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On the Finite Dimensionality of Closed Subspaces in $L_p(M, d\mu) \cap L_q(M, d\nu)$

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Abstract: Finding effective finite-dimensional criteria for closed subspaces in L_p, endowed with some additional functional constraints, is a well-known and interesting problem. In this work, we are interested in some sufficient constraints on closed functional subspaces, $S_p \subset L_p$, whose finite dimensionality is not fixed a priori and can not be checked directly. This is often the case in diverse applications, when a closed subspace $S_p \subset L_p$ is constructed by means of some additional conditions and constraints on L_p with no direct exemplification of the functional structure of its elements. We consider a closed topological subspace, $S_p^{(q)}$, of the functional Banach space, $L_p(M, d\mu)$, and, moreover, one assumes that additionally, $S_p^{(q)} \subset L_q(M, d\nu)$ is subject to a probability measure ν on M. Then, we show that closed subspaces of $L_p(M, d\mu) \cap L_q(M, d\nu)$ for $q > \max\{1, p\}, p > 0$ are finite dimensional. The finite dimensionality result concerning the case when $q > p > 0$ is open and needs more sophisticated techniques, mainly based on analysis of the complementary subspaces to $L_p(M, d\mu) \cap L_q(M, d\nu)$.

Keywords: closed Banach subspace; isometry; embedding; finite dimensionality; probabilistic measure

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1. Introduction

The problems, concerned with the finite dimensionality of closed functional subspaces in L_p (in part, in $L_p(0, 1; \mathbb{C})$), are of long-time interest in analysis, being related to their many applications in operator and approximation theories [1–5], in dynamical systems theory [6–11] and other applied fields. As an example, one can recall a central problem in Banach space theory to classify the complemented subspaces of L_p up to isomorphism; the finite-dimensional analogue is to find for any given $S_p \subset L_p$ a description of the finite-dimensional spaces which are S_p-isomorphic to S_p-complemented subspaces of L_p. These problems were thoroughly studied before [12], in particular finite-dimensional versions of this complemented subspaces of the L_p problem, yet in both cases their classification is far from over.

It was observed that sometimes, the finite-dimensional version of an infinite-dimensional problem leads to a theory which is much more interesting than the infinite-dimensional theory. Here, one can recall the problem of describing the subspaces of L_p which embed isomorphically into a "smaller" L_p space; namely, the space l_p, for which there is a fairly good answer [12]. One can recall that density on a probability space, M, is a strictly positive measurable function $h : M \to \mathbb{R}_+$ for which $\int h d\mu = 1$. Such a density h induces for fixed $0 < p < \infty$ an isometry $J_h^{(p)}$ from $L_p(M, d\mu)$ onto $L_p(M, h d\mu)$. The next result, due to D. Lewis [13,14], gives useful information about chosen a priori finite-dimensional subspaces $S_p \subset L_p$.

Theorem 1. *Let μ be a probability measure on M, and let S_p be a N-dimensional subspace of $L_p(M, d\mu)$, $0 < p < \infty$, with full support. Then, there is a density $h > 0$ so that the image $J_h^{(p)} S_p$ has a basis $\{\varphi_1, \varphi_2, \dots, \varphi_N\} \subset L_2(M, hd\mu)$, which is orthonormal in $L_2(M, hd\mu)$ and such that $\sum_{j=\overline{1,N}} |\varphi_j|^2 = N$.*

Assuming that S_p is already a subspace of L_p for some finite $\dim S_p \in \mathbb{N}$, one can randomly pick a few coordinates and hope that the natural projection onto these coordinates restricted to S_p is a good isomorphism. If we do this with *no additional preparation*, this will not work. Indeed, the subspace S_p may contain a vector with small support, say one of the unit vector basis elements of l_∞^N, in which case, the chance that a coordinate in its support is picked is small. Of course, if no such coordinate is picked, the said projection cannot be an isomorphism on S_p. The point is that one wants to change S_p first to another isometric copy of S_p, in which each element of S_p is spread out. This can be performed by a change of density. This method was used with other tools to produce the best known results.

2. Finite Dimensionality of Closed Subspaces in $L_p \cap L_q$

As the imbedding structure of a priori taken finite-dimensional subspaces in L_p is in many cases very important and instructive, nonetheless finding the effective criteria for closed subspaces in L_p endowed with some additional functional constraints to be finite dimensional remains very important and hard both from theoretical and applied points of view. Below, we are interested in some sufficient constraints on functional closed subspaces $S_p \subset L_p$, whose finite dimensionality is not fixed a priori and cannot be checked directly. This is often the case in diverse applications, when a closed subspace $S_p \subset L_p$ is constructed by means of some additional conditions and constraints on L_p with no direct presentation of the functional structure of its elements. In particular, we consider a topological subspace, $S_p^{(q)}$, of the functional Banach space, $L_p(M, d\mu)$, where μ is a probability measure on measurable space M. Moreover, one assumes that additionally, $S_p^{(q)} \subset L_q(M, d\nu)$ is subject to a probability measure ν on M. Then, we prove the following theorem first announced in [15].

Theorem 2. *Let a closed topological subspace $S_p^{(q)} \subset L_p(M, d\mu)$ belong to $L_q(M, d\nu)$, $q > \max\{1, p\}$, $p > 0$, where measures μ, ν are probabilistic and the measure μ is absolutely continuous with respect to the measure ν on M. Then the subspace $S_p^{(q)}$ is finite dimensional, that is $\dim S_p^{(q)} < \infty$.*

Let us consider a closed topological subspace, $S_p^{(q)}$, of the functional Banach space, $L_p(M, d\mu)$, where μ is a probability measure absolutely continuous with respect to the measure ν on M, and satisfies, in addition, the constraint $S_p^{(q)} \subset L_q(M, d\nu)$ subject to a probability measure ν on M. In order to state Theorem 2, formulated above, we need some lemmas.

Lemma 1. *For any $q > p > 0$, there exists a bounded positive constant $K_{p,q}$, such that*

$$||f||_{q,\nu} \leq K_{p,q} ||f||_{1,\nu} \tag{1}$$

for any $f \in S_p^{(q)} \subset L_p(M, d\mu) \cap L_q(M, d\nu)$.

Proof. As the topological space $S_p^{(q)} \subset L_p(M, d\mu) \cap L_q(M, d\nu)$ is closed in $L_p(M, d\mu)$, one can define the identical imbedding

$$J : S_p^{(q)} \subset L_p(M, d\mu) \to L_q(M, d\nu) \tag{2}$$

If a sequence $\{f_n : n \in \mathbb{Z}_+\} \subset S_p^{(q)}$ converges in $S_p^{(q)}$ to an element $f \to S_p^{(q)}$ with respect to the norm on $L_p(M, d\mu)$ and simultaneously it converges to an element $g \in L_q(M, d\nu)$ with respect to the norm on $L_p(M, d\mu)$, owing to the absolute continuity of the measure μ with respect to ν, one can identify these limits $f \sim g$ almost everywhere. Then, we enter into conditions of the Banach closed graph theorem [16–18] and can infer that there exists such a positive constant $K < \infty$ that

$$||f||_{q,\nu} = ||Jf||_{q,\nu} \leq K||f||_{p,\mu} \tag{3}$$

for any $f \in S_p^{(q)} \cap L_q(M, d\nu)$, where as usual, we denote $||f||_{p,\mu} := \left(\int_M |f|^p d\mu\right)^{1/p}$, $||f||_{q,\nu} := \left(\int_M |f|^q d\nu\right)^{1/q}$. It is easy to check, using the classical Young inequality, that for $1 \geq p > 0$

$$||f||_{p,\mu} = ||f \cdot 1||_{p,\mu} \leq ||f||_{1,\mu}||1||_{(1-p),\mu} \leq ||h||^2_{\infty,\nu}||f||_{1,\nu} \leq ||f||_{1,\nu}, \tag{4}$$

giving rise to (1), where we take into account [19,20] that the Radon–Nikodym derivative $d\mu/d\nu = h \in L_1(M, d\nu) \cap L_\infty(M, d\nu)$ and $||h||_{\infty,\nu} \leq 1$. If $p > 1$, based on the inequality (3), one can also easily obtain that

$$||f||_{q,\nu} \leq K_{p,q}||f||_{1,\nu} \tag{5}$$

for any $f \in S_p^{(q)} \subset L_p(M, d\mu) \cap L_q(M, d\nu)$, if $q > p > 1$. Indeed, consider the next norm transformations, once more based on the Young inequality:

$$\begin{aligned} ||f||^p_{p,\mu} &= \left(\int_M |f|^{\frac{q(p-1)}{(q-1)}} \cdot |f|^{\frac{(q-p)}{(q-1)}} d\mu\right) \leq \\ &\leq ||h||_{\infty,\nu}\left(\int_M |f|^q d\nu\right)^{\frac{p-1}{q-1}}\left(\int_M |f| d\nu\right)^{\frac{q-p}{q-1}} \leq \\ &\leq ||f||_{q,\nu}^{\frac{q(p-1)}{(q-1)}} \cdot ||f||_{1,\nu}^{\frac{q-p}{q-1}}. \end{aligned} \tag{6}$$

Now, making use of the inequality (3), it ensues from (6) that

$$||f||^p_{p,\mu} \leq K^{\frac{q(p-1)}{(q-1)}}||f||_{q,\mu}^{\frac{q(p-1)}{(q-1)}} \cdot ||f||_{1,\nu}^{\frac{q-p}{q-1}}, \tag{7}$$

which reduces, using (3) once more, in the inequality

$$||f||_{q,\nu} \leq K^{\frac{p(q-1)}{(q-p)}}||f||_{1,\nu} := K_{p,q}||f||_{1,\nu} \tag{8}$$

for all $f \in S_p^{(q)} \subset L_q(M, d\nu) \cap L_p(M, d\mu)$, proving the lemma. □

As a usuful consequence from Lemma 1 and the obvious norm property $||f||_1 \leq ||f||_q$ for any $f \in L_1(M, d\nu)$, we can deduce that $S_p^{(q)} \subset L_1(M, d\nu) \cap L_q(M, d\nu)$, which makes it possible to single out from the subspace $S_p^{(q)} \subset L_q(M, d\nu)$ linear-independent functions $\varphi_j \in S_p^{(q)} \subset L_1(M, d\nu), j = \overline{1, N}$, for some $N \in \mathbb{N}$ and construct the closed N-diemensional subspace:

$$S_{p,N}^{(q)} := \mathrm{span}_{\mathbb{C}}\{\varphi_j \in S_p^{(q)} \subset L_1(M, d\nu) : ||\varphi_j||_{1,\nu} = 1, j = \overline{1, N}\}. \tag{9}$$

For fixed $N \in \mathbb{N}$, the subspace $S_{p,N}^{(q)} \subset S_p^{(q)} \subset L_1(M, d\nu), q > \max\{1, p\}, p > 0$, characterizes the next lemma.

Lemma 2. *Given the N-dimensional subspace* $S^{(q)}_{p,N} \subset S^{(q)}_{p} \subset L_1(M, d\nu) \cap L_q(M, d\nu)$*, defined by (9),* $q > \max\{1, p\}, p > 0$*. Then, there exists an N-dimensional subspace* $\tilde{S}^{(q),*}_{p,N} \subset L_\infty(M, d\nu)$ *such that*

$$\tilde{S}^{(q),*}_{p,N} := \mathrm{span}_{\mathbb{C}}\{\psi_j \in S^{(q),*}_{p,N} \subset L_\infty(M, d\nu) : j = \overline{1, N}\}, \tag{10}$$

$\dim S^{(q),*}_{p,N} = N$*, and whose basis functions satisfy the biorthogonality condition*

$$\left\{\int_M \psi_k \varphi_j d\nu = \delta_{jk} : j, k = \overline{1, N}\right\} \tag{11}$$

for all $j, k = \overline{1, N}$*. Moreover, owing to the canonical isomorphisms* $L_q(M, d\nu) \simeq L_{\tilde{q}}(M, d\nu)$*,* $1/\tilde{q} + 1/q = 1$*, and* $L_1(M, d\nu) \simeq L_\infty(M, d\nu)$*, the corresponding subspace* $S^{(q),*}_{p} \subset L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$ *is also closed and* $S^{(q),*}_{p,N} \subset S^{(q),*}_{p}$*.*

Remark 1. *It is interesting to note here [1] that the spaces* $L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$ *and* $L_\infty(M, d\nu) \cup L_{\tilde{q}}(M, d\nu)$ *are not isomorphic.*

Proof. Owing to Lemma 1, one can define linear bounded functionals $F_k : S^{(q)}_{p,N} \subset L_1(M, d\nu) \to \mathbb{C}, k = \overline{1, N}$, for which

$$F_k(\varphi_j) = \delta_{jk} \tag{12}$$

for all $j, k = \overline{1, N}$. They are well defined, as the basis function $\varphi_j \in S^{(q)}_{p,N} \subset L_1(M, d\nu), j = \overline{1, N}$ is linearly independent. Now, making use of the classical Hahn–Banach theorem [16,18], these functionals can be extended as bounded linear functionals on the whole space $L_1(M, d\nu)$, to which one can apply the Riesz representation theorem:

$$F_k(\varphi) := \int_M \varphi \psi_k d\nu \tag{13}$$

for all $\varphi \in L_1(M, d\nu)$, where $\psi_k \in L_\infty(M, d\nu)$, $||F_k|| = ||\psi_k||_\infty < \infty, k = \overline{1, N}$, are the corresponding functional elements, generating the subspace $S^{(q),*}_{p,N} \subset L_1(M, d\nu)^* \simeq L_\infty(M, d\nu)$ and satisfying the condition (11). As $q > \max\{1, p\}, p > 0$, and the closed subspace $S^{(q)}_{p,N} \subset S^{(q)}_{p} \subset L_q(M, d\nu) \cap L_1(M, d\nu)$, owing to the canonical isomorphisms $L_q(M, d\nu) \simeq L_{\tilde{q}}(M, d\nu)$, $1/\tilde{q} + 1/q = 1$, and $L_1(M, d\nu) \simeq L_\infty(M, d\nu)$ one easily finds that the subspace $S^{(q),*}_{p} \subset L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$ is also closed and $S^{(q),*}_{p,N} \subset S^{(q),*}_{p}$, thus proving the lemma. □

Proof of Theorem 2. As follows from Lemma 2, the closed subspace $S^{(q),*}_{p} \subset L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$ a priori contains the finite-dimensional subspace $S^{(q),*}_{p,N} \subset L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$, $\dim S^{(q),*}_{p,N} = N$. The latter makes it possible to reduce the finite dimensionality problem subject to the closed subspace $S^{(q)}_{p} \subset L_1(M, d\nu) \cap L_q(M, d\nu)$ to the one of the closed subspace $S^{(q),*}_{p} \subset L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu)$, following the Grothendieck [21] scheme. First, we observe that the embedding mapping $S^{(q),*}_{p} \subset L_{\tilde{q}}(M, d\nu) \to S^{(q),*}_{p} \subset L_\infty(M, d\nu)$ is a closed operator, giving rise owing to the Banach closed operator theorem to the inequality

$$||g||_\infty \le R||g||_{\tilde{q}} \tag{14}$$

for any $g \in S_p^{(q),*} \subset L_\infty(M, d\nu)$ and some positive and bounded number $R < \infty$. Moreover, making use of the Young inequality, for any $\infty > \tilde{q} > 0$ one can find such a positive constant $R_{\tilde{q}} < \infty$ that

$$||g||_\infty \le R_{\tilde{q}}||g||_2 \quad (15)$$

for any $g \in S_p^{(q),*} \subset L_\infty(M, d\nu)$. Taking into account that, according to (14), any $g \in S_p^{(q),*} \subset L_2(M, d\nu) \cap L_\infty(M, d\nu)$, one can choose the finite dimensional subspace (10) such that the set of functions $\psi := \{\psi_j \in S_p^{(q),*} : j = \overline{1, N}\}$ can be ortonormal, that is

$$\int_M \bar{\psi}_j \psi_k d\nu = \delta_{jk} \quad (16)$$

for all $j, k = \overline{1, N}$. Let now $Q \subset \mathbb{D}_1(0)$ be a countable everywhere dense subset of the unit disc $\mathbb{D}_1(0)$ of the Euclidean space $\mathbb{E}^N := (\mathbb{C}^N; \langle\cdot|\cdot\rangle)$. Then, for every vector $c \in \mathbb{D}_1(0)$, one finds that the function $g_c := \langle c|\psi\rangle \in L_2(M, d\nu)$, that is $||g_c||_2 \le 1$, owing to (15)

$$||g_c||_\infty \le R_{\tilde{q}}. \quad (17)$$

Taking into account the fact that the set $Q \subset \mathbb{D}_1(0)$is countable, one can find such a measurable subset $M' \subset M$ that the measure $\nu(M') = 1$ and $|g_c(u)| \le R_{\tilde{q}}$ for all vectors $c \in Q \subset \mathbb{D}_1(0)$ and all points $u \in M'$. Since at a fixed point $u \in M'$ the mapping $\mathbb{D}_1(0) \ni c \to |g_c(u)| \in \mathbb{R}$ is continuous on $\mathbb{D}_1(0) \subset \mathbb{E}^N$, one can extend this function on the whole disc $\mathbb{D}_1(0)$, obtaining the inequality

$$|g_c(u)| \le R_{\tilde{q}} \quad (18)$$

already for all $c \in \mathbb{D}_1(0)$ and $u \in M'$. Making use of the arbitrariness of the vector $c \in \mathbb{D}_1(0)$, it can be chosen as $c := \frac{\bar{\psi}(u)}{|\psi(u)|} \in \mathbb{D}_1(0) \cap S_p^{(q),*}, u \in M'$, giving rise to the following inequality: $|\psi(u)| \le R_{\tilde{q}}$, or

$$|\psi(u)|^2 \le R_{\tilde{q}}^2. \quad (19)$$

Having integrated the inequality (19) over $M \simeq M'$, one finds that $N \le R_{\tilde{q}}^2 < \infty$. The latter means that $\dim S_p^{(q),*} \le \max N < \infty$, being equivalent to the condition that $\dim S_p^{(q)} \le \max N < \infty$, thus proving the theorem. □

As a consequence, we also state that the closed subspace $S_p^{(q)} \subset L_p(M, d\mu) \cap L_q(M, d\nu)$ is isomorphic to the L_2-subspace of $L_\infty(M, d\nu) \cap L_{\tilde{q}}(M, d\nu), 1/q + 1/\tilde{q} = 1$.

3. Conclusions

We studied a classical problem of finding finite-dimensional effective criteria for closed subspaces in L_p, endowed with some additional functional constraints. We considered a closed topological subspace $S_p^{(q)}$ of the functional Banach space $L_p(M, d\mu)$ and, moreover, assumed that additionally, $S_p^{(q)} \subset L_q(M, d\nu)$ is subject to a probability measure ν on M. Then, we showed that closed subspaces of $L_p(M, d\mu) \cap L_q(M, d\nu)$ for $q > \max\{1, p\}, p > 0$, are finite dimensional, if the measures μ, ν are probabilistic on M and the measure μ is absolutely continuous with respect to the measure ν on M. The finite dimensionality result concerning the case when $q > p > 0$ is open and needs more sophisticated techniques, mainly based on analysis of the complementary subspaces to $L_p(M, d\mu) \cap L_q(M, d\nu)$.

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