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Article

A Generalization of Multifractional Brownian Motion

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Abstract: In this article, some properties of multifractional Brownian motion (MFBM) are discussed. It is shown that it has persistence of signs long range dependence (LRD) and persistence of magnitudes LRD properties. A generalization called here nth order multifractional Brownian motion (n-MFBM) that allows to take the functional parameter $H(t)$ values in the range $(n-1, n)$ is discussed. Two representations of the n-MFBM are given and their relationship with each other is obtained.

Keywords: multifractional Brownian motion; long range dependence; harmonizable representation; Hurst parameter; Hölder continuity

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1. Introduction

The fractional Brownian motion (FBM) introduced in [1] is a generalization of Brownian motion which exhibit long range dependence (LRD) property with normally distributed marginal distributions. However, many real life time-series display both LRD and heavy tailed increments characteristics. An example of a stochastic process which exhibits both LRD and heavy tailed marginals is the linear fractional stable motion (LFSM). In LFSM, the integral is taken with respect to the α-stable random measure instead of Gaussian measure unlike in case of FBM (see e.g., [2]). The constant value of the Hurst exponent of FBM is a strong limitation for example in simulating artificial mountains [3], detection of cancer tumors in medical images [4]. There are examples where Hurst exponent H values above 1 have been observed as, for example, in Neil River data [5]. To allow the Hurst exponent in the interval $(n-1, n)$, the nth order FBM (n-FBM) is discussed in [6]. A generalization of FBM is discussed in [7] where the Hurst parameter of FBM is replaced by a stochastic process. This generalization also have wide applications since the pointwise irregularity varies stochastically with time. Multifractional Brownian motion (MFBM) is another generalization of FBM in which the Hurst parameter is a deterministic function of time and was introduced in [8]. In MFBM, the local roughness varies which is measured by the local Hölder exponent and is defined by [8]

$$\begin{aligned} B_{H(t)}(t) &= \frac{1}{\Gamma(H(t)+1/2)} \int_{-\infty}^{0} \left[(t-s)^{H(t)-1/2} - (-s)^{H(t)-1/2} \right] dB(s) \\ &\quad + \frac{1}{\Gamma(H(t)+1/2)} \int_{0}^{t} (t-s)^{H(t)-1/2} dB(s) \\ &= \frac{1}{\Gamma(H(t)+1/2)} \int_{-\infty}^{\infty} \left[((t-s)_+)^{H(t)-1/2} - ((-s)_+)^{H(t)-1/2} \right] dB(s), \end{aligned} \tag{1}$$

where $B(t)$ is the standard Brownian motion and the integration is taken in the mean square sense and H is a Hölder function of exponent $\beta > 0$, such that,

$$H : [0, \infty) \to [a, b] \subset (0, 1) \tag{2}$$

and $|H(t) - H(s)| \leq c|t-s|^{\beta}$ for some $c > 0$. Further,

$$u_+ = \begin{cases} u \text{ if } u \geq 0, \\ 0 \text{ if } u < 0. \end{cases} \tag{3}$$

Choosing appropriate values for the parameters (see e.g., [9,10]) the following functions are considered for Hölder exponent $H(t)$ in literature:

$$\begin{aligned}
\text{linear} \quad & H(t) = at + b, \; t \in [0, T]; \\
\text{logistic} \quad & H(t) = \frac{c-b}{1+\exp(-d(\frac{t-t_0}{T}))} + b, \; t \in [0, T]; \\
\text{periodic} \quad & H(t) = a\sin\left(4\pi\frac{t}{T}\right) + b, \; t \in [0, T]; \\
\text{arctan} \quad & H(t) = \frac{1}{\pi}\arctan(t) + \frac{1}{2}, \; t > 0; \\
& H(t) = \frac{at}{b+at}, \; t > 0; \\
& H(t) = a\left|\sin\left(\frac{\pi}{2}t^{\alpha}\right)\right|, \; 0 < a < 1, \; 0 < \alpha < 1, t > 0; \\
& H(t) = \frac{1}{b+t^{\alpha}}, 1 < b < \infty, 0 < \alpha < 1, \; t > 0.
\end{aligned}$$

The MFBM is also analogously defined with harmonizable representation of FBM as follows (see [11])

$$\tilde{B}_{H(t)}(t) = \int_{-\infty}^{\infty} \frac{e^{it\xi} - 1}{|\xi|^{H(t)+1/2}} d\hat{B}(\xi), \tag{4}$$

where $\hat{B}$ denotes a complex-valued Gaussian measure. We refer to [12] for the precise definition and for the fact that $\tilde{B}_{H(t)}$ is real valued. Further, it is proved in [10] that two representations given in (1) and (4) neither have equality in distribution up to a multiplicative constant nor equality of the correlations which is claimed in Theorem 1 of [12]. Form Theorem 3.1 of [10], it follows

$$\begin{aligned}
B_{H(t)}(t) = & \frac{\cos((H(t)+1/2)\pi/2)}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{e^{it\xi}-1}{|\xi|^{H(t)+1/2}} d\hat{B}(\xi) \\
& + \frac{\sin((H(t)+1/2)\pi/2)}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{e^{it\xi}-1}{i\xi|\xi|^{H(t)-1/2}} d\hat{B}(\xi).
\end{aligned} \tag{5}$$

Form (5), it is evident that processes $B_{H(t)}(t)$ and $\tilde{B}_{H(t)}(t)$ have the same laws up to a multiplicative deterministic function if $H(t) = 1/2$. The MFBM is called a standard MFBM if its variance at time 1 is 1. The covariance function of standard MFBM is obtained in [13] and is given by

$$\mathbb{E}(\tilde{B}_{H(t)}(t)\tilde{B}_{H(s)}(s)) = D(H(t), H(s))\left(t^{H(t)+H(s)} + s^{H(t)+H(s)} - |t-s|^{H(t)+H(s)}\right), \tag{6}$$

where

$$D(x, y) = \frac{\sqrt{\Gamma(2x+1)\Gamma(2y+1)\sin(\pi x)\sin(\pi y)}}{2\Gamma(x+y+1)\sin(\pi(x+y)/2)}. \tag{7}$$

Note that for $x = y$, D(x,y) = 1/2. This natural extension of FBM however results in some sense a loss of properties. Unlike FBM, the MFBM does not have stationary increments and the process is no more self-similar. However, the choice of H as a Hölder continuous function ensures continuity of its sample paths.

The LFSM is further generalized to linear multifractional stable motion which allows the scaling exponent (or functional exponent) to be a deterministic function of time as well as heavy tailedness of marginals, see e.g., [14]. In [15], the LFSM is generalized to tempered fractional stable motion. Moreover, the higher order fractional stable motion introduced in [16], allows the Hurst exponent to vary in the interval $(n-1, n)$. The higher order fractional stable motion generalizes LFSM and n-FBM. In addition, MFBM is extended to so called generalized multifractional Brownian motion (GMFBM) [17]. Using the idea in [6,16], we generalize the MFBM to an nth order multifractional Brownian motion (n-MFBM) that allows the functional Hurst parameter $H \in (n-1, n)$. In this article, we show that MFBM has persistence of signs LRD and persistence of magnitudes LRD properties. Further, two representations of the introduced process n-MFBM are given and their relationship with each other is obtained. The rest of the article is organized as follows. In Section 2, the LRD properties of MFBM are discussed. Section 3, deals with the n-MFBM and its properties like variance, covariance and harmonizable representation. We concludes the paper in Section 4. The last section discusses the possible generalizations of n-MFBM and other related processes.

2. Long Range Dependence (LRD)

A stationary finite variance process $X(t)$ is said to exhibit short or long-range dependence (SRD or LRD) properties according as $\sum_{k=0}^{\infty} \gamma_k$ converges or diverges, where $\gamma_k = \mathrm{Cov}(X(t), X(t+k))$ for $k \geq 1$, being the autocovariance at lag k [5]. A stationary process $X(t)$ has persistence of signs LRD property if LRD holds for the process $\mathrm{sign}(X(t))$, where

$$\mathrm{sign}(x) = \begin{cases} +1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0, \end{cases} \tag{8}$$

and persistence of magnitudes LRD property if LRD holds for the process $|X(t)|$ [18]. The increments of FBM possess persistence of signs LRD, when $1/2 < H < 1$ and also exhibit persistence of magnitudes LRD when $3/4 \leq H < 1$, see e.g., [18]. Note that MFBM is neither a stationary process nor does it have stationary increments and hence the definition of LRD given above is not applicable. However, for a non-stationary process $Y(t)$ an equivalent definition is given below.

Definition 1. *Let $s > 0$ be fixed and $t > s$. Then the process $Y(t)$ is said to have LRD property if*

$$\mathrm{Corr(Y(s), Y(t))} \approx \mathrm{c(s)t^{-d(s)}}, \quad \text{as } \mathrm{t} \to \infty, \tag{9}$$

where $c(s)$ is a constant depending on s and $d(s) \in (0,1)$. Further, $f(h) \approx g(h)$, as h tends to infinity, when there exists $0 < a < b < \infty$ such that for all sufficiently large h, $a \leq \frac{f(h)}{g(h)} \leq b$.

In analogy to the definition of [18], we have following definitions for the persistence of signs LRD and persistence of magnitudes LRD.

Definition 2. *(i) A second order process $X(t)$ is said to have persistence of signs LRD property if for fixed s and large t, there exist functions $c_1(s)$ and $d_1(s) \in (0,1)$:*

$$\mathrm{Corr}(\mathrm{sign}X(s), \mathrm{sign}X(t)) \approx c_1(s)t^{-d_1(s)} \text{ as } t \to \infty. \tag{10}$$

(ii) It has persistence of magnitudes LRD property if for fixed s and large t, there exist functions $c_2(s)$ and $d_2(s) \in (0,1)$:

$$\mathrm{Corr}(|X(s)|, |X(t)|) \approx c_2(s)t^{-d_2(s)} \text{ as } t \to \infty. \tag{11}$$

Proposition 1. *Let $\tilde{B}_{H(t)}(t)$ be a standard MFBM given in (4). Then,*

(i) For all admissible $H(t)$, MFBM has persistence of signs LRD in the sense of Definition 2.
(ii) Also, it has persistence of magnitudes LRD taken in the sense of Definition 2, when

(a) $\forall s > 0$, $H(t) + H(s) < 1$ *and* $H(t) < 1/2$ *for large* t.
(b) $H(t) + H(s) > 1$, $H(s) > 1/2$ *for all* $s > 0$ *and large* t.

Proof. The proof follows on the line of [18]. Let

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N_2(\mu, \Sigma), \tag{12}$$

be the bivariate normal vector with mean $(0,0)^T$ and covariance matrix

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}. \tag{13}$$

Then, we have (see e.g., [19,20])

$$\mathbb{P}(X > 0, Y > 0) = \mathbb{P}(X < 0, Y < 0) = \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho, \tag{14}$$

and

$$\mathbb{P}(X > 0, Y < 0) = \mathbb{P}(X < 0, Y > 0) = \frac{1}{4} - \frac{1}{2\pi}\sin^{-1}\rho. \tag{15}$$

Further,

$$\mathbb{E}(|X||Y|) = \frac{2}{\pi}\sigma_1\sigma_2(\sqrt{1-\rho^2} + \rho\ \sin^{-1}\rho),\ \mathbb{E}|X| = \sqrt{\frac{2}{\pi}}\sigma_1,\ \ \mathbb{E}|Y| = \sqrt{\frac{2}{\pi}}\sigma_2, \tag{16}$$

which implies

$$\mathrm{Cov}(|X|, |Y|) = \frac{2}{\pi}\sigma_1\sigma_2(\sqrt{1-\rho^2} - 1 + \rho\sin^{-1}\rho).$$

Hence

$$\mathrm{Corr}(|X|, |Y|) = \frac{2}{(\pi-2)}\left(\sqrt{1-\rho^2} - 1 + \rho\sin^{-1}\rho\right).$$

In addition, $\mathbb{E}(\mathrm{sign}X) = \mathbb{E}(\mathrm{sign}Y) = 0$ and $\mathrm{Var}(\mathrm{sign}X) = \mathrm{Var}(\mathrm{sign}Y) = 1$, see [18]. Moreover, using (14) and (15)

$$\begin{aligned}\mathrm{Cov}(\mathrm{sign}X, \mathrm{sign}Y) &= \mathbb{E}(\mathrm{sign}X\ \mathrm{sign}Y) - \mathbb{E}(\mathrm{sign}X)\mathbb{E}(\mathrm{sign}Y) = \mathbb{E}(\mathrm{sign}X\ \mathrm{sign}Y) \\ &= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho + \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho - \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho - \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho \\ &= \frac{2}{\pi}\sin^{-1}\rho.\end{aligned}$$

It follows that

(i)

$$\mathrm{Corr}(\mathrm{sign}\tilde{B}_{H(s)}(s), \mathrm{sign}\tilde{B}_{H(t)}(t)) = \mathrm{Cov}(\mathrm{sign}\tilde{B}_{H(s)}(s), \mathrm{sign}\tilde{B}_{H(t)}(t)), \tag{17}$$

follows by the fact that $\tilde{B}_{H(t)}(t) \sim N(0, t^{2H(t)})$ for all $t \geq 0$. We have by (14) and from the fact that $\mathrm{Corr}(\tilde{B}_{H(t)}(t), \tilde{B}_{H(s)}(s)) \downarrow 0$ as $t \to \infty$ for all $s \geq 0$,

$$\mathrm{Corr}(\mathrm{sign}\tilde{B}_{H(s)}(s), \mathrm{sign}\tilde{B}_{H(t)}(t)) = \frac{2}{\pi}\sin^{-1}(\rho(t,s)) \sim \frac{2}{\pi}\rho(t,s)\ \text{ as } t \to \infty, \tag{18}$$

where $\rho(t,s) = \mathrm{Corr}(\tilde{B}_{H(t)}(t), \tilde{B}_{H(s)}(s))$. From Proposition 7 in [13], we have

$$\begin{aligned}\mathrm{Corr}(\mathrm{sign}\tilde{B}_{H(s)}(s), \mathrm{sign}\tilde{B}_{H(t)}(t)) &\approx t^{-H(t)},\ \text{if } H(t) + H(s) < 1 \\ &\approx t^{H(s)-1},\ \text{if } H(t) + H(s) > 1,\end{aligned} \tag{19}$$

for all fixed $s > 0$ and as $t \to \infty$. Since $H(t)$ and $1 - H(s)$ both belong to $(0, 1)$, we conclude that MFBM has persistence of signs LRD in the sense of Definition 2, for all values of $H(t)$.

(ii) In addition,

$$\text{Corr}(|\tilde{B}_{H(t)}(t)|, |\tilde{B}_{H(s)}(s)|) = \frac{2}{\pi - 2}(\sqrt{1 - \rho(t,s)^2} - 1 + \rho(t,s)\sin^{-1}\rho(t,s)) \sim \frac{\rho(t,s)^2}{\pi - 2}, \tag{20}$$

as $t \to \infty$, where $\rho(t,s) = \text{Corr}(\tilde{B}_{H(t)}(t), \tilde{B}_{H(s)}(s))$. So we have,

$$\begin{aligned}\text{Corr}(|\tilde{B}_{H(t)}(t)|, |\tilde{B}_{H(s)}(s)|) &\approx t^{-2H(t)}, \text{ if } H(t) + H(s) < 1 \\ &\approx t^{2H(s)-2}, \text{ if } H(t) + H(s) > 1.\end{aligned} \tag{21}$$

If $H(t) + H(s) < 1$ and $H(t) < 1/2$ for large t and for all $s > 0$, then we have persistence of magnitudes LRD property. In case $H(t) + H(s) > 1$, persistence of magnitudes occurs when $H(s) > 1/2$ for all $s > 0$.

□

Remark 1. *If for all s, $H(t) + H(s) > 1$ for all sufficiently large t, then by (19) the MFBM with functional parameter H has persistence of signs LRD in the sense of Definition 2, with functional LRD exponent $\alpha(s) = H(s) - 1$. Similarly by (21), we have persistence of magnitude LRD in sense of Definition 2, with functional LRD exponent $\alpha(s) = 2H(s) - 2$, with $H(s) < 1/2$.*

In case $H(t) + H(s) < 1$, MFBM may not have persistence of signs and persistence of magnitudes LRD properties when there exist a sequence $\{t_n\}$ tends to infinity s.t. $H(t_n) + H(s) < 1$, for every n and $H(t_n)$ does not have a limit. For example consider the functional parameter $H(t) = \frac{1}{6} + \frac{1}{6}|\sin(\frac{\pi}{2}t^\alpha)|$, $\alpha \in (0,1)$. Here $H(t) + H(s) \le 2/3$ for all t and s, but $\lim_{n\to\infty} H(t_n)$ does not exist for any sequence $t_n \uparrow \infty$. In this case, there does not exist $d(s)$, s.t. $Corr(\tilde{B}_{H(t)}(t), \tilde{B}_{H(t)}(s)) = O(t^{-d(s)})$. Similarly, when $H(t) + H(s) < 1$ MFBM will not have persistence of magnitude LRD in general in sense of Definition 2.

In definition of persistence of magnitudes, absolute values could be replaced by rth powers of absolute values for any $r > 0$, similar to the result given in [18].

Proposition 2. *Let $\tilde{B}_{H(t)}(t)$ be a standard MFBM given in (4), then LRD holds in sense of Definition 2 for $|X(t)|^r, r > 0$. When,*

(i) $\forall s \ge 0$, $H(t) + H(s) < 1$ and $H(t) < 1/2$ for sufficiently large t.
(ii) $H(t) + H(s) > 1$, $H(s) > 1/2$ for all $s \ge 0$ and sufficiently large t.

Proof. The result again follows on the line of [18] for FBM. Let $(X, Y)^T$ be a bivariate Gaussian random vector as taken in Proposition 1. Then, for $r > 0$ (see e.g., [19]),

$$\mathbb{E}|XY|^r = \frac{2^r}{\pi}\sigma_1^r\sigma_2^r\left(\Gamma\left(\frac{r+1}{2}\right)\right)^2 F_1\left(-\frac{1}{2}r, -\frac{1}{2}r; \frac{1}{2}; \rho^2\right), \tag{22}$$

where F_1 is the hypergeometric function [21]. In addition,

$$\mathbb{E}|X|^r = \sqrt{\frac{2^r}{\pi}}\sigma_1^r\Gamma\left(\frac{r+1}{2}\right), \ \mathbb{E}|Y|^r = \sqrt{\frac{2^r}{\pi}}\sigma_2^r\Gamma\left(\frac{r+1}{2}\right).$$

Moreover,

$$F_1\left(-\frac{1}{2}r, -\frac{1}{2}r; \frac{1}{2}; \rho^2\right) = 1 + \frac{1}{2}r^2\rho^2 + O(\rho^4) \text{ as } \rho \to 0, \tag{23}$$

and $\mathrm{Var}(|X|^r) = a_r\sigma_1^{2r}$, $\mathrm{Var}(|Y|^r) = a_r\sigma_2^{2r}$, where,

$$a_r = \frac{2^r}{\sqrt{\pi}}\Gamma\left(\frac{2r+1}{2}\right) - \frac{2^r}{\pi}\left(\Gamma\left(\frac{r+1}{2}\right)\right)^2.$$

Thus,

$$\begin{aligned}\mathrm{Corr}(|\tilde{B}_{H(t)}(t)|^r, |\tilde{B}_{H(s)}(s)|^r) &= \frac{2^r}{\pi a_r}\Gamma\left(\frac{r+1}{2}\right)^2\left[F_1\left(-\frac{1}{2}r, -\frac{1}{2}r; \frac{1}{2}; \rho(t,s)^2\right) - 1\right] \\ &\approx \rho(t,s)^2 \text{ as } \rho(t,s) \to 0,\end{aligned}$$

and the result follows with the help of Proposition 1(ii). □

3. The *n*th Order Multifractional Brownian Motion (*n*-MFBM)

Similar to *n*th-order fractional Brownian motion (*n*-FBM) [6], we define *n*th-order multifractional Brownian motion (*n*-MFBM) by subtracting n terms in the definition of MFBM. Precisely, that will allow functional parameter H in the range $(n-1, n)$. Using [6], for the harmonizable representation of *n*-FBM denoted by $\tilde{B}_H^n(t)$, we have

$$\begin{aligned}\mathbb{E}(\tilde{B}_H^n(t)\tilde{B}_H^n(s)) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{|\xi|^{2H+1}}\left(e^{it\xi} - \sum_{k=0}^{n-1}\frac{(it\xi)^k}{k!}\right)\left(e^{-is\xi} - \sum_{k=0}^{n-1}\frac{(-is\xi)^k}{k!}\right)d(\xi) \\ &= (-1)^n\frac{C_H^n}{2}\left\{|t-s|^{2H} - \sum_{j=0}^{n-1}(-1)^j\binom{2H}{j}\left[\left(\frac{t}{s}\right)^j|s|^{2H} + \left(\frac{s}{t}\right)^j|t|^{2H}\right]\right\},\end{aligned} \tag{24}$$

where $C_H^n = \frac{1}{\Gamma(2H+1)|\sin\pi H|}$. Analogous to the definition of *n*-FBM, we define *n*-MFBM as follows.

Definition 3. *Let $H : [0,\infty) \to [a,b] \subset (n-1,n)$ be a Hölder function of exponent $\beta > 0$. For $t \geq 0$ the following random function, denoted by $B_{H(t)}^n(t)$, is called nth-order multifractional Brownian motion with functional parameter H:*

$$\begin{aligned}B_{H(t)}^n(t) = &\frac{1}{\Gamma(H(t)+\frac{1}{2})}\Big\{\int_{-\infty}^{0}\Big[(t-u)^{H(t)-1/2} - (-u)^{H(t)-1/2} - \left(H(t)-\frac{1}{2}\right)(-u)^{H(t)-3/2}t \\ &- \cdots - \left(H(t)-\frac{1}{2}\right)\cdots\left(H(t)-\frac{2n-3}{2}\right)(-u)^{H(t)-n+1/2}\frac{t^{n-1}}{(n-1)!}\Big]dB(u) \\ &+ \int_0^t (t-u)^{H(t)-1/2}dB(u)\Big\}. \\ = &\int_{\mathbb{R}} f_+(t,u)dB(u),\end{aligned} \tag{25}$$

where

$$f_+(t,u) = \frac{1}{\Gamma(H(t)+\frac{1}{2})}\left\{(t-u)_+^{H(t)-1/2} - \sum_{k=0}^{n-1}(-u)_+^{H(t)-k-1/2}\frac{t^k}{k!}\prod_{i=1}^{k}(H(t)-i+1/2)\right\}. \tag{26}$$

Remark 2. *It is easy to show that*

$$f_+(t,u) \sim \begin{cases} \binom{H(t)-\frac{1}{2}}{n}t^n(-u)^{H(t)-\frac{1}{2}-n}, \text{ as } u \to -\infty, \\ t^{H(t)-\frac{1}{2}}, \text{ as } u \to 0, \\ (t-u)_+^{H(t)-\frac{1}{2}}, \text{ as } u \to t \end{cases} \tag{27}$$

Thus, we have $\int_{-\infty}^{\infty}|f_+(t,u)|^2du < \infty$.

Remark 3. *The following Hölder functions can be used for* $H(t) \in (n-1, n),\ t > 0$*:*

$$H(t) = (n-1) + \frac{1}{\pi}\arctan(t) + \frac{1}{2},\ t > 0;$$
$$H(t) = (n-1) + \frac{at}{b+at},\ t > 0;$$
$$H(t) = (n-1) + a\left|\sin\left(\frac{\pi}{2}t^{\alpha}\right)\right|,\ 0 < a < 1,\ 0 < \alpha < 1, t > 0.$$

In addition, the definition of n-MFBM can be given in following harmonizable representation form.

Definition 4. *Let* $H : [0, \infty) \to [a, b] \subset (n-1, n)$ *be a Hölder function of exponent* $\beta > 0$*. For* $t \geq 0$ *the following random function, denoted by* $\tilde{B}^n_{H(t)}(t)$*, we call nth-order multifractional Brownian motion with functional parameter H:*

$$\tilde{B}^n_{H(t)}(t) = \int_{\mathbb{R}} \frac{1}{|\xi|^{H(t)+1/2}}\left[e^{it\xi} - \sum_{k=0}^{n-1}\frac{(it\xi)^k}{k!}\right] d\hat{B}(\xi), \tag{28}$$

where $\hat{B}$ *denotes a complex-valued Gaussian measure.*

For $n = 1$, the definitions given in Definitions 3 and 4 correspond to the definition of MFBM given in Definitions 1 and 4.

Proposition 3. *Let* $\hat{f}_+(\xi) = \mathcal{F}(f_+)(\xi)$ *be the Fourier transform of the function* $f_+(u) \in \mathbf{L}^2(du)$*, which is defined in (26). For all* $t \in \mathbb{R}$*,* $H(t) \in (n-1, n)$*, we have*

$$\hat{f_+}(t, \xi) = \frac{e^{-isign(\xi)(H(t)+1/2)\pi/2}}{\sqrt{(2\pi)}|\xi|^{H(t)+1/2}}\left[e^{it\xi} - \sum_{k=0}^{n-1}\frac{(it\xi)^k}{k!}\right],\ \xi \in \mathbb{R}\backslash\{0\}.$$

Proof. The Fourier transform of function $g(u) = u^j_+, u \in \mathbb{R},\ j \in (-1/2, 1/2)$ is given by (see [22], p. 171)

$$\mathcal{F}(u^j_+)(\xi) = \frac{\Gamma(j+1)}{\sqrt{(2\pi)}}e^{isign(\xi)(j+1)\pi/2},\ \ \xi \neq 0. \tag{29}$$

This is shown for $-1 < \mathcal{R}(j) < 0$ by regularizing the function u^j_+ at infinity by considering $u^j_+e^{-\tau u}$ taking the Fourier transform and letting $\tau \downarrow 0$. This result can be extend to $\mathcal{R}(j) \in \mathbb{R}\backslash\{-1, -2, -3, \ldots\}$ due to analytic continuation.

By Using (29) with $H(t) \in (k, k+1)$, we have

$$\mathcal{F}\Big((u_+)^{H(t)-k-1/2}\Big)(\xi) = \frac{\Gamma(H(t)-k+1/2)}{\sqrt{(2\pi)}}\frac{e^{isign(\xi)(H(t)+1/2)\pi/2}}{|\xi|^{H(t)+1/2}}(-i\xi)^k.$$

Then we get (see Proposition 3.1 [10])

$$\begin{aligned}\mathcal{F}(f_+(t,u))(\xi) &= \frac{1}{\Gamma(H(t)+1/2)}\Big[\mathcal{F}((t-u)^{H(t)-1/2}_+)(\xi) \\ &\quad - \sum_{k=0}^{n-1}\mathcal{F}((-u)^{H(t)-k-1/2}_+)(\xi)\frac{t^k}{k!}\prod_{i=1}^{k}(H(t)-i+1/2)\Big] \\ &= \frac{e^{-isign(\xi)(H(t)+1/2)\pi/2}}{\sqrt{(2\pi)}|\xi|^{H(t)+1/2}}\left[e^{it\xi} - \sum_{k=0}^{n-1}\frac{(i\xi t)^k}{k!}\right],\ \xi \neq 0.\end{aligned} \tag{30}$$

□

Proposition 4. *The moving average representation of the n-MFBM (25) can also be represented as*

$$B^n_{H(t)}(t) = \frac{\cos((H(t)+1/2)\pi/2)}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{1}{|\xi|^{H(t)+1/2}} \left[e^{it\xi} - \sum_{k=0}^{n-1} \frac{(i\xi t)^k}{k!}\right] d\hat{B}(\xi) + \frac{\sin((H(t)+1/2)\pi/2)}{\sqrt{2\pi}} \int_{\mathbf{R}} \frac{1}{i\xi|\xi|^{H(t)-1/2}} \left[e^{it\xi} - \sum_{k=0}^{n-1} \frac{(i\xi t)^k}{k!}\right] d\hat{B}(\xi). \tag{31}$$

Proof. By using the Proposition 2.1 in [10], the result follows. □

Remark 4. *Alternatively, using the definition of FBM given in ([23], p. 328), the n-MFBM can be defined as*

$$\bar{B}^n_{H(t)}(t) = \int_{\mathbb{R}} \frac{1}{i\xi|\xi|^{H(t)-1/2}} \left[e^{it\xi} - \sum_{k=0}^{n-1} \frac{(it\xi)^k}{k!}\right] d\hat{B}(\xi). \tag{32}$$

The representations for n-MFBM given in (28) and (32) are equivalent since both are centered Gaussian process and

$$\mathbb{E}(\bar{B}^n_{H(t)}(t)\bar{B}^n_{H(s)}(s)) = \mathbb{E}(\tilde{B}^n_{H(t)}(t)\tilde{B}^n_{H(s)}(s)) = \int_{\mathbb{R}} \frac{1}{|\xi|^{H(t)+H(s)+1}} \left[\left(e^{it\xi} - \sum_{k=0}^{n-1} \frac{(it\xi)^k}{k!}\right)\left(e^{-is\xi} - \sum_{k=0}^{n-1} \frac{(-is\xi)^k}{k!}\right)\right] d\xi.$$

Remark 5. *When $H(t) = n - 1/2$, the moving average and harmonizable representations of n-MFBM have the same laws up to a multiplicative deterministic function. For $n = 2m+1$, $\sin(2m+1)\pi/2 = (-1)^m$ and $\cos(2m+1)\pi/2 = 0$, which gives*

$$B^n_{2m+1/2}(t) = \frac{(-1)^m}{\sqrt{2\pi}} \bar{B}_{2m+1/2}(t).$$

For $n = 2m$, we have $\sin(m\pi) = 0$ and $\cos(m\pi) = (-1)^m$, which leads to

$$B^n_{2m-1/2}(t) = \frac{(-1)^m}{\sqrt{2\pi}} \tilde{B}_{2m-1/2}(t).$$

Covariance Structure of n-MFBM

MFBM is a zero mean non-stationary Gaussian process with continuous sample paths. From definition it is obvious that n-MFBM is also a zero mean Gaussian process. Since n-MFBM is Gaussian, we will only be interested in statistical properties up to the second order, namely, the covariance function and variance of the process.

Proposition 5. *Let $\tilde{B}^n_{H(t)}(t)$ be the n-MFBM given in (28) with functional parameter $H(t)$. Then,*

$$\mathrm{Cov}(\tilde{B}^n_{H(t)}(t), \tilde{B}^n_{H(s)}(s)) = (-1)^n \frac{C(t,s)}{2} \left[|t-s|^{H(t)+H(s)} - \sum_{j=0}^{n-1} (-1)^j \binom{H(t)+H(s)}{j} \times \left[\left(\frac{t}{s}\right)^j |s|^{H(t)+H(s)} + \left(\frac{s}{t}\right)^j |t|^{H(t)+H(s)}\right]\right], \tag{33}$$

where $C(t,s)$ is given by

$$C(t,s) = \frac{2\pi}{\Gamma(H(t)+H(s)+1)|\sin(\frac{\pi}{2}(H(t)+H(s)))|}. \tag{34}$$

Proof. By definition,

$$\begin{aligned}\mathbb{E}(\tilde{B}^n_{H(t)}(t)\tilde{B}^n_{H(s)}(s)) &= \mathbb{E}\Big[\int_{\mathbb{R}} \frac{1}{|\xi|^{H(t)+1/2}}\Big(e^{it\xi} - \sum_{k=0}^{n-1}\frac{(it\xi)^k}{k!}\Big)d\hat{B}(\xi) \\ &\quad \times \int_{\mathbb{R}} \frac{1}{|\xi|^{H(s)+1/2}}\Big(e^{-is\xi} - \sum_{k=0}^{n-1}\frac{(-is\xi)^k}{k!}\Big)d\hat{B}(\xi)\Big] \\ &= \int_{\mathbb{R}} \frac{1}{|\xi|^{H(t)+H(s)+1}}\Big[\Big(e^{it\xi} - \sum_{k=0}^{n-1}\frac{(it\xi)^k}{k!}\Big)\Big(e^{-is\xi} - \sum_{k=0}^{n-1}\frac{(-is\xi)^k}{k!}\Big)\Big]d\xi.\end{aligned} \tag{35}$$

Covariance function of standard n-FBM $B^n_H(t)$ with Hurst parameter H, is given in (24). For fixed t and s, let $B^n_H(t)$ be a standard n-FBM with Hurst parameter $H = \frac{H(t)+H(s)}{2}$. Using (24), it follows

$$\begin{aligned}&\int_{\mathbb{R}} \frac{1}{|\xi|^{H(t)+H(s)+1}}\Big(e^{it\xi} - \sum_{k=0}^{n-1}\frac{it\xi}{k!}\Big)\Big(e^{-is\xi} - \sum_{k=0}^{n-1}\frac{(-is\xi)}{k!}\Big)d\xi \\ &= (-1)^n\frac{C(t,s)}{2}\left[|t-s|^{H(t)+H(s)} - \sum_{j=0}^{n-1}(-1)^j\binom{H(t)+H(s)}{j}\Big[\Big(\frac{t}{s}\Big)^j|s|^{H(t)+H(s)} + \Big(\frac{s}{t}\Big)^j|t|^{H(t)+H(s)}\Big]\right],\end{aligned} \tag{36}$$

where

$$C(t,s) = \frac{2\pi}{\Gamma(H(t)+H(s)+1)|\sin(\frac{\pi}{2}(H(t)+H(s)))|}.$$

Using (35) and (36), the desired result follows. □

Remark 6. *For $n = 1$ and $H(t) = H$ for all $t > 0$, we have*

$$\mathbb{E}(\tilde{B}^n_{H(t)}(t)\tilde{B}^n_{H(s)}(s)) = \frac{C(t,s)}{2}\Big\{t^{2H} + s^{2H} - |t-s|^{2H}\Big\} \tag{37}$$

where $C(t,s) = \frac{2\pi}{H\Gamma(2H)\sin(H\pi)}$, which is the covariance function of FBM [6].

Remark 7. *Using (33), it follows*

$$\mathrm{Var}(\tilde{\mathrm{B}}^{\mathrm{n}}_{\mathrm{H(t)}}(\mathrm{t})) = (-1)^{n-1}C(t,t)|t|^{2H(t)}\sum_{j=0}^{n-1}(-1)^j\binom{2H(t)}{j} = C(t,t)|t|^{2H(t)}\binom{2H(t)-1}{n-1}, \tag{38}$$

where $C(t,t) = \frac{\pi}{H(t)\Gamma(H(t))|\sin(\pi H(t))|}$.

Proposition 6. *Let $\tilde{B}^n_{H(t)}(t)$ be the n-MFBM defined in (28). Then, for all admissible $H(t)$, the n-MFBM has LRD property in the sense of Definition 1.*

Proof. We have

$$
\begin{aligned}
&\mathrm{Corr}(\tilde{\mathrm{B}}^{\mathrm{n}}_{\mathrm{H(t)}}(\mathrm{t})), \tilde{\mathrm{B}}^{\mathrm{n}}_{\mathrm{H(s)}}(\mathrm{s}))) \\
&= \frac{1}{\sqrt{C(t,t)C(s,s)\binom{2H(t)-1}{n-1}\binom{2H(s)-1}{n-1}t^{H(t)}s^{H(s)}}}\Big[t^{H(t)+H(s)}(1-\frac{s}{t})^{H(t)+H(s)} \\
&- \sum_{j=0}^{n-1}(-1)^j\binom{H(t)+H(s)}{j}\Big[t^j s^{H(t)+H(s)-j} + s^j t^{H(t)+H(s)-j}\Big]\Big] \\
&= D(t,s)\left(\frac{t}{s}\right)^{H(s)}\Big[\Big(1-\frac{s}{t}\Big)^{H(t)+H(s)} - \sum_{j=0}^{n-1}(-1)^j\binom{H(t)+H(s)}{j}\Big[\Big(\frac{s}{t}\Big)^{H(t)+H(s)-j} + \Big(\frac{s}{t}\Big)^j\Big]\Big] \\
&= D(t,s)\left(\frac{t}{s}\right)^{H(s)}\Big[\sum_{j=n}^{\infty}(-1)^j\binom{H(t)+H(s)}{j}\Big(\frac{s}{t}\Big)^j - \sum_{j=0}^{n-1}(-1)^j\binom{H(t)+H(s)}{j}\Big(\frac{s}{t}\Big)^{H(t)+H(s)-j}\Big] \\
&\approx \begin{cases} \left(\frac{s}{t}\right)^{H(t)-n+1} & \text{if } \frac{H(t)+H(s)}{2} \le n-1/2 \\ \left(\frac{s}{t}\right)^{n-H(s)} & \text{if } \frac{H(t)+H(s)}{2} \ge n-1/2. \end{cases}
\end{aligned}
$$

□

4. Conclusions

Brownian motion is a central object in stochastic calculus and stochastic differential equations. However due to independent increments, Brownian motion has limitations in modeling random phenomena exhibiting dependence. Fractional Brownian motion (FBM) is a generalization of Brownian motion possessing LRD, persistence of signs LRD and persistence of magnitude LRD properties. The pointwise irregularities in FBM do not vary with time and hence it cannot be used to simulate artificial mountains for example. To overcome these limitations MFBM was introduced in literature where pointwise irregularities represented by a Hölder continuous function. In this article, we generalize the MFBM to n-MFBM which allows the Hölder continuous function to take values in the interval $(n-1, n)$. The basic properties like covariance and variance of the introduced process are obtained. It is shown that n-MFBM has LRD property. Further, the harmonizable representation of n-MFBM is given. We believe that n-MFBM will be useful in modeling of natural phenomena where the pointwise irregularities vary with time and the Hurst parameter does not necessarily belong to the interval $(0, 1)$.

5. Discussion

Note that the tempered fractional stable motion introduced in [15] adds an exponential tempering to the power-law kernel in the LFSM. In the line of higher order fractional stable motion [16], it is very natural to define a higher order tempered fractional stable motion. Furthermore, the higher order fractional stable motion and the tempered fractional stable motion can be extended further by choosing the Hurst parameter as a stochastic process instead of deterministic function of time as done in [7]. In addition, the GMFBM can be further theorized to higher order GMFBM and higher order generalized MFBM.

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Abbreviations

The following abbreviations are used in this manuscript:

MFBM	Multifractional Brownian motion
FBM	Fractional Brownian motion
LRD	Long range dependence
SRD	Short range dependence
n-MFBM	nth order multifractional Brownian motion.

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