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One-dimensional strain-limiting viscoelasticity with an arctangent type nonlinearity

Yasemin Şengül [a,*]

Sabanci University, Faculty of Engineering and Natural Sciences, Orta Mahalle, Tuzla, 34956, Istanbul, Turkey

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ABSTRACT

In this note a one-dimensional nonlinear partial differential equation, which has been recently introduced by the author and co-workers, describing the response of viscoelastic solids showing limiting strain behaviour in strain and stress-rate cases is investigated. The model results from an implicit constitutive relation between the linearized strain and the stress. For this viscoelastic model, a specific form of the nonlinearity that has been investigated only in the elastic case in the literature is studied and it is shown that traveling wave solutions can be found analytically or numerically for various approximations of the nonlinearity, as well as the nonlinearity itself. Moreover, the analysis is carried out for both small and larger values of the stress, the latter being the first time in the literature within the current context.

1. Introduction

In the present work, our main aim is to investigate traveling wave solutions for a one-dimensional nonlinear viscoelastic model introduced by Erbay and Şengül in Erbay and Şengül (2015) and Erbay and Şengül (2020) in the context of strain-limiting theory. Here, we analyze the model with a specially chosen nonlinearity that has been proposed very recently by Meneses, Orellana and Bustamante in Meneses et al. (2018). The novelty of the current work is to investigate this nonlinearity in the viscoelastic setting within the context of strain-limiting theory.

Rajagopal in Rajagopal (2003, 2007) introduced some models to study response of elastic bodies where an implicit constitutive relation is specified between the stress and the strain (or the gradient of the displacement). This was a consequence of the idea that there might exist more complicated relationships between the stress and the kinematical variables than just the fact that stress is an explicit function of them. Furthermore, if one considers the cases where the strain is small, then it would be possible to obtain nonlinear relationships between the linearized strain and the stress. This approach, which is called the strain-limiting theory, has attracted a serious amount of attention in the recent years in various contexts (see e.g. Bulíček et al. (2014); Bustamante (2009); Bustamante and Rajagopal (2011); Erbay and Şengül (2015); Rajagopal (2014); Rajagopal and Saccomandi (2014)) due to the fact that it is capable of explaining some phenomena such as what happens at a crack tips (see e.g. Rajagopal and Walton (2011)), and also that it has been observed in some experiments (see e.g. Rajagopal (2014); Saito et al. (2003)). One can also refer to Şengül (2021) for a detailed overview.

Even though many studies exist in elastic setting, there are very few results exploring viscoelastic phenomena. In order to study the response of viscoelastic materials, Muliana et al. Muliana et al. (2013) developed a quasi-linear viscoelastic model where the linearized strain is expressed as an integral of a nonlinear measure of the stress (see also Rajagopal (2009), Bulíček et al. (2012)). Rajagopal and Saccomandi Rajagopal and Saccomandi (2014), on the other hand, modelled rate-type viscoelasticity by considering a special subclass of the general implicit constitutive relations. Using these constitutive relations, Erbay and Şengül Erbay and Şengül (2015) derived a one-dimensional nonlinear strain-rate viscoelastic model and studied conditions on the nonlinearity for the existence of traveling wave solutions. Later, in Erbay and Şengül (2020), Erbay and Şengül thermomechanically derived the corresponding stress-rate type viscoelastic model and showed that the travelling wave solutions for this and the strain-rate type model coincide.

* Corresponding author.
E-mail address: yaseminsengul@sabanciuniv.edu

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Very recently, Meneses et al. Meneses et al. (2018) obtained some self-similar solutions for a new type of nonlinear wave equation in the case of a one-dimensional elastic straight bar. In this article, we are interested in investigating traveling wave solutions for the viscoelastic model proposed by Erbay and Şengül Erbay and Şengül (2015) with a nonlinearity studied in Meneses et al. (2018). Our results also apply to the stress-rate type model studied in Erbay and Şengül (2020) due to the remark above. The novelty in this work is that the constitutive relation is given by an arctangent type nonlinear function which exhibits limiting strain behaviour as expected and has not been studied in the context of viscoelasticity before. Additionally, traveling wave profiles are given for large values of the stress which is required in the strain-limiting context, which has not been done before.

The structure of the paper is as follows. In Section 2 we mention the governing equations for one-dimensional strain-limiting viscoelasticity and state the model to be studied in this work. In Section 3 we construct traveling wave solutions analytically or numerically for various Taylor polynomials corresponding to the series expansion of the nonlinearity. Finally, a similar analysis is done for the arctangent type nonlinearity. Also in Section 4, we show that for large values of the stress travelling wave solutions exist and the profiles for the nonlinearity and the stress behave as expected. Throughout the work, we use dimensionless variables and parameters.

2. Strain-limiting viscoelasticity

2.1. One-dimensional strain-limiting viscoelasticity

In one space dimension, the implicit constitutive relation introduced by Rajagopal Rajagopal (2003, 2007) takes the form

$$\phi(\epsilon, \mathbf{T}) = 0, \tag{2.1}$$

where $\epsilon(x,t)$ is the linearized strain given by $\epsilon = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ for displacement $\mathbf{u}(x,t)$, and $\mathbf{T}(x,t)$ is the Cauchy stress. In some cases we can write (2.1) as

$$\epsilon = g(\mathbf{T}), \tag{2.2}$$

for a nonlinear function g satisfying $g(0) = 0$. Rajagopal and Saccomandi Rajagopal and Saccomandi (2014) extended this approach to viscoelastic materials by considering the relation $\phi(\epsilon, \epsilon_t, \mathbf{T}) = 0$, where ϵ_t stands for the time derivative of the strain. Recently, Erbay and Şengül Erbay and Şengül (2015) considered the constitutive relation

$$\epsilon + \nu \epsilon_t = g(\mathbf{T}), \tag{2.3}$$

where $\nu > 0$ is the viscosity constant, and $g(\cdot)$ is a nonlinear function of the stress. In Erbay and Şengül (2015), using the equation of motion for the displacement, the constitutive relation (2.3) and the definition of the linearized strain, Erbay and Şengül obtained, in one space dimension, the third-order semilinear equation for the stress $T = T(x,t)$ as

$$T_{xx} + \nu T_{xxt} = g(T)_{tt}, \tag{2.4}$$

and studied traveling wave solutions of this equation with some nonlinear functions g widely studied in the literature. Recently, in Erbay et al. (2020), Erbay, Erkip and Şengül proved local-in-time existence of solutions for (2.4). Also, Meneses et al. Meneses et al. (2018) considered the elastic case together with the relation (2.2), which results in (see Bustamante and Sfyris (2015))

$$\mathrm{grad}\left(\frac{1}{\rho}\mathrm{div}\mathbf{T}\right) + \left[\mathrm{grad}\left(\frac{1}{\rho}\mathrm{div}\mathbf{T}\right)\right]^T + \mathrm{grad}(\mathbf{b}) + [\mathrm{grad}(\mathbf{b})]^T = 2\frac{\partial^2}{\partial t^2}[\mathbf{g}(\mathbf{T})], \tag{2.5}$$

where $\mathbf{b}$ is the body force and ρ is the density of the body. In Meneses et al. (2018), (2.5) is studied in the particular case when $\mathbf{b} = \mathbf{0}$, and ρ is a constant. In one space dimension with constant ρ, (2.5) reduces to $T_{xx} = \rho g(T)_{tt}$. Note that (2.4) is equivalent to this equation in the elastic setting (that is, when $\nu = 0$) with all the quantities made dimensionless.

Also, in Erbay and Şengül (2020), Erbay and Şengül considered the stress-rate type model given by

$$\epsilon + \gamma \mathbf{T}_t = h(\mathbf{T}), \tag{2.6}$$

and instead of (2.4) they obtained the resulting equation as

$$T_{xx} + \gamma T_{ttt} = h(T)_{tt}. \tag{2.7}$$

As remarked in Erbay and Şengül (2020), the travelling wave solutions for (2.4) and (2.7) coincide. Therefore, the travelling wave profiles we provide are valid for the latter case as well and hence we will not investigate this case separately.

2.2. Traveling wave solutions

In this section we define the form of the solution we seek and state its properties to obtain an equation corresponding to (2.4). Traveling waves are solutions of the form

$$T = T(\xi), \quad \xi = x - ct, \tag{2.8}$$

where the wave propagation speed c is a constant to be determined later. Substitution of (2.8) into (2.4) gives

$$T'' - \nu c T''' = c^2[g(T)]'', \tag{2.9}$$

where the symbol $'$ stands for differentiation with respect to ξ. We also assume that

$$\lim_{\xi \to -\infty} T(\xi) = T_\infty^-, \quad \lim_{\xi \to +\infty} T(\xi) = T_\infty^+, \tag{2.10}$$

with $T_\infty^- \neq T_\infty^+$, where T_∞^- and T_∞^+ are the two constant states corresponding to heteroclinic traveling waves which will be specified later. We can integrate (2.9) using the fact that $T'(\xi), T''(\xi) \to 0$ as $\xi \to \pm\infty$. Also using (2.10), we obtain

$$T' = \frac{1}{\nu c}\left\{T - \frac{T_\infty^- + T_\infty^+}{2} - c^2\left[g(T) - \frac{g(T_\infty^-) + g(T_\infty^+)}{2}\right]\right\}, \tag{2.11}$$

where

$$c^2 = \frac{T_\infty^- - T_\infty^+}{g(T_\infty^-) - g(T_\infty^+)}. \tag{2.12}$$

From (2.12), since c^2 is always positive, one concludes that either $T_\infty^- > T_\infty^+$ and $g(T_\infty^-) > g(T_\infty^+)$, or $T_\infty^- < T_\infty^+$ and $g(T_\infty^-) < g(T_\infty^+)$ must hold. We will work with either of these conditions depending on the case we consider.

3. Traveling wave solutions for the arctangent model

Our main aim in this section is to consider the nonlinearity proposed by Meneses et al. Meneses et al. (2018) which, in general, can be written as

$$g(T) = \aleph \arctan(\vartheta T), \tag{3.1}$$

for positive constants $\aleph$ and ϑ. In fact, as explained by Meneses et al. Meneses et al. (2018), relation (3.1) is proposed as an approximation of the expression

$$g(T) = \alpha\left[\left(-1 + \frac{1}{1 + \beta T}\right) + \frac{\gamma}{(1 + \iota T^2)^{1/2}} T\right], \tag{3.2}$$

which was introduced and studied by Bustamante and Rajagopal Bustamante and Rajagopal (2011), where α, β, γ and ι are constants. Clearly, as explained in Meneses et al. (2018), equation (3.2) models the response of an elastic body for which the strain remains small independently of the magnitude of the stress. Moreover, it is relatively easier to deal with the relation (3.1) rather than (3.2) in terms of the calculations so as to study the qualitative properties of the implicit solutions. It would be a great achievement to carry out a similar analysis for (3.2). However, since (3.1) and (3.2) show similar qualitative properties in terms of implicit constitutive modelling, it is more reasonable to start with the

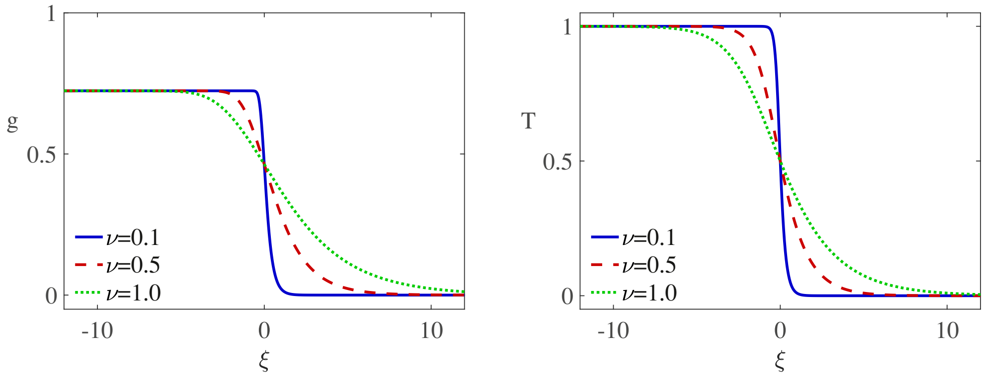

Fig. 1. Second Taylor polynomial case.

analysis of (3.1). This type of relation, together with an exponential and polynomial type nonlinearities, are discussed by Meneses et al. Meneses et al. (2018) where the particular case of a one-dimensional bar is considered and boundary-value problems are analyzed in an elastic setting. In this work, we study (3.1) by taking the constants $\aleph$ and ϑ as unity for simplification, and show existence of traveling wave solutions for equation (2.11).

To this end, we will investigate the sequence of Taylor polynomials approximating the arctangent function. Since this is valid only around zero, we divide the analysis into two parts; firstly, we consider small values of T and solve (2.11) directly, and secondly, we put $T = 1/z$ and solve a new equation corresponding to (2.11) so that the theory is valid away from zero as well.

3.1. First Taylor polynomial

Looking at the first Taylor polynomial of (3.1), namely, $g(T) = T$, equation (2.11) reduces to $T' = 0$ giving the constant solution which contradicts $T_\infty^- > T_\infty^+$. Therefore, we do not obtain a heteroclinic travelling wave in this case.

3.2. Second Taylor polynomial

In this section we will look at the approximation of (3.1) by its second Taylor polynomial, namely $g(T) = T - \frac{T^3}{3}$. Choosing $T_\infty^- = 1$ and $T_\infty^+ = 0$, and using $g(0) = 0$, from (2.11) we obtain

$$T' + kT = kT^3, \quad k = 1/(2\nu c). \tag{3.3}$$

This is a Bernoulli type differential equation which can be solved analytically. In order to fix the travelling wave, we assume that

$$T(0) = 1/2. \tag{3.4}$$

As a result, we obtain

$$T(\xi) = \frac{1}{\sqrt{1 + 3e^{2k\xi}}}, \tag{3.5}$$

where (3.4) is used. Note that the equilibrium points are $T = 0$ and $T = \pm 1$. Fig. 1 shows the variation of (3.5) with T for three different values of the viscosity parameter ν. We observe that, as it is expected, the traveling wave profile becomes smoother as the viscosity increases. We also deduce that the values of the profile for $g(T)$ are reduced which is due to the non-linear dependence. Moreover, since the effective width of the wave does not decrease to zero, the denominator of T' is never zero, and consequently no shock can occur.

It is worth mentioning that equation (3.3) was also studied by Jordan and Puri in Jordan and Puri (2005) as the mathematical analysis of the experimental observation of a shock transverse wave propagating in an elastic medium done by Catheline, et al. Catheline et al. (2003). In Catheline et al. (2003), authors investigate a soft tissue model called *Agar-gelatin phantom* and they applied a sinusoidal driving signal at the boundary. They model their stress-strain relation by a cubic (which was corrected in Catheline et al. (2005)) that becomes (3.3) for the consideration of travelling wave solutions as shown in Jordan and Puri (2005). This provides another motivation to study models (2.4) and (2.7). Also, the profile for traveling wave solutions agree with those in Jordan and Puri (2005) in terms of behaviour against changing viscosity.

3.3. Third Taylor polynomial

in this case we take $g(T) = T - \frac{T^3}{3} + \frac{T^5}{5}$. Choosing the constant stated as $T_\infty^- = \sqrt{2/3}$ and $T_\infty^+ = 0$, from (2.11) we obtain the differential equation

$$T' = -\frac{1}{13\nu c} T(1 - T^2)(2 - 3T^2). \tag{3.6}$$

It is clear that this equation admits five equilibrium solutions, namely $T = 0$, $T = \pm 1$ and $T = \pm\sqrt{2/3}$. In fact, this is the reason to initially choose T_∞^- to be equal to $\sqrt{2/3}$. Equation (3.6) is a nonlinear, first order differential equation with a polynomial type nonlinearity, and hence it must have an analytic solution. Solving (3.6) gives the closed form solution

$$\frac{T^{1/2}(1 - T^2)^{1/2}}{(2 - 3T^2)^{3/4}} = Ce^{-\frac{\xi}{13\nu c}}, \tag{3.7}$$

where the constant $C = \sqrt{3}/5^{3/4}$ can be found using (3.4). The profiles for the traveling wave T and the nonlinearity g can be seen in Fig. 2 for different values of ν. As in the previous case, the traveling wave profile becomes smoother as the viscosity increases. Moreover, as expected, the stress converges to the preset constant states as $\xi \to \pm\infty$.

3.4. Fourth Taylor polynomial

Here we take $g(T) = T - \frac{T^3}{3} + \frac{T^5}{5} - \frac{T^7}{7}$ together with $T_\infty^- = 1$, $T_\infty^+ = 0$. In this case, equation (2.11) gives

$$T' = \frac{1}{76\nu c}\left(-29T + 35T^3 - 21T^5 + 15T^7\right). \tag{3.8}$$

As in the third Taylor polynomial case, this is a first order differential equation with a polynomial type nonlinearity. Simplifying the right-hand side gives

$$T' = \frac{1}{76\nu c} T(T^2 - 1)(15T^4 - 6T^2 + 29).$$

From this from of the equation, it is clear that there are seven equilibrium points where three of them are real, namely $T = 0$ and $T = \pm 1$, and four of them are complex. It can be calculated that the analytic solution exist in a closed form similar to the third approximation case as in (3.7), and since the coefficients involved are very big and have no significant effect on the form of the solution we prefer to give the numerical solution in Fig. 3.

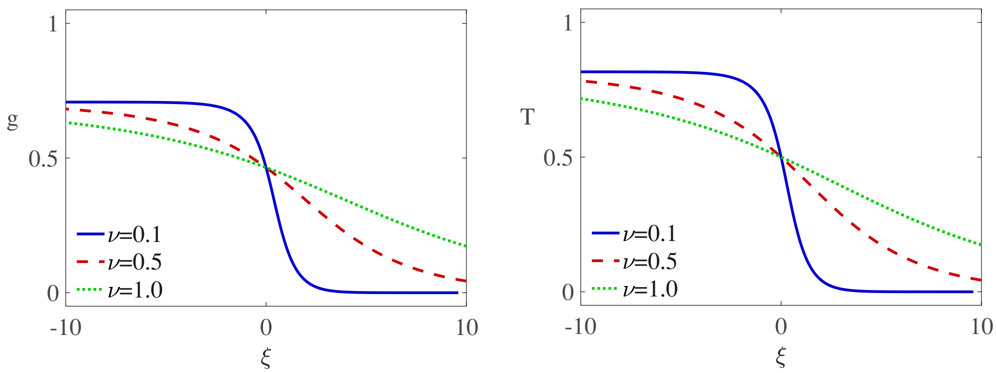

Fig. 2. Third Taylor polynomial case.

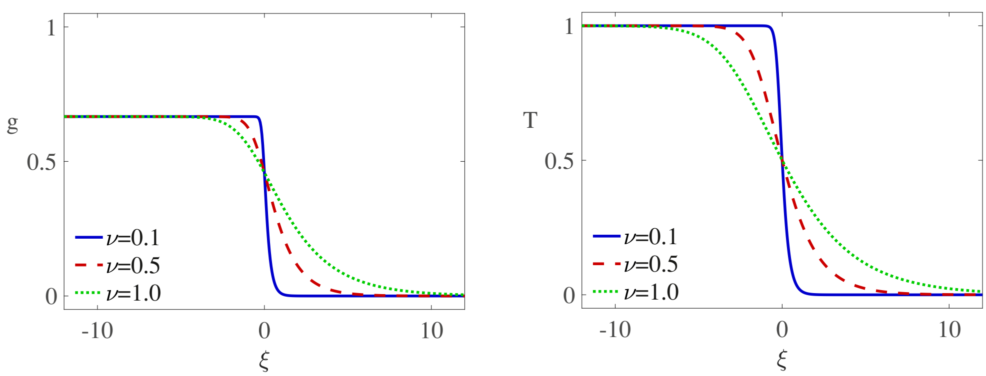

Fig. 3. Fourth Taylor polynomial case.

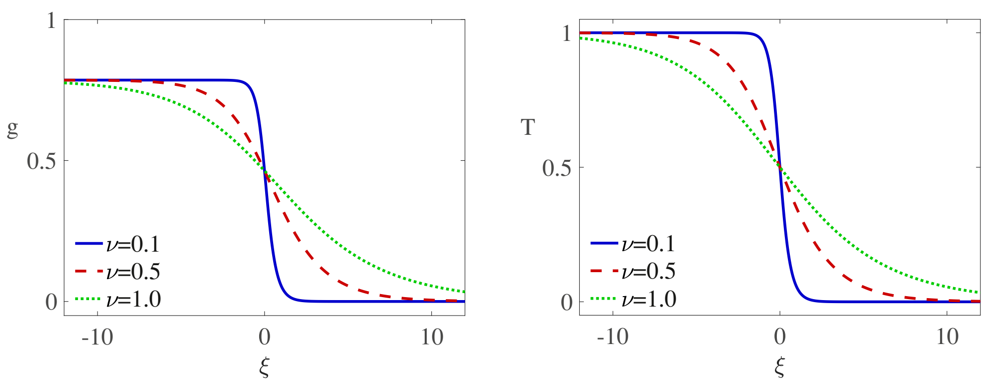

Fig. 4. Numerical solution for the arctangent function.

3.5. Higher order Taylor polynomials

One can do similar calculations for approximations of (3.1) with higher order Taylor polynomials. Since they are all polynomials, the differential equation obtained would be a first order ordinary differential equation with a polynomial type nonlinearity. Therefore, they all must have an analytic solution, explicit or implicit. However, these solutions will not be significantly different from what we obtain in the third and fourth Taylor polynomial cases. Therefore, we prefer to skip them here and look for a solution with (3.1) as the nonlinearity. Nonetheless, it is possible to plot the profiles for each Taylor approximation corresponding to the arctangent function.

3.6. The arctan function

Here we take $g(T)$ as in (3.1) together with $T_\infty^- = 1, T_\infty^+ = 0$ and $T(0) = 1/2$. In this case (2.11) becomes

$$T' = \frac{4}{\pi \nu c}\left(\frac{\pi}{4}T - \arctan(T)\right),$$

which is a separable equation whose solution involves a nontrivial integral. Therefore, we focus on the numerical solution instead. Using MATLAB function `ode45` to solve this differential equation, which is the standard solver of MATLAB for ordinary differential equations, we present in Fig. 4 the numerical solutions for three different values of the viscosity parameter ν.

4. Approximating the arctangent for large values of T

As mentioned before, Taylor series approximation for the arctangent function is only valid for small values of T. However, in strain-limiting theory we would like to have large T values. In order to look for a travelling wave solutions when T gets large, we would put $T = 1/z$ in (2.11). This requires $g(1/z)$ on the right-hand side. Since we would like to investigate large values of T, z must be small. Hence we use the equality

$$\arctan(1/z) + \arctan(z) = \pi/2,$$

so that we take $g(1/z) = \pi/2 - z + z^3/3 - \ldots$. Considering the first Taylor polynomial, we will take $g(1/z) = \pi/2 - z$ so that from (2.12) we obtain $c^2 = T_\infty^- T_\infty^+$.

$$z' = -\frac{1}{\nu c}\left(z + T_\infty^- T_\infty^+ z^3 - (T_\infty^- + T_\infty^+) z^2\right),$$

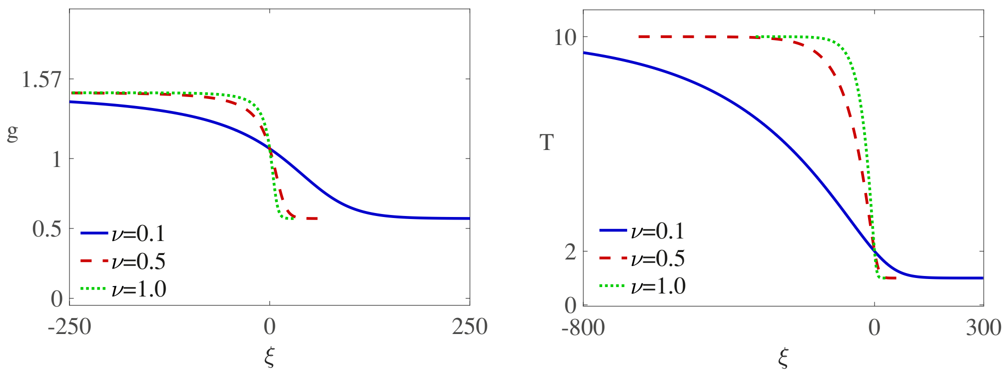

Fig. 5. Large values of T with $T(0) = 2$.

which can be written as

$$z' = -\frac{1}{\nu c} z(1 - T_\infty^- z)(1 - T_\infty^+ z). \tag{4.1}$$

Clearly, the equilibrium points for this first order ordinary differential equation are $z = 0$, $z = 1/T_\infty^-$ and $z = 1/T_\infty^+$. Solving (4.1) gives

$$-\frac{1}{\nu c}\xi + C = \frac{T_\infty^-}{T_\infty^+ - T_\infty^-}\log(1 - T_\infty^- z) - \frac{T_\infty^+}{T_\infty^+ - T_\infty^-}\log(1 - T_\infty^+ z) + \log(z). \tag{4.2}$$

where C is the integration constant which can be found using $z(0) = 1/2$ (which corresponds to $T(0) = 2$). Going back to variable T by substituting $z = 1/T$, we can look at the profiles for g and T changing with viscosity ν in Fig. 5, where we choose $T_\infty^- = 10$ and $T_\infty^+ = 1$.

5. Conclusions

We have investigated heteroclinic traveling wave solutions corresponding to a viscoelastic model within the context of strain-limiting theory. Our aim was to study a nonlinearity of arctangent type which has been studied in the elastic setting recently, and see how the wave profiles change by the viscosity. With this respect, we analyzed traveling waves corresponding to Taylor polynomial approximations of the nonlinearity as well as the nonlinearity itself, and obtained solutions analytically or numerically both for small and large values of the stress. This is the first time in the literature that a traveling wave solution in the context of strain-limiting theory is illustrated for large values of the stress which is believed to be a further step in the understanding of response of different types of materials in nature.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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