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VISCOELASTICITY WITH LIMITING STRAIN

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Dedicated to Alexander Mielke on the occasion of his 60th birthday

Abstract. A self-contained review is given for the development and current state of implicit constitutive modelling of viscoelastic response of materials in the context of strain-limiting theory.

1. **Introduction.** It is essential to study realistic mathematical models and properties of solutions of the governing equations in order to get a good understanding of nature. Elasticity theory allows us to use this information so as to gain some knowledge about properties of materials. A class of materials we encounter frequently in our daily lives is the one showing viscoelastic response. Viscoelasticity, by definition, involves the material response of both elastic solids and viscous fluids, which can be modelled linearly or nonlinearly (cf. [74]). The main purpose of this article is to discuss nonlinear viscoelasticity in solids within the context of strain-limiting theory. There is a considerable amount of literature dealing with the analysis of elastic and viscoelastic models based in a hyperelastic setting such as [34, 35, 36, 37]. However, the strain-limiting theory goes much beyond this and calls for new analytical tools.

It is not unexpected to believe that implicit constitutive theories allow for a much more general structure than explicit ones in order to study response of materials. In fact, explicit constitutive models, where the stress is given explicitly as a function of the strain (or other kinematical variables) can be seen as a sub-class of implicit relations, where the inverse might also be true. That is, as a result of implicit constitutive modelling, the strain could be given as a function of the stress. In fact, as Truesdell [73] explains in his seminal work, in a constitutive equation, a relation between the force, which is the cause, and the deformation, which is the effect, is given. Therefore, since in continuum mechanics the force is specified by the stress, in a constitutive relation the strain should be given in terms of the stress. This is clearly not what has been adopted so far. Recently, Rajagopal [46] introduced a new framework allowing for such relations to describe mechanics of continuous media through implicit constitutive theories. As he explains, within the context of viscoelasticity, implicit constitutive theories arise naturally. However, some of the models that are in place are not truly implicit in the sense that an explicit relation is possible to obtain by integrating the equation for the stress

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under a suitable initial condition. For this reason a more careful study of some notions should be carried out, an example of which is that of material symmetry, so that the resulting constitutive theory would be responsible to represent all present and possible situations. The theory of limiting strain provided by such implicit constitutive relations is able to explain some experimental observations that have not been understood rigorously before such as Gum metal (see e.g. [40, 67, 72, 75]), and other titanium alloys (see e.g. [68, 20, 21, 76])(see also [12, 28] and references therein), as well as describing the response of soft solids (see e.g. [10] and references therein).

2. **Strain-limiting theory.** The concept of constructing implicit constitutive models that not only mathematically contain a limiting strain but also are thermodynamically consistent was introduced by Rajagopal [46]. As he points out, the implicit theory he considers is for the bodies that do not dissipate and for which the stored energy is not a function of only the deformation gradient. Since for such bodies the stress cannot be expressed in terms of the derivative of the stored energy with respect to the deformation gradient, and hence the body is not hyperelastic, he calls such a response the *implicit non-hyperelastic response*. The non-dissipative response where the stress cannot be expressed explicitly as a function of the strain can be illustrated as in Figure 1, where σ stands for the one-dimensional stress and ϵ is the linearized strain (cf. [46]). As the stress increases, the slope of the curve tends to infinity as the strain reaches a critical value (cf. [43]).

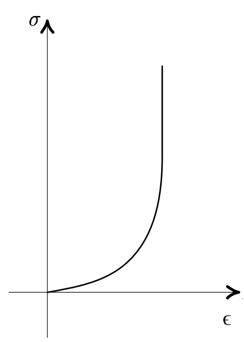


FIGURE 1. Limiting strain behaviour

Using the implicit relation between the stress and the strain, as a result of the linearization under the assumption that the strain, or the gradient of the displacement, is small (see condition (1)), one can obtain a model where a nonlinear relationship between the linearized strain and the stress is specified (see Section 4 for the modelling). Moreover, it becomes possible that when the strain reaches a certain limiting value, any further increase in stress will not cause any changes in strain. Such models are called *strain-limiting models*. The advantage of this new theory is that it allows for the gradient of the displacement to stay small so that one could treat the linearized strain, even for arbitrary large values of the stress. This is, in fact, observed in many experiments today as there are many experimental results proving that both engineered (such as composite materials with flexible microstructures) and natural materials (such as biological tissues composed of collagen fibrils, see Figure 2) might show limiting strain response. Also, such a phenomenon occurs during the fracture of brittle materials where the strain could be bounded at the crack tip as the stress gets infinitely large (see e.g. [23], [64]). A related concept is that of *strain-locking materials* which are hyperelastic and firstly mentioned by Prager [45] in 1957 (see also [11] and [44]). For him, plastic behaviour was an extreme type of soft behaviour, while strain locking response was the opposite, an extreme type of hard behaviour. Therefore, he thought the best way to do modelling was to use the analogy with perfectly plastic solids as a guide to generalize stress-strain laws which show promise to be adoptable to various experiments once they become available.

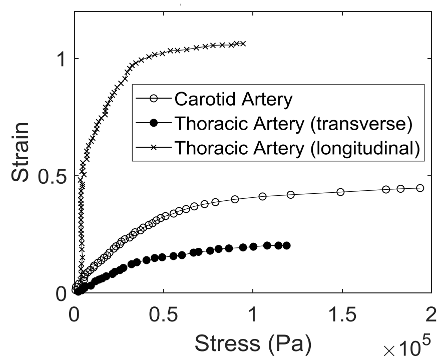


FIGURE 2. Experimental data for the stress-strain relationship for porcine carotid and thoracic artery tissues (cf. [43]).

3. **Kinematics.** Let $\mathbf{u}(\mathbf{x}, t)$ be the displacement of the body at the current position $\mathbf{x} \in \mathbb{R}^3$ of a particle $\mathbf{X}$ in the reference configuration at time t. That is, $\mathbf{u} = \mathbf{x} - \mathbf{X}$. We can define the deformation of the body, which is assumed to be stress free initially, as $\boldsymbol{\chi}(\mathbf{X}, t)$ so that the deformation gradient is defined as $\mathbf{F} = \partial\boldsymbol{\chi}/\partial\mathbf{X}$. This is the mathematical tool to describe choices of functions as models of deformations. Once we have the deformation gradient, by the polar decomposition theorem (cf. [19]) we can ensure existence of positive definite, symmetric tensors $\mathbf{U}$ and $\mathbf{V}$, and a rotation $\mathbf{R}$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R},$$

where $\mathbf{U}$ is *the right* and $\mathbf{V}$ is *the left Cauchy-Green stretch tensor*. Moreover, we know that each of these decompositions is unique and

$$\mathbf{C} = \mathbf{U}^2 = \mathbf{F}^T\mathbf{F}, \quad \mathbf{B} = \mathbf{V}^2 = \mathbf{F}\mathbf{F}^T,$$

where $\mathbf{B}, \mathbf{C}$ are called *the right* and *the left Cauchy-Green deformation tensors*, respectively. We can define the velocity as $\mathbf{v} = \partial\boldsymbol{\chi}/\partial t$, as well as $\mathbf{D}$, which is the symmetric part of the gradient of the velocity field $\mathbf{L} = \partial\mathbf{v}/\partial\mathbf{x}$. That is,

$$\mathbf{D} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T).$$

Under the assumption that

$$\max_{\mathbf{x},t} |\nabla\mathbf{u}| = O(\delta), \qquad \delta \ll 1, \tag{1}$$

one can obtain the linearized strain as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right],$$

where $\nabla\mathbf{u} := \partial\mathbf{u}/\partial\mathbf{x}$ and $|\cdot|$ stands for the usual trace norm. Using this definition, one can define the linearized counterpart of $\mathbf{D}$ as $\boldsymbol{\epsilon}_t = \partial\boldsymbol{\epsilon}/\partial t$.

4. **Modelling.** In Cauchy elasticity, the relation between the stress and the strain would be given by the explicit relation $\mathbf{T} = \mathbf{G}(\mathbf{F}, \mathbf{X})$. Consideration of homogeneous bodies allows one to drop $\mathbf{X}$ as a variable so that the stress is given only as a function of the deformation gradient as

$$\mathbf{T} = \mathbf{G}(\mathbf{F}). \tag{2}$$

As mentioned before, even though these kind of explicit constitutive relations are quite successful in describing the response of a wide variety of solids, they are not able to capture many important observed features such as the nonlinear relationship between the stress and the strain even when the strains are so small. Even though some implicit models for describing the elastic response of solids had existed for a considerable amount of time, the importance of the cause and the effect in those descriptions was not realized until recently when Rajagopal [46] introduced a more general framework to describe material response, namely by means of an implicit relation of the form

$$\mathbf{G}(\mathbf{T}, \mathbf{F}) = \mathbf{0}. \tag{3}$$

Rajagopal [53] studies the implicit constitutive relations for frame-indifferent and isotropic bodies. Taking the general invariance requirements into account, from (3) he derives

$$\mathbf{G}(\mathbf{T}, \mathbf{B}) = \mathbf{0} \tag{4}$$

as the implicit relation for an isotropic body. A classical reference where a detailed calculation of representation of $\mathbf{G}$ in terms of material moduli and the principal invariants is the work of Spencer [70] (one can also refer to [4], [52] or [56] and references therein). There, one can find explicit calculations for invariants of matrices based on the Cayley-Hamilton theorem (cf. [70, Sections 2.4 and 2.5]), which says that every matrix satisfies its own characteristic equation, so for any 3×3 matrix $\mathbf{A}$ we have

$$\mathbf{A}^{\mathbf{3}} - (\operatorname{tr}\mathbf{A})\mathbf{A}^2 + \frac{1}{2}[(\operatorname{tr}\mathbf{A})^2 + \operatorname{tr}\mathbf{A}^2]\mathbf{A} - [\frac{1}{3}\operatorname{tr}\mathbf{A}^3 - \frac{1}{2}\operatorname{tr}\mathbf{A}\operatorname{tr}\mathbf{A}^2 + \frac{1}{6}(\operatorname{tr}\mathbf{A})^3]\mathbf{I} = \mathbf{0}.$$

However, Spencer himself gives credit to Rivlin [66] for the calculation related to the special case we are interested in. In fact, Rivlin [66, pages 698-701] explicitly states that (4) can be written as

$$\begin{aligned}\mathbf{G}(\mathbf{T}, \mathbf{B}) = {} & \chi_0\mathbf{I} + \chi_1\mathbf{T} + \chi_2\mathbf{B} + \chi_3\mathbf{T}^2 + \chi_4\mathbf{B}^2 + \chi_5(\mathbf{T}\mathbf{B} + \mathbf{B}\mathbf{T}) \\ & + \chi_6(\mathbf{T}^2\mathbf{B} + \mathbf{B}\mathbf{T}^2) + \chi_7(\mathbf{B}^2\mathbf{T} + \mathbf{T}\mathbf{B}^2) + \chi_8(\mathbf{T}^2\mathbf{B}^2 + \mathbf{B}^2\mathbf{T}^2),\end{aligned} \tag{5}$$

where the χ's are depending on the scalar invariants of $\mathbf{T}$ and $\mathbf{B}$ expressible in terms of

$$\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{B}, \operatorname{tr}\mathbf{T}^{\mathbf{2}}, \operatorname{tr}\mathbf{B}^{\mathbf{2}}, \operatorname{tr}\mathbf{T}^{\mathbf{3}}, \operatorname{tr}\mathbf{B}^{\mathbf{3}}, \operatorname{tr}\mathbf{T}\mathbf{B}, \operatorname{tr}\mathbf{T}^{\mathbf{2}}\mathbf{B}, \operatorname{tr}\mathbf{T}\mathbf{B}^{\mathbf{2}}, \operatorname{tr}\mathbf{T}^{\mathbf{2}}\mathbf{B}^{\mathbf{2}}.$$

Clearly, depending on the properties of $\mathbf{G}$, relation (5) might allow one to write the stress as a function of the strain, as it is done in classical elasticity, or the strain as a function of the stress, as expected by the causality argument by Truesdell that was mentioned before.

There is another important advantage of the framework introduced by Rajagopal that should be pointed out. Namely, under the assumption (1), as a result of linearization of the strain in the classical Cauchy elasticity, we obtain

$$\mathbf{T} = \mathbb{C}\colon \boldsymbol{\epsilon}, \tag{6}$$

where ': ' stands for matrix product and $\mathbb{C}$ is the fourth order elasticity tensor. As Saravanan [69] explains, (6) is obtained since the stress tensor, which is a second order tensor, is linearly related to another second order tensor, which is the strain, giving the action of a fourth order tensor $\mathbb{C}$ on the strain. Moreover, one could invert (6) and write the strain in terms of the stress as

$$\boldsymbol{\epsilon} = \mathbb{D}\colon \mathbf{T}. \tag{7}$$

Here, $\mathbb{D}$ is the fourth order elastic compliance tensor (cf. [69]). In conclusion, one could only obtain a linear relation between the stress and the strain. On the other hand, as Rajagopal [49] explains, equation (5) leads to an approximation for a different small displacement gradient theory allowing for a nonlinear relationship between the linearized strain and the stress (see also [3, 4, 6, 46, 48, 52]). We use exactly the same small displacement gradient approximation leading to the classical linearized theory of elasticity as in (6) and (7), but within the context of (5). Under the assumption (1), $\mathbf{B}$ becomes $\mathbf{B} = \mathbf{I} + 2\,\boldsymbol{\epsilon} + O(\delta^2)$, and hence (5) reduces to

$$\boldsymbol{\epsilon} + \hat{\chi}_0\mathbf{I} + \hat{\chi}_1\mathbf{T} + \hat{\chi}_2\mathbf{T}^2 + \hat{\chi}_3(\mathbf{T}\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}) + \hat{\chi}_4(\mathbf{T}^2\boldsymbol{\epsilon} + \boldsymbol{\epsilon}\mathbf{T}^2) = \mathbf{0}, \tag{8}$$

where the $\hat{\chi}_i$'s for $i = 0, 1, 2$, are scalar-valued functions depending at most linearly on $\boldsymbol{\epsilon}$ and arbitrarily on the invariants of $\mathbf{T}$, while for $i = 3, 4$, they are functions of the invariants of $\mathbf{T}$ only (cf. [10, 54]). Obviously, this is a nonlinear relationship between the linearized strain and the stress. Moreover, there is no restriction on the stress, which can be arbitrarily large, where the linearized strain $\boldsymbol{\epsilon}$ is necessarily small. As mentioned before, this is observed in many experiments and could not be given an explanation due to lack of such a mathematical tool. In most of the studies, since (5) is unnecessarily complicated to work with, a simpler sub-class given by

$$\mathbf{B} = \tilde{\chi}_0\mathbf{I} + \tilde{\chi}_1\mathbf{T} + \tilde{\chi}_2\mathbf{T}^2 \tag{9}$$

is considered. Here, the $\tilde{\chi}$'s depend on the scalar invariants $\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{T^2}, \operatorname{tr}\mathbf{T^3}$. Under the assumption (1), (9) becomes

$$\boldsymbol{\epsilon} = \bar{\chi}_0\mathbf{I} + \bar{\chi}_1\mathbf{T} + \bar{\chi}_2\mathbf{T}^2, \tag{10}$$

with some invariant-dependent coefficients $\bar{\chi}$'s.

5. **Elasticity.** A considerable amount of literature is present on the analysis of models resulting from (4) with limiting small strain in an elastic setting. To start with, Rajagopal [49, 50, 51] proposed several constitutive relations exhibiting strain-limiting behaviour some of which can be described by (see also [5] and [26])

$$\boldsymbol{\epsilon} = \beta_0\mathbf{I} + \frac{\mathbf{T}}{\alpha_0\big(1 + \gamma_0(\operatorname{tr}\mathbf{T^2})^{r/2}\big)^{1/r}}, \tag{11}$$

where β_0 depends on $\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{T^2}$, α_0, γ_0 and r are positive constants. The first study of an existence and uniqueness theory in a multi-dimensional setting for a strain-limiting nonlinear elastic model is by Bulíček, Málek and Süli [5] where problem

$$\begin{aligned} -\operatorname{div}\ \mathbf{T} &= \mathbf{f}, \\ \mathbf{G}(\mathbf{T}, \mathbf{B}) &= \mathbf{0} \end{aligned} \tag{12}$$

is analyzed. For the constitutive relation in (12), they consider (11) and neglect the spherical part by taking $\beta_0 = 0$, which helps to simplify the calculations while keeping the essential mathematical difficulties. Also, they set $\alpha_0 = 1$ for simplicity. Moreover, they consider an axiparallel parallelepiped as their domain with spatially periodic boundary conditions. This is an important simplification since it allows the authors to introduce the concept of a *renormalized* solution, which is a weak solution under some conditions on the regularity of the stress tensor. Moreover, they heavily use the properties of this spatially periodic setting in order to construct the solution via the numerical method called the Fourier spectral method. For different values of r they prove existence and uniqueness of weak and only existence of renormalized solutions to the problem. Similar results for general bounded domains in multi-dimensional setting are obtained by Bulíček et al. [4]. With the same special $\boldsymbol{\epsilon}$, Gelmetti and Süli [16] look at the spectral approximation for problem (12). Before these, however, many authors considered special cases such as simple shear, torsion, extension, etc. For example, Bustamante and Rajagopal [7] derive equations governing plane stress and plane strain within the context of implicit constitutive theories. Moreover, since these equations are too complicated even in two-dimensional setting, they develop a weak formulation for which numerical analysis is possible to be done. Bustamante [6] extends these calculations in terms of both the development of equations and the numerical analysis. None of these works, however, solve any specific boundary value problems. Rajagopal [50] does this for some simple problems such as extension, shear and torsion. Also, Bustamante and Rajagopal [8] solve some one-dimensional problems for constitutive relations of the form (10) where the stresses are both homogeneous and inhomogeneous.

A different approach is to consider related wave propagation problems. As an example, Kannan et al. [26] consider the constitutive relation (11) with $\beta_0 = \beta(\mathrm{tr}\mathbf{T})$ where $\beta \leq 0$, $\alpha_0 = \alpha$, $\gamma_0 = \gamma/2$, and $r = 2 = -1/n$, for constants β, α, γ, r and n. Then, on the slab $\{(X,Y,Z)| -\infty \leq X \leq \infty, 0 \leq Y \leq y_0, -\infty \leq Z \leq \infty\}$, which has thickness y_0, the motion of the unsteady shear can be represented as $x = X + f(Y,t), y = Y, z = Z$. In one space dimension, the balance of linear momentum (with unit density of the reference configuration) together with the constitutive relation (11) gives the system of equations $u_{tt} = T_y$ and $u_y = \alpha(1+\gamma T^2)^n T$. Eliminating the coupling between the displacement and the shear stress leads to the equation

$$T_{yy} = \alpha\big[\big(1+\gamma T^2\big)^n T\big]_{tt},$$

which is a higher order equation for the stress highlighting the fact that the shear stress is governed by a nonlinear hyperbolic equation. Some other wave propagation related studies also exist such as [9, 26, 30, 32, 55], and they will be investigated in more detail in Section 6 since they are dissipative.

Fracture, as mentioned in the introduction, is one of the principle issues urging the necessity to understand the nonlinear relationship between the stress and the strain when the strain remains bounded, possibly infinitesimal, while the stress can become arbitrarily large. The first work investigating fracture within the context of strain-limiting theories of elasticity is by Rajagopal and Walton [64] where the simplest fracture mode, anti-plane shear, is considered. For that they take

$$\boldsymbol{\epsilon} = \phi(|\mathbf{T}|)\mathbf{T}, \tag{13}$$

where ϕ is a monotone decreasing function such that $r\phi(r) \to 1$ as $r \to \infty$. In that work, authors perform an asymptotic analysis of the solution of a suitably posed boundary-value problem in a neighbourhood of the crack tip using a variational approach. Following this, Gou et al. [18] obtained similar results with (13) for the plane-strain fracture problem where they use strong ellipticity properties of such implicit strain-limiting theories which was investigated by Mai and Walton [31]. Similarly, Itou et al. [22, 23, 24] investigated crack problems. The most recent one on fracture for small strain elasticity is by Kulvait et al. [29] where the state of anti-plane stress of the body with a smoothened V-notch is considered (for the numerical study of the same problem see [27]). Some numerical studies with different geometry of the domains also exist (see e.g. [77]). More generally than fracture, in many studies, strain limiting implicit constitutive equations in regions close to geometric discontinuities such as holes, notches and cracks have been investigated by solving boundary value problems using finite element analysis, see e.g., [38] and [41].

As mentioned above, the so-called strain-locking materials show a similar stress-strain relation. There are many studies in the literature investigating problems related to such materials in the elastic setting. For example, Benešová et al. [2] use gradient Young measures to show existence of minimizers for variational problems modelling ideal locking in elasticity. They study relaxation for the locking constraint which models the fact that once the strain gets too large, the material leaves the elastic regime under strong tension. Some other recent studies for strain-locking materials also exist such as the work by Golay and Seppecher [17], where the equilibrium of locking materials is studied in connection with a problem of structural optimisation. More precisely, authors show that the problem of fictitious material optimisation is equivalent to the equilibrium problem of a perfect locking material via techniques from convex minimisation, regularisation and numerical analysis.

6. **Viscoelasticity.** In order to incorporate viscoelastic behaviour into the implicit constitutive relation (4) one needs to consider

$$\mathbf{G}(\mathbf{T}, \mathbf{B}, \mathbf{D}) = \mathbf{0}. \tag{14}$$

If the stress is expressible as a function of the kinematics, instead of (14) one considers

$$\mathbf{T} = \mathbf{f}(\mathbf{B}, \mathbf{D}), \tag{15}$$

where $\mathbf{f}$ is a (possibly) nonlinear function. As explained by Merodio and Rajagopal [33] model (15) is a generalization of an isotropic Kelvin-Voigt solid model (see also [56]). Moreover, it is mentioned that due to the presence of the stretching tensor $\mathbf{D}$, models of the form (15) have some serious disadvantages with regard to describing the response of viscoelastic solids. One of them is its incapability for stress relaxation (as in the classical neo-Hookean elastic solid), and the second one is that a finite jump in the strain leads to an infinite response in the stress (as in the classical viscous fluid model). Nevertheless, model (15) has been widely used to describe response of viscoelastic solids in geomechanics and biomechanics.

Considering similar objectivity requirements as before, one can represent (15) as

$$\begin{aligned}\mathbf{T} =& \alpha_0\mathbf{I} + \alpha_1\mathbf{B} + \alpha_2\mathbf{D} + \alpha_3\mathbf{B}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{B}\mathbf{D} + \mathbf{D}\mathbf{B}) \\ &+ \alpha_6(\mathbf{B}^2\mathbf{D} + \mathbf{D}\mathbf{B}^2) + \alpha_7(\mathbf{D}^2\mathbf{B} + \mathbf{B}\mathbf{D}^2) + \alpha_8(\mathbf{B}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{B}^2),\end{aligned} \tag{16}$$

where α's are functions of the set of invariants expressed in terms of $\mathbf{B}$ and $\mathbf{D}$. Note that classical nonlinear versions of the Kelvin-Voigt model such as

$$\mathbf{T} = k_0\mathbf{I} + k_1\mathbf{B} + k_2\mathbf{D} \tag{17}$$

in the compressible case, where the coefficients are constants, are special subclasses of (16). Similar models of type (17) and corresponding versions in the incompressible case have been widely studied in the literature (see e.g. [56]).

We are, on the other hand, interested in models where the strain is given as a function of the stress. Following Rivlin [66] as was done for (4), we obtain the representation for (14) as

$$\begin{aligned}\mathbf{0} = &\Big(\mathbf{TBD} + \mathbf{TDB} + \mathbf{BDT} + \mathbf{BTD} + \mathbf{DTB} + \mathbf{DBT}\Big)\\ &- \mathbf{T}\big(\mathrm{tr}\,\mathbf{B}\,\mathbf{D} - \mathrm{tr}\,\mathbf{B}\,\mathrm{tr}\,\mathbf{D}\big) - \mathbf{B}\big(\mathrm{tr}\,\mathbf{DT} - \mathrm{tr}\,\mathbf{D}\,\mathrm{tr}\,\mathbf{T}\big) - \mathbf{D}\big(\mathrm{tr}\,\mathbf{TB} - \mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\mathbf{B}\big)\\ &- \big(\mathbf{BD} + \mathbf{DB}\big)\mathrm{tr}\,\mathbf{T} - \big(\mathbf{DT} + \mathbf{TD}\big)\mathrm{tr}\,\mathbf{B} - \big(\mathbf{TB} + \mathbf{BT}\big)\mathrm{tr}\,\mathbf{D}\\ &- \mathbf{I}\big(\mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\mathbf{B}\,\mathrm{tr}\,\mathbf{D} - \mathrm{tr}\,\mathbf{T}\,\mathrm{tr}\,\mathbf{BD} - \mathrm{tr}\,\mathbf{B}\,\mathrm{tr}\,\mathbf{DT} - \mathrm{tr}\,\mathbf{D}\,\mathrm{tr}\,\mathbf{TB} + \mathrm{tr}\,\mathbf{TBD} + \mathrm{tr}\,\mathbf{DBT}\big).\end{aligned} \tag{18}$$

As one can clearly observe, (18) is highly nonlinear and not of a manageable size in terms of calculations. One can, however, work on a subclass while keeping essential features about modelling. In order to do that, we can consider a special subclass of (14) which is given by

$$\gamma\mathbf{B} + \nu\mathbf{D} = \beta_0\mathbf{I} + \beta_1\mathbf{T} + \beta_2\mathbf{T}^2, \tag{19}$$

where $\beta_i = \beta_i(I_1, I_2, I_3)$, $i = 0, 1, 2$, $I_1 = \mathrm{tr}\mathbf{T}, I_2 = \frac{1}{2}\mathrm{tr}\mathbf{T}^2, I_3 = \frac{1}{3}\mathrm{tr}\mathbf{T}^3$, and non-negative constants γ and ν represent the behaviour of the material. To be more precise, when $\nu = 0$, (19) reduces to the elastic case (9). For $\gamma = 0$ we obtain the behaviour of a fluid (see e.g., [1] for the treatment of (19) in the incompressible case). To ensure that the material is a viscoelastic solid, one can take γ to be equal to 1 and ν to be nonzero. Then, under assumption (1), from (19) one obtains

$$\boldsymbol{\epsilon} + \nu\boldsymbol{\epsilon}_t = \hat{\beta}_0\mathbf{I} + \hat{\beta}_1\mathbf{T} + \hat{\beta}_2\mathbf{T}^2. \tag{20}$$

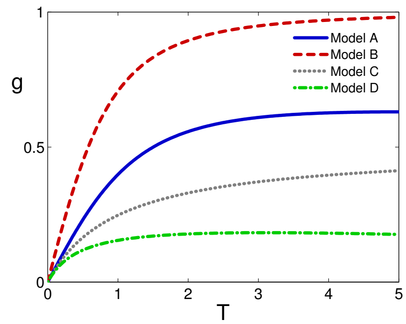
Under further restrictions Rajagopal and Saccomandi [55] studied (20) and investigated propagation of circularly polarized transverse stress waves, standing shear stress waves and oscillatory shear stress waves within the context of strain-limiting theory. Erbay and Şengül [13] consider

$$\boldsymbol{\epsilon} + \nu\boldsymbol{\epsilon}_t = \mathbf{g}(\mathbf{T}) \tag{21}$$

as a generalization of (20), where $\mathbf{g}$ is a (possibly) nonlinear function of the stress, and they derive the nonlinear partial differential equation given by

$$T_{xx} + \nu T_{xxt} = g(T)_{tt}. \tag{22}$$

As explained in detail in [13], in order to obtain (22), they differentiate the one-dimensional form of (21) twice with respect to t, and the balance of momentum equation $u_{tt} = T_x$ both with respect to x and t, under sufficient smoothness assumptions and after the introduction of dimensionless quantities. Adopting widely studied forms of the nonlinear constitutive relation $g(T)$, they investigate traveling wave solutions of (22). This equation differs from classical models of viscoelasticity in the sense that it is given in terms of the stress rather than the displacement (or the deformation, or the strain). Moreover, the nonlinearity is on the inertia term, which not only makes (22) difficult to attack analytically, but also interesting on its own right. The choices they make for the nonlinearity include general quadratic and cubic models, which, depending on the choice of the parameters, might give rise to negative or large positive values of $g(T)$ with increasing values of the stress, see Figure 3. Therefore, even though the quadratic and cubic models may give rise



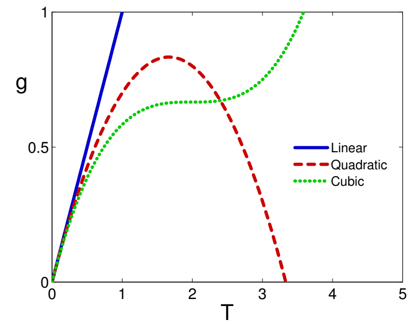


FIGURE 3. Left. Model A: $g(T) = \beta T + \alpha\left(1+\frac{\gamma}{2}T^2\right)^n T$; Model B: $g(T) = \frac{T}{(1+|T|^r)^{1/r}}$; Model C: $g(T) = \alpha\left\{\left[1-\exp\left(-\frac{\beta T}{1+\delta|T|}\right)\right]+\frac{\gamma T}{1+|T|}\right\}$; Model D: $g(T) = \alpha\left(1-\frac{1}{1+\frac{T}{1+\delta|T|}}\right)+\beta\left(1+\frac{1}{1+\gamma T^2}\right)^n T$, where $\alpha,\beta,\gamma,\delta,n$ and $r>0$ are constants. Right. General linear, quadratic and cubic nonlinearities.

to physically unacceptable strain values and the small strain assumption of strain-limiting viscoelastic solid is violated for large and positive values of such $g(T)$, it is worth considering them as the simplest representatives of the nonlinear models. Moreover, it is possible to find traveling wave solutions analytically in these cases. They also consider some other nonlinearities, namely Models A, B, C and D, which are commonly studied in the literature in the elastic setting. Note the similarities between the experimental data in Figure 2 and models A, B, C and D in Figure 3. Also note that due to the constitutive relation (21), the vertical axis in Figure 3 measures the sum of the linearized strain and the strain rate, and Model B is related to the one-dimensional version of (11). Moreover, all of these models are related to one another, e.g., when $\beta = 0, n = -1/2, \alpha = 1$ and $\gamma = 2$, Model A becomes equivalent to Model B with $r = 2$.

Similarly, Şengül [71] investigates travelling wave solutions for (22) for an arctangent type g, which, in general, can be written as

$$g(T) = \aleph \arctan(\vartheta T), \tag{23}$$

where $\aleph$ and ϑ are positive constants. In fact, as explained by Bustamante et al. [32], relation (23) is proposed as an approximation of the expression

$$g(T) = \alpha\left[\left(-1+\frac{1}{1+\beta T}\right)+\frac{\gamma}{(1+\iota T^2)^{1/2}}T\right], \tag{24}$$

which was introduced and studied in [8, 42], where α,β,γ and ι are constants. Clearly, as explained in [32], equation (24) models the response of an elastic body for which the strain remains small independently of the magnitude of the stress. Moreover, the relation (23) is more reasonable than (24) to study the qualitative properties of the implicit solutions. This type of relation, together with an exponential and polynomial type nonlinearities, are discussed by Bustamante et al. [32] where the particular case of a one-dimensional bar is considered and boundary-value problems are analyzed in an elastic setting.

Recently, Erbay et al. [15] prove local-in-time well-posedness of strong solutions for the Cauchy problem arising in one-dimensional viscoelasticity corresponding to (22) under some assumptions on the nonlinearity g. In their work, assuming that g is strictly increasing, authors convert the problem to a new form for the strain variable. Using techniques from the theory of elliptic operators, they prove local-in-time existence and uniqueness of solutions in the Sobolev space H^s, where $s > 5/2$, both for the unknown and its time derivative, with initial data given in the same space. Their main result includes linearization around a given state, definition of a contractive mapping and the usage of Banachs fixed point theorem. They state their results in terms of the strain, as well as the displacement variables, and for several constitutive functions widely used in the literature, they show that the assumption on which the proof of existence is based is not violated.

It is also possible to describe nonlinear response of viscoelastic solids through an expression in terms of the history of the stress. Muliana et al. [39], for example, develop a new class of viscoelastic models in one-space dimension, for homogeneous and isotropic bodies where the linearized strain is expressed as an integral of a nonlinear measure of the stress. More specifically, they consider (see also [65]) the constitutive relation

$$\epsilon(t) = f(T(0), t) + \int_0^t \frac{\partial f(T(s), t-s)}{\partial T} \frac{dT}{ds} ds, \tag{25}$$

with $f(T(t), t) = \{G(T)\}J(t)$, where the function $G(T)$ is a nonlinear function of the stress, and $J(t)$ can be viewed as a 'generalized creep function' which is chosen to have a special form. It is worth mentioning that when $G(T)$ is a linear function, the model reduces to the constitutive relation for a linear viscoelastic material. The authors call model (25) 'quasi-linear', since the stress is expressed as a linear function of a nonlinear measure of the strain. Such a model, as explained in the article, cannot be obtained as a proper linearization with respect to the gradient of the displacement of a nonlinear model. Recently, Itou et al. [25] considered an implicit viscoelastic constitutive relation of the form (25) in the context of fracture.

One should also not leave out fluid-related studies since they share some characteristics with viscoelasticity of solids. In fact, Rajagopal [47] goes through the same procedure he carries out for solids in order to define implicit constitutive relations for fluids. This time, instead of (4), he starts with

$$\mathbf{F}(\mathbf{T}, \mathbf{D}) = \mathbf{0} \tag{26}$$

in order to incorporate dissipative nature of fluids. Considering similar objectivity requirements as before, instead of (5) he obtains

$$\begin{aligned} &\gamma_0 \mathbf{I} + \gamma_1 \mathbf{T} + \gamma_2 \mathbf{D} + \gamma_3 \mathbf{T}^2 + \gamma_4 \mathbf{D}^2 + \gamma_5(\mathbf{TD} + \mathbf{DT}) \\ &+ \gamma_6(\mathbf{T}^2\mathbf{D} + \mathbf{DT}^2) + \gamma_7(\mathbf{D}^2\mathbf{T} + \mathbf{TD}^2) + \gamma_8(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2) = \mathbf{0}, \end{aligned} \tag{27}$$

where the γ's are depending on the scalar invariants of $\mathbf{T}$ and $\mathbf{D}$ expressible in terms of

$$\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}, \operatorname{tr}\mathbf{T^2}, \operatorname{tr}\mathbf{B^2}, \operatorname{tr}\mathbf{T^3}, \operatorname{tr}\mathbf{B^3}, \operatorname{tr}\mathbf{TD}, \operatorname{tr}\mathbf{T^2D}, \operatorname{tr}\mathbf{TD^2}, \operatorname{tr}\mathbf{T^2D^2}.$$

As he points out (27) includes the classical incompressible Navier-Stokes model as well as many other commonly studied models as special cases. More importantly, he mentions the fact that one could achieve more general rate-type models that are used to describe viscoelastic fluids by incorporating the material time derivatives into the implicit constitutive relation giving

$$\mathbf{F}(\mathbf{T}, \dot{\mathbf{T}}, \cdots, \overset{(n)}{\mathbf{T}}, \mathbf{D}, \dot{\mathbf{D}}, \cdots, \overset{(n)}{\mathbf{D}}) = \mathbf{0}, \tag{28}$$

where the superscript (n) stands for the n-th material time derivative. Similar to that of solids, a thermodynamical derivation is also possible for fluids which is investigated by Rajagopal and Srinivasa [57, 61, 62].

As done by Rajagopal and Srinivasa [63] one can also incorporate thermal effects into the constitutive relation to describe the response of thermoviscoelastic solids. Also, in the light of (28), other thermodynamically consistent models for viscoelastic solids in the context of strain-limiting theory are possible to explore (see e.g., [14]).

7. **Conclusion.** Having touched upon various problems surrounding viscoelasticity in the context of strain-limiting theory, it is natural to ask for possible open problems for the future. It should, now, be clear that implicit constitutive theory accommodates a huge class of models describing different responses of materials and what has been studied so far should be thought to be only a very small portion of the whole range of questions that could be asked.

Considering the studies mentioned throughout the manuscript, an immediate open problem is related to well-posedness of strain-limiting viscoelasticity in three space dimensions. Another mathematical problem to be posed could be generalization of implicit modelling for solids (as in (28) for fluids) so that different types of behaviour are included in the constitutive relation.

By the guidance of new experimental results leading to a good grasp of mechanics behind physically observed phenomena, mathematicians would be able to enlighten the way that nature is understood.

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