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Nonlinear viscoelasticity of strain-rate type: an overview

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There are some materials in nature which experience deformations that are not elastic. Viscoelastic materials are some of them. We come across with many of such materials in our daily lives through a number of interesting applications in engineering, material science and medicine. The present paper concerns itself with modelling of nonlinear response of a class of viscoelastic solids. In particular, nonlinear viscoelasticity of strain-rate type, which can be described by a constitutive relation for the stress function depending not only on the strain but also on the strain-rate, is considered. This particular case is not only favourable from a mathematical analysis point of view but also, due to experimental observations, knowledge on strain-rate sensitivity of viscoelastic properties is crucial for accurate predictions of mechanical behaviour of solids in different areas of applications. Firstly, a brief introduction of some basic terminology and preliminaries, including kinematics, material frame-indifference and thermodynamics, are given. Then, considering the governing equations with constitutive relationships between the stress and the strain for the modelling of nonlinear viscoelasticity of strain-rate type, the most general model of interest is obtained. After that the long-time behaviour of solutions is discussed. Finally, some applications of the model are presented.

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1. Introduction

Real materials exhibit a variety of inelastic phenomena. Viscoelasticity, plasticity and fracture are just a few to mention. This kind of response might be observed as the restoration of the material slowly, or partially, once the forces causing the deformation are removed. It is also possible that the deformation depends on the history of the applied forces. Viscoelastic materials have three important characteristics: *stress relaxation* (constant strain resulting in time-dependent decreasing stress), *creep* (constant stress resulting in time-dependent decreasing strain) and *hysteresis* (the difference between loading and unloading processes) (cf. [15]). As the word "viscoelasticity" suggests, this kind of mechanical response combines the response of elastic solids and viscous fluids. As a result, not only solids but also fluids can possess such a property. However, the way they respond is quite different. In particular, the response of a fluid to a given deformation would be the same starting from any two states, whereas a solid would respond differently, for example, in its initial configuration and after being deformed. More generally, for solids, pure strains might effect the behaviour of the material while rotations might not have an influence (cf. [102]). In this manuscript, we would like to focus on the viscoelastic response of solids. Furthermore, even though classical linear theories of solid mechanics can be applied to a larger class of materials simply because many different nonlinear constitutive equations can actually possess the same linear first approximation (see e.g. [102]), most natural processes are nonlinear. Therefore, nonlinear theories are able to provide much more accurate explanations for the behaviour of materials. This will be the main motivation in this work while discussing possible models.

A deformation is termed *elastic* if the undeformed (or reference) shape restores itself completely, once all the external forces are removed (see e.g., [99]). Underlying the constitutive laws of classical elasticity theory is the basic assumption that the stress-strain curve is the same for the loading and unloading process, and the restoring force (stress) is a single-valued function of the current deformation (strain), not its history. In order to quantify elastic restoring forces, it is possible to use potential energies of deformation, which is the characterization that we employ in the formulation of our models. Similar to an ideal spring, an elastic model stores potential energy during deformation and releases the energy entirely as it recovers the reference shape. On the contrary, a perfect (Newtonian) fluid stores no deformation energy, hence it exhibits no resilience. In the present survey, we are interested in models representing this very common inelastic deformation phenomena that is intermediate between perfectly elastic solids, on the one hand, and viscous fluids, on the other. In particular, we are interested in the case when the relation between the stress, the strain and the strain-rate is nonlinear.

There are some solids in nature which experience deformations that are not elastic. Examples of such materials are metals at certain temperatures and more familiar ones, certainly, are plastics. We explain a phenomenon observed in some solids by the following experiment suggested by Spencer [94]. Firstly, let us take a solid rod with a certain length. Suppose that we hang a weight on the end of it and wait for a certain period of time. If we measure the length of the rod during this time, we will find out that it gradually extends. How much it extends depends on the material that the rod is made of. If now we remove the weight, we will see that the rod slowly gets shorter again. After a long enough time, it might or might not go back to its original length. This again depends on the particular material we are using. This experiment demonstrates rather strikingly that in some circumstances the way in which a body deforms is determined not only by the size of the forces which are applied to it, but also by the length of time they are allowed to act. This phenomenon is the so-called *viscoelasticity*. As mentioned by Banks, Hu and Kenz in [15], even though there are different definitions, in general, viscoelasticity is the property of materials exhibiting both viscous (dashpot-like) and elastic (spring-like) characteristics when undergoing deformation. In accordance with the effect of time in their mechanical behaviour, viscoelastic materials can also be called *time-dependent materials*. The experimental study of such materials is more difficult compared to time-independent ones, basically because one cannot keep time constant or eliminate it during an experiment (cf. [37]). Metals at high temperatures, some

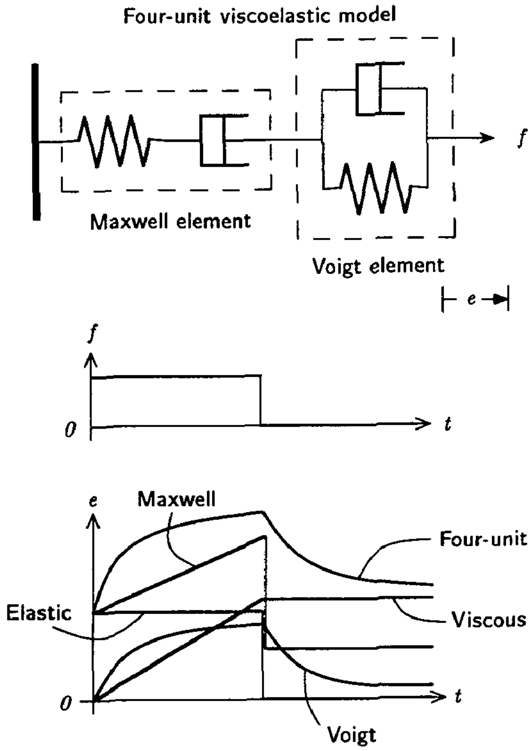

Figure 1. Top: A model connecting a Maxwell viscoelastic unit and a Voigt viscoelastic unit. Middle: The graph of the force applied to the top model. Bottom: Response of various components. [99]

crystals, wood, some types of polymers, soil and biological soft tissue are some of the materials showing viscoelastic response.

The one-dimensional mechanical response of a linear elastic solid can be represented by a mechanical analog, namely, a linear spring. Similarly, the ideal linear viscous unit is the dashpot. As mentioned before, viscoelastic response can be interpreted as a combination of elastic solid and viscous fluid response. Therefore, it is a meaningful attempt to try to represent viscoelastic properties by combining the mechanical analogs of these simpler responses into more complicated mechanisms. In fact, for linear viscoelasticity, the Maxwell model, represented by a dashpot and an elastic spring connected in series, the Kelvin-Voigt (or Voigt) model, represented by a dashpot and an elastic spring connected in parallel (see [18] and [85] for detailed explanation and generalizations), or the standard linear model, combination of Maxwell model and an elastic spring in parallel, can be used (see Figure 1). These models are only useful in investigating the macroscopic behaviour and they do not provide a molecular basis for the viscoelastic response (see [107] for more information).

In a dashpot, the rate of increase in elongation (or contraction) is proportional to applied force f. This can be represented as $\eta\dot{e} = f$, where η is the viscosity constant, the rate of elongation is e, and the dot denotes a time derivative. The elastic and viscous units are combined to model linear viscoelasticity, so that the internal forces depend not just on the magnitude of deformation, but also on the rate of deformation.

As mentioned before, viscoelasticity can also be characterized by the phenomenon of creep, which can be described as a time dependent deformation under constant applied force (and it can be influenced by the temperature). In addition to the instantaneous deformation, creep deformations develop which generally increase with the duration of the force. Whereas an elastic model, by definition, is one which has the memory only of its reference shape, the instantaneous deformation of a viscoelastic model is a function of the entire history of applied force. Conversely, the instantaneous restoring force is a function of the entire history of deformation. As noted in [15], while Kelvin-Voigt solids are very accurate in modelling creep, they are unable to describe stress relaxation as well as solids modelled by viscoelasticity of strain-rate type, which is the fundamental subject of this work.

2. Preliminaries

(a) Kinematics

In continuum mechanics, in order to formulate problems one can use either material coordinates as independent variables, which corresponds to the *Lagrangian* description, or spatial coordinates, which corresponds to the *Eulerian* description of the problem. In the material description, attention is fixed on a given material particle of the solid and study how it moves. In the spatial description, on the other hand, the focus is on a particular point in space. For fluids, it is common to use the Eulerian description since the governing equations take a relatively simple form. For solids, however, it is more convenient to use Lagrangian description (see e.g. [95]). Even though it is possible to convert a problem described in Lagrangian coordinates into one with an Eulerian description, the former is commonly accepted as a natural choice for nonlinear solid mechanics problems (cf. [5]).

For the purpose of the classical mechanics we assume that a three-dimensional body can be informally defined as a set that can occupy regions of $\mathbb{R}^3$, that has volume, that has mass, and that can maintain forces. The elements of a body are called *material points*. We distinguish one configuration of the body, $\Omega \subset \mathbb{R}^3$, and call it *the reference configuration*. This configuration can be a natural stress-free configuration as well as one which is occupied by the body at a certain instant of time. It might even be some ideal configuration that is unlikely to be occupied by the body. Using the Lagrangian description, we denote the position of a point $\mathbf{x} \in \Omega$ at time t in a typical deformed configuration $\mathbf{y}(\mathbf{x}, t)$.

For a homogeneous elastic body with a reference configuration Ω and with unit reference density, a motion is an evolution of diffeomorphisms $\mathbf{y}(\cdot, t) : \Omega \to \mathbb{R}^3$, where $t \in I \subset \mathbb{R}$. The gradient of the deformation at time t is written as $\nabla \mathbf{y}(\mathbf{x}, t)$, or equivalently $D\mathbf{y}$ or $\mathbf{F}$, and can be identified with the $n \times n$ matrix of partial derivatives

$$(Dy)_{i\alpha} = y_{i,\alpha} = \frac{\partial y_i}{\partial x_\alpha}.$$

This is called *the deformation gradient*.

To be physically acceptable it is required that for (almost) every t, the actual position field $\mathbf{y}(\cdot, t)$ is injective, that is, the deformation $\mathbf{y}$ is invertible in Ω. We make this assumption to avoid interpenetration of matter so that two distinct material points cannot simultaneously occupy the same position in space. Nevertheless, we can still allow some cases where, for example, self-contact occurs on the boundary (see [9] for more information). We assume that the admissible deformations satisfy the constraint

$$\det \nabla \mathbf{y}(\mathbf{x}, t) > 0. \tag{2.1}$$

Condition (2.1) ensures that the admissible deformations are *orientation-preserving* and *locally invertible* (by the Inverse Function Theorem, at least if they are smooth enough). However, local invertibility does not imply global invertibility (see [9] for examples).

An elastic material is *hyperelastic* if there exists a function $W : \Omega \times \mathbb{M}^{3\times 3}_+ \to \mathbb{R}$ differentiable with respect to the variable $\mathbf{F} \in \mathbb{M}^{3\times 3}_+$ for each $x \in \Omega$ such that the *first Piola-Kirchhoff stress tensor* is given by

$$\mathbf{T}_R(\mathbf{x}, \mathbf{F}) = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{x}, \mathbf{F}), \tag{2.2}$$

that is, componentwise,

$$(T_R)_{ij}(\mathbf{x}, \mathbf{F}) = \frac{\partial W}{\partial F_{ij}}(\mathbf{x}, \mathbf{F}).$$

Here $\mathbb{M}^{3\times 3}$ denotes the space of real $n \times n$ matrices with positive determinant. The function W is called the *stored-energy function*. Naturally, if the material is *homogeneous*, it is a function of $\mathbf{F}$ only (cf. [23], [24]), which is the case we consider. As noted by Ball [9], this is more restrictive than saying that Ω is occupied by the same material at each point, since it is possible to have some pre-existing stresses. We can also define the *second Piola-Kirchhoff stress tensor* as

$$\hat{\mathbf{T}}(\mathbf{x}, \mathbf{F}) = \mathbf{F}^{-1}\mathbf{T}_R(\mathbf{x}, \mathbf{F}), \tag{2.3}$$

and the *Cauchy stress tensor* as

$$\mathbf{T}(\mathbf{x}, \mathbf{F}) = (\det \mathbf{F})^{-1}\mathbf{T}_R(\mathbf{x}, \mathbf{F})\mathbf{F}^{\mathrm{T}}. \tag{2.4}$$

The elastic energy corresponding to the deformation $\mathbf{y}$ is defined as

$$I(\mathbf{y}) = \int_\Omega W(\nabla \mathbf{y}(\mathbf{x}, t))d\mathbf{x}. \tag{2.5}$$

Unless stated otherwise, we will make the following convention that the initial free energy is finite,

$$\int_\Omega W(\nabla \mathbf{y}(x, 0))d\mathbf{x} < \infty.$$

The matrix

$$\mathbf{C} = \nabla \mathbf{y}^{\mathrm{T}}\nabla \mathbf{y} \tag{2.6}$$

is called *the right Cauchy-Green strain tensor*. It is symmetric and is positive-definite where $\nabla \mathbf{y}$ is nonsingular.

The displacement field $\mathbf{u} : \Omega \times [0, \infty] \to \mathbb{R}^3$ of a typical particle $\mathbf{x}$ at time t is defined as $\mathbf{u}(\mathbf{x}, t) = \mathbf{y}(\mathbf{x}, t) - \mathbf{x}$. The advantage of using the displacement while modelling is that it vanishes in the reference configuration. Nevertheless, the notion of deformation is more commonly used in nonlinear elasticity. If all points in a given body experience the same displacement, then neither the shape nor the size of the body is changed. In this case, we say that it has been given a rigid body displacement. Deformation, on the other hand, occurs if there is a relative displacement between the particles of the body (cf. [47]). It also worths mentioning that the deformation gradient and the displacement gradient are related by the formula $\nabla \mathbf{y} = \mathbf{I} + \nabla \mathbf{u}$.

We now state the so-called polar decomposition theorem as a result of which we can decompose any deformation gradient tensor into a stretch tensor $\mathbf{U}$, which describes distortion, followed by a rotation tensor $\mathbf{R}$, which describes the orientation. This result is the main tool in the analysis of the strain (see e.g. [23], [46] and also [25, pg. 242] for a version for arbitrary positive operators).

Theorem 2.1. *(Polar Decomposition Theorem). Let* $\mathbf{F} \in \mathbb{M}^{3\times 3}$, $\det \mathbf{F} > 0$. *Then, there exist positive-definite and symmetric matrices* $\mathbf{U}, \mathbf{V}$ *and* $\mathbf{R} \in \mathrm{SO}(3)$ *such that* $\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$. *These representations (right and left respectively) are unique.*

Using Thm. 2.1 we can rewrite $\mathbf{C}$ in (2.6) as $\mathbf{C} = \mathbf{U}^2$. Similarly, we have *the left Cauchy-Green strain tensor* defined as $\mathbf{B} = \nabla \mathbf{y}\nabla \mathbf{y}^{\mathrm{T}} = \mathbf{V}^2$. The matrices $\mathbf{U}$ and $\mathbf{V}$ are called *the right and left stretch tensors* respectively. We denote the set of rotations as $\mathrm{SO}(3) = \{\mathbf{R} \in \mathbb{M}^{3\times 3} : \mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}, \det \mathbf{R} = 1\}$.

(b) Material Frame-Indifference

The mechanical behaviour of materials is governed by some general principles one of which is the *principle of frame-indifference* or *objectivity*. In some texts it is referred as the *axiom of invariance under a change of observer*. It restricts the form of the constitutive functions and thus plays an important role in nonlinear continuum mechanics (see e.g. [102, pg. 36], [23, pg. 100], [64, pg. 194]). In addition to this, as Šilhavý [96] explains, frame-indifference also has a theoretical role, which is basically forming a link between the general dynamical statements (e.g. the equation of balance of energy) and the specific continuum dynamical concepts (e.g. the equations of balance of linear and angular momentum).

As a general axiom in physics, it states that the response of a material must be independent of the observer. In other words, any observable quantity must be independent of the particular orthogonal basis in which it is computed. Rather than stating it in its most general form we would like to adopt the version for elastic materials which says (cf. [6]):

The Principle of Frame-Indifference: Constitutive functions are invariant under rigid motions and time shifts.

In order to express this principle as a mathematical statement, first we note that a change of observer (or equivalently the orthogonal basis in which the observable quantity is computed) can be seen as application of rigid-body motions on the current configuration (see e.g. [55, Sec. 5.2], [100], [101]). The following is the definition of a rigid-body motion (or rigid-body deformation) adopted from [6]:

If a body undergoes a motion $\mathbf{p}$, then a motion differing from $\mathbf{p}$ by a rigid motion relative to a different clock is given by

$$\tilde{\mathbf{p}}(\mathbf{x},\tilde{t}) = \mathbf{c}(t) + \mathbf{R}(t)\cdot\mathbf{p}(\mathbf{x},t), \quad \tilde{t} = t + a, \ \ a \in \mathbb{R}, \ \ \mathbf{c} \in \mathbb{R}^3, \ \ \mathbf{R} \in \mathrm{SO}(3),$$

for each point $\mathbf{x} \in \Omega$ and time t. In other words, a rigid-body motion consists of a translation and a rotation. In each of these motions, the relative positions of the points of the material remain the same. As the deformation gradient is not affected by the translations of the origin, if a body undergoes the motion $\tilde{\mathbf{q}}$, the corresponding expression for the stress becomes

$$\tilde{\mathbf{T}}_R(\mathbf{x},\tilde{t}) = \mathbf{R}(t)\mathbf{T}_R(\mathbf{x},t).$$

Adopting the prescription of Noll [71], (see also [6, Chapter 12]) we know that $\mathbf{T}_R$ is invariant under rigid motions and time shifts if and only if it is independent of $\mathbf{y}$ and t. Moreover, it takes the form

$$\mathbf{T}_R(\mathbf{x},\mathbf{F}) = \mathbf{R}(\mathbf{x},t)\cdot\mathbf{T}_R(\mathbf{x},\mathbf{U}). \tag{2.7}$$

As shown by Şengül [91], for homogeneous materials, for stresses of the form $\mathbf{T}_R(\mathbf{F},\dot{\mathbf{F}})$, where $\dot{\mathbf{A}}$ denotes differentiation with respect to time for $\mathbf{A} \in \mathbb{M}^{3\times3}$, (2.7) takes the form

$$\mathbf{T}_R(\mathbf{F},\dot{\mathbf{F}}) = \mathbf{R}\,\mathbf{T}_R(\mathbf{U},\dot{\mathbf{U}}). \tag{2.8}$$

A more convenient form of (2.8) is obtained in [91] using the second Piola-Kirchhoff stress tensor (2.3) as

$$\hat{\mathbf{T}}(\mathbf{F},\dot{\mathbf{F}}) = \mathbf{G}(\mathbf{C},\dot{\mathbf{C}}), \tag{2.9}$$

where $\mathbf{G}$ is a symmetric, matrix-valued function of the right strain tensor $\mathbf{C}$ and its time derivative (or $\mathbf{U}$ and its time derivative). This is equivalent to say that

$$\mathbf{T}_R(\mathbf{F},\dot{\mathbf{F}}) = \mathbf{F}\,\mathbf{G}(\mathbf{C},\dot{\mathbf{C}}). \tag{2.10}$$

This is the constitutive relation for frame-indifferent stress corresponding to nonlinear viscoelastic materials of strain-rate type.

(c) Thermodynamics

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In this section we derive the general model for nonlinear thermoviscoelasticity of strain-rate type using balance laws. We refer to [10] and [55] for details (see also [8] and [96]). The fundamental balance laws in thermomechanics are those of the balance of linear momentum, the balance of angular momentum and the balance of energy. Denoting by $\mathbf{t}_R$ the Piola-Kirchhoff stress vector, $\rho_R > 0$ the constant density in the reference configuration, $\mathbf{f}$ the body force, E the internal energy, $\mathbf{q}_R$ the heat flux vector and r the heat supply, we can state these laws, respectively, as

$$\frac{d}{dt}\int_{\bar{\Omega}} \rho_R\, \mathbf{y}_t d\mathbf{x} = \int_{\partial\bar{\Omega}} \mathbf{t}_R dS + \int_{\bar{\Omega}} \rho_R \mathbf{f}\, d\mathbf{x}, \tag{2.11}$$

and

$$\frac{d}{dt}\int_{\bar{\Omega}} \rho_R \mathbf{x}\wedge \mathbf{y}_t d\mathbf{x} = \int_{\partial\bar{\Omega}} \mathbf{x}\wedge \mathbf{t}_R dS + \int_{\bar{\Omega}} \mathbf{x}\wedge \mathbf{f} d\mathbf{x}, \tag{2.12}$$

and as

$$\frac{d}{dt}\int_{\bar{\Omega}} \left(\frac{1}{2}\rho_R|\mathbf{y}_t|^2 + E\right) d\mathbf{x} = \int_{\partial\bar{\Omega}} \mathbf{t}_R\cdot \mathbf{y}_t dS + \int_{\bar{\Omega}} \mathbf{f}\cdot \mathbf{y}_t d\mathbf{x} + \int_{\bar{\Omega}} r d\mathbf{x} + \int_{\partial\bar{\Omega}} \mathbf{q}_R\cdot \mathbf{n}\, dS. \tag{2.13}$$

Here $\bar{\Omega}$ denotes an arbitrary open subset of Ω, with sufficiently smooth boundary $\partial\bar{\Omega}$, and the unit outward normal to $\partial\bar{\Omega}$ is denoted by $\mathbf{n}$. It is worth noting that it is possible to derive the other balance laws from conservation of energy and the physical requirement that it has the same form for different observers. In addition to the balance laws, thermomechanical processes are required to obey the Second Law of Thermodynamics, which we assume to hold in the form of the Clausius-Duhem Inequality

$$\frac{d}{dt}\int_{\bar{\Omega}} \eta d\mathbf{x} \geq -\int_{\partial\bar{\Omega}} \frac{\mathbf{q}_R\cdot \mathbf{n}}{\theta} dS + \int_{\bar{\Omega}} \frac{r}{\theta} d\mathbf{x}, \tag{2.14}$$

for all $\bar{\Omega}$, where η is the entropy and θ is the temperature. Assuming that the processes are sufficiently smooth, the pointwise forms of (2.11), (2.13) and (2.14) can be written as

$$\operatorname{Div}\mathbf{T}_R + \mathbf{f} = \rho_R\, \mathbf{y}_{tt}, \tag{2.15a}$$

$$\frac{d}{dt}\left(\frac{1}{2}|\mathbf{y}_t|^2 + E\right) - \mathbf{f}\cdot \mathbf{y}_t - \operatorname{Div}(\mathbf{y}_t\mathbf{T}_R) + \operatorname{Div}\mathbf{q}_R - r = 0, \tag{2.15b}$$

$$\eta_t + \operatorname{Div}\left(\frac{\mathbf{q}_R}{\theta}\right) - \frac{r}{\theta} \geq 0. \tag{2.15c}$$

Also, (2.12) gives the symmetry of the Cauchy stress tensor (2.4). We have $\mathbf{t}_R = \mathbf{T}_R\mathbf{n}$, where, as before, $\mathbf{T}_R$ is the Piola-Kirchhoff stress tensor (2.2). The lack of symmetry of $\mathbf{T}_R$, which naturally results from the fact that it is involved in the Lagrangian equations of motion, suggests using the second Piola-Kirchhoff stress tensor (2.3), which is symmetric (see e.g. [6]).

Defining the *Helmholtz free energy* as $\psi = E - \theta\eta$, we can use (2.15*b*) and (2.15*a*) to get from (2.15*c*) the inequality

$$-\psi_t - \theta_t\eta + \mathbf{T}_R\cdot \nabla\mathbf{y}_t - \frac{\mathbf{q}_R\cdot \operatorname{grad}\theta}{\theta} \geq 0.$$

As mentioned by Ball in [10] given arbitrary deformation $\mathbf{y}$ and temperature field θ one can choose $\mathbf{f}$ and r to balance (2.15*a*) and (2.15*b*) so that this inequality becomes an equality. For thermoelastic materials (no viscous dissipation) where $\mathbf{T}_R$, η, ψ and $\mathbf{q}_R$ are assumed to be functions of $\nabla\mathbf{y}$, θ and grad θ, this leads to $\psi = \psi(\nabla\mathbf{y}, \theta)$, $\mathbf{T}_R = D\psi$ and $\eta = -D_\theta\psi$ so that one obtains

$$-\frac{\mathbf{q}_R\cdot \operatorname{grad}\theta}{\theta} \geq 0. \tag{2.16}$$

In the more general case of thermoviscoelastic materials of strain-rate type one obtains

$$\mathbf{S}\cdot \nabla\mathbf{y}_t - \frac{\mathbf{q}_R\cdot \operatorname{grad}\theta}{\theta} \geq 0,$$

where $\mathbf{S}$ is defined through

$$\mathbf{T}_R = D\psi(\nabla\mathbf{y}) + \mathbf{S}(\nabla\mathbf{y}, \nabla\mathbf{y}_t, \theta, \operatorname{grad}\theta).$$

The equations of isothermal thermoelasticity are obtained by assuming that $\theta(\mathbf{x}, t) = \theta_0$ is constant. As a result, in the case of thermoviscoelastic materials of strain-rate type, the balance of linear momentum becomes

$$\rho_R\,\mathbf{y}_{tt} - \operatorname{Div} D\psi(\nabla\mathbf{y}, \theta_0) - \operatorname{Div}\mathbf{S}(\nabla\mathbf{y}, \nabla\mathbf{y}_t, \theta_0, 0) - \mathbf{f} = 0. \tag{2.17}$$

For thermoelastic materials the balance of angular momentum (2.12) is satisfied as a consequence of the requirement that $\mathbf{T}_R$ is frame-indifferent, which can mathematically be expressed as (see also (2.7))

$$\mathbf{T}_R(\mathbf{RA}, \theta) = \mathbf{R}\,\mathbf{T}_R(\mathbf{A}, \theta) \text{ for all } \mathbf{R} \in \mathrm{SO}(3), \mathbf{A} \in \mathbb{M}^{3\times 3}.$$

3. Governing equations

In a purely longitudinal motion of a homogeneous bar of uniform cross-section and unit length, we denote by $y(x, t)$ the x-component of $\mathbf{y}(\mathbf{x}, t)$ (in accordance with Section 2) along which the motion occurs, and by $\tau(x, t)$ the stress on the section at time t. In this case, the equation of motion takes the form

$$\rho_R\, y_{tt} = \tau_x, \qquad (x, t) \in \Omega \times (0, \infty), \tag{3.1}$$

where ρ_R is the material density and cross-sectional area is assumed to be unity. The corresponding component of the deformation gradient in this case is y_x giving $\det \nabla\mathbf{y} = y_x$ so that condition (2.1) is expressed as $y_x > 0$. Assuming that the stress depends nonlinearly on y_x, that is, $\tau = \sigma(y_x)$, where σ is the first Piola-Kirchhoff stress component in one space dimension (see (2.2) for the definition), we obtain the nonlinear wave equation

$$\rho_R\, y_{tt} = \sigma(y_x)_x, \qquad (x, t) \in \Omega \times (0, \infty), \tag{3.2}$$

where $\sigma = W'$ (see (2.2) for the general definition) stands for the non-monotone stress as in Figure 2. By results of MacCamy and Mizel [63], we know that global solutions for equation (3.2), even for smooth initial data, do not exist in general due to the fact that second derivatives of the solutions may become infinite in finite time (see also the discussion in [45] for a hyperbolic-parabolic formulation related to Volterra equations). In particular, as Pego [75] explains, when y_x is in the ranges where σ is decreasing, the equation becomes elliptic and this makes the initial value problem ill-posed. A way of overcoming this problem is to consider a physically relevant regularization which can be done by adding capillarity (see e.g., [70] or [52]) or viscosity effects into the equation. We focus on the latter method in which the stress includes a viscosity term proportional to the strain rate y_{xt}, a general form of which can be written as

$$\rho_R\, y_{tt} = \sigma(y_x, y_{xt})_x. \tag{3.3}$$

This equation can be thought of as the simplest model of a solid with history dependence, and it has been treated by many authors two of whom are Dafermos [26] and Antman and Seidman [7] (with the presence of an external force). Dafermos proved the existence and uniqueness of the solutions for (3.3) under a parabolicity assumption on the stress which ensures that the viscosity is bounded away from zero. He made no assumption on the monotonicity of the stress but the condition on its growth was rather restrictive in the sense that it was suitable for shearing motions of solids but not for longitudinal ones which require that an infinite compressive force accompany a total compression. Moreover, as stated in his article, this growth condition alone was not able to guarantee asymptotic stability of the solutions and a further restriction was necessary. Antman and Seidman [7], on the other hand, obtained a global existence theory under some assumptions that are compatible with longitudinal motion of viscoelastic rods. Moreover, they managed to handle the physically natural requirement that "an infinite compressive force accompany a total compression", which is difficult to ensure in general basically because it causes the governing

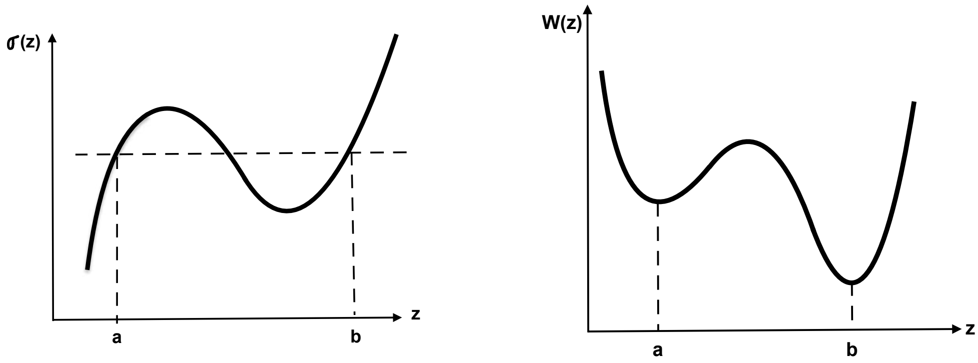

Figure 2. Typical non-monotone stress-strain relation (left) and the corresponding stored-energy function (right) with two local minima at a and b.

equations to be singular. They state that constitutive hypothesis related to total compression is given in the static case by

$$W'(y_x) = \sigma(y_x, 0) \to -\infty \qquad \text{as} \qquad y_x \to 0, \tag{3.4}$$

which they replace by three strengthened versions for dynamics. It is worth mentioning that it is not unreasonable to assume that an increase in the stretch results in an increase in the stress so that $\sigma' > 0$, which is monotonicity. However, this condition is not compatible with models for phase changes (e.g., [4], [26], [75]).

The equation

$$y_{tt} = \sigma(y_x)_x + \lambda y_{xxt}, \tag{3.5}$$

which is a special case of (3.3) with unit density is obtained when the material is assumed to be a nonlinear Kelvin solid. In this case the stress is given by the relation

$$\tau = \sigma(y_x) + \lambda y_{xt}, \tag{3.6}$$

where λ is a positive constant, which can be interpreted as the viscosity coefficient. Indeed, by applying the second law of thermodynamics one can show that λ has to be positive. Equation (3.5), which has been studied by numerous authors such as Greenberg, MacCamy and Mizel [44], Andrews [3], Andrews and Ball [4], Norton [72], Pego [75] and Yamada [108], is also directly related to equations for isothermal motions of van der Waals gas and shearing motions in polymeric fluids (see e.g. [56]). As explained in [44], the modified equation (3.5) is suggested as a result of two motivations. First one is that by the inclusion of the strain rate term λy_{xt} the past history of the strain is represented. Moreover, it is the simplest model to have such a feature. Therefore, one can view this equation as a step towards more general memory theories of rational mechanics. Secondly, introduction of the term λy_{xt} adds a damping mechanism to the process as shown in [44]. In the same article it is also shown that a unique smooth solution exists which decays to the zero solution as t goes to infinity under the assumption that the stress is monotone, that is, $\sigma'(y_x) > 0$ and the initial data is smooth. The method of proof is nonconstructive relying on some results on linear heat equation. One year later, Greenberg [42] revisits the same equation (3.5) and obtains existence and uniqueness of a generalized solution by showing that the solutions of certain finite difference approximations to (3.5) converge to the desired solution.

Andrews [3], on the other hand, proves existence of local and global weak solutions to (3.5) without imposing a monotonicity condition on the stress, and with two sets of boundary conditions; one with both ends of the bar fixed, and the other with one stress-free end. He first deals with local existence and then shows that under some additional mild hypotheses on σ, which do not imply monotonicity, it is possible to prove existence of global solutions. By doing this, he clearly shows the purpose of each restriction on σ. In his article, Andrews uses a fixed point method due to Krasnosel'skii [59] in order to get an existence theory.

Andrews and Ball [4] mainly focuses on the asymptotic behaviour of the solutions as time t goes to infinity (see also Section 4). As they explained in their paper, the main purpose of their work was to study the initial boundary-value problem in the case when σ is not a monotone increasing function, so that the stored-energy function

$$W(u_x) = \int_0^{u_x} \sigma(z)dz, \tag{3.7}$$

is not convex, which implies that the equilibrium problem of solving $\sigma(u_x)$ equals to a constant has infinitely many roots in general (here, in accordance with Section 2, $u_x = y_x - 1$). To see this, one can associate different phases of the material with suitable ranges of the values of the deformation gradient, which in one dimension would be the same as identifying a certain phase with the interval of the values of u_x where σ is monotone. If u_x is allowed to have finite discontinuities, then it can jump from one intersection point to another in one equilibrium configuration leading to infinitely many configurations. Ericksen [36] analyzed this problem in the context of one-dimensional equilibrium theory of elastic bars, which he said was an "elementary study" of phase transformations. Note that (3.7) is equivalent to say $\sigma = W'$ as in (3.2).

Pego [75] provided a simplified existence theory for (3.5) associated with mixed type boundary conditions. His analysis was based on the theory of abstract semilinear parabolic equations as presented by Henry [51]. His main tool was the transformation of the problem into a semilinear system coupling a parabolic partial differential equation to an ordinary differential equation as follows. He writes

$$p(x,t) = \int_1^x y_t(z,t)dz, \qquad q(x,t) = y_x(x,t) - p(x,t),$$

so that the solution y is recovered from

$$y(x,t) = \int_0^x (p+q)(z,t)dz.$$

Pego calls this p the "velocity potential" and q the "modified strain", and notes that in equilibrium q is equal to the strain y_x. Having these notions, instead of (3.5), one can solve the system

$$\begin{aligned} p_t &= p_{xx} + \sigma(p+q), \qquad x \in \Omega, t > 0, \\ q_t &= -\sigma(p+q), \end{aligned} \tag{3.8}$$

together with some initial and boundary data. In his article, he has many important conclusions about the asymptotic behaviour solutions including strong convergence of solutions to a stationary state as time goes to infinity, regularity results showing that y_x must remain discontinuous if it is initially so, and identification of uncountably many dynamically stable stationary states, existence of which is expected since the dynamic processes dissipate energy. His results show that it is possible for the stationary states with discontinuous strain to arise as time-asymptotic limits of solutions with smooth strain, that is, coexistence of phases in stable states might actually be true. As he points out, by the energy minimization arguments one can predict that phases can coexist only if the energy density in each phase is the same, at the absolute minimum. However, as in the example of twinned martensite, when the material is placed under a load favouring one of the twins, even though the energy densities of the phases differ, they are possibly observed to coexist (see Section 5 for more details).

Ball, Holmes, James, Pego and Swart [12] also provided some models in order to investigate the dynamical behaviour of small scale microstructure observed during phase transformations which are also interesting from the point of view of infinite-dimensional dynamical systems with infinitely many unstable modes. They were essentially motivated by the mechanical systems that dissipate energy as time t increases and their models were constructed in such a way that the underlying energy functions have minimizing sequences that converge weakly to non-minimizing states rather than attaining a minimum. The first model they investigate, which is a nonlinear partial differential equation closely related to (3.3), represents the behaviour of a one-dimensional nonlinear viscoelastic continuum that is bonded to a substrate with strength α, and is given by

$$u_{tt} = \left(u_x^3 - u_x + \beta u_{xt}\right)_x - \alpha u, \tag{3.9}$$

where the term βu_{xxt} represents viscoelastic damping. As they stated, the choice $(u_x^3 - u_x)$ in (3.9) for the stress is not crucial and any cubic-like strain-stress function would lead to similar results (see also [41]). The reason for having a cubic (or cubic-like) stress function lies within the fact that the corresponding energy function is a double-well potential, and such functions are used to model development of finer and finer microstructure that is observed in certain material phase transformations. As a second model, they replaced the local nonlinear term $u_x^2 u_{xx}$ by the spacially averaged term $\|u_x\|^2 u_{xx}$ obtaining a nonlinear nonlocal model. In their last model, they just replace the second order time derivative in their second model by a first order time derivative to obtain a pseudo-parabolic equation. After establishing global existence and uniqueness results for the first two models, they consider the long-time behavior of the systems to show that they differ dramatically. This is because solutions of the local model do not minimize energy, whereas almost all solutions of the other models do so. By doing this, they are able to obtain results for the structure of attracting sets for infinite-dimensional, dissipative evolution equations.

A related equation for (3.5) (see also (3.9)) is given by

$$y_{tt} - \Delta y - \alpha \Delta y_t = f(y),$$

where Δ is the Laplacian, $\alpha > 0$, and f is a nonlinear function of y satisfying certain conditions. This model has been studied many authors one of whom is Webb [106] (see also the references therein). By taking the advantage of the semilinear character of the equation and reformulating the problem as an ordinary differential equation in a Banach space, Webb proved existence of unique global solutions as well as stability to equilibrium points.

In the case of three space dimensions, the equation of motion takes the form

$$\rho_R\, \mathbf{y}_{tt} = \mathrm{Div}\, \mathbf{T}_R \tag{3.10}$$

and the corresponding model for (3.5) becomes

$$\rho_R\, \mathbf{y}_{tt} = \mathrm{Div}\big(DW(\nabla \mathbf{y}) + \nabla \mathbf{y}_t\big), \tag{3.11}$$

where the first Piola-Kirchhoff stress tensor is taken to be

$$\mathbf{T}_R = DW(\nabla \mathbf{y}) + \nabla \mathbf{y}_t.$$

This equation models the isothermal case and can be derived from the law of linear momentum by a constitutive assumption for the stress tensor and is therefore coherent with thermomechanics (see Section 2(c)). A theory of existence for (3.11) is available by Rybka [89], [90], and Friesecke and Dolzmann [39]. However, following from Section 2(b), the corresponding viscous stress $\nabla \mathbf{y}_t$ is not frame-indifferent (see also [32]), which is one of the properties necessary to exclude physically unreasonable effects. In [89], Rybka obtained existence of vector-valued solutions under a rather technical assumption that the stress is globally Lipschitz continuous (see also [97]). This assumption severely restricts the growth of the nonlinearity for large arguments and hence not favourable. In [90] he is able to replace this assumption with that of local Lipschitz continuity of the stress together with being close to a linear mapping for large arguments. He obtains results in the space of bounded mean oscillations (see [58] for its definition) which turns out to be the proper space due to instantaneous formation of singularities at the origin. Friesecke and Dolzmann [39], on the other hand, analyzed (3.11) with being able to get rid of the global Lipschitz continuity assumption as well as non-convexity assumption for the stored energy function. As they stated, the approach they used is based on implicit time-discretization and a compactness property of the discrete dynamical scheme not shared by energy-minimizing sequences. Their main contribution is the observation that the discretized counterpart of the damping term $\nabla \mathbf{y}_t$ provides convexity of the static problem despite the stored-energy function not being convex.

One could include temperature in (3.11) as an additional parameter in order to model nonlinear thermoviscoelastic materials by

$$\rho_R\, \mathbf{y}_{tt} = \mathrm{Div}\big(DW(\nabla \mathbf{y}, \theta) + \nabla \mathbf{y}_t\big), \tag{3.12}$$

where θ denotes the temperature. As explained by Zimmer [110] in order for the system to be well-posed, (3.12) is coupled with an equation for θ (see also [105]). Zimmer is able to deal with non-convex energy in three-dimensional setting as he proves global in time existence of solutions following an approach based on a fixed point argument using an implicit time-discretization and renormalized solutions for parabolic equations. Watson [105], on the other hand, looks at the one-dimensional, initial-boundary value problem corresponding to pinned endpoints held at constant temperature and proves global in time existence of solutions. He claims that his approach applies to all boundary conditions involving pinned or stress-free endpoints that could be held at constant temperature or insulated. Similarly, Dafermos and Hsiao [29], Dafermos [27], Chen and Hoffmann [22], Blanchard and Guibé [17], and Racke and Zheng [83] (see also [98]) investigated thermoviscoelasticity. The most recent work on thermoviscoelasticity is by Mielke and Roubíček [67]. In this work authors formulate thermodynamically-consistent frame-indifferent viscoelastic model by using the second grade non-simple materials. The energy contribution in this kind of materials includes a second-order gradient term which generates enough regularity to handle the geometric and physical nonlinearities. In [67], weak solutions for the quasistatic evolution (see below for definition of such models) is obtained.

The most general form of nonlinear viscoelasticity of strain-rate type can be written as

$$\rho_R\, \mathbf{y}_{tt} - \mathrm{Div}\, DW(\nabla \mathbf{y}) - \mathrm{Div}\, \mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t) = 0. \tag{3.13}$$

Note that this is, in fact, equation (2.17) with $W(\mathbf{A}) = \psi(\mathbf{A}, \theta_0)$ and $\mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t) = \mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t, \theta_0, 0)$.

In (3.13) the constitutive equation for the stress reads

$$\mathbf{T}_R(\nabla \mathbf{y}, \nabla \mathbf{y}_t) = DW(\nabla \mathbf{y}) + \mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t), \tag{3.14}$$

where the first part is called the *elastic* and the second part is called the *viscoelastic* part of the stress. We denote the viscoelastic part with $\mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t)$. Since $\mathbf{S}$ depends on the "rate of the strain", this model is called the *nonlinear viscoelasticity of strain-rate type*.

The first theory of existence of solutions for this problem with frame indifferent $\mathbf{S}$ in three dimensions is that of Potier-Ferry [81], [82], who established global existence and uniqueness of solutions for initial data close to a smooth equilibrium for pure displacement boundary conditions. Ball and Şengül [14] (see also [91]) studied the quasistatic version of (3.13) which is believed to be the key step towards the study of the full dynamics. Quasistatic problems in mechanics arise when the system observed evolves slowly in time. In this case the system is observed over a long time scale and the inertial terms in the equations of motion become negligible. This is never exact in real processes, but in many systems dissipative forces beat the acceleration term and the quasistatic approximation is useful, even though neither mass nor velocity is necessarily small (cf. [78]). In [91] the quasistatic version of (3.13) is given by

$$\mathrm{Div}\, DW(\nabla \mathbf{y}) + \mathrm{Div}\, \mathbf{S}(\nabla \mathbf{y}, \nabla \mathbf{y}_t) = 0. \tag{3.15}$$

This equation is obtained by neglecting the inertia term in (3.13). Şengül [91] introduces a variational method in three dimensions for (3.15) where, by a fixed-point argument, an existence theory with frame-indifference is obtained. In [38], on the other hand, Friedrich and Kružík investigated the quasistatic problem with higher order gradient terms and under the assumption of small strains. They rigorously showed that solutions to the nonlinear equations converge to the unique solution of the linear systems as the parameter defining the order of the deformation gradient converges to zero. In [14], on the other hand, quasistatic problem in one space dimension is considered with two sets of boundary conditions; one-end free, and both ends fixed. These boundary conditions are expressed mathematically as

$$y(0,t)=0 \qquad \text{and} \qquad (\sigma+S)(1,t)=0, \tag{3.16}$$

where $\sigma=W'$, as before, and as

$$y(0,t)=0 \qquad \text{and} \qquad y(1,t)=\mu>0, \tag{3.17}$$

respectively. Here, μ is a positive constant representing the position of the end of the bar corresponding to $x=1$. Under the assumption that the viscoelastic part of the stress is given by y_{xt}, the model reduces to

$$\big(\sigma(y_x)+y_{xt}\big)_x=0, \tag{3.18}$$

which is, in fact, the quasistatic version of equation (3.5) with unit viscosity coefficient (which can be obtained by rescaling time). When equation (3.18) is complemented with the boundary conditions (3.16), they obtain the system

$$\begin{aligned} &p_t(x,t)=-\sigma(p(x,t)), \qquad x\in(0,1),\\ &p(x,0)=p_0(x). \end{aligned} \tag{3.19}$$

Ball and Şengül [14] obtained well-posedness for (3.19) as well as stability of solutions (see Section 4 for details). When they considered the boundary conditions (3.17) for (3.18) instead, they obtained the system

$$\begin{aligned} &p_t(x,t)=-\sigma(p(x,t))+\int_0^1\sigma(p(x,t))dx,\\ &p(x,0)=p_0(x),\\ &\int_0^1 p(x,t)dx=\mu>0. \end{aligned} \tag{3.20}$$

System (3.20) is much more difficult to investigate due to the additional nonlocal integral term in the equation. Ball and Şengül [14] investigated (3.20) both for global and local Lipschitz continuity assumption on σ. In the former, they defined a contraction mapping and used Banach's fixed point theorem. In the latter, they had some additional assumptions on the behaviour of σ. By assuming that the initial data has finitely many values, they were able to obtain existence of global upper and lower bounds which helped to pass to the limit to get existence for $L^2(0,1)$ initial data. These upper and lower bounds were also useful in the investigation of asymptotic behaviour of solutions (see Section 4). In the same paper, they showed equivalence of the system (3.20) with gradient flow equation (see e.g. [2] for definition, and [68] for a variational approach as well as some useful references for the general theory) under the assumption that the stored-energy function W is λ-convex, that is, a quadratic perturbation of a convex function, for which the existence of solutions is already known in certain Hilbert spaces.

Similarly, Mielke, Ortner and Şengül [66] investigated the quastatic evolution corresponding to (3.13) but with a completely different approach. They formulate the problem as a gradient system and focus on nonlinear dissipation functionals and distances that are related to metrics on weak diffeomorphisms and that ensure time-dependent frame indifference of the viscoelastic stress. They perform time-discretization where, because of the missing compactness, the limit of vanishing time steps can be obtained only by proving some kind of strong convergence. They show that this is possible in the one-dimensional case by using a suitably generalized convexity in the sense of geodesic convexity of gradient flows. More recently, Krömer and Roubíček [60] looked at the quasistatic case in three-dimensional space while dealing with the requirement of frame-indifference. They obtain existence of weak solutions using the concept of second grade non-simple materials.

A different approach to the existence of solutions in elasticity, as Ball [10] points out, is to change the concept of solution by weakening it to that of a measure-valued solution. The unknown, in this case, is a Young measure $\nu_{x,t}$ in appropriate variables and, roughly speaking, it is obtained by passing to the weak limit in a sequence of approximate solutions. The global existence of such solutions for (3.13) has been proved by Demoulini [31] by using a variational time-discretization method. The strict monotonicity assumption she made on the viscoelastic part of the stress was preventing it from being frame-indifferent and she assumed a uniform dissipation condition, which is much weaker than monotonicity. However, she was unable to handle the constraint $\det \nabla \mathbf{y} > 0$, which is another important physical restriction (see (2.1)).

Another recent work on the problem (3.13) is by Tvedt [103], in which existence and uniqueness of weak solutions were obtained with mixed boundary conditions and suitable initial data for a potential energy which was a non-convex function of the strain. The critical hypothesis he made was that the dependence of the stress function on the strain rate be uniformly strictly monotone. As proven in [91], this hypothesis by itself is not compatible with frame-indifference. Lewicka and Mucha [61], on the other hand, handles the frame-invariance property but obtain existence of local-in-time regular solutions only for (3.13), using the theory of quasilinear parabolic equations. More precisely, they apply the maximal regularity estimates to control the nonlinearities in the equation by the dominating dissipative part.

4. Long-time behaviour of solutions

Given a dynamical system starting from an initial state, it is difficult to predict how the system will evolve as time increases. It might converge to an equilibrium state or there might exist some periodic states. Even though the dissipative character of the system may lead to the existence of absorbing sets, there are various difficulties one can encounter especially in infinite dimensions. The mathematical problem here is the study of the long-time behaviour of the system to determine which permanent state will be observed after a certain period of time.

When the initial data takes only a finite number of values, that is, $p_0(x) \in \{p_{01}, p_{02}, \dots, p_{0N}\}$, which can be expressed as

$$p_0(x) = \sum_{i=1}^{N} p_{0i} \chi_{E_i}(x),$$

where meas $E_i = \mu_i$, $\sum_i \mu_i = 1$, and E_i are mutually disjoint subsets of $(0, 1)$, the system (3.20) reduces to the following finite system of ordinary differential equations

$$\dot{p}_i = \sigma(p_i) + \sum_j \mu_j \sigma(p_j),$$

where $\sum \mu_j = 1$ and $\sum \mu_j p_j = \mu$. Pego [76] proved that for this system every bounded solution stabilizes to some equilibrium as $t \to \infty$. His proof was later clarified by Hale and Raugel in [49]. Pego uses a result due to Hale and Massatt [48] on stabilization of hyperbolic trajectories of systems of ordinary differential equations, which is only valid in the finite dimensional case. Trying to adapt the proof to the case of (3.20) encounters a serious difficulty which was already noted by Friesecke and McLeod [40], namely that for bounded $\sigma : \mathbb{R} \to \mathbb{R}$ the map $p \mapsto \sigma(p)$ is not C^1 from $L^2(0, 1)$ to $L^2(0, 1)$ unless σ is constant. Accordingly, a possible strategy seems to use the fact that one has a dense set of initial data for which convergence to a unique equilibrium holds, namely finite-dimensional initial data. However, this kind of argument fails even in finite dimensions, as an example in [14], given by Ball and Şengül, shows.

Andrews and Ball [4] studied the long-time behaviour of solutions to equation (3.5). They introduced an assumption which they called a *nondegeneracy condition*. In Pego's [76] words, assuming that σ is piecewise monotone and, in particular, that *for z in any bounded set of $\mathbb{R}$ the equation $\sigma(z) = s$ has a finite number $M = M(s)$ of roots $z_1(s) < z_2(s) < \dots < z_M(s)$, where M is piecewise continuous jumping a finite number of times*, the nondegeneracy condition asserts that

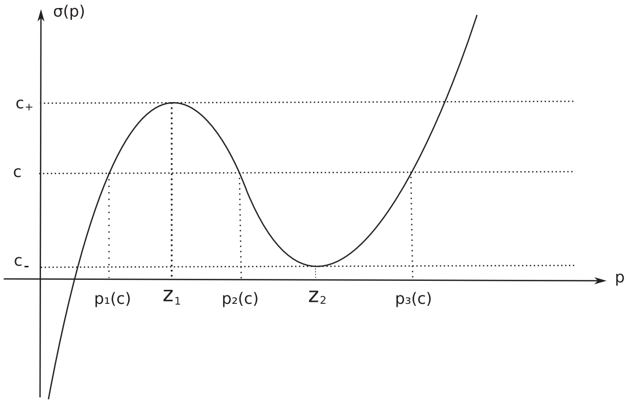

Figure 3. A general cubic-like σ with exactly two critical points $z_1 < z_2$ with $c_- = \sigma(z_2), c_+ = \sigma(z_1)$, and for any $c \in (c_-, c_+)$, $\sigma(p) = c$ has exactly three roots $p_i(c)$. [14]

Nondegeneracy Condition (NC): The derivatives $z_j'(s), j = 1, 2, \ldots, M$, are linearly independent on any common interval of definition.

As stated in [76] this is equivalent to saying that $1, z_1(s), z_2(s), \ldots, z_M(s)$ are linearly independent functions of s on any interval where $M(s)$ is constant (see also [77] for the most recent work using (NC) for the analysis of a reaction-diffusion system with Neumann boundary conditions, and [11] for group theoretic investigations about (NC)). Under the assumption (NC), Andrews and Ball [4] proved that $\sigma(p(\cdot, t))$ converges to a constant in $L^2(0,1)$. Ball and Şengül [14] adopted a slightly modified version of (NC) and proved convergence of solutions of (3.20) to equilibrium states under this assumption. They also gave a direct proof of stabilization when $\sigma(p) = p^3 - p$, which corresponds to the stored-energy function given by $W(p) = \frac{1}{4}(p^2 - 1)^2$, which as explained before (see e.g., the explanation about the choice of model (3.9)) is of interest in various applications. Moreover, from a practical point of view, this expression yields zero stress at infinite compression, that is when $p \to 0$, and in the absense of deformation, that is when $p = 1$. Generalizing this, they investigated cubic-like real analytic stresses, as in Figure 3, and showed that convergence to unique equilibrium holds under some weakened form of (NC). In addition to this, Ball and Şengül [14] defined the ω-limit set and showed that there is a global attractor in a certain subspace of $L^2(0,1)$ for the semiflow $\{T(t)\}_{t\geq 0}$ associated with problem (3.20). As they mention in their paper, another standard technique for proving convergence to a unique equilibrium is to use the Lojasiewicz-Simon inequality, introduced by Lojasiewicz in [62] in a finite-dimensional setting and later generalized by Simon in [93] (see also [57]) to infinite dimensions, under some assumptions on the analyticity of the nonlinear terms. This method, however, is not applicable for similar reasons to those mentioned above in connection with the Hale - Massatt theorem.

The regularized nonlinear diffusion equation

$$u_t = \Delta(f(u) + \nu u_t), \quad x \in \Omega,\ t \in \mathbb{R},\ \nu > 0 \text{ constant}, \tag{4.1}$$

motivated by the problem of phase separation in a viscous binary mixture, was studied by Novick-Cohen and Pego [73] who proved that under the assumption of either (NC) or other certain technical hypotheses, each solution approaches some steady state depending on the initial data. One such hypothesis was that f is cubic (see also Figure 3) with

$$f(u) = c_1(u - c_2)^3 + c_3(u - c_2) + c_4, \quad c_1 > 0,\ c_3 < 0.$$

In addition to this, they assumed that the mean concentration of the initial data is not equal to c_2. Equation (4.1) was also investigated by Plonikov [80] where he proves that the set of functions f corresponding to the values of the regularization parameter ν is relatively compact in the strong metric. He also shows that as $\nu \to 0$ the solutions of the regularized equation converge to the measure-valued solution of the equation with a variable direction of parabolicity. He makes assumptions involving the condition (NC) and in [77] his approach using the theory of Young-measures is carefully explained.

5. Applications

There are many areas where nonlinear viscoelasticity of strain-rate type has been applied in solid mechanics. One such context is the dynamics of microstructure observed in solids. Some materials like elastic crystals experience a certain class of phase transformations and as a result they possess a combination of different fine-scale spatial domains. Such transformations might be caused by various mechanical interactions including application of some forces, imposition of electric or magnetic fields, or change in their temperatures. The microstructure observed (see Figure 4) is due to different or differently oriented atomic lattice structures of the crystal and it develops as a consequence of the multi-well form of the energy density (cf. [16], [40]).

A considerable amount of literature has been published throughout the last two to four decades on the presence of such microstructures and its features, most of which follow the approach of minimization of the free energy in continuum models (see [13], [16] and references therein). Although extensive research has been carried out on variational integrals and their roles in modelling microstructures, very few studies exist which adequately cover the dynamic processes by which such microstructural patterns may be created or evolve (see e.g. [40], [75], [89]).

A *martensitic phase transformation* is a solid to solid phase transformation where the structure of the lattice suddenly changes at a certain temperature. *Austenite* is the phase associated with high temperature and the low-temperature phase is called *martensite*. Various materials can undergo martensitic phase transformations. Some of them are metals, alloys and ceramics. Studies on these transformations have proved the importance of their technological implications, some of which are due to the so-called *shape-memory effect* (see e.g. [16] for more information). The reason of interest in martensitic phase transformations is the observed microstructure they produce. Let us first understand what microstructure really means in the language of materials science. In a martensitic phase transformation, the symmetry of the austenite phase is more than that of the martensite phase. As an example, one can think of a crystal having a square lattice in the austenite phase and a rectangular lattice in the martensite phase. As a consequence of this property, martensite has multiple variants the number of which depends on the change in the symmetry during transformation. Due to various reasons such as nucleation events, the crystal forces the martensite to make a mixture of different variants which gives rise to some patterns at a very small length scale. These characteristic patterns are called the *microstructure of martensite*.

To give an energetic interpretation to these configurations, we only consider materials that are single crystals in the austenite state and suppose that the specimen is subject to a deformation $\mathbf{y}$. We assume that the stored-energy function W depends on the local change of the lattice which is measured by the deformation gradient $\nabla\mathbf{y}$. The total energy of the body, in this case, is then given by (2.5). We can think of the equilibrium state of the body as a minimizer of the total free energy which suggests that the behaviour of microstructure is completely determined by W. It is important to note that the stored-energy function W is allowed to have several potential wells (see Figure 4) and this is the main reason for the equilibrium states to have a mixture of phases. This can be made rigorous as follows.

As explained by Ball and James [13], the understanding of microstructure should be made in terms of minimizing sequences rather than minimizers of the energy. This is because when the ellipticity conditions are not satisfied, integrals of the calculus of variations do not attain a minimum among ordinary functions but in the space of generalized curves which are the limits of minimizing sequences that finely oscillate more and more. It is very typical of elasticity models for solids which change phase that the minimizing sequences converge weakly to deformations which are not minimizers of the total free energy. In other words, the energy of the weak limit of a sequence of deformations might be greater than the limit of the energy (for examples see [16, Chp. 6]). This implies that the total free energy function is not lower semicontinuous with respect to weak convergence in a suitable Sobolev space. Such a property is a result of failure of ellipticity of the energy functional which can be associated with the multi-well structure of the energy density. If, on the contrary, the energy of the limit is always smaller than the limit of the energy, then it will not be helpful for the material to make alternating gradients to minimize the energy. We see such a property when the energy density function has a one-well structure and in this case no microstructure is observed. In other words, microstructure occurs as a result of coexistence of several phases in a martensitic phase transformation which can be interpreted as W having several local minima. Normally, such W are not convex and thus the energy functional (2.5) is not sequentially weakly lower semicontinuous. Since minimizers of the energy might not exist, one is forced to study minimizing sequences instead (see also [50] for the concept of mutual recovery sequences).

As a result of the vanishing of the first variation of functional (2.5), any sufficiently smooth minimizer must satisfy the associated Euler-Lagrange equation given by $\operatorname{Div}\mathbf{T}_R(\nabla\mathbf{y}) = \mathbf{0}$ provided that the energy density W is also smooth enough. As explained in the above section, W is assumed to be non-convex. Thus, the Euler-Lagrange equation, complemented with suitable boundary conditions, typically has a multitude of minimizers corresponding to different phases. This non-uniqueness of the solution can be seen as a result of the fact that the dynamical process which is responsible for selecting from among the many possible equilibrium states, depending on the initial data, the particular one which is preferred by the body, is ignored (see e.g. [1], [40], [97]). Therefore, the corresponding inertial effects should also be included in the model, which can be done by adding the kinetic energy to the energy functional (2.5) giving the total energy as

$$I(\mathbf{y}, \mathbf{y}_t) = \frac{1}{2}\int_{\Omega} |\mathbf{y}_t|^2 d\mathbf{x} + \int_{\Omega} W(\nabla\mathbf{y}) d\mathbf{x}.$$

The corresponding equation of motion for this energy is given by (3.10). Since W is not quasiconvex, this leads to the loss of ellipticity in the stationary problem which corresponds to the failure of hyperbolicity for the dynamical problem. Due to the hyperbolic nature of the dynamical problem, spatial discontinuities may form in finite time which forces one to study weak solutions that allow for jump discontinuities in the deformation gradient, strain and stress. The lack of uniqueness for these weak solutions is an indication of incompleteness of the constitutive modelling and there are various possible ways to overcome this problem. The first approach is

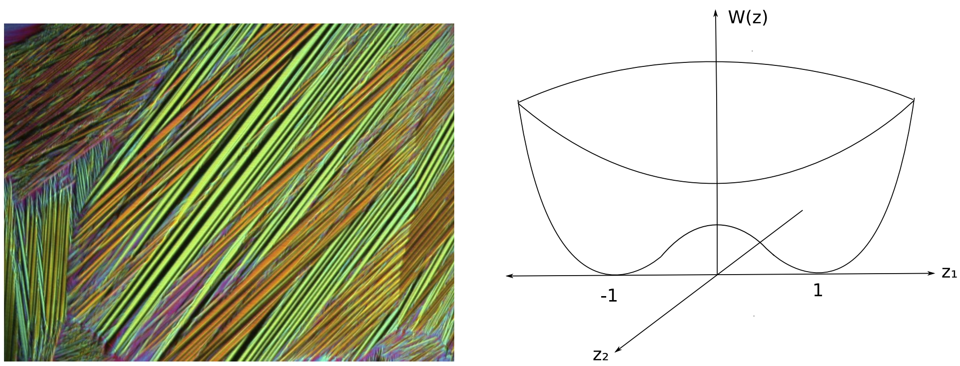

Figure 4. Left: Microstructure in CuZnAl (courtesy of Morin) Right: A double-well energy density [91]

that of constructing more detailed constitutive models which describe thermodynamics of multi-phase materials and the evolution of the microstructure observed. In the case of non-convex stored-energy functions W, the second law of thermodynamics in the form of the Clausius-Duhem inequality (see (2.14)) is not sufficient to provide a unique weak solution, and hence it is necessary to make additional constitutive assumptions, as was done by Abeyaratne and Knowles [1] in the context of one-dimensional isothermal bars. Another possible method of achieving well-posedness is adding to the stress tensor a higher order regularizing term corresponding to viscosity, which was explained in detail in Section 3 and onwards.

Another area where the theory of nonlinear viscoelasticity of strain-rate type has been applied is that of visco-hyperelastic soft materials. As discussed in (2.2), hyperelasticity theory assumes existence of a density function and is used in order to model large deformations with nonlinear stress-strain behaviour. Accordingly, there are many constitutive models, such as Mooney-Rivlin [69], [87], [88] and Ogden [74] models, explaining mechanical behaviour of soft materials like rubbers and biological tissues. However, time-dependent response is more complicated from modelling point of view and it is one of the fundamental characterizations of mechanical response of soft materials. As discussed by Upadhyay, Subhash and Spearot in [104], the required time-dependent modelling can be done in various ways. We are interested in short-term response of soft materials including different kinds of phenomena (e.g., softening of soft polymers), which play important roles in studying medical conditions like brain injuries, automobile crashes, etc. The main reason of our interest in such material response lies in the constitutive modelling of the stress. More precisely, Helmholtz free energy based models, which are thermodynamically meaningful, are possible to use for this kind of response. Among such models, which also include extended hyperelastic models and models with multiplicative decomposition of the deformation gradient (see also [53], and [54] for internal state variable modelling), we want to focus on external thermodynamic state variable driven viscous dissipation models, which were firstly introduced by Pioletti, et al. [79] in order to study human knee ligaments and tendons. As the authors state in [79], similar to the motivation explained in Sections 2(c) and 3, the realistic three-dimensional viscoelastic constitutive law they propose, which also takes into account the strain rate effect in soft tissue, verifies four fundamental necessities. Namely, it describes nonlinear stress-strain curves, the strain-rate is given as an explicit variable, large deformations are justified, and finally thermodynamic principles are satisfied. The entropy production is assumed to be entirely due to viscous dissipation and the external thermodynamic state variable is given by $\dot{\mathbf{C}}$ (recall (2.10) in relation to this) so that nonlinear strain-rate dependency of materials is captured. These models have been studied extensively modelling soft tissues such as brain, tongue tissue, tendons and skeletal muscles (see [104] and references therein). In these models, similar to the situation in (3.14), an additive relationship between the elastic (corresponding to the hyperelastic strain energy density) and viscoelastic stress (corresponding to a viscous dissipation potential) is introduced. In [104] experimental data of human brain tissue gray matter is compared with numerically fitted response in uniaxial and simple shear deformations showing that such models capture all features of stress-strain data.

Modelling of viscoelastic response of materials with limiting strain is yet another scientific phenomena which can be be seen as an application of nonlinear viscoelasticity of strain-rate type. Starting with Rajagopal [84], implicit constitutive modelling for the response of materials have been studied and used in order to express the nonlinear relationship between the stress and the strain differently (see also [86] for implicit thermomechanical modelling for thermoviscoelastic solids). Namely, by expressing the strain as a function of the stress, unlike what had been traditionally adopted. Moreover, as a result of a series of articles of Rajagopal and his co-authors (see [92] for all the related references) it was understood that, by using implicit constitutive modelling one can obtain a nonlinear stress-strain relation even when the strain is linearized, which is impossible to achieve in Cauchy elasticity. Additionally, it becomes possible that when the strain reaches a certain limiting value, any further increase in stress will not cause any change in strain. Such models are called *strain-limiting models*. The main advantage of such models is to be able to treat the linearized strain even for arbitrary large values of the stress since the theory allows for the gradient of the displacement to stay small. A general framework for the elastic setting has been studied extensively (see [19] for a survey as well as the treatment in three dimensions). For viscoelasticity, on the other hand, not many mathematical investigations were present since the work of Merodio and Rajagopal [65]. However, recently strain-limiting viscoelasticity has attracted a lot of attention due to the observed physical applications in mechanics and material science such as Gum metal and titanium alloys. Şengül in [92] gives a detailed overview of the theory as well as the references to consult for physical experiments. In [34], Erbay and Şengül considers the one-dimensional problem with a constitutive equation with strain-rate dependence. Using this constitutive relation and the equation of motion, they were able to obtain a partial differential equation which was new and interesting from two points of view. One was that the variable was the stress rather than the strain which was causing difficulties for the application of usual analytical methods in the analysis. Secondly, the inertia term was nonlinear unlike usual dynamic equations of motion and this was making the problem even more intractable. In this paper, authors were investigating traveling wave solutions for the nonlinear partial differential equation they deduced. In [33], authors proved local-in-time existence of strong solutions to the same model introduced in [34]. Very recently, Erbay and Şengül [35] also investigated constitutive equations with stress-rate dependence and proposed a new model for limiting strain behaviour. They not only showed that the one-dimensional model they proposed is thermodynamically consistent, but also that the traveling wave solutions coincide with that of the strain-rate model they studied before. Even more recently, Bulíček, Patel, Şengül and Süli [20], [21] proved existence and uniqueness of global weak solutions with periodic and Dirichlet boundary conditions.

Certainly, there are many more application areas in solid mechanics where nonlinear viscoelasticity of strain-rate type modelling is used. The range of such areas depend substantially on the technological advances resulting in performance of physical experiments (see e.g. [109] for mechanics of cellulose nanofibril composite hydrogels, [30] for applications in polymers which is motivated by molecular theories of viscoelasticity). Nevertheless, in order to capture all the physical phenomena observed during experiments, novel approaches and newly developed techniques are necessary which can only be achieved by significant improvements in mathematical analysis. Besides analytical achievements, numerical treatment of existing mathematical models also provides knowledge for the observed and the expected phenomena (see e.g. [53] for finite-element applications in polymeric structures, [15] for computational treatment of soil).

6. Conclusion

Having selected a big collection of studies on nonlinear viscoelasticity of strain-rate type, starting from many decades ago up to the present day, it is expected that one is convinced for great developments in the theory many of which are motivated by real observations via experiments. This shows that introduction of ideas from other disciplines is necessary in order to pose fundamental new questions. Additionally, it is clear that development of such mathematical theories allows perception of these new methods within other branches of mathematics, more generally, science. It must be a great pleasure for us to be able to contribute in the "written" book of nature by means of modelling and analysis. Having experienced unfamiliarized changes in our lives as human beings, we must hope that future will hold many more noble questions for us to explore and answer, so that the universe is slowly but surely understood even more.

Ethics. There are no ethical considerations associated with this review.

Data Accessibility. This article has no additional data.

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