$$\begin{array}{ccc} \mathrm{act}^*\mathcal{E} & \xrightarrow{\alpha} & \pi_2^*\mathcal{E} \\ {\scriptstyle \mathrm{act}^* f}\downarrow & & \downarrow{\scriptstyle \pi_2^* f} \\ \mathrm{act}^*\mathcal{F} & \xrightarrow[\alpha']{} & \pi_2^*\mathcal{F}. \end{array}$$

Definition 1.6.0.10. *Let $G \times Y \rightrightarrows Y$ be an action groupoid with a trivial action of G so that the action and projection maps σ_Y and π_Y coincide, and let $(\mathcal{E}, \alpha)$ be a coherent sheaf on this groupoid. This gives us two isomorphisms $\pi_Y^*\mathcal{E} \rightrightarrows \pi_Y^*\mathcal{E}$, given by the identity and α, and therefore two corresponding maps $\mathcal{E} \rightrightarrows (\pi_Y)_*\pi_Y^*\mathcal{E}$ by adjunction. Define $\mathcal{E}^G \subseteq \mathcal{E}$ to be the subsheaf given by the equaliser of this diagram,*

$$\mathcal{E}^G \longrightarrow \mathcal{E} \rightrightarrows (\pi_Y)_*\pi_Y^*\mathcal{E}.$$

We now slightly extend this idea to the situation where we have a $G \times H$ action on Y, where it is only the G component which acts trivially. Thus consider the following diagram

$$\begin{array}{ccccc} & & H \times Y & & \\ & & \downarrow{\scriptstyle \varepsilon_G} & & \\ G \times Y & \xrightarrow{\varepsilon_H} & G \times H \times Y & \xrightarrow{\pi_{23}} & H \times Y \\ & & \downarrow\downarrow & & {\scriptstyle \sigma}\downarrow\downarrow{\scriptstyle \pi} \\ & & G \times Y & \xrightarrow[p_2]{} & Y \end{array}$$

in which the commutative square depicts two fibre squares, ε_G and ε_H are the inclusions by the identity elements of G and H, and p_2 is the projection on the second factor. Denote the groupoids $A := (G \times H \times Y \rightrightarrows Y)$ and $B := (H \times Y \rightrightarrows Y)$ for brevity. We define a functor $\mathrm{Coh}(A) \to \mathrm{Coh}(B)$ by taking the "G-invariant part", in the following sense. Let $(\mathcal{F}, \beta) \in \mathrm{Coh}(A)$, so that

$(\mathcal{F}, \varepsilon_H^*\beta)$ is a coherent sheaf on the groupoid $G \times Y \to Y$, on which the action of G is trivial, so this defines an invariant part $\mathcal{F}^G$ via the equaliser

$$\mathcal{F}^G \longrightarrow \mathcal{F} \underset{(p_{2*}\varepsilon_H^*\beta)\circ\eta}{\overset{\eta}{\rightrightarrows}} p_{2*}p_2^*\mathcal{F} \tag{1.2}$$

where η is the morphism induced by the unit of adjunction map id $\to p_{2*}p_2^*$. As σ^* and π^* are left exact they preserve equalisers, and applying ε_G^* to β gives an isomorphism $\sigma^*\mathcal{F} \to \pi^*\mathcal{F}$, so that the following diagram commutes

$$\begin{array}{ccccc}
\sigma^*\mathcal{F}^G & \longrightarrow & \sigma^*\mathcal{F} & \underset{(\sigma^*p_{2*}\varepsilon_H^*\beta)\circ\sigma^*\eta}{\overset{\sigma^*\eta}{\rightrightarrows}} & \sigma^*p_{2*}p_2^*\mathcal{F} \\
{\scriptstyle\beta^G}\Big\downarrow & & \Big\downarrow{\scriptstyle\varepsilon_G^*\beta} & & \Big\downarrow{\scriptstyle(\pi_{23})_*\pi_{23}^*\varepsilon_G^*\beta} \\
\pi^*\mathcal{F}^G & \longrightarrow & \pi^*\mathcal{F} & \underset{(\pi^*p_{2*}\varepsilon_H^*\beta)\circ\pi^*\eta}{\overset{\pi^*\eta}{\rightrightarrows}} & \pi^*p_{2*}p_2^*\mathcal{F},
\end{array}$$

where we identify $\sigma^*p_{2*}p_2^*\mathcal{F} \simeq (\pi_{23})_*\pi_{23}^*\sigma^*\mathcal{F}$. Thus, by the universal properties of the two equalisers, β restricts to an isomorphism β^G. We define the functor of taking G-invariants

$$\begin{aligned}
\mathrm{Coh}(A) &\to \mathrm{Coh}(B) \\
(\mathcal{F}, \beta) &\mapsto (\mathcal{F}^G, \beta^G).
\end{aligned}$$

Lemma 1.6.0.11. *There is an adjunction*

$$\mathit{Hom}_{\mathit{Coh}(A)}\big((\mathcal{E}, \pi_{23}^*\alpha), (\mathcal{F}, \beta)\big) \overset{1:1}{\longleftrightarrow} \mathit{Hom}_{\mathit{Coh}(B)}\big((\mathcal{E}, \alpha), (\mathcal{F}^G, \beta^G)\big)$$

Proof. We construct the bijective function as follows. Let $f : (\mathcal{E}, \pi_{23}^*\alpha) \to (\mathcal{F}, \beta)$, i.e. a map $f : \mathcal{E} \to \mathcal{F}$ such that the following square commutes:

$$\begin{array}{ccc}
\pi_{23}^*\sigma^*\mathcal{E} & \xrightarrow{\pi_{23}^*\alpha} & \pi_{23}^*\pi^*\mathcal{E} \\
{\scriptstyle\pi_{23}^*\sigma^*f}\Big\downarrow & & \Big\downarrow{\scriptstyle\pi_{23}^*\pi^*f} \\
\pi_{23}^*\sigma^*\mathcal{F} & \xrightarrow[\beta]{} & \pi_{23}^*\pi^*\mathcal{F}.
\end{array} \tag{1.3}$$

Applying ε_H^* to this diagram yields

$$\begin{array}{ccc}
p_2^*\mathcal{E} & \xrightarrow{\mathrm{id}} & p_2^*\mathcal{E} \\
{\scriptstyle p_2^*f}\Big\downarrow & & \Big\downarrow{\scriptstyle p_2^*f} \\
p_2^*\mathcal{F} & \xrightarrow[\varepsilon_H^*\beta]{} & p_2^*\mathcal{F}.
\end{array}$$

Thus the two compositions

$$p_2^*\mathcal{E} \xrightarrow{p_2^* f} p_2^*\mathcal{F} \underset{\varepsilon_H^*\beta}{\overset{\mathrm{id}}{\rightrightarrows}} p_2^*\mathcal{F}$$

are equal and, by adjunction and using the universal property of the equaliser, there exists a unique map $f' : \mathcal{E} \to \mathcal{F}^G$ such that f filters through $\mathcal{F}^G$. Now applying ε_G^* to (1.3) and using the filtering property of f, we find

$$\begin{array}{ccccccc}
\sigma^*\mathcal{E} & & & \xrightarrow{\quad\alpha\quad} & & & \pi^*\mathcal{E} \\
& \searrow^{\sigma^* f'} & & & & \swarrow^{\pi^* f'} & \\
{\scriptstyle \sigma^* f}\Big\downarrow & & \sigma^*\mathcal{F}^G & \xrightarrow{\beta^G} & \pi^*\mathcal{F}^G & & \Big\downarrow{\scriptstyle \pi^* f} \\
& \swarrow & & & & \searrow & \\
\sigma^*\mathcal{F} & & & \xrightarrow[\varepsilon_G^*\beta]{} & & & \pi^*\mathcal{F}
\end{array}$$

and thus $f' \in \mathrm{Hom}_{\mathrm{Coh}(B)}\big((\mathcal{E}, \alpha), (\mathcal{F}^G, \beta^G)\big)$.
For the other direction, let $g \in \mathrm{Hom}_{\mathrm{Coh}(B)}\big((\mathcal{E}, \alpha), (\mathcal{F}^G, \beta^G)\big)$, i.e. $g : \mathcal{E} \to \mathcal{F}^G$ such that the following square commutes

$$\begin{array}{ccc}
\sigma^*\mathcal{E} & \xrightarrow{\alpha} & \pi^*\mathcal{E} \\
{\scriptstyle \sigma^* g}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi^* g} \\
\sigma^*\mathcal{F}^G & \xrightarrow[\beta^G]{} & \pi^*\mathcal{F}^G
\end{array}$$

Apply π_{23}^* to this diagram and postcompose with the pullback of the inclusion $\mathcal{F}^G \to \mathcal{F}$ to get

$$\begin{array}{ccc}
\pi_{23}^*\sigma^*\mathcal{E} & \xrightarrow{\pi_{23}^*\alpha} & \pi_{23}^*\pi^*\mathcal{E} \\
{\scriptstyle \pi_{23}^*\sigma^* g}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{23}^*\pi^* f} \\
\pi_{23}^*\sigma^*\mathcal{F}^G & \xrightarrow[\pi_{23}^*\beta^G]{} & \pi_{23}^*\pi^*\mathcal{F}^G \\
\Big\downarrow & & \Big\downarrow \\
\pi_{23}^*\sigma^*\mathcal{F} & \xrightarrow[\beta]{} & \pi_{23}^*\pi^*\mathcal{F}
\end{array}$$

which lies in $\mathrm{Hom}_{\mathrm{Coh}(A)}\big((\mathcal{E}, \pi_{23}^*\alpha), (\mathcal{F}, \beta)\big)$. These two functions are mutually inverse, and thus the functors in the statement of the lemma are adjoint. □

Corollary 1.6.0.12. *Let*

$$\begin{array}{ccccc}
G \times H \times X \times Y & \xrightarrow{\pi_{124}} & G \times H \times Y & \xrightarrow{\pi_{23}} & H \times Y \\
{\scriptstyle (\sigma_X, \sigma_Y)}\Big\downarrow\Big\downarrow{\scriptstyle (\pi_X, \pi_Y)} & & \Big\downarrow\Big\downarrow & & {\scriptstyle \sigma_Y}\Big\downarrow\Big\downarrow{\scriptstyle \pi_Y} \\
X \times Y & \xrightarrow[q]{} & Y & \xrightarrow[id]{} & Y
\end{array}$$

be a commutative diagram for some q. Denote the groupoids $A = (G \times H \times X \times Y \rightrightarrows X \times Y)$, $B = (G \times H \times Y \rightrightarrows Y)$, $C = (H \times Y \rightrightarrows Y)$. By slight abuse of notation denote

$$q^* : D^b(C) \to D^b(A) \qquad\qquad q_*^G : D^b(A) \to D^b(C)$$
$$(\mathcal{E}, \alpha) \mapsto (q^*\mathcal{E}, \pi_{124}^*\pi_{23}^*\alpha) \qquad\qquad (\mathcal{F}, \beta) \mapsto ([q_*\mathcal{F}]^G, [\pi_{124*}\beta]^G)$$

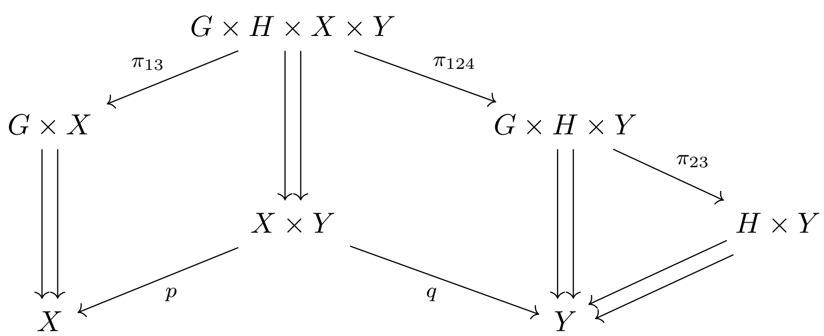
Then $q^* \dashv q_*^G$.

Proof.

$$\begin{aligned}\mathrm{Hom}_A((q^*\mathcal{E}, \pi_{124}^*\pi_{23}^*\alpha), (\mathcal{F}, \beta)) &= \mathrm{Hom}_B((\mathcal{E}, \pi_{23}^*\alpha), (q_*\mathcal{F}, \pi_{124*}\beta)) \\ &= \mathrm{Hom}_C((\mathcal{E}, \alpha), ([q_*\mathcal{F}]^G, [\pi_{124*}\beta]^G))\end{aligned}$$

where the existence of the first map follows from base change, and is a bijection as $q^* \dashv q_*$ and $\pi_{124}^* \dashv \pi_{124*}$. The second bijection is the statement of the lemma. □

Let $G \times X \rightrightarrows X$ and $H \times Y \rightrightarrows Y$ be quotient stacks, with G and H reductive groups, and let $(\mathcal{E}, \alpha)$ be a coherent sheaf on $G \times X \rightrightarrows X$. As in Definition 1.1.0.1, denote by $X \xleftarrow{p} X \times Y \xrightarrow{q} Y$ the projections. In addition, let (K, β) be a coherent sheaf on the product groupoid $G \times H \times X \times Y \rightrightarrows X \times Y$ with the support of K proper over both X and Y. The action of $G \times H$ on $X \times Y$ is given by the actions of G on X and H on Y. All maps denoted by a π with subscripts are the projection maps onto the factors indicated by these subscripts. The equivariant Fourier-Mukai transform works as follows.



i) By commutativity of the leftmost square, $(p^*E, \pi_{13}^*\alpha)$ is a coherent sheaf on $G \times H \times X \times Y \rightrightarrows X \times Y$.

ii) Tensor by the kernel to get $(p^*\mathcal{E} \otimes K, \pi_{13}^*\alpha \otimes \beta)$.

iii) Using the fibre square, by flat base change $(q_*(p^*\mathcal{E} \otimes K), (\pi_{124})_*(\pi_{13}^*\alpha \otimes \beta))$ is a coherent sheaf on $G \times H \times Y \rightrightarrows Y$, and taking the G-invariant part gives us a coherent sheaf on $H \times Y \rightrightarrows Y$.

Formally, we thus arrive at the following definition

Definition 1.6.0.13. *With notation as above, the equivariant Fourier-Mukai transform with kernel (K, β) is the functor*

$$\Phi_{(K,\beta)} : D^b([X/G]) \to D^b([Y/H])$$
$$(\mathcal{E}, \alpha) \mapsto ([q_*(p^*\mathcal{E} \otimes K)]^G, [(\pi_{124})_*(\pi_{13}^*\alpha \otimes \beta)]^G)$$

where the pushforwards, pullbacks and tensor product are derived. As we noted in Remark 1.1.0.2, p^* and π_{13}^* are the normal pullbacks as their respective underlying maps are flat. The derived tensor product is exact, and if a locally free resolution exists for K then the derived tensor product coincides with the underived version. Taking invariants is exact as G is a reductive group. Equivariant Fourier-Mukai transforms are therefore automatically exact functors.

Definition 1.6.0.14. *Given a group G acting on X, we define the* equivariant diagonal *to be*

$$\Delta G := \{(x, g \cdot x) \mid x \in X,\ g \in G\} \subseteq X \times X.$$

That is, ΔG is the image of $(\pi, \sigma) : G \times X \to X \times X$ where $\sigma, \pi : G \times X \rightrightarrows X$ is the action groupoid.

Given this action of G on X it is immediate that this induces a natural $(G \times G)$ action on $X \times X$, which we denote $\sigma_{G^2} : G \times G \times X \times X \to X \times X$. Define a $(G \times G)$ action on $G \times X$ by

$$\sigma' : G \times G \times G \times X \to G \times X \tag{1.4}$$
$$(g_1, g_2, g, x) \mapsto (g_2 g g_1^{-1}, g_1 \cdot x) \tag{1.5}$$

so that (π, σ) is $(G \times G)$-equivariant. There is a fibre diagram

$$\begin{array}{ccc} G \times G \times G \times X & \xrightarrow{(\mathrm{id}_{G\times G}, \pi, \sigma)} & (G \times G) \times (X \times X) \\ {\scriptstyle \sigma'} \downdownarrows {\scriptstyle \pi'} & & {\scriptstyle \sigma_{G^2}} \downdownarrows {\scriptstyle \pi_{G^2}} \\ G \times X & \xrightarrow[(\pi,\sigma)]{} & X \times X. \end{array}$$

We justify the name *equivariant diagonal* as $\mathcal{O}_{\Delta G} := ((\pi,\sigma)_*\mathcal{O}_{G\times X}, (\mathrm{id}_{G\times G},\pi,\sigma)_*\gamma)$ is the equivariant kernel for the identity functor, where γ is the canonical linearisation of $\mathcal{O}_{G\times X}$, as we show in the following example. Although this result is well-known, we are not aware of it being shown explicitly in this generality in the literature. For G a finite group this is Example 3.15 of [Plo05]; for $X = \mathrm{Spec}R$ and $G = \mathbb{G}_m$ this is Lemma 2.1.5 of [BDF17].

Lemma 1.6.0.15. *The equivariant Fourier-Mukai transform $\Phi_{\mathcal{O}_{\Delta G}}$ is the identity functor on $D^b([X/G])$.*

Proof. Let $(\mathcal{E},\alpha) \in D^b([X/G])$.

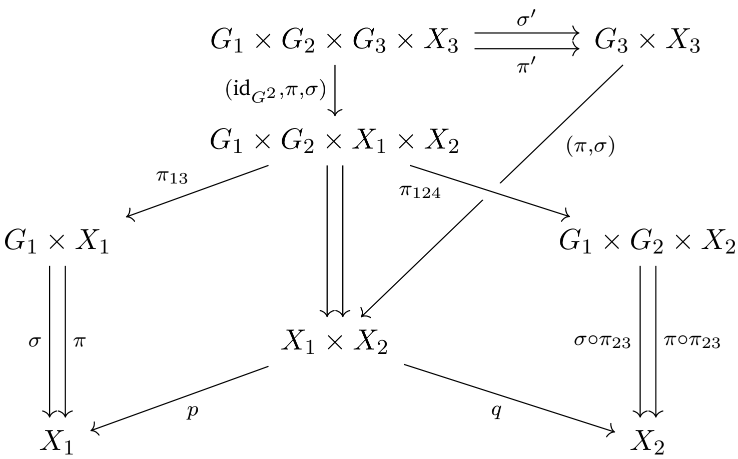
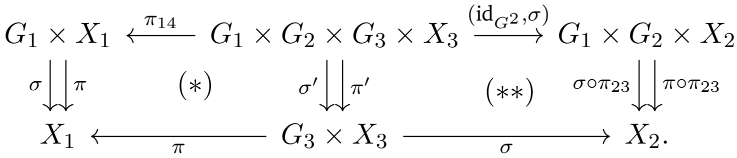


Figure 1.4: The Fourier-Mukai transform diagram for the equivariant diagonal.

Then

$$\begin{aligned}\Phi_{\mathcal{O}_{G\Delta}}(\mathcal{E},\alpha) &= \left([q_*(p^*\mathcal{E}\otimes\mathcal{O}_{G\Delta})]^G, [\pi_{124*}(\pi_{13}^*\alpha\otimes(\mathrm{id}_{G\times G},\pi,\sigma)_*\gamma)]^G\right)\\ &\simeq ([q_*(\pi,\sigma)_*(\pi,\sigma)^*p^*\mathcal{E}]^G, [\pi_{124*}(\mathrm{id}_{G^2},\pi,\sigma)_*(\mathrm{id}_{G^2},\pi,\sigma)^*\pi_{13}^*\alpha]^G)\\ &\simeq ([\sigma_*\pi^*\mathcal{E}]^G, [(\mathrm{id}_{G^2},\sigma)_*\pi_{14}^*\alpha]^G)\end{aligned}$$

where the second line follows by the projection formula and $(\mathrm{id}_{G^2},\sigma)$ and π_{14} are defined as the obvious compositions. In other words, we are pulling back along the left square in the following diagram and pushing forward along the right:



This gives a well-defined element $(\sigma_*\pi^*\mathcal{E}, (\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha) \in D^b([X_2/(G_1 \times G_2)])$ as we now show. To see that $(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha$ gives a suitable equivariant structure on this sheaf, note that it is an isomorphism $(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\sigma^*\mathcal{E} \to (\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\pi^*\mathcal{E}$ and that

$$(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\sigma^*\mathcal{E} \simeq (\mathrm{id}_{G^2}, \sigma)_*\sigma'^*\pi^*\mathcal{E} \simeq (\sigma \circ \pi_{23})^*\sigma_*\pi^*\mathcal{E} \quad \text{and} \quad (\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\pi^*\mathcal{E} \simeq (\mathrm{id}_{G^2}, \sigma)_*\pi'^*\pi^*\mathcal{E} \simeq (\pi \circ \pi_{23})^*\sigma_*\pi^*\mathcal{E}.$$

The first isomorphism in both cases is the commutativity of $(*)$, and the second isomorphisms follow by base-change around the respective fibre diagrams $(**)$. Therefore

$$(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha : (\sigma \circ \pi_{23})^*\sigma_*\pi^*\mathcal{E} \xrightarrow{\sim} (\pi \circ \pi_{23})^*\sigma_*\pi^*\mathcal{E}$$

as required. Letting ε_2 denote the inclusion by the identity element of G_2, the pullback around

$$\begin{array}{ccc} G_1 \times G_2 \times X_2 & \xleftarrow{\varepsilon_2} & G_1 \times X_2 \\ {\scriptstyle \sigma\circ\pi_{23}} \downarrow\downarrow {\scriptstyle \pi\circ\pi_{23}} & & \downarrow {\scriptstyle \pi} \\ X_2 & \xleftarrow{\mathrm{id}} & X_2 \end{array}$$

gives $(\sigma_*\pi^*\mathcal{E}, \varepsilon_2^*(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha)$. The commutativity of this square ensures that the equivariant structure here is an isomorphism

$$\varepsilon_2^*(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha : \pi^*\sigma_*\pi^*\mathcal{E} \longrightarrow \pi^*\sigma_*\pi^*\mathcal{E}. \tag{1.6}$$

By (1.2), the G-invariant part is determined as the equaliser of the two maps

$$\sigma_*\pi^*\mathcal{E} \underset{\pi_*\varepsilon_2^*(\mathrm{id}_{G^2},\sigma)_*\pi_{14}^*\alpha\circ\eta}{\overset{\eta}{\rightrightarrows}} \pi_*\pi^*\sigma_*\pi^*\mathcal{E} \tag{1.7}$$

where the lower map is the map obtained from (1.6) by adjunction. We define two maps:

$$p : \mathcal{E} \xrightarrow{\eta'} \sigma_*\sigma^*\mathcal{E} \xrightarrow{\sigma_*\alpha} \sigma_*\pi^*\mathcal{E} \xrightarrow{\eta} \pi_*\pi^*\sigma_*\pi^*\mathcal{E}$$

$$q : \pi_*\pi^*\sigma_*\pi^*\mathcal{E} \xrightarrow{i^*\varepsilon} \sigma_*\pi^*\mathcal{E} \xrightarrow{\sigma_*\alpha^{-1}} \sigma_*\sigma^*\mathcal{E} \xrightarrow{i^*\varepsilon'} \mathcal{E}$$

where $i : X \to G \times X$ is the inclusion by the identity and $\varepsilon' : \sigma^*\sigma_*\sigma^*\mathcal{E} \to \sigma^*\mathcal{E}$ is the counit map. We remark the following

i) $qp \simeq \mathrm{id}_{\mathcal{E}}$ by inspection, which follows from the pullbacks along i of the following compositions

$$\sigma^*\mathcal{E} \xrightarrow{\sigma^*\eta'} \sigma^*\sigma_*\sigma^*\mathcal{E} \xrightarrow{\varepsilon'} \sigma^*\mathcal{E}$$

$$\pi^*\sigma_*\pi^*\mathcal{E} \xrightarrow{\pi^*\eta} \pi^*\pi_*\pi^*\sigma_*\pi^*\mathcal{E} \xrightarrow{\varepsilon} \pi^*\sigma_*\pi^*\mathcal{E}$$

which, by standard facts about adjunctions, are both the identity.

ii) $pq \simeq \pi_*\varepsilon_2^*(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\alpha$ by inspection on stalks. In particular, $(\pi_*\varepsilon_2^*(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\sigma^*\mathcal{E})_{(g_1, g_3\cdot x)} \simeq \mathcal{E}_{g_1\cdot x}$ and $(\pi_*\varepsilon_2^*(\mathrm{id}_{G^2}, \sigma)_*\pi_{14}^*\pi^*\mathcal{E})_{(g_1, g_3\cdot x)} \simeq \mathcal{E}_x$, with the map between them given by $\sigma_*\alpha$. For the composition pq, the pullbacks along i of the two counit maps restrict the family of sections to the section over the identity element. Thus they do not "see" the map $\sigma_*\alpha^{-1}$ as this acts trivially over the identity element. The composition pq is therefore also determined completely by $\sigma_*\alpha$ and the result follows.

The following therefore equalises (1.7):

$$\mathcal{E} \xrightarrow{\eta'} \sigma_*\sigma^*\mathcal{E} \xrightarrow{\sigma_*\alpha} \sigma_*\pi^*\mathcal{E} \underset{pq\circ\eta}{\overset{\eta}{\rightrightarrows}} \pi_*\pi^*\sigma_*\sigma^*\mathcal{E}$$

and satisfies the requisite universal property by taking, for any $r : \mathcal{F} \to \sigma_*\pi^*\mathcal{E}$ which equalises (1.7), $i^*\varepsilon' \circ \sigma_*\alpha^{-1} \circ r : \mathcal{F} \to \mathcal{E}$. Thus $\Phi_{\mathcal{O}_{G\Delta}}(\mathcal{E}, \alpha) = (\mathcal{E}, \alpha)$, i.e. $\Phi_{\mathcal{O}_{G\Delta}}$ is the identity functor. □

1.7 Quivers and quiver representations

Definition 1.7.0.1. *A quiver $Q = (Q_0, Q_1, h, t)$ is a directed graph specified by a set of vertices Q_0, a set of arrows Q_1 and two functions $h, t : Q_1 \to Q_0$ taking an arrow to the vertex at its head or tail, respectively. A quiver is called finite if both Q_0 and Q_1 are finite sets. A quiver is called connected if the underlying graph obtained by forgetting the direction of the arrows in the quiver is connected.*

Definition 1.7.0.2. *A k-linear representation $V = (V_i, \varphi_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of a finite quiver Q is the data of a k-vector space V_i for each $i \in Q_0$, together with a linear map $\varphi_\alpha : V_{t(\alpha)} \to V_{h(\alpha)}$ for each $\alpha \in Q_1$. A representation is called finite dimensional if all the V_i are finite dimensional vector spaces. A morphism between two such representations V and V' consists of linear maps $f_i : V_i \to V_i'$ for all $i \in Q_0$ such that all the induced squares commute. In particular, a subrepresentation $V \hookrightarrow V'$ is a morphism of representations such that all the f_i are inclusion maps.*

We denote by $\text{Rep}(Q)$ the category of representations of Q, and by $\text{Rep}^{fd}(Q)$ the subcategory of finite dimensional representations.
For any quiver Q, a *path* in the quiver from i to j of length l is a concatenation of arrows $\alpha_1 \dots \alpha_l$ such that $t(\alpha_1) = i$, $h(\alpha_l) = j$ and $h(\alpha_k) = t(\alpha_{k+1})$ for all $1 \leq k \leq l-1$. For any vertex i we have the trivial path ϵ_i of length 0. The following simple lemma will prove useful.

Lemma 1.7.0.3. *Let Q be a quiver. Let V be a representation of Q with one-dimensional vector spaces V_i and V_j at vertices i, j respectively. Let $V' \subseteq V$ be a subrepresentation of the quiver. If $V_i \subset V'$ and there exists a path p from i to j in the quiver such that $\varphi_p : V_i \to V_j$ is a nonzero map, then $V_j \subset V'$ also.*

Proof. In order for V' to be a subrepresentation of V, the diagram

$$\begin{array}{ccc} V_i & \hookrightarrow & V_i \\ \downarrow & & \downarrow{\scriptstyle \varphi_p} \\ ? & \hookrightarrow & V_j \end{array}$$

must commute, where $? = V_j$ or 0 as $\dim(V_j) = 1$. But going round the top of the diagram is a nonzero map, so $? = V_j$. □

We define the *path algebra* as follows.

Definition 1.7.0.4. *The path algebra $\mathbb{C}Q$ is defined to be the k-algebra whose underlying vector space is the space of formal k-weighted sums of paths in the quiver. Multiplication of two paths $\alpha_1 \dots \alpha_l$ and $\beta_1 \dots \beta_{l'}$ is defined by*

$$(\alpha_1 \dots \alpha_l) \times (\beta_1 \dots \beta_{l'}) = \begin{cases} \alpha_1 \dots \alpha_l \beta_1 \dots \beta_{l'}, & \textit{if } h(\alpha_l) = t(\beta_1) \\ 0, & \textit{else.} \end{cases}$$

Remark 1.7.0.5. i) $\mathbb{C}Q$ is not commutative in general.

ii) It is an associative graded algebra, with grading by path length.

iii) It is a unital associative graded algebra iff Q is finite. In particular, if $Q_0 = \{0, \dots N\}$, then the identity is $1 = \sum_{i=0}^{N} \epsilon_i$. For more detail, see [Sos13].

Definition 1.7.0.6. *Let Q be a finite and connected quiver. The two-sided ideal of $\mathbb{C}Q$ generated by the arrows of Q is called the arrow ideal and is denoted by $\mathcal{R}_Q$.*

Definition 1.7.0.7. *Let Q be a finite quiver. A two-sided ideal $\mathcal{I}$ of $\mathbb{C}Q$ is called admissible if there exists an $m \geq 2$ such that $\mathcal{R}_Q^m \subseteq \mathcal{I} \subseteq \mathcal{R}_Q^2$.*

Definition 1.7.0.8. *Let Q be a finite quiver and let $V = (V_i, \varphi_\alpha)$ be a representation of Q. For a nontrivial path $\omega = \alpha_1 \dots \alpha_l$ of length l from i to j in Q, we define the evaluation of V on the path ω to be the $\mathbb{C}$-linear map from V_i to V_j given by*

$$\varphi_\omega = \varphi_{\alpha_l} \cdots \varphi_{\alpha_1}.$$

Given a set of paths $\{\omega_k\}_{k=1}^m$ from i to j, the definition formally extends to $\mathbb{C}$-linear combinations as follows. For a linear combination of paths $\Omega = \sum_{k=1}^m \lambda_k \omega_k$ with $\lambda_k \in \mathbb{C}$, we define the evaluation to be

$$\varphi_\Omega = \sum_{k=1}^{m} \lambda_k \varphi_{\omega_k}.$$

Definition 1.7.0.9. *Let Q be a finite quiver and $\mathcal{I}$ an admissible ideal of $\mathbb{C}Q$. A representation V is said to be bound by $\mathcal{I}$ if we have $\varphi_\Omega = 0$ for all $\Omega \in \mathcal{I}$. We denote by Rep$(Q, \mathcal{I})$ the full subcategory of Rep(Q) containing representations of Q which are bound by $\mathcal{I}$.*

Definition 1.7.0.10. *Let Q be a finite and connected quiver, and let $\mathcal{I}$ be an admissible ideal of $\mathbb{C}Q$. The bound quiver algebra is defined to be $\mathbb{C}Q/\mathcal{I}$.*

Proposition 1.7.0.11. *Let Q be a finite and connected quiver, and let $\mathcal{I}$ be an admissible ideal of $\mathbb{C}Q$. There exists a $\mathbb{C}$-linear equivalence of categories*

$$\mathit{Mod}(\mathbb{C}Q/\mathcal{I}) \xrightarrow{\sim} \mathit{Rep}(Q, \mathcal{I}),$$

which restricts to an equivalence

$$\mathit{Mod}^{fg}(\mathbb{C}Q/\mathcal{I}) \xrightarrow{\sim} \mathit{Rep}^{fd}(Q, \mathcal{I}).$$

Proof. See [Sos13] or [ASS06]. □

1.8 Notation and conventions

By an algebraic variety we mean a reduced scheme of finite type over an algebraically closed field $k = \overline{k}$, and this ground field should be taken to be the complex numbers $\mathbb{C}$ unless otherwise indicated. Similarly, unless otherwise indicated, we denote by $A := \mathbb{C}[x_1, \dots, x_n]$ the coordinate ring of affine space $\mathbb{A}^n$, and by $R := \mathbb{C}[f_0, \dots, f_5]/(f_0 f_1 = f_2 f_3 = f_4 f_5)$ the coordinate ring of the affine scheme of interest in Chapters 3 and 4. We denote by $\mathbb{G}_m :=$ Spec$\mathbb{C}[t, t^{-1}]$ the multiplicative group of $\mathbb{C}$, considered as a scheme.

— Chapter 2 —

Wall-Crossing Equivalences from VGIT

In this chapter we present historical results about stability in the sense of geometric invariant theory (GIT) [MFK94]. Morally, taking quotients of varieties by group actions should be the space of closed orbits of the group action, but this turns out to be too strict a notion and often removes much of the interesting geometry. GIT is one way to solve this. It works by removing the points which cause this bad behaviour, known as the "unstable" points, and keeping the complement, known as the "semistable" points. The space of closed orbits in this semistable locus exhibits more of this interesting geometry which was previously hidden (cf. the introduction to [Kin94]). Here stability is defined with respect to a stability parameter given by a G-equivariant line bundle.

By varying this stability parameter we get a wall-and-chamber decomposition of the stability space into cells [DH98], where the semistable loci are unchanged by moving within the cell. The study of how the quotient changes when the stability parameter is varied is known as variation of GIT (VGIT). In particular, crossing walls changes the GIT quotient, often via a birational modification, and a much-studied problem has been to determine when wall-crossings correspond to equivalences between the derived categories of the quotients on either side of the wall.

As we will see, the derived categories appearing in the schober we construct in Chapter 4 arise naturally from (the stacky version of) a VGIT problem, and we wish to use the heavyweight technology developed by Halpern-Leistner [Hal15] and Ballard, Favero and Katzarkov [BFK19] to construct derived equivalences between wall-crossings. This involves careful analysis of the *unstable* locus and a certain stratification of the unstable locus known as a Kempf-Ness (KN) stratification. In this chapter we summarise the main aspects of this theory.

2.1 Group actions and variation of GIT

In this section, we first construct an equivalence between the category of G-equivariant quasi-coherent sheaves on an affine scheme $X = \mathrm{Spec}R$ and the category of R-modules with a coaction. We present some results where $G = \mathbb{G}_m$, then go on to define GIT stability and give a numerical criterion for stability due to Mumford. To illustrate the technology of VGIT, we introduce a running example involving $\mathbb{G}_m$ acting on $\mathbb{A}^{n+1}$ by multiplication on the coordinates. In the wall-and-chamber decomposition of the stability space, the GIT quotients are $\mathbb{P}^1$ on one side of the wall and the empty set on the other, whilst on the wall itself it is a single point; in this case there is clearly no hope of the wall-crossing giving an equivalence. We will continue to use this running example in the subsequent sections in this chapter. For experts, it is hoped that this running example will also make apparent the sign conventions we are using, which can be inconsistent in the literature.

Definition 2.1.0.1. *Given a ring homomorphism $f : R \to S$, the restriction of scalars functor res$_f$: S-Mod $\to$ R-Mod is the one which takes an S-module N and considers it as an R-module via f. Explicitly, the action is given by $r \cdot_R s := f(r) \cdot_S s$.*
Given a morphism of S-modules, the restriction of scalars functor returns the same morphism, but now considered as a morphism of R-modules.

Definition 2.1.0.2. *Given a ring homomorphism $f : R \to S$, the extension of scalars functor ext$_f$: R-Mod $\to$ S-Mod is the one which takes an R-module M and takes it to the S-module $S \otimes_{R,f} M$. This tensor product is well-defined over R as we consider S as an R-module via restriction of scalars along f. The S-action on $S \otimes_{R,f} M$ is given by the action of S on itself, i.e. $s \cdot (\sum_i s_i \otimes m_i) = \sum_i ss_i \otimes m_i$.*
Given a morphism of R-modules $\alpha : M \to N$, the extension of scalars functor returns the morphism of S-modules

$$\begin{aligned} S \otimes_{R,f} M &\to S \otimes_{R,f} N \\ \sum_i s_i \otimes m_i &\mapsto \sum_i s_i \otimes \alpha(m_i). \end{aligned}$$

Lemma 2.1.0.3. *The extension and restriction of scalars functors form an adjoint pair, ext$_f \dashv$ res$_f$.*

Definition 2.1.0.4. *Let G be a linear algebraic group and let $S := \Gamma(G, \mathcal{O}_G)$. In addition, let $\hat{\mu} : S \to S \otimes_{\mathbb{C}} S$ and $\hat{\beta} : S \to \mathbb{C}$ be the $\mathbb{C}$-algebra homomorphisms corresponding to multiplication μ and the inclusion of a point as the identity in G,*

$\beta : Spec(\mathbb{C}) \hookrightarrow G$. *A co-action of* S *on* R *is a homomorphism of* $\mathbb{C}$*-algebras* $\hat{\sigma} : R \to S \otimes_{\mathbb{C}} R$ *such that*

i) The following diagram is commutative in the category of $\mathbb{C}$*-algebras*

$$\begin{array}{ccc} R & \xrightarrow{\hat{\sigma}} & S \otimes_{\mathbb{C}} R \\ {\scriptstyle \hat{\sigma}}\downarrow & & \downarrow{\scriptstyle \hat{\mu}\otimes id_R} \\ S \otimes_{\mathbb{C}} R & \xrightarrow[id_S \otimes \hat{\sigma}]{} & S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} R \end{array} \tag{CR1}$$

ii) The composition

$$R \xrightarrow{\hat{\sigma}} S \otimes_{\mathbb{C}} R \xrightarrow{\hat{\beta}\otimes id_R} R \tag{CR2}$$

is the identity.

Definition 2.1.0.5. *Given a co-action of* S *on a* $\mathbb{C}$*-algebra* R*,* $\hat{\sigma} : R \to S \otimes_{\mathbb{C}} R$*, an* R*-module with a co-action of* S *is an* R*-module* M *with a map of* R*-modules* $\hat{\sigma}_M : M \to S \otimes_{\mathbb{C}} M$ *such that*

i) The following diagram is commutative in the category of R*-modules*

$$\begin{array}{ccc} M & \xrightarrow{\hat{\sigma}_M} & S \otimes_{\mathbb{C}} M \\ {\scriptstyle \hat{\sigma}_M}\downarrow & & \downarrow{\scriptstyle \hat{\mu}\otimes id_M} \\ S \otimes_{\mathbb{C}} M & \xrightarrow[id_S \otimes \hat{\sigma}_M]{} & S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} M \end{array} \tag{CM1}$$

where we consider $S \otimes_{\mathbb{C}} M$ *and* $S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} M$ *as* R*-modules via restriction of scalars along* $\hat{\sigma}$ *and* $(\hat{\mu} \otimes id_R) \circ \hat{\sigma} = (id_S \otimes \hat{\sigma}) \circ \hat{\sigma}$*, respectively.*

ii) The composition

$$M \xrightarrow{\hat{\sigma}_M} S \otimes_{\mathbb{C}} M \xrightarrow{\hat{\beta}\otimes id_M} M \tag{CM2}$$

is the identity.

This defines the objects in the category of R*-modules with a co-action of* S*, and the morphisms in this category are defined to be the maps of* R*-modules* $f : M \to N$ *such that the following square commutes*

$$\begin{array}{ccc} M & \xrightarrow{\hat{\sigma}_M} & S \otimes_{\mathbb{C}} M \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle id_S \otimes f} \\ N & \xrightarrow[\hat{\sigma}_N]{} & S \otimes_{\mathbb{C}} N. \end{array}$$

Definition 2.1.0.6. *We say* $m \in M$ *is* invariant *if* $\hat{\sigma}_M(m) = 1_S \otimes m$*.*

Lemma 2.1.0.7. *Let $(f, f^{\#}) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces, $\mathcal{F}$ a sheaf of $\mathcal{O}_Y$-modules and $x \in X$. Then there is an isomorphism (as $\mathcal{O}_{X,x}$-modules) of stalks*

$$(f^*\mathcal{F})_x \simeq \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

where the tensor product is well-defined by using $(f^{\#})_x : \mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$.

Proposition 2.1.0.8. *Let R be a finitely-generated commutative $\mathbb{C}$-algebra. The category of G-equivariant quasi-coherent sheaves on $X = SpecR$ is equivalent to the category of R-modules with a co-action of S.*

Proof. The category of quasi-coherent sheaves on X is well-known to be equivalent to the category of R-modules. Explicitly, this equivalence is given in one direction by taking global sections, and in the other via the $\widetilde{(-)}$ functor of [Har77]. It therefore remains to translate between the G-equivariant structure on a sheaf and the co-action of S on its global sections.

Let $\mathcal{E}$ be a G-equivariant quasi-coherent sheaf on X. From the two maps $\pi_2, \mathrm{act} : G \times X \to X$ we get the corresponding maps of rings

$$\widehat{\pi_2} : R \to S \otimes_{\mathbb{C}} R \qquad\qquad \widehat{\mathrm{act}} : R \to S \otimes_{\mathbb{C}} R$$
$$r_i \mapsto 1_S \otimes r_i \qquad\qquad r_i \mapsto s_i \otimes r_i$$

where the r_i are the generators of R, as well as the maps

$$\hat{\mu} : S \to S \otimes_{\mathbb{C}} S \qquad\qquad \hat{\beta} : S \to \mathbb{C}$$
$$s \mapsto \hat{\mu}(s) \qquad\qquad s \mapsto s(1_G)$$

corresponding to the multiplication map $\mu : G \times G \to G$ and the inclusion of the identity, respectively. As $\mathcal{E}$ comes with a choice of isomorphism $\alpha : \mathrm{act}^*\mathcal{E} \xrightarrow{\sim} \pi_2^*\mathcal{E}$, this induces an isomorphism of $(S \otimes R)$-modules between global sections

$$\hat{\alpha} : \Gamma(G \times X, \mathrm{act}^*\mathcal{E}) \xrightarrow{\sim} \Gamma(G \times X, \pi_2^*\mathcal{E}).$$

As $\Gamma(G \times X, f^*\mathcal{E}) \simeq \mathrm{ext}_{\hat{f}}(E)$ for $f \in \{\mathrm{act}, \pi_2\}$, there is a corresponding map of R-modules under the adjunction of Lemma 2.1.0.3 given by the composition

$$\begin{array}{ccccc} \Gamma(X, \mathcal{E}) & \longrightarrow & \Gamma(G \times X, act^*\mathcal{E}) & \xrightarrow{\hat{\alpha}} & \Gamma(G \times X, \pi_2^*\mathcal{E}) \simeq S \otimes_{\mathbb{C}} E \\ E & \longrightarrow & (S \otimes_{\mathbb{C}} R) \otimes_{R,\widehat{act}} E & \longrightarrow & (S \otimes_{\mathbb{C}} R) \otimes_{R,\widehat{\pi_2}} E \simeq S \otimes_{\mathbb{C}} E \end{array} \tag{2.1}$$

where the first map[1] is given by $e \mapsto (1_S \otimes 1_R) \otimes e$ and we identify $(S \otimes_{\mathbb{C}} R) \otimes_{R,\widehat{\pi_2}} E$ with $S \otimes_{\mathbb{C}} E$ via the isomorphism $(s \otimes r) \otimes e = (s \otimes 1_R) \otimes re \mapsto s \otimes re$. Writing $\hat{\alpha}(1_S \otimes 1_R \otimes e) = \sum_i s_i \otimes 1_R \otimes e_i$, we call this composition $\hat{\sigma}_E : E \to S \otimes_{\mathbb{C}} E$, $e \mapsto \sum_i s_i \otimes e_i$. Taking global sections in diagram (CC), the commutativity of the diagram gives us two equal maps of $(S \otimes S \otimes R)$-modules

$$\text{ext}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} E \to \text{ext}_{(\hat{\mu} \otimes \text{id}_R) \circ \widehat{\pi_2}} E,$$

and their equality is equivalent to the equality of the image of these maps under the adjunction $\text{ext}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} \dashv \text{res}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}}$.
Examine first the anticlockwise map in (CC), which we consider as the composition

$$\begin{array}{ccccc} \text{ext}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} E & = & \text{ext}_{(\hat{\mu} \otimes \text{id}_R) \circ \widehat{\text{act}}} E & \xrightarrow{\sim} & \text{ext}_{\hat{\mu} \otimes \text{id}_R} \text{ext}_{\widehat{\text{act}}} E \\ & & & & \downarrow \text{ext}_{\hat{\mu} \otimes \text{id}_R} \hat{\alpha} \\ & & S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E & \xleftarrow{\sim} & \text{ext}_{\hat{\mu} \otimes \text{id}_R} \text{ext}_{\widehat{\pi_2}} E. \end{array} \tag{2.2}$$

The last map is an isomorphism as

$$\begin{aligned} \text{ext}_{\hat{\mu} \otimes \text{id}_R} \text{ext}_{\widehat{\pi_2}} E &\simeq \text{ext}_{(\hat{\mu} \otimes \text{id}_R) \circ \widehat{\pi_2}} E \\ &= (S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} R) \otimes_{R, (\hat{\mu} \otimes \text{id}_R) \circ \widehat{\pi_2}} E \\ &\simeq S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E \end{aligned}$$

where the final line is due to the fact that

$$(s_1 \otimes s_2 \otimes r) \otimes e = (s_1 \otimes s_2 \otimes 1_R)(1_S \otimes 1_S \otimes r) \otimes e = (s_1 \otimes s_2 \otimes 1_R) \otimes re.$$

By adjunction, (2.2) then corresponds canonically to the map $E \to \text{res}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} (S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E)$ which is given by applying $\text{res}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}}$ to (2.2) and precomposing with the unit of adjunction map $E \to \text{res}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} \text{ext}_{(id_S \otimes \widehat{\text{act}}) \circ \widehat{\text{act}}} E$. A check verifies that this is the map of R-modules given by the composition

$$\begin{aligned} & e \mapsto 1_{S \otimes S \otimes R} \otimes e \overset{\sim}{\mapsto} 1_{S \otimes S \otimes R} \otimes 1_{S \otimes R} \otimes e \mapsto 1_{S \otimes S \otimes R} \otimes \hat{\alpha}(1_{S \otimes R} \otimes e) = \\ & = 1_{S \otimes S \otimes R} \otimes (\sum_i s_i \otimes 1_R \otimes e_i) \overset{(*)}{=} \sum_i \hat{\mu}(s_i) \otimes 1_R \otimes 1_{S \otimes R} \otimes e_i \overset{\sim}{\mapsto} \\ & \overset{\sim}{\mapsto} \sum_i \hat{\mu}(s_i) \otimes 1_R \otimes e_i \overset{\sim}{\mapsto} \sum_i \hat{\mu}(s_i) \otimes e_i \in S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E, \end{aligned} \tag{2.3}$$

where $(*)$ follows by commuting past the tensor product in $\text{ext}_{\hat{\mu} \otimes \text{id}_R} \text{ext}_{\widehat{\pi_2}} E = (S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} R) \otimes_{S \otimes R, \hat{\mu} \otimes \text{id}_R} (S \otimes_{\mathbb{C}} R) \otimes_{R, \widehat{\pi_2}} E$. By the construction of the map $\hat{\sigma}_E$ above,

[1] This is the unit of adjunction map $E \to \text{res}_{\widehat{\text{act}}} \text{ext}_{\widehat{\text{act}}} E$.

the s_i and e_i appearing here are exactly those in $\hat{\sigma}_E(e) = \sum_i s_i \otimes e_i$, and so (2.3) is exactly the composition map $E \xrightarrow{\hat{\sigma}_E} S \otimes_{\mathbb{C}} E \xrightarrow{\hat{\mu}\otimes \mathrm{id}_E} S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E$ in (CM1).
Similarly, we now examine now the clockwise map in (CC). As a map of $(S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} R)$-modules, we consider this as the composition

$$
\begin{array}{ccccc}
\mathrm{ext}_{(\mathrm{id}_S \otimes \widehat{\mathrm{act}})\circ \widehat{\mathrm{act}}} E & \xrightarrow{\sim} & \mathrm{ext}_{\mathrm{id}_S \otimes \widehat{\mathrm{act}}} \mathrm{ext}_{\widehat{\mathrm{act}}} E & \xrightarrow{\mathrm{ext}_{\mathrm{id}_S \otimes \widehat{\mathrm{act}}} \hat{\alpha}} & \mathrm{ext}_{\mathrm{id}_S \otimes \widehat{\mathrm{act}}} \mathrm{ext}_{\widehat{\pi_2}} E \\
 & & & & \downarrow \wr \\
\mathrm{ext}_{\hat{p}_{23}} \mathrm{ext}_{\widehat{\mathrm{act}}} E & \xleftarrow{\sim} & \mathrm{ext}_{\hat{p}_{23}\circ \widehat{\mathrm{act}}} E & = & \mathrm{ext}_{(\mathrm{id}_S \otimes \widehat{\mathrm{act}})\circ \widehat{\pi_2}} E \\
\downarrow {\scriptstyle \mathrm{ext}_{\hat{p}_{23}} \hat{\alpha}} & & & & \\
\mathrm{ext}_{\hat{p}_{23}} \mathrm{ext}_{\widehat{\pi_2}} E & \xrightarrow{\sim} & \mathrm{ext}_{\hat{p}_{23}\circ \widehat{\pi_2}} E & = & \mathrm{ext}_{(\hat{\mu}\otimes \mathrm{id}_R)\circ \widehat{\pi_2}} E \\
 & & & & \downarrow \wr \\
 & & & & S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E
\end{array}
$$

which by adjunction induces the map of R-modules $E \to \mathrm{res}_{(\mathrm{id}_S \otimes \widehat{\mathrm{act}})\circ \widehat{\mathrm{act}}}(S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E)$ given by

$$
\begin{aligned}
& e \mapsto 1_{S\otimes S\otimes R} \otimes e \overset{\sim}{\mapsto} 1_{S\otimes S\otimes R} \otimes 1_{S\otimes R} \otimes e \mapsto 1_{S\otimes S\otimes R} \otimes \hat{\alpha}(1_{S\otimes R} \otimes e) = \\
& \quad = 1_{S\otimes S\otimes R} \otimes (\sum_i s_i \otimes 1_R \otimes e_i) = \sum_i s_i \otimes 1_{S\otimes R} \otimes 1_{S\otimes R} \otimes e_i \overset{\sim}{\mapsto} \\
& \quad \overset{\sim}{\mapsto} \sum_i s_i \otimes 1_{S\otimes R} \otimes e_i \overset{\sim}{\mapsto} \sum_i s_i \otimes 1_{S\otimes R} \otimes 1_{S\otimes R} \otimes e_i \mapsto \\
& \quad \mapsto \sum_i s_i \otimes 1_{S\otimes R} \otimes \hat{\alpha}(1_{S\otimes R} \otimes e_i) = \sum_i s_i \otimes 1_{S\otimes R} \otimes \sum_j s_j \otimes 1_R \otimes e_{ij} = \\
& \quad = \sum_{i,j} s_i \otimes s_j \otimes 1_R \otimes 1_{S\otimes R} \otimes e_{ij} \overset{\sim}{\mapsto} \sum_{i,j} s_i \otimes s_j \otimes 1_R \otimes e_{ij} \overset{\sim}{\mapsto} \sum_{i,j} s_i \otimes s_j \otimes e_{ij},
\end{aligned}
$$

and this is indeed the map of R-modules $E \xrightarrow{\hat{\sigma}_E} S \otimes_{\mathbb{C}} E \xrightarrow{\mathrm{id}_S \otimes \hat{\sigma}_E} S \otimes_{\mathbb{C}} S \otimes_{\mathbb{C}} E$ in (CM1). We have therefore shown that the equality of the two directions around (CC) implies the equality of the directions around (CM1). As this was an adjunction argument, the converse statement also holds. The cocycle condition (CC) therefore holds for a sheaf if and only if (CM1) holds for its global sections.
Given a G-equivariant sheaf $\mathcal{E}$, we now check that (CM2) holds. Consider the map $\beta \times \mathrm{id}_X : X \to G \times X$. On global sections, $(\beta \times \mathrm{id}_X)^* \alpha$ and the identifications

$\mathrm{ext}_{\hat\beta\otimes\mathrm{id}_R}\mathrm{ext}_{\widehat{\mathrm{act}}}E \simeq E \simeq \mathrm{ext}_{\hat\beta\otimes\mathrm{id}_R}\mathrm{ext}_{\widehat{\pi_2}}E$ give us the map of R-modules

$$\begin{array}{ccc} E & \xrightarrow{\sim} & R\otimes_{S\otimes_{\mathbb{C}}R,\hat\beta\otimes\mathrm{id}_R}(S\otimes_{\mathbb{C}}R)\otimes_{R,\widehat{\mathrm{act}}}E \\ & & \Big\downarrow \mathrm{ext}_{\hat\beta\otimes\mathrm{id}_R}\hat\alpha \\ E & \xleftarrow{\sim} & R\otimes_{S\otimes_{\mathbb{C}}R,\hat\beta\otimes\mathrm{id}_R}(S\otimes_{\mathbb{C}}R)\otimes_{R,\widehat{\pi_2}}E \end{array} \tag{2.4}$$

given by

$$e\mapsto 1_R\otimes 1_{S\otimes_{\mathbb{C}}R}\otimes e\mapsto 1_R\otimes\sum_i(s_i\otimes 1_R\otimes e_i)=\sum_i\hat\beta(s_i)1_R\otimes 1_{S\otimes_{\mathbb{C}}R}\otimes e_i=$$
$$=1_R\otimes 1_{S\otimes_{\mathbb{C}}R}\otimes\sum_i\hat\beta(s_i)e_i\mapsto\sum_i\hat\beta(s_i)e_i,$$

which is exactly the composition map from (CM2). Now define a map $\gamma : X \to G\times G\times X$ by $x\mapsto(1_G,1_G,x)$ and note that the three possible compositions

$$X\xrightarrow{\gamma}G\times G\times X\xrightarrow{f}G\times X,\qquad\qquad f\in\{\mathrm{id}_G\times\mathrm{act},p_{23},\mu\times\mathrm{id}_X\},$$

are all the same map, namely $\beta\times\mathrm{id}_X$. Note also that $\mathrm{act}\circ(\beta\times\mathrm{id}_X)=\pi_2\circ(\beta\times\mathrm{id}_X)=\mathrm{id}_X$. Pulling (CC) back along γ therefore gives the following commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{(\beta\times\mathrm{id}_X)^*\alpha} & \mathcal{E} \\ \Big\| & & \Big\downarrow (\beta\times\mathrm{id}_X)^*\alpha \\ \mathcal{E} & \xrightarrow[(\beta\times\mathrm{id}_X)^*\alpha]{} & \mathcal{E} \end{array}$$

and thus $(\beta\times\mathrm{id}_X)^*\alpha$ is idempotent. As it is the pullback of an isomorphism, and therefore an isomorphism itself, $(\beta\times\mathrm{id}_X)^*\alpha=\mathrm{id}_{\mathcal{E}}$. The composition in (2.4) is therefore the identity also, and so (CM2) is satisfied.

For the converse, assume M is an R-module with a coaction of S. From before, we know that (CC) holds for $\alpha:\mathrm{act}^*\tilde M\to\pi_2^*\tilde M$ and we now show that α is in fact an isomorphism. First, define a map

$$\delta : G\times X\to G\times G\times X$$
$$(g,x)\mapsto(g^{-1},g,x)$$

and pull (CC) back along δ to get

$$\begin{array}{ccc} \pi_2^*\tilde M & \xrightarrow{\epsilon^*\alpha} & \mathrm{act}^*\tilde M \\ \Big\| & & \Big\downarrow \alpha \\ \pi_2^*\tilde M & \xrightarrow[\zeta^*\alpha]{} & \pi_2^*\tilde M \end{array}$$

where ϵ, ζ are the precompositions by γ of $\mathrm{id}_G \times \mathrm{act}$ and $\mu \times \mathrm{id}_X$, respectively, i.e.

$$\begin{aligned} \epsilon : G \times X &\to G \times X & \zeta : G \times X &\to G \times X \\ (g, x) &\mapsto (g^{-1}, g \cdot x) & (g, x) &\mapsto (1_G, x). \end{aligned}$$

Using Lemma 2.1.0.7 to take stalks at the point $(g, x) \in G \times X$, this gives the commutative square

$$\begin{array}{ccc} (\tilde{M})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{G\times X,(g,x)} & \xrightarrow{(\alpha)_{(g^{-1}, g\cdot x)}} & (\tilde{M})_{g\cdot x} \otimes_{\mathcal{O}_{X,g\cdot x}} \mathcal{O}_{G\times X,(g,x)} \\ \| & & \downarrow{\scriptstyle (\alpha)_{(g,x)}} \\ (\tilde{M})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{G\times X,(g,x)} & \xrightarrow[(\alpha)_{(1_G,x)}]{} & (\tilde{M})_x \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{G\times X,(g,x)}. \end{array}$$

Using (CM2), $(\beta \times \mathrm{id}_X)^*\alpha = \mathrm{id}_{\tilde{M}}$, and thus the bottom map in this diagram is the identity. Therefore, $(\alpha)_{(g,x)}$ is an isomorphism at every point (g, x), and so α is an isomorphism as well. □

Example 2.1.0.9. Recall that an R-module M is of *finite presentation* if and only if there exist $n, m \in \mathbb{N}$ and an exact sequence $R^{\oplus m} \to R^{\oplus n} \to M \to 0$, and that a quasi-coherent sheaf $\mathcal{E}$ on $X = \mathrm{Spec} R$ is of *finite presentation* if and only if $\Gamma(X, \mathcal{E})$ is an R-module of finite presentation.

Let $\mathcal{E}, \mathcal{F}$ be two G-equivariant quasi-coherent sheaves and assume $\mathcal{E}$ is of finite presentation. As act, $\pi_2 : G \times X \to X$ are flat maps,

$$\begin{aligned} \mathrm{act}^*\mathcal{H}om_X(\mathcal{E}, \mathcal{F}) &\simeq \mathcal{H}om_{G\times X}(\mathrm{act}^*\mathcal{E}, \mathrm{act}^*\mathcal{F}) \\ &\simeq \mathcal{H}om_{G\times X}(\pi_2^*\mathcal{E}, \pi_2^*\mathcal{F}) \\ &\simeq \pi_2^*\mathcal{H}om_X(\mathcal{E}, \mathcal{F}), \end{aligned}$$

and we define $\alpha : \mathrm{act}^*\mathcal{H}om_X(\mathcal{E}, \mathcal{F}) \xrightarrow{\sim} \pi_2^*\mathcal{H}om_X(\mathcal{E}, \mathcal{F})$ to be the composition.

We define the map $\mathrm{Hom}_R(E, F) \to \mathrm{Hom}_R(E, F) \otimes_{\mathbb{C}} S \simeq \mathrm{Hom}_R(E, F \otimes_{\mathbb{C}} S)$ via composing with the coaction map of F,

$$E \longrightarrow F \longrightarrow F \otimes_{\mathbb{C}} S.$$

Example 2.1.0.10. Let R be a $\mathbb{Z}$-graded ring and let $X = \mathrm{Spec} R$. We have a $\mathbb{G}_m$ action on X corresponding to the coaction

$$\begin{aligned} R &\to S \otimes_{\mathbb{C}} R \\ r &\mapsto t^{\deg(r)} \otimes r, \end{aligned}$$

where $S = \mathbb{C}[t^{\pm}]$. Suppose M is a graded R-module. Then we obtain a $\mathbb{G}_m$-equivariant sheaf $\tilde{M}$ with $\alpha : \mathrm{act}^*\tilde{M} \xrightarrow{\sim} \pi_2^*\tilde{M}$ corresponding to the map of $(S \otimes_{\mathbb{C}} R)$-modules

$$(S \otimes_{\mathbb{C}} R) \otimes_{R,\widehat{act}} M \xrightarrow{\sim} (S \otimes_{\mathbb{C}} R) \otimes_{R,\widehat{\pi_2}} M$$
$$1_S \otimes 1_R \otimes m \mapsto t^{\deg(m)} \otimes 1_R \otimes m.$$

The following two lemmas show that this behaviour is indicative of something more general.

Lemma 2.1.0.11. *Suppose we have an action of $\mathbb{G}_m$ on an affine scheme $X = SpecR$. This action defines a $\mathbb{Z}$-grading $R = \bigoplus_n R_n$ such that the co-action map is the one in Example 2.1.0.10. That is, $r \in R_n$ if and only if*

$$\hat{\sigma} : R \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} R$$
$$r \mapsto t^n \otimes r.$$

Proof. The coaction map is given by some map

$$R \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} R$$
$$r \mapsto \sum_{n \in \mathbb{Z}} t^n \otimes r_n,$$

where only finitely many of the r_n are non-zero and the r_n are uniquely determined. It follows from condition (CR2) that $r = \sum_{n \in \mathbb{Z}} r_n$. From condition (CR1), we see that

$$\sum_{n,m \in \mathbb{Z}} t^n \otimes t^m \otimes (r_n)_m = \sum_{n \in \mathbb{Z}} t^n \otimes t^n \otimes r_n$$

and thus $(r_n)_m = \delta_{n,m} r_n$, where $\delta_{n,m}$ denotes the Kronecker delta. Defining $R_n := \{r \in R \mid r_n = r\}$, it is clear that $R_i R_j \subseteq R_{i+j}$, and so we obtain a direct sum decomposition $R = \bigoplus_{n \in \mathbb{Z}} R_n$. □

Lemma 2.1.0.12. *Let $\mathbb{G}_m$ act on $X = SpecR$, so that R has the grading given by the previous lemma. In addition, let $\mathcal{F}$ be a $\mathbb{G}_m$-equivariant quasi-coherent sheaf. Then $M := H^0(X, \mathcal{F})$ is a $\mathbb{Z}$-graded R-module with co-action*

$$\hat{\sigma}_M : M \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} M$$
$$m \mapsto t^{deg(m)} \otimes m.$$

Proof. The construction of the direct sum decomposition $M = \bigoplus_n M_n$ works in exactly the same way as in the proof of Lemma 2.1.0.11. To check that $R_i M_j \subseteq M_{i+j}$, let $r \in R_i$, $m \in M_j$. Then

$$\begin{aligned}\hat{\sigma}_M(rm) &= \hat{\sigma}(r)\hat{\sigma}_M(m)\\ &= (t^i \otimes r)(t^j \otimes m)\\ &= t^{i+j} \otimes rm.\end{aligned}$$

□

Thus, the notion to keep in mind is that a $\mathbb{G}_m$ action on an affine scheme $X = \text{Spec}R$ is nothing but a $\mathbb{Z}$-grading on R, with equivariant quasi-coherent sheaves on X being graded R-modules.

Definition 2.1.0.13. *Let $(X, \mathcal{O}_X)$ be a ringed space, and let $\mathcal{E}$ be an $\mathcal{O}_X$-module. Define*

$$T(\mathcal{E}) := \bigoplus_{n \geq 0} \mathcal{E}^{\otimes n}$$

and define Sym$(\mathcal{E})$ to be the quotient of $T(\mathcal{E})$ by the ideal generated locally by all elements of the form $s \otimes t - t \otimes s \in \mathcal{E}^{\otimes 2}(U)$.

We now recall the relative version of the Spec construction, which we denote $\underline{\text{Spec}}$. Given a scheme X and a quasi-coherent sheaf $\mathcal{E}$ of $\mathcal{O}_X$-algebras, we have associated to every open $\text{Spec}(A) = U \subseteq X$ an A-algebra, $\Gamma(U, \mathcal{E})$. The affine schemes $\text{Spec}(\Gamma(U, \mathcal{E}))$ are compatible [Stacks, Tag 01LL], and we can glue them together to form the scheme $\underline{\text{Spec}}_X(\mathcal{E})$. Note that if X is an affine scheme, $\underline{\text{Spec}}_X(\mathcal{E}) = \text{Spec}(\Gamma(X, \mathcal{E}))$.

Example 2.1.0.14. Let R be a $\mathbb{C}$-algebra and consider the affine scheme $X = \text{Spec}R$ and the structure sheaf $\mathcal{O}_X$. Then $\underline{\text{Spec}}_X(\mathcal{O}_X)$ is the total space of the trivial line bundle. This is indicative of more general behaviour, given by the following well-known result.

Proposition 2.1.0.15. *Let X be a scheme. There is a (covariant) equivalence between the category of vector bundles on X and the category of locally free sheaves on X. Given a locally free sheaf $\mathcal{E}$, this equivalence is given in one direction by the functor which returns the vector bundle $\tilde{E} := \underline{\text{Spec}}(\text{Sym}\mathcal{E}^\vee)$. Due to this equivalence, we will follow the standard convention of sometimes confusing the terms "vector bundle" and "locally free sheaf".*

We now have a third way of thinking of a G-equivariant sheaf $\mathcal{E}$, in this case under the additional hypothesis of $\mathcal{E}$ being locally free. In this setting, $\mathrm{act}^*\mathcal{E}$ and $\pi_2^*\mathcal{E}$ are also locally free, and isomorphic via the map of sheaves α. Thus, their induced geometric vector bundles are also isomorphic:

$$a : (G \times X) \times_{X,\mathrm{act}} \tilde{E} \xrightarrow{\sim} (G \times X) \times_{X,\pi_2} \tilde{E}.$$

These maps a correspond canonically to maps Σ such that the following square is commutative

$$\begin{array}{ccc} G \times \tilde{E} & \xrightarrow{\Sigma} & \tilde{E} \\ {\scriptstyle \mathrm{id}_G \times \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ G \times X & \xrightarrow[\mathrm{act}]{} & X, \end{array}$$

where Σ is a bundle isomorphism of $G \times \tilde{E}$ over $G \times X$ and $\tilde{E}$ over X. This correspondence is given by taking Σ to be the composition

$$G \times \tilde{E} \xrightarrow{\sim} (G \times X) \times_{X,\mathrm{act}} \tilde{E} \xrightarrow{a} (G \times X) \times_{X,\pi_2} \tilde{E} \xrightarrow{\sim} G \times \tilde{E} \xrightarrow{\text{proj.}} \tilde{E},$$

and this defines an *action* Σ of G on $\tilde{E}$ (Ch.1, §3 [MFK94]). That is, putting a G-equivariant structure on a locally free sheaf is the "same" as extending the action of G on X to an action of G on (the total space of) the vector bundle $\tilde{E}$.

Definition 2.1.0.16 (Definitions 1.7 & 1.8, [MFK94]). *Let $\mathcal{L}$ be a G-equivariant line bundle on X. Write $H^0(X, \mathcal{L}^n)^G$ for the G-invariant global sections of $\mathcal{L}^n$ and X_s for the open set given by the complement of the zero-locus of s. Letting $G \cdot x$ and G_x denote the orbit and stabiliser of x, respectively, define three subsets of X as*

$$\begin{aligned} X^{ss}(\mathcal{L}) &:= \{x \in X \mid \exists s \in H^0(X, \mathcal{L}^n)^G \text{ for some } n \in \mathbb{N}, \text{ with } s(x) \neq 0 \text{ and } X_s \text{ affine}\} \\ X^{s}(\mathcal{L}) &:= \{x \in X^{ss}(\mathcal{L}) \mid G \cdot x \subseteq X^{ss}(\mathcal{L}) \text{ is closed, and } G_x \text{ is finite}\} \\ X^{us}(\mathcal{L}) &:= X \setminus X^{ss}(\mathcal{L}), \end{aligned}$$

called the semistable, stable and unstable loci, respectively. If $X^{ss}(\mathcal{L}) = X^s(\mathcal{L})$, then we say the choice of linearisation is generic.

Remark 2.1.0.17. What is now almost universally called stable in the literature is referred to as *properly* stable in the original reference [MFK94].

We now give a topological criterion for (semi)-stability.

Proposition 2.1.0.18 (Proposition 2.2, [MFK94]). *Let $\mathcal{L}$ be a G-equivariant line bundle on X, i.e. an extension of the action of G on X to an action of G on $\tilde{L}$. Let $x \in X$ and let $\tilde{x} \in \tilde{L}$ be a point lying over x, outside the zero locus of $\tilde{L}$. Then:*

i) x *is semistable if and only if* $\overline{G\cdot\tilde{x}}$ *doesn't intersect the zero-locus of* $\tilde{L}$.

ii) x *is stable if and only if* $G\cdot\tilde{x}$ *is closed and* $G_{\tilde{x}}$ *is finite.*

Definition 2.1.0.19. *A* 1*-parameter subgroup of a reductive group* G *is a non-trivial group homomorphism* $\lambda : \mathbb{G}_m \to G$. *We will abuse notation by confusing the homomorphism* λ *with its image as a subgroup of* G.

Definition 2.1.0.20 (Definition 2.2, [MFK94]). *Let* $\mathcal{L}$ *be a* G*-equivariant line bundle,* $\lambda \subseteq G$ *a* 1*-parameter subgroup and* $x \in X$ *a closed point. If it exists, define* $x_0 := \lim_{t\to 0}(\lambda(t)\cdot x)$ *and note that* x_0 *is fixed under the action of* λ. *The action of* λ *on the fibre of* $\tilde{L}$ *over the fixed point* x_0 *is given by a character of* λ,

$$\begin{aligned} \chi : \lambda &\to \mathbb{G}_m \\ t &\mapsto t^r \end{aligned}$$

for some $r \in \mathbb{Z}$. *Define Mumford's numerical function* $\mu(x, \lambda, \mathcal{L}) := r$.

The key idea to keep in mind here comes from the topological criterion for semistability - if $\tilde{x}_0$ is a non-zero point of the fibre over the fixed point x_0, the closure of the orbit $\lambda\cdot\tilde{x}_0$ will intersect the zero-locus of $\tilde{L}$ if and only if $r > 0$. In fact, *Mumford's numerical criterion* (Proposition 2.1.0.24) essentially states that the topological condition for $x \in X$ to be semistable (Proposition 2.1.0.18) is equivalent to having $r \leq 0$ for all 1-parameter subgroups of G acting on the fibre of $\tilde{L}$ over x_0, where the limit point x_0 exists. We now generalise the function μ slightly, to arbitrary locally free sheaves $\mathcal{E}$.

Definition 2.1.0.21. *Let* V *be a finite-dimensional representation of* $\mathbb{G}_m$ *and decompose it as*

$$V \simeq \bigoplus_{\chi\in\mathbb{Z}} V_\chi$$

where $\chi \in Hom(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}$ *is a character of* $\mathbb{G}_m$*, and* V_χ *is the subspace of* V *where* $\mathbb{G}_m$ *acts by* χ. *We slightly abuse notation by saying that* $v \in V_\chi$ *has weight* $\chi \in \mathbb{Z}$*, and we call* $\{\chi \in \mathbb{Z} \mid V_\chi \neq 0\}$ *the set of weights of* V.

Definition 2.1.0.22. *Let* $\mathcal{E}$ *be a locally-free quasi-coherent* G*-equivariant sheaf on* X*,* λ *a* 1*-parameter subgroup of* G*, and assume that the limit point* $x_0 = \lim_{t\to 0}(\lambda(t)\cdot x) \in X^\lambda$ *exists. The set of* λ*-weights of* $\mathcal{E}$ *at* x*,* $\mu(x, \lambda, \mathcal{E})$*, is defined to be the set of weights of the* $\mathbb{G}_m$*-action on the fibre of* $\tilde{E}$ *over* x_0.

Lemma 2.1.0.23. *With the notation as above, if x, x' lie in the same connected component of X^λ, then $\mu(x, \lambda, \mathcal{E}) = \mu(x', \lambda, \mathcal{E})$.*

Proof. See e.g. [BFK19], Lemma 2.1.19. □

We will therefore adopt the notation $\mu(Z, \lambda, \mathcal{E}) := \mu(x, \lambda, \mathcal{E})$ for x a point of the connected component $Z \subseteq X^\lambda$, without ambiguity.

Proposition 2.1.0.24 (Mumford's numerical criterion, [BFK19])**.** *Let $X = \mathbb{A}^n$ and let $\mathcal{L}$ be a G-equivariant line bundle. A point $x \in X^{us}(\mathcal{L})$ if and only if there exists a 1-parameter subgroup λ such that $x_0 = \lim_{t\to 0}(\lambda(t)\cdot x)$ exists and $\mu(x_0, \lambda, \mathcal{L}) > 0$.*

Definition 2.1.0.25. *Let $\mathbb{G}_m$ act on an affine scheme $X = SpecR$ and let M be an R-module with a co-action $\hat{\sigma}_M : M \to \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} M$. We say $m \in M$ is homogeneous of degree $l \in \mathbb{Z}$ if $\hat{\sigma}_M(m) = t^l \otimes m$. Denote this by $deg_M(m) = l$. In particular, by Definition 2.1.0.6, m is invariant if and only if it is homogeneous of degree 0.*

Lemma 2.1.0.26. *Consider R as an R-module, and endow it with the two module coactions*

$$\begin{aligned} \hat{\sigma}_m : R_m &\to \mathbb{C}[t,t^{-1}] \otimes R_m \\ r &\mapsto t^{m+deg_R(r)} \otimes r \end{aligned} \qquad \begin{aligned} \hat{\sigma}_n : R_n &\to \mathbb{C}[t,t^{-1}] \otimes R_n \\ r &\mapsto t^{n+deg_R(r)} \otimes r. \end{aligned}$$

Then the R-module $Hom_R(R_m, R_n)$ has a natural coaction given by

$$\begin{aligned} Hom_R(R_m, R_n) &\to \mathbb{C}[t,t^{-1}] \otimes Hom_R(R_m, R_n) \\ (1 \mapsto r) &\mapsto t^{n-m+deg_R(r)} \otimes (1 \mapsto r) \end{aligned}$$

such that the homogeneous elements of degree 0 are exactly the morphisms $R_m \to R_n$ compatible with the coactions $\hat{\sigma}_m$ and $\hat{\sigma}_n$.

Proof. Let $f \in \mathrm{Hom}(R_m, R_n)$ given by $1 \mapsto r$. Then f is homogeneous of degree 0 $\Leftrightarrow m = n + \deg_R(r) \Leftrightarrow$ the requisite diagram

$$\begin{array}{ccc} R_m & \longrightarrow & \mathbb{C}[t,t^{-1}] \otimes R_m \\ \downarrow & & \downarrow \\ R_n & \longrightarrow & \mathbb{C}[t,t^{-1}] \otimes R_n \end{array}$$

commutes. □

We now introduce a lemma which will prove useful for computing weights $\mu(x, \lambda, \mathcal{E})$.

Lemma 2.1.0.27. *Let $\mathcal{E}$ be a G-equivariant locally free sheaf of rank n on $X = SpecR$. Then $\mu(x, \lambda, \mathcal{E})$ is equal to the set of the degrees of the homogeneous elements of the induced map*

$$E_{x_0} \to (\mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} E)_{x_0}$$

where $(-)_{x_0}$ denotes the localisation at the maximal ideal corresponding to the limit point x_0.

Proof. Sketch of proof: We have established a covariant equivalence of categories between the category of R-modules E with a co-action of G and the category of vector bundles $\tilde{E}$ on X with an action of G, compatible with the action of G on the base space X. Restricting to the fibre of $\tilde{E}$ over x_0, Σ induces an action $\lambda \times \mathbb{C}^n \to \mathbb{C}^n$, with weights $\chi_i \in \mathbb{Z}$. Under this equivalence (for the space $x_0 = \text{Spec}(\mathbb{C})$), this action map corresponds to the co-action map $\mathbb{C}^n \to \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^n$ with homogeneous elements of degree χ_i. □

We are now able to finally define what we mean by a GIT quotient. We will also consider taking GIT quotients as stacks, so we use the additional qualifier *classical* for the following concept.

Definition 2.1.0.28. *Let $\mathcal{L}$ be a G-equivariant line bundle on X with corresponding semistable locus X^{ss}. The classical GIT quotient is the set of points of X^{ss} modulo the following equivalence relation on the closures of their orbits in X^{ss}:*

$$x \sim y \Leftrightarrow \overline{G \cdot x} \cap \overline{G \cdot y} \neq \emptyset.$$

Definition 2.1.0.29. *As above, let $\mathcal{L}$ be a G-equivariant line bundle on X with corresponding semistable locus X^{ss}. By a GIT quotient stack we mean the quotient stack $[X^{ss}/G]$. As discussed in Section 1.5, the points of this are the G-orbits where we also remember the stabilisers of points in the orbit.*

Remark 2.1.0.30. From these definitions it's clear that the stacky GIT quotient is a refinement of the data of the corresponding classical GIT quotient. As we will be dealing with stabiliser subgroups which are not finite, we follow [Alp13] and call these classical GIT quotients the *good moduli space* of the corresponding stacky GIT quotient. This is not just a matter of labels - for the rigorous definition of what a good moduli space is, see [Alp13, Definition 4.1]; the fact that classical GIT quotients are good moduli spaces for the corresponding quotient stack is Theorem 13.6 *ibid*. The classical and stacky notions of GIT coincide precisely when the action of the group is free.

Example 2.1.0.31. Let us illuminate Proposition 2.1.0.24 with the standard example of $G := \mathbb{G}_m$ acting on $X := \mathbb{C}^{n+1}$ via

$$\begin{aligned} G \times X &\to X \\ (g, (x_0, x_1, \ldots, x_n)) &\mapsto (gx_0, gx_1, \ldots, gx_n). \end{aligned}$$

This gives us a solution to the VGIT problem by describing every possible GIT quotient with respect to this group action of $\mathbb{G}_m$ on $\mathbb{C}^{n+1}$. Setting $R := \mathbb{C}[f_0, \ldots f_n]$ and $S := \mathbb{C}[t, t^{-1}]$, this action corresponds to the coaction

$$\begin{aligned} \hat{\sigma} : R &\to S \otimes_{\mathbb{C}} R \\ f_i &\mapsto t \otimes f_i. \end{aligned}$$

By Lemma 2.1.0.12, a choice of linearisation of the structure sheaf $\mathcal{O}_X$ corresponds to a module coaction

$$\begin{aligned} \hat{\sigma}_M : M &\to S \otimes_{\mathbb{C}} M \\ f_i &\mapsto t^{m+1} \otimes f_i. \end{aligned}$$

for some $m \in \mathbb{Z}$, where we define $M := \Gamma(X, \mathcal{O}_X) = R$ to emphasise when we are viewing R as a graded R-module. Thinking of R as a graded ring, the different module coactions here are nothing but M as a graded R-module, where the grading of M is the grading of R shifted by m. Given a 1-parameter subgroup $\lambda : t \mapsto t^p$, we examine when limit points exist, in order to use Mumford's numerical criterion.

$$\begin{array}{ccccc} \mathbb{G}_m \times X & \xrightarrow{\lambda \times \mathrm{id}_X} & \mathbb{G}_m \times X & \xrightarrow{\text{act}} & X \\ & & & & \\ (g, (x_0, \ldots, x_n)) & \longmapsto & (g^p, (x_0, \ldots, x_n)) & \longmapsto & (g^p x_0, \ldots, g^p x_n) \end{array}$$

If $p > 0$, the limit point will exist for all points $x \in X$, and in fact is always the point $(0, \ldots, 0)$. If $p < 0$, then no limit point exists if at least one of the coordinates is non-zero, so $(0, \ldots, 0)$ is the only point for which a limit point exists, and the limit point is trivially itself. Choosing the linearisation of $\mathcal{O}_X$ induces the composition for stalks

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & S \otimes \mathbb{C} & \longrightarrow & S \otimes \mathbb{C} \\ & & & & \\ 1 & \longmapsto & t^m \otimes 1 & \longmapsto & t^{pm} \otimes 1 \end{array}$$

Consider the following cases:

1) Let $m < 0$.

   i) Pick the point $(x_0, \ldots, x_n)$ such that there exists $x_i \neq 0$. The limit point exists for $p > 0$, and thus 1 is homogeneous of degree $pm < 0$.

   ii) Pick $(0, \ldots, 0)$. The limit point exists for all p, thus pm can be either sign, so $(0, \ldots, 0)$ is unstable.

   iii) The semistable locus is the set of all non-zero points of $\mathbb{C}^{n+1}$, and points lying on the same line through the origin are GIT equivalent, so the quotient is $\mathbb{P}^n$.

2) Let $m = 0$. Pick $(x_0, \ldots, x_n)$ such that there exists $x_i \neq 0$. The limit point again exists for $p > 0$, and so 1 is homogeneous of degree 0. For $(0, \ldots, 0)$, the limit exists for all p, but 1 is again homogeneous of degree 1. Thus all points of $\mathbb{C}^{n+1}$ are semistable for $m = 0$. As the closure of all the orbits intersect at the origin, they are all GIT equivalent, and the quotient is a single point.

3) Finally, let $m > 0$.

   i) Let $(x_0, \ldots, x_n)$ be such that there exists $x_i \neq 0$. The limit exists for $p > 0$, so 1 is homogeneous of degree $pm > 0$.

   ii) Pick $(0, \ldots, 0)$. The limit point exists for all p, thus pm can be either sign, so $(0, \ldots, 0)$ is unstable.

   iii) As all points are unstable, the GIT quotient is $\emptyset$.

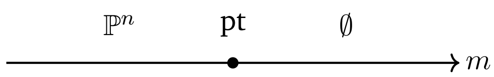


Figure 2.1: The wall-and-chamber decomposition for the VGIT problem given by describing all possible GIT quotients of $\mathbb{C}^n$ by $\mathbb{G}_m$, where the group action is given by multiplication in each coordinate.

2.2 Unstable loci and their KN stratifications

As alluded to in the introduction to this chapter, we wish to use the technology of Halpern-Leistner [Hal15] and Ballard, Favero and Katzarkov [BFK19] to construct equivalences between the derived categories of (stacky) GIT quotients. The referenced papers give certain conditions for equivalences to exist, which is phrased in terms of a careful analysis of the unstable loci for the different quotients. In particular, it involves a certain stratification of the unstable locus, known as a KN stratification. This should be viewed as an algorithmic way of decomposing the unstable locus into disjoint pieces, starting with the "most unstable" points, and is described in Algorithm 2.2.0.4. We first define what we mean by KN strata.

Definition 2.2.0.1. *Let X be a quasi-projective variety with a linearisable action of an algebraic torus $G \simeq \mathbb{G}_m^n$. Given a one-parameter subgroup $\lambda' : \mathbb{G}_m \to G$ and a connected component $Z' \subseteq X^{\lambda}$, define*

$$Y_{\lambda',Z'} := \{x \in X \mid \lim_{t \to 0} \lambda'(t) \cdot x \in Z'\} \subseteq X$$

and the canonical projection map

$$\begin{aligned} \pi : Y_{\lambda',Z'} &\to Z' \\ y &\mapsto \lim_{t \to 0} \lambda'(t) \cdot y. \end{aligned}$$

A closed KN stratum is a closed subvariety $S \subseteq X$ such that there is a one-parameter subgroup $\lambda : \mathbb{G}_m \to G$ and an open and closed subvariety $Z \subseteq X^{\lambda}$ satisfying the following properties:

i) *$g \cdot z \in Z$ and $g \cdot y \in Y_{\lambda,Z} = S$ for all $g \in G$, $z \in Z$ and $y \in Y_{\lambda,Z}$.*

ii) *The projection $\pi : Y_{\lambda,Z} \to Z$ is algebraic and affine.*

iii) *The conormal sheaf $\mathcal{I}_S/\mathcal{I}_S^2$, restricted to Z, has non-positive weights with respect to λ.*

iv) *If X is not smooth in a neighbourhood of Z, there exists a G-equivariant closed immersion $X \subset X'$ and a KN stratum $S' \subseteq X'$ such that S is a union of connected components of $S' \cap X$ and X' is smooth in a neighbourhood of Z'.*

Remark 2.2.0.2. We will refer to $Y_{\lambda,Z}$ as the *blade* over Z.

Definition 2.2.0.3. *Let $Y \subseteq X$ be a closed equivariant subvariety. A set of locally closed subvarieties $\{S_i \subseteq Y \mid i = 1, \dots, n\}$ is called a KN stratification of Y if*

i) $Y = \bigcup_{i=1}^{n} S_i$

ii) *$S_i \subseteq X \setminus \left(\bigcup_{j<i} S_j\right)$ is a closed KN stratum for all i.*

In [Hal15], Halpern-Leistner gives an iterative procedure for constructing a KN stratification of the unstable locus of a fixed choice of linearisation of a G-equivariant line bundle $\mathcal{L}$, where G is a reductive group. We now recall the version for G an algebraic torus. First, denote the cocharacter lattice of G by $\Lambda := \mathrm{Hom}(\mathbb{G}_m, G) \simeq \mathbb{Z}^n$. Fixing an inner product on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ allows us to define a norm $\|\cdot\|$ such that $\|\lambda\| > 0$ for any 1-parameter subgroup λ. The iterative procedure for constructing the KN strata S_i is as follows:

Algorithm 2.2.0.4 (§2.1, [Hal15]). *1) Choose a pair (λ^*, Z^*), for $Z^* \subseteq X^{\lambda^*}$ a connected component, which maximises the scaled version of Mumford's numerical invariant*

$$\mu^*(\lambda^*, Z^*) := \frac{1}{\|\lambda^*\|}\mu(\mathcal{L}, \lambda^*, Z^*).$$

If this is the second (or higher) iteration of this algorithm, impose the additional condition that $Z^* \nsubseteq \bigcup_i S_i$ for the strata S_i already defined.

2) If this is the first iteration, define $Z_1 := Z^*$. Otherwise, for the i^{th} iteration, define $Z_i := Z^* \setminus \bigcup_{j<i} S_j$.

3) Define the blade $Y^* := \{x \in X \mid \lim_{t\to 0} \lambda^*(t) \cdot x \in Z^*\}$ with projection map $\pi : Y^* \to Z^*$. Define the KN stratum $S_i := \pi^{-1}(Z_i)$.

4) Iterate the previous steps until there are no remaining suitable pairs (λ^*, Z^*) with $\mu^*(\lambda^*, Z^*) > 0$.

Remark 2.2.0.5. Note that, although the unstable locus is constant for any linearisation of $\mathcal{L}$ in a given cell of the GIT wall-and-chamber space, the KN stratification of it produced by this algorithm is not. In particular, for a cell with two or more different walls, it is likely that a different set of unstable points will become semistable as we move onto the walls. As we construct the strata by decreasing degree of instability, for linearisations near to the walls these points will be contained in the last KN stratum to be defined. We will see this effect in the KN stratifications we produce in Chapter 3.

Example 2.2.0.6. We now return to Example 2.1.0.31 to demonstrate a simple instance of this algorithm. Pick the standard Euclidean norm on $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}$.

i) For $m < 0$, choose the 1-parameter subgroup $g \mapsto g^p$ for $p < 0$. Then the fixed locus is the point at the origin and $\mu(\lambda^*, (0, \dots, 0)) = \frac{1}{-p}pm = -m > 0$. The blade over the origin is just the origin itself, and this is the whole unstable locus, so the algorithm terminates.

ii) For $m = 0$, the unstable locus is empty.

iii) For $m > 0$, choose any 1-parameter subgroup $g \mapsto g^p$ for $p > 0$. The fixed locus is again just the origin, and $\mu(\lambda^*, (0, \ldots, 0)) = \frac{1}{p}pm = m > 0$. This time, the blade over the origin is the whole unstable locus (i.e. the whole of $\mathbb{C}^{n+1}$), so the unstable locus is stratified by one stratum and the algorithm terminates.

2.3 Stacky GIT quotients and wall-crossing equivalences

Recall that a G-linearised ample line bundle $\mathcal{L}$ defines an open semistable locus $X^{ss}(\mathcal{L}) \subseteq X$. By "GIT quotient" we now refer to the *stack* given by the quotient of the semistable locus by the group action, $[X^{ss}(\mathcal{L})/G]$, rather than the varieties constructed in section 2.1. If confusion seems likely to occur, we will use the qualifier *classical* GIT quotient to refer to the quotient varieties.

This sits inside the quotient stack of the whole space by the group action, $[X/G]$. As we have discussed, for two different choices of linearisation $\mathcal{L}_\pm$ sitting on either side of a wall, a pertinent question is when we have derived equivalences

$$D^b([X^{ss}(\mathcal{L}_-)/G]) \simeq D^b([X^{ss}(\mathcal{L}_+)/G]).$$

If these equivalences exist, we require some way of constructing them. A possible general method is to identify some subcategory $G_w \subseteq D^b([X/G])$ such that the restriction functors $\mathrm{res}_\pm : G_w \to D^b([X^{ss}(\mathcal{L}_\pm)/G])$ simultaneously give an equivalence:

$$\begin{array}{ccc} & D^b([X/G]) & \\ & \cup\!\mathsf{I} & \\ & G_w & \\ \swarrow^{\sim} & & \searrow^{\sim} \\ D^b([X^{ss}(\mathcal{L}_-)/G]) & & D^b([X^{ss}(\mathcal{L}_+)/G]) \end{array}$$

Figure 2.2: A general recipe for constructing wall-crossing equivalences via window subcategories G_w.

A derived equivalence between the derived categories of the two GIT quotients is then given simply by $\mathrm{res}_+ \circ \mathrm{res}_-^{-1}$. We make two brief remarks about this general technique.

Remark 2.3.0.1. Note that there is no requirement here for the subcategory $G_w \subseteq D^b([X/G])$ to be *geometric* in origin, i.e. there is no requirement for it to be the derived category of any stack.

Remark 2.3.0.2. Note also that the key word in the statement of this technique is *simultaneously*. Indeed, as we shall see next, it is possible in high generality to construct subcategories of $D^b([X/G])$ which restrict to *one* of $D^b([X^{ss}(\mathcal{L}_\pm)/G])$ as an equivalence, but restricting to *both* as an equivalence at the same time is a far more constraining condition.

Let us now fix some notation. Consider a linearisation of a line bundle $\mathcal{L}$, let Z_i, S_i and X be as in the above algorithm and denote the inclusions

$$Z_i \xhookrightarrow{\sigma_i} S_i \xhookrightarrow{j_i} X \xhookleftarrow{\iota} X^{ss}(\mathcal{L}).$$

In addition, fix a choice of integer w_i for each Z_i and set $w := (w_1, \ldots, w_n)$. The main result of [Hal15] is in two parts. The first states that

$$G_w := \{F \in D^b([X/G]) \mid \text{for all } i,\ \mu(Z_i, \lambda_i, \mathcal{H}^*(\sigma_i^* j_i^* F)) \leq w_i < \mu(Z_i, \lambda_i, \mathcal{H}^*(\sigma_i^* j_i^! F))\} \tag{2.5}$$

gives a suitable subcategory of $D^b([X/G])$ such that the restriction functor

$$\iota^* : G_w \longrightarrow D^b([X^{ss}(\mathcal{L})/G])$$

is an equivalence. Here $\mathcal{H}^*(-) := \oplus \mathcal{H}^i(-)$ and $j_i^!$ denotes the twisted pullback functor $j_i^! F := \mathcal{H}om(\mathcal{O}_{S_i}, F|_U)$, regarded as an $\mathcal{O}_{S_i}$-module, where $[U/G] \subseteq [X/G]$ is an open substack containing $[S_i/G]$ as a closed substack. If X is smooth in a neighbourhood of Z_i then $j_i^! F \simeq j_i^! \mathcal{O}_X \otimes j_i^* F$. Note that, due to the sign conventions we have adopted, the inequalities in (2.5) are the opposite to those found in [Hal15]. Defining $D^b_{X^{us}}([X/G])$ to be the subcategory of $D^b([X/G])$ consisting of those complexes whose cohomology sheaves are supported on the unstable locus, the second part of the statement identifies this G_w as the middle component in the semi-orthogonal decomposition

$$D^b([X/G]) = \langle D^b_{X^{us}}([X/G])_{>w}, G_w, D^b_{X^{us}}([X/G]_{\geq w}) \rangle$$

where

$$D^b_{X^{us}}([X/G])_{>w} := \{F \in D^b_{X^{us}}([X/G]) \mid \text{ for all } i,\ w_i < \mu(Z_i, \lambda_i, \sigma_i^* j_i^! F)$$
$$D^b_{X^{us}}([X/G]_{\leq w} := \{F \in D^b_{X^{us}}([X/G]) \mid \text{ for all } i,\ \mu(Z_i, \lambda_i, \sigma_i^* j_i^* F) \leq w_i\}.$$

We are interested in the Fourier-Mukai kernels which describe this equivalence. For a semistable locus $X^{ss} \subseteq X$, the kernel of the restriction functor $\iota^* : D^b([X/G]) \to D^b([X^{ss}/G])$ is simply the (equivariant version of the) graph subvariety of ι,

$$\begin{aligned} G \times X^{ss} &\to X \times X^{ss} \\ (g, x) &\mapsto (g \cdot x, x), \end{aligned}$$

restricted to $G_w \subseteq D^b([X/G])$. The kernels of the inverse functors for different w are more subtle. The construction of these works by extending the equivariant diagonal inside $X^{ss} \times X^{ss}$ to $X^{ss} \times X$. These extensions are certainly not unique, and different extensions correspond to different choices of w. In Section 2.3.1 we explain how to construct a particular extension due to Ballard, Diemer and Favero [BDF17], and in Section 2.3.2 we recall an algorithm due to [Hal15] which gives a general recipe to modify this extension to give the correct extension for any given choice of w.

Before finishing this section, we state a result about the nicest type of wall-crossings, which are known as *balanced* wall-crossings. Let θ_0 be a stability condition on a wall, so that $\theta_\pm := \theta_0 \pm \varepsilon\theta_1$ for some small $\epsilon > 0$ are stability conditions on either side of the wall. Letting $S_i^\pm \subseteq X^{us}(\theta_\pm)$ denote the KN strata which become semi-stable on the wall, we can write

$$X^{ss}(\theta_0) = X^{ss}_{\theta_+} \cup \bigcup_i^{m_+} S_i^+$$

and symmetrically

$$X^{ss}(\theta_0) = X^{ss}_{\theta_-} \cup \bigcup_i^{m_-} S_i^-$$

From the construction of KN strata, these $S_i^\pm$ are blades over a connected component of the fixed locus for some one-parameter subgroup. We denote these connected components $Z_i^\pm$. The following definition and proposition are the torus versions of results stated in [Hal15].

Definition 2.3.0.3. *The wall-crossing is* balanced *if* $m_+ = m_-$ *and* $[Z_i^+/G] = [Z_i^-/G]$.

When we are in the situation of a balanced wall-crossing, the additional conditions we impose in the following proposition ensure that the window subcategories $G_w^\pm$ coincide. This gives a wall-crossing equivalence via the general recipe shown in Figure 2.2.

Proposition 2.3.0.4 (Proposition 4.5, [Hal15]). *If the following conditions hold, then the wall-crossing is an equivalence.*

i) The wall crossing is balanced.

ii) The stability conditions θ_+ and θ_- are generic.

iii) For all $Z_i^{\pm} \subseteq X^{us}(\theta^{\pm})$ which lie in $X^{ss}(\theta_0)$ with corresponding 1-parameter subgroup $\lambda^{\pm}$, the weight of the canonical bundle of X restricted to $Z_i^{\pm}$ is zero.

In particular, if X has $\omega_X \simeq \mathcal{O}_X$, then all generic chambers are derived equivalent [Hal15, Corollary 4.8].

2.3.1 Kernels from compactifications

In this section we state an important construction due to Ballard, Diemer and Favero [BDF17], whereby we extend the action of G on X to a *partial compactification* of the action, and use this partial compactification to produce a Fourier-Mukai kernel. These kernels give maps $D^b_{\mathrm{QCoh}}([X^{ss}/G]) \to D^b_{\mathrm{QCoh}}([X/G])$ and restrict to $X^{ss} \times X^{ss}$ as the equivariant diagonal.

Definition 2.3.1.1 ([BDF17]). *Let G act on X, and let $\tilde{X}$ be a variety equipped with a $G \times G$ action. In addition let $i : G \times X \to \tilde{X}$ be a $(G \times G)$-equivariant open immersion, where the action of $G \times G$ on $G \times X$ is given by* (1.4)*. If there exist maps $\tilde{\pi}, \tilde{\sigma}$ such that the inner and outer triangles in the following diagram commute*

$$\begin{array}{ccc} & & \tilde{X} \\ & \overset{i}{\nearrow} & \tilde{\sigma}\downarrow\downarrow\tilde{\pi} \\ G \times X & \underset{\pi}{\overset{\sigma}{\rightrightarrows}} & X \end{array} \tag{2.6}$$

and $(\tilde{\pi}, \tilde{\sigma}) : \tilde{X} \to X \times X$ is $(G \times G)$-equivariant, where the $(G \times G)$-action on $X \times X$ is the obvious one induced by the action of G on X, then we say that the quadruple $(\tilde{X}, i, \tilde{\pi}, \tilde{\sigma})$ is a partial compactification of the action of G on X. If the context is clear, we will simply refer to $\tilde{X}$ as a partial compactification.

We construct the Fourier-Mukai kernels defining maps from $D^b_{\mathrm{QCoh}}([X^{ss}/G])$ to $D^b_{\mathrm{QCoh}}([X/G])$ via the following general recipe.

Definition 2.3.1.2. *Given a partial compactification diagram* (2.6)*, define*

$$Q_{X,G} := (\tilde{\pi}, \tilde{\sigma})_* \mathcal{O}_{\tilde{X}} \in D^b_{QCoh}([X \times X/G \times G])$$

and restrict it to the semistable locus in the first component to obtain an object

$$Q^{ss}_{X,G} := (j, id)^* Q_{X,G} \in D^b_{QCoh}([X^{ss} \times X/G \times G]).$$

for $(j, id) : X^{ss} \times X \hookrightarrow X \times X$. *Thus* $Q^{ss}_{X,G}$ *defines a Fourier-Mukai transform*

$$\Phi_{Q^{ss}_{X,G}} : D^b_{QCoh}([X^{ss}/G]) \to D^b_{QCoh}([X/G]). \tag{2.7}$$

For $X = \mathrm{Spec} R$ and $G = \mathbb{G}_m$ we have a particularly explicit way of producing examples of such partial compactifications.

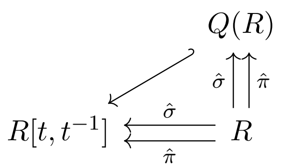
Definition 2.3.1.3. *Let* $X = \mathit{Spec} R$ *and* $G = \mathbb{G}_m$. *The action groupoid provides two maps* $\hat{\pi}, \hat{\sigma} : R \to R[t, t^{-1}]$. *Define* $Q(R) = \langle \hat{\pi}(R), \hat{\sigma}(R), t \rangle \subseteq R[t, t^{-1}]$ *to be the subalgebra generated by* t *and the images of* $\hat{\pi}$ *and* $\hat{\sigma}$.

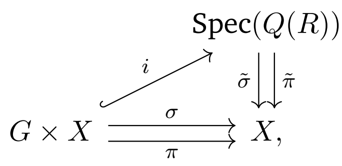
This generalises to the case of an algebraic torus acting on an affine scheme as follows.

Definition 2.3.1.4. *Let* $X = \mathit{Spec} R$ *and* $G = \mathbb{G}_m^n$, *with corresponding character lattice* $\chi(G) = \mathit{Hom}(G, \mathbb{G}_m)$. *Let* $C \subseteq \chi(G)$ *be a finitely generated submonoid. The action groupoid gives* $\hat{\sigma} : R \to R[t_1^{\pm}, \dots, t_n^{\pm}] \simeq R[\chi(G)]$. *We define* $Q(R)$ *to be the subalgebra generated by the monoid ring and the image of* R *under* $\hat{\sigma}$,

$$Q(R) := \langle R[C], \hat{\sigma}(R) \rangle \subseteq R[\chi(G)].$$

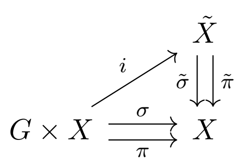
Thus we have the following two commutative diagrams, which are clearly equivalent to each other





and the right diagram is a partial compactification (Proposition 3.1.2, [BDF17]).

Definition 2.3.1.5. *Given a partial compactification diagram*



there is a natural notion of a boundary $\partial := \tilde{X} \setminus i(G \times X)$. *We define the* $\tilde{\sigma}$*-unstable locus to be* $X^{us}_{\tilde{\sigma}} := \tilde{\sigma}(\partial)$, *and the* $\tilde{\sigma}$*-semistable locus to be the complement* $X^{ss}_{\tilde{\sigma}} := X \setminus X^{us}_{\tilde{\sigma}}$.

This notion of stability is closely linked to that of stability in the sense of GIT for the action of an algebraic torus on an affine variety, as the following proposition shows. We use the same notation as in Definition 2.3.1.4.

Proposition 2.3.1.6 (Proposition 3.1.8, [BDF17]). *Assume Pic$(X) \otimes \mathbb{Q} = 0$. Choose the monoid C of integral points given by the closure of a GIT chamber, and construct the corresponding partial compactification. Choosing any linearisation $\mathcal{L}$ lying in the relative interior of this chamber, $X^{ss}(\mathcal{L}) = X^{ss}_{\tilde{\sigma}}$.*

Example 2.3.1.7. For our running $\mathbb{P}^n$ example, $A = \mathbb{C}[f_0, \ldots, f_n]$ and $Q(A) = \mathbb{C}[t, f_0, \ldots, f_n] \subset \mathbb{C}[t, t^{-1}, f_0, \ldots, f_n]$. Our partial compactification is therefore $\tilde{X} := \mathbb{A}^{n+2}$ and $i : G \times X \to \tilde{X}$ the naive inclusion[2], i.e. we partially compactify G at the point at the origin. Thus $\partial = \{0\} \times \mathbb{A}^{n+1}$ and the image of this under $\tilde{\sigma}$ is the point at the origin, i.e. the unstable locus as predicted by Proposition 2.3.1.6. As the following square is fibre

$$\begin{array}{ccccc} G \times X^{ss} & \hookrightarrow & G \times X & \xrightarrow{i} & \tilde{X} \\ \downarrow{\scriptstyle (\pi,\sigma)} & & & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\ X^{ss} \times X^{ss} & \hookrightarrow & X^{ss} \times X & \xrightarrow[(j,\mathrm{id})]{} & X \times X \end{array}$$

the restriction of $Q^{ss}_{X,G}$ to $X^{ss} \times X^{ss}$ is indeed the equivariant diagonal. We now show that the EFMT (2.7) for this kernel restricts to a functor between the equivariant bounded derived categories of coherent sheaves. As a scheme, $D^b(X^{ss})$ is generated by $\mathcal{O}_{X^{ss}}$. The derived category $D^b([X^{ss}/G])$ is therefore generated by copies of the structure sheaf with all possible equivariant structures, and in fact it suffices to take the ones whose equivariant structures lie in the weight window [BDF17, Lemma 4.1.4]. To show that (2.7) restricts to a functor between coherents, it is sufficient to show that these generating objects are all mapped to coherents. By the projection formula, $\Phi_{Q^{ss}_{X,G}}(\mathcal{O}_{X^{ss}}) \simeq (\tilde{\sigma}_* \mathcal{O}_{\mathbb{A} \times X^{ss}})_{(0,*)}$.

$$\begin{array}{ccccc} & \mathbb{A} \times X^{ss} & \hookrightarrow & \tilde{X} = \mathbb{A} \times X & \\ & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} & \\ & X^{ss} \times X & \hookrightarrow & X \times X & \\ \swarrow & & \searrow & & \\ X^{ss} & & & & X \end{array}$$

[2]In our example we are lucky and i can be taken to be the naive inclusion, but this need not always be the case. In particular, if G acted on the coordinates of X with the opposite sign, we would take a different i corresponding to the partial compactification of G at the point at infinity. See [BDF17] for further details.

- For $n = 1$, X and X^{ss} are both affine, so we can argue directly with modules. Note that the coordinate ring of $X^{ss} \times X$, denoted $\mathbb{C}[g^{\pm}, h]$, naturally has the $\mathbb{Z}^2$-grading with g in degree $(1,0)$ and h in degree $(0,1)$. Recall that equivariant coherent sheaves on $X^{ss} \times X$ are graded modules compatible with the ring grading. Our kernel $Q^{ss}_{X,G}$ has global sections $\mathbb{C}[t, f_0] \otimes_{\mathbb{C}[g,h],(\tilde{\pi},\tilde{\sigma})} \mathbb{C}[g^{\pm}, h]$ as a $\mathbb{C}[g^{\pm}, h]$-module, with $\mathbb{Z}^2$-grading as follows:

Element	$\mathbb{Z}^2$-grading
$1 \otimes g = f_0 \otimes 1$	$(1,0)$
$1 \otimes h = tf_0 \otimes 1$	$(0,1)$
$t \otimes 1$	$(-1,1)$

  Pushing the kernel forward to X^{ss} therefore corresponds to viewing the kernel as a $\mathbb{C}[h]$-module via $\mathbb{C}[h] \hookrightarrow \mathbb{C}[g^{\pm}, h]$ and taking the $(0,*)$ graded part. This is isomorphic to $\mathbb{C}[h]$, so $\Phi_{Q^{ss}_{X,G}}(\mathcal{O}_{X^{ss}}) \simeq \mathcal{O}_X$, which is coherent.

- For $n \geq 2$, X^{ss} is only quasi-affine. As $\tilde{\sigma}$ is affine, higher direct images vanish. The global sections of $\mathcal{O}_{\mathbb{A} \times X^{ss}}$ are $\mathbb{C}[t, f_0, \dots, f_n]$; therefore the global sections of $\Phi_{Q^{ss}_{X,G}}$ are the degree $(0,*)$ part of $\mathbb{C}[t, f_0, \dots, f_n]$, viewed as a $\mathbb{C}[g_0, \dots, g_n]$-module via

$$\begin{aligned} \mathbb{C}[g_0, \dots, g_n] &\to \mathbb{C}[t, f_0, \dots, f_n] \\ g_i &\mapsto tf_i. \end{aligned}$$

  Here t has degree $(-1,1)$ and f_i have degree $(1,0)$, so the degree $(0,*)$ part is $\mathbb{C}[tf_0, \dots, tf_n] \simeq \mathbb{C}[g_0, \dots, g_n]$. Therefore $\Phi_{Q^{ss}_{X,G}}(\mathcal{O}_{X^{ss}}) \simeq \mathcal{O}_X$, which is coherent.

Results of [BDF17] show that $\Phi_{Q^{ss}_{X,G}}$ is both full and faithful, and thus the essential image of the restricted functor

$$\Phi_{Q^{ss}_{X,G}} : D^b([X^{ss}/G]) \to D^b_{\text{QCoh}}([X/G])$$

is generated by copies of the structure sheaf with given equivariant structure. In particular, $\Phi_{Q^{ss}_{X,G}}$ restricts to a functor between the bounded derived categories of coherent sheaves.

Next, we recall an algorithm due to Halpern-Leistner which allows us to modify objects in such a way that it shifts them into different weight windows G_w. This can also be applied to the kernels themselves; this gives a way to construct the embedding functors corresponding to different windows.

2.3.2 The shifting algorithm

Assume for the moment that the unstable locus is comprised of only one stratum (necessarily closed), and consequently drop the subscripts i for brevity. Let $H \in D^b([X^{ss}(\mathcal{L})/G])$ and choose some $F \in D^b([X/G])$ such that $\iota^* F \simeq H$. Starting with this initial candidate F, our aim is to modify it so that it lies in $G_w \subseteq D^b([X/G])$. We make use of the following facts, which are Lemma 3.36 of [Hal15]:

i) $\mu(Z, \lambda, \sigma^* j^* F) \leq b$ for some minimal b

ii) $\mu(Z, \lambda, \sigma^* j^! F) > a$ for some maximal a

iii) $a \leq b$.

Note that $F \in G_w$ if and only if $a = b = w$. Step 1 of the following algorithm provides a method for decreasing b (resp. Step 2 increases a) by constructing new objects F' until we produce an object for which $a = b = w$. The steps are as follows:

Algorithm 2.3.2.1 ([Hal15]). *1) If $b \leq w$ there is no problem with the weights of the regular pullback, so go directly to Step 2. Thus, assume $b > w$ and let $E \in D^b([Z/G])$ be the non-trivial subcomplex of $\sigma^* j^* F$ with weight b. There is a morphism $j^* F \to \pi^* E$ and therefore, by adjunction, a morphism $F \to j_* \pi^* E$ which induces $\sigma^* j^* F \to \sigma^* j^* j_* \pi^* E$, which is an isomorphism[3] between the components in degree b. Define a new object F' by the exact triangle*

$$F' \longrightarrow F \longrightarrow j_* \pi^* E \longrightarrow F'[1].$$

This new object $F' \in D^b([X/G])$ satisfies $\iota^* F' \simeq H$ and $\mu(Z, \lambda, \sigma^* j^* F') \leq b - 1$. In addition, $\mu(Z, \lambda, \sigma^* j^! j_* \pi^* E) \geq b \geq a$, and so we have two possibilities:

i) $b > a$, in which case $\mu(Z, \lambda, \sigma^* j^! F') > a$

ii) $b = a$, in which case $\mu(Z, \lambda, \sigma^* j^! F') > w$.

By abuse of notation, iterating this step yields F such that $\iota^* F \simeq H$, $\mu(Z, \lambda, \sigma^* j^* F) \leq w$ and $\mu(Z, \lambda, \sigma^* j^! F) > \min\{a, w\}$.

2) If $a \geq w$ then we are done, so assume $a < w$. Let $E \in D^b([Z/G])$ be the non-trivial subcomplex of $\sigma^* j^! F$ with weight $a + 1$. By complete analogy with Step 1, there is a morphism $j_* \pi^* E \to F$, and so we define F' by the exact triangle

[3] Notice that this algorithm works for *any* morphism $F \to j_* \pi^* E$ which induces such an isomorphism.

$$j_*\pi^*E \longrightarrow F \longrightarrow F' \longrightarrow j_*\pi^*E[1].$$

This satisfies $\iota^*F' \simeq H$ and $\mu(Z,\lambda,\sigma^*j^*F') \leq w$, but now $\mu(Z,\lambda,\sigma^*j^!F') > a+1$. Iterate this step until we have $F \in D^b([X/G])$ such that $\mu(Z,\lambda,\sigma^*j^!F') > w$. If the unstable locus consists of only one stratum, then $F \in G_w$.

3) *If the unstable locus consists of multiple strata S_i for $i \in \{1,2,\dots k\}$, we must repeat Steps 1 & 2 for each of these strata in turn. We start with S_k, which is closed in $X \setminus \left(\bigcup_{j<k} S_j\right)$. The algorithm produces some element of $D^b([X \setminus \left(\bigcup_{j<k} S_j\right)/G])$ lying in the required weight window with respect to Z_k, and iterating through each of the strata in turn gives us our required element of $G_w \subseteq D^b([X/G])$.*

Let's return again to the situation considered in Examples 2.1.0.31 & 2.2.0.6 and run this algorithm with the aim of first computing which object of G_w restricts to $D^b([X^{ss}(\mathcal{L}_{m<0})/G])$ as $\mathcal{O}_{X^{ss}(\mathcal{L}_{m<0})}$. As there is only one unstable stratum, we resume our practice of dropping the subscripts i. Note also that $Z = S$, so both σ and π are the identity map and we omit mention of them. We choose our initial lift of $\mathcal{O}_{X^{ss}(\mathcal{L}_{m<0})}$ to $D^b([X/G])$ to be $\mathcal{O}_X$.

The weights of the structure sheaf $F = \mathcal{O}_X$

Consider first $F = \mathcal{O}_X$. Then $j^*F = \mathcal{O}_Z$, and the corresponding coaction for λ is

$$\begin{array}{ccccc} \mathbb{C} & \longrightarrow & \mathbb{C}[t,t^{-1}] \otimes \mathbb{C} & \longrightarrow & \mathbb{C}[t,t^{-1}] \otimes \mathbb{C} \\ 1 & \longmapsto & t^0 \otimes 1 & \longmapsto & t^{p0} \otimes 1 \end{array}$$

and thus $\mu(Z,\lambda,j^*F) = 0$. Choose $U := X$ and recall that we defined $A := \mathbb{C}[f_0,\dots,f_n]$ with the natural grading, with A_n denoting A as a graded A-module with grading shifted by n. Note that $\mathcal{O}_S$ is quasi-isomorphic to the equivariant Koszul complex

$$A_{n+1} \xrightarrow{(f_n,\dots,f_0)} \bigoplus_{i=0}^n A_n \longrightarrow \dots \longrightarrow \bigoplus_{i=0}^n A_1 \xrightarrow{\begin{pmatrix} f_0 \\ \vdots \\ f_n \end{pmatrix}} A_0 \tag{2.8}$$

and thus $j^!\mathcal{O}_X$ is quasi-isomorphic to the dual complex

$$A_0 \longrightarrow \bigoplus_{i=0}^n A_{-1} \longrightarrow \dots \longrightarrow \bigoplus_{i=0}^n A_{-n} \longrightarrow A_{-(n+1)},$$

where the grading on the coaction changes according to Lemma 2.1.0.26. The cohomology of this complex is zero in every degree except the terminal one, and so $j^!\mathcal{O}_X \simeq \mathcal{O}_S \otimes \chi_{-(n+1)}[-(n+1)]$. The corresponding coaction for the 1-parameter subgroup λ is

$$\mathbb{C} \longrightarrow \mathbb{C}[t,t^{-1}] \otimes \mathbb{C} \longrightarrow \mathbb{C}[t,t^{-1}] \otimes \mathbb{C}$$

$$1 \longmapsto t^{-(n+1)} \otimes 1 \longmapsto t^{-p(n+1)} \otimes 1$$

and so $\mu(Z,\lambda,j^!F) = -p(n+1) > 0$. Thus, $\mathcal{O}_X \in G_w$ for $0 \leq w < -p(n+1)$.

The case $w < 0$

Assume now that $w < 0$, and thus $\mathcal{O}_X \notin G_w$. We now demonstrate Step 1 of the shifting algorithm. The pullback j^*F is concentrated in a single weight, and so Step 1 of the shifting algorithm tells us to take the distinguished triangle

$$F' \longrightarrow \mathcal{O}_X \longrightarrow j_*\mathcal{O}_S \longrightarrow F'[1]$$

and thus $F' \simeq \mathcal{I}_S$, the ideal sheaf of $S \subset X$, by the standard short exact sequence for closed subvarieties. This has a locally free resolution given by the truncation of the Koszul resolution (2.8):

$$A_{n+1} \xrightarrow{(f_n,\ldots,f_0)} \bigoplus_{i=0}^n A_n \longrightarrow \ldots \longrightarrow \bigoplus_{i=0}^n A_1$$

Upon pulling back to Z, we find $\mu(Z,\lambda,j^*\mathcal{I}_S) = \{(n+1)p, np, \ldots, p\}$. The corresponding weights for the twisted pullback are $\mu(Z,\lambda,j^!\mathcal{I}_S) = \{0,-p,\ldots,-np\}$, and so $\mathcal{I}_S \in G_w$ for $p \leq w < 0$. If $w < p$, we iterate Step 1 of the algorithm.

The case $w \geq -p(n+1)$

Analogously, we now assume that $w \geq -p(n+1)$ and run Step 2 of the shifting algorithm. As $j^!\mathcal{O}_X \simeq \mathcal{O}_S \otimes \chi_{-(n+1)}[-(n+1)]$ is concentrated in a single weight, we choose E to be this and take the distinguished triangle

$$j_*\mathcal{O}_S \otimes \chi_{-(n+1)}[-(n+1)] \longrightarrow \mathcal{O}_X \longrightarrow F' \longrightarrow j_*\mathcal{O}_S \otimes \chi_{-(n+1)}[-n],$$

which yields $F' \simeq \mathcal{I}_S^\vee$ by the derived dual of the standard short exact sequence for closed subvarieties. The weights for this are

$$\mu(Z,\lambda,j^*\mathcal{I}_S^\vee) = \{-p,-2p,\ldots,-(n+1)p\}$$
$$\mu(Z,\lambda,j^!\mathcal{I}_S^\vee) = \{-(n+2)p,-(n+3)p,\ldots,-(2n+2)p\}.$$

and so $\mathcal{I}_S^\vee \in G_w$ for $-(n+1)p \leq w < -(n+2)p$. If $w \geq -(n+2)p$, we iterate Step 2 of the algorithm.

Fourier-Mukai kernels

Perhaps the most powerful use of the aforementioned shifting algorithm is in constructing Fourier-Mukai kernels. More explicitly, in the circumstances discussed above, there is an equivalence $G_w \xrightarrow{\sim} D^b([X^{ss}/G])$ given by the restriction map, and the shifting algorithm allows us to construct the FMK for the inverse functor $D^b([X^{ss}/G]) \xrightarrow{\sim} G_w$. Given a KN stratification $S_i \subseteq X$ for $i = 1, \dots, k$, we make use of the KN strata $X^{ss} \times S_i \subseteq X^{ss} \times X$. Take the equivariant diagonal $\mathcal{O}_{\Delta G} \in D^b([X^{ss} \times X^{ss}/G \times G])$ and, for a given $w \in \mathbb{Z}^k$, extend it to some object in $G'_w \subseteq D^b([X^{ss} \times X/G \times G])$ by running the shifting algorithm with respect to this stratification. Lemma 2.16 of [Hal15] tells us that the window subcategories G_w and G'_w are compatible in the following sense: if $Q' \in G'_w$ then $\Phi_{Q'}(F) \in G_w \subseteq D^b([X/G])$ for any $F \in D^b([X^{ss}/G])$, and the remark shortly after this lemma concludes that Q' is the kernel of the functor inverse to the restriction.

For our running example, we have only one stratum S and we can choose our initial extension of the equivariant diagonal to $D^b([X^{ss} \times X/G \times G])$ to be $Q^{ss}_{X,G}$ as in Example 2.3.1.7. We denote

$$S' := X^{ss}(\mathcal{L}_{m<0}) \times S \overset{j'}{\hookrightarrow} X^{ss}(\mathcal{L}_{m<0}) \times X =: X'.$$

Using a very similar trick to before, $j'_*\mathcal{O}_{S'} \in D^b([X'/G \times G])$ is quasi-isomorphic to the complex

$$\mathcal{O}_{X'} \otimes \chi_{0,n+1} \xrightarrow{(g_n, \dots, g_0)} \bigoplus \mathcal{O}_{X'} \otimes \chi_{0,n} \longrightarrow \dots \longrightarrow \bigoplus \mathcal{O}_{X'} \otimes \chi_{0,1} \xrightarrow{\begin{pmatrix} g_0 \\ \vdots \\ g_n \end{pmatrix}} \mathcal{O}_{X'} \otimes \chi_{0,0}$$

and thus $(j')^!\mathcal{O}_{X'}$ has the single weight $-p(n+1)$, via the coaction

$$\begin{array}{ccccc} R' & \longrightarrow & \mathbb{C}[t^\pm, s^\pm] \otimes R' & \longrightarrow & \mathbb{C}[t^\pm, s^\pm] \otimes R' \\ 1 & \longmapsto & t^0 s^{-(n+1)} \otimes 1 & \longmapsto & s^{-p(n+1)} \otimes 1 \end{array}$$

where $R' := \mathbb{C}[f_0, \dots, f_n, g_0, \dots, g_n]$ are the global sections of $\mathcal{O}_{X'}$. This is just the bigrading given by the gradings on each of X^{ss} and X.

By base change around the fibre diagram

$$
\begin{array}{ccccc}
S' & \hookrightarrow & & \longrightarrow & \tilde{X} \\
{\scriptstyle \mathrm{id}}\downarrow & & & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\
S' & \xrightarrow[j']{} & X' & \hookrightarrow & X \times X
\end{array}
$$

we see that $(j')^* Q^{ss}_{X,G} \simeq \mathcal{O}_{S'}$, and thus the only weight of $(j')^* Q^{ss}_{X,G}$ is zero. Therefore, $Q^{ss}_{X,G} \in G'_w$ for $0 \leq w < -p(n+1)$. We now run the shifting algorithm.

The case $w < 0$

The sole weight of $(j')^* Q^{ss}_{X,G} \simeq \mathcal{O}_{S'}$ is zero, so we want to take a distinguished triangle of the form

$$Q' \longrightarrow Q^{ss}_{X,G} \longrightarrow j'_* \mathcal{O}_{S'} \longrightarrow Q'[1]$$

We obtain this by observing the commutativity of

$$
\begin{array}{ccc}
 & & \mathbb{A} \times X^{ss} \\
 & {\scriptstyle j''}\nearrow & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\
X^{ss} \times S & \xhookrightarrow[j']{} & X^{ss} \times X
\end{array}
$$

and that the image of $S' = X^{ss} \times S$ in $\mathbb{A} \times X^{ss}$ is closed. We therefore take the standard short exact sequence in $D^b(\mathbb{A} \times X^{ss})$

$$0 \longrightarrow \mathcal{I}_{S'} \longrightarrow \mathcal{O}_{\mathbb{A}\times X^{ss}} \longrightarrow j''_* \mathcal{O}_{S'} \longrightarrow 0. \tag{2.9}$$

As $(\tilde{\pi}, \tilde{\sigma})$ is affine, higher direct images vanish and we can push this SES forward to obtain the distinguished triangle

$$(\tilde{\pi}, \tilde{\sigma})_* \mathcal{I}_{S'} \longrightarrow Q^{ss}_{X,G} \longrightarrow j'_* \mathcal{O}_{S'} \longrightarrow (\tilde{\pi}, \tilde{\sigma})_* \mathcal{I}[1]$$

Thus the new kernel is $Q' = (\pi, \sigma)_* \mathcal{I}_{S'}$. If this still doesn't lie in G'_w, iterate the algorithm until we obtain something that lies in the weight window.

The case $w \geq -p(n+1)$

Conversely, in this case we want an exact triangle of the form

$$(j')_* \mathcal{O}_{S'} \otimes \chi_{-p(n+1)} \longrightarrow Q^{ss}_{X,G} \longrightarrow Q' \longrightarrow (j')_* \mathcal{O}_{S'} \otimes \chi_{-p(n+1)}[1]$$

We obtain this by first dualising (2.9) to get

$$0 \longrightarrow \mathcal{H}om(j''_* \mathcal{O}_{S'}, \mathcal{O}_{\mathbb{A}\times X^{ss}}) \longrightarrow \mathcal{O}_{\mathbb{A}\times X^{ss}} \longrightarrow \mathcal{I}^{\vee}_{S'} \longrightarrow 0$$

and note that $\mathcal{H}om(j''_*\mathcal{O}_{S'}, \mathcal{O}_{\mathbb{A}\times X^{ss}}) \simeq (j'')_*\mathcal{O}_{S'} \otimes \chi_{-p(n+1)}$. Pushing this SES forward along $(\tilde{\pi}, \tilde{\sigma})$ gives the required distinguished triangle with new kernel $Q' \simeq (\tilde{\pi}, \tilde{\sigma})_*\mathcal{I}_{S'}^{\vee}$.

— Chapter 3 —

VGIT Stacks for 2-dim. A_n-type McKay Correspondence

The main aim of this chapter is to provide the geometry necessary for the schober we construct in Chapter 4. In Section 3.1 we remind the reader of some results from the representation theory of finite groups G and give the definition of objects known as G-constellations for $G \subset GL(n, \mathbb{C})$ due to Craw [Cra01]. These G-constellations have a particularly nice formulation of GIT stability due to King [Kin94], known as θ-stability, as well as a natural group action. This gives a description of the moduli space of θ-semistable G-constellations as a VGIT problem, and there is a natural decomposition of the stability space into cells via a wall-and-chamber description. In Section 3.2 we give an example of this by giving a description of the θ-semistable G-constellations and the wall-and-chamber decomposition for the specific case $G = \mathbb{Z}_3 := \mathbb{Z}/3\mathbb{Z}$. In Section 3.3 we describe the geometry of these semistable loci, viewing them as quotient stacks. In Section 3.5 we study the unstable loci and give corresponding KN stratifications.

3.1 G-constellations as McKay quiver representations

Recall that we work over the base field $\mathbb{C}$ and use A to denote the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]$ of $\mathbb{C}^n$. We first remind the reader of some some well-known results from the representation theory of finite groups, which can be found in [FH91]. The main references for the subsequent part of this section are [Cra01] and [Kin94]. In this section G denotes a finite group.

Proposition 3.1.0.1. *For any representation V of a finite group G, there is a decomposition $V = W_1^{\oplus a_1} \oplus \cdots \oplus W_k^{\oplus a_k}$ where the W_i are distinct irreducible representations. The decomposition of V into a direct sum of k factors is unique, as are the W_i that occur and their multiplicities a_i.*

Lemma 3.1.0.2. *Schur's Lemma*
If V and W are irreducible representations of G and $\varphi : V \to W$ is a G-equivariant linear map, then either φ is an isomorphism, or $\varphi = 0$. If $V = W$ then $\varphi = \lambda id_V$ for $\lambda \in \mathbb{C}$.

For G a finite abelian group, all irreducible representations are one-dimensional, i.e. they are characters $\chi_i \in \mathrm{Hom}(G, \mathbb{G}_m)$ of the group. For any representation V we can therefore say that there is a decomposition into irreducible representations given by:

$$V = \chi_0^{\oplus a_0} \oplus \cdots \oplus \chi_k^{\oplus a_k}$$

and define $V_i := \chi_i^{\oplus a_i}$.
The given representation of $G \subseteq GL(n, \mathbb{C})$ is defined to be the vector space $V_{giv} = \mathbb{C}^n$ with action given by $G \hookrightarrow GL(n, \mathbb{C})$.

Definition 3.1.0.3. *The McKay quiver $\mathcal{Q}_M$ of a finite group $G \subseteq GL(n, \mathbb{C})$ is the quiver with vertices indexed by the irreducible representations $\rho_0, \ldots, \rho_N$ of G, and $dim_{\mathbb{C}} Hom_G(\rho_i, V_{giv} \otimes \rho_j)$ arrows from vertex i to vertex j.*

Having chosen vector spaces V_i of dimension vector $v = (v_i)_{i=0}^N$ for $v_i = \dim V_i$, isomorphism classes of representations of the McKay quiver are in natural bijection with orbits in the *representation space*

$$R(\mathcal{Q}_M, v) := \bigoplus_{\alpha \in Q_1} \mathrm{Hom}_{\mathbb{C}}(V_{t(\alpha)}, V_{h(\alpha)})$$

under the natural faithful group action of

$$PGL(v) := \big(\prod_{i \in Q_0} GL(V_i) \big) / \Delta$$

acting[1] by $(g \cdot \varphi)_\alpha = g_{t(\alpha)} \varphi_\alpha g_{h(\alpha)}^{-1}$. We factor out the diagonal 1-parameter subgroup $\Delta = \{(c(\mathrm{id}_{V_i})_i) \mid c \in \mathbb{G}_m\}$ as it acts trivially. There are $n(N+1)$ arrows in this quiver. For one-dimensional V_i this action corresponds to a rescaling of the basis element for each of the V_i, modulo rescaling them all by the same thing. For dimension vector $v = (1)_{i=0}^N$, $R(\mathcal{Q}_M, v) \simeq \mathbb{C}^{n(N+1)}$. This group action also provides a natural interpretation of the moduli space of representations as a GIT problem. In this case the line bundle is necessarily trivial ($R(\mathcal{Q}_M, v)$ is an affine space) and the different linearisations correspond to different choices of characters of $PGL(v)$ [Kin94]. King

[1] Note that some authors use the opposite convention, i.e. that $(g \cdot \varphi)_\alpha = g_{h(\alpha)} \varphi_\alpha g_{t(\alpha)}^{-1}$.

gives a translation of the notion of GIT stability, known as θ-stability, for this problem *ibid.*:

Definition 3.1.0.4. *Given $\theta \in \mathbb{Z}^{N+1}$ and a representation V of the McKay quiver with dimension vector $(v_i)_{i=0}^N$, let $\Theta(V) := \sum_{i=0}^N v_i\theta_i$. A representation is said to be θ-stable if $\Theta(V) = 0$ and every proper subrepresentation $0 \subset W \subset V$ has $\Theta(W) > 0$. θ-semistability is defined likewise, but substituting the weak inequality $\Theta(W) \geq 0$ for all $0 \subset W \subset V$. A representation is said to be θ-unstable if it is not θ-semistable. For a fixed choice of θ, if representations are stable if and only if they are semistable, we say θ is generic.*

As indicated, the notion of θ-stability coincides with that of GIT stability. The exact statement is as follows:

Proposition 3.1.0.5 (Proposition 3.1, [Kin94]). *Let $x \in R(\mathcal{Q}_M, v)$ be a point corresponding to a representation V. The point x is GIT semistable (resp. GIT stable) if and only if V is θ-semistable (resp. θ-stable).*

In addition to this, there is also a nice translation of two points x and y being GIT equivalent to this new language of θ-stability due to King. We will state this as Proposition 3.1.0.20. For any finite group G, denote by V_{reg} the regular representation of G. Recall that the McKay Correspondence [McK80] is the study of the geometry of quotient singularities where a finite group G acts on $\mathbb{C}^n$.

Definition 3.1.0.6. *Let $G \subset GL(n, \mathbb{C})$ be finite. A G-cluster $Z \subseteq \mathbb{C}^n$ is a G-invariant finite length subscheme with $\Gamma(\mathcal{O}_Z) \cong V_{reg}$ as representations of G.*

The space parametrising G-clusters on $\mathbb{C}^n$ goes by the name G-Hilb$(\mathbb{C}^n)$ and there is a natural Hilbert-Chow morphism $\eta : G\text{-Hilb}(\mathbb{C}^n) \to \mathbb{C}^n/G$ given on closed points by sending the corresponding G-cluster to its defining subscheme (this subscheme is G-invariant by assumption). In dimension 2, this map is the minimal resolution [IN96]. In some higher dimensional cases it is a crepant resolution, e.g. [BKR01]. The notion of a G-cluster admits a generalisation due to Craw known as a *G-constellation*. These were introduced and first studied in his thesis [Cra01] and his subsequent paper with Ishii [CI04].

Definition 3.1.0.7 ([Cra01]). *Let $G \subset GL(n, \mathbb{C})$ be finite. A G-constellation is a G-equivariant coherent sheaf M on $\mathbb{C}^n$ with $\Gamma(M) \cong V_{reg}$ as representations of G.*

Remark 3.1.0.8. We think of G-clusters as being the G-invariant finite-length subschemes for which $\mathcal{O}_Z$ is a G-constellation. In this sense, G-constellations are a natural generalisation of G-clusters.

Definition 3.1.0.9. *The cross-product algebra $A \rtimes G$.*
Viewing A and $\mathbb{C}[G]$ as vector spaces over $\mathbb{C}$, the underlying vector space of $A \rtimes G$ is that of $A \otimes_{\mathbb{C}} \mathbb{C}[G]$. Multiplication in $A \rtimes G$ is given, for all $f_1, f_2 \in A$ and $g_1, g_2 \in \mathbb{C}[G]$, by

$$(f_1 \otimes g_1) \times (f_2 \otimes g_2) = \big(f_1(g_1 \cdot f_2)\big) \otimes (g_1 g_2),$$

where the action of $\mathbb{C}[G]$ on A is given by linearly extending the action of G on A.

Remark 3.1.0.10. The (equivariant versions of the) global section functor $\Gamma(-)$ and the $(\tilde{-})$ functor [Har77, p.110] give mutually inverse equivalences between the abelian categories of G-equivariant coherent sheaves, $\mathrm{Coh}^G(\mathbb{C}^n)$, and finitely generated $A \rtimes G$-modules, $\mathrm{Mod}^{fg}(A \rtimes G)$:

$$\Gamma : \mathrm{Coh}^G(\mathbb{C}^n) \xrightarrow{\sim} \mathrm{Mod}^{fg}(A \rtimes G)$$
$$(\tilde{-}) : \mathrm{Mod}^{fg}(A \rtimes G) \xrightarrow{\sim} \mathrm{Coh}^G(\mathbb{C}^n).$$

In this way, we can think of a G-constellation as being either a G-equivariant coherent sheaf, or its corresponding $A \rtimes G$-module.

Remark 3.1.0.11. Take any G-equivariant coherent sheaf M on $\mathbb{C}^n$. As a G-representation, $M = \bigoplus M_\rho \otimes \rho$. The A-module structure on M is given by G-equivariant maps $V_{giv}^* \otimes M \to M$, where V_{giv}^* is the dual of V_{giv}, i.e. $V_{giv}^* = \langle x_1, \dots x_n \rangle = \bigoplus \mathbb{C} x_i$.

$$\begin{aligned}\mathrm{Hom}_G(V_{giv}^* \otimes M, M) &= \mathrm{Hom}_G(\bigoplus M_\rho \otimes \rho \otimes V_{giv}^*, \bigoplus M_\rho \otimes \rho) \\ &= \bigoplus_{\rho, \rho'} \mathrm{Hom}_{\mathbb{C}}(M_\rho, M_{\rho'}) \otimes \mathrm{Hom}_G(V_{giv}^* \otimes \rho, \rho')\end{aligned}$$

The $\mathrm{Hom}_G(V_{giv}^* \otimes \rho, \rho')$ are the actions of $x_1, \dots x_n$ on ρ, which define the arrows in our quiver diagram. In this way we can think of G-constellations as being certain representations of the McKay quiver. We make this precise via the following proposition. The admissible ideal $\mathcal{I}$ appearing in the statement can be found in [CMT07a]. The proposition is well-known, but the proof is short so we include it for completeness. The key result is that $\mathbb{C}\mathcal{Q}_M/\mathcal{I} \cong A \rtimes G$, which is proved in Proposition 2.8 of [CMT07b].

Proposition 3.1.0.12. *For a finite abelian subgroup $G \subset GL(n, \mathbb{C})$, there is an equivalence of categories*

$$Coh^G(\mathbb{C}^n) \xrightarrow{\sim} Rep^{fd}(\mathcal{Q}_M, \mathcal{I})$$

where the admissible ideal $\mathcal{I} \subseteq \mathbb{C}\mathcal{Q}_M$ is given by

$$\mathcal{I} = \langle a_j^{\rho\rho_i} a_i^{\rho} - a_i^{\rho\rho_j} a_j^{\rho} \mid \rho \in Hom_G(G, \mathbb{G}_m),\ 1 \leq i, j \leq n \rangle \tag{3.1}$$

where we label the arrows in the McKay quiver $a_i^{\rho} : \rho\rho_i \to \rho$.

Proof. We note that the McKay quiver is connected. This is because, as V_{giv} is a faithful representation, every irreducible representation of G is contained in $V_{giv}^{\otimes n}$ for some n (see [CR62], Theorem 32.9). Therefore, by the definition of arrows in $\mathcal{Q}_M$, there exists a path in the quiver from a given vertex to every other vertex. As $\mathcal{Q}_M$ is a finite quiver, Proposition 1.7.0.11 gives that $\text{Mod}^{fg}(\mathbb{C}\mathcal{Q}_M/\mathcal{I}) \simeq \text{Rep}^{fd}(\mathcal{Q}_M, \mathcal{I})$. For $G \subset GL(n, \mathbb{C})$ abelian, it is known that $\mathbb{C}\mathcal{Q}_M/\mathcal{I} \cong A \rtimes G$ for this ideal (Proposition 2.8, [CMT07b]), so the result follows by Remark 3.1.0.10. □

For finite *abelian* $G \subseteq GL(n, \mathbb{C})$, we can therefore think of G-constellations as being representations of the McKay quiver with the linear maps satisfying the path relations given by $\mathcal{I}$. The relations specified by $\mathcal{I}$ correspond to the commutativity of $x_i, x_j \in A$ when we view our G-constellation as an $A \rtimes G$-module (Remark 3.7, [CMT07b]). In fact, the G-constellations are precisely the representations of the McKay quiver which obey these relations and are isomorphic to V_{reg}.

Remark 3.1.0.13. The analogous result also holds for arbitrary finite $G \subseteq SL(n, \mathbb{C})$ [BSW10, p.7] for an appropriate choice of $\mathcal{I}$; here it is no longer true in general that $\mathbb{C}\mathcal{Q}_M/\mathcal{I} \cong A \rtimes G$, but they are still Morita equivalent, so the result follows by the same logic.

Viewing a G-constellation as a representation of the McKay quiver via Proposition 3.1.0.12, we have a natural notion of θ-stability for G-constellations. For a given choice of θ, the category of θ-semistable G-constellations, $G\text{-Const}_\theta^{ss}$, is an abelian subcategory of the abelian category of G-constellations. We remind the reader of the following trivial definition.

Definition 3.1.0.14. *An object X in an abelian category with a zero object is simple if there are precisely two subobjects of X, namely 0 and X itself.*

Lemma 3.1.0.15 ([Kin94]). *For a given choice of θ, the simple objects in $G\text{-Const}_\theta^{ss}$ are precisely the θ-stable G-constellations.*

Proof. Let $M \in G\text{-Const}_\theta^{ss}$ be a simple object. Then $\Theta(M) = 0$ and there are no non-trivial proper subrepresentations to check, so M is stable.
Conversely, let $M \in G\text{-Const}_\theta^{ss}$ be stable. Then $\Theta(E) > 0$ for any nontrivial subrepresentations $E \subset M$. M cannot properly contain any nontrivial semistable representations, as this would require $\Theta(E) = 0$. M is therefore simple. □

Definition 3.1.0.16. *Given a θ-semistable G-constellation M, we call a filtration*

$$M_0 = 0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$$

a composition series of M, where the M_i are θ-semistable G-constellations for $i < n$ and the quotients of successive terms are simple. We call

$$M_1/M_0 \oplus M_2/M_1 \oplus \cdots \oplus M_n/M_{n-1}$$

the composition factor.

Remark 3.1.0.17. We can always find such a filtration for a θ-semistable G-constellation. Indeed, if M is a strictly θ-semistable G-constellation, then $\Theta(M) = 0$ and $\Theta(E) = 0$ for some $E \subset M$ a proper subrepresentation. Then E is either θ-stable or strictly θ-semistable itself. If E is strictly θ-semistable, then repeat the process. If E is θ-stable then by Lemma 3.1.0.15, it is simple. Therefore there do not exist non-trivial $E' \subset E$ such that $\Theta(E') = 0$, and so the filtration terminates. In fact, the filtration must always terminate as M is finite dimensional as a vector space (indeed, M is a G-representation for G a finite group by definition).

Remark 3.1.0.18. Theorem 2.1 in [Ses67] tells us that the Jordan-Hölder theorem holds. Therefore, although the composition series of a θ-semistable G-constellation M may not be unique, the composition factor is (up to reordering of the direct summands).

Definition 3.1.0.19 ([Ses67])**.** *We call two θ-semistable representations S-equivalent if they have the same composition factors in the category of θ-semistable representations.*

A result of King gives the promised translation of the idea of GIT equivalence:

Proposition 3.1.0.20 (Proposition 4.2, [Kin94])**.** *Let x and y be two points of $R(\mathcal{Q}_M, v)$ with corresponding $\mathcal{Q}_M$-representations V and W. Then x and y are GIT equivalent if and only if V and W are S-equivalent.*

We now note a general result due to Kronheimer that we will observe specific instances of in the rest of this chapter. From the definition of θ-stability, for a representation V with dimension vector $(v_0, \ldots, v_N)$ to be θ-semistable, θ must lie on the hyperplane given by $\sum_{i=0}^{N} v_i\theta_i = 0$, and the following result describes the wall-and-chamber decomposition of this hyperplane, and what the moduli space of θ-semistable G-constellations is in the generic open chambers. Let v be the dimension vector of V_{reg}. The space of G-constellations inside $R(\mathcal{Q}_M, v) \simeq \mathbb{C}^{n(N+1)}$ is given by the subset such that the coordinates satisfy the commutativity relations imposed by (3.1). Denote this space X, with corresponding θ-semistable loci X_θ^{ss} for each value of θ.

Proposition 3.1.0.21 ([Kro86]). *Let $G \subseteq SL(2, \mathbb{C})$ be finite. Then the classical GIT quotient $X_\theta^{ss}/PGL(v)$ for any generic θ is the minimal resolution of the quotient singularity $\mathbb{C}^2/G$. Moreover, the wall-and-chamber decomposition of the hyperplane given by $\sum_{i=0}^{N} v_i\theta_i = 0$ is given by a root system of the same ADE-type as $\mathbb{C}^2/G$.*

An observation due to Ito and Nakajima [IN00, §3] shows that the choice of θ lying in one of the generic cells gives $X_\theta^{ss}/PGL(v) \simeq G$-Hilb$(\mathbb{C}^2)$.

We finish off this section by making a key observation. Start with a generic stability condition θ lying in some open chamber, so that the classical GIT quotient $X_\theta^{ss}/PGL(v)$ is the minimal resolution with exceptional locus $E = \bigcup_i E_i$ given by some collection of rational curves E_i, intersecting transversally. Whenever you move your stability condition $\theta \rightsquigarrow \theta'$ to a neighbouring lower dimensional cell (e.g. a bordering wall), the new GIT quotient $X_{\theta'}^{ss}/PGL(v)$ corresponds to a *contraction* of some number of these E_i (i.e. it is a *partial resolution* of $\mathbb{C}^2/G$). Moving to the central cell $\theta = (0, \ldots, 0)$, the GIT quotient is the contraction of *all* the E_i, i.e. the GIT quotient is the original singularity $\mathbb{C}^2/G$. We prove this final statement now.

Proposition 3.1.0.22. *Let $G \subseteq SL(2, \mathbb{C})$ be finite and abelian, $\theta_0 = (0, \ldots, 0)$ and let v be the dimension vector of V_{reg}. The good moduli space of $[X_{\theta_0}^{ss}/PGL(v)]$ is $\mathbb{C}^2/G$.*

Proof. From [Alp13, Theorem 13.6], the good moduli space of $[X_{\theta_0}^{ss}/PGL(v)]$ is $X_{\theta_0}^{ss}/PGL(v)$. There is a Hilbert-Chow morphism $\eta : [X_{\theta_0}^{ss}/PGL(v)] \to \mathbb{C}^2/G$ given by taking (the isomorphism class of) a representation to the support of the corresponding G-constellation. By the universal property of good moduli spaces for maps to schemes [Alp13, Theorem 4.16], there exists a unique map π such that the following diagram commutes

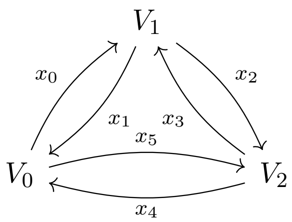
$$\begin{array}{ccc} [X^{ss}_{\theta_0}/PGL(v)] & \xrightarrow{\eta'} & X^{ss}_{\theta_0}/PGL(v) \\ & \searrow^{\eta} & \downarrow \pi \\ & & \mathbb{C}^2/G. \end{array}$$

There are two cases to consider. First, assume $\eta(F) \neq 0$. As G acts freely away from the origin, there is only one $PGL(v)$-orbit which F could belong to, i.e. a single point in the quotient stack. The map π is therefore an isomorphism away from the origin $0 \in \mathbb{C}^2/G$.
As $X^{ss}_{\theta_0}/PGL(v)$ is separable, if $x, y \in [X^{ss}_{\theta_0}/PGL(v)]$ are two points such that the closures of their orbits intersect, they are sent to the same point by η'. Assume $\eta(F) = 0$. We now show that the closures of the orbits of all such points intersect. Let $\mathcal{Q}_F$ denote the McKay quiver of G where we remove all the arrows corresponding to zero maps in F. The condition $\eta(F) = 0$ implies that $\mathcal{Q}_F$ has no loops. If $\mathcal{Q}_F$ has no arrows left, F was the zero representation. Assuming $\mathcal{Q}_F$ has at least one arrow, choose a vertex which has arrows $\{a_i\}$ going out but none coming in. Now, choose the one-parameter subgroup $\lambda : \mathbb{G}_m \to PGL(v)$ for which $t \in \mathbb{G}_m$ rescales the linear maps corresponding to $\{a_i\}$ by t. Sending $t \to 0$, we find that F is inseparable from some representation F' with strictly fewer non-zero maps. Proceeding by induction on the (finite) number of non-zero maps, we find that F is inseparable from the zero representation. All F such that $\eta(F) = 0$ are therefore mapped to the same point in $X^{ss}_{\theta_0}/PGL(v)$. Thus π is an isomorphism in this case as well, and so an isomorphism. □

Remark 3.1.0.23. This result is an instance of the following mantra, which it may be useful to keep in mind in the sequel: when we move to lower dimensional cells in the stability space, the classical GIT quotient gets *smaller*, whereas the stacky GIT quotient gets *bigger*.

3.2 The GIT wall-and-chamber decomposition for $G = \mathbb{Z}_3$

Consider $\mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z} \subset GL(2, \mathbb{C})$. As the number of irreducible representations of a finite group G is equal to the number of conjugacy classes of elements, this McKay quiver has three vertices 0, 1 and 2, with single arrows $i \to i+1$ and $i \to i-1$ for all i, where we count modulo 3. The $\mathbb{Z}_3$-constellations are precisely the representations of this McKay quiver where the V_i are one-dimensional and the commutativity relations induced by (3.1) are satisfied. By choosing a basis for each of the one-dimensional V_i, these maps are fully determined by where we send the basis element, and so they may be represented by $x_i \in \mathbb{C}$. Therefore we represent these $\mathbb{Z}_3$-constellations diagrammatically as



where $\dim(V_i) = 1$. For $x, y \in \mathbb{C}[x, y]$, the clockwise arrows correspond to the action of x, and the anti-clockwise arrows correspond to the action of y. A direct check verifies that condition (3.1) implies the requirement that $x_0x_1 = x_2x_3 = x_4x_5$. Denote $V := \bigoplus_{i=0}^{2} V_i$. Given a fixed choice of basis, the representation space of these $\mathbb{Z}_3$-constellations is then given by $X := \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_0x_1 = x_2x_3 = x_4x_5\} \subset \mathbb{C}^6$. However, we wish to consider isomorphism classes of these $\mathbb{Z}_3$-constellations, i.e. to consider them up to *any* choice of basis, and by rescaling the basis elements we get the natural action of the projective general linear group $(\mathbb{G}_m)^3/(\lambda, \lambda, \lambda) =: G \simeq \mathbb{G}_m^2$ we described on p.60. This action is given by

$$(a, b, c) \cdot (x_0, x_1, x_2, x_3, x_4, x_5) = \Big(\frac{a}{b}x_0, \frac{b}{a}x_1, \frac{b}{c}x_2, \frac{c}{b}x_3, \frac{c}{a}x_4, \frac{a}{c}x_5\Big). \tag{3.2}$$

Thus, the fine moduli space of $\mathbb{Z}_3$-constellations is given by the stacky GIT quotient $[X/G]$, and we examine the geometry of this stack in Section 3.3. Recalling Definition 3.1.0.4, we investigate the space of stability conditions $\theta = (\theta_0, \theta_1, \theta_2)$. Note that $\dim(V) = (1, 1, 1)$, and so

$$\Theta(V) = \sum_{i=0}^{2} \theta_i.$$

In order for a representation V to be semistable, we must lie on the hyperplane given by $\Theta(V) = 0$. Thus, we consider stability conditions of the form $\theta = (\theta_0, \theta_1, -\theta_0 - \theta_1)$. All possible non-trivial strict subrepresentations of V are given by $\{V_0, V_1, V_2, V_0 \oplus V_1, V_0 \oplus V_2, V_1 \oplus V_2\}$, and so we have a natural wall-and-chamber decomposition of our stability space by the three walls $\theta_0 = 0$, $\theta_1 = 0$ and $\theta_2 = -\theta_0 - \theta_1 = 0$. We project the $\Theta(V) = 0$ plane to the (θ_1, θ_2) coordinate plane for ease of presentation. This gives a linear hyperplane arrangement, and we label the thirteen cells of this as follows:

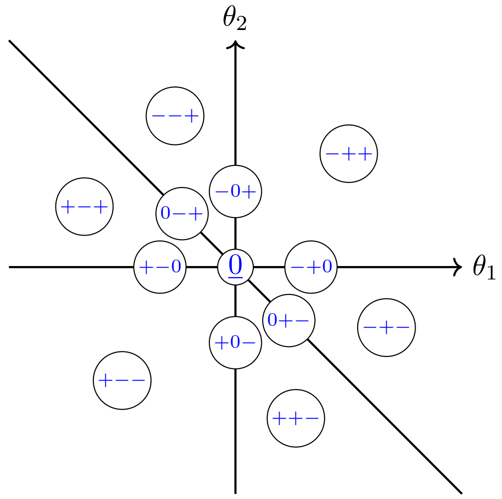


Figure 3.1: The wall-and-chamber decomposition of the hyperplane $\sum_{i=0}^{2} \theta_i = 0$ for $G = \mathbb{Z}_3$. As the cells are solely determined by the parity of the θ_i, we label the cells C_i, with i a string of three characters as indicated in the diagram. In Chapter 4, we will introduce a partial ordering on these cells by inclusion of closures.

This is the wall-and-chamber decomposition promised by Lemma 3.1.0.21, and is indeed an A_2 root system[2], which is of the same type as the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_3$. In the open two-dimensional cells of this hyperplane arrangement, V is stable if and only if it is semistable, and so the θ in these connected components are generic; θ lying in any of the zero- or one-dimensional cells are not generic. Varying the stability parameter θ inside these cells does not change which $\mathbb{Z}_3$-constellations are (semi)stable, i.e. $X^{ss}(\theta) = X^{ss}(\theta')$ for all θ, θ' lying in the same cell.

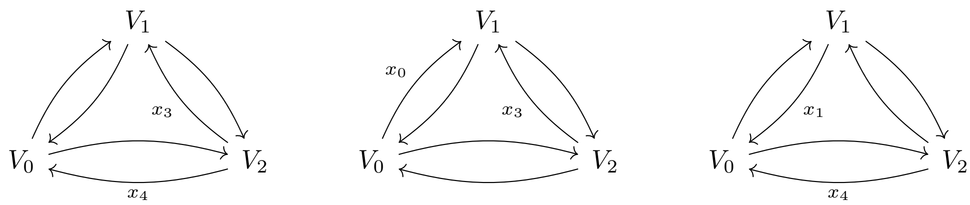
3.3 Geometry of the semistable loci

In this section we explore the geometry of the semistable loci for the thirteen different cells in Figure 3.1. As we consider lower-dimensional cells in the stability space, more G-constellations become semistable and the geometry becomes slightly more involved. We therefore start by describing the geometry for the two-dimensional generic chambers, then for the one-dimensional walls, before finally describing the geometry for $\theta = (0, 0, 0)$, for which all G-constellations are semistable.

[2]The cells in this diagram are in fact evenly spaced on the hyperplane $\sum_{i=0}^{2} \theta_i = 0$, so this really is an A_2 root system; the 'skewness' of this presentation is due to the projection down to the (θ_1, θ_2) plane.

3.3.1 The semistable locus in generic chambers

Take θ lying in Cell C_{++-}, e.g. $\theta = (1, 1, -2)$. Then G-constellations are unstable if and only if they have at least one of V_2, $V_0 \oplus V_2$ or $V_1 \oplus V_2$ as a subrepresentation. By Lemma 1.7.0.3, in order for a G-constellation to be semistable we therefore require at least one non-zero map out of each of these. Thus, a G-constellation is semistable if and only if at least one of the labelled maps in each of the following three diagrams is non-zero:



and so, as a simple corollary,

$$X^{ss}_{++-} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_1x_3 \neq 0 \text{ or } x_3x_4 \neq 0 \text{ or } x_0x_4 \neq 0\} \subseteq X.$$

Note that, as two coordinates must always be non-zero, the stabiliser of every point in this semistable locus is trivial. The stacky GIT quotient therefore coincides with the good moduli space given by the classical GIT quotient, and is indeed the minimal resolution of the A_2 singularity $\mathbb{C}^2/\mathbb{Z}_3$, as promised by Proposition 3.1.0.21.

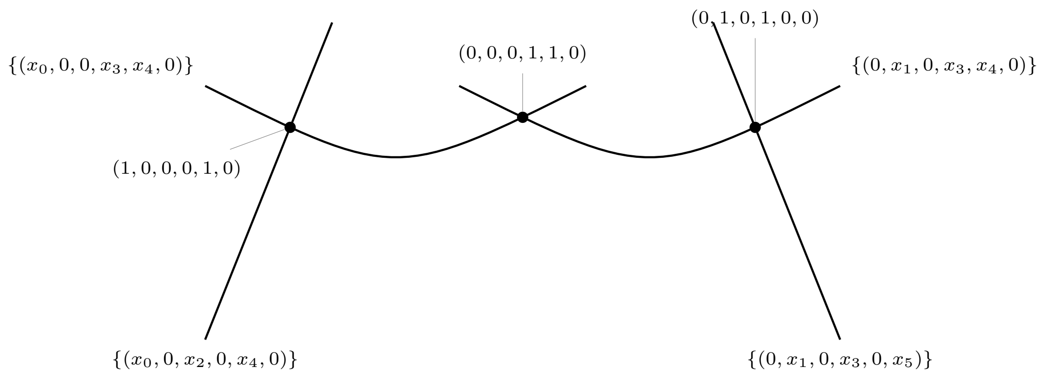


Figure 3.2: The orbit space of the semistable locus for generic θ in Cell C_{++-}.

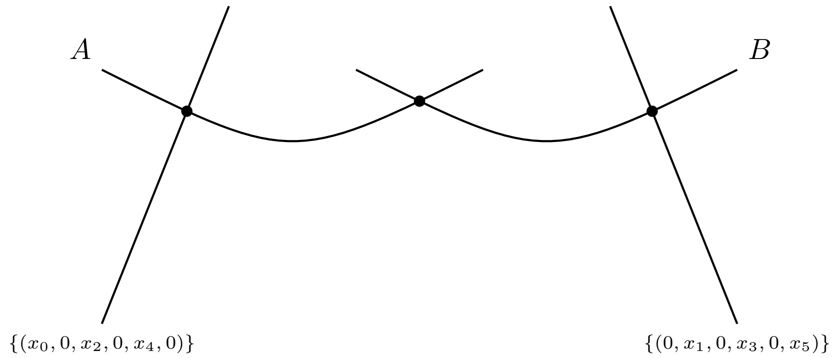
The argument for the remaining five generic open chambers is identical[3], and in each case the stacky quotient gives us something isomorphic to the same minimal

[3] This is due to the incredibly strong Weyl group symmetry of the problem, which manifests here as a clear symmetry in the coordinates we are about to give of the semistable loci in the different generic chambers.

resolution. We give their coordinates here for reference, and plot their orbits in Figure 3.3.

$$X^{ss}_{+--} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_1x_3 \neq 0 \text{ or } x_1x_4 \neq 0 \text{ or } x_2x_4 \neq 0\} \tag{3.3}$$
$$X^{ss}_{+-+} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_1x_2 \neq 0 \text{ or } x_1x_5 \neq 0 \text{ or } x_2x_4 \neq 0\} \tag{3.4}$$
$$X^{ss}_{--+} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_0x_2 \neq 0 \text{ or } x_2x_5 \neq 0 \text{ or } x_1x_5 \neq 0\} \tag{3.5}$$
$$X^{ss}_{-++} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_0x_2 \neq 0 \text{ or } x_0x_5 \neq 0 \text{ or } x_3x_5 \neq 0\} \tag{3.6}$$
$$X^{ss}_{-+-} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_0x_3 \neq 0 \text{ or } x_0x_4 \neq 0 \text{ or } x_3x_5 \neq 0\} \tag{3.7}$$

where, in each case, the defining relations of X hold.

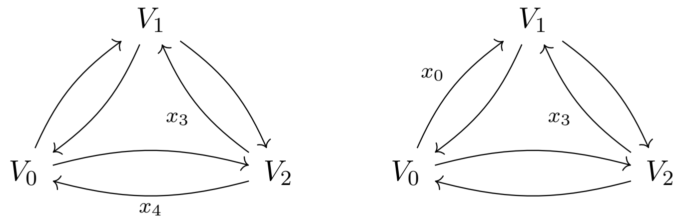


	A	B
X^{ss}_{++-}	$\{(x_0, 0, 0, x_3, x_4, 0)\}$	$\{(0, x_1, 0, x_3, x_4, 0)\}$
X^{ss}_{+--}	$\{(0, x_1, x_2, 0, x_4, 0)\}$	$\{(0, x_1, 0, x_3, x_4, 0)\}$
X^{ss}_{+-+}	$\{(0, x_1, x_2, 0, x_4, 0)\}$	$\{(0, x_1, x_2, 0, 0, x_5)\}$
X^{ss}_{--+}	$\{(x_0, 0, x_2, 0, 0, x_5)\}$	$\{(0, x_1, x_2, 0, 0, x_5)\}$
X^{ss}_{-++}	$\{(x_0, 0, x_2, 0, 0, x_5)\}$	$\{(x_0, 0, 0, x_3, 0, x_5)\}$
X^{ss}_{-+-}	$\{(x_0, 0, 0, x_3, x_4, 0)\}$	$\{(x_0, 0, 0, x_3, 0, x_5)\}$

Figure 3.3: The orbit space of the semistable locus for each generic chamber.

3.3.2 The semistable locus on a one-dimensional wall

We now take θ lying on a codimension one wall, e.g. $\theta = (0, 1, -1)$ in Cell C_{0+-}. Now G constellations are unstable if and only if they contain V_2 or $V_0 \oplus V_2$ as a subrepresentation. To be semistable, we therefore require at least one of the labelled maps in each of the following diagrams to be non-zero



and thus

$$X_{0+-}^{ss} = \{(x_0, x_1, x_2, x_3, x_4, x_5) \mid x_3 \neq 0 \text{ or } x_0 x_4 \neq 0\} \subseteq X$$

The stabiliser of any point of the form $(0, 0, 0, x_3, 0, 0) \in X_1^{ss}$ is non-trivial (it is isomorphic to $\mathbb{G}_m$), and so, in contrast to the generic chamber cases, the stack $[X_1^{ss}/G]$ is not a scheme. Following the same process as before, we plot the orbits in Figure 3.4.

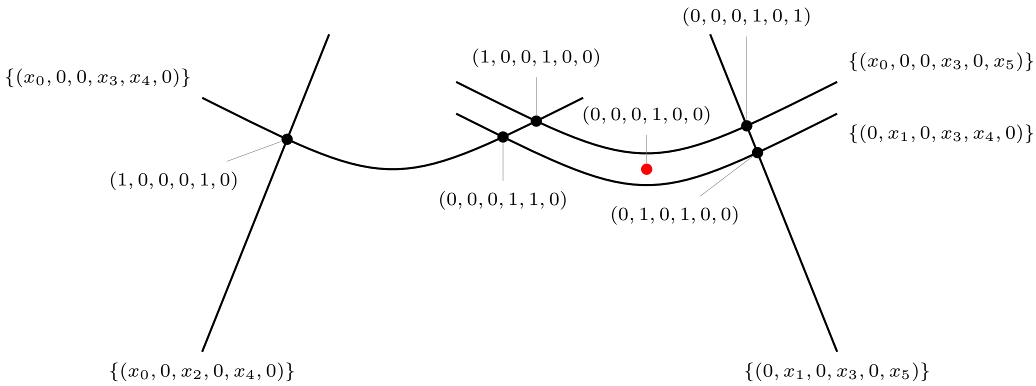


Figure 3.4: The orbit space of the semistable locus for θ in Cell C_{0+-}. The stacky point is shown in red.

The five components with three non-zero coordinates are each isomorphic to $\mathbb{C}^\times$ when we take the stacky quotient, and the compactifications of each of these by the indicated black points in the diagram give three $\mathbb{P}^1$ curves, as well as the original two coordinate axes.

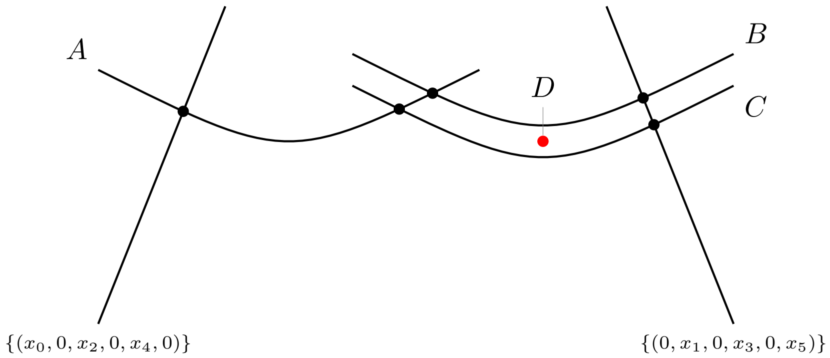
As this is a genuinely stacky quotient due to the non-trivial stabiliser of the point $(0, 0, 0, 1, 0, 0)$, the behaviour of the good moduli space is a little more interesting than in the case for the generic open cells. Taking $t \in \mathbb{G}_m$ and a point each in $\{(0, x_1, 0, x_3, x_4, 0)\}$ and $\{(x_0, 0, 0, x_3, 0, x_5)\}$ with all $x_i \neq 0$, we note that

$$(1, tx_3^2, tx_3) \cdot (0, x_1, 0, x_3, x_4, 0) = (0, tx_3^2 x_1, 0, 1, tx_3 x_4, 0)$$
$$(1, tx_3^2, tx_3) \cdot (x_0, 0, 0, x_3, 0, x_5) = (tx_0/x_3^2, 0, 0, 1, 0, tx_5/x_3)$$

and that the limit of these two points as $t \to 0$ is $(0,0,0,1,0,0)$. As the closures of the orbits of these two points therefore intersect, they are mapped to the same point in the good moduli space given by the classical GIT quotient. This good moduli space is thus the partial resolution obtained from the minimal resolution of $\mathbb{C}^2/\mathbb{Z}_3$ by contracting one of the exceptional curves. The argument for θ lying on each of the other dimension 1 cells is identical - in each case the stacky quotient has three $\mathbb{P}^1$ curves and the closures of the orbits corresponding to any two points on the "doubled" $\mathbb{P}^1$ intersect. The good moduli space is therefore the corresponding partially contracted space where we contract this doubled $\mathbb{P}^1$. We describe the semistable loci of the remaining one-dimensional cells for reference.

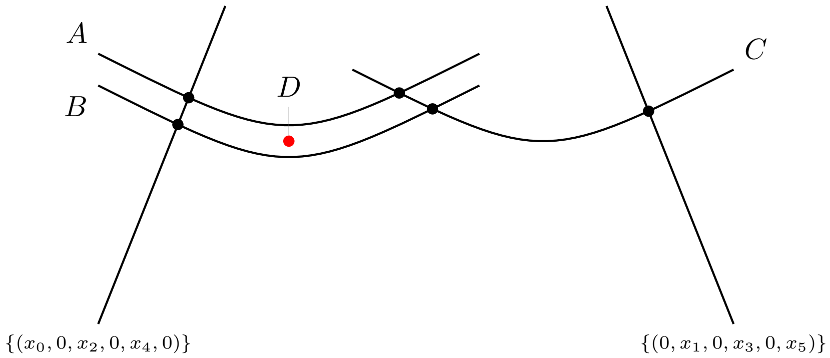
$$X^{ss}_{+0-} = \{(x_0,x_1,x_2,x_3,x_4,x_5) \mid x_4 \neq 0 \text{ or } x_1x_3 \neq 0\}$$
$$X^{ss}_{+-0} = \{(x_0,x_1,x_2,x_3,x_4,x_5) \mid x_1 \neq 0 \text{ or } x_2x_4 \neq 0\}$$
$$X^{ss}_{0-+} = \{(x_0,x_1,x_2,x_3,x_4,x_5) \mid x_2 \neq 0 \text{ or } x_1x_5 \neq 0\}$$
$$X^{ss}_{-0+} = \{(x_0,x_1,x_2,x_3,x_4,x_5) \mid x_5 \neq 0 \text{ or } x_0x_2 \neq 0\}$$
$$X^{ss}_{-+0} = \{(x_0,x_1,x_2,x_3,x_4,x_5) \mid x_0 \neq 0 \text{ or } x_3x_5 \neq 0\}$$

Plotting the orbits, we find two symmetric cases depending on which chamber we are in. These two cases correspond to whether the associated wall crossing from generic chamber to generic chamber involves replacing a $\mathbb{P}^1$ curve on the left or the right of the diagram. The geometry in the case of replacing the $\mathbb{P}^1$ on the right is shown in Figure 3.5, and for the left in Figure 3.6.



	A	B	C	D
X^{ss}_{0+-}	$\{(x_0,0,0,x_3,x_4,0)\}$	$\{(x_0,0,0,x_3,0,x_5)\}$	$\{(0,x_1,0,x_3,x_4,0)\}$	$(0,0,0,1,0,0)$
X^{ss}_{+-0}	$\{(0,x_1,x_2,0,x_4,0)\}$	$\{(0,x_1,x_2,0,0,x_5)\}$	$\{(0,x_1,0,x_3,x_4,0)\}$	$(0,1,0,0,0,0)$
X^{ss}_{-0+}	$\{(x_0,0,x_2,0,0,x_5)\}$	$\{(0,x_1,x_2,0,0,x_5)\}$	$\{(x_0,0,0,x_3,0,x_5)\}$	$(0,0,0,0,0,1)$

Figure 3.5: The orbit space of the semistable locus for θ in Cells C_{0+-}, C_{+-0} and C_{-0+}. The stacky point D is shown in red.



	A	B	C	D
X^{ss}_{+0-}	$\{(x_0,0,0,x_3,x_4,0)\}$	$\{(0,x_1,x_2,0,x_4,0)\}$	$\{(0,x_1,0,x_3,x_4,0)\}$	$(0,0,0,0,1,0)$
X^{ss}_{0-+}	$\{(x_0,0,x_2,0,0,x_5)\}$	$\{(0,x_1,x_2,0,x_4,0)\}$	$\{(0,x_1,x_2,0,0,x_5)\}$	$(0,0,1,0,0,0)$
X^{ss}_{-+0}	$\{(x_0,0,x_2,0,0,x_5)\}$	$\{(x_0,0,0,x_3,x_4,0)\}$	$\{(x_0,0,0,x_3,0,x_5)\}$	$(1,0,0,0,0,0)$

Figure 3.6: The orbit space of the semistable locus for θ in Cells C_{+0-}, C_{0-+} and C_{-+0}. The stacky point D is shown in red.

The good moduli spaces of the semistable loci appearing in Figure 3.5 are therefore the partial resolutions where we contract the right $\mathbb{P}^1$, with the left $\mathbb{P}^1$ being contracted for those semistable loci appearing in Figure 3.6.

3.3.3 The geometry of $[X/G]$

For Cell $C_{\underline{0}}$, all G-constellations are trivially semistable. The geometry of the stack is therefore an amalgamation of all of the preceding diagrams, as follows:

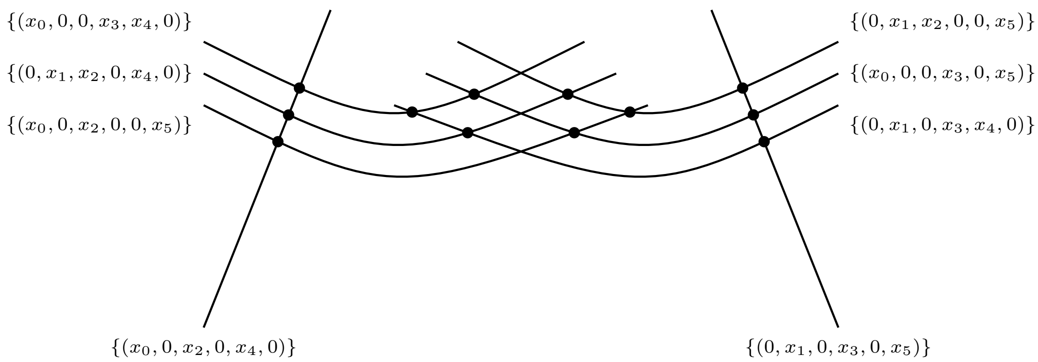


Figure 3.7: The geometry of $[X/G]$.

There are also seven stacky points not pictured. Considering the three $\mathbb{P}^1$ curves on the left of the diagram, there is a stacky point corresponding to the intersection of each pair of curves. Similarly, there are three stacky points corresponding to pairs of $\mathbb{P}^1$ curves on the right of the diagram. These six stacky points all have stabiliser isomorphic to $\mathbb{G}_m$. The seventh stacky point is $(0,0,0,0,0,0)$, which has stabiliser the whole of $G \simeq \mathbb{G}_m^2$. Picking any pair of points on these six $\mathbb{P}^1$ curves, the closures of their orbits intersect and the good moduli space is the fully contracted space, $\mathbb{C}^2/\mathbb{Z}_3$, by Proposition 3.1.0.22.

3.4 The Weyl group action

When considering the semistable loci in each of the above cells, there is an obvious $\mathbb{Z}_3$-action given by cyclically permuting the roles of the V_i (i.e., rotating the triangular diagram representing the G-constellation). Thus, if a given G-constellation $(x_0, x_1, x_2, x_3, x_4, x_5)$ is semistable with respect to the stability condition $\theta = (\theta_0, \theta_1, \theta_2)$, then $(x_2, x_3, x_4, x_5, x_0, x_1)$ is automatically semistable with respect to $\theta' = (\theta_1, \theta_2, \theta_0)$. Therefore, if the original G-constellation is semistable in a particular chamber, the new G-constellation is semistable in the chamber obtained by an order three rotation clockwise around the origin on the hyperplane $\Theta(V) = 0$.

Similarly, if $(x_0, x_1, x_2, x_3, x_4, x_5)$ is semistable with respect to $\theta = (\theta_0, \theta_1, \theta_2)$, then

$(x_1, x_0, x_3, x_2, x_5, x_4)$ is semistable with respect to $\theta' = (-\theta_0, -\theta_1, -\theta_2)$. That is, the new G-constellation is semistable with respect to the stability condition diametrically opposite.

These two facts encode how the Weyl group of the hyperplane arrangement acts on G-constellations.

3.5 KN stratifications of the unstable loci

Recall the iterative construction of the Kempf-Ness stratification of the unstable locus given by Algorithm 2.2.0.4, and also the action of G on X given in (3.2). All 1-parameter subgroups of G can be written as $\lambda(t) = (t^{n_0}, t^{n_1}, t^{n_2})$ for $n_i \in \mathbb{Z}$, under the equivalence relation that $(t^{n_0}, t^{n_1}, t^{n_2}) = (t^{n_0+n}, t^{n_1+n}, t^{n_2+n})$ for any $n \in \mathbb{Z}$. The cocharacter lattice is correspondingly $\Lambda = \mathbb{Z}^3/\sim$, where the equivalence is as indicated. We choose the standard three-dimensional Euclidean norm $\|\lambda\| := \sqrt{n_0^2 + n_1^2 + n_2^2}$ on the hyperplane $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \simeq \{(n_0, n_1, n_2) \in \mathbb{R}^3 \mid n_0 + n_1 + n_2 = 0\} \subseteq \mathbb{R}^3$. We now construct our stratification iteratively by choosing pairs (λ^*, Z^*) which maximise the numerical invariant

$$\mu^*(\lambda^*, Z^*) := \frac{1}{\|\lambda^*\|}\mu(\mathcal{L}, \lambda^*, Z^*).$$

Now, choose a stability parameter $\theta = (\theta_0, \theta_1, \theta_2)$. The corresponding weight function for this linearisation is simply the integer associated to the evaluation of the character θ on the 1-parameter subgroup:

$$t \mapsto (t^{n_0}, t^{n_1}, t^{n_2}) \mapsto (t_0^n)^{\theta_0}(t_1^n)^{\theta_1}(t_2^n)^{\theta_2} = t^{n_0\theta_0 + n_1\theta_1 + n_2\theta_2},$$

i.e. the weight of the one-parameter subgroup $(t^{n_0}, t^{n_1}, t^{n_2})$ with respect to θ is $n_0\theta_0 + n_1\theta_1 + n_2\theta_2$, and so we obtain

$$\mu^*(\lambda^*, Z^*) = \frac{n_0\theta_0 + n_1\theta_1 + n_2\theta_2}{\sqrt{n_0^2 + n_1^2 + n_2^2}} \tag{3.8}$$

for $Z^* \subseteq X^{\lambda^*}$. We label the 13 cells of the GIT wall and chamber decomposition as shown in Figure 3.1, and for stability conditions θ in each of the cells we compute KN stratifications of the unstable locus. Note that the unstable locus for each cell is essentially given by some rational curves with some extra stacky points. What the KN stratification process does is pick some connected component of the fixed locus with respect to a one-parameter subgroup, which is necessarily a (union of) stacky point(s). The algorithm then takes the stratum to be the blade over this fixed locus; this is a (union of these) rational curve(s).

3.5.1 Cell $C_{\underline{0}}$

For $\theta = (0,0,0)$ there are no unstable $\mathbb{Z}_3$-constellations, and so there is no unstable locus to stratify.

3.5.2 One-dimensional cells

First, choose the one-dimensional cell C_{0+-}. The relevant 1-parameter subgroups to consider in this case are

	1-PS	μ
1)	$\lambda_1(t) = (1, t^n, t^{-n})$	$\sqrt{2}\theta_1$
2)	$\lambda_2(t) = (t^{-n}, t^{2n}, t^{-n})$	$\frac{\sqrt{6}}{2}\theta_1$
3)	$\lambda_3(t) = (t^n, t^n, t^{-2n})$	$\frac{\sqrt{6}}{2}\theta_1$

and this leads to the following KN stratification.

1) Max μ given by $\lambda_1(t) = (1, t^n, t^{-n})$.

$$\begin{aligned} X^{\lambda_1} &= \{(0,0,0,0,0,0)\} =: Z_1 \\ \text{Blade} &= \{(0, x_1, x_2, 0, 0, x_5)\} =: S_1 \end{aligned}$$

2) Max μ given by $\lambda_2(t) = (t^{-n}, t^{2n}, t^{-n})$.

$$\begin{aligned} X^{\lambda_2} &= \{(0,0,0,0,x_4,0)\} \cup \{(0,0,0,0,0,x_5)\} \\ X^{\lambda_2} \setminus S_1 &= \{(0,0,0,0,x_4,0) \mid x_4 \neq 0\} =: Z_2 \\ \text{Blade} &= \{(0, x_1, x_2, 0, x_4, 0) \mid x_4 \neq 0\} =: S_2 \end{aligned}$$

3) Max μ given by $\lambda_3(t) = (t^n, t^n, t^{-2n})$.

$$\begin{aligned} X^{\lambda_3} &= \{(x_0,0,0,0,0,0)\} \cup \{(0,x_1,0,0,0,0)\} \\ X^{\lambda_3} \setminus \bigcup_{i=1}^{2} S_i &= \{(x_0,0,0,0,0,0) \mid x_0 \neq 0\} =: Z_3 \\ \text{Blade} &= \{(x_0, 0, x_2, 0, 0, x_5) \mid x_0 \neq 0\} =: S_3 \end{aligned}$$

Note that we could have considered either λ_2 or λ_3 first, and that the algorithm would have produced the same strata (up to relabelling S_2 and S_3).
Now, a moment's thought shows that the value for μ^* is unchanged by simultaneously cyclically permuting the n_i and θ_i in (3.8). Similarly, simultaneously negating all of the n_i and θ_i has no effect on the value of μ^*.

Utilising the Weyl-symmetry of the problem, as discussed in §3.4, we can therefore write down the corresponding KN stratifications for the other one-dimensional cells. These are shown in Figure 3.8.

Cell	S_1	S_2	S_3
C_{0+-}	$\{(0, x_1, x_2, 0, 0, x_5)\}$	$\{(0, x_1, x_2, 0, x_4, 0) \mid x_4 \neq 0\}$	$\{(x_0, 0, x_2, 0, 0, x_5) \mid x_0 \neq 0\}$
C_{+0-}	$\{(x_0, 0, x_2, 0, 0, x_5)\}$	$\{(0, x_1, x_2, 0, 0, x_5) \mid x_1 \neq 0\}$	$\{(x_0, 0, 0, x_3, 0, x_5) \mid x_3 \neq 0\}$
C_{+-0}	$\{(x_0, 0, 0, x_3, 0, x_5)\}$	$\{(x_0, 0, x_2, 0, 0, x_5) \mid x_2 \neq 0\}$	$\{(x_0, 0, 0, x_3, x_4, 0) \mid x_4 \neq 0\}$
C_{0-+}	$\{(x_0, 0, 0, x_3, x_4, 0)\}$	$\{(x_0, 0, 0, x_3, 0, x_5) \mid x_5 \neq 0\}$	$\{(0, x_1, 0, x_3, x_4, 0) \mid x_1 \neq 0\}$
C_{-0+}	$\{(0, x_1, 0, x_3, x_4, 0)\}$	$\{(x_0, 0, 0, x_3, x_4, 0) \mid x_0 \neq 0\}$	$\{(0, x_1, x_2, 0, x_4, 0) \mid x_2 \neq 0\}$
C_{-+0}	$\{(0, x_1, x_2, 0, x_4, 0)\}$	$\{(0, x_1, 0, x_3, x_4, 0) \mid x_3 \neq 0\}$	$\{(0, x_1, x_2, 0, 0, x_5) \mid x_5 \neq 0\}$

Figure 3.8: The KN stratifications of the unstable loci, for the one-dimensional cells.

3.5.3 Two-dimensional cells

Now, consider the two-dimensional cell C_{++-}, in which θ_0 and θ_1 are both strictly positive. The relevant 1-parameter subgroups to consider in this case are

	1-PS	μ
1)	$\lambda_\alpha(t) = (t^{n\theta_0/\theta_1}, t^n, t^{-n(\theta_0/\theta_1+1)})$	$\sqrt{2\theta_0^2 + 2\theta_0\theta_1 + 2\theta_1^2}$
2)	$\lambda_\beta(t) = (t^n, t^n, t^{-2n})$	$\frac{\sqrt{6}}{2}(\theta_0 + \theta_1)$
3)	$\lambda_\gamma(t) = (t^{2n}, t^{-n}, t^{-n})$	$\frac{\sqrt{6}}{2}\theta_0$
4)	$\lambda_\delta(t) = (t^{-n}, t^{2n}, t^{-n})$	$\frac{\sqrt{6}}{2}\theta_1$

and we encourage the reader to think for a moment about the choice of ordering in the corresponding KN strata. Note that λ_α always has the largest value for μ in this cell, and also that it coincides with λ_β when $\theta_0 = \theta_1$. Whether λ_γ or λ_δ provides the larger value for μ clearly depends on which of θ_0 and θ_1 is larger. Therefore, we have three subtly different KN stratifications in C_{++-}, corresponding to the cases $\theta_0 < \theta_1$, $\theta_0 = \theta_1$ and $\theta_0 > \theta_1$.[4] We now compute these three stratifications, then use the Weyl group symmetry of the problem to obtain the stratifications for all of the generic cells.

The case $\theta_0 < \theta_1$

In this case, we have

[4]To gain intuition as to why this is the case, consider that one of the KN strata will become semistable as we pass onto one adjoining one-dimensional wall, and that, near to the wall, this will be the least unstable stratum. Similarly, the stratum that becomes semistable as we pass onto the other wall bordering the cell will be the least unstable stratum near to this wall. The behaviour when $\theta_0 = \theta_1$ is the "cross-over point", equidistant from these two walls, when these two KN strata are equally unstable.

1) Max μ is given by $\lambda_1(t) = \lambda_\alpha(t)$.

$$X^{\lambda_1} = \{(0,0,0,0,0,0)\} =: Z_1$$
$$\text{Blade} = \{(0, x_1, x_2, 0, 0, x_5)\} =: S_1$$

2) Max μ given by $\lambda_2(t) = \lambda_\beta(t)$.

$$X^{\lambda_2} = \{(x_0,0,0,0,0,0)\} \cup \{(0,x_1,0,0,0,0)\}$$
$$X^{\lambda_2}(t) \setminus S_1 = \{(x_0,0,0,0,0,0) \mid x_0 \neq 0\} =: Z_2$$
$$\text{Blade} = \{(x_0,0,x_2,0,0,x_5) \mid x_0 \neq 0\} =: S_2$$

3) Max μ given by $\lambda_3(t) = \lambda_\delta(t)$.

$$X^{\lambda_3} = \{(0,0,0,0,x_4,0)\} \cup \{(0,0,0,0,0,x_5)\}$$
$$X^{\lambda_3} \setminus \bigcup_{i=1}^{2} S_i = \{(0,0,0,0,x_4,0) \mid x_4 \neq 0\} =: Z_3$$
$$\text{Blade} = \{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\} =: S_3$$

4) Max μ given by $\lambda_4(t) = \lambda_\gamma(t)$.

$$X^{\lambda_4} = \{(0,0,x_2,0,0,0)\} \cup \{(0,0,0,x_3,0,0)\}$$
$$X^{\lambda_4} \setminus \bigcup_{i=1}^{3} S_i = \{(0,0,0,x_3,0,0) \mid x_3 \neq 0\} =: Z_4$$
$$\text{Blade} = \{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\} =: S_4$$

The case $\theta_0 = \theta_1$

In this case, we have

1) Max μ is given by $\lambda_1(t) = \lambda_\alpha(t) = \lambda_\beta(t)$.

$$X^{\lambda_1} = \{(x_0,0,0,0,0,0)\} \cup \{(0,x_1,0,0,0,0)\} =: Z_1$$
$$\text{Blade} = \{(x_0,0,x_2,0,0,x_5)\} \cup \{(0,x_1,x_2,0,0,x_5)\} =: S_1$$

2) Max μ given by $\lambda_2(t) = \lambda_\gamma(t)$.

$$X^{\lambda_2} = \{(0,0,x_2,0,0,0)\} \cup \{(0,0,0,x_3,0,0)\}$$
$$X^{\lambda_2}(t) \setminus S_1 = \{(0,0,0,x_3,0,0) \mid x_3 \neq 0\} =: Z_2$$
$$\text{Blade} = \{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\} =: S_2$$

3) Max μ given by $\lambda_3(t) = \lambda_\delta(t)$.

$$X^{\lambda_3} = \{(0,0,0,0,x_4,0)\} \cup \{(0,0,0,0,0,x_5)\}$$
$$X^{\lambda_3} \setminus \bigcup_{i=1}^{2} S_i = \{(0,0,0,0,x_4,0) \mid x_4 \neq 0\} =: Z_3$$
$$\text{Blade} = \{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\} =: S_3$$

The case $\theta_0 > \theta_1$
In this case, we have

1) Max μ is given by $\lambda_1(t) = \lambda_\alpha(t)$.

$$X^{\lambda_1} = \{(0,0,0,0,0,0)\} =: Z_1$$
$$\text{Blade} = \{(x_0,0,x_2,0,0,x_5)\} =: S_1$$

2) Max μ given by $\lambda_2(t) = \lambda_\beta(t)$.

$$X^{\lambda_2} = \{(x_0,0,0,0,0,0)\} \cup \{(0,x_1,0,0,0,0)\}$$
$$X^{\lambda_2}(t) \setminus S_1 = \{(0,x_1,0,0,0,0) \mid x_1 \neq 0\} =: Z_2$$
$$\text{Blade} = \{(0,x_1,x_2,0,0,x_5) \mid x_1 \neq 0\} =: S_2$$

3) Max μ given by $\lambda_3(t) = \lambda_\gamma(t)$.

$$X^{\lambda_3} = \{(0,0,x_2,0,0,0)\} \cup \{(0,0,0,x_3,0,0)\}$$
$$X^{\lambda_3} \setminus \bigcup_{i=1}^{2} S_i = \{(0,0,0,x_3,0,0) \mid x_3 \neq 0\} =: Z_3$$
$$\text{Blade} = \{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\} =: S_3$$

4) Max μ given by $\lambda_4(t) = \lambda_\delta(t)$.

$$X^{\lambda_4} = \{(0,0,0,0,x_4,0)\} \cup \{(0,0,0,0,0,x_5)\}$$
$$X^{\lambda_4} \setminus \bigcup_{i=1}^{3} S_i = \{(0,0,0,0,x_4,0) \mid x_4 \neq 0\} =: Z_4$$
$$\text{Blade} = \{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\} =: S_4$$

Utilising the Weyl group symmetry of the problem, as previously discussed, we can write down the corresponding KN stratifications for the other generic cells. These KN stratifications are shown in Figure 3.9.

Cell	Case	S_1	S_2	S_3	S_4
C_{++-}	$\theta_0 < \theta_1$	$\{(0,x_1,x_2,0,0,x_5)\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_0 \neq 0\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\}$
	$\theta_0 = \theta_1$	$\{(0,x_1,x_2,0,0,x_5)\} \cup \{(x_0,0,x_2,0,0,x_5)\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\}$	–
	$\theta_0 > \theta_1$	$\{(x_0,0,x_2,0,0,x_5)\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_1 \neq 0\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\}$
C_{+--}	$\theta_2 < \theta_1$	$\{(x_0,0,x_2,0,0,x_5)\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_1 \neq 0\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_4 \neq 0\}$
	$\theta_2 = \theta_1$	$\{(x_0,0,x_2,0,0,x_5)\} \cup \{(x_0,0,0,x_3,0,x_5)\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_1 \neq 0\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_4 \neq 0\}$	–
	$\theta_2 > \theta_1$	$\{(x_0,0,0,x_3,0,x_5)\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_2 \neq 0\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_4 \neq 0\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_1 \neq 0\}$
C_{+-+}	$\theta_2 < \theta_0$	$\{(x_0,0,0,x_3,0,x_5)\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_4 \neq 0\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_2 \neq 0\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_1 \neq 0\}$
	$\theta_2 = \theta_0$	$\{(x_0,0,0,x_3,0,x_5)\} \cup \{(x_0,0,0,x_3,x_4,0)\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_2 \neq 0\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_1 \neq 0\}$	–
	$\theta_2 > \theta_0$	$\{(x_0,0,0,x_3,x_4,0)\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_5 \neq 0\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_1 \neq 0\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_2 \neq 0\}$
C_{--+}	$\theta_1 < \theta_0$	$\{(x_0,0,0,x_3,x_4,0)\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_1 \neq 0\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_5 \neq 0\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_2 \neq 0\}$
	$\theta_1 = \theta_0$	$\{(x_0,0,0,x_3,x_4,0)\} \cup \{(0,x_1,0,x_3,x_4,0)\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_5 \neq 0\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_2 \neq 0\}$	–
	$\theta_1 > \theta_0$	$\{(0,x_1,0,x_3,x_4,0)\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_0 \neq 0\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_2 \neq 0\}$	$\{(x_0,0,0,x_3,0,x_5) \mid x_5 \neq 0\}$
C_{-++}	$\theta_1 < \theta_2$	$\{(0,x_1,0,x_3,x_4,0)\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_2 \neq 0\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_0 \neq 0\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_5 \neq 0\}$
	$\theta_1 = \theta_2$	$\{(0,x_1,0,x_3,x_4,0)\} \cup \{(0,x_1,x_2,0,x_4,0)\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_0 \neq 0\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_5 \neq 0\}$	–
	$\theta_1 > \theta_2$	$\{(0,x_1,x_2,0,x_4,0)\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_3 \neq 0\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_5 \neq 0\}$	$\{(x_0,0,0,x_3,x_4,0) \mid x_0 \neq 0\}$
C_{-+-}	$\theta_0 < \theta_2$	$\{(0,x_1,x_2,0,x_4,0)\}$	$\{(0,x_1,x_2,0,0,x_5) \mid x_5 \neq 0\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_3 \neq 0\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_0 \neq 0\}$
	$\theta_0 = \theta_2$	$\{(0,x_1,x_2,0,x_4,0)\} \cup \{(0,x_1,x_2,0,0,x_5)\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_3 \neq 0\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_0 \neq 0\}$	–
	$\theta_0 > \theta_2$	$\{(0,x_1,x_2,0,0,x_5)\}$	$\{(0,x_1,x_2,0,x_4,0) \mid x_4 \neq 0\}$	$\{(x_0,0,x_2,0,0,x_5) \mid x_0 \neq 0\}$	$\{(0,x_1,0,x_3,x_4,0) \mid x_3 \neq 0\}$

Figure 3.9: The KN stratifications of the unstable loci, for the two-dimensional cells.

— Chapter 4 —

An $\mathcal{H}$-Schober for 2-dim. A_n-type McKay Correspondence

In this final chapter we give the construction of the $\mathcal{H}$-schober. We do this by considering the hyperplane arrangement $\mathcal{H}$ given by the wall-and-chamber decomposition of the space of stability conditions in Figure 3.1. Recall that we order the thirteen cells induced by $\mathcal{H}$ by inclusion of closures: $C_i \leq C_j$ if and only if $C_i \subseteq \overline{C_j}$. We now remind the reader of the definition of an $\mathcal{H}$-schober as given in Definition 1.4.0.3. To construct a schober on this particular hyperplane arrangement we need the data of a triangulated category $\mathcal{E}_{C_i}$ for each cell C_i and, for each $C_i \leq C_j$, an adjoint pair of functors $\gamma_{ij} : \mathcal{E}_{C_i} \to \mathcal{E}_{C_j}$ and $\delta_{ji} : \mathcal{E}_{C_j} \to \mathcal{E}_{C_i}$ such that

i) All the induced triangles commute for the γ and δ functors.

ii) For all $C_i \leq C_j$, $\gamma_{ij}\delta_{ji} \simeq \mathrm{id}_{\mathcal{E}_{C_j}}$; thus, for any two cells C_i and C_j, the flopping functors $\varphi_{ij} := \gamma_{kj}\delta_{ik}$ are well-defined, where C_k is any choice of cell such that $C_k \leq C_i, C_j$.

iii) Collinear transitivity holds.

iv) For generic cells separated by a dimension one cell, and for one-dimensional cells and their opposite one-dimensional cell, the flopping functors are equivalences.

In our schober, we take the triangulated categories to be the derived categories of the quotient stacks for the semistable loci in each of the thirteen cells. The maps γ_{ij} in this schober are given by the open restrictions. In Section 4.1 we construct partial compactifications for each cell and use these to construct the maps δ_{ji}. In Section 4.2 we consider for which window subcategories G_w the extension of the structure sheaf as the structure sheaf is the correct one to take, and conjecture that the δ_{ji} we have constructed are the isomorphisms $\delta_{ji} : D^b([X_i^{ss}/G]) \xrightarrow{\sim} G_w$ where we choose $w_k = 0$ for each unstable stratum S_k. In Section 4.3 we give the formal construction of the schober we've just described, and in Section 4.4 we conclude this thesis and give some possible directions for future work.

As all points of the pertinent group $G = \{(a,b,c)\}/\sim$ can be written as $(1,b,c)$ without loss of generality, we choose this identification $G \simeq \mathbb{G}_m^2 = \{(b,c)\}$ in this chapter. Under this identification, the action of G on X given in (3.2) becomes

$$(b,c)\cdot(x_0,x_1,x_2,x_3,x_4,x_5) = \Big(\frac{1}{b}x_0, bx_1, \frac{b}{c}x_2, \frac{c}{b}x_3, cx_4, \frac{1}{c}x_5\Big). \tag{4.1}$$

As the labelling of the cells given in Chapter 3 will prove notationally cumbersome in the work presented in this chapter, we relabel the cells in the following way:

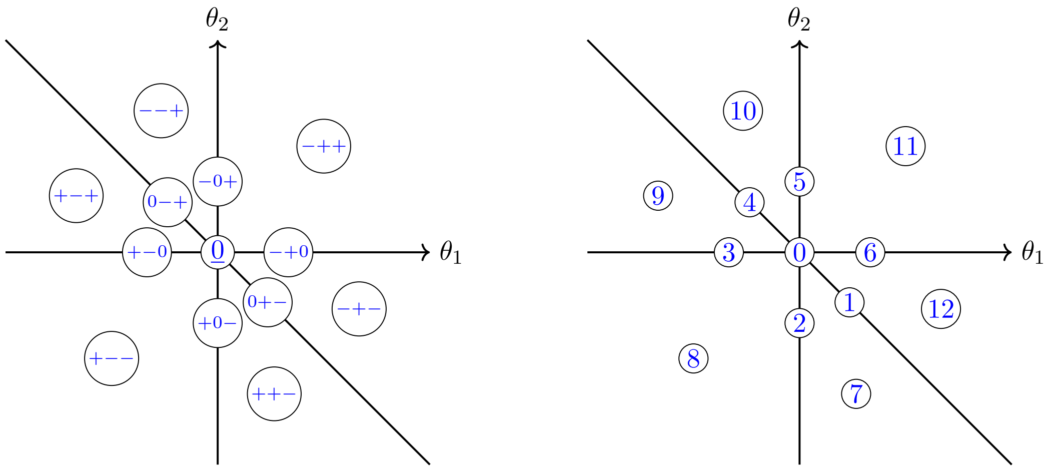


Figure 4.1: Left: the labelling of the cells used in Chapter 3, as given in Figure 3.1. Right: the relabelling used in this chapter.

4.1 Partial compactifications and kernels

In this section we construct the Fourier-Mukai kernels of the maps δ_{ji} we will have in our schober; we do this by constructing partial compactifications of the group action.

4.1.1 Partial compactifications and the kernels for maps into the central stack

In this subsection we construct the Fourier-Mukai kernels for the functors going into the central stack in our schober. We do this by extending the equivariant

diagonal $\mathcal{O}_{\Delta G} \in D^b([X_i^{ss} \times X_i^{ss}/G \times G])$ to an object in $D^b_{QCoh}([X_i^{ss} \times X/G \times G])$. It might be tempting to choose this extension to be the (equivariant) graph; however, the functor going into the big stack would then be the pushforward along the open immersion $X_i^{ss} \hookrightarrow X$. In general, there is no reason for such a pushforward to land in the *coherent* derived category. We therefore construct partial compactifications of the group action, and show that the functors given by the corresponding kernels do indeed land in the coherent derived category. We now construct a partial compactification for each non-zero cell C_i.
For an arbitrary group G acting on some X with corresponding action/projection maps $\pi, \sigma : G \times X \to X$, recall from Definition 1.6.0.14 that $\mathcal{O}_{\Delta G} := (\pi, \sigma)_* \mathcal{O}_{G \times X}$. This is the pushforward along a $G \times G$-equivariant map, where the $G \times G$ action on $G \times X$ was given by

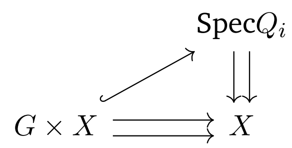
$$\begin{aligned}(G \times G) \times (G \times X) &\to G \times X \\ (g_1, g_2, g, x) &\mapsto (g_2 g g_1^{-1}, g_1 \cdot x).\end{aligned}$$

For $X = \text{Spec} R$ and $G = \text{Spec} S$, where $R = \mathbb{C}[f_0, \ldots, f_5]/(f_0 f_1 = f_2 f_3 = f_4 f_5)$ and $S = \mathbb{C}[s^{\pm}, t^{\pm}]$, the corresponding coaction $R[s^{\pm}, t^{\pm}] \to R[s^{\pm}, t^{\pm}, s_1^{\pm}, t_1^{\pm}, s_2^{\pm}, t_2^{\pm}]$ is a $\mathbb{Z}^4$ grading on the elements of $R[s^{\pm}, t^{\pm}]$ by Lemma 2.1.0.11. We give this grading in the following table. We also introduce rescalings of the f_i which will naturally appear in the calculations to come. It will be convenient for these to have a label, so we denote the rescaled version of f_i by g_i.

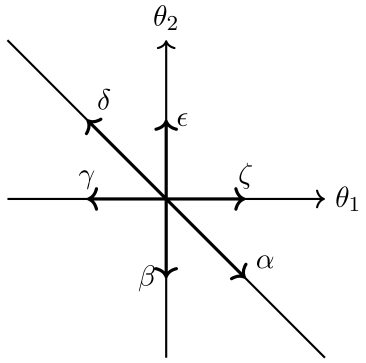
Element	Grading	Element	Grading
s	$(-1, 0, 1, 0)$	t	$(0, -1, 0, 1)$
f_0	$(-1, 0, 0, 0)$	$g_0 = s^{-1} f_0$	$(0, 0, -1, 0)$
f_1	$(1, 0, 0, 0)$	$g_1 = s f_1$	$(0, 0, 1, 0)$
f_2	$(1, -1, 0, 0)$	$g_2 = s t^{-1} f_2$	$(0, 0, 1, -1)$
f_3	$(-1, 1, 0, 0)$	$g_3 = s^{-1} t f_3$	$(0, 0, -1, 1)$
f_4	$(0, 1, 0, 0)$	$g_4 = t f_4$	$(0, 0, 0, 1)$
f_5	$(0, -1, 0, 0)$	$g_5 = t^{-1} f_5$	$(0, 0, 0, -1)$

Figure 4.2: The $\mathbb{Z}^4$-grading on elements of $R[s^{\pm}, t^{\pm}]$, corresponding to the $G \times G$-action on $G \times X$.

Recalling Definition 2.3.1.4, we will construct our partial compactifications as the spectra of certain subalgebras $Q_i \subseteq R[s^{\pm}, t^{\pm}]$ corresponding to finitely generated submonoids of the space of characters. This gives an extension of the group action and projection maps as follows:



These subalgebras have the $\mathbb{Z}^4$ grading induced by the grading of $R[s^{\pm}, t^{\pm}]$. We first identify the space of characters of G with the monomial lattice in $S = \mathbb{C}[s^{\pm}, t^{\pm}]$, and so cones in the stability space give cones in the monomial lattice. We choose the submonoids for the generic cells in the obvious way - the submonoids given by the generators of each of the cells themselves in the wall-and-chamber decomposition of the space of characters, under this identification with the monomial lattice in S. As our group action is that of $G = \mathbb{G}_m^2$, we are compactifying the action of two copies of $\mathbb{G}_m$ in each case and we require a choice of two generators for each cell. For each one-dimensional cell we therefore take the generators for the union of the two generic cells bordering it. The choice of generators is depicted in Figure 4.3.



$$\alpha = (0, 1, -1) \rightsquigarrow s^{-1}t$$
$$\beta = (1, 0, -1) \rightsquigarrow t$$
$$\gamma = (1, -1, 0) \rightsquigarrow s$$
$$\delta = (0, -1, 1) \rightsquigarrow st^{-1}$$
$$\epsilon = (-1, 0, 1) \rightsquigarrow t^{-1}$$
$$\zeta = (-1, 1, 0) \rightsquigarrow s^{-1}$$

Figure 4.3: Left: the wall-and-chamber decomposition of the space of characters, with the generators of each generic cell depicted. Right: the corresponding element of $R[s^{\pm}, t^{\pm}]$.

Recalling the G action on X given by (4.1):

$$(b, c) \cdot (x_0, x_1, x_2, x_3, x_4, x_5) = \Big(\frac{1}{b}x_0, bx_1, \frac{b}{c}x_2, \frac{c}{b}x_3, cx_4, \frac{1}{c}x_5\Big),$$

the corresponding coaction map $\hat{\sigma} : R \to R[s^{\pm}, t^{\pm}]$ is given by

$$f_0 \mapsto s^{-1}f_0 \qquad f_3 \mapsto s^{-1}tf_3$$
$$f_1 \mapsto sf_1 \qquad f_4 \mapsto tf_4$$
$$f_2 \mapsto st^{-1}f_2 \qquad f_5 \mapsto t^{-1}f_5$$

We thus write down the partial compactifications SpecQ_i to use in the following table.

Cell	$Q_i \subseteq R[s^{\pm}, t^{\pm}]$	Cell	$Q_i \subseteq R[s^{\pm}, t^{\pm}]$
C_1	$R[s^{-1}, t, \hat{\sigma}(R)]$	C_7	$R[s^{-1}t, t, \hat{\sigma}(R)]$
C_2	$R[s, s^{-1}t, \hat{\sigma}(R)]$	C_8	$R[s, t, \hat{\sigma}(R)]$
C_3	$R[st^{-1}, t, \hat{\sigma}(R)]$	C_9	$R[s, st^{-1}, \hat{\sigma}(R)]$
C_4	$R[s, t^{-1}, \hat{\sigma}(R)]$	C_{10}	$R[st^{-1}, t^{-1}, \hat{\sigma}(R)]$
C_5	$R[s^{-1}, st^{-1}, \hat{\sigma}(R)]$	C_{11}	$R[t^{-1}, s^{-1}, \hat{\sigma}(R)]$
C_6	$R[s^{-1}t, t^{-1}, \hat{\sigma}(R)]$	C_{12}	$R[s^{-1}, s^{-1}t, \hat{\sigma}(R)]$

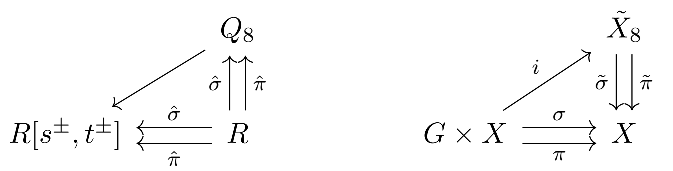
In this section, we investigate what these partial compactifications are geometrically, and what the extended action and projection maps are. We do the generic Cell 8 as our first example as the generators of the submonoid for Q_8 have the nicest description. We then proceed with the remaining cells.

4.1.1.1 Cell 8

Recall that $R = \mathbb{C}[f_0, \ldots, f_5]/J$ for $J = (f_0f_1 = f_2f_3 = f_4f_5)$. In this case we take

$$\begin{aligned} Q_8 &= R[s, t, \hat{\sigma}(R)] \subseteq R[s^{\pm}, t^{\pm}] \\ &\simeq \mathbb{C}[s, t, f_0, s^{-1}f_0, f_1, sf_1, f_2, st^{-1}f_2, f_3, s^{-1}tf_3, f_4, tf_4, f_5, t^{-1}f_5]/J \\ &\simeq \mathbb{C}[s, t, s^{-1}f_0, f_1, f_2, st^{-1}f_2, f_3, s^{-1}tf_3, f_4, t^{-1}f_5]/J \\ &\simeq \mathbb{C}[s, t, g_0, f_1, f_2, g_2, f_3, g_3, f_4, g_5]/K_8 \end{aligned} \tag{4.2}$$

where K_8 is the ideal of relations $(sg_0f_1 = f_2f_3 = g_2g_3 = tf_4g_5, tg_2 = sf_2, sg_3 = tf_3)$. The obvious ring maps induce the partial compactification diagram for $\tilde{X}_8 :=$ SpecQ_8



Using the coordinates corresponding to (4.2), the partial compactification is the following closed subvariety of $\mathbb{A}^{10}$:

$$\begin{aligned} \tilde{X}_8 = \{(h_1, h_2, c_0, x_1, x_2, c_2, x_3, c_3, x_4, c_5) \in \mathbb{A}^{10} \mid h_1c_0x_1 = x_2x_3 = c_2c_3 = h_2x_4c_5, \\ h_2c_2 = h_1x_2, \\ h_1c_3 = h_2x_3\} \end{aligned}$$

and the maps (of schemes) are given on closed points by

$$\begin{aligned} i(b,c,x_0,x_1,x_2,x_3,x_4,x_5) &= (b,c,b^{-1}x_0,x_1,x_2,bc^{-1}x_2,x_3,b^{-1}cx_3,x_4,c^{-1}x_5) \\ \tilde{\pi}(h_1,h_2,c_0,x_1,x_2,c_2,x_3,c_3,x_4,c_5) &= (h_1c_0,x_1,x_2,x_3,x_4,h_2c_5) \\ \tilde{\sigma}(h_1,h_2,c_0,x_1,x_2,c_2,x_3,c_3,x_4,c_5) &= (c_0,h_1x_1,c_2,c_3,h_2x_4,c_5). \end{aligned}$$

A direct check verifies that everything in $\tilde{X}_8$ with $h_1h_2 \neq 0$ is in the image of i. Clearly nothing with $h_1h_2 = 0$ is in this image, so the boundary $\partial = \tilde{X}_8 \setminus i(G \times X)$ is exactly this set of points. Under $\tilde{\sigma}$, all points in the boundary land in the unstable locus X_8^{us}, as expected.

We take our initial kernel to be the open restriction of $(\tilde{\pi},\tilde{\sigma})_*\mathcal{O}_{\tilde{X}_8}$ to $X_8^{ss} \times X$. Defining $\tilde{X}_8^{ss}$ as the fibre product of the following square on the right, we see that our initial kernel is isomorphic to $(\tilde{\pi},\tilde{\sigma})_*\mathcal{O}_{\tilde{X}_8^{ss}}$ by flat base change.

$$\begin{array}{ccccc} G \times X_8^{ss} & \hookrightarrow & \tilde{X}_8^{ss} & \hookrightarrow & \tilde{X}_8 \\ \downarrow{\scriptstyle(\pi,\sigma)} & & \downarrow{\scriptstyle(\tilde{\pi},\tilde{\sigma})} & & \downarrow{\scriptstyle(\tilde{\pi},\tilde{\sigma})} \\ X_8^{ss} \times X_8^{ss} & \hookrightarrow & X_8^{ss} \times X & \hookrightarrow & X \times X \end{array} \tag{4.3}$$

Referring to the description of X_8^{ss} given on p.70, the set of points in $\tilde{X}_8^{ss}$ which land in X_8^{ss} under $\tilde{\pi}$ gives an explicit description of the closed points of $\tilde{X}_8^{ss} \subseteq \tilde{X}_8$ as

$$\tilde{X}_8^{ss} = \{(h_1,h_2,c_0,x_1,x_2,c_2,x_3,c_3,x_4,c_5) \in \tilde{X}_8 \mid x_1x_3 \neq 0 \text{ or } x_1x_4 \neq 0 \text{ or } x_2x_4 \neq 0\}. \tag{4.4}$$

As all points in ∂ are mapped to the unstable locus under $\tilde{\sigma}$, the left square in (4.3) is also fibre. Therefore the restriction of our initial kernel to $X_8^{ss} \times X_8^{ss}$ is the equivariant diagonal. We have therefore constructed an extension of the equivariant diagonal $\mathcal{O}_{\Delta G} \in D^b([X_8^{ss} \times X_8^{ss}/G \times G])$ to the space of Fourier-Mukai kernels, $D^b([X^{ss} \times X/G \times G])$. The construction of the kernels for each of the other cells is identical, and in the rest of this section we state what each of the partial compactifications $\tilde{X}_i$ are, as well as the corresponding extensions of the action and projection maps.

4.1.1.2 Cell 1

In this cell our subalgebra $Q_1 \subseteq R[s^{\pm},t^{\pm}]$ is given by

$$\begin{aligned} Q_1 &= R[s^{-1},t,\hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s^{-1},t,f_0,g_1,g_2,f_3,f_4,g_5]/K_1 \end{aligned}$$

for $K_1 = (s^{-1}f_0g_1 = s^{-1}tg_2f_3 = tf_4g_5)$, and thus

$$\tilde{X}_1 = \{(h_1, h_2, x_0, c_1, c_2, x_3, x_4, c_5) \mid h_1x_0c_1 = h_1h_2c_2x_3 = h_2x_4c_5\},$$

where the maps in the partial compactification diagram are given by

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b^{-1}, c, x_0, bx_1, bc^{-1}x_2, x_3, x_4, c^{-1}x_5) \\ \tilde{\pi}(h_1, h_2, x_0, c_1, c_2, x_3, x_4, c_5) &= (x_0, h_1c_1, h_1h_2c_2, x_3, x_4, h_2c_5) \\ \tilde{\sigma}(h_1, h_2, x_0, c_1, c_2, x_3, x_4, c_5) &= (h_1x_0, c_1, c_2, h_1h_2x_3, h_2x_4, c_5). \end{aligned}$$

4.1.1.3 Cell 2

In this case we have

$$\begin{aligned} Q_2 &= R[s^{-1}t, s, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s, s^{-1}t, g_0, f_1, g_2, f_3, f_4, g_5]/K_2 \end{aligned}$$

for $K_2 = (sg_0f_1 = s^{-1}tg_2f_3 = (s)(s^{-1}t)f_4g_5)$, and thus

$$\tilde{X}_2 = \{(h_1, h_2, c_0, x_1, c_2, x_3, x_4, c_5) \mid h_1c_0x_1 = h_2c_2x_3 = h_1h_2x_4c_5\},$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b, b^{-1}c, b^{-1}x_0, x_1, bc^{-1}x_2, x_3, x_4, c^{-1}x_5) \\ \tilde{\pi}(h_1, h_2, c_0, x_1, c_2, x_3, x_4, c_5) &= (h_1c_0, x_1, h_2c_2, x_3, x_4, h_1h_2c_5) \\ \tilde{\sigma}(h_1, h_2, c_0, x_1, c_2, x_3, x_4, c_5) &= (c_0, h_1x_1, c_2, h_2x_3, h_1h_2x_4, c_5). \end{aligned}$$

4.1.1.4 Cell 3

In this case we have

$$\begin{aligned} Q_3 &= R[st^{-1}, t, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[st^{-1}, t, g_0, f_1, f_2, g_3, f_4, g_5]/K_3 \end{aligned}$$

for $K_3 = ((st^{-1})tg_0f_1 = (st^{-1})f_2g_3 = tf_4g_5)$, and thus

$$\tilde{X}_3 = \{(h_1, h_2, c_0, x_1, x_2, c_3, x_4, c_5) \mid h_1h_2c_0x_1 = h_1x_2c_3 = h_2x_4c_5\},$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (bc^{-1}, c, b^{-1}x_0, x_1, x_2, b^{-1}cx_3, x_4, c^{-1}x_5) \\ \tilde{\pi}(h_1, h_2, c_0, x_1, x_2, c_3, x_4, c_5) &= (h_1h_2c_0, x_1, x_2, h_1c_3, x_4, h_2c_5) \\ \tilde{\sigma}(h_1, h_2, c_0, x_1, x_2, c_3, x_4, c_5) &= (c_0, h_1h_2x_1, h_1x_2, c_3, h_2x_4, c_5). \end{aligned}$$

4.1.1.5 Cell 4

In this case we have

$$\begin{aligned} Q_4 &= R[s, t^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s, t^{-1}, g_0, f_1, f_2, g_3, g_4, f_5]/K_4 \end{aligned}$$

for $K_4 = (sg_0f_1 = st^{-1}f_2g_3 = t^{-1}g_4f_5)$, and thus

$$\tilde{X}_4 = \{(h_1, h_2, c_0, x_1, x_2, c_3, c_4, x_5) \mid h_1c_0x_1 = h_1h_2x_2c_3 = h_2c_4x_5\},$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b, c^{-1}, b^{-1}x_0, x_1, x_2, b^{-1}cx_3, cx_4, x_5) \\ \tilde{\pi}(h_1, h_2, c_0, x_1, x_2, c_3, c_4, x_5) &= (h_1c_0, x_1, x_2, h_1h_2c_3, h_2c_4, x_5) \\ \tilde{\sigma}(h_1, h_2, c_0, x_1, x_2, c_3, c_4, x_5) &= (c_0, h_1x_1, h_1h_2x_2, c_3, c_4, h_2x_5). \end{aligned}$$

4.1.1.6 Cell 5

In this case we have

$$\begin{aligned} Q_5 &= R[s^{-1}, st^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s^{-1}, st^{-1}, f_0, g_1, f_2, g_3, g_4, f_5]/K_5 \end{aligned}$$

for $K_5 = (s^{-1}f_0g_1 = st^{-1}f_2g_3 = s^{-1}(st^{-1})g_4f_5)$, and thus

$$\tilde{X}_5 = \{(h_1, h_2, x_0, c_1, x_2, c_3, c_4, x_5) \mid h_1x_0c_1 = h_2x_2c_3 = h_1h_2c_4x_5\},$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b^{-1}, bc^{-1}, x_0, bx_1, x_2, b^{-1}cx_3, cx_4, x_5) \\ \tilde{\pi}(h_1, h_2, x_0, c_1, x_2, c_3, c_4, x_5) &= (x_0, h_1c_1, x_2, h_2c_3, h_1h_2c_4, x_5) \\ \tilde{\sigma}(h_1, h_2, x_0, c_1, x_2, c_3, c_4, x_5) &= (h_1x_0, c_1, h_2x_2, c_3, c_4, h_1h_2x_5). \end{aligned}$$

4.1.1.7 Cell 6

In this case we have

$$\begin{aligned} Q_6 &= R[s^{-1}t, t^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s^{-1}t, t^{-1}, f_0, g_1, g_2, f_3, g_4, f_5]/K_6 \end{aligned}$$

for $K_6 = ((s^{-1}t)t^{-1}f_0g_1 = s^{-1}tg_2f_3 = t^{-1}g_4f_5)$, and thus

$$\tilde{X}_6 = \{(h_1, h_2, x_0, c_1, c_2, x_3, c_4, x_5) \mid h_1h_2x_0c_1 = h_1c_2x_3 = h_2c_4x_5\},$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b^{-1}c, c^{-1}, x_0, bx_1, bc^{-1}x_2, x_3, cx_4, x_5) \\ \tilde{\pi}(h_1, h_2, x_0, c_1, c_2, x_3, c_4, x_5) &= (x_0, h_1h_2c_1, h_1c_2, x_3, h_2c_4, x_5) \\ \tilde{\sigma}(h_1, h_2, x_0, c_1, c_2, x_3, c_4, x_5) &= (h_1h_2x_0, c_1, c_2, h_1x_3, c_4, h_2x_5). \end{aligned}$$

4.1.1.8 Cell 7

In this case we have

$$\begin{aligned} Q_7 &= R[s^{-1}t, t, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s^{-1}t, t, f_0, g_0, f_1, g_1, g_2, f_3, f_4, g_5]/K_7 \end{aligned}$$

for $K_7 = (f_0f_1 = g_0g_1 = s^{-1}tg_2f_3 = tf_4g_5,\ s^{-1}tf_0 = tg_0,\ tf_1 = s^{-1}tg_1)$, and thus

$$\begin{aligned} \tilde{X}_7 = \{(h_1, h_2, x_0, c_0, x_1, c_1, c_2, x_3, x_4, c_5) \mid x_0x_1 = c_0c_1 = h_1c_2x_3 &= h_2x_4c_5, \\ h_1x_0 &= h_2c_0, \\ h_1c_1 &= h_2x_1\}, \end{aligned}$$

with

$$\begin{aligned} i(b, c, x_0, x_1, x_2, x_3, x_4, x_5) &= (b^{-1}c, c, x_0, b^{-1}x_0.x_1, bx_1, bc^{-1}x_2, x_3, x_4, c^{-1}x_5) \\ \tilde{\pi}(h_1, h_2, x_0, c_0, x_1, c_1, c_2, x_3, x_4, c_5) &= (x_0, x_1, h_1c_2, x_3, x_4, h_2c_5) \\ \tilde{\sigma}(h_1, h_2, x_0, c_0, x_1, c_1, c_2, x_3, x_4, c_5) &= (c_0, c_1, c_2, h_1x_3, h_2x_4, c_5). \end{aligned}$$

4.1.1.9 Cell 9

In this case we have

$$\begin{aligned} Q_9 &= R[s, st^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s, st^{-1}, g_0, f_1, f_2, g_3, f_4, g_4, f_5, g_5]/K_9 \end{aligned}$$

for $K_9 = (sg_0f_1 = st^{-1}f_2g_3 = f_4f_5 = g_4g_5,\ st^{-1}g_4 = sf_4,\ sg_5 = st^{-1}f_5)$, and thus

$$\begin{aligned} \tilde{X}_9 = \{(h_1, h_2, c_0, x_1, x_2, c_3, x_4, c_4, x_5, c_5) \mid h_1c_0x_1 = h_2x_2c_3 = x_4x_5 &= c_4c_5, \\ h_2c_4 &= h_1x_4, \\ h_1c_5 &= h_2x_5\}, \end{aligned}$$

with

$$\begin{aligned} i(b,c,x_0,x_1,x_2,x_3,x_4,x_5) &= (b, bc^{-1}, b^{-1}x_0, x_1, x_2, b^{-1}cx_3, x_4, cx_4, x_5, c^{-1}x_5) \\ \tilde{\pi}(h_1,h_2,c_0,x_1,x_2,c_3,x_4,c_4,x_5,c_5) &= (h_1c_0, x_1, x_2, h_2c_3, x_4, x_5) \\ \tilde{\sigma}(h_1,h_2,c_0,x_1,x_2,c_3,x_4,c_4,x_5,c_5) &= (c_0, h_1x_1, h_2x_2, c_3, c_4, c_5). \end{aligned}$$

4.1.1.10 Cell 10

In this case we have

$$\begin{aligned} Q_{10} &= R[st^{-1}, t^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[st^{-1}, t^{-1}, f_0, g_0, f_1, g_1, f_2, g_3, g_4, f_5]/K_{10} \end{aligned}$$

for $K_{10} = (f_0f_1 = g_0g_1 = st^{-1}f_2g_3 = t^{-1}g_4f_5,\ st^{-1}g_0 = t^{-1}f_0,\ t^{-1}g_1 = st^{-1}f_1)$, and thus

$$\begin{aligned} \tilde{X}_{10} = \{(h_1,h_2,x_0,c_0,x_1,c_1,x_2,c_3,c_4,x_5) \mid x_0x_1 = c_0c_1 = h_1x_2c_3 = h_2c_4x_5,& \\ h_1c_0 = h_2x_0,& \\ h_2c_1 = h_1x_1\},& \end{aligned}$$

with

$$\begin{aligned} i(b,c,x_0,x_1,x_2,x_3,x_4,x_5) &= (bc^{-1}, c^{-1}, x_0, b^{-1}x_0, x_1, bx_1, x_2, b^{-1}cx_3, cx_4, x_5) \\ \tilde{\pi}(h_1,h_2,x_0,c_0,x_1,c_1,x_2,c_3,c_4,x_5) &= (x_0, x_1, x_2, h_1c_3, h_2c_4, x_5) \\ \tilde{\sigma}(h_1,h_2,x_0,c_0,x_1,c_1,x_2,c_3,c_4,x_5) &= (c_0, c_1, h_1x_2, c_3, c_4, h_2x_5). \end{aligned}$$

4.1.1.11 Cell 11

In this case we have

$$\begin{aligned} Q_{11} &= R[t^{-1}, s^{-1}, \hat{\sigma}(R)] \\ &\simeq \mathbb{C}[t^{-1}, s^{-1}, f_0, g_1, f_2, g_2, f_3, g_3, g_4, f_5]/K_{11} \end{aligned}$$

for $K_{11} = (s^{-1}f_0g_1 = f_2f_3 = g_2g_3 = t^{-1}g_4f_5,\ s^{-1}g_2 = t^{-1}f_2,\ t^{-1}g_3 = s^{-1}f_3)$, and thus

$$\begin{aligned} \tilde{X}_{11} = \{(h_1,h_2,x_0,c_1,x_2,c_2,x_3,c_3,c_4,x_5) \mid h_2x_0c_1 = x_2x_3 = c_2c_3 = h_1c_4x_5,& \\ h_2c_2 = h_1x_2,& \\ h_1c_3 = h_2x_3\},& \end{aligned}$$

with

$$\begin{aligned} i(b,c,x_0,x_1,x_2,x_3,x_4,x_5) &= (c^{-1},b^{-1},x_0,bx_1,x_2,bc^{-1}x_2,x_3,b^{-1}cx_3,cx_4,x_5) \\ \tilde{\pi}(h_1,h_2,x_0,c_1,x_2,c_2,x_3,c_3,c_4,x_5) &= (x_0,h_2c_1,x_2,x_3,h_1c_4,x_5) \\ \tilde{\sigma}(h_1,h_2,x_0,c_1,x_2,c_2,x_3,c_3,c_4,x_5) &= (h_2x_0,c_1,c_2,c_3,c_4,h_1x_5). \end{aligned}$$

4.1.1.12 Cell 12

In this case we have

$$\begin{aligned} Q_{12} &= R[s^{-1},s^{-1}t,\hat{\sigma}(R)] \\ &\simeq \mathbb{C}[s^{-1},s^{-1}t,f_0,g_1,g_2,f_3,f_4,g_4,f_5,g_5]/K_{12} \end{aligned}$$

for $K_{12} = (s^{-1}f_0g_1 = s^{-1}tg_2f_3 = f_4f_5 = g_4g_5,\ s^{-1}g_4 = s^{-1}tf_4,\ s^{-1}tg_5 = s^{-1}f_5)$, and thus

$$\begin{aligned} \tilde{X}_{12} = \{(h_1,h_2,x_0,c_1,c_2,x_3,x_4,c_4,x_5,c_5) \mid h_1x_0c_1 = h_2c_2x_3 = x_4x_5 &= c_4c_5, \\ h_1c_4 &= h_2x_4, \\ h_2c_5 &= h_1x_5\}, \end{aligned}$$

with

$$\begin{aligned} i(b,c,x_0,x_1,x_2,x_3,x_4,x_5) &= (b^{-1},b^{-1}c,x_0,bx_1,bc^{-1}x_2,x_3,x_4,cx_4,x_5,c^{-1}x_5) \\ \tilde{\pi}(h_1,h_2,x_0,c_1,c_2,x_3,x_4,c_4,x_5,c_5) &= (x_0,h_1c_1,h_2c_2,x_3,x_4,x_5) \\ \tilde{\sigma}(h_1,h_2,x_0,c_1,c_2,x_3,x_4,c_4,x_5,c_5) &= (h_1x_0,c_1,c_2,h_2x_3,c_4,c_5). \end{aligned}$$

4.1.2 The functors restrict to coherent derived categories

We now show that the functors given by the kernels we've just constructed restrict to functors between the coherent derived categories.

Lemma 4.1.2.1. *The partial compactifications we have just constructed each give a kernel which is an extension of the relevant equivariant diagonal $\mathcal{O}_{\Delta G} \in D^b([X_i^{ss} \times X_i^{ss}/G \times G])$.*

Proof. The argument for each of the kernels is the same as we saw for $\tilde{X}_8$. In each case the kernel is given by $(\tilde{\pi},\tilde{\sigma})_*\mathcal{O}_{\tilde{X}_i^{ss}}$, where $\tilde{X}_i^{ss}$ is defined by the relevant fibre diagram. As the boundary ∂ is mapped to the unstable locus under $\tilde{\sigma}$ in each case, the following square is fibre

$$\begin{array}{ccc} G \times X_i^{ss} & \hookrightarrow & \tilde{X}_i^{ss} \\ {\scriptstyle (\pi,\sigma)}\downarrow & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\ X_i^{ss} \times X_i^{ss} & \hookrightarrow & X_i^{ss} \times X \end{array}$$

and the result follows by flat base change. □

We now prove a useful lemma about the functors given by the kernels we have just constructed.

Lemma 4.1.2.2. *Each of the Fourier-Mukai transforms given by the kernels we have just constructed maps the structure sheaf to the structure sheaf, i.e.*

$$\Phi_{(\tilde{\pi},\tilde{\sigma})_*\mathcal{O}_{\tilde{X}_i^{ss}}}(\mathcal{O}_{X_i^{ss}}) = \mathcal{O}_X$$

for all cells C_i.

Proof. Using the projection formula on the following Fourier-Mukai diagram

$$\begin{array}{ccccc} & & \tilde{X}_i^{ss} & & \\ & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} & & \\ & & X_i^{ss} \times X & & \\ & \swarrow & & \searrow & \\ X_i^{ss} & & & & X \end{array}$$

we see that $\Phi_{(\tilde{\pi},\tilde{\sigma})_*\mathcal{O}_{\tilde{X}_i^{ss}}}(\mathcal{O}_{X_i^{ss}}) \simeq \tilde{\sigma}_*^G\mathcal{O}_{\tilde{X}_i^{ss}}$, where $(-)^G$ refers to taking invariants with respect to the left copy of G in the $G \times G$-action. Observe that $\Gamma(\tilde{X}_i^{ss}, \mathcal{O}_{\tilde{X}_i^{ss}}) \simeq Q_i$ and X is affine, so computing the zeroth direct image $\mathbb{R}^0\tilde{\sigma}_*^G\mathcal{O}_{\tilde{X}_i^{ss}}$ corresponds to taking the degree $(0,0,*,*)$ part of Q_i with respect to the grading shown in Figure 4.2. A direct check verifies that this is $\mathbb{C}[g_0,\ldots,g_5]/(g_0g_1 = g_2g_3 = g_4g_5) \simeq R$ in each case, and so $\mathbb{R}^0\tilde{\sigma}_*^G\mathcal{O}_{\tilde{X}_i^{ss}} \simeq \mathcal{O}_X$. Computing the Čech resolution for generic cells using the affine cover of $\tilde{X}_i^{ss}$ by three pieces[1], we find that it is exact in degree $p \neq 0, 1$ and thus all higher direct images $\mathbb{R}^p\tilde{\sigma}$ for $p \geq 2$ vanish [Har77, III, Proposition 8.7], so in particular their degree $(0,0,*,*)$ parts are also zero. In degree 0 the cohomology is Q_i, which recovers our earlier computation. In degree 1 the cohomology is non-zero, but nothing in it has degree $(0,0,*,*)$; therefore all direct images $\mathbb{R}^p\tilde{\sigma}^G$ for $p \geq 1$ vanish. For one-dimensional cells the computation is similar, with the affine cover given by two pieces. □

[1]c.f. the description of $\tilde{X}_8^{ss}$ on p.86. Here the affine cover by three pieces is given by $x_1x_3 \neq 0$, $x_1x_4 \neq 0$ and $x_2x_4 \neq 0$.

Corollary 4.1.2.3. *The functors given by these kernels restrict to functors between the coherent derived categories.*

Proof. As $D^b([X_i^{ss}/G])$ is generated by the structure sheaf with different equivariant structures, we need only see where these are mapped to in $D^b_{QCoh}([X/G])$. The pullback of these structure sheaves to $\tilde{X}_i^{ss}$ corresponds to shifts of the grading in the first two entries of the $\mathbb{Z}^4$-grading on Q_i. The computation of the direct images is now the same as the preceding lemma, except we take the $(i, j, *, *)$-graded part of the cohomologies of the Čech resolution, for $i, j \in \mathbb{Z}$. These are finitely generated, hence the corresponding sheaves are coherent, and the result follows. □

4.1.3 Kernels for maps from generic cells onto the walls

In the preceding section we determined kernels for the Fourier-Mukai transform going from any cell into $D^b([X/G])$. To construct our schober we also require kernels for the transforms $D^b([X_i^{ss}/G]) \to D^b([X_j^{ss}/G])$ for C_i generic and C_j an adjacent one-dimensional wall, and we obtain these by restrictions of the kernels we have already determined, as follows. As we define these functors by mapping into the big stack and then restricting, these kernels automatically restrict to functors between the coherent derived categories. The geometry of these kernels is particularly nice, as we now see.

4.1.3.1 Cell 7 to Cell 1

We do the example of $D^b([X_7^{ss}/G]) \to D^b([X_1^{ss}/G])$. Recall that the kernel of the transform $D^b([X_7^{ss}/G]) \to D^b([X/G])$ was given by the open restriction of $(\tilde{\pi}, \tilde{\sigma})_* \mathcal{O}_{\tilde{X}_7}$ to $X_7^{ss} \times X$. We now further restrict this kernel along the open immersion to $X_7^{ss} \times X_1^{ss}$ and define $\tilde{X}_{7,1}^{ss}$ by the following fibre diagram. By base change, this new kernel is the same as $(\tilde{\pi}, \tilde{\sigma})_* \mathcal{O}_{\tilde{X}_{7,1}^{ss}}$.

$$\begin{array}{ccccc} \tilde{X}_{7,1}^{ss} & \hookrightarrow & \tilde{X}_7^{ss} & \hookrightarrow & \tilde{X}_7 \\ \downarrow{\scriptstyle(\tilde{\pi},\tilde{\sigma})} & & \downarrow{\scriptstyle(\tilde{\pi},\tilde{\sigma})} & & \downarrow{\scriptstyle(\tilde{\pi},\tilde{\sigma})} \\ X_7^{ss} \times X_1^{ss} & \hookrightarrow & X_7^{ss} \times X & \hookrightarrow & X \times X \end{array}$$

i.e. we define the map $D^b([X_7^{ss}/G]) \to D^b([X_1^{ss}/G])$ by the composition

$$D^b([X_7^{ss}/G]) \to D^b([X/G]) \to D^b([X_1^{ss}/G])$$

where the second map is the obvious open restriction map. As $\tilde{X}_{7,1}^{ss}$ can be intuitively thought of as the points of $\tilde{X}_7$ which land in $X_7^{ss} \times X_1^{ss}$ under $(\tilde{\pi}, \tilde{\sigma})$, an explicit description of the closed points of $\tilde{X}^{ss}_{7,1}$ is as the following subset of the closed points of $\tilde{X}_7$:

$$\tilde{X}^{ss}_{7,1} = \{(h_1, h_2, x_0, c_0, x_1, c_1, c_2, x_3, x_4, c_5) \mid i)x_1x_3 \neq 0 \text{ or } x_3x_4 \neq 0 \text{ or } x_0x_4 \neq 0 \\ ii)h_1x_3 \neq 0 \text{ or } h_2c_0x_4 \neq 0\}$$

where the defining relations of $\tilde{X}_7$ must also hold. The restricted morphism $(\tilde{\pi}, \tilde{\sigma})$: $\tilde{X}^{ss}_{7,1} \to X^{ss}_7 \times X^{ss}_1$ is a closed immersion. In the notation of Figure 3.5, which we reproduce here for convenience, its image is given by $\Delta_G \cup (C \times (B \cup D)) \subseteq X^{ss}_7 \times X^{ss}_1$.

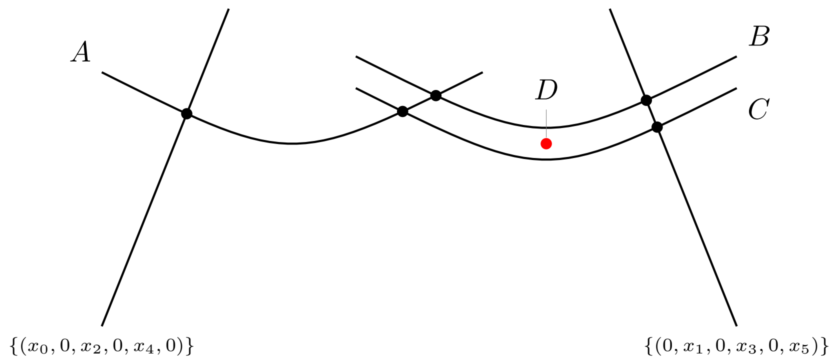


Figure 4.4: A reminder of the geometry of X^{ss}_1. When you move off the wall into Cell 7, B and D become unstable.

The kernels for the remaining 11 maps from a generic cell to a bordering one-dimensional wall are constructed by restricting in exactly the same way, and we omit the precise details for brevity.

4.2 Weights of pullbacks and the shifting algorithm for $\mathcal{O}_{X^{ss}}$

Let $C_j \leq C_i$ be two cells in the hyperplane arrangement shown in Figure 3.1. Then $X^{ss}_i \subseteq X^{ss}_j$, and we now illustrate the process of extending structure sheaves $\mathcal{O}_{X^{ss}_i}$ to the derived category $D^b([X^{ss}_j/G])$. We do the example of Cell 7, and drop any mention of the cell number in our notation where it's not likely to cause confusion, e.g. we may write X^{ss} for X^{ss}_7. Computing possible extensions is a more involved process than it was for the $\mathbb{P}^n$ example we did in Chapter 2, as there are four unstable strata and we must extend iteratively for each of these in turn. Recall that in Section 3.5 we stratified the unstable locus into four disjoint strata

$$X = X^{ss} \cup S_1 \cup S_2 \cup S_3 \cup S_4.$$

We extend iteratively by including one more unstable stratum each time:

$$D^b([X^{ss}/G]) \rightsquigarrow D^b([X^{ss} \cup S_4/G]) \rightsquigarrow D^b([X^{ss} \cup S_3 \cup S_4/G]) \rightsquigarrow \cdots \rightsquigarrow D^b([X/G]).$$

The first extension, where we extend our structure sheaf to some object living on $X_7^{ss} \cup S_4 = X_1^{ss}$, is the extension corresponding to moving from the generic cell onto the wall given by Cell 1. The remaining three extensions correspond to moving from this one-dimensional wall onto the dimension zero cell at the origin.

4.2.1 Extending the structure sheaf when moving onto a one-dimensional wall

In this section we make the obvious initial choice of extension of $\mathcal{O}_{X_7^{ss}}$ to $D^b(X_1^{ss})$ given by $\mathcal{O}_{X_1^{ss}}$, then determine its weights and conclude for which values of w this extension lies in the window subcategory

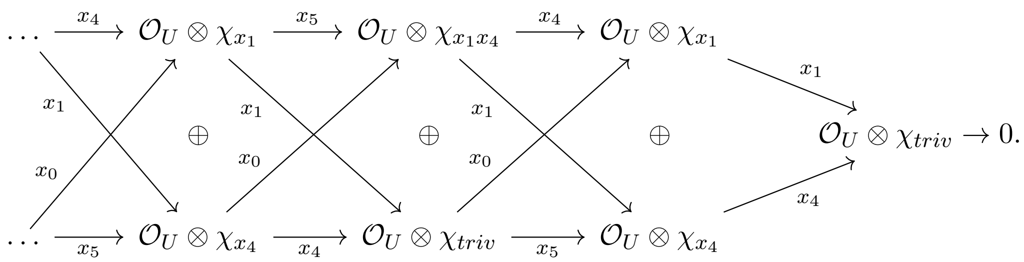
$$G_w = \{F \in D^b([X_1^{ss}/G]) \mid \mu(Z, \lambda, \mathcal{H}^*(\sigma^* j^* F)) \leq w < \mu(Z, \lambda, \mathcal{H}^*(\sigma^* j^! F))\}$$

first defined on p.46. As we are dealing with one KN stratification, this window is determined by a single choice of integer $w \in \mathbb{Z}$. We now calculate the weights of $\mathcal{O}_{X_1^{ss}}$ to see for which values of w this is the correct extension to take. Take the following notation: $Z = \{(0,0,0,x_3,0,0) \mid x_3 \neq 0\}$, $S = \{(x_0,0,0,x_3,0,x_5) \mid x_3 \neq 0\}$, $\lambda = (t^{2n}, t^{-n}, t^{-n})$, with maps

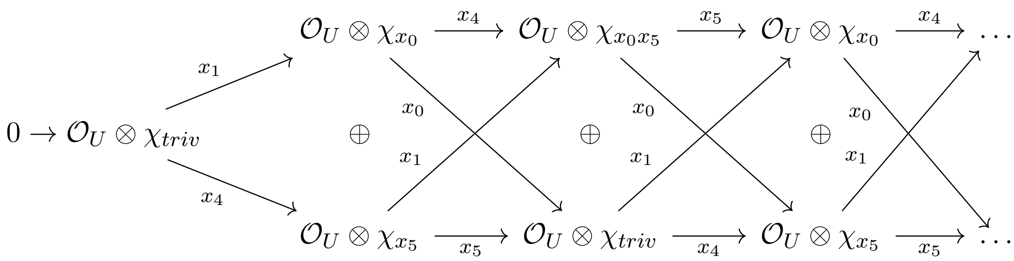
$$Z \xhookrightarrow{\sigma} S \xhookrightarrow{j} X_1^{ss}$$

and $j_i^!(-) := \mathbb{R}\mathcal{H}om_{D^b([U/G])}(\mathcal{O}_{S_i}, (-)|_U)$, where $U \subseteq X_1^{ss}$ is open and contains S_4 as a closed substack. As $S \subseteq X_1^{ss}$ is closed, we take $U = X_1^{ss}$.
As $\sigma^* j^* \mathcal{O}_{X_1^{ss}} = \mathcal{O}_Z$, this is a complex concentrated in degree zero, so there is only one component of $\mathcal{H}^*(\sigma^* j^* \mathcal{O}_X)$ to consider. As $Z = \mathrm{Spec}(\mathbb{C}[c_3^{\pm}])$, we work at the level of global sections. For the 1-parameter subgroup λ, the coaction corresponding to the action map $\lambda \times Z \to Z$ sends $1 \mapsto t^0$, i.e. the only λ-weight of $\mathcal{H}^*(\sigma^* j^* \mathcal{O}_X)$ is zero.
We now consider $j^! \mathcal{O}_X = \mathbb{R}\mathcal{H}om_{[U/G]}(\mathcal{O}_S, \mathcal{O}_X|_U)$. In $D^b([X_1^{ss}/G])$ we have the following bounded-above[2] twisted equivariant Koszul complex, which is quasi-isomorphic to $\mathcal{O}_S$:

[2]Here we are implicitly invoking the equivalence of categories between the *bounded* derived category and the full subcategory of the unbounded derived category with *bounded cohomology*. See e.g. Proposition 2.30 in [Huy06].



As this is a locally free resolution, $j^!\mathcal{O}_{X_1^{ss}}$ is the dual complex



where we view this as a complex of $\mathcal{O}_S$-modules. When we dualise, the characters are inverted. The cohomology of this complex is zero except in degree 1, where it's $\mathcal{O}_S \otimes \chi_{x_0x_5}$; thus $j^!\mathcal{O}_X \simeq (\mathcal{O}_S \otimes \chi_{x_0x_5})[-1]$. This is locally free as an $\mathcal{O}_S$-module, and so pulling back along σ, we get the coaction with respect to the one-parameter subgroup

$$\begin{aligned} \mathbb{C}[f_3^{\pm}] &\to \mathbb{C}[f_3^{\pm}] \otimes \mathbb{C}[t^{\pm}] \\ 1 &\mapsto 1 \otimes t^{2n} \end{aligned}$$

and thus the sole weight of $\mathcal{H}^*(\sigma^* j^! \mathcal{O}_{X_1^{ss}})$ is $2n$. Thus $\mathcal{O}_{X_1^{ss}}$ lies in G_w for $0 \leq w < 2n$.

We have just observed that the correct lifting of $\mathcal{O}_{X_7^{ss}}$ to $D^b([X_1^{ss}/G])$ is the obvious choice $\mathcal{O}_{X_1^{ss}}$ when $0 \leq w < 2n$. In fact the correct lifting to take when lifting the structure sheaf to a larger derived category is always the structure sheaf when we choose $w_k = 0$ for all the unstable loci in the KN stratification. As we know that this is precisely what the Fourier-Mukai kernels we have constructed do, we now conjecture the following result.

$$D^b([X_7^{ss}/G]) \xrightarrow{\sim} G_w$$

for $0 \leq w < 2n$.

Conjecture 4.2.1.1. *The kernels we constructed via partial compactification diagrams in Section 4.1 correspond to the equivalences*

$$D^b([X_i^{ss}/G]) \xrightarrow{\sim} G_w$$

where we choose $w_k = 0$ for each of the relevant number of unstable strata.

Remark 4.2.1.2. The idea for proving this seems clear. In particular, we need to show that our kernels live in the weight window $G'_w \subseteq D^b_{QCoh}([X_i^{ss} \times X_j^{ss}/G \times G])$ with respect to the KN stratification given by $X_i^{ss} \times S_k$, where the $S_k \subseteq X_i^{us}$ are the strata which become semistable in X_j^{ss}. What is currently missing is a (e.g. locally free) resolution of our kernels, which would allow us to compute the pullbacks to the fixed loci $X_i^{ss} \times Z_k$ needed to determine the weights. As we know that the functors defined by our kernels restrict to functors between the bounded derived categories of coherents, if we can verify that the kernels lie in G'_w for $w = (0, \ldots, 0)$, then the conjecture follows by [Hal15, Lemma 2.16].

4.2.2 The shifting algorithm for kernels

In Section 4.1 we wrote down partial compactifications, and used these to produce Fourier-Mukai kernels for maps to larger GIT quotients. Once we have one such kernel, we can obtain a whole family of them by running the shifting algorithm for each choice of integers w_k. As we have seen, actually running this algorithm in practice is non-trivial, especially where there are multiple KN strata. We now outline the process for these Fourier-Mukai kernels.

i) Pick a cell C_i with corresponding partial compactification $\tilde{X}_i$, semistable locus X_i^{ss} and KN stratification $\{S_i\}_{i=1}^k$. To summarise what we have already seen: for our first extension, we have the diagram

$$\begin{array}{ccc} \tilde{X}_i^{ss} & \hookrightarrow & \tilde{X}_i \\ \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\ X_i^{ss} \times (X_i^{ss} \cup S_k) & \hookrightarrow & X \times X \end{array} \tag{4.5}$$

where $\tilde{X}_i^{ss}$ is defined to be the fibre product. Our initial candidate for the kernel is $(\tilde{\pi}, \tilde{\sigma})_* \mathcal{O}_{\tilde{X}_i^{ss}}$ in $D^b_{QCoh}(X_i^{ss} \times (X_i^{ss} \cup S_k))$. This is a valid initial candidate by Lemma 4.1.2.1.

ii) In the running of the shifting algorithm, we pull our kernels back to the fixed locus $X_i^{ss} \times Z_k$ and take the highest weight subcomplex E as appropriate. Doing this in a practical sense requires some locally free resolution of our initial kernel.

$$X_i^{ss} \times Z_k \underset{i_k}{\overset{\pi_k}{\leftrightarrows}} X_i^{ss} \times S_k \xhookrightarrow{j_k} X_i^{ss} \times (X_i^{ss} \cup S_k)$$

iii) Once we have shifted our kernel to live in the correct weight window with respect to $w_k \in \mathbb{Z}$, we use the following situation

$$X_i^{ss} \times Z_{k-1} \underset{i_{k-1}}{\overset{\pi_{k-1}}{\leftrightarrows}} X_i^{ss} \times S_{k-1} \xhookrightarrow{j_{k-1}} X_i^{ss} \times (X_i^{ss} \cup S_k)$$

to run the algorithm with respect to w_{k-1}. Repeat this for all remaining w_k.

If Conjecture 4.2.1.1 does *not* hold, we can obtain the correct kernel for $w = 0$ via this process.

4.3 The $\mathcal{H}$-schober

We are now in a position to construct our proposed schober. Recall the cells C_i given by the wall and chamber decomposition in Figure 3.1 and the partial ordering of the cells by inclusion of closures $C_i \leq C_j \Leftrightarrow C_i \subseteq \overline{C_j}$, along with the corresponding GIT quotient stacks $[X_i^{ss}/G]$. Our candidate for the schober is shown in Figure 4.5. Let the γ_{ij} be given by open subset restrictions and, for now, let the $\delta_{ji} : D^b([X_i^{ss}/G]) \xrightarrow{\sim} G_w \subseteq D^b([X_j^{ss}/G])$ be given by the inclusion as the weight window with *any* choice of integers w_i. By the definition of these functors, it is clear that $\gamma_{ij}\delta_{ji} \simeq \mathrm{id}_{D^b([X_i^{ss}/G])}$ and $(\delta_{ji}\gamma_{ij})|_{G_w} \simeq \mathrm{id}_{G_w}$. As the γ_{ij} are given by open restrictions, the commutativity relations on the γ_{ij} in Definition 1.4.0.3 are all clear. The corresponding commutativity statement for the δ_{ji} is only slightly more complicated, and is the subject of the following lemma. In order for this to work, we must globally decide on the values of the w_k for each unstable stratum so that the δ_{ji} are compatible with one another.

Lemma 4.3.0.1. *For the inclusion $\delta_{ji} : D^b([X_j^{ss}/G]) \xrightarrow{\sim} G_w \subseteq D^b([X_i^{ss}/G])$ by any globally fixed choice of numbers w_k on the unstable strata, the commutativity condition on the δ_{ji} holds.*

Proof. Let C_j be a generic cell with corresponding unstable strata $\{S_k\}$, so that $X = X_j^{ss} \cup \bigcup_k S_k$. For C_i a neighbouring cell of dimension one, the map δ_{ji} extends any object to the semistable locus on the wall, which is the semistable locus in the generic chamber union one unstable stratum, and runs the shifting algorithm so that the extension lies in the correct weight window with respect to this one unstable stratum. The subsequent extension given by δ_{i0} extends the object to $D^b([X/G])$

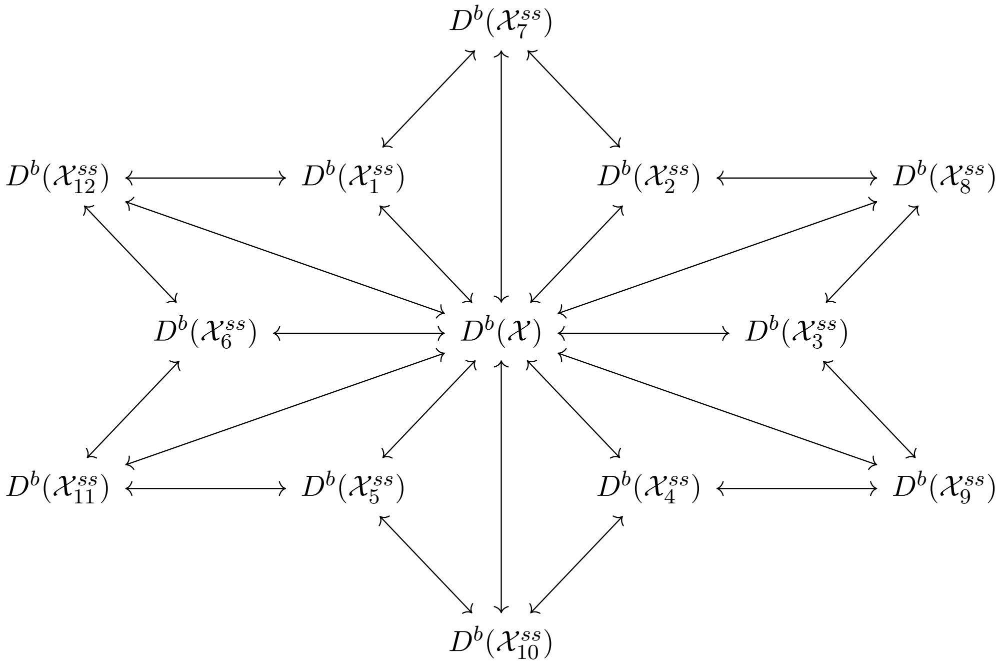
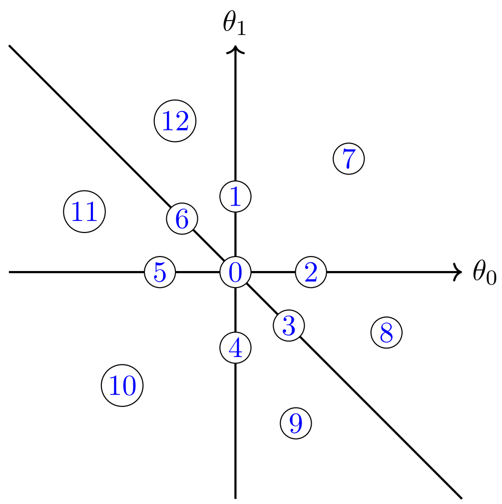


Figure 4.5: Our candidate for constructing a schober from the McKay correspondence. We denote the quotient stacks $\mathcal{X}_i^{ss} := [X_i^{ss}/G]$. Each arrow $\leftrightarrow$ denotes a pair of functors, with one going in each direction. The functors going outwards are restrictions to open subsets, and the functors going inwards are the Kirwan surjectivity maps given as Fourier-Mukai transforms by the kernels constructed in Section 4.1.

with respect to the remaining w_k. The functor δ_{j0} extends objects with respect to the same w_k as we have globally fixed these, so commutativity follows. $\square$

Lemma 4.3.0.2. *Fixing a global choice of w, let C_i, C_j and C_k be three collinear cells. Then there is a natural isomorphism between the flopping functors $\varphi_{ik} \simeq \varphi_{jk}\varphi_{ij}$, i.e. the collinear transitivity property holds.*

Proof. The proof of this requires a case-by-case analysis. As we have globally fixed the choices of w_i, Lemma 4.3.0.1 holds. We reproduce the diagram showing the labelling of the cells for convenience:



Throughout, let C_i, C_j, C_k be three collinear cells. We shall refer to Case $(\dim C_i, \dim C_j, \dim C_k)$, where the dimension refers to the dimension of the cell. Denote by $G^i_w \subseteq D^b([X/G])$ the window subcategory for Cell C_i for the globally determined choice of weights w. By the inductive argument presented in [BKS18, §8] it is sufficient to consider the following subcases:

i) The simplest of these subcases is where there is a direct relation on two of the cells: for $C_i \geq C_j$ and any C_k,

$$\begin{aligned}\varphi_{jk}\varphi_{ij} &\simeq \gamma_{0k}\delta_{j0}\delta_{ij}\\ &\simeq \gamma_{0k}\delta_{i0}\\ &\simeq \varphi_{ik}\end{aligned}$$

and for $C_j \leq C_k$ and any C_i then similarly

$$\begin{aligned}\varphi_{jk}\varphi_{ij} &\simeq \gamma_{jk}\gamma_{0j}\delta_{i0}\\ &\simeq \gamma_{0k}\delta_{i0}\\ &\simeq \varphi_{ik}\end{aligned}$$

ii) Case $(2,1,2)$ where the cells are neighbours, e.g. (C_7, C_2, C_8). In this case it is clear that the window subcategories $G^i_w, G^k_w \subseteq G^j_w$. Thus

$$\begin{aligned}\varphi_{jk}\varphi_{ij} &\simeq \gamma_{0k}\delta_{j0}\gamma_{0j}\delta_{i0}\\ &\simeq \gamma_{0k}\delta_{i0}\\ &\simeq \varphi_{ik}\end{aligned}$$

where the second isomorphism follows as $(\delta_{j0}\gamma_{0j})|_{G^j_w} \simeq \mathrm{id}_{G^j_w}$.

iii) Case $(1, 2, 1)$ where the cells are neighbours, e.g. (C_1, C_7, C_2). In this case, a look at the strata involved confirms that $G_w^j = G_w^i \cap G_w^k$. Thus

$$\begin{aligned}\varphi_{jk}\varphi_{ij} &\simeq \gamma_{0k}\delta_{j0}\gamma_{0j}\delta_{i0} \\ &\simeq \gamma_{0k}\delta_{i0} \\ &\simeq \varphi_{ik}.\end{aligned}$$

The crucial step here is clearly this second isomorphism. Here $\delta_{j0}\gamma_{0j}$ is the identity on elements of $G_w^j = G_w^i \cap G_w^k$. On elements of $G_w^i \setminus G_w^j$, $\delta_{j0}\gamma_{0j}$ is in general not the identity, but the only modification it can make is on the support of the single stratum in $X_j^{us} \setminus X_i^{us}$. But this is exactly the stratum we restrict away from when we apply γ_{0k}.

iv) The three remaining cases are: Case $(1, 1, 1)$ where the cells are neighbours, e.g. (C_1, C_2, C_3); Case $(2, 2, 2)$ where the cells are neighbours, e.g. (C_7, C_8, C_9); Case $(2, 2, 2)$ where the cells are not neighbours, e.g. (C_7, C_8, C_{10}). In each of these cases the argument is the same as for the previous case: the only modifications we make by doing this intermediate restriction and re-embedding are on the support of a stratum we subsequently restrict away from. Thus

$$\begin{aligned}\varphi_{jk}\varphi_{ij} &\simeq \gamma_{0k}\delta_{j0}\gamma_{0j}\delta_{i0} \\ &\simeq \varphi_{ik}\end{aligned}$$

in all the remaining cases. Invoking the induction argument of [BKS18, §8] concludes the proof.

□

Form now on, let the δ_{ji} be given by the Fourier-Mukai kernels we constructed in Section 4.1. We conjecture the following result.

Conjecture 4.3.0.3. *The Fourier-Mukai transforms given by the kernels we constructed in Section 4.1 are the right adjoints of the open restriction maps γ_{ij}.*

Remark 4.3.0.4. In the definition of an $\mathcal{H}$-schober we require that the pairs of functors $(\gamma_{ij}, \delta_{ji})$ form an adjoint pair. As adjoints are unique (up to isomorphism), the isomorphisms

$$D^b([X_i^{ss}/G]) \xrightarrow{\sim} G_w$$

can only be the right adjoint of the open restriction functor γ_{ij} for at most one choice of w.

The following result shows that the flopping functors φ_{ij} are well-defined, even if the δ_{ji} we have constructed are not the adjoints of the open restrictions.

Corollary 4.3.0.5. *For δ_{ji} given by the kernels we have constructed, the composition $\gamma_{ij}\delta_{ji}$ for any $C_i \leq C_j$ is the identity.*

Proof. This is a corollary to Lemma 4.1.2.1. The Fourier-Mukai kernel of $\gamma_{ij}\delta_{ji}$ is just the restriction of the Fourier-Mukai kernel of δ_{ji}. For δ_{j0} this is the equivariant diagonal by Lemma 4.1.2.1, and so the result follows. As the FMK for going from a generic cell C_i onto a wall C_j was defined to be the restriction of the kernel for going from C_i into C_0, base change around the following diagram gives the remaining cases.

$$\begin{array}{ccccc} G \times X_i^{ss} & \hookrightarrow & \tilde{X}_{ij}^{ss} & \hookleftarrow & \tilde{X}_i^{ss} \\ \downarrow{\scriptstyle (\pi,\sigma)} & & \downarrow & & \downarrow{\scriptstyle (\tilde{\pi},\tilde{\sigma})} \\ X_i^{ss} \times X_i^{ss} & \hookrightarrow & X_i^{ss} \times X_j^{ss} & \hookrightarrow & X_i^{ss} \times X \end{array}$$

□

The schober conditions also require the flopping functors φ_{ij} to be equivalences for certain wall-crossings; we now consider these. In particular, we require wall-crossing equivalences in the following two situations:

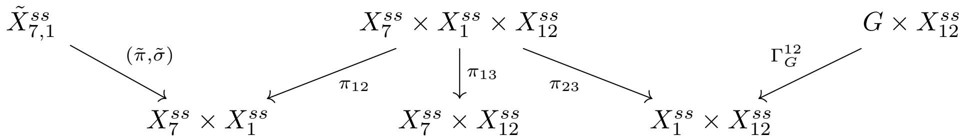
i) Wall-crossings from a generic chamber to a neighbouring generic chamber via the intervening one-dimensional wall.

ii) Wall-crossings from a one-dimensional cell to the opposite one-dimensional cell via the 'wall' given by the zero-dimensional cell at the origin.

We consider these two situations now.
The flopping functors for going from a generic chamber to an adjacent generic chamber via the one-dimensional wall are known to be equivalences as this is a balanced wall-crossing, and so *any* choice of integer w_k for the one unstable stratum gives an equivalence. The following lemma tells us what this equivalence is for the kernels we have constructed.

Lemma 4.3.0.6. *The flopping functors for crossing from a generic chamber to an adjacent generic chamber via a one-dimensional wall are geometric spherical twists. In particular, they are the twists around the spherical object $\mathcal{O}_E$, where E is the exceptional $\mathbb{P}^1$ which becomes unstable as we cross the wall. In particular, these flopping functors are equivalences.*

Proof. We do the example of crossing from Cell 7 to Cell 12 via Cell 1, with the other generic wall crossings following by the same argument. We determine the convolution of the Fourier-Mukai kernels. Recall the geometry of X_1^{ss} given by Figure 3.5 on p.73. In the notation of this figure, B is the exceptional curve which is semistable in Cell 12 but not Cell 7, and vice-versa for C.



As we have already noted, $\tilde{X}_{7,1}^{ss} = \Delta_G^7 \cup (C \times (B \cup D)) \subseteq X_7^{ss} \times X_1^{ss}$. Thus the kernel of the convolution is the pushforward along π_{13} of the intersection of $\tilde{X}_{7,1}^{ss} \times X_{12}^{ss}$ and $X_7^{ss} \times \Delta_G^{12}$ inside $X_7^{ss} \times X_1^{ss} \times X_{12}^{ss}$. But this is just the structure sheaf of $\Delta_G \cup (C \times B) \subseteq X_7^{ss} \times X_{12}^{ss}$. Now, we invoke the fact that G equivariant sheaves on X_i^{ss} are the same as sheaves on $[X_i^{ss}/G]$ and, when i is a generic chamber there is a geometric isomorphism

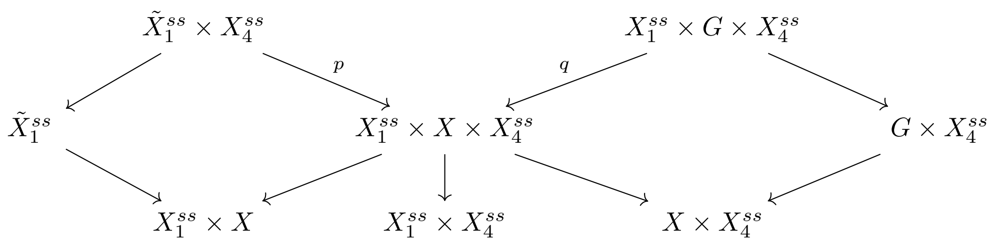
$$[X_i^{ss}/G] \xrightarrow{\sim} Y_{\text{min}}$$
$$\hat{x} \mapsto x$$

where $\hat{x}$ denotes the orbit of a point x. Applying this isomorphism to both components, $\Delta_G \cup (C \times B) \subseteq X_7^{ss} \times X_{12}^{ss}$ becomes $\Delta \cup (\mathbb{P}^1 \times \mathbb{P}^1)$, which is known to be the Fourier-Mukai kernel of the geometric spherical twist. □

Similarly, we expect the following result:

Conjecture 4.3.0.7. *The flopping functors for crossing from a one-dimensional wall to the opposite one-dimensional wall via the dimension zero cell are equivalences.*

We now describe the kernel for the flopping functor corresponding to going from a one-dimensional wall to the opposite one-dimensional wall via the zero-dimensional cell at the origin, doing the example of Cell 1 to Cell 4. The kernel for this transform is given by the convolution of the kernel we determined in Section 4.1 and the equivariant graph:



Base changing around the two squares, the convolution of the two kernels is given by

$$\pi_{13}^{G}(p_*\mathcal{O}_{\tilde{X}_1^{ss}\times X_4^{ss}} \otimes q_*\mathcal{O}_{X_1^{ss}\times G\times X_4^{ss}})$$

where π_{13}^{G} refers to taking G-invariants with regard to the middle copy of G in the $G \times G \times G$-action.

The main difficulty in showing that the functor induced by this kernel is an equivalence is likely to be in showing that it is fully faithful. A possible way of doing this would be to show that an analogue of the Bondal-Orlov criterion for fully faithfulness holds for these quotient stacks; unfortunately the author is not aware of such a statement existing in the literature in sufficient generality. In the stacky setting, recent work of Lim and Polishchuk [LP20] has shown that the Bondal-Orlov criterion holds for smooth and proper Deligne-Mumford stacks with projective coarse moduli spaces. This is clearly not sufficient for our case as the stacks corresponding to non-generic cells are not smooth, proper or Deligne-Mumford, and the good moduli space is not projective. In the non-equivariant setting, the most general statement of the Bondal-Orlov criterion the author could find [Mar17] would still not be sufficient for our purposes, even with a suitable equivariant modification. Here the statement holds for the source variety being smooth quasi-projective, and still requires the target variety to be projective, so both of our varieties cause issues.

A possible route forward is to take our flopping functor for the wall crossing, e.g. φ_{14}, compose it with its left/right adjoints and show that the kernel of this composition is given by the equivariant diagonal.

4.4 Conclusions and further work

In this thesis we have considered the A_2 quotient singularity $\mathbb{C}^2/\mathbb{Z}_3$, constructed a candidate for an $\mathcal{H}$-schober on the wall-and-chamber decomposition induced by the corresponding VGIT problem, and verified the majority of the schober conditions. There are three things that remain to be checked before we can definitively say that our proposed schober is indeed one. These are given by the conjectures in this chapter: it remains to show that the kernels we have constructed give the inclusion as the weight window where we choose $w_k = 0$ for all strata, that the adjoints condition holds, and that the dimension one to dimension one wall-crossings via the central stack are equivalences.

Assuming that our expectations are correct and that these remaining conditions do indeed hold, some possible avenues for future research directions are as follows:

i) Considering A_n singularities for $n \geq 2$, there is again an obvious collection of derived categories of semistable loci. In this case the semistable loci in generic chambers are again the minimal resolution, which now involves a chain of n rational curves. There is a corresponding wall-and-chamber decomposition given by A_n root system lying on a hyperplane in $\mathbb{R}^n$, and it seems reasonable to expect that this situation would also produce $\mathcal{H}$-schobers.

ii) In addition to this, it also seems likely that we should be able to produce schobers for the remaining ADE-type surface singularities, i.e. to consider constructing $\mathcal{H}$-schobers on the wall-and-chamber space given by the root systems for D_n ($n \geq 4$) and the exceptional E_6, E_7 and E_8.

iii) In the same way as Seidel and Thomas [ST01] observed a faithful action of the braid group B_{n+1} on the group of autoequivalences of the minimal resolution of an A_n surface singularity, there should be a similar statement to be made about the partial resolutions appearing in the corresponding A_n schober. This should take the form of a statement about *generalised braids*, see e.g. [DeB19]. Generalised braids behave like the ordinary braid group, but we count strands with multiplicities and allow them to join and split while keeping track of this multiplicity. The endpoints of generalised braids therefore correspond to partitions of $n + 1 \in \mathbb{N}$; as they have multiple endpoints, generalised braids form a category rather than a group. In our A_n schober situation the minimal resolution would correspond to the trivial partition $(1, \dots, 1)$, with the various partial resolutions corresponding to the different non-trivial partitions.

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