Cogent Economics & Finance

ISSN: (Print) 2332-2039 (Online) Journal homepage: https://www.tandfonline.com/loi/oaef20

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To cite this article: Nneka Umeorah & Phillip Mashele | (2019) A Crank-Nicolson finite difference approach on the numerical estimation of rebate barrier option prices, Cogent Economics & Finance, 7:1, 1598835, DOI: 10.1080/23322039.2019.1598835

To link to this article: https://doi.org/10.1080/23322039.2019.1598835

Published online: 04 Apr 2019.

Umeorah & Mashele, *Cogent Economics & Finance* (2019), 7: 1598835
https://doi.org/10.1080/23322039.2019.1598835

cogent economics & finance

FINANCIAL ECONOMICS | RESEARCH ARTICLE

A Crank-Nicolson finite difference approach on the numerical estimation of rebate barrier option prices

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Received: 22 October 2018
Accepted: 15 March 2019
First Published: 25 March 2019

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Reviewing editor:
David McMillan, University of Stirling, Stirling, UK

Additional information is available at the end of the article

Abstract: In modelling financial derivatives, the pricing of barrier options are complicated as a result of their path-dependency and discontinuous payoffs. In the case of rebate knock-out barrier options, discount factors known as rebates are introduced, which are payable to the option holder when the barrier level is breached. The analytical closed-form solution for the vanilla options are known but the barrier options, owing to their discontinuous nature, can be obtained analytically using the extended Black-Scholes formula. This research work captures the solution of the corresponding option pricing partial differential equation on a discrete space-time grid. We employ the Crank-Nicolson finite difference scheme to estimate the prices of rebate barrier options, as well as to discuss the effect of rebate on barrier option values. This work will further investigate the spurious oscillations which arise from the sensitivity analysis of the Greeks of the barrier options using the Crank-Nicolson scheme. The theoretical convergence of the Crank-Nicolson discretisation scheme will be analysed. Furthermore, our research will compare the results from

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PUBLIC INTEREST STATEMENT

Barrier options are path-dependent exotic options whose terminal values depend on the movement of the underlying asset prices against specified barrier levels. Investors can use them to reduce potential risks in the market and the option either becomes valid (knocks-in) or void (knocks-out) once the barrier is breached. They are very attractive to the investors because with the rebates, some compensations are obtained when the contract terminates prematurely. The values of these financial contracts (options) are to be known and paid upfront before the contract can be sealed by the parties involved. In our research, we used the Crank-Nicolson method (CNM) to numerically estimate the prices of these barrier options and then compared these numerical values to the analytical prices. We further analysed the sensitivity and the convergence scheme of the CNM. Finally, we observed that there exists a positive correlation between the rebates and the premium of these options.

the extended Black-Scholes model based on continuous time monitoring, together with the finite difference results from the Crank-Nicolson method.

Subjects: Analysis - Mathematics; Computational Numerical Analysis; Mathematical Economics

Keywords: Black-Scholes model; barrier options; rebate barrier options; spurious oscillations; Crank-Nicolson method; finite difference method

1. Introduction and literature review

Derivative securities over the years have offered investors an increased expected return, as well as a reduction in risk exposures. In comparison to other exotic options in the financial markets, the barrier options tend to be very popular. Their relative cheapness and marketability offer investors a more flexible approach to hedging and speculation. Speculators can choose a variety of barrier options that can help them to monitor possible asset price movements and this will, in turn, reduce their potential loss. For example, if a speculator perceives that the underlying asset price will stay at a specific price range, then the knock-out barrier option will offer more profit potential compared to the vanilla option. If they perceive a significant fluctuation of the underlying price in such a way that the possibility of hitting a specified level is high, then the knock-in option becomes attractive. Barrier options are class of path-dependent exotic options whose payoffs depend on whether the underlying asset price reaches a specified barrier level.

Generally, they are classified as knock-in options or knock-out options. The former becomes activated once the underlying price reaches the barrier level, and the latter becomes extinguished if the barrier is breached. A lot of research work exists on barrier options, and we shall adopt the popular Black-Scholes model as the basis for our option valuation. Barrier options are generally priced using the partial differential equation (PDE) approach and the expectation approach. Consider Buchen (2012); he discussed the valuation of barrier options in the Black-Scholes framework, using different formats of Method of Images which originates from option pricing PDE. He applied the barrier condition in a continuous monitoring time. Merton (1973) obtained the exact prices for the down-and-out barrier options using the PDE approach. Broadie, Glasserman, and Kou (1997) applied the concept of simple continuity correction on the barrier level to approximate the values of discrete barrier options, and Boyle and Lau (1994) used the binomial method to value barrier options.

Based on the South African market, Kotzé (1999) gave a brief overview of the exotic and vanilla options that are traded mainly in the South African financial markets. He focused on continuously monitored barrier options, as well as discussing the risk parameters and the hedging of barrier options. Pelsser and Vorst (1994) and Ioffe and Ioffe (2003) employed the binomial method and the implicit finite difference methods (FDM) respectively, to value barrier options without a rebate. Our recent work in barrier option pricing considered a comparative study of the zero-rebate knock-out barrier options with European features, to the theoretical extended Black–Scholes prices (Umeorah & Mashele, 2018). We employed the antithetic Monte-Carlo simulation (MCS) and quasi MCS approach in the estimation of the option prices, which was an improvement of the standard MCS. The underlying asset prices were simulated using the log-normal concept of option pricing, the payoffs were discounted using the risk-free interest rates, the barrier condition was applied and finally, the mean of the discounted payoffs was obtained which gave the estimated option prices.

A significant problem of trading barrier option is that if the option is knocked-out or if the option refused to knock-in prior to the contract's expiration, the holder will face a greater risk of losing the value of the contract. Thus, the introduction of rebate serves as compensation to the option holder. Rebate is a positive discount which is paid to the option holder by the option writer in the event of the problem stated above. It can be a time dependent or a constant function (usually a specific

percentage of the underlying asset's value). Lots of research work had been done on the barrier option pricing, but without a rebate. The incorporation of rebates in barrier options give an avenue for more research. Few researchers like Rich (1994) and Le, Zhu, and Lu (2016) focused on barrier options with rebate. The former derived the closed-form solutions and the latter applied the concept of continuous Fourier sine transform to solve the option PDE.

This paper, however, incorporates the rebate features (paid at knock-out and at expiry) of the barrier options using the CN discretisation scheme. The significance of this research work is not to present a new formula for pricing the barrier options. However, we seek to estimate the prices of these exotic rebate barrier option (basing our estimated results on the theoretical prices found in existing works of literature) using the Crank-Nicolson (CN) method. We base the choice of this method on the fact that its accuracy is of second-order in space-time discretisation, as well as its unconditional stability in time. We will further investigate the convergence analysis and sensitivity analysis of the CN scheme in the barrier option pricing. We shall equally discuss and graphically explain the effects of rebate on the down-and-out barrier options. The organisation of this research work is as follows: Section 2 considers the problem valuation and the closed-form valuation of the rebate barrier options. Section 3 discusses the numerical approximations of the option pricing PDE by introducing the CN FDM, as well as examining the sensitivity analysis and the convergence analysis of the scheme. Section 4 outputs some results obtained and then discussion follows. Finally, Section 5 summarizes and concludes the study.

2. Valuation of barrier options

2.1. Problem formulation

Specific parameters like the underlying price S, the barrier level B, the time to expiration T, the current time t, the strike price K, the risk-free interest rate r, the inherent volatility σ, and the rebate R, all affect the price of a rebate barrier option. Consider an asset price dynamics that follows a geometric Brownian motion below:

$$\mathrm{d}S(t) = S(r\mathrm{d}t + \sigma \mathrm{d}W(t)), \tag{2.1}$$

where $W(t)$ is the standard Brownian motion. The above equation can be solved using Ito's calculus. Let $V(t,S)$ be the value of a non-dividend paying down-and-out (DO) barrier option which pays a rebate when the barrier is breached. Under the Black-Scholes framework (Black & Scholes, 1973), the option price $V(t,S)$ satisfies the Black–Scholes PDE below:

$$\frac{\partial V(t,S)}{\partial t} + rS\frac{\partial V(t,S)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 V(t,S)}{\partial S^2} = rV(t,S), \tag{2.2}$$

Subject to:

$$V(T,S) = \max\{S(T) - K, 0\}, \tag{2.3}$$

$$V(t,B) = R, \tag{2.4}$$

$$V(t,\infty) = S - K\mathrm{e}^{-r(T-t)}. \tag{2.5}$$

We express the domain of the above PDE as $\mathcal{D} = \{(t,S) : B \leq S \leq \infty,\ t \in [0,T]\}$. Equation (2.3) is the payoff; Equation (2.4) occurs when the barrier is breached and finally, (2.5) when the asset price becomes very large. Applying the risk-neutrality concept of option pricing, we obtain the solution of Equation (2.2) as

$$V(t,S) = \mathrm{e}^{-r(T-t)}\int_K^\infty (S(T) - K)^+ f(T,S(T);t,S)\mathrm{d}S(T) + R\int_t^T \mathrm{e}^{-r(p-t)} g(p;B,t,S)\mathrm{d}p, \tag{2.6}$$

where f is the density function for the underlying and g is the first passage time density of the underlying S, at which a downstream barrier B is first hit by the Brownian motion $W(t)$. The function

$g(p;B)\mathrm{d}p = \mathbb{P}(\tau_B \in \mathrm{d}p)$, where $\tau_B = \inf\{t : S(t) = B\}$. The first part of the integral occurs when the barrier is not breached and the second follows from the rebate features. The rebate payment is expected to be redeemed over the interval $[p, p + \mathrm{d}p]$. The functions f and g are known and the integrals above can be valued by applying the concept of reflection principle and method of images (Yue-Kuen, 1998).

2.2. Closed-form valuation

The closed-form solution for the vanilla options are known, but the closed-form prices for the barrier option and the rebate barrier option are valued using the extended Black-Scholes pricing formula, as given in Equation (2.7). Let ZRDO represent the value of the zero-rebate down-and-out call option, RDOE is the value of down-and-out call option that pays a rebate $Re^{-r(T-t)}$ at expiry, and RDO is the down-and-out call option that pays a rebate R at knock-out. Then, the closed-form valuation for RDO call option, defined on continuous time monitoring is known[1] and it is given as

$$\begin{aligned}\mathrm{RDO} = SN(d_1) - Ke^{-r(T-t)}N(d'_1) - \left[S\left(\frac{B}{S}\right)^{2\lambda}N(d_2) - Ke^{-r(T-t)}\left(\frac{B}{S}\right)^{2\lambda-2}N(d'_2)\right] \\ + R\left[\left(\frac{B}{S}\right)^{2\lambda-1}N(d_3) + \left(\frac{S}{B}\right)N(d_3 - 2\lambda\sigma\sqrt{T-t})\right],\end{aligned} \tag{2.7}$$

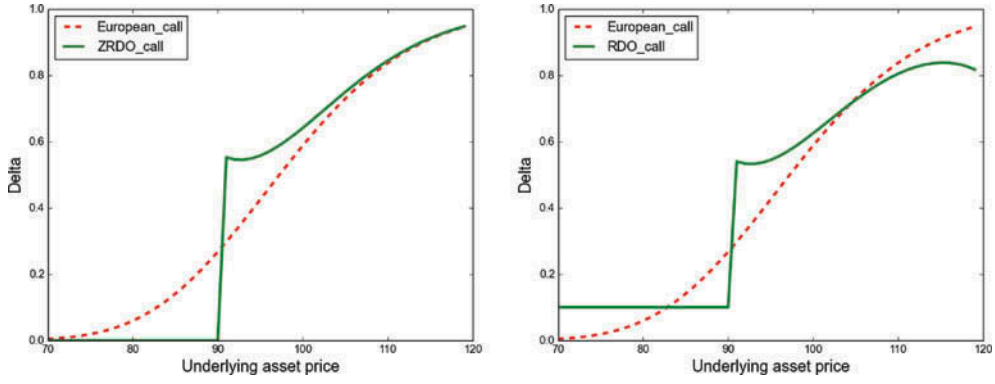
where

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad d_2 = \frac{\ln\left(\frac{B^2}{SK}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_3 = \frac{\ln\left(\frac{B}{S}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \quad d'_{1,2} = d_{1,2} - \sigma\sqrt{T-t} \quad \text{and} \quad \lambda = \frac{2r + \sigma^2}{2\sigma^2}.$$

The above conditions occur for $K \geq B$. For $K < B$, substitute $K = B$ in d_1 and d_2 above. If moreover, $R = 0$ and $R = Re^{-r(T-t)}$, then Equation (2.7) outputs the values of ZRDO and RDOE, respectively. From Equation (2.7), we can deduce that the value of a rebate down-and-out call which pays a rebate immediately at knock-out is equal to the value for ZRDO + American cash-or-nothing put with payoff R and strike B. Similarly, the value of up-and-out call that pays a rebate immediately at knock-out is equal to the value for ZRDO + American cash-or-nothing call option with payoff R and strike B. Thus, the rebate term in the barrier options with a constant rebate is the same as the value of the American digital option which pays a specific amount once the underlying price crosses a specified value. The rebate is always less than the premium paid upfront, and once the barrier is triggered before option's expiration, there is always a loss for the option holder. The rebate barrier option is American in nature since early exercise can be optimal.

Figure 1. Discontinuities of the delta for barrier options.



(a) Delta for European call and ZRDO call

(b) Delta for European call and RDO call

If the underlying is heading towards the barrier and the option is deep-in-the-money, then it can be worthwhile to exercise.

2.3. Delta for rebate barrier options

In risk management, we always encounter problem in hedging barrier options owing to their discontinuities at the barrier. Figure 1 shows the delta of the rebate knock-out barrier options. The parameters used are $K = 100, B = 90, r = 0.08, \sigma = 0.25, T = 0.25$. As soon as the barrier is breached by the underlying, the curve twirled, thus becoming discontinuous. Below the barrier, the delta of the ZRDO call is 0 and that of the RDO call remains positively non-zero as a result of the rebate features.[2] We also observed that as the underlying becomes increasingly large, the effect of the barrier becomes insubstantial and thus, the delta of the ZRDO call option gradually coincides with the delta of the European call option, as shown in Figure 1a. Moreover, when the rebate is 0, Figure 1b becomes 1a. The exotic nature of the barrier options is noted in the discontinuity of the curve at the barrier which affects the payoffs and thus complicates the hedging of these options.

3. Finite difference methods

In pricing financial derivatives using numerical methods, the PDE approach like the FDM, the binomial or the trinomial approach and the MCS are basically employed (Brandimarte, 2013). The FDM for solving the Black–Scholes PDE which describes the exotic option pricing involves solving the associated PDE on a discrete space-time grid. The computational domain is $[0, S_{max}] \times [0, T]$ and we discretise this domain by the uniform asset mesh and time mesh with steps ΔS and Δt. The payoff at time T is known and hence the solution involves applying the concept of backward iteration on the square or rectangular grid up until time $t = 0$. With regards to the Black-Scholes formula, the option price is a function of the underlying price and time, and this can be obtained via iteration.

Consider the discretisations below:

$$S = 0, \Delta S, 2\Delta S, \cdots, (m-1)\Delta S, m\Delta S = S_{\max} \quad \text{and}$$

$$t = 0, \Delta t, \Delta t, \cdots, (n-1)\Delta t, n\Delta t = T$$

Let $S_k = k\Delta S$ and $T_n = i\Delta t$, then the option price $V(t, S)$ can be denoted in grid form by $V_{i,k} = V(t_i, S_k)$, where $k = 0, 1, 2, \cdots, m$ and $i = 0, 1, \cdots, n$. Let $S_{\max}$ be the largest value that the underlying can possibly have. The corresponding terminal and boundary conditions of the PDE which give the values of the option prices at time $t = T$, $S = 0$ and $S = S_{\max}$ are known. Thus, it suffices to use the known values at the extreme end of the nodes to calculate the values for the other interior nodes.

3.1. Terminal and boundary conditions

Rebate knock-out barrier options: The terminal and boundary conditions of the DO barrier call options with rebate at knock-out, represented in Equations (2.3), (2.4) and (2.5) can be written in discrete form as follows:

$$V_{n,k} = \max\{k\Delta S - K, 0\}, \tag{3.1}$$

$$V_{i,B} = R, \tag{3.2}$$

$$V_{i,m\Delta S} = m\Delta S - Ke^{-r(n-i)\Delta t}. \tag{3.3}$$

For the rebate paid at expiry, all other conditions remain the same except for Equation (3.2) which becomes $Re^{-r(n-i)\Delta t}$.

3.2. Crank-Nicolson FDM

John Crank and Phyllis Nicolson developed the Crank-Nicolson method as a numerical solution of a PDE which arises from the heat-conduction problems (Crank & Nicolson, 1996). It was introduced to curb the instability, as well as to increase the efficiency and the accuracy of the implicit and the

explicit method. Consider the Black–Scholes PDE defined in Equation (2.2). The discretised PDE for the implicit FDM or the forward difference method is given as

$$\frac{V_{i,k}-V_{i-1,k}}{\Delta t}+rk\Delta t\left[\frac{V_{i-1,k+1}-V_{i-1,k-1}}{2\Delta S}\right]+\frac{(\sigma k\Delta S)^2}{2}\left[\frac{V_{i-1,k+1}-2V_{i-1,k}+V_{i-1,k-1}}{\Delta S^2}\right]=rV_{i-1,k}. \tag{3.4}$$

Let the approximations of the underlying at node $(i-1,k)$ and the node (i,k) be the same (Hull, 2006), then the discretised PDE for the explicit FDM or the backward difference method is

$$\frac{V_{i,k}-V_{i-1,k}}{\Delta t}+rk\Delta S\left[\frac{V_{i,k+1}-V_{i,k-1}}{2\Delta S}\right]+\frac{(\sigma k\Delta S)^2}{2}\left[\frac{V_{i,k+1}-2V_{i,k}+V_{i,k-1}}{\Delta S^2}\right]=rV_{i,k}. \tag{3.5}$$

The Crank-Nicolson (CN) method thus aims at combining and averaging the forward and the backward difference method, using the same boundary conditions. Taking the average and re-arranging Equations (3.4) and (3.5), we have

$$V_{i-1,k-1}\left[\frac{-rk\Delta t}{4}+\frac{\sigma^2k^2\Delta t}{4}\right]+V_{i-1,k}\left[-1-\frac{\Delta t}{2}(\sigma^2k^2+r)\right]+V_{i-1,k+1}\left[\frac{rk\Delta t}{4}+\frac{\sigma^2k^2\Delta t}{4}\right]$$
$$=V_{i,k-1}\left[\frac{rk\Delta t}{4}-\frac{\sigma^2k^2\Delta t}{4}\right]+V_{i,k}\left[-1+\frac{\Delta t}{2}(\sigma^2k^2+r)\right]+V_{i,k+1}\left[\frac{-rk\Delta t}{4}-\frac{\sigma^2k^2\Delta t}{4}\right]$$

This can be written as

$$-\lambda_kV_{i-1,k-1}+(-1-\beta_k)V_{i-1,k}-\eta_kV_{i-1,k+1}=\lambda_kV_{i,k-1}+(-1+\beta_k)V_{i,k}+\eta_kV_{i,k+1} \tag{3.6}$$

for $i=n-1,n-2,\cdots,1,0$ and $k=1,2,\cdots,m-1$, where

$$\lambda_k=\frac{\Delta t}{4}[rk-\sigma^2k^2],\qquad \beta_k=\frac{\Delta t}{2}(\sigma^2k^2+r)\qquad \text{and}\qquad \eta_k=\frac{-\Delta t}{4}[rk+\sigma^2k^2].$$

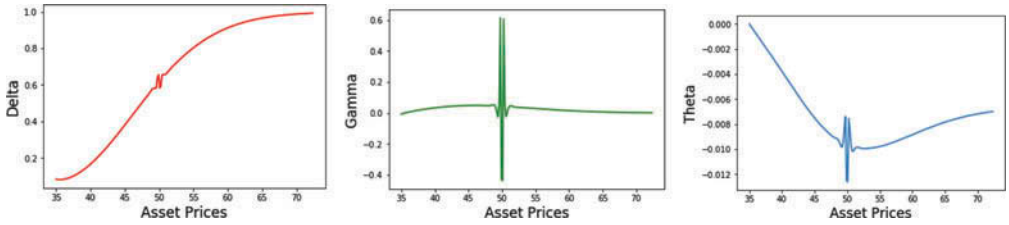
CN discretisation results in tridiagonal scheme which is solvable at each time step. The CN method provides the best approximate value in comparison to the other FDM, like the explicit and the implicit FDM (Hull, 2006).

3.3. Sensitivity analysis of the CN scheme

We observe some spurious oscillations when the coefficient of the diffusion term of the Black–Scholes PDE is minimal or when the coefficient of the advection term of the PDE is substantial, or when both occurs (Duffy, 2004). One of the critiques of the CN scheme on barrier option pricing is the fact when the barrier condition is applied; it induces discontinuities in the numerical solution. If we monitor the barrier options discretely, then discontinuities of the option values can exist at each monitoring dates. These discontinuities, however, result in spurious oscillations which affect the sensitivities of the option value. Spurious oscillations are known to affect the hedging parameters or the Greeks of the derivatives, even though the option values might appear to be correct and smooth. Figure 2 below shows the plot for the Greeks of the continuously monitored barrier options with zero rebate, using the CN scheme and Figure 3 considered the same Greeks but discretely monitored.

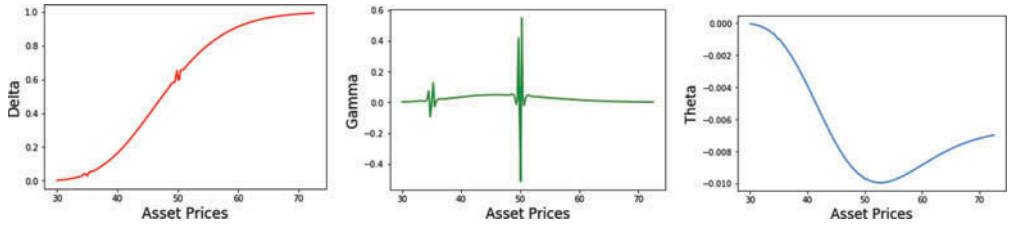
Figures 2 and 3 consider the parameters: $S=60,K=50,B=35,r=0.05,\sigma=0.2,T=0.75,m=150,n=25$ and $S_{\max}=140$. The figures show oscillations at the strike, owing to the discontinuous payoffs. The delta for the barrier option approaches one as the underlying moves increasingly away from the barrier position. Gamma, on the other hand, tends to assume the highest value when the option is at-the-money. However, when we price the discrete barrier options, the Greeks exhibit oscillations at the strike and also close to the barrier, as observed in Figure 3. Similar findings were obtained by Tavella and Randall (2000), which explained that when $\Delta t\geq\tau_d$, where $\tau_d=\frac{(\Delta S)^2}{(\sigma S)^2}$ is the characteristic grid diffusion, then spurious oscillations close to the barrier arise for

Figure 2. Greeks for continuously monitored barrier options.



(a) Plot of Delta (b) Plot of Gamma (c) Plot of Theta

Figure 3. Greeks for discretely monitored barrier options with 5 monitoring periods.



(a) Plot of Delta (b) Plot of Gamma (c) Plot of Theta

the discretely monitoring case. When we choose our parameters such that $\Delta t \geq \tau_d$ as depicted in the figures below, oscillations are observed.

To correct this problem, however, series of research had been conducted. Pooley, Vetzal, and Forsyth (2003) applied the concept of averaging the initial data, grid shifting and projection method to smoothen the discontinuities. Khaliq, Voss, and Yousuf (2007) developed a strong stable (L-Stable) Padé scheme on exotic options with discontinuous payoffs. Duffy (2001) developed a more robust difference scheme, also known as the exponentially fitted scheme which solves the general two-point boundary value problem, with applications to the Black–Scholes PDE. Rannacher suggested a modification of the CN method from the idea of smoothing the discontinuities in the initial and at the barrier conditions for barrier option pricing (Giles and Carter (2006) and Rannacher (1984)). The idea of Rannacher time stepping involves the discretisation of the initial timesteps using backward Euler integration, with the aim of recovering the second order convergence of the CN scheme.

Furthermore, Zvan, Forsyth, and Vetzal (1998) explained that the presence of spurious oscillations which results from the use of the central weighting scheme can be avoided if the Peclet conditions are satisfied:

$$\frac{1}{\Delta S_{i-\frac{1}{2}}} > \frac{r}{\sigma^2 S_i}$$

and

$$\frac{2}{\Delta t} > \frac{\sigma^2 S_i^2}{2}\left(\frac{1}{\Delta S_{i-\frac{1}{2}}\Delta S_i} + \frac{1}{\Delta S_{i+\frac{1}{2}}\Delta S_i}\right) + r,$$

where $\Delta S_{i+\frac{1}{2}} = S_{i+1} - S_i$ and $\Delta S_i = \frac{S_{i+1} - S_{i-1}}{2}$.

The two conditions guarantee the positivity of the solution and the maximum principle has to be applied to ensure that if the initial and boundary conditions of a continuous system in a given domain is positive, then the solution in the interior domain will remain positive (Duffy, 2001). The positivity of the domain will further make the parameters λ_k in Equation (3.6) to stay

strictly positive, and thus, Fusai, Sanfelici, and Tagliani (2002) explained that the positive eigenvalues of a given iteration matrix is a guarantee against such oscillations.

In this research, we employ the special time-step restriction on the CN method to avoid the emergence of spurious oscillations. We first applied the concept of Tavella and Randall (2000) with regards to the characteristic grid diffusion, and then applied the concept of maximum principle. With CN method, we observe the presence of some numerical spikes which do not quickly decay as a result of the discontinuities at the payoffs, monitoring dates (for discrete monitoring barriers) and the barrier conditions. The spikes do not show the instability of the CN scheme, but they explain that the sharp gradient which results when the option value is set to R, for $S \leq B$ on the rebate barrier options, takes time to decay completely in the CN scheme. Consider re-writing Equation (3.6) in matrix form as

$$WV_{i-1,k} = XV_{i,k}, \qquad \text{where}$$

$$W = \text{triad}\left\{\frac{\Delta t}{4}\left(\frac{rS_k}{\Delta S} - \left(\frac{\sigma S_k}{\Delta S}\right)^2\right); 1 + \frac{\Delta t}{2}\left(\left(\frac{\sigma S_k}{\Delta S}\right)^2 + r\right); \frac{-\Delta t}{4}\left(\frac{rS_k}{\Delta S} - \left(\frac{\sigma S_k}{\Delta S}\right)^2\right)\right\}$$

$$X = \text{triad}\left\{\frac{-\Delta t}{4}\left(\frac{rS_k}{\Delta S} - \left(\frac{\sigma S_k}{\Delta S}\right)^2\right); 1 - \frac{\Delta t}{2}\left(\left(\frac{\sigma S_k}{\Delta S}\right)^2 + r\right); \frac{\Delta t}{4}\left(\frac{rS_k}{\Delta S} - \left(\frac{\sigma S_k}{\Delta S}\right)^2\right)\right\},$$

with $k = 0, \cdots, m$ and $i = 0, \cdots, n$.

We impose the following conditions found in (Milev & Tagliani, 2010), which explains that if the discretisation scheme strictly follows the hypothesis $\frac{\sigma^2}{r} > 1$, then the following properties are satisfied:

- The solution is positive, that is, $V_{i-1,k} > 0$ and with $r < \sigma^2$, we have $\frac{rS_k}{\Delta S} - \left(\frac{\sigma S_k}{\Delta S}\right)^2 < 0$. Furthermore, the matrix W is an irreducible row diagonally dominant $M-$ matrix, with properties $W^{-1} > 0$, $||W^{-1}||_\infty \leq \frac{1}{1+\frac{r\Delta t}{2}}$.
- The maximum principle is satisfied with $X \geq 0$ and $||X||_\infty = 1 - \frac{r\Delta t}{2} > 0$. That is,

$$||V_{i-1,k}||_\infty = ||W^{-1}XV_{i,k}||_\infty = ||W^{-1}||_\infty ||X||_\infty ||V_{i,k}||_\infty \leq \frac{1 - \frac{r\Delta t}{2}}{1 + \frac{r\Delta t}{2}} ||V_{i,k}||_\infty \leq ||V_{i,k}||_\infty .$$

- Together with the time-step restriction $\Delta t < \frac{2}{r + 2(\sigma m)^2}$, the eigenvalues of the iteration matrix W are real and distinct. The time-step becomes prohibitively small, with an increase in the asset space grid m. Thus, from Gerschgorin's theorem, the eigenvalues of the matrix

$$W^{-1}X \in \left(\frac{1}{1 + \frac{r\Delta t}{2}}; \frac{1}{1 + \Delta t\left(\frac{r}{2} + (\sigma m)^2\right)}\right) \subset (0, 1).$$

Hence, if the time restriction and $\sigma^2 > r$ are ignored, positivity of the solution does not ensue, and some eigenvalues will become negative or complex, leading to spurious oscillations.[3]

When we apply the special time step restriction described above to the CN scheme on the Greeks of the continuously monitored barrier options in Figure 2, then the resulting plots in Figure 4 are obtained. The discontinuities which occur at the payoff conditions were smoothed out, and this eliminates the spurious oscillations that were present before. Close to the strike price, gamma assumes the highest positive value and theta assumes the highest negative value. This is absolutely important in the construction of delta-neutral portfolio.

Figure 4. Greeks for continuously monitored barrier options without oscillations.

(a) Plot of Delta (b) Plot of Gamma (c) Plot of Theta

3.4. Convergence analysis

For simplicity, we transform the original Black–Scholes PDE in Equation (2.2) into heat equation of the form

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty, \quad \tau > 0, \tag{3.7}$$

using the change of parameters

$$x = \log S + (r - 0.5\sigma^2)(T - t) \quad \text{and} \quad \tau = 0.5\sigma^2(T - t), \quad \text{such that}$$

$$u(x, \tau) = e^{-r\left(T - \frac{2\tau}{\sigma^2}\right)} V\left(e^{x - \left(\frac{2r}{\sigma^2} - 1\right)\tau}, T - \frac{2\tau}{\sigma^2}\right).$$

Let $u_{i,j} \approx u(x_i, \tau_j)$, where $i = 1, \cdots, M$ and $j = 1, \cdots, N$.

Applying the CN discretisation scheme on Equation (3.7) results to

$$\frac{u_{i,j+1} - u_{i,j}}{\Delta\tau} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{2\Delta x^2} + \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2\Delta x^2} \tag{3.8}$$

Substituting $\zeta = \frac{\Delta\tau}{2\Delta x^2}$ into Equation (3.8) and rearranging gives the difference equation:

$$Au_{i,j+1} = Bu_{i,j}, \tag{3.9}$$

where $A_{M-1\times M-1} = \text{tridiag}\{-\zeta, 1 + 2\zeta, -\zeta\}$ and $B_{M-1\times M-1} = \text{tridiag}\{\zeta, 1 - 2\zeta, \zeta\}$.

Next, we use the Lax Equivalence theorem to analyse the convergence of the CN scheme. The theorem states that given a well-posed initial value problem, a consistent FDM is convergent if and only if the scheme is stable. In investigating the numerical stability of the CN scheme, we use the eigenvalues of the tridiagonal Toeplitz matrix (LeVeque, 2007). The two matrices A and B can be written inform of constant tridiagonal matrices, as $A = I + \zeta C$ and $B = I - \zeta C$, where $C = \text{tridiag}\{-1, 2, -1\}$. Thus, Equation (3.9) becomes $(I + \zeta C)u_{i,j+1} = (I - \zeta C)u_{i,j} = (2I - A)u_{i,j}$. Thus, we have $u_{i,j+1} = (2A^{-1} - I)u_{i,j}$. Denote the eigenvalues of A with $\lambda_j(A)$. Since C satisfies the properties of Toeplitz matrix, the eigenvalues are given by:

$$\lambda_j(A) = \lambda_j(I) + \zeta.\lambda_j(C)$$
$$= 1 + 4\zeta \sin^2\left(\frac{\pi j}{2M}\right), \qquad \text{for } j = 1, \cdots, M - 1.$$

For stability, we ensure that for all j

$$\left|\frac{2}{\lambda_j(A)} - 1\right| < 1 \Rightarrow \left|\frac{1 - 4\zeta \sin^2\left(\frac{\pi j}{2M}\right)}{1 + 4\zeta \sin^2\left(\frac{\pi j}{2M}\right)}\right| < 1.$$

Since all the eigenvalues of A are greater than 1, the stability condition of the CN scheme is guaranteed and hence, for all $\zeta > 0$ and with $\Delta\tau > 0$. We observe the consistency of the CN scheme in the analysis of the local truncation errors. LeVeque (2007) explains that the scheme exhibits quadratic convergence in both space and time, that is, the truncation error is $u(x,\tau) = (x_i,\tau_j) = \mathcal{O}(i^2 + j^2)$. Thus, by Lax Equivalence theorem, the CN is convergent and the numerical experiments are displayed in the result section.

4. Results and discussion

The results computed in this section considered the non-dividend rebate down-and-out call options, with European features. In the numerical computation, we obtain our results using the program ipython notebook. Let N be the discretisation steps of time; M, the discretisation steps for the underlying asset and let CNV be the Crank-Nicolson values obtained. We use the following parameters below to output the results in Table 1:

$S = 50, S_{\max} = 140, K = 40, B = 20, r = 0.04, \sigma = 0.3$ and $T = 0.5$. The exact value of the option with rebate of 2.5 paid at knock-out is obtained using Equation (2.7), and the value is 11.3777.

In Table 1, the option pays a rebate value of 2.5 when the barrier is breached. We observe the effect of increasing the space-time discretisation steps on the values obtained using the Crank-Nicolson method. As space and time steps increase simultaneously, the observed values converge to the true solution. In the FDM, the choice of $S_{\max}$, (maximum underlying price) which is an artificial limit is yet to be known and a proper choice will lead to a faster convergence and more accurate result. Finally, we observe that the computation time for the execution of the finite difference algorithm increases with an increase in the discretisation steps.

In Table 2, we compare the values of rebate at knock-out barrier options obtained using both the CN method, MCS and the AMCS. The parameters $S, S_{\max}, K, B, r, \sigma, R$ and T, remain the same as in Table 1. We differ the asset prices and restrict the asset and time steps to be $N = M = 400$ for the CN method. We further considered 100000 simulations with 400 time steps. The values are seen in the following table:

In Table 2, we observe that the values obtained using the CN method are fairly close to the exact values. The probability of the values from the CN method approaching the exact value becomes high when the asset and the time-step sizes are increased. With regards to the simulated values from the MCS and the AMCS, the results are relatively accurate in comparison to the exact values. The AMCS performs better than the MCS and this is based on the fact that the former is a variance reduction method.

Table 1. Crank-Nicolson values on knock-out barrier options with rebate at knock-out

N	M	CNV	Computation time (secs)
150	150	11.4090	0.1722
200	200	11.3875	0.3572
250	250	11.3818	0.7278
300	300	11.3787	1.1481
350	350	11.3786	2.3930
400	400	11.3780	3.8790
450	450	**11.3777**	6.0094
500	500	**11.3777**	8.7626

Table 2. Rebate at knock-out barrier option values using CN, MCS and the antithetic MCS

S	Exact	CNV	MCS	AMCS
70	30.8026	30.5945	30.8960	30.8853
65	25.8226	25.7657	25.7487	25.7761
60	20.8777	20.8655	20.8328	20.8870
55	16.0225	16.0208	16.0336	16.0103
50	11.3777	11.3780	11.3847	11.3768
45	7.1737	7.1755	7.1558	7.1622
40	3.7590	3.7635	3.7883	3.7408
35	1.4876	1.4866	1.4655	1.4670

For Table 3, we consider the parameters below: $S = 100, B = 60, K = 100, r = 0.08, \sigma = 0.1, T = 0.5$ and $S_{max} = 260$. The exact value of the rebate barrier option with rebate ($R = 4\%$ of the underlying) at knock-out is 5.1563 (using Equation (2.7)). The outputs obtained are displayed below:

Table 3 considered the non-dividend DO rebate call option, with rebate paid at knock-out. It compares the values obtained using the Crank-Nicolson method with differing discretisation steps. It also outputs some results using the same increasing step sizes and different increasing step sizes. We observe equality (up to 4 dp) in the estimated values when the time steps are twice that of asset steps and when the asset steps are twice that of time steps. The convergence is very slow, but when the asset steps and time steps are the same and increasing, the estimated values converge to the true solution. Here, at $\mathbf{N = M = 500}$.

Figure 5 compares the values of the zero-rebate DO; rebate DO which pays rebate at knockout and rebate DO call options with rebate at expiry, to the vanilla call options. The parameters considered here are $B = 120, K = 125, r = 0.06, \sigma = 0.5, T = 2$ and a rebate $R = 5\%$ of the underlying. Since call option is been applied to all, we observe a decrease in the option value as the underlying price decrease. The vanilla call option is generally more expensive than the barrier options since the former is a combination of the knock-in and the knock-out barrier components.[4] We positioned the barrier at $B = 120$ and we observe that as the underlying price is tending towards the barrier, the probability of it being knocked-out is increased. The ZRDO call option pays nothing when the barrier is finally breached. For the rebate barrier options, their values are generally more expensive compared to the ZRDO options because of the presence of the rebate terms. Both the RDO and the RDOE pay 5% of the underlying as a rebate at knock-out and at expiry respectively. The rebate term is added to the ZRDO option value to give the final option value for the RDO and the RDOE. The value of the RDOE is always cheaper than that of the RDO because the rebate term of the former is always discounted at a risk-free interest rate.

Table 3. Crank-Nicolson values with different increasing step sizes on DO rebate call options

N	M	CNV	N	M	CNV	M	N	CNV
300	600	5.1556	300	300	5.2787	300	600	5.1556
325	650	5.1556	325	325	5.2393	325	650	5.1556
350	700	5.1557	350	350	5.2057	350	700	5.1557
375	750	5.1558	375	375	5.1832	375	750	5.1558
400	800	5.1558	400	400	5.1700	400	800	5.1558
425	850	5.1559	425	425	5.1628	425	850	5.1559
450	900	5.1559	450	450	5.1590	450	900	5.1559
475	950	5.1560	475	475	5.1572	475	950	5.1560
500	1000	5.1560	500	500	5.1563	500	1000	5.1560

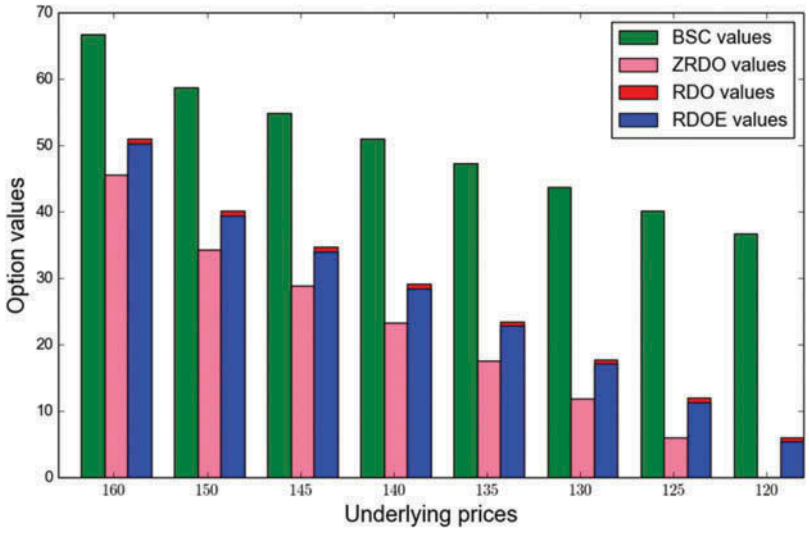


Figure 5. Comparing barrier option prices with European Black-Scholes call values on different underlying prices.

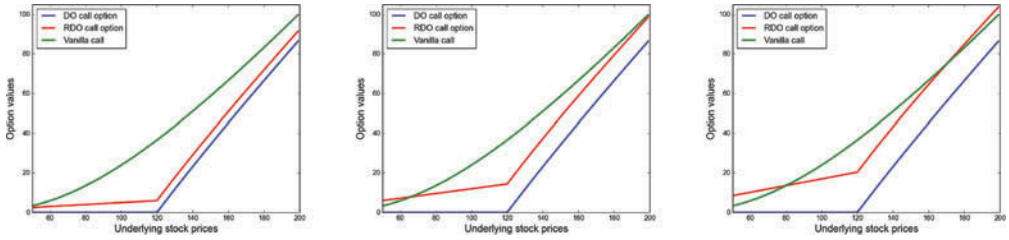
Next, we consider the effect of increase in rebates on barrier option values. Table 4 depicts the effects of an increase in rebates on option values. The parameters considered here are $B = 120, K = 125, r = 0.06, \sigma = 0.5, T = 2$ and different increasing rebate terms. The ZRDO option values remain the same at each step of the underlying. Increasing the rebate (i.e., 5%, 12%, and 17%) increases the rebate value, which in turn increases the option values. The above result is in harmony with Zhang (1998) who explained that the presence of large rebates will tend to de-leverage the transaction and thus, leads to more expensive option prices. We observe that the rebate term for the RDOE is lesser than that of RDO and this makes the value of the RDOE to be cheaper than that of RDO. We also observed that at knock-out, the RDO pays a specified percentage of the underlying, but it has to be discounted if the rebate is paid at expiry. For instance, consider when $S = B = 120$ and suppose we choose a rebate of 5%, then the rebate value becomes 6.0000 at knock-out. For the rebate at expiry, the rebate term becomes $Re^{-r(T-t)} = 5.3215$. We add these values are added to the value of the ZRDO to get the true rebate barrier option value.

These are illustrated graphically in Figures 6 and 7. The green and the blue curve denote the European call option value and the ZRDO call options, respectively. Both are constant with respect to the rebate since they are unaffected by the rebate values. The red curve denotes the RDO and

Table 4. Effect of increase in rebates on option values

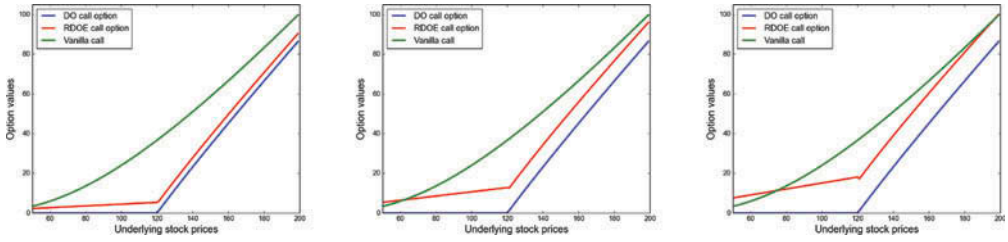
S	Zero rebate option value	Rebate terms					
		Rebate at knockout			Rebate at expiry		
		5%	12%	17%	5%	12%	17%
200	87.3962	5.0691	12.1659	17.2350	4.4959	10.7902	15.2861
190	77.0044	5.2365	12.5676	17.8041	4.6443	11.1464	15.7908
180	66.5247	5.3973	12.9535	18.3507	4.7870	11.4887	16.2756
170	55.9353	5.5484	13.3161	18.8644	4.9210	11.8103	16.7313
160	45.2082	5.6860	13.6464	19.3324	5.0430	12.1033	17.1463
150	34.3070	5.8057	13.9338	19.7395	5.1492	12.3582	17.5074
140	23.1841	5.9023	14.1654	20.0677	5.2348	12.5636	17.7985
130	11.7765	5.9694	14.3266	20.2960	5.2944	12.7065	18.0009
120	0.0000	6.0000	14.4000	20.4000	5.3215	12.7717	18.0932

Figure 6. Effect of increase in rebates on option value: Rebate at knock-out.



(a) Rebate: 5% of S (b) Rebate: 12% of S (c) Rebate: 17% of S

Figure 7. Effect of increase in rebates on option value: Rebate at expiry.



(a) Rebate: 5% of S (b) Rebate: 12% of S (c) Rebate: 17% of S

RDOE option values at Figures 6 and 7 respectively, and furthermore, increasing the rebate terms results in the shifting of the ZRDO option value, as denoted by the red curve. Figure 6 occurs when the rebate at knock-out and Figure 7, when the rebate is paid is expiry.

5. Conclusion

This research work had considered the numerical valuation of the rebate barrier options which pays a rebate at knock-out, as well as, at expiry. We employed the Crank-Nicolson finite difference methods in the discretisation of the option pricing PDE, which was solved using the corresponding boundary and the terminal value conditions. We observed that the increase in the discretisation steps of both the underlying asset and time resulted in the convergence of the estimated values to the true solution. Thus, path-dependent options like the barrier options can be successfully priced using the Crank-Nicolson method. With regards to the effect of a rebate, there is a positive correlation between the premium and the rebate, as the premium entirely depends on the size of the specified rebate. We equally observed that since the rebate assumes a positive value, it serves as compensation to the investors and as such, they are attractive to investors since they do not lose out in event of an early knock-out. We observed the presence of spurious oscillations when the Greeks of the barrier options are analysed using the CN scheme. This research, however, applied a special time step restriction to dissolve the existing numerical spikes.

Future work will consider a higher order numerical approach, which handles the discontinuities on the valuation of barrier options. We equally aim at extending the numerical valuation to double barrier options.

Acknowledgements
The authors wish to acknowledge the financial support from the Deutscher Akademischer Austauschdienst (DAAD) in collaboration with the African Institute for Mathematical Sciences (AIMS), South Africa and the Centre for Business Mathematics and Informatics, North-West University, Potchefstroom, South Africa.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: A Crank-Nicolson finite difference approach on the numerical estimation of rebate barrier option prices, Nneka Umeorah & Phillip Mashele, *Cogent Economics & Finance* (2019), 7: 1598835.

Notes

1. See Derman and Kani (1997); Douady (1999).
2. At Figure 1b, we choose the rebate value to be the value of delta at 0.1.
3. For analysis of $\sigma^2 < r$, see (Milev & Tagliani, 2013).
4. This occurs only when the zero-rebate barrier options are considered.

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***Cogent Economics & Finance* (ISSN: 2332-2039) is published by Cogent OA, part of Taylor & Francis Group.**