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SOJOURN FUNCTIONALS FOR SPATIOTEMPORAL GAUSSIAN RANDOM FIELDS WITH LONG–MEMORY

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Abstract

This paper addresses the asymptotic analysis of sojourn functionals of spatiotemporal Gaussian random fields with long-range dependence (LRD) in time also known as long memory. Specifically, reduction theorems are derived for local functionals of nonlinear transformation of such fields, with Hermite rank $m \geq 1$, under general covariance structures. These results are proven to hold, in particular, for a family of non–separable covariance structures belonging to Gneiting class. For $m = 2$, under separability of the spatiotemporal covariance function in space and time, the properly normalized Minkowski functional, involving the modulus of a Gaussian random field, converges in distribution to the Rosenblatt type limiting distribution for a suitable range of the long memory parameter.

Keywords: Asymptotic normality; excursion sets; LRD; Rosenblatt–type distribution; spatiotemporal random fields

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1. Introduction

Geometric characteristics of random surfaces play a critical role in areas such as geostatistics, environmetrics, astrophysics, and medical imaging. There exists an extensive literature on data analysis based on Gaussian random field modeling. Minkowski functionals have played an important role in the geometrical analysis of their sample paths. In Novikov, Schmalzing and Mukhanov [35], Minkowski functionals are applied to the characterization of hot regions (i.e., the excursion sets), where the normalized

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temperature fluctuation field exceeds a given threshold. The normalized temperature fluctuation field, associated with CMB temperature on the sky, is represented in terms of a spherical random field (see also Linde and Mukhanov [27]; Novikov, Schmalzing and Mukhanov [35]). Furthermore, Minkowski functionals are attractive due to their geometrical interpretation in two dimensions, in relation to the total area of all hot regions, the total length of the boundary between hot and cold regions, and the Euler characteristic, which counts the number of isolated hot regions minus the number of isolated cold regions. Minkowski functionals have also been applied to brain mapping analysis, and, in general, to the description of texture models in medical imaging analysis, in relation to anatomy segmentation, and pathology detection and diagnosis (see, e.g., Steele [40]). Truncated Gaussian processes or sequential indicator simulation play a crucial role in geosciences to model the spatial distribution of the materials. Here, Minkowski functionals are used as morphological measures (see, e.g., Mosser, Dubrule and Blunt [34]; Pyrcz and Deutsch [38]). In that sense, a wide research area has been developed in the multiscale analysis of media with complex internal structures (see Armstrong *et al.* [3]), including soils, sedimentary rocks, foams, ceramics and composite materials (see, e.g., Gregorová *et al.* [14]; Ivonin *et al.* [17]; [36], and Tsukanov *et al.* [43]). Also, a good overview and introduction to some of these applications can be found in Adler and Taylor [1] and Marinucci and Peccati [31].

Since the nighties sojour functionals were extensively analyzed in the context of weak–dependent random fields (see, e.g., Bulinski *et al.* [6]; Ivanov and Leonenko [16], among others). A parallel literature has also been developed in the long–range dependence random field context (see Leonenko [21]; Leonenko and Olenko [22]; Makogin and Spodarev [30]; Marinucci, Rossi and Vidotto [32], just to mention a few). Particularly, limit theorems for level functionals of stationary Gaussian processes and fields constitute a major topic in this literature (see, e.g., Azäis and Wschebor [4]; Estrade and León [11]; Kratz and León [18]; [19]; Marinucci and Vadlamani [33]). The approach adopted in this paper continues this research line.

There has been a growing interest on covariance function modeling for spatiotemporal random fields. Marinucci, Rossi and Vidotto [32] consider isotropic in space and stationary in time Gaussian random fields on the two–dimensional unit sphere, and investigate the asymptotic behaviour of the empirical measure or excursion area, as time goes to infinity, covering both cases when the underlying field exhibits short and long memory in time. It turns out that the limiting distribution is not universal, depending both of the memory parameter and the threshold or level of sojourn functional. Marinucci, Rossi and Vidotto [32] adopt an intrinsic spherical isotropic random field methodology based on Karhunen-Loéve expansion in terms of spherical harmonics. As given in their Condition 2, a semiparametric model characterizes the resulting stationary time-varying angular spectrum involving a memory parameter depending on the spatial resolution level. As reflected in their Condition 3, the smallest exponent corresponding to the largest memory range, and the exponent at the coarsest spatial scale $l = 0$ are involved in the scaling to determine the asymptotic variance in time of the sojourn functional. Different scenarios are considered, distinguishing between null and non–null threshold parameter. Under these scenarios, one can find the first, second or third chaos domination, respectively leading to Gaussian, and non–Gaussian (so–called composite Rosenblatt 2 and 3) asymptotic probability distributions.

This paper analyzes the asymptotic behavior in time of local nonlinear functionals of LRD Gaussian random fields restricted to a spatial convex compact set. Specifically, the spectral diagonalization of isotropic continuous covariance kernels on sphere, in terms of spherical harmonics, applied in Marinucci, Rossi and Vidotto [32], is replaced here by the isonormal representation of a homogeneous and isotropic spatiotemporal Gaussian random field. Our main result, reduction principle given in Theorem 1, holds beyond the first Minkowski functional. The particular cases of this general reduction principle analyzed in Theorems 2 and 3 could be extended to the more general framework of spatial frequency varying long–memory parameters in time, in the spirit of Marinucci, Rossi and Vidotto [32] results. The same assertions hold regarding Proposition 1 below, derived in a separable covariance framework in space and time, that will be extended to the non–separable case in a subsequent paper. Note that the nonseparable covariance modeling assumed in Marinucci, Rossi and Vidotto [32] is given in terms of the tensorial product of a spatial basis (spherical harmonics), and a temporal basis (complex exponentials) that do not provide a diagonal representation. While Proposition 1 works under the diagonal representation in terms of complex exponentials of the covariance function of the underlying Gaussian random field.

To focus the topic and better describe the contributions of this work, we have to noting that our starting model is a spatially homogeneous and isotropic Gaussian random field, displaying stationarity and LRD in time, defined on $\mathbb{R}^d \times \mathbb{R}$. Its restriction to a convex compact set in space is then considered. An increasing sequence of temporal intervals is involved in the increasing domain asymptotic approach adopted. Note that our methodology is applicable, in particular, to considering the restriction to a compact two–points homogeneous space, like the sphere, of our original family of spatiotemporal Gaussian random fields on $\mathbb{R}^d \times \mathbb{R}$ (see, e.g., Leonenko and Ruiz–Medina [24]).

We present a general reduction principle (Theorem 1), discovered first by Taqqu [41] (see also Dobrushin and Major [9]; Leonenko, Ruiz–Medina and Taqqu [25]; [26]; Taqqu [42]), obtaining the limiting distributions of properly normalised integrals of non-linear transformations of spatiotemporal Gaussian random fields, from the asymptotic distribution of Hermite polynomial type functionals of such Gaussian random fields. The method of the proof is standard. Indeed, we use the expansion of the local functional of a Gaussian field into series of Hermite polynomials of such a field. But the novelty of the paper is that we consider spatiotemporal random fields beyond the regularly varying condition on the spatiotemporal covariance function. Hence, we can analyze a larger class of spatiotemporal covariance functions, including Gneiting class (see Gneiting [13]). This class of covariance functions is popular in many applications, including Meteorology or Earth sciences, among others. Theorems 2 shows that, under very general conditions on the decaying of covariance function to zero in time, the limiting distribution of normalized first Minkowski functional is asymptotically normal for large classes of covariances, including Gneiting class. For the modulus of a Gaussian random field, the limiting distribution is given in the form of a multiple Wiener–Itô stochastic integral, assuming separability in space and time of the covariance function. We also assume that the covariance function is a regularly varying function in time. The derived limiting distribution is of Rosenblatt type.

The outline of the paper is as follows. We first review some results on geometric probabilities in Section 2. The general reduction theorem, Theorem 1, for subordinated Gaussian spatiotemporal random fields with LRD in time is presented in Section 3. These results are applied to sojourn functionals introduced in Section 4, providing the asymptotic normality of the first Minkowski functional of a Gaussian random field, and limiting distribution of Rosenblatt type, for the sojourn functional, given by the modulus of a Gaussian random field. In Section 5.1 we provide examples in terms of separable covariance structures. While in Section 5.2 we present examples of covariance structures for which main results hold for non-separable covariance structures. We restrict out exposition by the covariances known as Gneting class of covariance functions.

2. Geometric probability

Some fundamental elements and basic results on geometric probability are now introduced (see Ahronyan and Khlatayan [2]; Ivanov and Leonenko [16]; Lellouche and Souries [20]; Lord [28], and the references therein).

Let ν_d be the Lebesgue measure on $\mathbb{R}^d$, $d \geq 1$, and $\mathcal{K}$ be a convex body in $\mathbb{R}^d$, i.e., a compact convex set with non empty interior. We will denote by $\mathcal{D}(\mathcal{K}) = \{\max \|\mathbf{x}-\mathbf{y}\|,\ \mathbf{x},\mathbf{y} \in \mathcal{K}\}$ the diameter of $\mathcal{K}$. Let $\nu_d(\mathcal{K}) = |\mathcal{K}|$ be the volume of $\mathcal{K}$, and for $d \geq 2$, $\nu_{d-1}(\delta\mathcal{K}) = \mathcal{U}_{d-1}(\mathcal{K})$ be the surface area of $\mathcal{K}$, where $\delta\mathcal{K}$ denotes the boundary of $\mathcal{K}$. For $d = 1$, we put $\mathcal{U}_0(\mathcal{K}) = 0$. For example, let $\mathcal{K} = \mathcal{B}(1) = \left\{\mathbf{x} \in \mathbb{R}^d;\ \|\mathbf{x}\| \leq 1\right\}$ be the unit ball. Hence, $\delta\mathcal{K} = \delta\mathcal{B}(1) = \mathcal{S}_{d-1} = \left\{\mathbf{x} \in \mathbb{R}^d;\ \|\mathbf{x}\| = 1\right\}$ is the unit sphere. Thus,

$$\mathcal{D}(\mathcal{B}(1)) = 2, \quad |\mathcal{B}(1)| = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}+1\right)},\ \mathcal{U}_{d-1}\left(\mathcal{B}(1)\right) = |\mathcal{S}_{d-1}| = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{1}$$

Let $\mathcal{Q}$ be the *stellate* space in $\mathbb{R}^d$, and $d\Gamma$ is an element of a locally finite measure in the space $\mathcal{Q}$, which is invariant with respect to the group $\mathcal{M}$ of all Euclidean motions in the space $\mathbb{R}^d$. Let now consider a chord length distribution function of body $\mathcal{K}$, given by

$$F_{\mathcal{K}}(v) = \frac{2(d-1)}{|\mathcal{S}_{d-2}|}\int_{\chi(\Gamma)\leq v} d\Gamma, \tag{2}$$

where $\chi(\Gamma) = \Gamma \cap \mathcal{K}$ is a chord in $\mathcal{K}$. For example, if $\mathcal{K} = \mathcal{B}(1)$, then

$$F_{\mathcal{B}(1)}(v) = \begin{cases} 0, \quad v \leq 0 \\ 1 - \left(1-\left(\frac{v}{2}\right)^2\right)^{\frac{d-1}{2}}, \quad 0 \leq v \leq 2 \\ 1, \quad v \geq 2 \end{cases} \tag{3}$$

(see Ahoronyan and Khalatyan [2] for details).

Let now consider two points $P_1, P_2 \in \mathcal{K}$ randomly and independently selected, with uniform distribution in $\mathcal{K}$. We consider the probability density $\psi_{\rho_{\mathcal{K}}}$ of the random variable $\rho_{\mathcal{K}} = \|P_1 - P_2\|$, given by

$$\psi_{\rho_{\mathcal{K}}}(z) = \frac{d}{dz}\mathbb{P}\left(\rho_{\mathcal{K}} \leq z\right).$$

In the particular case $\mathcal{K} = [-1, 1]$, $d = 1$, we have

$$\psi_{\rho_{\mathcal{K}}}(u) = 1 - \frac{u}{2}, \quad 0 \leq u \leq 2,$$

while for $d \geq 2$ (see Ivanov and Leonenko, 1989; Lord, 1954)

$$\psi_{\rho_{\mathcal{B}(1)}}(z) = \mathcal{I}_{1-\left(\frac{z}{2}\right)^2}\left(\frac{d+1}{2}, \frac{1}{2}\right), \quad 0 \leq z \leq 2, \tag{4}$$

where $\mathcal{I}_\mu(p,q)$ denotes the incomplete Beta function, given by

$$\mathcal{I}_\mu(p,q) = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}\int_0^\mu t^{p-1}(1-t)^{q-1}dt, \quad \mu \in [0,1]. \tag{5}$$

It is also known (see, e.g., equation (2.6) in Ahoronyan and Khalatyan [2]) that

$$\begin{aligned}\psi_{\rho_{\mathcal{K}}}(z) &= \frac{1}{|\mathcal{K}|^2}\left[z^{d-1}\left|\mathcal{S}_{d-1}\right|\left|\mathcal{K}\right|\right.\\ &- \left. z^{d-1}|\mathcal{S}_{d-2}|\mathcal{U}_{d-1}(\mathcal{K})\frac{1}{d-1}\int_0^z\left(1 - F_{\mathcal{K}}(v)\right)dv\right], \quad 0 \leq z \leq \mathcal{D}(\mathcal{K}).\end{aligned} \tag{6}$$

In particular, for the ball $\mathcal{K} = \mathcal{B}(1)$, we obtain an alternative to equation (4), given by, for $0 \leq z \leq 2$,

$$\psi_{\rho_{\mathcal{B}(1)}}(z) = z^{d-1}\left[\frac{2\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{2d-1}{2}}} - \frac{4\Gamma\left(\frac{d}{2}+1\right)}{\pi^{\frac{d-1}{2}}\Gamma\left(\frac{d+1}{2}\right)(d-1)}\int_0^z\left(1 - \left(\frac{u}{2}\right)^2\right)^{\frac{d-1}{2}}du\right]. \tag{7}$$

3. Reduction theorems for spatiotemporal random fields with LRD in time

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be the basic probability space, where the random components of the spatiotemporal real–valued Gaussian random field $\{Z(\mathbf{x},t),\ \mathbf{x} \in \mathbb{R}^d,\ t \in \mathbb{R}\}$ are defined. That is, $Z : (\Omega, \mathcal{A}, \mathbb{P}) \times \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}$.

Condition 1. Let Z be a measurable mean–square continuous homogeneous and isotropic in space, and stationary in time Gaussian random field with $\mathbb{E}[Z(\mathbf{x},t)] = 0$,

$\mathbb{E}[Z^2(\mathbf{x},t)] = 1$, and covariance function $\widetilde{C}(\|\mathbf{x}-\mathbf{y}\|, |t-s|) = \mathbb{E}\left[Z(\mathbf{x},t)Z(\mathbf{y},s)\right] \geq 0$, for every $t, s \in \mathbb{R}$, and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$. In spherical coordinates we denote

$$C(z,\tau) = \widetilde{C}(\|\mathbf{x}-\mathbf{y}\|, |t-s|), \quad z = \|\mathbf{x}-\mathbf{y}\| \geq 0,\ \tau = |t-s| \geq 0. \tag{8}$$

For simplicity, we will use $d\mathbf{x}$ instead of $\nu_d(d\mathbf{x})$, and dt instead of $\nu(dt)$. In the spatiotemporal isotropic spherical random field case sojourn functionals have been analyzed in Marinucci, Rossi and Vidotto [32]. We now introduce the following sojourn functional motivated by the first Minkowski functional. For each time t fixed, the random area

$$\begin{aligned}\mathcal{A}_u(t) &= \left|Z^{-1}(\cdot,t)\left([u,\infty)\right)\right| = |\{\mathbf{x} \in \mathcal{K};\ Z(\mathbf{x},t) \geq u\}| \\ &= \int_{\mathcal{K}} 1_{\mathcal{S}_Z(u)}(\mathbf{x},t)d\mathbf{x}\end{aligned}$$

provides the empirical measure (i.e., the excursion area) of $Z(\cdot,t)$ corresponding to the level u, $u \in \mathbb{R}$. The integrated area over the temporal interval $[0,T]$ is then computed as

$$M_T^{(1)}(u) = |\{0 \leq t \leq T;\ \mathbf{x} \in \mathcal{K},\ Z(\mathbf{x},t) \geq u\}| = \int_0^T \int_{\mathcal{K}} 1_{\mathcal{S}_Z(u)}(\mathbf{x},t)d\mathbf{x}dt, \tag{9}$$

where $1_{\mathcal{S}_Z(u)}(\cdot,\cdot)$ denotes the indicator function of the set $\mathcal{S}_Z(u) = \{(\mathbf{y},s) \in \mathcal{K}\times[0,T];\ Z(\mathbf{y},s) \geq u\}$. Similarly, we can define, for $u \geq 0$, the random area

$$\begin{aligned}\widetilde{\mathcal{A}}_u(t) &= \left|Z^{-1}(\cdot,t)\left[(-\infty,-u]\cup[u,\infty)\right]\right| = |\{\mathbf{x} \in \mathcal{K};\ |Z(\mathbf{x},t)| \geq u\}| \\ &= \int_{\mathcal{K}} 1_{\mathcal{S}_{|Z|}(u)}(\mathbf{x},t)d\mathbf{x},\end{aligned}$$

temporally integrated over $[0,T]$, defining the functional

$$M_T^{(2)}(u) = |\{0 \leq t \leq T;\ \mathbf{x} \in \mathcal{K},\ |Z(\mathbf{x},t)| \geq u\}| = \int_0^T \int_{\mathcal{K}} 1_{\mathcal{S}_{|Z|}(u)}(\mathbf{x},t)d\mathbf{x}dt. \tag{10}$$

Let $Z \sim \mathcal{N}(0,1)$ be a standard Gaussian random variable with probability density ϕ, and distribution function Φ given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{z^2}{2}\right), \quad \Phi(u) = \int_{-\infty}^{u} \phi_Z(z)dz, \quad z, u \in \mathbb{R}.$$

Let now $\mathcal{G}$ be a Borel measurable function such that

$$\int_{\mathbb{R}} [\mathcal{G}(z)]^2\phi(z)dz < \infty.$$

Then, $\mathcal{G}$ has an expansion with respect to the normalized Hermite polynomials that converges in $L_2(\mathbb{R}, \phi(z)dz)$:

$$\mathcal{G}(z) = \sum_{q=0}^{\infty} \frac{\mathcal{G}_q}{q!} H_q(z), \quad z \in \mathbb{R}, \quad \mathcal{G}_q = \int_{\mathbb{R}} H_q(\xi)\mathcal{G}(\xi)\phi(\xi)d\xi, \quad q \geq 1, \tag{11}$$

where the Hermite polynomial of order $q \geq 1$, denoted as H_q satisfies the equation:

$$\frac{d^n \phi}{dz^n}(z) = (-1)^n H_n(z)\phi(z). \tag{12}$$

Note that

$$\begin{aligned} H_0(x) &= 1, \ H_1(x) = x, \ H_2(x) = x^2 - 1 \\ H_3(x) &= x^3 - 3x, \ H_4(x) = x^4 - 6x^2 + 3, \dots. \end{aligned} \tag{13}$$

Particularly, if $\mathcal{G}_u(z) = 1_{\{z \geq u\}}$, we then obtain

$$\begin{aligned} \mathcal{G}_u(Z(\mathbf{x}, t)) &= \mathbb{E}[1_{\mathcal{S}_{Z(u)}}(\mathbf{x}, t)] \\ &+ \sum_{q=1}^{\infty} \frac{\mathcal{G}_q(u)}{q!} H_q(Z(\mathbf{x}, t)), \quad \forall (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}. \end{aligned} \tag{14}$$

Here, for every $(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}$,

$$\begin{aligned} \mathcal{G}_0(u) &= \mathbb{E}\left[1_{\mathcal{S}_{Z(u)}}(\mathbf{x}, t)\right] = [1 - \Phi(u)] = \int_u^{\infty} \phi(\xi)d\xi \\ \mathcal{G}_q(u) &= \phi(u)H_{q-1}(u), \quad q \geq 1. \end{aligned} \tag{15}$$

For the second functional corresponding to $\widetilde{\mathcal{G}}_u(z) = 1_{\{|z| \geq u\}}$, we have

$$\begin{aligned} \widetilde{\mathcal{G}}_0(u) &= \mathbb{E}[1_{\mathcal{S}_{|Z|(u)}}(\mathbf{x}, t)] = 2[1 - \Phi(u)] = 2\int_u^{\infty} \phi(\xi)d\xi \\ \widetilde{\mathcal{G}}_q(u) &= 2\phi(u)H_{q-1}(u), \end{aligned} \tag{16}$$

for any even $q \geq 0$, and $\widetilde{\mathcal{G}}_q(u) = 0$, for odd $q \geq 1$.

In what follows, from (14)–(16), we will consider the induced expansions of the functionals $M_T^{(i)}(u)$, $i = 1, 2$, given by

$$M_T^{(1)}(u) = (1 - \Phi(u))T|\mathcal{K}| + \phi(u) \sum_{n=1}^{\infty} \frac{H_{n-1}(u)}{n!} \eta_n \tag{17}$$

$$M_T^{(2)}(u) = 2(1 - \Phi(u))T|\mathcal{K}| + 2\phi(u) \sum_{n=1}^{\infty} \frac{H_{2n-1}(u)}{(2n)!} \eta_n, \tag{18}$$

where

$$\eta_n = \int_0^T \int_{\mathcal{K}} H_n(Z(\mathbf{x},t))d\mathbf{x}dt,$$

and

$$\begin{aligned} \mathbb{E}[\eta_n] &= 0, \quad \mathbb{E}[\eta_n \eta_l] = 0, \quad n \neq l, \\ \sigma^2_{n,\mathcal{K}}(T) &= \mathbb{E}[\eta_n^2] = 2n!T \int_0^T \left(1 - \frac{\tau}{T}\right) \int_{\mathcal{K}\times\mathcal{K}} \widetilde{C}^n(\|\mathbf{x}-\mathbf{y}\|, \tau) d\tau d\mathbf{x} d\mathbf{y} \\ &= 2n!T|\mathcal{K}|^2 \int_0^T \left(1 - \frac{\tau}{T}\right) \mathbb{E}\left[\widetilde{C}^n\left(\|P_1 - P_2\|, \tau\right)\right] d\tau \\ &= 2n!T|\mathcal{K}|^2 \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{\mathcal{D}(\mathcal{K})} \psi_{\rho_{\mathcal{K}}}(z) C^n(z,\tau) dz d\tau. \end{aligned} \tag{19}$$

Condition 2. Assume that

(i) $\sup_{z\in[0,\mathcal{D}(\mathcal{K})]} |C(z,\tau)| = \sup_{z\in[0,\mathcal{D}(\mathcal{K})]} C(z,\tau) \to 0$, $\tau \to \infty$

(ii) For certain fixed $m \in \{1, 2, \dots\}$, there exists $\delta \in (0,1)$ such that

$$\lim_{T\to\infty} \frac{1}{T^\delta} \int_0^T \left(1 - \frac{\tau}{T}\right) \int_0^{\mathcal{D}(\mathcal{K})} C^m(z,\tau)\psi_{\rho_{\mathcal{K}}}(z) dz d\tau = \infty. \tag{20}$$

3.1. Reduction theorem

In this section, we extend the results by Taqqu [41];[42] to the case of spatiotemporal random fields with LRD in time. For a function $\mathcal{G} \in L_2(\mathbb{R}, \phi(u)du)$, under **Condition 1**, we consider the following local functional

$$\begin{aligned} A_T &= \int_0^T \int_{\mathcal{K}} \mathcal{G}(Z(\mathbf{x},t))d\mathbf{x}dt \\ &= T|\mathcal{K}|\mathcal{G}_0 + \sum_{n=1}^{\infty} \frac{\mathcal{G}_n}{n!} \int_0^T \int_{\mathcal{K}} H_n(Z(\mathbf{x},t))d\mathbf{x}dt, \end{aligned} \tag{21}$$

where, for $n \geq 0$, $\mathcal{G}_n$ denotes the Fourier coefficient of function $\mathcal{G}$ with respect to H_n, and the series (21) converges in $\mathcal{L}_2(\Omega, \mathcal{A}, P)$. Denote as in (19), $\sigma^2_{n,\mathcal{K}}(T) = \mathbb{E}[\eta_n^2]$, hence, we obtain

$$\sigma_T^2 = \mathrm{Var}(A_T) = \mathbb{E}\left[A_T - E[A_T]\right]^2 = \sum_{n=0}^{\infty} \sigma^2_{n,\mathcal{K}}(T).$$

Definition. We say that an integer $m \geq 1$ is the Hermite rank of function $\mathcal{G}$, if for $m = 1$, $\mathcal{G}_1 \neq 0$, or for $m \geq 2$, $\mathcal{G}_1 = \cdots = \mathcal{G}_{m-1} = 0$, $\mathcal{G}_m \neq 0$ (see also Taqqu [41]).

Theorem 1. *Under **Conditions 1** and **2**, assume that function $\mathcal{G}$ in (21) has Hermite rank m, the random variables*

$$Y_T = \frac{A_T - \mathbb{E}[A_T]}{|\mathcal{G}_m|\sigma_{m,\mathcal{K}}(T)(1/m!)} \tag{22}$$

and

$$Y_{m,T} = \frac{sgn\{\mathcal{G}_m\}\int_0^T \int_{\mathcal{K}} H_m(Z(\mathbf{x},t))d\mathbf{x}dt}{\sigma_{m,\mathcal{K}}(T)} \tag{23}$$

have the same limiting distributions (if one of it exists).

Proof. We split

$$A_T - \mathbb{E}[A_T] = S_{1,T} + S_{2,T},$$

where using notation (19), and applying Parseval identity,

$$S_{1,T} = \frac{\mathcal{G}_m}{m!}\xi_m, \quad S_{2,T} = \sum_{n=m+1}^{\infty} \frac{\mathcal{G}_n}{n!}\xi_n, \quad \sum_{n=m}^{\infty} \frac{\mathcal{G}_n^2}{n!} < \infty \quad \text{a.s.} \tag{24}$$

From (19) and (24) we get

$$\text{Var}\,(A_T) = \text{Var}(S_{1,T}) + \text{Var}(S_{2,T}), \tag{25}$$

and we have to show that

$$\frac{\text{Var}(S_{2,T})}{\sigma^2_{m,\mathcal{K}}(T)} \to 0, \quad T \to \infty.$$

Under **Condition 2**(i),

$$\sup_{z\in[0,\mathcal{D}(\mathcal{K})],\tau\geq T^{\delta}} C(z,\tau) \to 0, \quad T \to \infty, \tag{26}$$

where δ satisfies **Condition** 2(ii). Note that, for $0 \leq \tau \leq T^{\delta}$, the unit variance of Z allows to work with the uniform estimate $|C(z,\tau)|^{m+1} \leq 1$, $z \in \mathbb{R}_+$. From (24), we then have

$$\begin{aligned} \text{Var}(S_{2,T}) &\leq \sum_{n=m+1}^{\infty} \frac{\mathcal{G}_n^2}{(n!)^2}\sigma^2_{n,\mathcal{K}}(T) \\ &\leq M_1 \left\{ 2T \left[\int_0^{T^{\delta}} + \int_{T^{\delta}}^{T} \right] \right\} \left(1 - \frac{\tau}{T}\right) \int_0^{\mathcal{D}(\mathcal{K})} C^{m+1}(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau, \end{aligned} \tag{27}$$

for $M_1 > 0$, whose value follows from (19). In addition, from (6),

$$\psi_{\rho_{\mathcal{K}}}(z) \leq \frac{z^{d-1}}{|\mathcal{K}|}|S_{d-1}|, \quad 0 \leq z \leq \mathcal{D}(\mathcal{K}), \tag{28}$$

leading to

$$\begin{aligned} \mathrm{Var}(S_{2,T}) &\leq M_1 \left\{ M_2 T^{\delta+1} + 2T \int_{T^\delta}^{T} \left(1 - \frac{\tau}{T}\right) \int_{\mathcal{K}} C^{m+1}(z,\tau)\psi_{\rho_{\mathcal{K}}}(z) dz d\tau \right\} \\ &\leq M_3 \left\{ T^{\delta+1} + 2T \sup_{z\in[0,\mathcal{D}(\mathcal{K})],\tau\geq T^\delta} C(z,\tau) \right. \\ &\qquad \left. \times \int_{T^\delta}^{T} \left(1 - \frac{\tau}{T}\right) \int_{\mathcal{K}} C^{m}(z,\tau)\psi_{\rho_{\mathcal{K}}}(z) dz d\tau \right\}. \end{aligned} \tag{29}$$

Hence,

$$\begin{aligned} \frac{\mathrm{Var}(S_{2,T})}{\sigma^2_{m,\mathcal{K}}(T)} \leq M_4 &\left\{ \frac{1}{T^{-(\delta+1)}\sigma^2_{m,\mathcal{K}}(T)} \right. \\ &\left. + M_5 \sup_{z\in[0,\mathcal{D}(\mathcal{K})],\tau\geq T^\delta} C(z,\tau) \frac{\int_{T^\delta}^{T}\left(1 - \frac{\tau}{T}\right)\int_{\mathcal{K}} C^m(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau}{\int_0^T \left(1 - \frac{\tau}{T}\right)\int_{\mathcal{K}} C^m(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau} \right\}. \end{aligned} \tag{30}$$

From (19), under **Condition 2**(ii),

$$\frac{\sigma^2_{m,\mathcal{K}}(T)}{T^{\delta+1}} \to \infty, \quad T \to \infty, \tag{31}$$

and under **Condition 2(i)**,

$$\sup_{z\in[0,\mathcal{D}(\mathcal{K})],\tau\geq T^\delta} C(z,\tau) \to 0, \quad T \to \infty.$$

Note that,

$$\frac{\int_{T^\delta}^{T}\left(1 - \frac{\tau}{T}\right)\int_{\mathcal{K}} C^m(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau}{\int_0^T \left(1 - \frac{\tau}{T}\right)\int_{\mathcal{K}} C^m(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau} \leq 1. \tag{32}$$

The convergence to zero of $\frac{\mathrm{Var}(S_{2,T})}{\sigma^2_{m,\mathcal{K}}(T)}$ then follows from equation (30) under **Condition 2**, leading to $\mathbb{E}[Y_T - Y_{m,T}]^2 = \frac{\mathrm{Var}(S_{2,T})}{\sigma^2_{m,\mathcal{K}}(T)} \to 0$, $T \to \infty$, as we wanted to prove. □

Remark 1. We have applied in (32) that, under **Condition 1**, the correlation function $C(z,\tau) \geq 0$, for every $\tau, z \in \mathbb{R}_+$.

Remark 2. For a short memory case, one can assume that for a fixed $m \geq 1$,

$$\int_0^\infty \int_0^{\mathcal{D}(\mathcal{K})} \psi_{\rho_{\mathcal{K}}}(\mathbf{z}) \left|C(\mathbf{z},\tau)\right|^m d\mathbf{z}d\tau < \infty.$$

Then, once can show, using standard arguments that, as $T \to \infty$, the asymptotic variance satisfies

$$\sigma_T^2 = \mathrm{Var}(A_T) = \sum_{n=m}^{\infty} \frac{\mathcal{G}_n^2}{(n!)^2} \sigma_{n,\mathcal{K}}^2(T) = TB(1+o(1)), \tag{33}$$

where

$$B = \sum_{n=m}^{\infty} \frac{\mathcal{G}_n^2}{(n!)^2} \lim_{T\to\infty} \frac{\sigma_{n,\mathcal{K}}^2(T)}{T} < \infty.$$

Then, using the method of moments and diagram formulae (see for details Theorem 2.3.1 in Ivanov and Leonenko [16]), one can prove under condition (33) that $(A_T - E[A_T])/\sqrt{T}$ converges to a normal distribution with zero mean and variance B.

4. Sojourn functionals

As an application of reduction Theorem 1, the following result proves the convergence to a standard normal distribution for the case of Hermite rank $m = 1$.

Theorem 2. *Under **Conditions 1** and **2**, for $m = 1$, the random variables*

$$X_{1,T} = \frac{M_T^{(1)}(u) - T|\mathcal{K}|(1-\Phi(u))}{\phi(u)\left[2T\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(\mathcal{K})} C(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau\right]^{1/2}}, \tag{34}$$

and

$$\frac{\int_0^T \int_{\mathcal{K}} Z(x,t)dxdt}{\left[2T\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(\mathcal{K})} C(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau\right]^{1/2}} \tag{35}$$

have the same limit as $T \to \infty$. Namely, the convergence to a standard normal distribution holds.

The analogous result to Theorem 2 for functional $M_T^{(2)}$ is now formulated.

Theorem 3. *Under **Conditions 1** and **2**, with $m = 2$, the random variables*

$$X_{2,T} = \frac{M_T^{(2)}(u) - 2T|\mathcal{K}|(1-\Phi(u))}{[\phi(u)]^2\left[T|\mathcal{K}|^2\int_0^T \left(1-\frac{\tau}{T}\right)\int_0^{\mathcal{D}(\mathcal{K})} C^2(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau\right]^{1/2}}, \tag{36}$$

and

$$Y_{2,T} = \frac{\int_0^T \int_{\mathcal{K}} (Z^2(x,t)-1)dxdt}{2\left[T|\mathcal{K}|^2 \int_0^T \left(1-\frac{\tau}{T}\right) \int_0^{\mathcal{D}(\mathcal{K})} C^2(z,\tau)\psi_{\rho_{\mathcal{K}}}(z)dzd\tau\right]^{1/2}} \tag{37}$$

have the same limit distribution in the sense that if one exists then so does the other and the two are equal.

The proof of Theorem 3 is obtained from Theorem 1.

5. Examples

In Sections 5.1 and 5.2, we will present some examples of covariance functions displaying long–range dependence in time for which Conditions 2(i)–(ii) hold true.

5.1. Separable covariance structures

Under **Condition 1**, the covariance function $\widetilde{C}(\|\mathbf{x}-\mathbf{y}\|, |t-s|) = C(z,\tau)$ is said to be separable if it can be factorized as the product of a spatial $C_{\mathcal{S}}$ and temporal $C_{\mathcal{T}}$ covariance functions (see Cressie and Huang [8], and Christakos [7]). That is,

$$\widetilde{C}(\|\mathbf{x}-\mathbf{y}\|, |t-s|) = C(z,\tau) = C_{\mathcal{S}}(z)C_{\mathcal{T}}(\tau), \tag{38}$$

where, as before, $z \geq 0$, and $\tau \geq 0$.

Condition 4. Consider the covariance function

$$C_{\mathcal{T}}(\tau) = \frac{\mathcal{L}(\tau)}{\tau^{\alpha}}, \quad \tau \geq 0, \quad \alpha \in (0,1), \tag{39}$$

where $\mathcal{L}$ is a slowly varying function at infinity locally bounded, i.e., bounded at each bounded interval.

Under **Conditions 1** and **4**, for $\alpha \in \left(0, \frac{1}{n}\right)$, for separable covariance functions as given in (38), we obtain

$$\begin{aligned} \sigma_n^2(T) &= 2n!T \int_0^T \left(1-\frac{\tau}{T}\right) C_{\mathcal{T}}^n(\tau)d\tau \\ &= T^{2-n\alpha}\mathcal{L}^n(T)\left[2n!\int_0^1 (1-\tau)\tau^{-n\alpha}d\tau\right](1+o(1)). \end{aligned} \tag{40}$$

From (19) and (40), as $T \to \infty$,

$$\sigma_{n,\mathcal{K}}^2(T) = c_{\mathcal{K}}(n,\alpha)T^{2-n\alpha}\mathcal{L}^n(T)\left(1+o(1)\right),$$

where

$$c_{\mathcal{K}}(n,\alpha) = 2n!\left[\int_0^1 (1-\tau)\frac{d\tau}{\tau^{\alpha n}}\right]|\mathcal{K}|^2\int_0^{\mathcal{D}(\mathcal{K})} C_{\mathcal{S}}(z)\psi_{\rho_{\mathcal{K}}}(z)dz.$$

Proposition 1. *Under **Conditions 1** and **4**, for separable covariance functions (38), **Condition 2(ii)** holds for $\alpha \in (0,1)$, if $m=1$, and for $\alpha \in (0,1/2)$ if $m=2$. Moreover, for $\alpha \in (0,1/2)$, the random variables (36) and (37) have, as $T\to\infty$, the limiting distribution $\mathcal{R}$ of Rosenblatt type, given by the following Wiener-Itô integral representation, with respect to spatiotemporal complex Gaussian white noise random measure W on $\mathbb{R}^2\times\mathbb{R}^{2d}$ (integration over hyperdiagonals are excluded, see, e.g., Dobrushin and Major [9])*

$$\begin{aligned}\mathcal{R} &= \frac{c_T(\alpha)}{\sqrt{c_{\mathcal{K}}(2,\alpha)}}\int_{\mathbb{R}^2}^{\prime}\frac{\exp\{i(\mu_1+\mu_2)\}-1}{i(\mu_1+\mu_2)}\frac{1}{|\mu_1\mu_2|^{\frac{1-\alpha}{2}}}\\ &\times \int_{\mathbb{R}^{2d}}^{\prime}\left[\int_{\mathcal{K}}\exp\{i\langle \mathbf{x},\boldsymbol{\omega}_1+\boldsymbol{\omega}_2\rangle\}d\mathbf{x}\right]\left[\prod_{j=1}^2 f_S(\boldsymbol{\omega}_j)\right]^{1/2}\\ &\times W(d\mu_1,d\boldsymbol{\omega}_1)W(d\mu_2,d\boldsymbol{\omega}_2), \qquad (41)\end{aligned}$$

where

$$f_{\mathcal{S}}(\boldsymbol{\omega}) = \frac{1}{[2\pi]^d}\int_{\mathbb{R}^d}\exp\left(-i\langle\boldsymbol{\omega},\mathbf{x}\rangle\right)\widetilde{C}_{\mathcal{S}}(\|\mathbf{x}\|)d\mathbf{x},$$

and

$$c_T(\alpha) = \frac{\Gamma\left(\frac{1-\alpha}{2}\right)}{2^{\alpha}\Gamma\left(\frac{\alpha}{2}\right)\sqrt{\pi}} \qquad (42)$$

is the Tauberian constant.

Remark 3. Note that $E\left[\mathcal{R}^2\right]<\infty$.

Proof. The proof of Proposition 1 is standard (see, e.g., Leonenko and Olenko [22]). A sketch of the proof is now given. Note that the spectral density of a spatiotemporal random field with separable covariance function (38) is also separable, i.e.,

$$\begin{aligned}f(\boldsymbol{\omega},\mu) &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}\times\mathbb{R}^d}\exp\left(-i\mu\tau\right)\exp\left(-i\langle\boldsymbol{\omega},\mathbf{x}\rangle\right)\widetilde{C}_{\mathcal{S}}(\|\mathbf{x}\|)\widetilde{C}_{\mathcal{T}}(|\tau|)d\mathbf{x}d\tau\\ &= \left[\frac{1}{[2\pi]^d}\int_{\mathbb{R}^d}\exp\left(-i\langle\boldsymbol{\omega},\mathbf{x}\rangle\right)\widetilde{C}_{\mathcal{S}}(\|\mathbf{x}\|)d\mathbf{x}\right]\left[\frac{1}{2\pi}\int_{\mathbb{R}}\exp\left(-i\mu\tau\right)\widetilde{C}_{\mathcal{T}}(|\tau|)d\tau\right]\\ &= f_{\mathcal{S}}(\boldsymbol{\omega})f_{\mathcal{T}}(\mu), \quad \boldsymbol{\omega}\in\mathbb{R}^d, \quad \mu\in\mathbb{R}. \qquad (43)\end{aligned}$$

From Tauberian Theorems (see Leonenko and Olenko [23]), under **Condition 4**, we get convergence

$$f_{\mathcal{T}}(\mu) \sim c_T(\alpha) \frac{\mathcal{L}\left(\frac{1}{\mu}\right)}{|\mu|^{1-\alpha}}, \quad \mu \to 0, \tag{44}$$

for $0 < \alpha < \frac{1}{2}$, where the Tauberian constant $c_T(\alpha)$ has been introduced in (42). From (43), applying the Wiener–Itô stochastic integral representation (see, e.g., Major [29], and Section 4.4.2 in Marinucci and Peccati [31]), we obtain isonormal representation:

$$Z(\mathbf{x},t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \exp(i\mu t) \exp(i \langle \boldsymbol{\omega}, \mathbf{x} \rangle) \sqrt{f_{\mathcal{T}}(\mu) f_{\mathcal{S}}(\boldsymbol{\omega})} W(d\mu, d\boldsymbol{\omega}), \tag{45}$$

with W denoting complex–valued white noise measure.

For $\underset{d}{=}$ denoting the identity in probability distribution, applying now the self–similarity of Gaussian white noise random measure

$$W(ad\mu, bd\boldsymbol{\omega}) \underset{d}{=} \sqrt{a} b^{d/2} W(d\mu, d\boldsymbol{\omega}), \quad \forall \mu \in \mathbb{R},\ \boldsymbol{\omega} \in \mathbb{R}^d, \tag{46}$$

and the Itô formula (see, e.g., Dobrushin and Major [9]; Major [29]), from equation (45), we obtain

$$\begin{aligned}
Y_{2,T} &= \frac{\int_0^T \int_{\mathcal{K}} \left(Z^2(\mathbf{x},t) - 1\right) d\mathbf{x} dt}{T^{1-\alpha} \mathcal{L}(T) \sqrt{c_{\mathcal{K}}(2,\alpha)}} \\
&\underset{d}{=} \int'_{\mathbb{R}^2} \left[\int_0^1 \exp(i(\mu_1 + \mu_2)t)\, dt\right] \left[\prod_{j=1}^{2} f_{\mathcal{T}}\left(\frac{\mu_j}{T}\right)\right]^{1/2} \frac{1}{T^{1-\alpha} \mathcal{L}(T) \sqrt{c_{\mathcal{K}}(2,\alpha)}} \\
&\times \int'_{\mathbb{R}^{2d}} \left[\int_{\mathcal{K}} \exp(i \langle \mathbf{x}, \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \rangle)\, d\mathbf{x}\right] \left[\prod_{j=1}^{2} f_{\mathcal{S}}(\boldsymbol{\omega}_j)\right]^{1/2} W(d\mu_1, d\boldsymbol{\omega}_1)\, W(d\mu_2, d\boldsymbol{\omega}_2).
\end{aligned} \tag{47}$$

We denote

$$\begin{aligned}
\mathcal{I}_{\mathcal{K}} &= \frac{1}{c_{\mathcal{K}}(2,\alpha)} \int_{\mathbb{R}^{2d}} \left| \int_{\mathcal{K}} \exp(i \langle \mathbf{x}, \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \rangle)\, d\mathbf{x} \right|^2 \left[\prod_{j=1}^{2} f_{\mathcal{S}}(\boldsymbol{\omega}_j)\right] d\boldsymbol{\omega}_1 d\boldsymbol{\omega}_2 \\
&= \frac{|\mathcal{K}|^2}{c_{\mathcal{K}}(2,\alpha)} \mathbb{E}\left[C_{\mathcal{S}}^2 (\|P_1 - P_2\|)\right] \\
&= \frac{|\mathcal{K}|^2}{c_{\mathcal{K}}(2,\alpha)} \int_0^{\mathcal{D}(\mathcal{K})} C_{\mathcal{S}}^2(z)\, \psi_{\rho_{\mathcal{K}}}(z) dz,
\end{aligned} \tag{48}$$

where in the last identity, we have applied similar steps to (19).

From (47) and (48), we then obtain

$$\mathbb{E}\left[Y_{2,T}-\mathcal{R}\right]^2 = [c_T(\alpha)]^2\mathcal{I}_{\mathcal{K}}\int_{\mathbb{R}^2}\left|\int_0^1 \exp\left(i(\mu_1+\mu_2)t\right)dt\right|^2 \times\frac{1}{|\mu_1\mu_2|^{1-\alpha}}Q_T(\mu_1,\mu_2)d\mu_1 d\mu_2, \tag{49}$$

where

$$Q_T(\mu_1,\mu_2) = \left(\frac{1}{c_T(\alpha)}|\mu_1\mu_2|^{\frac{1-\alpha}{2}}\prod_{j=1}^{2} f_{\mathcal{T}}^{1/2}\left(\frac{\mu_j}{T}\right)\frac{1}{T^{1-\alpha}\mathcal{L}(T)}-1\right)^2. \tag{50}$$

Applying Tauberian Theorems (see (44)), and Dominated Convergence Theorem, as $T\to\infty$, (49) converges to zero for $\alpha\in(0,1/2)$. Hence, the convergence in probability distribution of the random variable $Y_{2,T}$ to $\mathcal{R}$ holds (see Leonenko and Olenko [22], for more details). □

5.2. Non–separable covariance functions

Let $\varphi(v)\geq 0$ be a completely monotone function. That is, an infinite differentiable function satisfying

$$(-1)^n\frac{d^n\varphi}{dv^n}(v)\geq 0,\quad v>0,\quad n\geq 0.$$

By Bernstein's Theorem

$$\varphi(v)=\int_0^v \exp\left(-v\xi\right)\mu(d\xi),$$

where μ is a positive measure over $[0,\infty)$ (see Gneiting [13]).

Suppose further that $\psi:[0,\infty)\to[0,\infty)$ has completely monotone derivatives, i.e., it is a Bernstein function. The Gneiting class of spatiotemporal covariance functions is defined as follows (see Gneiting [13])

$$\begin{aligned}\widetilde{C}(\|\mathbf{x}-\mathbf{y}\|,|t-s|) &= \frac{1}{[\psi(|t-s|^2)]^{d/2}}\varphi\left(\frac{\|\mathbf{x}-\mathbf{y}\|^2}{\psi(|t-s|^2)}\right)\\ &= C(z,\tau)=\frac{1}{[\psi(\tau^2)]^{d/2}}\varphi\left(\frac{z^2}{\psi(\tau^2)}\right)\\ &\qquad \mathbf{x},\mathbf{y}\in\mathbb{R}^d,\ t,s\in\mathbb{R},\ \tau,z\geq 0.\end{aligned} \tag{51}$$

It is known that the one–parameter Mittag–Leffler function E_ν, for $0<\nu\leq 1$, is a completely monotone function (see Feller [12], p. 147), given by

$$E_\nu(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+1)},\quad z\in\mathbb{C},\quad 0<\beta<1$$

(see Erdélyi *et al.* [10]; Haubold, Mathai and Saxena [15]).

For every $\nu \in (0,1)$, uniformly in $x \in \mathbb{R}_+$, the following two–sided estimates are obtained with optimal constants (see Simon [39], Theorem 4):

$$\frac{1}{1+\Gamma(1-\nu)x} \quad \leq \quad E_\nu(-x) \leq \frac{1}{1+[\Gamma(1+\nu)]^{-1}x}. \tag{52}$$

Note that the function

$$\psi(u) = (1+au^\alpha)^\beta,\ a>0,\ 0<\alpha\leq 1,\ 0<\beta\leq 1,\ u\geq 0$$

has completely monotone derivatives (as well as the functions, for $b > 1$, $\psi_2(u) = \frac{\log(b+au^\alpha)}{log(b)}$, and $\psi_3(u) = \frac{(b+au^\alpha)}{b(1+au^\alpha)}$, for $0<b\leq 1$). Thus, we consider the Gneiting class of covariance functions

$$\begin{aligned} C_Z(z,\tau) \quad &= \quad \frac{1}{(a\tau^{2\alpha}+1)^{\beta d/2}} E_\nu\left(-\frac{z^{2\gamma}}{(a\tau^{2\alpha}+1)^{\beta\gamma}}\right). \\ & \quad z,\tau \geq 0, \quad \nu,\alpha,\beta,\gamma \in (0,1), \quad a>0. \end{aligned} \tag{53}$$

From (52), the following proposition is derived.

Proposition 2. *Under **Condition 1**, and for the Gneiting class of covariance functions introduced in (53), **Condition 2(ii)** holds if* $m=1$, *for* $0<2\alpha\beta(d/2-\gamma)<1$, *and for* $0<2\alpha\beta(d/2-\gamma)<1/2$ *if* $m=2$.

Proof. The proof follows straightforward from equations (52) and (53). Specifically, for $m=1$,

$$\begin{aligned} \sigma^2_{1,\mathcal{K}}(T) &= 2T^2|\mathcal{K}|^2 \int_{[0,1]} (1-\tau) \int_0^{\mathcal{D}(\mathcal{K})} C_Z(z,T\tau)\psi_{\rho_\mathcal{K}}(z)dzd\tau \\ &= 2T^2|\mathcal{K}|^2 \int_{[0,1]} (1-\tau) \int_0^{\mathcal{D}(\mathcal{K})} \frac{1}{(a[T\tau]^{2\alpha}+1)^{\beta d/2}} \\ & \qquad \times E_\nu\left(-\frac{z^{2\gamma}}{(a[T\tau]^{2\alpha}+1)^{\beta\gamma}}\right)\psi_{\rho_\mathcal{K}}(z)dzd\tau \\ &\geq 2T^2|\mathcal{K}|^2 \int_{[0,1]} (1-\tau) \int_0^{\mathcal{D}(\mathcal{K})} \frac{1}{(a[T\tau]^{2\alpha}+1)^{\beta d/2}} \\ & \qquad\qquad \times \frac{1}{\left[1+\Gamma(1-\nu)\frac{z^{2\gamma}}{[1+aT^{2\alpha}\tau^{2\alpha}]^{\beta\gamma}}\right]}\psi_{\rho_\mathcal{K}}(z)dzd\tau \end{aligned}$$

$$\begin{aligned}
&\geq 2T^2|\mathcal{K}|^2 \int_{[0,1]} (1-\tau) \int_0^{\mathcal{D}(\mathcal{K})} \frac{1}{(a[T\tau]^{2\alpha}+1)^{\beta d/2}} \\
&\qquad \times \frac{a^{\beta\gamma} T^{2\alpha\beta\gamma} \tau^{2\alpha\beta\gamma}}{\left[[1+aT^{2\alpha}\tau^{2\alpha}]^{\beta\gamma} + \Gamma(1-\nu) z^{2\gamma}\right]} \psi_{\rho_{\mathcal{K}}}(z) dz d\tau \\
&= 2T^{2\left(1-\alpha\beta\left(\frac{d}{2}-\gamma\right)\right)} |\mathcal{K}|^2 \int_{[0,1]} (1-\tau) \int_0^{\mathcal{D}(\mathcal{K})} \frac{1}{\left(a\tau^{2\alpha} + \frac{1}{T^{2\alpha}}\right)^{\beta d/2}} \\
&\qquad \times \frac{a^{\beta\gamma} \tau^{2\alpha\beta\gamma}}{\left[[1+aT^{2\alpha}\tau^{2\alpha}]^{\beta\gamma} + \Gamma(1-\nu) z^{2\gamma}\right]} \psi_{\rho_{\mathcal{K}}}(z) dz d\tau.
\end{aligned} \tag{54}$$

From (54), **Condition(ii)** holds for $\alpha\beta\left(\frac{d}{2}-\gamma\right) < 1$.

In a similar way to (54), it can be proved that for $m=2$, **Condition(ii)** also holds for $\alpha\beta\left(\frac{d}{2}-\gamma\right) < 1/2$. □

As a direct consequence of Proposition 2, we obtain that Theorems 2 and 3 hold for the family of spatiotemporal Gaussian random fields with covariance function (53).

Similar assertions hold for the family of spatiotemporal covariance functions

$$\begin{aligned}
\widetilde{C}_Z(\mathbf{z},\tau) &= \frac{\sigma^2}{[\psi(\tau^2)]^{d/2}} \varphi\left(\frac{\|\mathbf{z}\|^2}{\psi(\tau^2)}\right), \ \sigma^2 \geq 0, \ (\mathbf{z},\tau) \in \mathbb{R}^d \times \mathbb{R} \\
\varphi(u) &= \frac{1}{(1+cu^\gamma)^\nu}, \ u \geq 0, \ c > 0, \ 0 < \gamma \leq 1, \ \nu > 0 \\
\psi(u) &= (1+au^\alpha)^\beta, \ a > 0, \ 0 < \alpha \leq 1, \ 0 < \beta \leq 1, \ u \geq 0,
\end{aligned} \tag{55}$$

for $\alpha\beta\left(\frac{d}{2}-\gamma\nu\right) < 1$ if $m=1$, and for $\alpha\beta\left(\frac{d}{2}-\gamma\nu\right) < 1/2$ if $m=2$.

6. Discussion

As commented in the Introduction, the main contribution of this paper relies on deriving a general reduction principle in Theorem 1, beyond the regularly varying condition on the spatiotemporal covariance function of the underlying Gaussian random field. Hence, we can analyze a larger class of spatiotemporal covariance functions. Particularly, some examples of Gneiting class are considered (see Gneiting [13]). This class of covariance functions is popular in many applications, including Meteorology or Earth sciences, among others.

By considering homogeneous and isotropic Gaussian random fields restricted to a spatial convex compact set evolving over time, this paper applies an extrinsic random field approach, alternatively to the intrinsic spherical one adopted in Marinucci, Rossi and Vidotto [32]. Thus, the isonormal representation of the underlying spatiotemporal Gaussian random field on $\mathbb{R}^d \times \mathbb{R}$, and the characteristic function of the uniform probability distribution on a temporal interval and a spatial convex compact set allow the consideration of a continuous spectral based approach, in the derivation of limit results in our framework (see, e.g., Proposition 1).

A time–varying pure point spectral approach is considered in Marinucci, Rossi and Vidotto [32], based on projection onto the orthonormal basis of spherical harmonics. Different ranges of dependence are then assumed at different spatial resolution levels in the sphere. In our paper, under the temporal decay velocity of the space–time covariance function established in **Condition 2**, a general reduction principle is derived in Theorem 1, providing the limiting distribution of properly normalised integrals of non-linear transformations of spatiotemporal Gaussian random fields. Theorem 2 constitutes a particular case of Theorem 1, where the scaling also depends on the threshold u, that provides a similar scenario to Theorem 1 in Marinucci, Rossi and Vidotto [32], when zero-th order multipole component is long–memory, and all the other multipoles have asymptotically smaller variance.

Note also that Proposition 2 of this paper corresponds to the separable case in time and space which is different situation to the non–separable case addressed in Marinucci, Rossi and Vidotto [32]. Furthermore, Marinucci, Rossi and Vidotto [32] consider different orthonormal bases for space and time, respectively in terms of the spherical harmonics and the complex exponentials. These bases do not provide a diagonal spectral representation of the space–time covariance function of the underlying Gaussian random field. The composite Rosenblatt distribution then arises from the set of multipoles where the larger dependence range (long–memory) is displayed (as in reduction theorems). This non–diagonal representation induces a similar effect to considering Hermite rank $m = 2$, in the case of separable covariance functions in space and time, admitting a diagonal representation in terms of the complex exponentials in space and time (see Proposition 1).

In a subsequent paper, our results can be extended to the non–separable case, in terms of a bounded spatially varying long–memory parameter satisfying **Condition 2**(ii). This could be the case, for example, of an extended version of Proposition 1, in terms of non–separable covariance functions, involving a bounded spatial frequency varying $\alpha(\cdot)$ parameter. In that case, when the supremum of $\alpha(\cdot)$ over the spatial frequencies satisfies **Condition 2**(ii), Theorem 1 holds under **Condition 2**(i).

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