Invariant subspaces of elliptic systems I: pseudodifferential projections

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Abstract

Consider an elliptic self-adjoint pseudodifferential operator A acting on m-columns of half-densities on a closed manifold M, whose principal symbol is assumed to have simple eigenvalues. We show existence and uniqueness of m orthonormal pseudodifferential projections commuting with the operator A and provide an algorithm for the computation of their full symbols, as well as explicit closed formulae for their subprincipal symbols. Pseudodifferential projections yield a decomposition of $L^2(M)$ into invariant subspaces under the action of A modulo $C^\infty(M)$. Furthermore, they allow us to decompose A into m distinct sign definite pseudodifferential operators. Finally, we represent the modulus and the Heaviside function of the operator A in terms of pseudodifferential projections and discuss physically meaningful examples.

Keywords: pseudodifferential projections, elliptic systems, invariant subspaces, pseudodifferential operators on manifolds.

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Contents

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1 Statement of the problem

Let M be a closed connected manifold of dimension $d \ge 2$. We denote local coordinates on M by $x = (x^1, \dots, x^d)$.

Let $C^\infty(M)$ be the vector space of m-columns of smooth complex-valued half-densities over M equipped with inner product

$$\langle v, w\rangle := \int_M v^* w \, dx \,, \tag{1.1}$$

where $dx := dx^1 \dots dx^d$. Here and further on the star stands for Hermitian conjugation when applied to matrices and for adjunction with respect to (1.1) when applied to operators. By $L^2(M)$ we denote the closure of $C^\infty(M)$ with respect to (1.1). Of course, our function spaces depend on the choice of the natural number m but in order to simplify notation we suppress this dependence. Thus, throughout this paper $m \ge 2$ is a fixed natural number.

By $H^s(M)$ we denote the usual Sobolev space, i.e. the vector space of m-columns of half-densities that are square integrable together with their partial derivatives up to order s. By Ψ^s we denote the space of classical pseudodifferential operators of order s with polyhomogeneous symbols, acting from $H^s(M)$ to $L^2(M)$. For an operator $P \in \Psi^s$ we denote its matrix-valued principal and subprincipal symbols by P_{prin} and P_{sub} respectively. Of course, these are scalar matrix-functions in $C^\infty(T^*M \setminus \{0\}; \mathrm{Mat}(m, \mathbb{C}))$ of degree of homogeneity in momentum s and $s-1$. We also introduce refined notation for the principal symbol. Namely, we denote by $(\,\cdot\,)_{\mathrm{prin},s}$ the principal symbol of the expression within brackets, regarded as an operator in Ψ^{-s}. To appreciate the need for such notation, consider the following example. Let B and C be pseudodifferential operators in Ψ^{-s} with the same principal symbol. Then, as an operator in Ψ^{-s}, $B-C$ has vanishing principal symbol: $(B-C)_{\mathrm{prin},s} = 0$. But this tells us that $B-C$ is, effectively, an operator in Ψ^{-s-1} and, as such, it may have nonvanishing principal symbol $(B-C)_{\mathrm{prin},s+1}$. This refined notation will be used whenever there is risk of confusion.

Definition 1.1. We say that $P \in \Psi^0$ is an *orthogonal pseudodifferential projection* if

$$P^2 = P \mod \Psi^{-\infty}, \tag{1.2}$$

$$P^* = P \mod \Psi^{-\infty}. \tag{1.3}$$

Definition 1.2. We call a set of m orthogonal pseudodifferential projections $\{P_j\}$ an *orthonormal pseudodifferential basis* if their principal symbols are rank 1 matrix-functions and

$$P_j P_k = 0 \mod \Psi^{-\infty} \quad \forall j \neq k, \tag{1.4}$$

$$\sum_j P_j = \mathrm{Id} \mod \Psi^{-\infty}, \tag{1.5}$$

where $\mathrm{Id} \in \Psi^0$ is the identity operator.

It is natural to ask the following questions.

Question 1 Does there exist a nontrivial operator P satisfying Definition 1.1?

Question 2 Assuming that the answer to Question 1 is positive, can we choose the P_j's so that they satisfy Definition 1.2?

The issue here is that in order to construct these pesudodifferential projections one has to determine the lower order (of degree of homogeneity $-1, -2, \ldots$) components of the symbols of the P_j's so as to satisfy (1.2)–(1.5). This requires solving an infinite sequence of heavily overdetermined systems of algebraic equations, and it is not *a priori* clear that these systems have solutions. We would like to point out that great care is needed in performing this analysis because our operators have matrix-valued symbols which in general do not commute.

Dealing with projections in infinite-dimensional spaces is known to be a challenging task and we believe that addressing Questions 1 and 2 is of interest in its own right. However, these pseudodifferential projections reveal their true potential when applied to the study of elliptic and hyperbolic systems of partial differential equations.

Let $A \in \Psi^s$, $s \in \mathbb{R}$, $s > 0$, be an elliptic self-adjoint linear pseudodifferential operator, where ellipticity means that

$$\det A_{\mathrm{prin}}(x,\xi) \neq 0, \qquad \forall (x,\xi) \in T^*M \setminus \{0\}.$$

We impose the following crucial assumption.

Assumption 1.3. The matrix-function $A_{\mathrm{prin}}(x,\xi)$ has simple eigenvalues.

We denote by m^+ (resp. m^-) the number of positive (resp. negative) eigenvalues of $A_{\mathrm{prin}}(x,\xi)$. We denote by $h^{(j)}(x,\xi)$ the eigenvalues of $A_{\mathrm{prin}}(x,\xi)$ enumerated in increasing order, with positive index $j=1,2,\dots,m^+$ for positive $h^{(j)}(x,\xi)$ and negative index $j=-1,-2,\dots,-m^-$ for negative $h^{(j)}(x,\xi)$. Clearly, self-adjointness, ellipticity and connectedness of M imply that

(i) the $h^{(j)}$ are scalar nonvanishing smooth real-valued functions on $T^*M \setminus \{0\}$,

(ii) m^+ and m^- are constant and

(iii) $m^+ + m^- = m$.

The spectrum of our operator $A : H^s(M) \to L^2(M)$ is discrete and accumulates to infinity. More precisely, if $m^+ \ge 1$ the spectrum accumulates to $+\infty$, if $m^- \ge 1$ the spectrum accumulates to $-\infty$, and if $m^+ \ge 1$ and $m^- \ge 1$ the spectrum accumulates to $\pm\infty$.

By $P^{(j)}(x,\xi)$ we denote the eigenprojection of $A_{\mathrm{prin}}(x,\xi)$ corresponding to the eigenvalue $h^{(j)}(x,\xi)$. Assumption 1.3 tells us that the matrix-functions $P^{(j)}(x,\xi)$ are rank 1. These eigenprojections satisfy

$$A_{\mathrm{prin}} = \sum_j h^{(j)} P^{(j)} \tag{1.6}$$

and

$$\sum_j P^{(j)} = I, \tag{1.7}$$

where I is the $m \times m$ identity matrix.

Question 3 Assuming that the answer to Question 2 is positive, can we choose the P_j's so that they commute with the operator A

$$[A, P_j] = 0 \mod \Psi^{-\infty} \tag{1.8}$$

and

$$(P_j)_{\mathrm{prin}} = P^{(j)}? \tag{1.9}$$

Here, in addition to the issues highlighted in relation to Questions 1 and 2, the extra difficulty is that the infinite sequence of overdetermined systems involves the $h^{(j)}$'s as well as the lower order components of the full symbol of the operator A.

Question 3 is important in that an affirmative answer would yield a collection of projections compatible with A, leading to two further natural questions.

Question 4 Can we exploit the pseudodifferential projections P_j to advance the current understanding of spectral asymptotics for elliptic systems?

Question 5 Can we exploit the pseudodifferential projections P_j to advance the current understanding of propagation of singularities for hyperbolic systems?

The goal of this paper is develop a comprehensive theory of pseudodifferential projections so as to positively answer Questions 1, 2 and 3, building upon earlier results. Questions 4 and 5 are addressed in the companion paper [16].

2 Main results

Pseudodifferential projections have been used over the years, under different names and with varying degree of awareness, in many areas of mathematical analysis.

Mathematicians working with Toeplitz operators found in 'pseudodifferential subspaces' a useful tool [3, 25], and Birman and Solomyak [4] studied and characterised subspaces of sections of vector bundles which can be the range of a pseudodifferential projection. Birman and Solomyak also provided an abstract formula for a single pseudodifferential projection with given principal symbol, obtained by integrating an appropriate resolvent along a carefully chosen path in the complex plane, see [4, Lemma 3].

Amongst applications in topology and index theory it is worth mentioning the works of Wojciechowski [40, 41], who analysed the topological properties of the space of equivalence classes of projections with the same principal symbol (i.e. modulo a compact operator) by means of Fredholm theory.

A flourishing avenue of research involving pseudodifferential projections in various forms is the study of boundary value problems for elliptic operators. A distinguished example is the celebrated Calderón projector [10, 30, 31, 24]. It is known [26, Vol. III][37] that for an arbitrary elliptic operator on a manifold with boundary one cannot, in general, impose boundary conditions satisfying the Shapiro–Lopatinski condition. This leads to ill-defined (non Fredholm) boundary value problems. The use of pseudodifferential projections proved useful in attempts to generalise elliptic theory to such operators so as to obtain Fredholm boundary value problems, see, e.g., [6, 7, 8, 37].

A considerable advancement in the understanding of pseudodifferential projections is due to Bolte and Glaser [5], who, relying on a strategy by Cordes [19], in the setting of semiclassical analysis, construct pseudodifferential projections in the spirit of Riesz projections. Their results establish existence and identify the minimal set of conditions that guarantee uniqueness. There are a number of differences between our approach and that of Bolte and Glaser.

(i) They work with semiclassical operators on $\mathbb{R}^d$ as opposed to classical ones on a manifold M.

(ii) The (local) construction of the symbol using Riesz projections requires one to compute the symbol of the resolvent $(A-\lambda\mathrm{Id})^{-1} \mod \Psi^{-\infty}$, which, in turn, necessitates parameter-dependent pseudodifferential calculus. Our algorithm at every step uses and produces invariant objects and it does not involve parameter-dependent symbol classes or partitions of unity.

(iii) Whereas the Riesz projection method is essentially linked to an operator A, our approach regards pseudodifferential projections as abstract objects, which exist and can be constructed independently of A. This has the advantage of shedding light on the structure of their symbols, as well as clarifying how the degrees of freedom are used up when imposing the defining conditions (1.2)–(1.9).

For specific operators, such as the Dirac operator or matricial versions of the Klein–Gordon operator, alternative approaches to the problem have been proposed, ones that avoid dealing with pseudodifferential projections by constructing almost-unitary operators that 'microlocally diagonalise' the matrix operator at hand, see, for example, [20, 22, 9, 34, 35]. Pseudodifferential projections are, in a sense, 'more fundamental' objects than almost-unitary operators: we refer the reader to our companion paper [16] for further comments on this matter.

We should also mention that pseudodifferential projections appeared in the form of (approximate) spectral projections in publications on the spectral theory of elliptic systems, though some of these publications are known to contain mistakes, see [17, Sec. 11].

Our main results can be summarised in the form of six theorems stated in this section.

Theorem 2.1. *Given a family of m orthonormal rank 1 projections*

$$P^{(j)} \in C^\infty(T^*M \setminus \{0\}; \mathrm{Mat}(m;\mathbb{C}))$$

positively homogeneous in momentum of degree zero, there exists an orthonormal pseudodifferential basis $\{P_j\} \subset \Psi^0$ as per Definition 1.2 with $(P_j)_\mathrm{prin} = P^{(j)}$.

One can show[1] that the operators P_j in Theorem 2.1 can be modified, by adding $\Psi^{-\infty}$ terms, in such a way that conditions (1.2)–(1.4) are satisfied exactly, and not merely modulo $\Psi^{-\infty}$. Indeed, consider the zero order pseudodifferential operator $B := \frac{1}{2}\sum_j b_j(P_j + P_j^*)$, where the b_j are some distinct real numbers. Its essential spectrum is the set of m points b_j. For a given j, choose a contour C_j in the complex plane which encircles the point b_j and no other points of the essential spectrum and avoids isolated eigenvalues of finite multiplicity. Then integration of $(B - \lambda\,\mathrm{Id})^{-1}$ over the contour C_j will produce the modified pseudodifferential projection P_j. Furthermore, if we choose our contours C_j in such a way that they do not intersect and, in total, encircle the whole spectrum of B, then condition (1.5) will also be satisfied exactly.

Theorem 2.2. *Under Assumption 1.3, there exist m pseudodifferential operators $P_j \in \Psi^0$ satisfying Definition 1.2 and conditions* (1.8), (1.9)*, and these are uniquely determined, modulo $\Psi^{-\infty}$, by the operator A.*

Of course, Theorem 2.1 follows from Theorem 2.2, but we listed them as separate results for the sake of logical clarity. Unlike Theorem 2.1, we do not believe that for a general operator A it is possible to adjust the choice of our pseudodifferential projections P_j in Theorem 2.2 so as to satisfy the commutation conditions (1.8) exactly whilst maintaining exact conditions (1.2)–(1.5).

Note that Theorem 2.2 cannot be obtained by elementary function-analytic arguments involving an expansion over eigenvalues and eigenfunctions of the operator A. Theorem 2.2 is to do with the structure of the principal symbol of the operator A, an object which is not detected by the Spectral Theorem. A semiclassical version of Theorem 2.2 was obtained, with the *caveat*s discussed above, in [5].

The m orthogonal projections P_j from Theorem 2.2 effectively decompose $L^2(M)$ into m infinite-dimensional subspaces which are invariant under the action of the operator A. Of

[1] We are grateful to Daniel Grieser for raising this issue.

course, since the construction of our projections is approximate, modulo $\Psi^{-\infty}$, the resulting decomposition of $L^2(M)$ is also approximate, modulo $C^\infty(M)$:

$$AP_jL^2(M) \subseteq P_jL^2(M) \mod C^\infty(M).$$

Remarkably, Theorem 2.2 will be established by devising an explicit algorithm leading to the determination of the full symbols of the pseudodifferential projections P_j's, see subsections 3.4 or 4.3. In particular, we will obtain the following result.

Theorem 2.3. *The explicit formula for the subprincipal symbol of the pseudodifferential projection P_j reads*

$$\begin{aligned}(P_j)_{\mathrm{sub}} = \frac{i}{2}\{P^{(j)}, P^{(j)}\} - i\, P^{(j)}\{P^{(j)}, P^{(j)}\}P^{(j)} \\ + \sum_{l \ne j} \frac{P^{(j)}(A_{\mathrm{sub}} - iQ^{(j)})P^{(l)} + P^{(l)}(A_{\mathrm{sub}} + iQ^{(j)})P^{(j)}}{h^{(j)} - h^{(l)}}\,,\end{aligned} \tag{2.1}$$

where

$$Q^{(j)} := \frac{1}{2}\big(\{A_{\mathrm{prin}}, P^{(j)}\} - \{P^{(j)}, A_{\mathrm{prin}}\}\big). \tag{2.2}$$

In formulae (2.1) and (2.2) curly brackets denote the Poisson bracket

$$\{B, C\} := \sum_{\alpha=1}^{d}(B_{x^\alpha}C_{\xi_\alpha} - B_{\xi_\alpha}C_{x^\alpha}) \tag{2.3}$$

on matrix-functions on the cotangent bundle. Further on in the paper we will also make use of the generalised Poisson bracket

$$\{B, C, D\} := \sum_{\alpha=1}^{d}(B_{x^\alpha}CD_{\xi_\alpha} - B_{\xi_\alpha}CD_{x^\alpha}).$$

Let us emphasise that the order of terms in matrix-valued Poisson brackets matters; for example, the usual properties $\{f, f\} = 0$ and $\{f, g\} = -\{g, f\}$ from Hamiltonian mechanics no longer hold if the scalar functions f and g are replaced by matrix-functions.

Let us point out that having an explicit formula for the subprincipal symbol of projections P_j is important for applications. For example, the matrix trace of $(P_j)_{\mathrm{sub}}$ appears in the second Weyl coefficient of the eigenvalue counting function(s) of the operator A, see [17]. Failure to appreciate this fact led to a number of incorrect publications. For a long time it was assumed that Safarov [36] did obtain the formula for the second Weyl coefficient, fixing previous mistakes, but his formula also turned out to be wrong. With the benefit of hindsight, Safarov's mistake can be traced back to the incorrect assumption that $(P_j)_{\mathrm{sub}} = 0$. A brief account of the troubled history of the subject is given in [17, Section 11].

Definition 2.4. We say that a symmetric pseudodifferential operator B is *nonnegative (resp. nonpositive) modulo* $\Psi^{-\infty}$ and write

$$B \ge 0 \mod \Psi^{-\infty} \qquad (\text{resp. } B \le 0 \mod \Psi^{-\infty})$$

if there exists a symmetric operator $C \in \Psi^{-\infty}$ such that $B + C \ge 0$ (resp. $B + C \le 0$).

Theorem 2.5. *We have*

$$P_j^* A P_j \geq 0 \mod \Psi^{-\infty} \quad \textit{for} \quad j = 1, \ldots, m^+, \tag{2.4}$$

$$P_j^* A P_j \leq 0 \mod \Psi^{-\infty} \quad \textit{for} \quad j = -1, \ldots, -m^-. \tag{2.5}$$

Note that the operators $P_j^* A P_j$ appearing in Theorem 2.5 are not elliptic,

$$\det(P_j^* A P_j)_{\mathrm{prin}}(x, \xi) = 0 \qquad \forall (x, \xi) \in T^*M \setminus \{0\},$$

therefore, proving that they are sign semidefinite modulo $\Psi^{-\infty}$ is a delicate matter. The fact that their principal symbols are sign semidefinite does not, on its own, imply that the operators are sign semidefinite — it does not even imply that they are semibounded.

Let λ_k be the eigenvalues of the operator A enumerated with account of multiplicity and v_k be the corresponding orthonormal eigenfunctions. The choice of a particular enumeration is irrelevant for our purposes.

Consider the operator *modulus* of A defined in accordance with

$$|A| := \sum_k |\lambda_k| \, \langle v_k \, , \, \cdot \, \rangle \, v_k \, . \tag{2.6}$$

Theorem 2.6. *The operator $|A|$ is pseudodifferential and*

$$|A| = \sum_{j=1}^{m^+} A P_j - \sum_{j=1}^{m^-} A P_{-j} \mod \Psi^{-\infty}. \tag{2.7}$$

Furthermore, the explicit formula for the subprincipal symbol of the operator $|A|$ reads

$$\begin{aligned} |A|_{\mathrm{sub}} = & \sum_{j,k} \frac{h^{(j)} + h^{(k)}}{|h^{(j)}| + |h^{(k)}|} P^{(j)} A_{\mathrm{sub}} P^{(k)} \\ & + \frac{i}{2} \sum_{j,k} \frac{1}{|h^{(j)}| + |h^{(k)}|} P^{(j)} (\{A_{\mathrm{prin}}, A_{\mathrm{prin}}\} - \{|A|_{\mathrm{prin}}, |A|_{\mathrm{prin}}\}) P^{(k)}. \end{aligned} \tag{2.8}$$

Let $\theta : \mathbb{R} \to \mathbb{R}$,

$$\theta(z) := \begin{cases} 0 & \text{if} \quad z \leq 0, \\ 1 & \text{if} \quad z > 0 \end{cases}$$

be the Heaviside function. Consider the operator

$$\theta(A) := \sum_{k: \ \lambda_k > 0} \langle v_k \, , \, \cdot \, \rangle \, v_k \, . \tag{2.9}$$

Theorem 2.7. *The operator $\theta(A)$ is pseudodifferential and*

$$\theta(A) = \sum_{j=1}^{m^+} P_j \mod \Psi^{-\infty}. \tag{2.10}$$

Of course, Theorem 2.3 immediately gives us an explicit formula for $[\theta(A)]_{\mathrm{sub}}$.

Remark 2.8. We should point out that the setting of our paper is not the most general setting in which one can construct pseudodifferential projections satisfying conditions (1.2)–(1.5), (1.8)–(1.9). Note, for example, that the ellipticity of A or the fact that A is of positive order are not really needed in the proof of Theorem 2.2 or in the construction algorithm leading up to Theorem 2.3 (but they are needed in Theorem 2.5). Furthermore, our algorithm can be extended without much effort to operators acting on vector bundles — and, in particular, to operators acting on differential forms (e.g., the operator curl). Different techniques may allow one to generalise the results even further. The reason why we refrain from carrying out such generalisations in the current paper is twofold: (i) we aim to write a paper accessible to a wide audience, not necessarily limited to (microlocal) analysts, and (ii) we are motivated by applications in spectral theory (see Theorems 2.3, 2.5 and [16, 11]). We will address certain generalisations elsewhere.

The paper is structured as follows.

In Section 3 we develop the general theory of pseudodifferential projections: in subsection 3.1 we construct a single pseudodifferential projection, in subsection 3.2 we construct an orthonormal basis of pseudodifferential projections and in subsection 3.3 we show that the latter can be chosen in such a way that it commutes with our elliptic operator A and that this determines the projections uniquely. The results of Section 3 are summarised in subsection 3.4 in the form of an algorithm for the construction of the full symbol of pseudodifferential projections.

In Section 4 we show that a set of m pseudodifferential projections commuting with an elliptic operator $A \in \Psi^s$, $s > 0$, are automatically orthonormal and sum to the identity operator, modulo $\Psi^{-\infty}$. This leads to a simplified algorithm for the construction of their full symbols, presented in subsection 4.3.

In Section 5 we carry out the first step of our algorithm and obtain a closed explicit formula for the subprincipal symbol of pseudodifferential projections.

Section 6 is concerned with the proof of Theorem 2.5, which consists in a rigorous formulation of the fact that one can use pseudodifferential projections to construct m distinct sign definite operators (modulo $\Psi^{-\infty}$) out of A.

Results from Sections 4 and 5 are employed in Section 7 to represent modulus and Heaviside function of A in terms of pseudodifferential projections. This yields a simpler – compared to those available in the literature – algorithm for the calculation of the full symbols of $|A|$ and $\theta(A)$, as well as explicit formulae for $|A|_{\mathrm{sub}}$ and $[\theta(A)]_{\mathrm{sub}}$.

Lastly, in Section 8 we discuss three applications of our results: to the massless Dirac operator on a closed 3-manifold, to the operator of linear elasticity (Lamé operator) on a 2-torus and to the Dirichlet-to-Neumann map of linear elasticity in 3D.

3 Pseudodifferential projections: general theory

The goal of this section is to develop a comprehensive and self-contained theory of pseudodifferential projections in $L^2(M)^2$, including an explicit construction of their full symbols. This analysis, which we believe to be of interest in its own right, will answer Questions 1, 2 and 3 from Section 1, and lay rigorous foundations for the use of pseudodifferential projections in the study of spectral asymptotics of elliptic systems carried out in our companion paper [16].

3.1 Construction of a single pseudodifferential projection

In this subsection we prove the existence and establish the general structure of an operator $P_j \in \Psi^0$ satisfying conditions (1.2)–(1.3). We do this by constructing a sequence $P_{j,k} \in \Psi^0$, $k = 0, 1, 2, \dots$, of pseudodifferential operators such that

$$P_{j,k+1} - P_{j,k} \in \Psi^{-k-1}, \tag{3.1}$$

$$P_{j,k}^2 = P_{j,k} \mod \Psi^{-k-1}, \tag{3.2}$$

$$P_{j,k}^* = P_{j,k} \mod \Psi^{-\infty} \tag{3.3}$$

for $k = 0, 1, 2, \dots$. For $P_{j,0}$ we choose an arbitrary pesudodifferential operator satisfying (3.2) and (3.3), and construct subsequent $P_{j,k}$ by solving (3.1)–(3.3) recursively.

To this end, we seek $P_{j,k}$, $k = 1, 2, \dots$, in the form

$$P_{j,k} = P_{j,k-1} + X_{j,k},$$

where $X_{j,k} \in \Psi^{-k}$ is an unknown pseudodifferential operator such that

$$X_{j,k} = X_{j,k}^* \mod \Psi^{-\infty}. \tag{3.4}$$

Then condition (3.1) is automatically satisfied, whereas solving (3.2) and (3.3) reduces to solving

$$[(P_{j,k-1} + X_{j,k})^2 - P_{j,k-1} - X_{j,k}]_{\mathrm{prin,k}} = 0,$$

$$[(P_{j,k-1} + X_{j,k})^* - P_{j,k-1} - X_{j,k}]_{\mathrm{prin,k}} = 0,$$

which gives us a system of equations for the unknown $(X_{j,k})_{\mathrm{prin}}$. This system of equations reads

$$P^{(j)}(X_{j,k})_{\mathrm{prin}} + (X_{j,k})_{\mathrm{prin}} P^{(j)} - (X_{j,k})_{\mathrm{prin}} = R_{j,k}\,, \tag{3.5}$$

$$(X_{j,k})_{\mathrm{prin}}^* - (X_{j,k})_{\mathrm{prin}} = 0\,, \tag{3.6}$$

where

$$R_{j,k} := -[(P_{j,k-1})^2 - P_{j,k-1}]_{\mathrm{prin,k}}\,. \tag{3.7}$$

In fact, once one has determined $(X_{j,k})_{\mathrm{prin}}$ satisfying (3.5) and (3.6), it is always possible to choose lower order terms in the symbol of $X_{j,k}$ so as to satisfy (3.4).

[2] Recall that in this paper $L^2(M)$ denotes the space of m-columns of square integrable complex-valued half-densities.

Lemma 3.1. *The general solution of the system (3.5), (3.6) reads*

$$(X_{j,k})_{\mathrm{prin}} = -R_{j,k} + P^{(j)} R_{j,k} + R_{j,k} P^{(j)} + Y_{j,k} + Y^*_{j,k}, \tag{3.8}$$

where $Y_{j,k}$ is an arbitrary matrix-function positively homogeneous in momentum of degree $-k$ such that

$$Y_{j,k} = P^{(j)} Y_{j,k} (I - P^{(j)}). \tag{3.9}$$

Proof. From the inductive assumption

$$P_{j,k-1} = P^*_{j,k-1} \mod \Psi^{-\infty}$$

it follows that $R_{j,k}$ is Hermitian,

$$R_{j,k} = R^*_{j,k}. \tag{3.10}$$

Therefore, (3.8) satisfies (3.6).

Direct inspection of (3.5) tells us that the system (3.5), (3.6) has a solution only if

$$P^{(j)} R_{j,k} (I - P^{(j)}) = 0. \tag{3.11}$$

Of course, (3.11) and (3.10) imply

$$(I - P^{(j)}) R_{j,k} P^{(j)} = 0. \tag{3.12}$$

Let us show that (3.11) is satisfied. We have

$$\begin{aligned} P^{(j)} R_{j,k} (I - P^{(j)}) &= -P^{(j)} [(P_{j,k-1})^2 - P_{j,k-1}]_{\mathrm{prin,k}} (I - P^{(j)}) \\ &= -[P_{j,k-1}]_{\mathrm{prin,0}} [(P_{j,k-1})^2 - P_{j,k-1}]_{\mathrm{prin,k}} [\mathrm{Id} - P_{j,k-1})]_{\mathrm{prin,0}} \\ &= -[P_{j,k-1}((P_{j,k-1})^2 - P_{j,k-1})(\mathrm{Id} - P_{j,k-1})]_{\mathrm{prin,k}} \\ &= [((P_{j,k-1})^2 - P_{j,k-1})^2]_{\mathrm{prin,k}} \\ &= 0\,. \end{aligned}$$

In the last step we used the fact that since $(P_{j,k-1})^2 - P_{j,k-1} \in \Psi^{-k}$ by inductive assumption, then $((P_{j,k-1})^2 - P_{j,k-1})^2 \in \Psi^{-2k}$, hence its k-principal symbol is zero.

It remains only to substitute (3.8) and (3.9) into (3.5) with account of (3.11)–(3.12) and observe that $Y_{j,k} + Y^*_{j,k}$ is the general solution of the homogeneous system

$$P^{(j)} Y + Y P^{(j)} - Y = 0, \qquad Y = Y^*.$$

□

All in all, the above argument establishes the following result.

Theorem 3.2. *Given a rank 1 orthogonal projection $P^{(j)} \in C^\infty(T^*M; \mathrm{Mat}(m, \mathbb{C}))$, the associated orthogonal pseudodifferential projection $P_j \in \Psi^0$ in the sense of Definition 1.1 exists and is given by*

$$P_j \sim P_{j,0} + \sum_{k=1}^{+\infty} X_{j,k}, \tag{3.13}$$

where $P_{j,0} \in \Psi^0$ is an arbitrary operator satisfying $(P_{j,0})_{\mathrm{prin}} = P^{(j)}$, $P_{j,0} = P^*_{j,0}$, and the operators $X_{j,k} \in \Psi^{-k}$, $k = 1, 2, \dots$, are constructed iteratively from $P_{j,0}$ by means of Lemma 3.1. Here $\sim$ stands for asymptotic expansion in smoothness.

Formula (3.13) allows one to explicitly determine the symbol of P_j with arbitrarily high accuracy.

Note that for each P_j at every stage of the iterative process we have $m-1$ complex-valued scalar degrees of freedom, see (3.9). Because we have m different P_j's, at every step of the iterative process we have a total of $m(m-1)$ complex-valued scalar degrees of freedom.

3.2 Construction of a basis of pseudodifferential projections

In this subsection we establish the existence and the general structure of an orthonormal pseudodifferential basis, in the sense of Definition 1.2, thus proving Theorem 2.1.

Suppose we are given m orthonormal rank 1 projections $P^{(j)}(x,\xi)$ satisfying (1.7), not necessarily coinciding with the eigenprojections of A_{prin}. They determine, via Theorem 3.2, a corresponding family of pseudodifferential projections P_j, satisfying

$$\left(\sum_j P_j\right)_{\mathrm{prin}} = I. \tag{3.14}$$

The task at hand is to exploit the degrees of freedom left in the symbols of the P_j's to satisfy conditions (1.4) and (1.5).

Firstly, with the help of (1.7), let us rewrite formulae (3.8) and (3.9) in the equivalent form

$$(X_{j,k})_{\mathrm{prin}} = -R_{j,k} + P^{(j)}R_{j,k} + R_{j,k}P^{(j)} + \sum_{l\ne j}[Y_{j,l,k} + Y^*_{j,l,k}], \tag{3.15}$$

where $Y_{j,l,k}$ is an arbitrary matrix-function positively homogeneous in momentum of degree $-k$ such that

$$Y_{j,l,k} = P^{(j)}Y_{j,l,k}P^{(l)}. \tag{3.16}$$

Subsection 3.2 gives us, for each j, a sequence of operators $P_{j,k} \in \Psi^0$, $k = 1,2,\ldots$, satisfying (3.1)–(3.3) of the form

$$P_{j,k} = P_{j,k-1} + X_{j,k},$$

where the principal symbol of $X_{j,k} \in \Psi^{-k}$, $X_{j,k} = X^*_{j,k} \mod \Psi^{-\infty}$, is given by (3.15). Satisfying (1.4) reduces to determining $Y_{j,l,k}$ such that

$$P_{n,k}P_{j,k} = 0 \mod \Psi^{-k-1} \qquad \forall n \ne j. \tag{3.17}$$

To this end, let $\widetilde{X}_{j,k} \in \Psi^{-k}$, $\widetilde{X}_{j,k} = \widetilde{X}^*_{j,k}$, be such that

$$(\widetilde{X}_{j,k})_{\mathrm{prin}} = -R_{j,k} + P^{(j)}R_{j,k} + R_{j,k}P^{(j)}. \tag{3.18}$$

Then satisfying (3.17) reduces to solving

$$\sum_{l\ne j} P^{(n)}(Y_{j,l,k} + Y^*_{j,l,k}) + \sum_{l\ne n}(Y_{n,l,k} + Y^*_{n,l,k})P^{(j)} = R_{n,j,k} \tag{3.19}$$

for all $j \ne n$, where $Y_{j,l,k}$ is of the form (3.16) and

$$R_{n,j,k} := -[(P_{n,k-1} + \widetilde{X}_{n,k})(P_{j,k-1} + \widetilde{X}_{j,k})]_{\mathrm{prin,k}}\,. \tag{3.20}$$

The system (3.19) amounts to a total of $m(m-1)$ algebraic equations.

Lemma 3.3. *The general solution of the system (3.19) reads*

$$Y_{j,l,k} = \frac{1}{2} R_{j,l,k} + Z_{j,l,k}\,, \tag{3.21}$$

where $Z_{j,l,k}$ are arbitrary matrix-functions positively homogeneous in momentum of degree $-k$ such that

$$Z_{j,l,k} = P^{(j)} Z_{j,l,k} P^{(l)}, \tag{3.22}$$

$$Z^*_{j,l,k} = -Z_{l,j,k}\,. \tag{3.23}$$

Proof. Formula (3.20) implies

$$R^*_{j,l,k} = R_{l,j,k}.$$

Direct inspection of (3.19) tells us that a necessary solvability condition reads

$$R_{j,l,k} = P^{(j)} R_{j,l,k} P^{(l)}. \tag{3.24}$$

Let us show that (3.24) is satisfied. We have

$$\begin{aligned}
P^{(j)} R_{j,l,k} P^{(l)} &= -P^{(j)} [(P_{j,k-1} + \widetilde{X}_{j,k})(P_{l,k-1} + \widetilde{X}_{l,k})]_{\mathrm{prin,k}} P^{(l)} \\
&= -[P_{j,k-1} + \widetilde{X}_{j,k}]_{\mathrm{prin,0}}\, [(P_{j,k-1} + \widetilde{X}_{j,k})(P_{l,k-1} + \widetilde{X}_{l,k})]_{\mathrm{prin,k}}\, [P_{l,k-1} + \widetilde{X}_{l,k}]_{\mathrm{prin,0}} \\
&= -[(P_{j,k-1} + \widetilde{X}_{j,k})^2 (P_{l,k-1} + \widetilde{X}_{l,k})^2]_{\mathrm{prin,k}} \\
&= -[(P_{j,k-1} + \widetilde{X}_{j,k})(P_{l,k-1} + \widetilde{X}_{l,k})]_{\mathrm{prin,k}} \\
&= R_{j,l,k}\,.
\end{aligned}$$

In the above argument we used the fact that

$$(P_{j,k-1} + \widetilde{X}_{j,k})^2 = P_{j,k-1} + \widetilde{X}_{j,k} \mod \Psi^{-k-1}$$

for all j.

Of course, (3.24) implies

$$R_{j,l,k} = P^{(j)} R_{j,l,k} = R_{j,l,k} P^{(l)}. \tag{3.25}$$

It remains only to substitute (3.21) into (3.19) with account of (3.25) and observe that $Z_{j,l,k}$ is the general solution of the homogeneous system

$$\sum_{l \neq j} P^{(n)} (Z_{j,l,k} + Z^*_{j,l,k}) + \sum_{l \neq n} (Z_{n,l,k} + Z^*_{n,l,k}) P^{(j)} = 0. \tag{3.26}$$

□

As it turns out, condition (1.5) is automatically satisfied.

Theorem 3.4. *Let $\{P_j\}$ be a family of m pseudodifferential operators of order zero with rank 1 principal symbols, satisfying (1.2), (1.4) and*

$$\sum_j (P_j)_{\mathrm{prin}} = I. \tag{3.27}$$

Then (1.5) is also satisfied.

Proof. Let us define
$$\widetilde{\mathrm{Id}} := \sum_j P_j$$
and let us put
$$B := \widetilde{\mathrm{Id}} - \mathrm{Id}. \tag{3.28}$$
The task at hand is to show that $B \in \Psi^{-\infty}$. Arguing by contradiction, suppose there exists a natural number k such that
$$B \in \Psi^{-k}$$
but
$$B \notin \Psi^{-k-1}.$$

The principal symbol of the operator B is positively homogeneous in momentum of degree $-k$ and has the property
$$B_{\mathrm{prin}}(x,\xi) \neq 0 \quad \text{for some} \quad (x,\xi) \in T^*M \setminus \{0\}. \tag{3.29}$$

On account of (1.2) and (1.4), formula (3.28) implies
$$\widetilde{\mathrm{Id}}\, B \in \Psi^{-\infty}. \tag{3.30}$$

We have
$$(\widetilde{\mathrm{Id}}\, B)_{\mathrm{prin}} = \widetilde{\mathrm{Id}}_{\mathrm{prin}} B_{\mathrm{prin}}\,. \tag{3.31}$$
But by (3.27)
$$\widetilde{\mathrm{Id}}_{\mathrm{prin}} = I, \tag{3.32}$$
so formulae (3.31) and (3.32) imply
$$(\widetilde{\mathrm{Id}}\, B)_{\mathrm{prin}} = B_{\mathrm{prin}}\,. \tag{3.33}$$

Formulae (3.29) and (3.33) imply
$$(\widetilde{\mathrm{Id}}\, B)_{\mathrm{prin,k}}(x,\xi) \neq 0 \quad \text{for some} \quad (x,\xi) \in T^*M \setminus \{0\},$$
which, in turn, implies
$$\widetilde{\mathrm{Id}}\, B \notin \Psi^{-k-1}.$$
The latter contradicts (3.30). □

All in all, the above arguments establish the following result.

Theorem 3.5. *Given m orthonormal rank 1 projections*
$$P^{(j)} \in C^\infty(T^*M; \mathrm{Mat}(m,\mathbb{C})),$$
there exists an orthonormal pseudodifferential basis $\{P_j\}$ in the sense of Definition 1.2 satisfying $(P_j)_{\mathrm{prin}} = P^{(j)}$. Furthermore, we have
$$P_j \sim P_{j,0} + \sum_{k=1}^{+\infty} X_{j,k}, \tag{3.34}$$

where $P_{j,0} \in \Psi^0$ is an arbitrary operator satisfying $(P_{j,0})_{\mathrm{prin}} = P^{(j)}$, $P_{j,0} = P_{j,0}^*$, and the operators $X_{j,k} \in \Psi^{-k}$, $X_{j,k} = X_{j,k}^* \mod \Psi^{-\infty}$, $k = 1, 2, \dots$, are constructed iteratively from $P_{j,0}$ in accordance with (3.15), (3.7), Lemma 3.3 and (3.20). Here $\sim$ stands for asymptotic expansion in smoothness.

Note that examination of formulae (3.22) and (3.23) shows that at every stage of the iterative process we have a total of $\frac{m(m-1)}{2}$ complex-valued scalar degrees of freedom left in our construction.

3.3 Commutation with an elliptic operator

In this subsection we will exploit the remaining degrees of freedom left in the symbols of our pseudodifferential basis to impose that individual projections commute with the operator A, in accordance with (1.8). We shall then show that this uniquely determines our pseudodifferential projections modulo $\Psi^{-\infty}$, thus completing the proof of Theorem 2.2.

Let $A \in \Psi^s$ be as in Section 1. Suppose we are given an orthonormal pseudodifferential basis constructed in accordance with subsection 3.2, whose principal symbols are the eigenprojections of A_{prin}.

Imposing condition (1.8) is equivalent to requiring, for every j,

$$[A, P_{j,k}] = 0 \mod \Psi^{-k-1} \tag{3.35}$$

recursively, for $k = 0, 1, \dots$.

For $k = 0$, condition (3.35) is automatically satisfied. In fact,

$$[A, P_{j,0}] = 0 \mod \Psi^{-1} \quad \Longleftrightarrow \quad [A, P_{j,0}]_{\mathrm{prin}} = 0$$

and

$$\begin{aligned} [A, P_{j,0}]_{\mathrm{prin}} &= A_{\mathrm{prin}} P^{(j)} - P^{(j)} A_{\mathrm{prin}} \\ &= \sum_l h^{(l)} [P^{(l)}, P^{(j)}] = 0. \end{aligned}$$

Let $\widetilde{R}_{j,k} \in \Psi^{-k}$, $\widetilde{R}_{j,k} = \widetilde{R}_{j,k}^* \mod \Psi^{-\infty}$, be such that

$$[\widetilde{R}_{j,k}]_{\mathrm{prin}} = [\widetilde{X}_{j,k}]_{\mathrm{prin}} + \frac{1}{2} \sum_{l \neq j} (R_{j,l,k} + R_{l,j,k}), \tag{3.36}$$

see (3.18), (3.20) and (3.21), and define

$$M_{j,k} := [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}}. \tag{3.37}$$

Then, for $k \geq 1$, satisfying (3.35) reduces to determining $Z_{j,l,k}$ such that

$$\sum_{l \neq j} (h^{(j)} - h^{(l)}) [Z_{j,l,k} + Z_{l,j,k}] = -M_{j,k}, \tag{3.38}$$

$$Z_{j,l,k} = P^{(j)} Z_{j,l,k} P^{(l)}, \qquad Z_{j,l,k} = -Z_{l,j,k}^*. \tag{3.39}$$

Lemma 3.6. *A solution to* (3.38), (3.39) *is given by*

$$Z_{j,l,k} = -\frac{P^{(j)} M_{j,k} P^{(l)}}{h^{(j)} - h^{(l)}}. \tag{3.40}$$

Proof. Formula (3.40) clearly satisfies (3.39), because $M_{j,k}$ is skew-Hermitian.

It is easy to see that necessary solvability conditions are

$$P^{(j)} M_{j,k} P^{(j)} = 0 \tag{3.41}$$

and

$$P^{(r)} M_{j,k} P^{(n)} = 0 \quad \text{for} \quad r, n \neq j. \tag{3.42}$$

Let us show that (3.41) and (3.42) are satisfied. We have

$$\begin{aligned} P^{(j)} M_{j,k} P^{(j)} &= P^{(j)} [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}} P^{(j)} \\ &= [P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin},0} [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}} [P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin},0} \\ &= 0. \end{aligned}$$

In the above argument we used the fact that

$$(P_{j,k-1} + \widetilde{R}_{j,k})^2 = P_{j,k-1} + \widetilde{R}_{j,k} \mod \Psi^{-k-1}, \tag{3.43}$$

as established in subsection 3.1. Similarly, we have

$$\begin{aligned} P^{(r)} M_{j,k} P^{(n)} &= P^{(r)} [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}} P^{(r)} \\ &= [P_{r,k-1} + \widetilde{R}_{r,k}]_{\mathrm{prin},0} [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}} [P_{n,k-1} + \widetilde{R}_{n,k}]_{\mathrm{prin},0} \\ &= 0. \end{aligned}$$

In the above argument we used the fact that

$$(P_{j,k-1} + \widetilde{R}_{j,k})(P_{l,k-1} + \widetilde{R}_{l,k}) = (P_{l,k-1} + \widetilde{R}_{l,k})(P_{j,k-1} + \widetilde{R}_{j,k}) = 0 \mod \Psi^{-k-1}, \tag{3.44}$$

as established in subsection 3.2.

Formulae (3.41) and (3.42) imply

$$M_{j,k} = \sum_{l \neq j} (P^{(j)} M_{j,k} P^{(l)} + P^{(l)} M_{j,k} P^{(j)}). \tag{3.45}$$

Furthermore, the matrix-functions $M_{j,k}$ satisfy the identity

$$P^{(l)} M_{l,k} P^{(j)} = -P^{(l)} M_{j,k} P^{(j)} \quad \text{for} \quad j \neq l. \tag{3.46}$$

In fact, for $j \neq l$ we have

$$\begin{aligned} P^{(l)} M_{l,k} P^{(j)} &= P^{(l)} [A, P_{l,k-1} + \widetilde{R}_{l,k}]_{\mathrm{prin,k-s}} P^{(j)} \\ &= [P_{l,k-1} + \widetilde{R}_{l,k}]_{\mathrm{prin},0} [A(P_{l,k-1} + \widetilde{R}_{l,k}) - (P_{l,k-1} + \widetilde{R}_{l,k})A]_{\mathrm{prin,k-s}} [P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin},0} \\ &= -[(P_{l,k-1} + \widetilde{R}_{l,k}) A (P_{j,k-1} + \widetilde{R}_{j,k}])]_{\mathrm{prin,k-s}} \\ &= -[P_{l,k-1} + \widetilde{R}_{l,k}]_{\mathrm{prin},0} [A(P_{j,k-1} + \widetilde{R}_{j,k}) - (P_{j,k-1} + \widetilde{R}_{j,k})A]_{\mathrm{prin,k-s}} [P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin},0} \\ &= -P^{(l)} [A, P_{j,k-1} + \widetilde{R}_{j,k}]_{\mathrm{prin,k-s}} P^{(j)} \\ &= -P^{(l)} M_{j,k} P^{(j)}. \end{aligned}$$

In the above argument we used (3.43) and (3.44).

It remains only to substitute (3.40) into (3.38) and use (3.45)–(3.46). □

Proposition 3.7. *The solution found in Lemma 3.6 is the unique solution to* (3.38)–(3.39).

Proof. It suffices to show that the homogeneous system does not admit nontrivial solutions.

Suppose $\widetilde{Z}_{j,l}$ is a solution of the homogeneous system

$$\sum_{l\neq j}(h^{(j)}-h^{(l)})[\widetilde{Z}_{j,l}+\widetilde{Z}_{l,j}]=0. \tag{3.47}$$

Then, multiplying (3.47) by $P^{(j)}$ on the left and by $P^{(n)}$, $n\neq j$, on the right, we obtain

$$(h^{(j)}-h^{(n)})\widetilde{Z}_{j,n}=0.$$

□

Combining Theorem 3.5, Lemma 3.6 and Proposition 3.7 we obtain Theorem 2.2.

3.4 The algorithm

Let us summarise the results from subsections 3.1–3.3 in the form of a concise algorithm for the construction of the full symbol of pseudodifferential projections.

Step 1. Given the m eigenprojections $P^{(j)}(x,\xi)$ of A_prin, choose m arbitrary pseudodifferential operators $P_{j,0}\in\Psi^0$ satisfying

(i) $(P_{j,0})_\mathrm{prin}=P^{(j)}$,

(ii) $P_{j,0}=P_{j,0}^* \mod \Psi^{-\infty}$.

Step 2. For $k=1,2,\dots$ define

$$P_{j,k}:=P_{j,0}+\sum_{n=1}^{k}X_{j,n},\qquad X_{j,n}\in\Psi^{-n}.$$

Assuming we have determined the pseudodifferential operator $P_{j,k-1}$, compute, one after the other, the following quantities:

(a) $R_{j,k}=-((P_{j,k-1})^2-P_{j,k-1})_\mathrm{prin,k}$,

(b) $S_{j,k}=-R_{j,k}+P^{(j)}R_{j,k}+R_{j,k}P^{(j)}$,

(c) $V_{j,l,k}=-\frac{1}{2}((P_{j,k-1}P_{l,k-1})_\mathrm{prin,k}+P^{(j)}S_{l,k}+S_{j,k}P^{(l)})$,

(d) $Z_{j,l,k}=(h^{(l)}-h^{(j)})^{-1}P^{(j)}\left([A\,,P_{j,k-1}]_\mathrm{prin,k-s}+[A_\mathrm{prin}\,,S_{j,k}+\sum_{n\neq j}(V_{j,n,k}+V_{n,j,k})]\right)P^{(l)}$, for $l \neq j$.

Step 3. Choose a pseudodifferential operator $X_{j,k} \in \Psi^{-k}$ satisfying

(i) $(X_{j,k})_{\mathrm{prin}} = S_{j,k} + \sum_{l\neq j}(V_{j,l,k} + V_{l,j,k}) + \sum_{l\neq j}(Z_{j,l,k} - Z_{l,j,k})$,

(ii) $X_{j,k} = X_{j,k}^* \mod \Psi^{-\infty}$.

Step 4. Put

$$P_j \sim P_{j,0} + \sum_{n=1}^{+\infty} X_{j,n}\,.$$

3.5 Proof of Theorem 2.2

Proof. In subsections 3.1–3.3 we established existence of our pseudodifferential projections. It remains to prove that they are unique.

Suppose there exist two sets of pseudodifferential projections, P_j and P_j', satisfying (1.2)–(1.9) whose difference is not in $\Psi^{-\infty}$. Then there exists a natural number k such that

$$P_j - P_j' \in \Psi^{-k} \quad \forall j,$$

but

$$P_j - P_j' \notin \Psi^{-k-1} \quad \text{for some} \quad j.$$

Consider the pseudodifferential operators

$$B_j := P_j - P_j' \in \Psi^{-k}.$$

The fact that the operators $P_j = B_j + P_j'$ satisfy (1.2)–(1.9) yields the following system of equations for $(B_j)_{\mathrm{prin}}$:

$$P^{(j)}(B_j)_{\mathrm{prin}} + (B_j)_{\mathrm{prin}}P^{(j)} - (B_j)_{\mathrm{prin}} = 0, \tag{3.48}$$

$$(B_j)_{\mathrm{prin}}^* = (B_j)_{\mathrm{prin}}, \tag{3.49}$$

$$P^{(l)}(B_j)_{\mathrm{prin}} + (B_l)_{\mathrm{prin}}P^{(j)} = 0, \qquad l \neq j, \tag{3.50}$$

$$A_{\mathrm{prin}}(B_j)_{\mathrm{prin}} - (B_j)_{\mathrm{prin}}A_{\mathrm{prin}} = 0. \tag{3.51}$$

The matrix-function $(B_j)_{\mathrm{prin}}$ can be uniquely represented in the form

$$(B_j)_{\mathrm{prin}} = \sum_{l,n} B_{j,l,n}, \tag{3.52}$$

where the matrix-functions $B_{j,l,n}$ satisfy

$$B_{j,l,n} = P^{(l)} B_{j,l,n} P^{(n)}. \tag{3.53}$$

Substituting (3.52) into (3.48) we immediately get

$$B_{j,j,j} = 0,$$

$$B_{j,l,n} = 0 \quad \text{for} \quad l \neq j, \ n \neq j,$$

so that (3.52) can be equivalently recast as

$$(B_j)_{\mathrm{prin}} = \sum_{l \neq j} (B_{j,j,l} + B_{j,l,j}). \tag{3.54}$$

Substituting (3.54) into (3.51) and taking into account (3.53) we get

$$\sum_{l \neq j} (h^{(j)} - h^{(l)})(B_{j,j,l} - B_{j,l,j}) = 0.$$

But the latter only admits the trivial solution. □

4 Commutation with an elliptic operator: revisited

4.1 An abstract theorem on pseudodifferential projections

The argument presented in subsection 3.5 shows that conditions (1.9), (1.2) and (1.8) alone force uniqueness of our orthonormal pseudodifferential basis commuting with A. This observation motivates us to formulate the following abstract result.

Theorem 4.1. *Suppose that we are given m pseudodifferential operators $P_j \in \Psi^0$ satisfying conditions*

$$P_j^2 = P_j \mod \Psi^{-\infty} \tag{4.1}$$

and (1.8), (1.9)*. Then these operators satisfy*

$$P_j^* = P_j \mod \Psi^{-\infty} \tag{4.2}$$

and (1.4), (1.5).

Proof. To begin with, let us show that we have (4.2). Suppose there exists a j such that, for some natural k, $P_j - P_j^* \in \Psi^{-k}$ but

$$P_j - P_j^* \notin \Psi^{-k-1}. \tag{4.3}$$

Put

$$B := P_j - P_j^*. \tag{4.4}$$

Then conditions (4.1), (1.8) and (1.9) give us the following equations for B_{prin}:

$$B_{\mathrm{prin}} P^{(j)} + P^{(j)} B_{\mathrm{prin}} - B_{\mathrm{prin}} = 0, \tag{4.5}$$

$$A_{\mathrm{prin}} B_{\mathrm{prin}} - B_{\mathrm{prin}} A_{\mathrm{prin}} = 0. \tag{4.6}$$

Formula (4.5) implies

$$P^{(j)}B_{\mathrm{prin}}P^{(j)}=0,$$

so that we have

$$B_{\mathrm{prin}}=\sum_{l\neq j}(P^{(l)}B_{\mathrm{prin}}P^{(j)}+P^{(j)}B_{\mathrm{prin}}P^{(l)})\,. \tag{4.7}$$

Substituting (4.7) into (4.6) we get

$$\sum_{l\neq j}(h^{(j)}-h^{(l)})(P^{(j)}B_{\mathrm{prin}}P^{(l)}-P^{(l)}B_{\mathrm{prin}}P^{(j)})=0,$$

which implies

$$B_{\mathrm{prin}}=0.$$

But the latter contradicts (4.3).

Next, let us show that (1.4) holds. Arguing by contradiction, suppose there exist j and l, $j\neq l$, such that P_j and P_l satisfy the assumptions of the theorem and, for some natural k, $P_jP_l\in\Psi^{-k}$ but

$$P_jP_l\notin\Psi^{-k-1}. \tag{4.8}$$

Put $C:=P_jP_l$. Then conditions (4.1), (1.8) and (1.9) give us the following constraints for C_{prin}:

$$P^{(j)}C_{\mathrm{prin}}-C_{\mathrm{prin}}=0, \tag{4.9}$$

$$C_{\mathrm{prin}}P^{(l)}-C_{\mathrm{prin}}=0, \tag{4.10}$$

$$A_{\mathrm{prin}}C_{\mathrm{prin}}-C_{\mathrm{prin}}A_{\mathrm{prin}}=0. \tag{4.11}$$

Formulae (4.9) and (4.10) imply

$$C_{\mathrm{prin}}=P^{(j)}C_{\mathrm{prin}}P^{(l)}. \tag{4.12}$$

Substituting (4.12) into (4.11) we obtain

$$(h^{(j)}-h^{(l)})C_{\mathrm{prin}}=0$$

which, in turn, yields

$$C_{\mathrm{prin}}=0.$$

The latter contradicts (4.8).

The fact that we have (1.5) now follows from Theorem 3.4. □

Note that an alternative proof of Theorem 4.1 can be obtained by arguing as in subsection 3.5 and using the constructive proof of the existence of projections P_j provided in subsections 3.1–3.3. However, we decided to give here an abstract self-contained proof which does not rely on the explicit construction of the P_j's.

4.2 Developing a simplified algorithm

Theorem 4.1 opens the way to the formulation of a new, shorter version of our algorithm.

In this subsection we will show directly what was argued at the end of subsection 3.5, namely that a family of orthogonal pseudodifferential projections in the sense of Definition 1.1 satisfying the commutation relation (1.8) is automatically an orthonormal pseudodifferential basis in the sense of Definition 1.2.

Suppose that, in accordance with subsection 3.1, we have constructed m orthogonal pseudodifferential projections of the form

$$P_j \sim P_{j,0} + \sum_{k=1}^{+\infty} X_{j,k} \tag{4.13}$$

where $(X_{j,k})_\mathrm{prin}$ is given by

$$(X_{j,k})_\mathrm{prin} = -R_{j,k} + P^{(j)} R_{j,k} + R_{j,k} P^{(j)} + \sum_{l\ne j} (Y_{j,l,k} + Y^*_{j,l,k}), \tag{4.14}$$

$$Y_{j,l,k} = P^{(j)} Y_{j,l,k} P^{(l)}, \qquad Y^*_{j,l,k} = P^{(l)} Y^*_{j,l,k} P^{(j)}, \tag{4.15}$$

cf. (3.8) and (3.9).

Choose pseudodifferential operators $\widetilde X_{j,k} \in \Psi^{-k}$, $\widetilde X_{j,k} = \widetilde X^*_{j,k} \mod \Psi^{-\infty}$, such that

$$(\widetilde X_{j,k})_\mathrm{prin} = -R_{j,k} + P^{(j)} R_{j,k} + R_{j,k} P^{(j)} \tag{4.16}$$

and put

$$T_{j,k} := -[A, P_{j,k-1} + \widetilde X_{j,k}]_{\mathrm{prin},k-s}. \tag{4.17}$$

Then, in view of (4.14) and (4.15), satisfying (1.8) reduces to solving the system of equations

$$\sum_{l\ne j} (h^{(j)} - h^{(l)})(Y_{j,l,k} - Y^*_{j,l,k}) = T_{j,k} \tag{4.18}$$

for $Y_{j,l,k}$.

Lemma 4.2. *The unique solution to* (4.18) *is given by*

$$Y_{j,l,k} = \frac{P^{(j)} T_{j,k} P^{(l)}}{h^{(j)} - h^{(l)}}. \tag{4.19}$$

Proof. It is easy to see that necessary solvability conditions are

$$P^{(j)} T_{j,k} P^{(j)} = 0, \tag{4.20}$$

$$T^*_{j,k} = -T_{j,k}. \tag{4.21}$$

Condition (4.21) is clearly satisfied. Let us check that the same is true for (4.20). We have

$$\begin{aligned} P^{(j)} T_{j,k} P^{(j)} &= -P^{(j)} [A, P_{j,k-1} + \widetilde X_{j,k}]_{\mathrm{prin},k-s} P^{(j)} \\ &= -(P_{j,k-1} + \widetilde X_{j,k})_{\mathrm{prin},0} [A, P_{j,k-1} + \widetilde X_{j,k}]_{\mathrm{prin},k-s} (P_{j,k-1} + \widetilde X_{j,k})_{\mathrm{prin},0} \\ &= 0. \end{aligned}$$

In the last step of the above calculation we used the fact that

$$(P_{j,k-1}+\widetilde{X}_{j,k})^2=P_{j,k-1}+\widetilde{X}_{j,k}\mod\Psi^{-k-1},\tag{4.22}$$

which was established in subsection 3.1.

Formula (4.20) implies

$$T_{j,k}=\sum_{l\ne j}(P^{(j)}T_{j,k}P^{(l)}+P^{(l)}T_{j,k}P^{(j)}).\tag{4.23}$$

By substituting (4.19) into (4.18) and taking into account (4.21) and (4.23) one shows that (4.19) is a solution.

To complete the proof it remains only to observe that the homogeneous system

$$\sum_{l\ne j}(h^{(j)}-h^{(l)})(Y_{j,l,k}-Y^*_{j,l,k})=0$$

complemented with (4.15) admits only the trivial solution. □

Lemma 4.2 completely determines $(X_{j,k})_{\mathrm{prin}}$. Let us show that for $j\ne l$ we automatically have

$$P_{j,k}P_{l,k}\in\Psi^{-k-1}.\tag{4.24}$$

In view of our recursive construction, condition (4.24) can be equivalently rewritten as

$$\frac{P^{(j)}(T_{l,k}+T_{j,k})P^{(l)}}{h^{(j)}-h^{(l)}}=-((P_{j,k-1}+\widetilde{X}_{j,k})(P_{l,k-1}+\widetilde{X}_{l,k}))_{\mathrm{prin,k}}.\tag{4.25}$$

In view of (4.22), we have

$$\begin{aligned}P^{(j)}(T_{l,k}+T_{j,k})P^{(l)}=&-(P_{j,k-1}+\widetilde{X}_{j,k})_{\mathrm{prin},0}[A,P_{l,k-1}+\widetilde{X}_{l,k}]_{\mathrm{prin},k-s}(P_{l,k-1}+\widetilde{X}_{l,k})_{\mathrm{prin},0}\\&-(P_{j,k-1}+\widetilde{X}_{j,k})_{\mathrm{prin},0}[A,P_{j,k-1}+\widetilde{X}_{j,k}]_{\mathrm{prin},k-s}(P_{l,k-1}+\widetilde{X}_{l,k})_{\mathrm{prin},0}\\=&-P^{(j)}[A,(P_{j,k-1}+\widetilde{X}_{j,k})(P_{l,k-1}+\widetilde{X}_{l,k})]_{\mathrm{prin,k-s}}P^{(l)}\\=&-P^{(j)}[A_{\mathrm{prin}},((P_{j,k-1}+\widetilde{X}_{j,k})(P_{l,k-1}+\widetilde{X}_{l,k}))_{\mathrm{prin,k}}]P^{(l)}\\=&-(h^{(j)}-h^{(l)})((P_{j,k-1}+\widetilde{X}_{j,k})(P_{l,k-1}+\widetilde{X}_{l,k}))_{\mathrm{prin,k}}.\end{aligned}\tag{4.26}$$

Substituting (4.26) into (4.25), we see that (4.25) is satisfied.

The fact that the operators (4.13), (4.14), (3.40) satisfy (1.5) now follows from Theorem 3.4.

4.3 The simplified algorithm

The construction from the previous subsection can be summarised as follows.

Step 1. Given the m eigenprojections $P^{(j)}$ of A_{prin}, choose m arbitrary pseudodifferential operators $P_{j,0}\in\Psi^0$ satisfying $(P_{j,0})_{\mathrm{prin}}=P^{(j)}$.

Step 2. For $k = 1, 2, \dots$ define

$$P_{j,k} := P_{j,0} + \sum_{n=1}^{k} X_{j,n}, \qquad X_{j,n} \in \Psi^{-n}. \tag{4.27}$$

Assuming we have determined the pseudodifferential operator $P_{j,k-1}$, compute, one after the other, the following quantities:

(a) $R_{j,k} := -((P_{j,k-1})^2 - P_{j,k-1})_{\mathrm{prin,k}}$,

(b) $S_{j,k} := -R_{j,k} + P^{(j)} R_{j,k} + R_{j,k} P^{(j)}$,

(c) $T_{j,k} := [P_{j,k-1}, A]_{\mathrm{prin},k-s} + [S_{j,k}, A_{\mathrm{prin}}]$.

Step 3. Choose a pseudodifferential operator $X_{j,k} \in \Psi^{-k}$ satisfying

$$(X_{j,k})_{\mathrm{prin}} = S_{j,k} + \sum_{l \neq j} \frac{P^{(j)} T_{j,k} P^{(l)} - P^{(l)} T_{j,k} P^{(j)}}{h^{(j)} - h^{(l)}}. \tag{4.28}$$

Step 4. Put

$$P_j \sim P_{j,0} + \sum_{n=1}^{+\infty} X_{j,n} \, . \tag{4.29}$$

Remark 4.3. We now have at our disposal two versions of the construction algorithm: the one just given above and that presented in subsection 3.4. Each has its own advantages.

The one given here is more concise and requires relatively simple calculations to obtain the final formulae. We will have an opportunity to appreciate this in Section 5, when we will compute the subprincipal symbol of pseudodifferential projections.

Our original algorithm given in subsection 3.4, despite being longer and entailing more complicated calculations, provides more refined information. On the one hand, it allows one to single out

(i) contributions to the symbols making P_j an orthogonal projection (the $S_{j,k}$'s),

(ii) those responsible for the orthonormality condition (the $V_{j,l,k}$'s) and

(iii) those ensuring the commutation with A (the $Z_{j,l,k}$'s).

On the other hand, it allows one to keep track of how many degrees of freedom are left in the symbol at each stage of the construction algorithm, at the same time shedding some light on how the available degrees of freedom are used up. This makes the first version of the algorithm more suitable if one is only interested in constructing a single pseudodifferential projection or a pseudodifferential basis, without necessarily relating the P_j's to an elliptic operator A.

5 Subprincipal symbol of pseudodifferential projections

In this section we will carry out the first iteration of the above algorithm explicitly, to obtain a closed formula for the subprincipal symbol of our projections and prove Theorem 2.3.

Let us choose the initial operators $P_{j,0} \in \Psi^0$ satisfying the additional property

$$(P_{j,0})_{\mathrm{sub}} = 0. \tag{5.1}$$

Then (5.1) and (4.29) imply

$$(P_j)_{\mathrm{sub}} = (X_{j,1})_{\mathrm{prin}}.$$

The task at hand reduces to computing $(X_{j,1})_{\mathrm{prin}}$.

Recall that the formula for the subprincipal of a composition of pseudodifferential operators reads [21, Eqn. (1.4)]

$$[BC]_{\mathrm{sub}} = B_{\mathrm{prin}} C_{\mathrm{sub}} + B_{\mathrm{sub}} C_{\mathrm{prin}} + \frac{i}{2}\{B_{\mathrm{prin}}, C_{\mathrm{prin}}\}. \tag{5.2}$$

Note that [21] adopts the opposite sign convention for the Poisson bracket.

Using (5.1) and (5.2), we obtain

$$R_{j,1} = -\frac{i}{2}\{P^{(j)}, P^{(j)}\}. \tag{5.3}$$

Formula (5.3) gives us

$$S_{j,1} = \frac{i}{2}(\{P^{(j)}, P^{(j)}\} - P^{(j)}\{P^{(j)}, P^{(j)}\} - \{P^{(j)}, P^{(j)}\}P^{(j)}). \tag{5.4}$$

In view of (5.2) and with account of (5.1), we have

$$\begin{aligned}
[P_{j,0}, A]_{1-s} &= (P_{j,0}A - AP_{j,0})_{\mathrm{sub}} \\
&= P^{(j)} A_{\mathrm{sub}} + \frac{i}{2}\{P^{(j)}, A_{\mathrm{prin}}\} - A_{\mathrm{sub}} P^{(j)} - \frac{i}{2}\{A_{\mathrm{prin}}, P^{(j)}\} \\
&= [P^{(j)}, A_{\mathrm{sub}}] + \frac{i}{2}(\{P^{(j)}, A_{\mathrm{prin}}\} - \{A_{\mathrm{prin}}, P^{(j)}\}).
\end{aligned} \tag{5.5}$$

Observe that

$$P^{(j)}\{P^{(j)}, P^{(j)}\} = \{P^{(j)}, P^{(j)}\}P^{(j)} = P^{(j)}\{P^{(j)}, P^{(j)}\}P^{(j)}.$$

Therefore, by means of (5.4), we obtain

$$\begin{aligned}
[S_{j,1}, A_{\mathrm{prin}}] &= \frac{i}{2}[\{P^{(j)}, P^{(j)}\} - P^{(j)}\{P^{(j)}, P^{(j)}\} - \{P^{(j)}, P^{(j)}\}P^{(j)}, A_{\mathrm{prin}}] \\
&= \frac{i}{2}(\{P^{(j)}, P^{(j)}\}A_{\mathrm{prin}} - A_{\mathrm{prin}}\{P^{(j)}, P^{(j)}\}).
\end{aligned} \tag{5.6}$$

By combining (5.5) and (5.6) we get

$$T_{j,1} = \frac{i}{2}(\{P^{(j)}, A_{\mathrm{prin}}\} - \{A_{\mathrm{prin}}, P^{(j)}\} + [\{P^{(j)}, P^{(j)}\}, A_{\mathrm{prin}}]) + [P^{(j)}, A_{\mathrm{sub}}].$$

We can then compute

$$\begin{aligned} P^{(j)}T_{j,1}P^{(l)} &= \frac{i}{2}(P^{(j)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(l)} - P^{(j)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(l)}) \\ &+ \frac{i}{2}(h^{(l)}-h^{(j)})P^{(j)}\{P^{(j)},P^{(j)}\}P^{(l)} + P^{(j)}A_{\mathrm{sub}}P^{(l)} \\ &= \frac{i}{2}(P^{(j)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(l)} - P^{(j)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(l)}) + P^{(j)}A_{\mathrm{sub}}P^{(l)}. \end{aligned} \tag{5.7}$$

In the above calculation we used that

$$P^{(j)}\{P^{(j)},P^{(j)}\}P^{(l)} = 0 \quad \text{for} \quad j \ne l.$$

Similarly, we have

$$\begin{aligned} P^{(l)}T_{j,1}P^{(j)} &= \frac{i}{2}(P^{(l)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(j)} - P^{(l)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(l)}) \\ &+ \frac{i}{2}(h^{(j)}-h^{(l)})P^{(l)}\{P^{(j)},P^{(j)}\}P^{(j)} - P^{(l)}A_{\mathrm{sub}}P^{(j)} \\ &= \frac{i}{2}(P^{(l)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(j)} - P^{(l)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(j)}) - P^{(l)}A_{\mathrm{sub}}P^{(j)}\,. \end{aligned} \tag{5.8}$$

Substituting (5.4), (5.7) and (5.8) into (4.28) we arrive at

$$\begin{aligned} (X_{j,1})_{\mathrm{prin}} &= \frac{i}{2}\{P^{(j)},P^{(j)}\} - i\,P^{(j)}\{P^{(j)},P^{(j)}\}P^{(j)} \\ &+ \frac{i}{2}\sum_{l\ne j}\frac{P^{(j)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(l)} - P^{(l)}\{P^{(j)},A_{\mathrm{prin}}\}P^{(j)}}{h^{(j)}-h^{(l)}} \\ &+ \frac{i}{2}\sum_{l\ne j}\frac{P^{(l)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(j)} - P^{(j)}\{A_{\mathrm{prin}},P^{(j)}\}P^{(l)})}{h^{(j)}-h^{(l)}} \\ &+ \sum_{l\ne j}\frac{P^{(l)}A_{\mathrm{sub}}P^{(j)} + P^{(j)}A_{\mathrm{sub}}P^{(l)}}{h^{(j)}-h^{(l)}}\,, \end{aligned}$$

which completes the proof of Theorem 2.3.

6 A positivity result

One of the most useful properties of pseudodifferential projections is that they can be used to construct sign definite operators (modulo $\Psi^{-\infty}$) out of A. A rigorous formulation of this statement is provided by Theorem 2.5, whose proof is given below.

Proof of Theorem 2.5. For definiteness, will prove the first part of the theorem, namely, formula (2.4). Formula (2.5) is proved similarly.

Suppose $j \in \{1,\dots,m^+\}$ and put

$$\widetilde{A} := A - 2\sum_{k=1}^{m^-} P_{-k}^* A P_{-k}\,.$$

Then
$$P_j^* A P_j = P_j^* \widetilde{A} P_j + C, \tag{6.1}$$
where $C \in \Psi^{-\infty}$ is the symmetric operator given by the explicit formula
$$C = 2\sum_{k=1}^{m^-} P_j^* P_{-k}^* A P_{-k} P_j \,.$$

The operator $\widetilde{A}$ is elliptic, self-adjoint and semibounded below. Let $\widetilde{\lambda}_l$, $l = 1, \dots, n$, be its negative eigenvalues and $\widetilde{\Pi}_l$ the corresponding eigenprojections. Put
$$\widehat{A} := \widetilde{A} - \sum_{l=1}^{n} \widetilde{\lambda}_l \, \widetilde{\Pi}_l \,.$$
Then
$$\widehat{A} \ge 0. \tag{6.2}$$

We have
$$P_j^* \widetilde{A} P_j = P_j^* \widehat{A} P_j + D \tag{6.3}$$
where $D \in \Psi^{-\infty}$ is the symmetric operator given by the explicit formula
$$D = \sum_{l=1}^{n} \widetilde{\lambda}_l P_j^* \widetilde{\Pi}_l P_j \,.$$

Formula (6.2) implies
$$P_j^* \widehat{A} P_j \ge 0. \tag{6.4}$$
Combining formulae (6.1), (6.3) and (6.4), we get
$$P_j^* A P_j \ge C + D \in \Psi^{-\infty}.$$

□

Theorem 2.5 will turn out to be quite useful in obtaining spectral-theoretic results in [16].

7 Modulus and Heaviside function of an elliptic system

In this section we represent the modulus and the Heaviside function of a self-adjoint elliptic matrix pseudodifferential operator $A \in \Psi^s$, $s > 0$, in terms of pseudodifferential projections.

It is well-known that $|A|$ and $\theta(A)$ are pseudodifferential operators of order s and 0, respectively, see, e.g., [2, Sec. 2]. Seeley's calculus [38] allows one, in principle, to compute locally the symbol of $|A|$ and $\theta(A)$ in terms of the symbol of the resolvent $(A - \lambda\,\mathrm{Id})^{-1}$. Carrying out such calculations involves dealing with pseudodifferential operators depending on a parameter, which makes it impractical to push the calculations beyond the very first few terms. In fact, we are unaware of any explicit formulae for $|A|_{\mathrm{sub}}$ or $[\theta(A)]_{\mathrm{sub}}$.

It is worth mentioning that an abstract analysis of the subprincipal symbol of elliptic operators of Laplace and Dirac type acting in vector bundles was performed in [32]. The

Heaviside function of an elliptic system is mentioned in [32], though the authors stop short of computing its subprincipal symbol.

Our contribution to the study of the operators $|A|$ and $\theta(A)$ is to establish, via Theorems 2.6 and 2.7, a relation between such operators and pseudodifferential projections. This yields, in turn, in view of subsection 4.3, an explicit algorithm for the calculation of the full symbol of $|A|$ and $\theta(A)$, arguably simpler and more straightforward than Seeley's. Note that, in particular, our approach does not involve either complex analysis or pseudodifferential operators depending on a parameter.

Proof of Theorem 2.6. We have $|A| = \sqrt{A^2}$, so the fact that $|A|$ is a pseudodifferential operator from the class Ψ^s follows from [38]; see also [39, §9–§11] for a more detailed exposition. The basic construction presented in [38, 39] requires the operator A^2 to be strictly positive, but this assumption is not necessarily satisfied for our operator A. However, one can deal with this issue as follows. Suppose that zero is an eigenvalue of A and denote by

$$\Pi_0 := \sum_{k:\ \lambda_k=0} \langle v_k,\, \cdot\,\rangle\, v_k \tag{7.1}$$

the eigenprojection onto the kernel of A. Then one can define the modulus of A via the identity $|A| = \sqrt{A^2+\Pi_0} - \Pi_0$ which only involves extracting the square root of a strictly positive operator.

In order to prove (2.7), let us argue by contradiction. Suppose there exists a positive integer k such that

$$|A| = \sum_{j=1}^{m^+} AP_j - \sum_{j=1}^{m^-} AP_{-j} - B, \tag{7.2}$$

where

$$B \in \Psi^{s-k} \quad \text{but} \quad B \notin \Psi^{s-k-1}. \tag{7.3}$$

Then B_{prin} has to satisfy

$$|A|_{\mathrm{prin}} B_{\mathrm{prin}} + B_{\mathrm{prin}} |A|_{\mathrm{prin}} = 0, \tag{7.4}$$

because

$$\left(\sum_{j=1}^{m^+} AP_j - \sum_{j=1}^{m^-} AP_{-j} \right)^2 = A^2 \mod \Psi^{-\infty}.$$

Since $|A|_{\mathrm{prin}} = \sum_j |h^{(j)}| P^{(j)}$, multiplying (7.4) by $P^{(n)}$ on the left and by $P^{(l)}$ on the right we obtain

$$(|h^{(n)}| + |h^{(l)}|)\, P^{(n)} B_{\mathrm{prin}} P^{(l)} = 0,$$

which implies $P^{(n)} B_{\mathrm{prin}} P^{(l)} = 0$ for every n and l, i.e. $B_{\mathrm{prin}} = 0$. But this contradicts (7.3).

Formula (2.8) follows from formula (2.7) and Theorem 2.3. Alternatively, it can be derived from the identity $|A|^2 = A^2$. □

Proof of Theorem 2.7. Let

$$B := \sum_{k:\, \lambda_k \neq 0} \frac{1}{\lambda_k} \langle v_k,\, \cdot\,\rangle\, v_k$$

be the pseudoinverse of A [29, Chapter 2 Section 2]. Then $AB = BA = \mathrm{Id} - \Pi_0$, where Π_0 is defined by (7.1). Thus, $B \in \Psi^{-s}$ is a parametrix for A.

We have

$$\theta(A) = \frac{\mathrm{Id} + B|A| - \Pi_0}{2},$$

and formula (2.10) follows now from Theorem 2.6 and formula (1.5). □

8 Applications

In this section we discuss some applications of the above results. The most important application — the partition of the spectrum of a positive order pseudodifferential system — will be the subject of a separate paper [16], where, among other things, results from [17, 13, 12, 15] will be refined and improved.

Throughout this section we adopt Einstein's summation convention over repeated indices.

8.1 Massless Dirac operator

Let (M, g) be a closed connected orientable and oriented Riemannian 3-manifold. We denote by ∇ the Levi-Civita connection, by $\Gamma^\alpha{}_{\beta\gamma}$ the Christoffel symbols, by $\rho(x) := \sqrt{g_{\alpha\beta}(x)}$ the Riemannian density and by

$$s^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = s_1, \quad s^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = s_2, \quad s^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = s_3$$

the standard Pauli matrices.

Let e_j, $j = 1, 2, 3$, be a positively oriented orthonormal global framing. In chosen local coordinates x^α, $\alpha = 1, 2, 3$, we denote by $e_j{}^\alpha$ the α-th component of the j-th vector field.

The massless Dirac operator $W : H^1(M) \to L^2(M)$ acting on 2-columns of complex-valued half-densities is the differential operator defined by

$$W := -i\sigma^\alpha \left(\frac{\partial}{\partial x^\alpha} + \frac{1}{4}\sigma_\beta \left(\frac{\partial \sigma^\beta}{\partial x^\alpha} + \Gamma^\beta{}_{\alpha\gamma}\,\sigma^\gamma \right) - \frac{1}{2}\Gamma^\beta{}_{\alpha\beta} \right), \tag{8.1}$$

where

$$\sigma^\alpha(x) := \sum_{j=1}^{3} s^j\, e_j{}^\alpha(x).$$

The goal of this subsection is to compute $|W|_{\mathrm{sub}}$. Note that $[\theta(W)]_{\mathrm{sub}}$ was calculated in [15, subsection 5.2].

Let us begin by observing that a global framing e_j, $j = 1, 2, 3$, defines a curvature-free metric compatible affine connection ∇^W on M, known as *Weitzenböck connection*, whose connection coefficients $\Upsilon^\alpha{}_{\beta\gamma}$ read

$$\Upsilon^\alpha{}_{\beta\gamma} = -\, e^j{}_\gamma \frac{\partial e_j{}^\alpha}{\partial x^\beta} = e_j{}^\alpha \frac{\partial e^j{}_\gamma}{\partial x^\beta}.$$

Here

$$e^j{}_\alpha := \delta^{jk}\, g_{\alpha\beta}\, e_k{}^\beta \tag{8.2}$$

and δ^{jk} is the Kronecker symbol. We refer the reader to [15, Appendix A] and references therein for further details.

The *contorsion* of ∇^W is defined to be the (1,2)-tensor

$$K^\alpha{}_{\beta\gamma} := \Upsilon^\alpha{}_{\beta\gamma} - \Gamma^\alpha{}_{\beta\gamma}. \tag{8.3}$$

Contorsion can be expressed in terms of the — perhaps more familiar — torsion of the affine connection ∇^W, see [15, Equations (A.3) and (A.5)], and is equivalent to it. Working with contorsion, as opposed to torsion, is just a matter of convenience.

Lowering the first index in (8.3) by means of the Riemannian metric g, we obtain a (0,3)-tensor antisymmetric in the first and third indices, $K_{\alpha\beta\gamma} = -K_{\gamma\beta\alpha}$. Because in dimension 3 antisymmetric tensors of order two are equivalent to vectors, instead of working with contorsion (8.3) we can equivalently work with

$$\overset{*}{K}_{\alpha\beta} := \frac{1}{2} K^\mu{}_\alpha{}^\nu\, E_{\mu\nu\beta}, \tag{8.4}$$

where

$$E_{\alpha\beta\gamma}(x) := \rho(x)\, \varepsilon_{\alpha\beta\gamma} \tag{8.5}$$

and ε is the totally antisymmetric symbol, $\varepsilon_{123} = +1$.

From (8.1) it is easy to see that

$$W_{\mathrm{prin}}(x,\xi) = \sigma^\alpha \xi_\alpha. \tag{8.6}$$

By direct computation one can establish that the eigenvalues of W_{prin} are simple and read $h^{(\pm1)} = \pm h$, where

$$h(x,\xi) := \sqrt{g^{\alpha\beta}(x)\xi_\alpha\xi_\beta}. \tag{8.7}$$

Combining [18, Lemma 6.1] with identities from [15, Appendix A] one can show that

$$W_{\mathrm{sub}}(x) = -\frac{1}{2}\overset{*}{K}{}^\alpha{}_\alpha(x)\, I\,. \tag{8.8}$$

Theorem 8.1. *The subprincipal symbol of the modulus of the massless Dirac operator $|W|$ is given by*

$$|W|_{\mathrm{sub}} = -\frac{1}{2h}\overset{*}{K}{}^\alpha{}_\beta\, \xi_\alpha\, (W_{\mathrm{prin}})_{\xi_\beta}. \tag{8.9}$$

Proof. When A is chosen to be the massless Dirac operator W, formula (2.7) from Theorem 2.6 can be simplified to read

$$|W|_{\mathrm{sub}} \;=\; \sum_{j=\pm1} \operatorname{sgn}(h^{(j)}) P^{(j)} W_{\mathrm{sub}} P^{(j)} \;+\; \frac{i}{4h}\big(\{W_{\mathrm{prin}}, W_{\mathrm{prin}}\} \;-\; \{|W|_{\mathrm{prin}}, |W|_{\mathrm{prin}}\}\big). \tag{8.10}$$

The task at hand is to compute the RHS of (8.10).

In view of (8.8), the first term on the RHS of (8.10) can be evaluated as

$$\begin{aligned}\sum_{j=\pm1}\operatorname{sgn}(h^{(j)})P^{(j)}W_{\mathrm{sub}}P^{(j)}&=-\frac{1}{2}\overset{*}{K}{}^{\alpha}{}_{\alpha}\,(P^{(1)}-P^{(-1)})\\&=-\frac{1}{2h}\overset{*}{K}{}^{\alpha}{}_{\alpha}\,W_{\mathrm{prin}}.\end{aligned}\tag{8.11}$$

Direct calculations involving (8.6) and (2.3) give us

$$\{W_{\mathrm{prin}},W_{\mathrm{prin}}\}=[\sigma^{\alpha}_{x^\mu},\sigma^\mu]\,\xi_\alpha.\tag{8.12}$$

Let us choose a point $y\in M$. Let $\{\widetilde{e}_j\}_{j=1}^3$ be the Levi-Civita framing generated by $\{e_j\}_{j=1}^3$ at y defined in accordance with [15, Definition 7.1], and let $\widetilde{W}$ be the massless Dirac operator associated with the latter. Then, formula (8.12) and [15, Corollary 7.3] imply

$$\{\widetilde{W}_{\mathrm{prin}},\widetilde{W}_{\mathrm{prin}}\}=0.\tag{8.13}$$

Now, there exists a smooth matrix-function $G:M\to SU(2)$ such that the framings $\{e_j\}_{j=1}^3$ and $\{\widetilde{e}_j\}_{j=1}^3$ are related in accordance with

$$e_j{}^\alpha(x)=\frac12\operatorname{tr}(s_j\,G^*(x)\,s^k\,G(x))\,\widetilde{e}_k{}^\alpha(x),\qquad G(y)=\mathrm{Id}.$$

It is easy to see that the corresponding Dirac operators and their principal symbols satisfy

$$W=G^*\,\widetilde{W}\,G\qquad\text{and}\qquad W_{\mathrm{prin}}=G^*\,\widetilde{W}_{\mathrm{prin}}\,G,\tag{8.14}$$

respectively.

Substituting (8.14) into (8.12) and using (8.13) we obtain

$$\{W_{\mathrm{prin}},W_{\mathrm{prin}}\}(y)=[G^*_{x^\alpha}\widetilde{W}_{\mathrm{prin}}+\widetilde{W}_{\mathrm{prin}}G_{x^\alpha},\sigma^\alpha]\Big|_{x=y}.\tag{8.15}$$

Formula [15, Eqn. (7.53)] tells us that

$$G_{x^\alpha}(y)=-\frac{i}{2}\overset{*}{K}{}_{\alpha\beta}(y)\,\sigma^\beta(y),\qquad G^*_{x^\alpha}(y)=\frac{i}{2}\overset{*}{K}{}_{\alpha\beta}(y)\,\sigma^\beta(y).\tag{8.16}$$

Substituting (8.16) into (8.15) and using elementary properties of Pauli matrices we get

$$\{W_{\mathrm{prin}},W_{\mathrm{prin}}\}(y)=-2i\left(\overset{*}{K}{}^{\alpha}{}_{\alpha}\,W_{\mathrm{prin}}-\overset{*}{K}{}^{\alpha}{}_{\beta}\,\xi_\alpha\,\sigma^\beta\right)\Big|_{x=y}.\tag{8.17}$$

We are left with computing $\{|W|_{\mathrm{sub}},|W|_{\mathrm{sub}}\}$. It follows from formula (8.6) that

$$|W|_{\mathrm{prin}}=|W_{\mathrm{prin}}|=h\,I\,,$$

which, in turn, immediately implies

$$\{|W|_{\mathrm{prin}},|W|_{\mathrm{prin}}\}=\{h,h\}\,I=0.\tag{8.18}$$

After observing that the point y chosen above is arbitrary, it only remains to substitute (8.11), (8.17) and (8.18) into (8.10). □

It is instructive to compare formulae (8.9) and (8.8): we see that $|W|_{\mathrm{sub}}$ is trace-free whereas W_{sub} is pure trace, i.e. proportional to the identity matrix.

Let us specialise further to the case of the 3-sphere,

$$M = \mathbb{S}^3 := \{\mathbf{x} \in \mathbb{R}^4 \,|\, \|\mathbf{x}\| = 1\},$$

equipped with the standard round metric $g_{\mathbb{S}^3}$, with orientation prescribed in accordance with [23, Appendix A]. If we choose our positively oriented framing e_j, $j = 1, 2, 3$, to be the restriction to $\mathbb{S}^3$ of the vector fields in $\mathbb{R}^4$

$$\begin{aligned}
\mathbf{e}_1 &:= -\mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^1} - \mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^2} + \mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^4},\\
\mathbf{e}_2 &:= +\mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^1} - \mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^2} - \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^4},\\
\mathbf{e}_3 &:= -\mathbf{x}^2 \frac{\partial}{\partial \mathbf{x}^1} + \mathbf{x}^1 \frac{\partial}{\partial \mathbf{x}^2} - \mathbf{x}^4 \frac{\partial}{\partial \mathbf{x}^3} + \mathbf{x}^3 \frac{\partial}{\partial \mathbf{x}^4},
\end{aligned}$$

then we obtain

$$\overset{*}{K} = -g_{\mathbb{S}^3}\,.$$

Therefore, Theorem 8.1 yields

$$|W_{\mathbb{S}^3}|_{\mathrm{sub}} = \frac{1}{2h}(W_{\mathbb{S}^3})_{\mathrm{prin}}\,. \tag{8.19}$$

8.2 Elasticity operator

Let (M, g) be a Riemannian manifold without boundary. Consider a diffeomorphism $\varphi : M \to M$ which is sufficiently close to the identity, so that it can be represented in terms of a vector field of displacements v, see [14, formula (4.1)]. Let $h := \varphi^* g$ be the pullback of g via φ. We define the *strain tensor* to be

$$\mathcal{S}^\alpha{}_\beta := \frac{1}{2}(g^{\alpha\gamma} h_{\gamma\beta} - \delta^\alpha{}_\beta). \tag{8.20}$$

Note that in [14] the factor $\frac{1}{2}$ was dropped for the sake of convenience, but in the current paper we stick with the more traditional definition (8.20), see also [27, formulae (1.2) and (1.3)].

Following [14], let us introduce the scalar invariants $e_k(\varphi)$, $k = 1, \dots, d$, where $e_k(\varphi)$ is the k-th elementary symmetric polynomial in the eigenvalues of $\mathcal{S}$ viewed as a linear operator in the tangent fibre. In particular, we have

$$e_1(\varphi) := \operatorname{tr} \mathcal{S}, \tag{8.21}$$

$$e_2(\varphi) := \frac{1}{2}\left[(\operatorname{tr} \mathcal{S})^2 - \operatorname{tr}(\mathcal{S}^2)\right], \tag{8.22}$$

where tr is the matrix trace.

The theory of linear elasticity is based on the following two assumptions.

(i) The potential energy of elastic deformation is quadratic (homogeneous of degree two) in strain.

(ii) The strain tensor has been linearised in displacements v.

Under the above assumptions the potential energy of elastic deformation reads

$$E(v) = \int_M (a(e_1)^2 + be_2)\,\rho\,dx \tag{8.23}$$

and formula (8.20) becomes

$$\mathcal{S}^\alpha{}_\beta = \frac{1}{2}(\nabla^\alpha v_\beta + \nabla_\beta v^\alpha). \tag{8.24}$$

The operator of linear elasticity L, acting on vector fields, is defined via

$$\frac{1}{2}\int_M v^\alpha (Lv)^\beta g_{\alpha\beta}\,\rho\,dx = E(v) \tag{8.25}$$

and (8.21)–(8.24). Here ρ is the Riemannian density, ∇ is the Levi-Civita connection associated with g and $a, b \in \mathbb{R}$ are parameters.

The tradition in elasticity theory is to express the parameters a and b in terms of the so-called *Lamé parameters* λ and μ,

$$a = \tfrac{1}{2}\lambda + \mu, \quad b = -2\mu, \tag{8.26}$$

which are assumed to satisfy conditions

$$\mu > 0, \qquad \lambda + \frac{2}{d}\,\mu > 0, \tag{8.27}$$

that guarantee strong convexity, see, for example, [33]. Formula (8.23) takes now the more familiar form

$$\begin{aligned} E(v) = \int_M \bigl(\tfrac{1}{2}\lambda(\operatorname{tr}\mathcal{S})^2 + \mu\operatorname{tr}(\mathcal{S}^2)\bigr)\,\rho\,dx \\ = \frac{1}{2}\int_M \bigl(\lambda(\nabla_\alpha v^\alpha)^2 + \mu(\nabla_\alpha v_\beta + \nabla_\beta v_\alpha)\nabla^\alpha v^\beta\bigr)\,\rho\,dx\,, \end{aligned} \tag{8.28}$$

see also [27, formula (4.1)]. Substituting (8.28) into (8.25) and integrating by parts we arrive at the explicit formula for the operator of linear elasticity (Lamé operator)

$$(Lv)^\alpha = -\mu(\nabla_\beta\nabla^\beta v^\alpha + \operatorname{Ric}^\alpha{}_\beta v^\beta) - (\lambda+\mu)\nabla^\alpha\nabla_\beta v^\beta, \tag{8.29}$$

where Ric is Ricci curvature.

The eigenvalues of L_{prin} are as follows: simple eigenvalue $(\lambda + 2\mu)h^2$ and eigenvalue μh^2 of multiplicity $d-1$, where h is defined by (8.7). These correspond to longitudinal and transverse waves, respectively. Our method requires the eigenvalues of the principal symbol to be simple, see Assumption 1.3, so further on in this subsection we restrict ourselves to the case $d = 2$.

The operator L does not fit into our scheme because it acts on 2-vectors as opposed to 2-columns of half-densities. In order to address this issue, we recast it as follows.

We first switch from 2-vectors to 2-columns of scalar functions by projecting onto a framing, which requires the manifold M to be parallelizable. Poincaré's theorem [28] tells us that a closed connected 2-manifold is parallelizable if and only if it is, topologically, a 2-torus, so further on in this subsection we restrict ourselves to the case $M = \mathbb{T}^2$.

Let e_j, $j = 1, 2$, be a positively oriented orthonormal global framing on $\mathbb{T}^2$. In chosen local coordinates x^α, $\alpha = 1, 2$, we denote by ${e_j}^\alpha$ the α-th component of the j-th vector field. Then the operator

$$(Bu)^\alpha := {e_j}^\alpha u^j$$

maps scalars to vectors and the operator

$$(B^{-1}v)^j := {e^j}_\alpha v^\alpha$$

maps vectors to scalars, see (8.2). We define the operator L_{scal} acting on 2-columns of scalar functions as

$$L_{\mathrm{scal}} := B^{-1} L B.$$

Finally, we turn L_{scal} into the operator

$$L_{1/2} := \rho^{1/2} L_{\mathrm{scal}}\, \rho^{-1/2}$$

acting on 2-columns of half-densities. Here ρ is the Riemannian density.

The operator $L_{1/2}$ is now of the type considered in this paper, so we can apply to it our results.

We have

$$(L_{1/2})_{\mathrm{prin}} = \mu h^2 I + (\lambda + \mu) h^2 p p^T, \tag{8.30}$$

where

$$p := h^{-1} \begin{pmatrix} {e_1}^\alpha \xi_\alpha \\ {e_2}^\alpha \xi_\alpha \end{pmatrix}. \tag{8.31}$$

The eigenvalues of $(L_{1/2})_{\mathrm{prin}}$ are $h^{(1)} = \mu h^2$ and $h^{(2)} = (\lambda + 2\mu) h^2$, and, in view of (8.27), these eigenvalues are simple. Here h is defined by formula (8.7). The corresponding eigenprojections are $P^{(1)} = I - pp^T$ and $P^{(2)} = pp^T$. Note that formula (8.27) implies

$$h^{(2)}/h^{(1)} > 2(1 - d^{-1}). \tag{8.32}$$

The subprincipal symbol of the operator $L_{1/2}$ is expressed via the torsion tensor

$$T^\alpha{}_{\beta\gamma} = \Upsilon^\alpha{}_{\beta\gamma} - \Upsilon^\alpha{}_{\gamma\beta}$$

of the Weitzenböck connection associated with our framing, see previous subsection for details. In dimension two the torsion tensor is equivalent to a covector field $t_\alpha := \frac{1}{2} T_\alpha{}^{\beta\gamma} E_{\beta\gamma}$, where $E_{\alpha\beta}(x) := \rho(x)\, \varepsilon_{\alpha\beta}$, $\varepsilon_{12} = +1$, compare with (8.5). It is easy to see that t is a closed 1-form: the fact that the exterior derivative of t is zero is a consequence of the fact that the Weitzenböck connection is flat. Namely, let $\mathfrak{R}^\alpha{}_{\beta\gamma\delta} = \partial_\gamma \Upsilon^\alpha{}_{\delta\beta} - \partial_\delta \Upsilon^\alpha{}_{\gamma\beta} + \Upsilon^\alpha{}_{\gamma\kappa} \Upsilon^\kappa{}_{\delta\beta} - \Upsilon^\alpha{}_{\delta\kappa} \Upsilon^\kappa{}_{\gamma\beta}$

be the curvature tensor of the Weitzenböck connection (which is zero by definition). Then $\mathfrak{R}$ and dt are related as $\mathfrak{R}_{\alpha\beta\gamma\delta} = E_{\alpha\beta}(dt)_{\gamma\delta} = (dt)_{\alpha\beta}E_{\gamma\delta}$.

Straightforward but lengthy calculations give us

$$(L_{1/2})_{\mathrm{sub}} = i(\lambda + 3\mu)\, t^\alpha \xi_\alpha\, \epsilon\,, \tag{8.33}$$

where $\epsilon := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The matrix ϵ admits a simple geometric interpretation. Namely, let $u = \begin{pmatrix} u^1 \\ u^2 \end{pmatrix} \in \mathbb{R}^2$ be a 2-column of scalars. Then $u \mapsto \epsilon u$ is the operator of rotation by $\pi/2$, clockwise, in the $\mathbb{R}^2$ plane.

One can show that

$$\{P^{(j)}, P^{(k)}\} = 0, \qquad j,k = 1,2,$$

$$P^{(1)}\epsilon P^{(2)} + P^{(2)}\epsilon P^{(1)} = \epsilon\,,$$

$$P^{(2)}Q^{(j)}P^{(1)} - P^{(1)}Q^{(j)}P^{(2)} = (-1)^j(\lambda + 3\mu)\, t^\alpha \xi_\alpha\, \epsilon\,, \qquad j = 1,2,$$

so that Theorem 2.3 gives us

$$(P_1)_{\mathrm{sub}} = (P_2)_{\mathrm{sub}} = 0. \tag{8.34}$$

There is an underlying reason for the subprincipal symbols of our pseudodifferential projections being zero in this particular case, i.e. for linear elasticity in dimension two. In dimension two a 1-form v can be written, locally, in terms of two scalar potentials φ and ψ as

$$v = d\varphi + *d\psi, \tag{8.35}$$

where $*$ is the Hodge dual (rotation by $\pi/2$ in the cotangent fibre). The operator of linear elasticity L agrees well with the substitution (8.35) in that it decouples L, modulo lower order terms involving curvature, into a pair of scalar operators acting on φ and ψ separately. In the scalar case ($m = 1$) the pseudodifferential projection from Theorem 2.2 would simply be the identity operator whose subprincipal symbol is, obviously, zero.

8.3 The Dirichlet-to-Neumann map of linear elasticity

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain (connected open set) with smooth boundary M. One could consider the more general case of a Riemannian 3-manifold with boundary but we refrain from doing this in the current paper for the sake of simplicity.

We denote Cartesian coordinates in $\mathbb{R}^3$ by $y = (y^1, y^2, y^3)$ and local coordinates on M by $x = (x^1, x^2)$. By x^3 we shall denote the signed distance from a point $P \in \mathbb{R}^3$ to M, positive for $P \notin \overline{\Omega}$ and negative for $P \in \Omega$. Clearly, (x^1, x^2, x^3) are local coordinates in $\mathbb{R}^3$. We assume that the orientation of (x^1, x^2, x^3) agrees with that of $y = (y^1, y^2, y^3)$, and this defines an orientation of M.

Consider the variational functional of 3-dimensional linear elasticity

$$E(v) = \frac{1}{2}\int_\Omega \big(\lambda(\partial_\alpha v^\alpha)^2 + \mu(\partial_\alpha v_\beta + \partial_\beta v_\alpha)\partial^\alpha v^\beta\big)\, dy\,. \tag{8.36}$$

Integration by parts gives

$$E(v) = \frac{1}{2}\int_\Omega v_\alpha (Lv)^\alpha \, dy \, + \frac{1}{2}\int_M v_\alpha (Tv)^\alpha \, dS \,, \tag{8.37}$$

where L is the operator of linear elasticity (see previous subsection) and T is *traction*. Traction T is a first order linear partial differential operator mapping a 3-dimensional vector field in Ω to a 3-dimensional vector field defined on M.

Let w be a 3-dimensional vector field defined on M. It is known that the Dirichlet problem for the elasticity operator

$$Lv = 0, \tag{8.38}$$

$$v|_M = w \tag{8.39}$$

has a unique solution. Let L_D^{-1} be the linear operator mapping w to v, a solution to (8.38), (8.39).

The Dirichlet-to-Neumann map of linear elasticity is the linear operator

$$DN := TL_D^{-1}. \tag{8.40}$$

This operator acts in the linear space of 3-dimensional vector fields defined on M.

Further on when working with 3-dimensional vector fields defined on M we will use local coordinates (x^1, x^2, x^3), so that our vector fields have the structure $v = (v^1, v^2, v^3)$, where (v^1, v^2) is a vector field on M and v^3 is the normal component of v which can be viewed as a scalar function. The inner product on such vector fields is defined as

$$(v, w) := \int_M (g_{\alpha\beta} v^\alpha w^\beta + v^3 w^3) \, dS \tag{8.41}$$

where g is the Riemannian metric on M induced by the Euclidean metric on $\mathbb{R}^3$ and summation is carried out over $\alpha, \beta = 1, 2$.

The operator DN is a first order 3×3 matrix pseudodifferential operator, symmetric with respect to the inner product (8.41). Furthermore, it is self-adjoint as an operator from $H^1(M)$ to $L^2(M)$, elliptic, nonnegative and has a 6-dimensional kernel (rigid translations and rotations).

Let $\nu = \frac{\lambda}{2(\lambda+\mu)}$ be Poisson's ratio. The eigenvalues of the principal symbol of the operator DN, enumerated in increasing order $0 < h^{(1)} \le h^{(2)} < h^{(3)}$, are expressed via Poisson's ratio as follows: if $-1 < \nu < 1/4$ then

$$h^{(1)} = \frac{2\mu h}{3 - 4\nu} \,, \qquad h^{(2)} = \mu h \,, \qquad h^{(3)} = 2\mu h \,, \tag{8.42}$$

and if $1/4 < \nu < 1/2$ then

$$h^{(1)} = \mu h \,, \qquad h^{(2)} = \frac{2\mu h}{3 - 4\nu} \,, \qquad h^{(3)} = 2\mu h \,. \tag{8.43}$$

For $\nu = 1/4$ we get a double eigenvalue $h^{(1)} = h^{(2)} = \mu h$. Here h is defined by formula (8.7), with summation carried out over $\alpha, \beta = 1, 2$.

Formulae (8.42) and (8.43) can be derived by considering an elastic half-space $x^3 < 0$ and performing separation of variables. Alternatively, one can compute the principal symbol of the operator DN and its eigenvalues by expressing this operator in terms of single and double layer potentials, see [1]. Note that for all $-1 < \nu < 1/2$ we have $2 \le h^{(3)}/h^{(1)} < 7$, compare with (8.32).

As explained in the previous subsection, in order to construct our pseudodifferential projections we need eigenvalues of the principal symbol to be simple and M to be parallelizable, which means that our construction would work when $\nu \ne 1/4$ and M is, topologically, a disjoint union of 2-tori. Admissible examples of 3-dimensional domains Ω would be, say, a doughnut or a toroidal shell.

Performing a detailed construction of pseudodifferential projections for the operator DN is a lengthy and technical enterprise which would shift the focus of this paper away from its core topic. We, therefore, decided not to carry out a full analysis of this meaningful example in the current paper and plan to address this matter comprehensively elsewhere.

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