Weighted domination models and randomized heuristics

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We consider the minimum weight and smallest weight minimum-size dominating set problems in vertex-weighted graphs. The latter is a two-objective optimization problem, which is concerned with optimizing both weight and cardinality of the dominating set. First, we reduce the two-objective optimization problem to the minimum weight dominating set problem by using Integer Linear Programming (ILP) formulations. Then, under different assumptions, we employ the probabilistic method to obtain new upper bounds on the minimum weight dominating sets in graphs. We also describe the corresponding randomized algorithms for finding small-weight dominating sets in graphs and use computational experiments to illustrate the results for two types of random graphs.

Weighted domination in graphs and networks can be used, for example, for modelling a problem of the placement of a number of transmitters in a communication network such that every site in the network either has a transmitter or is connected by a direct communication link to a site that has such a transmitter. There are usually some ‘costs’ associated with placing a transmitter in each particular location of the network, i.e. a vertex of the corresponding graph. The minimum weight dominating set problem usually does not place any restrictions on the size of the dominating set, i.e. the number of transmitters in this case – we only need to find a smallest weight/cost dominating set in a vertex-weighted graph. However, the total emitted radiation in the environment would be smaller with fewer transmitters installed. Similar facility location problems in road networks [2] and social networks [4] can be generalized to vertex-weighted graphs and two-criteria optimization as well. See also network problems in [5].

Given a simple graph G of order n, the weight assigned to each vertex v_i is denoted by w_i, $i = 1, ..., n$. The total weight of the graph, the minimum, maximum, and average vertex weights of G are denoted by w_G, $w_{\min}$, $w_{\max}$, and w_{ave}, respectively. The minimum vertex degree of G is denoted by $\delta = \delta(G)$. A set X of vertices of G is called a *dominating set* of G if every vertex not in X is adjacent to at least one vertex in X. The minimum cardinality of a dominating set of G is called the *domination number* of G and is denoted by $\gamma(G)$. We denote by $\gamma_w(G)$ the smallest weight of a dominating set in a graph G, and by $\gamma_w^*(G)$ the smallest weight of a minimum-cardinality dominating set X in G.

The problem of finding an exact value of $\gamma_w^*(G)$ and the corresponding dominating set X can be formulated as an ILP problem:

$$\begin{aligned} &\text{minimize} && z(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{w_i}{w_G} x_i && \\ &\text{subject to:} && \sum_{v_i \in N[v_j]} x_i \geq 1, && j = 1, \ldots, n, \\ & && x_i \in \{0, 1\}, && i = 1, \ldots, n, \end{aligned} \tag{1}$$

where a $(0,1)$-decision variable $x_i \in \{0,1\}$ is associated with each vertex $v_i \in G$ to indicate whether the vertex is in the solution set X or not, having $x_i = 1$ if and only if v_i is in X, $i = 1, ..., n$. Reassigning the graph vertex weights to $w_i' = 1 + \frac{w_i}{w_G}$, $i = 1, ..., n$, the ILP formulation becomes the single-objective optimization problem of finding $\gamma_{w'}(G)$ and the corresponding minimum weight dominating set in G with respect to the vertex weights w_i', $i = 1, ..., n$. At optimum, we have $\gamma_{w'}(G) = z^* = z(x_1^*, x_2^*, ..., x_n^*)$ and as $\sum_{i=1}^{n} \frac{w_i}{w_G} x_i^* < 1$, we also have $\gamma_w^*(G) = \sum_{i=1}^{n} w_i x_i^*$. In light of these results, the problems of finding $\gamma(G)$ and $\gamma_w^*(G)$ in G can be considered as particular cases of the more general problem of finding $\gamma_w(G)$ in G.

We use the probabilistic method to find several new upper bounds for $\gamma_w(G)$ in a graph G. These results are generalizations of the probabilistic method for the following classic upper bound for $\gamma(G)$:

Theorem 1. *[1, 3] For any graph G with $\delta \geq 1$,*

$$\gamma(G) \leq \left(1 - \frac{\delta}{(1+\delta)^{1+1/\delta}}\right) n.$$

The generalizations are based on the following general probabilistic construction and randomized algorithmic framework. First, given a certain probability p_i, $i = 1, ..., n$, we decide whether to include each vertex v_i of G into a set A, $i = 1, 2, ..., n$. Next, we consider the set of vertices that are not in A and do not have a neighbour in A, which is denoted by B. The set $D = A \cup B$ will then be a dominating set of G. By computing the expected total weight of the vertices in D, we obtain an upper bound for $\gamma_w(G)$. We use different ways to compute p_i to obtain the upper bounds and corresponding randomized algorithms. As the dominating set should have both small size and weight, we set $p_i = p \cdot x$, where p depends on vertex degrees in G, and x depends on vertex weights in G. By trying different expressions for x and optimizing the expected weight of D for p, we obtain the following upper bounds:

Theorem 2. *For any graph G with $\delta \geq 1$,*

$$\gamma_w(G) \leq \left(1 - \frac{\delta}{(1+\delta)^{1+1/\delta}}\right) w_G.$$

Theorem 3. *For a graph G with $\delta \geq 1$, $k = w_{\max}/w_{\text{ave}} \leq \delta + 1$, and $p = 1 - \left(\frac{k}{\delta+1}\right)^{1/\delta} \leq w_{\min}/w_{\max}$,*

$$\gamma_w(G) \;\leq\; npw_{\max} + \sum_{i=1}^{n} w_i \,(1-p)^{d_i+1} \;\leq\; \left(1 - \frac{\delta k^{1/\delta}}{(\delta+1)^{1+1/\delta}}\right) kw_G.$$

Theorem 4. *For a graph G with $\delta \geq 1$, $z = w_{\max}/w_{\min} \leq \delta + 1$, and $q = 1 - \left(\frac{z}{\delta+1}\right)^{1/\delta}$,*

$$\gamma_w(G) \;\leq\; qzw_G + \sum_{i=1}^{n} w_i \,(1-q)^{d_i+1} \;\leq\; \left(1 - \frac{\delta z^{1/\delta}}{(\delta+1)^{1+1/\delta}}\right) zw_G.$$

There are problem instances where the conditions of Theorem 3 are satisfied, but not those of Theorem 4 and vice versa. Also, Theorem 2 implicitly assumes that the ratio $w_{\max}/w_{\min}$ is reasonably close to 1.

We implemented and tested the deterministic and randomized heuristic solution methods for both problems on random graph instances of two types, one of which is the classic Erdös-Rényi random graph type, and the other is a random graph type used to prove asymptotic sharpness of the upper bounds of Theorem 1. Using the ILP formulations and a generic ILP solver (FICO® Xpress), the exact deterministic solutions to the problems of computing $\gamma_w^*(G)$ and $\gamma_w(G)$ were found in a reasonable amount of time of at most three hours for Erdös-Rényi random graphs of only at most 200 vertices, and the other type of graphs of at most 560 vertices. Then, three randomized heuristics based on Theorems 2, 3, and 4 were run on each of the random graph instances. In the case of the Erdös-Rényi random graphs, the three randomized algorithms performed similarly by the dominating set size, but the algorithm based on Theorem 2 was less successful when searching for better solutions by weight. For the other type of graphs, the algorithms corresponding to Theorems 3 and 4 performed better than that based on Theorem 2 by both parameters. Therefore, Theorems 3 and 4, whilst requiring stronger conditions, provide better randomized heuristics.

References

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