Journal of Philosophical Logic
https://doi.org/10.1007/s10992-022-09672-6

# Elementary Belief Revision Operators

Jake Chandler[1] · Richard Booth[2]

Received: 4 February 2022 / Accepted: 1 June 2022



## Abstract

Discussions of the issue of iterated belief revision are commonly accompanied by the presentation of three "concrete" operators: natural, restrained and lexicographic. This raises a natural question: What is so distinctive about these three particular methods? Indeed, the common axiomatic ground for work on iterated revision, the AGM and Darwiche-Pearl postulates, leaves open a whole range of alternative proposals. In this paper, we show that it is satisfaction of an additional principle of "Independence of Irrelevant Alternatives", inspired by the literature on Social Choice, that unites and sets apart our three "elementary" revision operators. A parallel treatment of iterated belief contraction is also given, yielding a family of elementary contraction operators that includes, besides the well-known "conservative" and "moderate" operators, a new contraction operator that is related to restrained revision.


**Keywords** Belief revision · Belief contraction · Iterated belief change · Irrelevant alternatives · Social choice

## 1 Introduction

One key unresolved question in the theory of belief dynamics is the proper handling of *iterated* revision: the computation of the impact on an agent's total set of beliefs of a *sequence* of successive local revisions.

There now exist a number of approaches to the issue (see for instance Section 5.2 of [14]). These are typically consistent with two popular sets of baseline principles, respectively proposed by Alchourrón, Gärdenfors and Makinson [1] and by Darwiche

This research was partially supported by the Australian Government through an ARC Future Fellowship (project number FT160100092) awarded to Jake Chandler.

✉ Jake Chandler
jake.chandler@cantab.net

1 La Trobe University, Melbourne, Australia

2 Cardiff University, Cardiff, UK

Published online: 01 December 2022 

& Pearl [13], which we shall henceforth call the "AGM" and the "DP" postulates. Quite strikingly, however, one finds, in the case of iterated *revision* at least, which is the most widely considered question, that a particular trio of proposals is ubiquitous in presentations of the issue: the lexicographic, restrained and natural revision operators respectively associated with Nayak et al. [22], Booth & Meyer [7] and Boutillier [9]. These notably make up three of the four iterated revision operators mentioned in Rott's influential survey [24]. (The remaining operator that he discusses, the irrevocable revision operator of [25], has the unusual and arguably undesirable characteristic of ensuring that the inputs to any revision are retained in the belief set after any subsequent revision).

It is not immediately obvious what the distinctive appeal of these particular three possibilities might be. For instance, they do share the feature of satisfying the AGM and DP postulates. But they are not alone in doing so. They are also alike in ensuring that the result of any future sequence of revisions is determined by a prior total preorder (TPO) over the set of propositional worlds, a property that we shall call "TPO-Reductionism". But again, so do other possible ways of proceeding, including some that satisfy the AGM and DP postulates. In this paper we identify precisely what it is that sets these three operators apart from the rest, by providing an additional principle that, in the presence of the AGM and DP postulates, is satisfied by them, and them alone.

In what follows, we first offer, in Section 2, some technical preliminaries that recapitulate some existing work on iterated belief change. There, we also introduce a novel tabular presentation of TPO-based belief dynamics, which simplifies both the subsequent exposition and its associated proofs. In Section 3, we then introduce a new property of "Independence of Irrelevant Alternatives", inspired by the literature on Social Choice. We show that this property, against the backdrop of the AGM and DP postulates, unites and sets apart the "elementary" lexicographic, restrained and natural revision operators. In the process, we also prove the soundness, for elementary revision, of a number of interesting further principles, including "Zero Symmetry" and "Representation Invariance". In Section 4, we consider what happens when these various principles are strengthened in obvious ways, noting that, in each case, the resulting strengthening leave us with lexicographic revision as a sole candidate. In Section 5 we offer a parallel discussion regarding the other main type of belief change operation discussed in the literature: belief *contraction*. As with revision, there exist a number of proposals that have been made, which typically satisfy a set of analogues of the DP postulates, initially proposed by Chopra et al. [12]. Here we consider an analogous characterisation of a family of elementary contraction operators. It turns out that this family includes the well known "conservative" and "moderate" contraction operators, as well as a new "restrained" contraction operator, for which we offer both semantic and syntactic characterisations. We close the section on contraction with a brief discussion of the relation between elementary revision and contraction operators from the vantage point of recent work on extensions of the Levi and Harper identities to the iterated case. Finally, we wrap up the paper with some comments on the possible weakening of one key characteristic principle of elementary revision.

With the exception of the proof of the main result, as well as that of Proposition 14, the proofs of the various propositions, lemmas and theorems have been relegated to a substantial technical Appendix A.[1]

## 2 Preliminaries

The beliefs of an agent are represented by a *belief state* $\Psi$. The latter determines a *belief set* $[\Psi]$, a deductively closed set of sentences, drawn from a propositional, truth-functional language $L$, generated by a finite set of $n$ atomic sentences. The set of classical logical consequences of $\Gamma \subseteq L$ will be denoted by $\mathrm{Cn}(\Gamma)$. When $\Gamma = \{C\}$, we write $\mathrm{Cn}(C)$. We write $A \equiv B$ for $A \leftrightarrow B \in \mathrm{Cn}(\varnothing)$ and $A \equiv_C B$ for $A \leftrightarrow B \in \mathrm{Cn}(C)$. We shall say that $A \in L$ is *complete* iff it is a maximally strong consistent sentence, i.e. such that $A \not\equiv \bot$ and, for any $B \in L$ such that $A \in \mathrm{Cn}(B)$, if $B \notin \mathrm{Cn}(A)$, then $B \equiv \bot$. The set of $2^n$ propositional worlds or valuations will be denoted by $W$, and the set of models of a given sentence $A$ by $[\![A]\!]$. Where $x \in W$, we will occasionally abuse notation and use $x$ to denote an arbitrary sentence that has $x$ as its unique model. This usage will be clear from context and occurs only in the proofs.

It will be useful in what follows to define, for every sentence $A \in L$ an associated *total preorder* (i.e. a connected and transitive binary relation; henceforth a "*TPO*") $\preccurlyeq_A$ over $W$, with asymmetric and symmetric parts denoted by $\prec_A$ and $\sim_A$ respectively, such that $x \prec_A y$ iff $x \in [\![A]\!]$ and $y \notin [\![A]\!]$ and $x \sim_A y$ iff $x, y \in [\![A]\!]$ or $x, y \notin [\![A]\!]$. In other words, $\preccurlyeq_A$ is the "two-level" TPO whose lower level is given by $[\![A]\!]$. For $S \subseteq W$ and TPO $\preccurlyeq$, we define $\min(\preccurlyeq, S)$ as $\{w \in W \mid \forall w' \in W, w \preccurlyeq w'\}$.

We consider the two classic belief change operations mapping a prior state $\Psi$ and consistent input sentence $A$ in $L$ onto a posterior state. The operation of *revision* $*$ returns the posterior state $\Psi * A$ that results from an adjustment of $\Psi$ to accommodate the inclusion of $A$, in such a way as to maintain consistency of the resulting belief set when $\neg A \in [\Psi]$. The operation of *contraction* $\div$ returns the posterior state $\Psi \div A$ that results from an adjustment of $\Psi$ to accommodate the retraction of $A$.

When considering a revision or contraction by $A$, we call $\preccurlyeq_\Psi$ the "*prior*" TPO, $\preccurlyeq_{\Psi */\div A}$ the "*posterior*" TPO and $\preccurlyeq_A$ the "*input sentence*" TPO.

### 2.1 Single-step Change

In terms of single-step change, revision and contraction are assumed to satisfy the postulates of Alchourrón, Gärdenfors and Makinson outlined in [1]. The AGM postulates for revision ensure a useful order-theoretic representability of the single-shot revision dispositions of an agent, given in [16], such that each $\Psi$ is associated with a TPO $\preccurlyeq_\Psi$ over $W$, such that $[\![[\Psi]]\!] = \min(\preccurlyeq_\Psi, W)$ and:

[1]This paper is a substantially extended and updated version of a portion of [11], presented at the LORI-19 conference. Among other things, it corrects a mistake found in the proof one of its key results, namely Theorem 1, which also provided unnecessarily strong characteristic principles for the class of elementary revision operators.

(KM$^{*}_{\preccurlyeq}$) $\min(\preccurlyeq_{\Psi * A}, W) = \min(\preccurlyeq_{\Psi}, [\![A]\!])$

The AGM postulates for contraction allow for an entirely analogous representation result, given in [10], in which the associated TPO is required to satisfy:

(KM$^{\div}_{\preccurlyeq}$) $\min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A]\!])$

We denote by TPO($W$) the set of all TPOs over $W$ and shall assume the following "Unrestricted Domain" condition:

(UD$_{\preccurlyeq}$) For every $\preccurlyeq\, \in \mathrm{TPO}(W)$, there exists a state $\Psi$ such that $\preccurlyeq\, = \,\preccurlyeq_{\Psi}$

For ease of exposition, it will be useful to help ourselves to the following concept and notation:

**Definition 1** Where $\preccurlyeq_i$ is a TPO, we define the corresponding *relative rank* function as follows

$$\rho_i(x, y) = \begin{cases} 1, \text{ if } x \prec_i y \\ 0, \text{ if } x \sim_i y \\ -1, \text{ if } y \prec_i x \end{cases}$$

We note in passing that $\rho_i(y, x) = -\rho_i(y, x)$ and, for $A \in L$, $\rho_A(x, y) = -\rho_{\neg A}(x, y)$. The requirement that $\preccurlyeq_{\Psi}$ be a TPO translates into the following constraints on $\rho_{\Psi}$:

- Reflexivity: $\rho_{\Psi}(x, x) = 0$
- Completeness: $\mathrm{Dom}(\rho_{\Psi}) = W \times W$
- Transitivity: If $\{\rho_{\Psi}(x, y), \rho_{\Psi}(y, z)\} \neq \{-1, 1\}$, then, for $S = \{\langle x, y\rangle, \langle y, z\rangle\}$, $\rho_{\Psi}(x, z) = \rho_{\Psi}(\arg\max_{S} |\rho_{\Psi}|)$

The transitivity condition tells us, for example, that, if $\rho_{\Psi}(x, y) = 1$ and $\rho_{\Psi}(y, z) = 0$, then $\rho_{\Psi}(x, z) = 1$, or again, if $\rho_{\Psi}(x, y) = -1$ and $\rho_{\Psi}(y, z) = 0$, then $\rho_{\Psi}(x, z) = -1$. It remains silent, when $\rho_{\Psi}(x, y) = 1$ and $\rho_{\Psi}(y, z) = -1$, or $\rho_{\Psi}(x, y) = -1$ and $\rho_{\Psi}(y, z) = 1$. It can be more straightforwardly represented in tabular form. See Table 1.

Following convention, we shall call principles couched in terms of belief sets "syntactic", and call "semantic" those principles presented in terms of TPOs, denoting the latter by subscripting the corresponding syntactic principle with "$\preccurlyeq$". The bulk of our discussion will focus on semantic principles, although we will also provide syntactic counterparts of a number of these.

**Table 1** Transitivity of $\preccurlyeq_{\Psi}$ represented as a matrix of constraints on values of $\rho_{\Psi}(x, z)$, depending on the values of $\rho_{\Psi}(x, y)$ and $\rho_{\Psi}(y, z)$

| $\rho_{\Psi}(x, y)$ \ $\rho_{\Psi}(y, z)$ | **1** | **0** | **−1** |
|---|---|---|---|
| **1** | 1 | 1 | $\{-1, 0, 1\}$ |
| **0** | 1 | 0 | −1 |
| **−1** | $\{-1, 0, 1\}$ | −1 | −1 |

## 2.2 Iterated Change

In terms of iterated revision, we focus our attention on the three "concrete" operators most commonly found in the literature: the lexicographic revision operator $*_{\mathrm{L}}$ [22], the restrained revision operator $*_{\mathrm{R}}$ [7] and the natural revision operator $*_{\mathrm{N}}$ [9]. They can be defined as follows (Fig. 1):

**Definition 2** If $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$ or $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, then:

$$\rho_{\Psi *_{\mathrm{R}}/*_{\mathrm{N}}/*_{\mathrm{L}} A}(x, y) = \begin{cases} 1, \text{ if } x \in \min(\preccurlyeq_\Psi, [\![A]\!]),\ y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \\ 0, \text{ if } x, y \in \min(\preccurlyeq_\Psi, [\![A]\!]) \\ -1, \text{ if } x \notin \min(\preccurlyeq_\Psi, [\![A]\!]),\ y \in \min(\preccurlyeq_\Psi, [\![A]\!]) \end{cases}$$

If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ , then:

$$\rho_{\Psi *_{\mathrm{L}} A}(x, y) = \begin{cases} \rho_A(x, y), \text{ if } \rho_A(x, y) \neq 0 \\ \rho_\Psi(x, y), \text{ if } \rho_A(x, y) = 0 \end{cases}$$

$$\rho_{\Psi *_{\mathrm{R}} A}(x, y) = \begin{cases} \rho_\Psi(x, y), \text{ if } \rho_\Psi(x, y) \neq 0 \\ \rho_A(x, y), \text{ if } \rho_\Psi(x, y) = 0 \end{cases}$$

$$\rho_{\Psi *_{\mathrm{N}} A}(x, y) = \rho_\Psi(x, y)$$

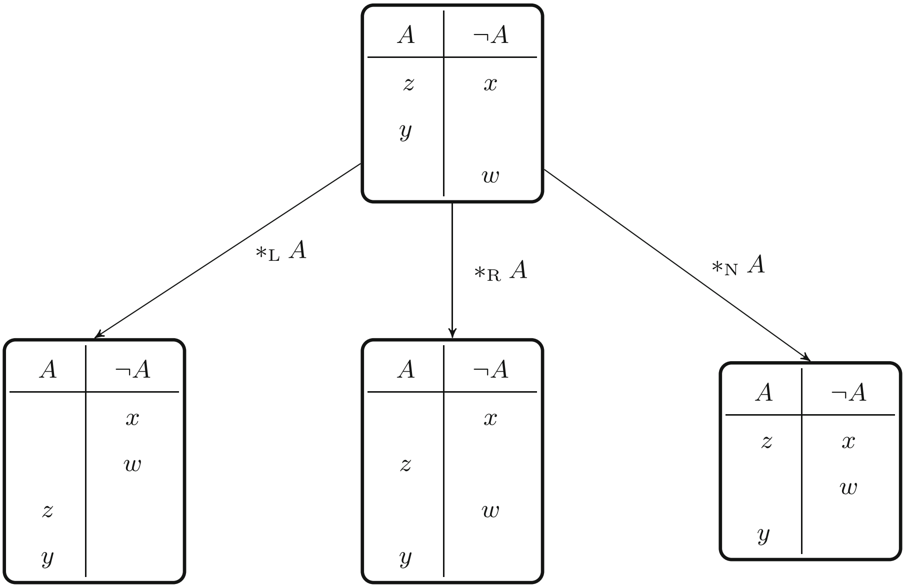


**Fig. 1** Revision by $A$ according to the operators $*_{\mathrm{L}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{N}}$. The boxes represent states and associated TPOs. The lower case letters, which represent worlds, are arranged in such a way that the lower the letter, the lower the corresponding world in the relevant ordering. The columns group worlds according to the sentences that they validate. So, for example, in the initial ordering, we have $w \prec y \prec x \sim z$, with $y, z \in [\![A]\!]$ and $x, w \in [\![\neg A]\!]$ and then, after lexicographic revision by $A$, $y \prec z \prec w \prec x$

**Table 2** Mappings from $\rho_\Psi(x, y)$ and $\rho_A(x, y)$ to $\rho_{\Psi * A}(x, y)$ for the operators $*_{\mathrm{N}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$, where $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **0** | 0 | 0 | 0 |
| **-1** | -1 | -1 | -1 |

(**a**) The natural revision operator $*_{\mathrm{N}}$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **0** | 1 | 0 | -1 |
| **-1** | -1 | -1 | -1 |

(**b**) The restrained revision operator $*_{\mathrm{R}}$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | -1 |
| **0** | 1 | 0 | -1 |
| **-1** | 1 | -1 | -1 |

(**c**) The lexicographic revision operator $*_{\mathrm{L}}$.

These operators can be conveniently presented in the form of matrices that represent, for all states $\Psi$, sentences $A$ and worlds $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, the value of the posterior rank $\rho_{\Psi * A}(x, y)$ as a function of the values of the prior relative rank $\rho_\Psi(x, y)$ and input sentence relative rank $\rho_A(x, y)$. (Indeed, ($\mathrm{KM}^*_{\preccurlyeq}$) takes care of the posterior relative rank when $x$ or $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$: If $x$ (respectively $y$) alone is in that set, then $\rho_{\Psi * A}(x, y) = 1$ (respectively $= -1$) and if both are in it, then $\rho_{\Psi * A}(x, y) = 0$.) See Table 2.

All three suggestions operate on the assumption that, for the purposes of iterated revision, a state $\Psi$ can essentially be identified with its corresponding TPO $\preccurlyeq_\Psi$ and that belief change functions effectively map pairs of TPOs and sentences onto TPOs. In other words, they entail:

($\mathrm{TPOR}^*_{\preccurlyeq}$) If $\preccurlyeq_\Psi = \preccurlyeq_\Theta$, then, for any $A$, $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Theta * A}$

We note, however, that this assumption is not uncontroversial and has been criticised at some length in [4].

Beyond this, the proposals all ensure that $*$ satisfies the postulates of Darwiche & Pearl [13]. Semantically, framed in terms of the $\rho$-notation, these are given as follows:[2]

[2] In the standard presentation of the Darwiche-Pearl axioms in terms of the $\preccurlyeq$-notation, ($\mathrm{C1}, 2^*_{\preccurlyeq}$) is broken down into two parts that cannot be expressed in terms of $\rho$:

($\mathrm{C1}^*_{\preccurlyeq}$) If $x, y \in [\![A]\!]$ then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_\Psi y$
($\mathrm{C2}^*_{\preccurlyeq}$) If $x, y \in [\![\neg A]\!]$ then $x \preccurlyeq_{\Psi * A} y$ iff $x \preccurlyeq_\Psi y$

This is worth noting, since ($\mathrm{C2}^*_{\preccurlyeq}$), but not ($\mathrm{C1}^*_{\preccurlyeq}$), has been the subject of some controversy (see, for instance, [12]).

(C1, $2^*_{\preccurlyeq}$) If $\rho_A(x, y) = 0$ then $\rho_{\Psi * A}(x, y) = \rho_\Psi(x, y)$

($C3^*_{\preccurlyeq}$) If $\rho_A(x, y) = 1$ and $\rho_\Psi(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$

($C4^*_{\preccurlyeq}$) If $\rho_A(x, y) = 1$ and $\rho_\Psi(x, y) \geq 0$, then $\rho_{\Psi * A}(x, y) \geq 0$

As such, they can be presented in the form of a matrix of constraints on the relation between the prior relative rank $\rho_A(x, y)$ and input sentence relative rank $\rho_\Psi(x, y)$ and the posterior relative rank $\rho_{\Psi * A}(x, y)$, for all states $\Psi$, sentences $A$ and worlds $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$. (Indeed, as noted above, ($KM^*_{\preccurlyeq}$) takes care of the posterior relative rank when $x$ or $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$.) See Table 3.

Regarding $\div$, we assume that it satisfies the postulates of Chopra et al [12], given semantically by:

(C1, $2^{\div}_{\preccurlyeq}$) If $\rho_A(x, y) = 0$ then $\rho_{\Psi \div A}(x, y) = \rho_\Psi(x, y)$

($C3^{\div}_{\preccurlyeq}$) If $\rho_A(x, y) = -1$ and $\rho_\Psi(x, y) = 1$, then $\rho_{\Psi \div A}(x, y) = 1$

($C4^{\div}_{\preccurlyeq}$) If $\rho_A(x, y) = -1$ and $\rho_\Psi(x, y) \geq 0$, then $\rho_{\Psi \div A}(x, y) \geq 0$

Again we can present these principles in the form of a matrix of constraints, this time on the relation between the prior relative rank $\rho_A(x, y)$ and input sentence relative rank $\rho_\Psi(x, y)$ and the posterior relative rank $\rho_{\Psi \div A}(x, y)$, for all states $\Psi$, sentences $A$ and worlds $x, y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$ (with ($KM^{\div}_{\preccurlyeq}$) handling the case in which $x$ or $y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$). See Table 4.

### 2.3 From Revision to Contraction and Back Again

The operations $*$ and $\div$ are assumed to be related in the single-shot case by the Levi and Harper identities, given semantically by:

($LI_{\preccurlyeq}$) $\min(\preccurlyeq_{\Psi * A}, W) = \min(\preccurlyeq_{\Psi \div \neg A}, [\![A]\!])$

($HI_{\preccurlyeq}$) $\min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_{\Psi * \neg A}, W)$

Concerning the relations between the belief revision and contraction operators in the iterated case, a proposal for extending (HI) to the two-step case was recently floated in [3]. It involved the characterisation of a particular binary TPO combination operator (a "TeamQueue combinator") $\oplus$, such that $\preccurlyeq_{\Psi \div A} = \preccurlyeq_\Psi \oplus \preccurlyeq_{\Psi * \neg A}$:

**Table 3** Mapping from $\rho_\Psi(x, y)$ and $\rho_A(x, y)$ to $\rho_{\Psi * A}(x, y)$, as constrained by (C1, $2^*_{\preccurlyeq}$) – ($C4^*_{\preccurlyeq}$)

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **–1** |
|---|---|---|---|
| **1** | 1 ($C3^*_{\preccurlyeq}$) | 1 (C1, $2^*_{\preccurlyeq}$) | $\{-1, 0, 1\}$ |
| **0** | $\{0, 1\}$ ($C4^*_{\preccurlyeq}$) | 0 (C1, $2^*_{\preccurlyeq}$) | $\{-1, 0\}$ ($C4^*_{\preccurlyeq}$) |
| **–1** | $\{-1, 0, 1\}$ | –1 (C1, $2^*_{\preccurlyeq}$) | –1 ($C3^*_{\preccurlyeq}$) |

**Table 4** Mapping from $\rho_\Psi(x, y)$ and $\rho_A(x, y)$ to $\rho_{\Psi \div A}(x, y)$, as constrained by (C1, $2^{\div}_{\preccurlyeq}$)-($C4^{\div}_{\preccurlyeq}$). This matrix is obtained from the one depicted in Table 3 by simply flipping the values across the middle column

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **–1** |
|---|---|---|---|
| **1** | $\{-1, 0, 1\}$ | 1<br>($C1/2^{\div}_{\preccurlyeq}$) | 1<br>($C3^{\div}_{\preccurlyeq}$) |
| **0** | $\{-1, 0\}$<br>($C4^{\div}_{\preccurlyeq}$) | 0<br>($C4^{\div}_{\preccurlyeq}$) | $\{0, 1\}$<br>($C4^{\div}_{\preccurlyeq}$) |
| **–1** | –1<br>($C3^{\div}_{\preccurlyeq}$) | –1<br>($C1/2^{\div}_{\preccurlyeq}$) | $\{-1, 0, 1\}$ |

**Definition 3** $\oplus$ is a *TeamQueue (TQ) combinator* iff, for each ordered pair $\langle \preccurlyeq_1, \preccurlyeq_2 \rangle$ of TPOs, there exists a sequence $\langle a_{\preccurlyeq_1, \preccurlyeq_2}(i) \rangle_{i \in \mathbb{N}}$ such that:

(a1) $\emptyset \neq a_{\preccurlyeq_1, \preccurlyeq_2}(i) \subseteq \{1, 2\}$ for each $i$
(a2) $a_{\preccurlyeq_1, \preccurlyeq_2}(1) = \{1, 2\}$

and the ordered partition $\langle T_1, T_2, \ldots, T_m \rangle$ of indifferences classes corresponding to $\preccurlyeq_{1 \oplus 2}$ is constructed inductively as follows:

$$T_i = \bigcup_{j \in a_{\preccurlyeq_1, \preccurlyeq_2}(i)} \min(\preccurlyeq_j, \bigcap_{k<i} T_k^c)$$

where "$T^c$" denotes the complement of set $T$ and $m$ is minimal such that $\bigcup_{i \leq m} T_i = W$.

Informally, the procedure takes the TPOs respectively associated with $\Psi$ and $\Psi * \neg A$ and processes them step by step to form a new TPO. At the first step, it removes the minimal elements of *both* TPOs and places them in the minimal rank of the output TPO, before deleting any copies of these elements that might remain in the input TPOs. At each step, it then repeats the process with the minimal elements of *one or both* of the remaining pruned input TPOs (depending on the specifics of the procedure, i.e. on the value(s) in $a_{\preccurlyeq_1, \preccurlyeq_2}(i)$ for the relevant step $i$), until both input TPOs have been processed entirely.

Among the particular proposals considered was an extension of ($HI_{\preccurlyeq}$) obtained by a specific TQ combinator, $\oplus_{STQ}$, that takes $a_{\preccurlyeq_1, \preccurlyeq_2}(i) = \{1, 2\}$ for all ordered pairs $\langle \preccurlyeq_1, \preccurlyeq_2 \rangle$ and all $i$. This suggestion was partly syntactically motivated by an appeal to the notion of "rational closure", introduced in [17].

*Example 1* Suppose that $W = \{x, y, z, w\}$, that $\preccurlyeq_1$ is the TPO represented by the ordered partition $\langle \{z\}, \{w\}, \{x, y\} \rangle$ and $\preccurlyeq_2$ is represented by $\langle \{x, z\}, \{y\}, \{w\} \rangle$. Then the ordered partition corresponding to $\preccurlyeq_{1 \oplus_{STQ} 2}$ is $\langle T_1, T_2 \rangle = \langle \{x, z\}, \{w, y\} \rangle$.

There are of course other possible TQ combinators. One example would be the combinator $\oplus_{TQ2}$, such that $a_{\preccurlyeq_1, \preccurlyeq_2}(1) = \{1, 2\}$ but $a_{\preccurlyeq_1, \preccurlyeq_2}(i) = \{2\}$ for all $i \geq 2$. This method yields an extension of the Harper Identity that maximally prioritises $\preccurlyeq_{\Psi * \neg A}$, so that with the exception of the resulting minimal worlds, $\preccurlyeq_{\Psi \div A}$ is simply given by $\preccurlyeq_{\Psi * \neg A}$.

*Example 2* Let $\preccurlyeq_1$ and $\preccurlyeq_2$ be given as in Example 1 above. Then $\preccurlyeq_{1\oplus_{\mathrm{TQ}2}2}=\preccurlyeq_2$.

In [11], two extensions of (LI$_{\preccurlyeq}$) were considered, the first essentially due to [21] and the second motivated syntactically, again via the notion of rational closure:

$$\begin{array}{ll} (\mathrm{iLI}*_{\preccurlyeq}) & \rho_{\Psi * A}(x, y) = \rho_{(\Psi \div \neg A) * A}(x, y) \\ (\mathrm{iLIRC}_{\preccurlyeq}) & \rho_{\Psi * A}(x, y) = \rho_{(\Psi \div \neg A) *_{\mathrm{N}} A}(x, y) \end{array}$$

We note that, while the second proposal effectively allows one to define two-step revision from two-step contraction, the first does not, since the very same revision operator appears on both sides of the equality (Fig. 2).

## 3 A Characterisation of Elementary Revision

### 3.1 Semantic Characterisation

#### 3.1.1 Introducing Independence of Irrelevant Alternatives

We define elementary revision operators semantically as follows:

**Definition 4** $*$ is an *elementary* revision operator iff it satisfies (KM$^*_{\preccurlyeq}$), (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$) and the following principle of "Independence of Irrelevant Alternatives":

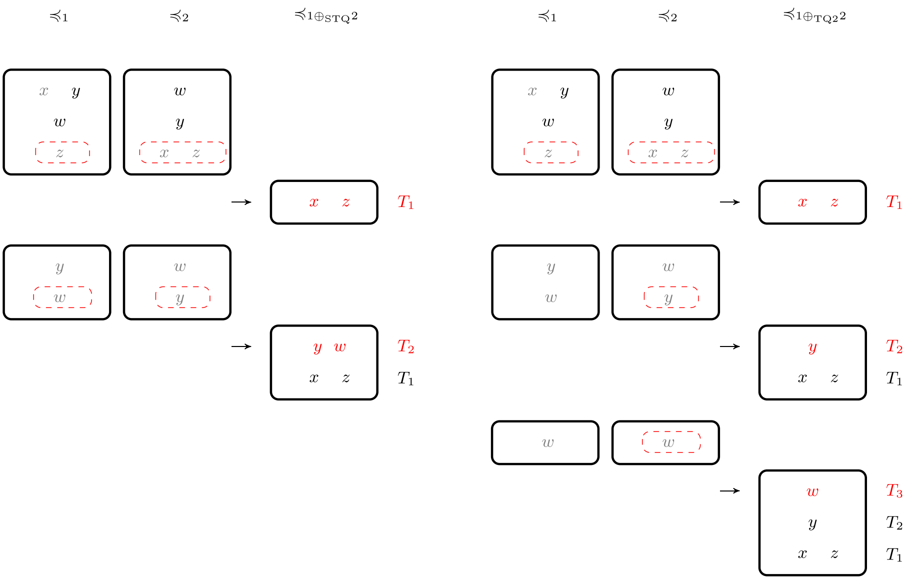


**Fig. 2** Illustration of the construction of $\preccurlyeq_{1\oplus 2}$ in Example 1 (left) and Example 2 (right). This construction in each case is illustrated chronologically from top to bottom

(IIA$^*_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, then, if $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and $\rho_A(x, y) = \rho_B(x, y)$, then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$

We have already introduced (KM$^*_{\preccurlyeq}$) and (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$). The new principle (IIA$^*_{\preccurlyeq}$) is named after a well known analogous precept in Social Choice [2]. In the presence of (KM$^*_{\preccurlyeq}$), it tells us that that the posterior relative rank $\rho_{\Psi * A}(x, y)$ of a pair of worlds $\{x, y\}$ is determined by the prior relative rank $\rho_\Psi(x, y)$ and input sentence relative rank $\rho_A(x, y)$ (although this mapping may be different for different pairs of worlds). Its prima facie appeal is similar to that of its Social Choice counterpart, substituting a doxastic interpretation of the ordering for a preferential one.

It can be easily checked that, given (KM$^*_{\preccurlyeq}$), (IIA$^*_{\preccurlyeq}$) and (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$) are logically independent, so that these principles play a non-redundant part in the definition:

**Proposition 1** *Given* (KM$^*_{\preccurlyeq}$), (IIA$^*_{\preccurlyeq}$) *does not imply any of* (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$).

**Proposition 2** *Given* (KM$^*_{\preccurlyeq}$), (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$) *do not jointly imply* (IIA$^*_{\preccurlyeq}$).

(IIA$^*_{\preccurlyeq}$) is a very strong principle and indeed turns out to ensure that elementary operators identify states with TPOs. We can more precisely pinpoint the locus of blame for this implication by breaking (IIA$^*_{\preccurlyeq}$) down into two "halves":

**Proposition 3** *Given* (UD$_{\preccurlyeq}$), (IIA$^*_{\preccurlyeq}$) *is equivalent to the conjunction of the following principles of "Independence of Irrelevant Alternatives" with respect to the "Prior" and the "Input", respectively:*

(IIAP$^*_{\preccurlyeq}$) *If* $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![A]\!])$, *then, if* $\rho_\Psi(x, y) = \rho_\Theta(x, y)$, *then* $\rho_{\Psi * A}(x, y) = \rho_{\Theta * A}(x, y)$

(IIAI$^*_{\preccurlyeq}$) *If* $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Psi, [\![B]\!])$, *then, if* $\rho_A(x, y) = \rho_B(x, y)$, *then* $\rho_{\Psi * A}(x, y) = \rho_{\Psi * B}(x, y)$

These sub-principles are demonstrably independent, even in the presence of the remainder of the principles that characterise elementary revision:

**Proposition 4** (IIAI$^*_{\preccurlyeq}$) *does not imply* (IIAP$^*_{\preccurlyeq}$) *or vice versa, even in the presence of* (KM$^*_{\preccurlyeq}$) *and* (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$).

It is the first of the sub-principles, (IIAP$^*_{\preccurlyeq}$), that forces the identification of states with TPOs:

**Proposition 5** *Given* (KM$^*_{\preccurlyeq}$), (IIAP$^*_{\preccurlyeq}$) *entails* (TPOR$^*_{\preccurlyeq}$).

The second sub-principle does not have this implication. Indeed, although it is new to the literature in the form in which it is presented, it can be shown to be equivalent, under our assumptions, to the conjunction of a pair of principles that were recently defended in [6] and which relate the prior TPO and pairs of posterior TPOs obtained by revisions by different sentences:

**Proposition 6** *Given* $(\mathrm{KM}^*_{\preccurlyeq})$, $(\mathrm{C1}, 2^*_{\preccurlyeq})$–$(\mathrm{C4}^*_{\preccurlyeq})$, $(\mathrm{IIAI}^*_{\preccurlyeq})$ *is equivalent to the conjunction of:*

$(\beta 1^*_{\preccurlyeq})$ *If* $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $\rho_A(x, y) = 1$, *and* $\rho_{\Psi * A}(x, y) \leq 0$, *then* $\rho_{\Psi * C}(x, y) \leq 0$
$(\beta 2^*_{\preccurlyeq})$ *If* $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $\rho_A(x, y) = 1$, *and* $\rho_{\Psi * A}(x, y) = -1$, *then* $\rho_{\Psi * C}(x, y) = -1$

These principles are known to be satisfied by a range of operators that is broad enough to include, beyond $*_{\mathrm{N}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$, the entire family of so-called "proper ordinal interval" (POI) revision operators, which do not generally satisfy $(\mathrm{TPOR}^*_{\preccurlyeq})$ (see [6]).

### 3.1.2 Some Derived Principles of Elementary Revision

We now prove a series of lemmas that demonstrate the soundness, for elementary operators, of a number of useful principles which we will later make use of in the derivation of our main technical contribution.

We first note without proof that $(\mathrm{C1}, 2^*_{\preccurlyeq})$ trivially entails the following principle of "Pareto Indifference" (so-named by analogy with a corresponding principle in Social Choice) which will make an appearance in some of the subsequent results:[3]

**Lemma 1** $(\mathrm{C1}, 2^*_{\preccurlyeq})$ *entails:*

$(\mathrm{PI}^*_{\preccurlyeq})$ *If* $\rho_\Psi(x, y) = \rho_A(x, y) = 0$, *then* $\rho_{\Psi * A}(x, y) = 0$

Given this weak principle, $(\mathrm{IIA}^*_{\preccurlyeq})$ has some surprisingly strong consequences. First we can show that:

**Lemma 2** *The conjunction of* $(\mathrm{KM}^*_{\preccurlyeq})$, *transitivity of* $\preccurlyeq_{\Psi * A}$, $(\mathrm{PI}^*_{\preccurlyeq})$ *and* $(\mathrm{IIA}^*_{\preccurlyeq})$ *implies the following principle of "Zero Symmetry":*

$(\mathrm{ZS}^*_{\preccurlyeq})$ *If* $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ *and* $\rho_A(x, y) = -\rho_B(x, y)$, *then* $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$

---

[3] Some other "Paretian" principles of possible interest, with analogues in Social Choice, are the conditions of "Pareto Weak Preference", "Weak Pareto" and "Strict Pareto", respectively given by:

$(\mathrm{PWP}^*_{\preccurlyeq})$ If $\rho_\Psi(x, y), \rho_A(x, y) \geq 0$, then $\rho_{\Psi * A}(x, y) \geq 0$
$(\mathrm{WP}^*_{\preccurlyeq})$ If $\rho_\Psi(x, y) = \rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$
$(\mathrm{SP}^*_{\preccurlyeq})$ If $\rho_\Psi(x, y) + \rho_A(x, y) \geq 1$, then $\rho_{\Psi * A}(x, y) = 1$

Clearly, $(\mathrm{PWP}^*_{\preccurlyeq})$ entails $(\mathrm{PI}^*_{\preccurlyeq})$, while $(\mathrm{SP}^*_{\preccurlyeq})$ entails $(\mathrm{WP}^*_{\preccurlyeq})$. $(\mathrm{WP}^*_{\preccurlyeq})$ just *is* $(\mathrm{C3}^*_{\preccurlyeq})$ and so is sound for elementary operators. Besides $(\mathrm{WP}^*_{\preccurlyeq})$, a large further chunk of $(\mathrm{SP}^*_{\preccurlyeq})$ does hold for all three operators. Indeed, the following is sound for elementary revision: if $\rho_\Psi(x, y) = 1$ and $\rho_A(x, y) \geq 0$, then $\rho_{\Psi * A}(x, y) = 1$. The remaining part of $(\mathrm{SP}^*_{\preccurlyeq})$ is the following: if $\rho_\Psi(x, y) = 0$ and $\rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$. This is simply the so-called principle (P) (see [7]) that holds for $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$ but not for $*_{\mathrm{N}}$. Indeed, let $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, $\rho_\Psi(x, y) = 0$ and $\rho_A(x, y) = 1$. Then $\rho_{\Psi *_{\mathrm{N}} A}(x, y) = 0$, in contradiction with this principle. Finally, it is easy to show that $(\mathrm{C1}, 2^*_{\preccurlyeq})$-$(\mathrm{C4}^*_{\preccurlyeq})$ jointly entail $(\mathrm{PWP}^*_{\preccurlyeq})$. Indeed: If $\rho_\Psi(x, y) = 1$ and $\rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$, by $(\mathrm{C3}^*_{\preccurlyeq})$. If $\rho_\Psi(x, y) = 1$ and $\rho_A(x, y) = 0$, then $\rho_{\Psi * A}(x, y) = 1$, by $(\mathrm{C1}, 2^*_{\preccurlyeq})$. If $\rho_\Psi(x, y) = 0$ and $\rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) \geq 0$, by $(\mathrm{C4}^*_{\preccurlyeq})$. Finally, if $\rho_\Psi(x, y) = 0$ and $\rho_A(x, y) = 0$, then $\rho_{\Psi * A}(x, y) = 0$, by $(\mathrm{C1}, 2^*_{\preccurlyeq})$.

Like (IIA$^*_\preccurlyeq$), (ZS$^*_\preccurlyeq$) is analogous to a principle of Social Choice, namely the principle of "Neutrality" introduced in [19]. It can arguably be read as saying that the revision process does not "favour" any world over another, in the sense that reversing the orders of preference corresponding to both the prior state and the input to revision simply yields a reversal of the ordering corresponding to the posterior state.

The proof of the preceding result can be co-opted to establish the derivation of a second principle. To introduce the latter, we need the following definition:

**Definition 5** $\pi$ is an *order isomorphism* from $\preccurlyeq_\Psi$ to $\preccurlyeq_\Theta$ iff it is a 1:1 mapping from $W$ onto itself such that $\rho_\Psi(x, y) = \rho_\Theta(\pi(x), \pi(y))$.

We can extend $\pi$ to sentences in $L$ in such a way that $[\![\pi(A)]\!] = \{x \in W \mid \exists y \in [\![A]\!],\ \text{such that } x = \pi(y)\}$. With this in hand, we can then offer:

**Lemma 3** *The conjunction of* (KM$^*_\preccurlyeq$), *transitivity of* $\preccurlyeq_{\Psi * A}$, (PI$^*_\preccurlyeq$) *and* (IIA$^*_\preccurlyeq$) *implies the following "Representation Invariance" principle:*

(RI$^*_\preccurlyeq$) $\rho_{\Psi * A}(x, y) = \rho_{\Theta * \pi(A)}(\pi(x), \pi(y))$, *for any order isomorphism* $\pi$ *from* $\preccurlyeq_\Psi$ *to* $\preccurlyeq_\Theta$[4]

(ZS$^*_\preccurlyeq$) turns out to be rather strong, and could in fact have been used in our characterisation instead of (IIA$^*_\preccurlyeq$). Indeed, the following holds:

**Proposition 7** *In the presence of* (UD$_\preccurlyeq$), (ZS$^*_\preccurlyeq$) *implies* (IIA$^*_\preccurlyeq$).

(RI$^*_\preccurlyeq$), however, does not share this implication:

**Proposition 8** (RI$^*_\preccurlyeq$) *does not imply either* (IIAI$^*_\preccurlyeq$) *or* (IIAP$^*_\preccurlyeq$), *even in the presence of* (KM$^*_\preccurlyeq$) *and* (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$).

In spite of this, we do note that it still trivially retains (TPOR$^*_\preccurlyeq$) as a consequence:

**Proposition 9** (RI$^*_\preccurlyeq$) *entails* (TPOR$^*_\preccurlyeq$).

### 3.1.3 Main Result

With Lemmas 1, 2 and 3 in hand, we can now offer our main result, which we prove in the main body of the article rather than the Appendix A:

**Theorem 1** *The only elementary revision operators are lexicographic, restrained and natural revision.*

[4]In Theorem 1 of [11], the set of operators $\{*_\mathrm{L}, *_\mathrm{R}, *_\mathrm{N}\}$ was characterised by (KM$^*_\preccurlyeq$), (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$) and (IIA$^*_\preccurlyeq$), supplemented with a weakening of (RI$^*_\preccurlyeq$) that requires the isomorphism $\pi$ to be "$A$-preserving". As Lemma 3 shows, this principle was simply derivable from the remainder of the postulates (since (RI$^*_\preccurlyeq$) was).

*Proof* We decompose the result into its two obvious parts. The soundness result, which states that lexicographic, restrained and natural revision operators are elementary operators, does not require much commentary. Indeed, it is well known that these operators satisfy (C1, $2^*_{\preccurlyeq}$)-(C$4^*_{\preccurlyeq}$), as well as (KM$^*_{\preccurlyeq}$). We have also, in noting their matrix representability as per Table 2, pointed out that they satisfy (IIA$^*_{\preccurlyeq}$).

We now establish the completeness part, which tells us that, if an operator is elementary, then it is a lexicographic, restrained or natural revision operator.

As we have seen, in the presence of (KM$^*_{\preccurlyeq}$), (IIA$^*_{\preccurlyeq}$) tells us that the posterior relative rank $\rho_{\Psi * A}(x, y)$ of a pair $\langle x, y\rangle$ of worlds is determined by the prior relative rank $\rho_{\Psi}(x, y)$ and input sentence relative rank $\rho_A(x, y)$. By virtue of (KM$^*_{\preccurlyeq}$), the mappings for pairs $\langle x, y\rangle$ such that $x$ or $y \in \min(\preccurlyeq_{\Psi}, [\![A]\!])$ are the same for all operators, so that any differences between them occur at the level of the matrices for the various pairs $\langle x, y\rangle$ such that $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$.

Although (KM$^*_{\preccurlyeq}$) and (IIA$^*_{\preccurlyeq}$) allow this mapping to be different for different non-minimal pairs of worlds, we know from Lemma 3 that the addition of (PI$^*_{\preccurlyeq}$), which is a consequence of (C1, $2^*_{\preccurlyeq}$) (see Lemma 1) gives us (RI$^*_{\preccurlyeq}$), which *does* secure the identity of the matrices for different non-minimal pairs.

We have seen that (C1, $2^*_{\preccurlyeq}$)-(C$4^*_{\preccurlyeq}$) constrain the range of values for the matrix for non-minimal pairs to the values specified in Table 3 above. Since we also know from Lemma 2 that, given our assumptions, (ZS$^*_{\preccurlyeq}$) holds, we are left with 6 matrices to consider. Three of these are the ones associated with the operators $*_{\mathrm{N}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$ and depicted in Table 2.

We now show that the remaining three are inconsistent with the transitivity of $\preccurlyeq_{\Psi * A}$. These are given in Table 5.

Since we assume $A$ to be consistent and hence that $\min(\preccurlyeq_{\Psi}, [\![A]\!])$ is non-empty, we must have strictly more than two, and therefore (since $|W| = 2^n$) at least four,

**Table 5** Mappings from prior relative rank and input sentence relative rank to posterior relative rank for the three new operators referenced in the proof of Theorem 1

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | -1 |
| **0** | 0 | 0 | 0 |
| **-1** | 1 | -1 | -1 |

**(d)**

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 0 |
| **0** | 1 | 0 | -1 |
| **-1** | 0 | -1 | -1 |

**(e)**

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 0 |
| **0** | 0 | 0 | 0 |
| **-1** | 0 | -1 | -1 |

**(f)**

worlds. Let these worlds be $x, y, z$ and $w$, and $A$ be such that $y \in [\![\neg A]\!]$ and $x, z, w \in [\![A]\!]$. We then have $\rho_A(x, y) = 1$ and $\rho_A(x, z) = 0$.

Regarding the operators associated with (e) and (f), assume that $w \prec_\Psi y \prec_\Psi z \prec_\Psi x$. Note that $x, y, z \notin \min(\preccurlyeq_\Psi, [\![A]\!])$. On the one hand, since $\rho_\Psi(x, z) = -1$ and $\rho_A(x, z) = 0$, we have $\rho_{\Psi * A}(x, z) = -1$. However, on the other hand, since $\rho_\Psi(x, y) = -1$ and $\rho_A(x, y) = 1$, we have $\rho_{\Psi * A}(x, y) = 0$ and, since $\rho_\Psi(y, z) = 1$ and $\rho_A(y, z) = -1$, we have $\rho_{\Psi * A}(y, z) = 0$. It then follows, by transitivity of $\preccurlyeq_{\Psi * A}$, that $\rho_{\Psi * A}(x, z) = 0$, contradicting our finding that $\rho_{\Psi * A}(x, z) = -1$.

Regarding the operator associated with (d), assume $w \prec_\Psi \{x, y\} \prec_\Psi z$. Again, note that $x, y, z \notin \min(\preccurlyeq_\Psi, [\![A]\!])$. On the one hand, since $\rho_\Psi(z, x) = -1$ and $\rho_A(z, x) = 0$, we have $\rho_{\Psi * A}(z, x) = -1$. However, on the other hand, since $\rho_\Psi(y, x) = 0$ and $\rho_A(y, x) = -1$, we have $\rho_{\Psi * A}(x, y) = 0$ and, since $\rho_\Psi(z, y) = -1$ and $\rho_A(z, y) = 1$, we have $\rho_{\Psi * A}(z, y) = 1$. By transitivity of $\preccurlyeq_{\Psi * A}$, it then follows that $\rho_{\Psi * A}(z, x) = 1$, contradicting our finding that $\rho_{\Psi * A}(z, x) = -1$. $\square$

### 3.2 Syntactic Characterisation

In the following section we provide syntactic counterparts for the various semantic principles introduced above. The syntactic versions of the definitions of our operators $*_{\mathrm{N}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$ are well known (see [24]) and are given as follows, against the background assumption of AGM:

$$[(\Psi *_{\mathrm{N}} A) *_{\mathrm{N}} B] = \begin{cases} [\Psi *_{\mathrm{N}} A \wedge B], & \text{if } \neg B \notin [\Psi *_{\mathrm{N}} A] \\ [\Psi *_{\mathrm{N}} B], & \text{otherwise} \end{cases}$$

$$[(\Psi *_{\mathrm{R}} A) *_{\mathrm{R}} B] = \begin{cases} [\Psi *_{\mathrm{R}} A \wedge B], & \text{if } \neg A \notin [\Psi *_{\mathrm{R}} B] \text{ or } \neg B \notin [\Psi *_{\mathrm{R}} A] \\ [\Psi *_{\mathrm{R}} B], & \text{otherwise} \end{cases}$$

$$[(\Psi *_{\mathrm{L}} A) *_{\mathrm{L}} B] = \begin{cases} [\Psi *_{\mathrm{L}} A \wedge B], & \text{if } [\Psi *_{\mathrm{L}} A \wedge B] \text{ is consistent} \\ [\Psi *_{\mathrm{L}} B], & \text{otherwise} \end{cases}$$

The same applies to $(\mathrm{C1}, 2^*_{\preccurlyeq})$-$(\mathrm{C4}^*_{\preccurlyeq})$, whose counterparts are well known to be, respectively:

(C1, 2*) If $A \in Cn(B)$ or $\neg A \in Cn(B)$, then $[(\Psi * A) * B] = [\Psi * B]$
(C3*) If $A \in [\Psi * B]$, then $A \in [(\Psi * A) * B]$
(C4*) If $\neg A \notin [\Psi * B]$, then $\neg A \notin [(\Psi * A) * B]$

Regarding $(\mathrm{TPOR}^*_{\preccurlyeq})$, we have:

(TPOR) If, for all $A$, $[\Psi * A] = [\Theta * A]$, then, for all $A, B$, $[(\Psi * A) * B] = [(\Theta * A) * B]$

The Pareto Indifference condition $(\mathrm{PI}^*_{\preccurlyeq})$ can also be given a fairly straightforward, if admittedly not particularly enlightening, syntactic formulation:

**Proposition 10** *Given AGM,* $(\mathrm{PI}^*_{\preccurlyeq})$ *is equivalent to:*

(PI*) *If* $[\Psi * B] = Cn(B)$, *then* $[(\Psi * A) * A \wedge B] = [\Psi * A \wedge B]$ *and* $[(\Psi * A) * \neg A \wedge B] = [\Psi * \neg A \wedge B]$

Matters are a little less straightforward regarding (IIA$^*_{\preccurlyeq}$) (as well as the similar (IIAP$^*_{\preccurlyeq}$) and (IIAI$^*_{\preccurlyeq}$)), (ZS$^*_{\preccurlyeq}$) and (RI$^*_{\preccurlyeq}$). We handle these in what follows.

### 3.2.1 Independence of Irrelevant Alternatives

We first offer the following definitions of the syntactic notion of "agreement" between states, modulo a sentence:

**Definition 6** States $\Psi$ and $\Theta$ *agree modulo C* iff $[\Psi * B \wedge C] = [\Theta * B \wedge C]$ for all $B$

With this in hand, it is easy to show:

**Proposition 11** *Given AGM and* (C1, 2$^*_{\preccurlyeq}$)*,* (IIA$^*_{\preccurlyeq}$)*,* (IIAP$^*_{\preccurlyeq}$) *and* (IIAI$^*_{\preccurlyeq}$) *are respectively equivalent to:*

(IIA$^*$) *If* $\neg C \in [\Psi * A] \cap [\Theta * B]$*,* $A \equiv_C B$ *and* $\Psi$ *and* $\Theta$ *agree modulo C, then so do* $\Psi * A$ *and* $\Theta * B$

(IIAP$^*$) *If* $\neg C \in [\Psi * A] \cap [\Theta * A]$*, then, if* $\Psi$ *and* $\Theta$ *agree modulo C, so do* $\Psi * A$ *and* $\Theta * A$

(IIAI$^*$) *If* $\neg C \in [\Psi * A] \cap [\Psi * B]$ *and* $A \equiv_C B$*, then* $\Psi * A$ *and* $\Psi * B$ *agree modulo C*

### 3.2.2 Zero Symmetry

Here we have a similar, if somewhat more complicated, syntactic formulation to the one given in the previous subsection. We first define:

**Definition 7** A sentence $B \in L$ is *quasi-complete* iff for all $A \in L$, if $A \vdash B$ and $B \nvdash A$, then $A$ is either inconsistent or complete.

With this in hand, we can offer the following syntactic definition of two states being in "opposition", modulo a sentence:

**Definition 8** States $\Psi$ and $\Theta$ are *in opposition modulo C* iff, for all $B, D \in L$ such that $B$ is quasi-complete and $B \wedge C \nvdash D$, if $D \in [\Psi * B \wedge C]$, then $\neg D \in [\Theta * B \wedge C]$

While $\Psi$ and $\Theta$'s being in agreement modulo $C$ requires the belief sets obtained by revision by $B \wedge C$, *for all B*, to be *identical*, their being in opposition modulo the same sentence requires the sets obtained by revision by $B \wedge C$, *for certain specific B*'s, to be *antithetical*.

The requirement that $B \wedge C \nvdash D$ is obviously imposed to circumvent the risk of the concept of being in opposition modulo $C$'s being inapplicable for consistent $C$. Indeed, assume this caveat were not in place. Since, for any consistent $C$, we have both $C \in [\Psi * B \wedge C]$ and $\neg C \notin [\Theta * B \wedge C]$, it follows that $\Psi$ and $\Theta$ would not be classified as being in opposition modulo $C$. The requirement of quasi-completeness of $B$ is admittedly less immediately intuitive. It plays a fairly obvious and critical role, however, in the proof of the next proposition.

With this in hand, we can now state:

**Proposition 12** *Given AGM and* $(\text{C}1, 2^*_{\preccurlyeq})$, $(\text{ZS}^*_{\preccurlyeq})$ *is equivalent to:*

(ZS$^*$) *If* $\neg C \in [\Psi * A] \cap [\Theta * B]$, $A \equiv_C \neg B$ *and* $\Psi$ *and* $\Theta$ *are in opposition modulo* $C$, *then so are* $\Psi * A$ *and* $\Theta * B$

#### 3.2.3 Representation Invariance

Regarding $(\text{RI}^*_{\preccurlyeq})$, we first define:

**Definition 9** $\iota$ is a *c-belief isomorphism* from $\Psi$ to $\Theta$ iff it is a 1:1 mapping from $L$ onto itself such that:

(i) $\iota(\top) = \top$ and $\iota(\bot) = \bot$
(ii) for any connective $c$ of arity $m$, $\iota(c(A_1, \ldots, A_m)) = c(\iota(A_1), \ldots, \iota(A_m))$
(iii) if $A$ is complete, then so is $\iota(A)$
(iv) $B \in [\Psi * A]$ *iff* $\iota(B) \in [\Theta * \iota(A)]$

In other words, the conditional beliefs of an agent in state $\Theta$ are obtained by $\iota$ from those in state $\Psi$ by a permutation of formulae that maps complete sentences onto complete sentences. In the terminology of [18], $\iota$ is a special case of a *"belief amount preserving symbol translation"* from $L$ onto itself: (i) and (ii) ensure that it is a "symbol translation" and (iii) that is satisfies their constraint of "Belief Amount Preservation", with (iv) imposing an additional constraint.

With this in hand, we then can show that:

**Proposition 13** *Given AGM,* $(\text{RI}^*_{\preccurlyeq})$ *is equivalent to*

(RI$^*$) $B \in [(\Psi * A) * C]$ *iff* $\iota(B) \in [(\Theta * \iota(A)) * \iota(C)]$, *for any c-belief isomorphism* $\iota$ *from* $\Psi$ *to* $\Theta$

## 4 Strengthening the Characteristic Postulates?

$(\text{IIAP}^*_{\preccurlyeq})$ significantly weakens a principle introduced under the name of "(IIA)" in [15], which simply corresponds to the embedded conditional: If $\rho_\Psi(x, y) = \rho_\Theta(x, y)$, then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * A}(x, y)$. $(\text{IIAI}^*_{\preccurlyeq})$ amounts to a similar weakening of a condition found in [8]. An interesting question, therefore, arises as to why the stronger principles do not figure in our characterisation.

One first observation is that it can be shown, as a corollary of Arrow's famous impossibility result in Social Choice [2], that the following holds:

**Proposition 14** *Lexicographic revision is the only elementary revision operator that satisfies the conjunction of the following two principles*

(IIAP+$^*_\preccurlyeq$) *If* $\rho_\Psi(x, y) = \rho_\Theta(x, y)$*, then* $\rho_{\Psi*A}(x, y) = \rho_{\Theta*A}(x, y)$
(IIAI+$^*_\preccurlyeq$) *If* $\rho_A(x, y) = \rho_B(x, y)$*, then* $\rho_{\Psi*A}(x, y) = \rho_{\Psi*B}(x, y)$

*or equivalently*

(IIA+$^*_\preccurlyeq$) *If* $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ *and* $\rho_A(x, y) = \rho_B(x, y)$*, then* $\rho_{\Psi*A}(x, y) = \rho_{\Theta*B}(x, y)$

A central question in Social Choice is the aggregation, into a group-level preference ordering, of the preference orderings of a set of $n$ individuals. The formal framework is specified as follows: a set of alternatives $W$, a tuple $P = \langle \preccurlyeq_{P1}, \ldots, \preccurlyeq_{Pn}\rangle$ of TPOs over that set, representing the individual-level preferences (a *preference profile*) and a *social welfare function* $f$ mapping preference profiles onto TPOs representing aggregate, group-level preferences. We write $x \preccurlyeq_{Pf} y$ to denote $\langle x, y\rangle \in f(P)$. Arrow famously showed that the following conditions of "Weak Pareto" (aka "Unanimity"), "Independence of Irrelevant Alternatives" and "Non-Dictatorship" on $f$ are jointly inconsistent, on the further "Unrestricted Domain" assumption that the domain of $f$ is the set $\mathrm{TPO}W^n$ of all preference profiles:

(WPn$^*_\preccurlyeq$) For all $P$ and $x, y \in W$, if, for all $1 \leq i \leq n$, $\rho_{Pi}(x, y) = 1$, then $\rho_{Pf}(x, y) = 1$
(IIAIn$^*_\preccurlyeq$) For all $P$, $P'$ and $x, y \in W$, if $\rho_{Pi}(x, y) = \rho_{P'i}(x, y)$, then $\rho_{Pf}(x, y) = \rho_{P'f}(x, y)$
(ND$^*_\preccurlyeq$) There does not exist $1 \leq i \leq n$, such that for all $P$ and $x, y \in W$, if $\rho_{Pi}(x, y) = 1$, then $\rho_{Pf}(x, y) = 1$

Formally-speaking, our problem of interest is a special two-person case of this one. The set of alternatives is $W$, the preference profile is $\langle \preccurlyeq_\Psi, \preccurlyeq_A\rangle$, with the prior state and input sentence playing the role of the individuals, and the aggregate preference ordering is $\preccurlyeq_{\Psi*A}$. It is easy to see, furthermore, that our condition (IIA$^*_\preccurlyeq$) is simply the Arrovian condition (IIAIn$^*_\preccurlyeq$) for our group of two individuals and that (C3$^*_\preccurlyeq$) similarly corresponds to a two-person version of the Arrovian condition (WPn$^*_\preccurlyeq$).

Given all this, Arrow's result then tells us that we have a dictatorship: Either (a) for all $\Psi, A \in L$ and $x, y, \in W$, if $\rho_\Psi(x, y) = 1$, then $\rho_{\Psi*A}(x, y) = 1$ or (b) for all $\Psi, A \in L$ and $x, y, \in W$, if $\rho_A(x, y) = 1$, then $\rho_{\Psi*A}(x, y) = 1$. But only option (b) is consistent with the AGM condition of Success, which translates semantically into $\min(\preccurlyeq_{\Psi*A}, W) \subseteq [\![A]\!]$: the "dictator" here must be the input $A$ rather than the prior state $\Psi$. (Indeed, assume that $\Psi$ is such that $\min(\preccurlyeq_\Psi, W) \subseteq [\![\neg A]\!]$. If $\Psi$ were the dictator, we would have $\min(\preccurlyeq_{\Psi*A}, W) \subseteq [\![\neg A]\!]$, contradicting Success.)

But (b), i.e. dictatorship by the input, is none other than the principle of "Recalcitrance" of [22], which, given (KM$^*_\preccurlyeq$), (C1$^*_\preccurlyeq$) and (C2$^*_\preccurlyeq$), which are sound for elementary operators, is known to characterise lexicographic revision. Hence lexicographic revision is the only elementary revision operator that satisfies the unqualified version of (IIA$^*_\preccurlyeq$).

This result, however, still leaves open the question of whether at least one of the two "component" principles of (IIA$^*_\preccurlyeq$) could have been involved in its unqualified version. This question can be answered in the negative:

**Proposition 15** *Lexicographic revision is the only elementary revision operator that satisfies* (IIAP+$^*_{\preccurlyeq}$).

**Proposition 16** *Lexicographic revision is the only elementary revision operator that satisfies* (IIAI+$^*_{\preccurlyeq}$).

Moving from (IIA$^*_{\preccurlyeq}$) to (ZS$^*_{\preccurlyeq}$), we note that the latter is extremely strong in its unqualified form, namely:

(ZS+$^*_{\preccurlyeq}$) If $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and $\rho_A(x, y) = -\rho_B(x, y)$, then $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$

Indeed, we can show that:

**Proposition 17** *Lexicographic revision is the only elementary revision operator that satisfies* (ZS+$^*_{\preccurlyeq}$).

We note in passing that, in [15], Glaister implicitly offers a further characterisation of $*_{\mathrm{L}}$ in a similar ballpark. His result involves (KM$^*_{\preccurlyeq}$), (IIAP+$^*_{\preccurlyeq}$) and what one might call a "holistic" weakening of (ZS+$^*_{\preccurlyeq}$). Assuming a principle of irrelevance of syntax, according to which, if $A \equiv B$, then $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Psi * B}$, this condition, which he calls "Reversal", can be presented as follows:

(Rev$^*_{\preccurlyeq}$) If $\forall x, y \in W$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and $\rho_A(x, y) = -\rho_B(x, y)$, then $\forall x, y \in W$, $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$

However, as we have just seen in Proposition 15, (IIAP+$^*_{\preccurlyeq}$) is really quite a strong principle and it turns out that the effective contribution of (Rev$^*_{\preccurlyeq}$) to Glaister's result is simply to derive (C2$^*_{\preccurlyeq}$), which we already assume as part of the characteristic properties of elementary operators.[5]

## 5 Elementary Contraction Operators

### 5.1 Semantic Characterisation

Just as we have discussed elementary revision operators, one can also consider elementary *contraction* operators, whose characteristic properties are obtained by swapping (KM$^{\dot{-}}_{\preccurlyeq}$) for (KM$^*_{\preccurlyeq}$), substituting the postulates of Chopra et al. for the DP postulates (see Section 2.2) and adapting, in the obvious manner, our principle (IIA$^*_{\preccurlyeq}$):

[5] Indeed, Glaister notes that, in the presence of AGM, (IIAP+$^*_{\preccurlyeq}$) entails (C1, 2$^*_{\preccurlyeq}$) (see his Fact 2.2 (b)). He also remarks, as we do in the proof of Proposition 15, that in the presence of the AGM postulates, (IIAP+$^*_{\preccurlyeq}$) gives us Recalcitrance (see his Fact 2.2 (a)). Furthermore, given (C1, 2$^*_{\preccurlyeq}$) and Recalcitrance, (Rev$^*_{\preccurlyeq}$) entails (C2$^*_{\preccurlyeq}$) (this is a consequence of his Facts 2.3 and 2.4). Finally, as we note in the proof of Proposition 15, Recalcitrance characterises $*_{\mathrm{L}}$ against the backdrop of (C1, 2$^*_{\preccurlyeq}$).

(IIA$^{\div}_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Theta, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!]) \cup \min(\preccurlyeq_\Theta, [\![\neg B]\!])$, then, if $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and $\rho_A(x, y) = \rho_B(x, y)$, then $\rho_{\Psi \div A}(x, y) = \rho_{\Theta \div B}(x, y)$

One can show, adjusting the proof of Theorem 1, that these are limited to: (1) the natural, aka "conservative", contraction operator $\div_{\mathrm{N}}$, (2) the priority, aka "moderate", contraction operator $\div_{\mathrm{P}}$ (see Nayak et al. [20] for both varieties of operator) and (3) a contraction operator $\div_{\mathrm{R}}$ that stands to restrained revision as priority contraction stands to lexicographic revision, which, to the best of our knowledge, is new to the literature (accordingly, we shall call this operator the "restrained contraction" operator). The relevant definitions are given as follows:

**Definition 10** If $x \in$ or $y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$, then:

$$\rho_{\Psi \div_{\mathrm{N/R/P}} A}(x, y) = \begin{cases} 1, \text{ if } x \in \text{ and } y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!]) \\ 0, \text{ if } x \in \text{ and } y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!]) \\ -1, \text{ if } x \notin \text{ and } y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!]) \end{cases}$$

If $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$, then:

$$\rho_{\Psi \div_{\mathrm{P}} A}(x, y) = \begin{cases} \rho_{\neg A}(x, y), \text{ if } \rho_{\neg A}(x, y) \neq 0 \\ \rho_\Psi(x, y), \text{ if } \rho_{\neg A}(x, y) = 0 \end{cases}$$

$$\rho_{\Psi \div_{\mathrm{R}} A}(x, y) = \begin{cases} \rho_\Psi(x, y), \text{ if } \rho_\Psi(x, y) \neq 0 \\ \rho_{\neg A}(x, y), \text{ if } \rho_\Psi(x, y) = 0 \end{cases}$$

$$\rho_{\Psi \div_{\mathrm{N}} A}(x, y) = \rho_\Psi(x, y)$$

See Fig. 3 for a graphic representation. If we compare Definition 10 to Definition 2, we can see that, for $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$, elementary contraction by $A$ simply behaves like elementary revision by $\neg A$. More specifically: for such $x$ and $y$, where $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$, we have $\rho_{\Psi \div A}(x, y) = \rho_{\Psi * \neg A}(x, y)$.

As with $*_{\mathrm{N}}$, $*_{\mathrm{R}}$ and $*_{\mathrm{L}}$, we can also provide characteristic matrices giving us, for all states $\Psi$, sentences $A$ and worlds $x, y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$, the value of the posterior relative rank $\rho_{\Psi \div A}(x, y)$ as a function of the values of the prior relative rank $\rho_\Psi(x, y)$ and input sentence relative rank $\rho_A(x, y)$. (Again, (KM$^{\div}_{\preccurlyeq}$) takes care of the posterior relative rank when $x$ or $y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$.) See Table 6.

## 5.2 Syntactic Characterisation

Syntactic characterisations for $\div_{\mathrm{N}}$ and $\div_{\mathrm{P}}$ were provided in [23]. They are given as follows, against the background assumption of AGM:

$$[(\Psi \div_{\mathrm{N}} A) \div_{\mathrm{N}} B] = \begin{cases} [\Psi \div_{\mathrm{N}} A] \cap [\Psi \div_{\mathrm{N}} \neg A \vee B], \text{ if } A \vee B \in [\Psi \div_{\mathrm{N}} B] \\ [\Psi \div_{\mathrm{N}} A] \cap [\Psi \div_{\mathrm{N}} A \vee B], \text{ if } \neg A \vee B \in [\Psi \div_{\mathrm{N}} B] \\ [\Psi \div_{\mathrm{N}} A] \cap [\Psi \div_{\mathrm{N}} \neg A \vee B] \cap [\Psi \div_{\mathrm{N}} A \vee B], \text{ otherwise} \end{cases}$$

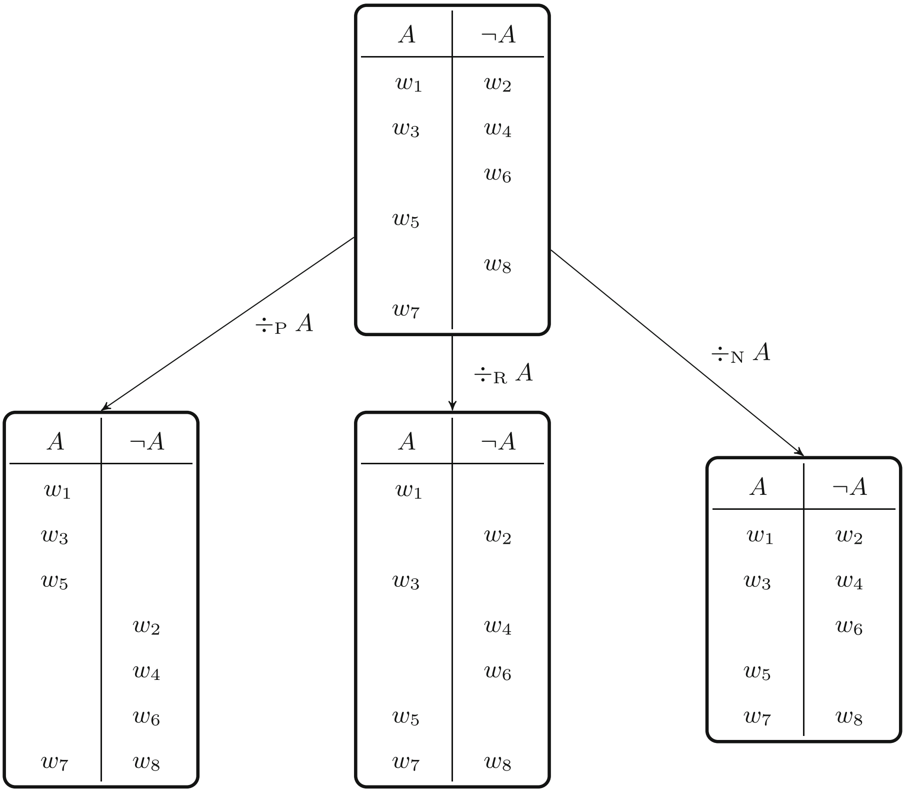


**Fig. 3** Elementary contraction by $A$

$$[(\Psi \div_{\mathrm{P}} A) \div_{\mathrm{P}} B] = \begin{cases} [\Psi \div_{\mathrm{P}} A] \cap [\Psi \div_{\mathrm{P}} \neg A \vee B], & \text{if } A \vee B \in \mathrm{Cn}(\varnothing) \\ [\Psi \div_{\mathrm{P}} A] \cap [\Psi \div_{\mathrm{P}} A \vee B], & \text{otherwise} \end{cases}$$

Regarding $\div_{\mathrm{R}}$, which is new to the literature, we can offer the following result:

**Proposition 18** *Given AGM, $\div_{\mathrm{R}}$ is characterised by the following property:*

$$[(\Psi \div_{\mathrm{R}} A) \div_{\mathrm{R}} B] = \begin{cases} [\Psi \div_{\mathrm{R}} A] \cap [\Psi \div_{\mathrm{R}} \neg A \vee B], & \textit{if } A \vee B \in [\Psi \div_{\mathrm{R}} B] \\ [\Psi \div_{\mathrm{R}} A] \cap [\Psi \div_{\mathrm{R}} A \vee B], & \textit{otherwise} \end{cases}$$

Interestingly, while we can see from the syntactic presentation of the elementary revision operators that, for any pair of inputs $A$ and $B$, there exists an input $C$ such that $[(\Psi * A) * B] = [\Psi * C]$, an analogous result does not appear to hold for their counterparts for contraction. And indeed, this suspicion turns out to be correct:

**Proposition 19** *Where $i \in \{P, R, N\}$, it is not the case that for any state $\Psi$ and sentences $A, B \in L$ there exists $C \in L$ such that $[(\Psi \div_i A) \div_i B] = [\Psi \div_i C]$.*

**Table 6** Mappings from prior and input sentence relative ranks to posterior relative rank for the operators $\div_{\mathrm{N}}$, $\div_{\mathrm{R}}$ and $\div_{\mathrm{P}}$

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **0** | 0 | 0 | 0 |
| **-1** | -1 | -1 | -1 |

(**a**) The natural contraction operator $\div_{\mathrm{N}}$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | 1 |
| **0** | -1 | 0 | 1 |
| **-1** | -1 | -1 | -1 |

(**b**) The restrained contraction operator $\div_{\mathrm{R}}$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | -1 |
| **0** | -1 | 0 | 1 |
| **-1** | -1 | -1 | 1 |

(**c**) The priority contraction operator $\div_{\mathrm{P}}$.

These matrices are simply obtained from the corresponding ones in Table 2 by flipping the values across the middle column

In [12], we find syntactic counterparts for (C1, $2^{\div}$)–($\mathrm{C4}^{\div}$), involving one contraction step followed by a revision step:

(C1, $2^{\div}$) If $\neg A \in \mathrm{Cn}(B)$ or $A \in \mathrm{Cn}(B)$, then $[(\Psi \div A) * B] = [\Psi * B]$
($\mathrm{C3}^{\div}$) If $\neg A \in [\Psi * B]$, then $\neg A \in [(\Psi \div A) * B]$
($\mathrm{C4}^{\div}$) If $A \notin [\Psi * B]$, then $A \notin [(\Psi \div A) * B]$

Finally, a syntactic version of ($\mathrm{IIA}^{\div}_{\preccurlyeq}$) can be straightforwardly given as follows, on the assumption that we can help ourselves to AGM and (C1, $2^{\div}_{\preccurlyeq}$):

($\mathrm{IIA}^{\div}$) If $\neg C \in [\Psi \div A] \cap [\Theta \div B]$, $A \equiv_C B$ and $\Psi$ and $\Theta$ agree modulo $C$, then so do $\Psi \div A$ and $\Theta \div B$

We note that, although agreement modulo $C$ is defined in terms of revision, we can simply use the syntactic counterpart of ($\mathrm{LI}_{\preccurlyeq}$), given as

(LI) $[\Psi * A] = \mathrm{Cn}([\Psi \div \neg A] \cup \{A\})$

to frame the concept in terms of contraction. We then have:

**Proposition 20** *Given* (HI), *states* $\Psi$ *and* $\Theta$ *agree modulo* $C$ *iff* $\mathrm{Cn}([\Psi \div \neg B \vee \neg C] \cup \{B \wedge C\}) = \mathrm{Cn}([\Theta \div \neg B \vee \neg C] \cup \{B \wedge C\})$ *for all* $B$

### 5.3 From Elementary Revision to Elementary Contraction and Back

We now consider the relation between elementary revision operators and elementary contraction operators, in the light of the recent work on extending the Harper and

Levi identities, ($\mathrm{HI}_{\preccurlyeq}$) and ($\mathrm{LI}_{\preccurlyeq}$), to the iterated case (see Section 2.2). The obvious question of interest here is whether or not the members of the pairs $\langle *_{\mathrm{L}}, \div_{\mathrm{P}}\rangle$, $\langle *_{\mathrm{N}}, \div_{\mathrm{N}}\rangle$ and $\langle *_{\mathrm{R}}, \div_{\mathrm{R}}\rangle$ of elementary revision and contraction operators might turn out to be interdefinable, moving (a) from contraction to revision by means of a plausible extension of ($\mathrm{LI}_{\preccurlyeq}$) and (b) from revision to contraction by means of a plausible extension of ($\mathrm{HI}_{\preccurlyeq}$).

Regarding (a) and the issue of moving from elementary contraction to elementary revision, the answer to our question is unfortunately mitigated. Indeed, while one can show that:

**Proposition 21** *Let* $\langle i, j\rangle \in \{\langle N, N\rangle, \langle R, R\rangle\}$. *Then, if* $*$ *is defined from* $\div_j$ *using* ($\mathrm{iLIRC}_{\preccurlyeq}$)*, then* $* = *_i$.

it remains the case that:

**Proposition 22** *If* $*$ *is defined from* $\div_{\mathrm{P}}$ *via* ($\mathrm{iLIRC}_{\preccurlyeq}$)*, then* $* \neq *_{\mathrm{L}}$.

So it is *not* the case that we can generally recover the elementary revision operators from their corresponding elementary contraction operators in the manner proposed in ($\mathrm{iLIRC}_{\preccurlyeq}$). Having said this, we note in passing that these revision/contraction operator pairs do nevertheless satisfy the non-reductive extension of ($\mathrm{LI}_{\preccurlyeq}$) that we mentioned, namely ($\mathrm{iLI}*_{\preccurlyeq}$):

**Proposition 23** *If* $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$*, then* $*_i$ *and* $\div_j$ *jointly satisfy* ($\mathrm{iLI}*_{\preccurlyeq}$).[6]

Turning now to (b) and the issue of the definability of elementary contraction from elementary revision, matters yet again do not look all that promising. The elementary contraction operators *can* be recovered from their corresponding elementary revision operators using a particular instance of the TeamQueue approach to extending ($\mathrm{HI}_{\preccurlyeq}$). Indeed, they can be so recovered by means of $\oplus_{\mathrm{TQ2}}$ TPO combination:

**Proposition 24** *Let* $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$. *Then, if* $\div$ *is defined from* $*_i$ *by* $\preccurlyeq_{\Psi\div A} = \preccurlyeq_{\Psi} \oplus_{\mathrm{TQ2}} \preccurlyeq_{\Psi *_i \neg A}$*, then* $\div = \div_j$.

*However*, as we have noted in Section 2.3, in [5], it was not $\oplus_{\mathrm{TQ2}}$ but an alternative TQ combinator, $\oplus_{\mathrm{STQ}}$, that was flagged out as being the most promising. What, then, would the ramifications of the use of $\oplus_{\mathrm{STQ}}$ be on the relation between elementary contraction and elementary revision? Well, first, neither $\div_{\mathrm{P}}$ nor $\div_{\mathrm{R}}$ can be *at all* defined from any revision operator using $\oplus_{\mathrm{STQ}}$. Indeed, Booth & Chandler [5, sec. 5.2] show that $\div_{\mathrm{P}}$ "cannot be recovered by combination of $\preccurlyeq_{\Psi}$ with *any* other ordering, by *any* combination method that satisfies" a condition that they call ($\mathrm{PAR}_{\oplus}$),

[6] In [11], it was suggested that the conjunction of ($\mathrm{iLI}*_{\preccurlyeq}$) and ($\mathrm{iLIRC}_{\preccurlyeq}$) entails that $* = *_{\mathrm{N}}$. This was incorrect, however, as evidenced by propositions 21 and 23: $*_{\mathrm{R}}$ and $\div_{\mathrm{R}}$ jointly satisfy both principles and yet $*_{\mathrm{R}} \neq *_{\mathrm{N}}$.

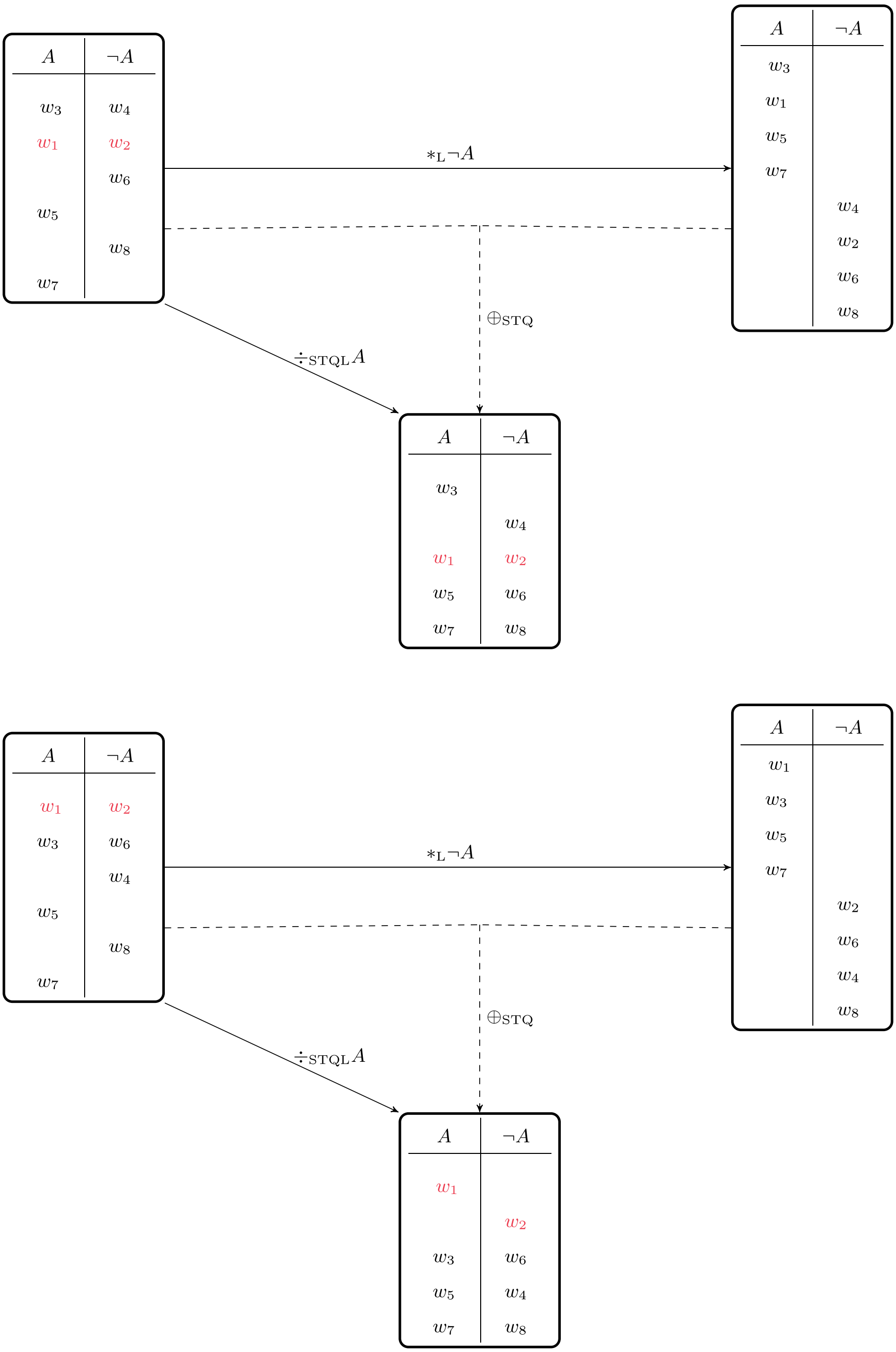


**Fig. 4** Failure of (IIAP$^{\div}_{\preccurlyeq}$) for $\div_{\mathrm{STQL}}$. The top diagram depicts $\preccurlyeq_{\Psi}$ and associated belief changes, while the second one pertains to $\preccurlyeq_{\Theta}$. While $\rho_{\Psi}(w_1, w_2) = 0$ and $\rho_{\Psi \div_{\mathrm{STQL}} A}(w_1, w_2) = 0$, we also have $\rho_{\Theta}(w_1, w_2) = 0$ but $\rho_{\Theta \div_{\mathrm{STQL}} A}(w_1, w_2) = -1$

which is satisfied by the proposal. It can be verified that their comments carry over to $\div_{\mathrm{R}}$.[7] Second, in using $\oplus_{\mathrm{STQ}}$, both $*_{\mathrm{N}}$ and $*_{\mathrm{R}}$ end up being mapped onto $\div_{\mathrm{N}}$ (see [5, Proposition 12]). Third, and finally, $*_{\mathrm{L}}$ gives us which they call the *STQ-lex contraction* operator ($\div_{\mathrm{STQL}}$) which is not elementary, since it violates ($\mathrm{IIAP}^{\div}_{\preccurlyeq}$). See Fig. 4.[8]

In our view, the above observations do point to somewhat of a quandary. Indeed, on the one hand, the plausibility of the principle of Independence of Irrelevant Alternatives for revision seems on par with that of its counterpart for contraction: those who find the one to be reasonable, would presumably have to find the other so too. So, assuming the compelling postulates of AGM, DP and Chopra et al., this then leaves elementary revision and elementary contraction on an equal footing in terms of plausibility. However, it would appear that some prima facie reasonable extensions of ($\mathrm{LI}_{\preccurlyeq}$) and ($\mathrm{HI}_{\preccurlyeq}$) make the coexistence of elementary revision and contraction problematic.

## 6 Concluding Comments

In this paper, we have notably shown how the three most popular "concrete" revision operators–natural, restrained and lexicographic revision–can be collectively characterised by supplementing the standard Darwiche-Pearl postulates with a principle of "Independence of Irrelevant Alternatives", ($\mathrm{IIA}^{*}_{\preccurlyeq}$), inspired by the Social Choice literature. A similar family of operators was found to be definable for iterated contraction.

As we have noted a number of times, however, elementary revision operators all satisfy the principle ($\mathrm{TPOR}^{*}_{\preccurlyeq}$), which problematically, in our view, forces an identification of doxastic states with TPOs. In the course of our brief discussion of ($\mathrm{IIA}^{*}_{\preccurlyeq}$), we saw, in Proposition 5, that the culprit here is the principle ($\mathrm{IIAP}^{*}_{\preccurlyeq}$). A natural question, then, is whether the latter can be weakened in a sensible manner, so as to avoid the problematic implication.

One promising avenue might be to investigate how much of ($\mathrm{IIAP}^{*}_{\preccurlyeq}$) is retained in the family of "basic ordinal interval" (BOI) revision operators, which includes the elementary operators among its members. BOI revision generalises the POI revision of [6], by relaxing a particular condition on its semantic representation.[9] The family of POI revision operators includes restrained and lexicographic revision but unfortunately excludes natural revision. POI revision violates ($\mathrm{IIAP}^{*}_{\preccurlyeq}$), while retaining the

[7] Indeed, they note that ($\mathrm{PAR}_{\oplus}$) gives us the following principle: If $x \prec_{\Psi \div A} y$ for every $x \in S^c, y \in S$, then $\min(\preccurlyeq_{\Psi}, S) \subseteq \min(\preccurlyeq_{\Psi \div A}, S)$. But this condition isn't satisfied by $\div_{\mathrm{R}}$. Suppose that $W = \{w, x, y, z\}$ and $x \prec_{\Psi} z \prec_{\Psi} \{y, w\}$. Let $[\![A]\!] = \{x, y\}$. Then, $\{x, z\} \prec_{\Psi \div_{\mathrm{R}} A} w \prec_{\Psi \div_{\mathrm{R}} A} y$. Let $S = \{y, w\}$. Then $u \prec_{\Psi \div_{\mathrm{R}} A} v$ for every $u \in S^c, v \in S, y \in \min(\preccurlyeq_{\Psi}, S)$ but $y \notin \min(\preccurlyeq_{\Psi \div_{\mathrm{R}} A}, S)$.

[8] The same comments apply, incidentally, to the *lexicographic contraction* operator $\div_{\mathrm{L}}$ [20], which also violates ($\mathrm{IIAP}^{\div}_{\preccurlyeq}$).

[9] In POI revision, doxastic states are associated with a binary relation $\leq$ over $W^{\pm} = \{w^i \mid w \in W \text{ and } i \in \{-, +\}\}$ that is notably required to satisfy the condition that $x^+ < x^-$. For BOI revision, this condition is relaxed to $x^+ \leq x^-$.

remaining properties of elementary revision, namely ($\mathrm{KM}^*_{\preccurlyeq}$), ($\mathrm{C1},2^*_{\preccurlyeq}$)-($\mathrm{C4}^*_{\preccurlyeq}$) and ($\mathrm{IIAI}^*_{\preccurlyeq}$). We conjecture that these properties will also be retained in BOI revision.

## Appendix A: Proofs

**Proposition 1** *Given* ($KM^{\dot{-}}_{\preccurlyeq}$)*,* ($\mathrm{IIA}^*_{\preccurlyeq}$) *does not imply any of* ($\mathrm{C1},2^*_{\preccurlyeq}$)-($\mathrm{C4}^*_{\preccurlyeq}$)

*Proof* Let $*$ be defined in such a way that, for all states $\Psi$, $x, y \in W$ and $A \in L$:

- $\min(\preccurlyeq_{\Psi * A}, W) = \min(\preccurlyeq_{\Psi}, [\![A]\!])$
- If $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$, then
  - If $\rho_{\Psi}(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = -1$
  - If $\rho_{\Psi}(x, y) = 0$, then $\rho_{\Psi * A}(x, y) = 0$

Essentially, this operator will set the $\preccurlyeq_{\Psi}$-minimal $A$-worlds as $\preccurlyeq_{\Psi * A}$-minimal (thus satisfying AGM) and simply "flip" the remainder of the ordering.

It is easy to see that $*$ satisfies ($\mathrm{IIA}^*_{\preccurlyeq}$): Assume that $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Theta}, [\![B]\!])$, that $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y)$ and that $\rho_A(x, y) = \rho_B(x, y)$. We need to show that $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$. For this, we consider two cases:

(a) Assume $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y) = 1$. Then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y) = -1$.
(b) Assume $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y) = 0$. Then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y) = 0$.

In either case, $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$, as required.

Clearly, however, each of ($\mathrm{C1}^*_{\preccurlyeq}$) to ($\mathrm{C4}^*_{\preccurlyeq}$) will be violated. A counterexample is provided in Fig. 5. There, ($\mathrm{C1}^*_{\preccurlyeq}$) fails because, for instance, $y \prec_{\Psi} x$ but $x \prec_{\Psi * A} y$. Regarding ($\mathrm{C2}^*_{\preccurlyeq}$), we have $t \prec_{\Psi} s$ but $s \prec_{\Psi * A} t$. Regarding ($\mathrm{C3}^*_{\preccurlyeq}$) and ($\mathrm{C4}^*_{\preccurlyeq}$), we have $z \prec_{\Psi} s$ but $s \prec_{\Psi * A} z$. □

**Proposition 2** *Given* ($KM^*_{\preccurlyeq}$)*,* ($\mathrm{C1},2^*_{\preccurlyeq}$)-($\mathrm{C4}^*_{\preccurlyeq}$) *do not jointly imply* ($\mathrm{IIA}^*_{\preccurlyeq}$)

*Proof* Consider the operator $*$ defined as follows: for all states $\Psi$ and $A \in L$, $\Psi * A = \Psi *_{\mathrm{L}} A$ if $\preccurlyeq_{\Psi}$ is a chain (i.e. an antisymmetric TPO, such that, for all $x, y \in W$, if $x \sim_{\Psi} y$, then $x \neq y$), and $\Psi * A = \Psi *_{\mathrm{R}} A$ otherwise.

It is easily verified that $*$ satisfies ($\mathrm{KM}^*_{\preccurlyeq}$) and ($\mathrm{C1},2^*_{\preccurlyeq}$)-($\mathrm{C4}^*_{\preccurlyeq}$). But $*$ does not satisfy ($\mathrm{IIA}^*_{\preccurlyeq}$), as can be seen from the following countermodel: Let $W = \{x, y, z, w\}$, $[\![A]\!] = \{z, x\}$, $z \prec_{\Psi} w \prec_{\Psi} x \prec_{\Psi} y$ and $\{z, w\} \prec_{\Theta} x \prec_{\Theta} y$, so that, notably, $\Psi * A = \Psi *_{\mathrm{L}} A$ but $\Theta * A = \Theta *_{\mathrm{R}} A$. Then $w, x \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Theta}, [\![A]\!])$, $\rho_{\Psi}(w, x) = \rho_{\Theta}(w, x) = 1$, but $\rho_{\Psi * A}(w, x) = -1$ and $\rho_{\Theta * A}(w, x) = 1$. □

**Proposition 3** *Given* ($\mathrm{UD}_{\preccurlyeq}$)*,* ($\mathrm{IIA}^*_{\preccurlyeq}$) *is equivalent to the conjunction of the following principles of "Independence of Irrelevant Alternatives" with respect to the "Prior" and the "Input", respectively:*

($\mathrm{IIAP}^*_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Theta}, [\![A]\!])$ then, if $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y)$ then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * A}(x, y)$

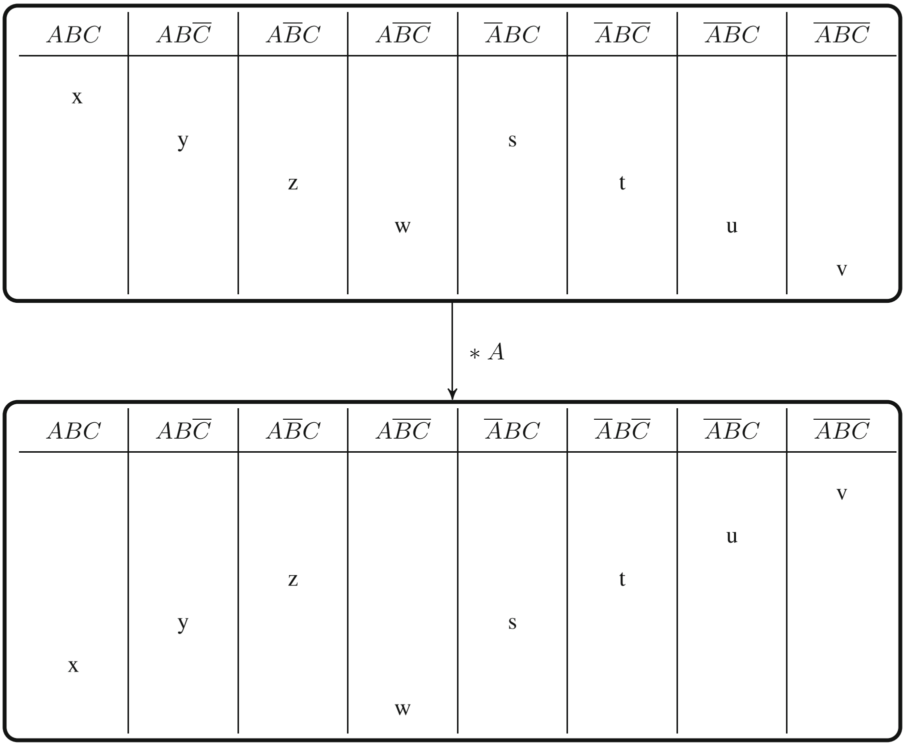


**Fig. 5** Model demonstrating the fact that the conjunction of ($\mathrm{KM}^*_{\preccurlyeq}$) and ($\mathrm{IIA}^*_{\preccurlyeq}$) does not imply any of ($\mathrm{C1}, 2^*_{\preccurlyeq}$) to ($\mathrm{C4}^*_{\preccurlyeq}$)

($\mathrm{IIAI}^*_{\preccurlyeq}$) If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Psi, [\![B]\!])$ then, if $\rho_A(x, y) = \rho_B(x, y)$ then $\rho_{\Psi * A}(x, y) = \rho_{\Psi * B}(x, y)$

*Proof* From ($\mathrm{IIA}^*_{\preccurlyeq}$) to ($\mathrm{IIAP}^*_{\preccurlyeq}$) and ($\mathrm{IIAI}^*_{\preccurlyeq}$): simply set $\Psi = \Theta$ in the former case and $A = B$ in the latter.

From ($\mathrm{IIAP}^*_{\preccurlyeq}$) and ($\mathrm{IIAI}^*_{\preccurlyeq}$) to ($\mathrm{IIA}^*_{\preccurlyeq}$): Suppose $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and $\rho_A(x, y) = \rho_B(x, y)$. We need to show $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$. Consider a state $\Phi$ such that, for all $z \notin \{x, y\}, z \prec_\Phi x, z \prec_\Phi y$ and $\rho_\Phi(x, y) = \rho_\Psi(x, y) = \rho_\Theta(x, y)$. Such a state exists by ($\mathrm{UD}_{\preccurlyeq}$). Since $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, by the construction of $\preccurlyeq_\Phi$, we must have $x, y \notin \min(\preccurlyeq_\Phi, [\![A]\!])$. Hence, by ($\mathrm{IIAP}^*_{\preccurlyeq}$), we have (1) $\rho_{\Psi * A}(x, y) = \rho_{\Psi * A}(x, y)$. Similarly, since $x, y \notin \min(\preccurlyeq_\Theta, [\![B]\!])$, we have $x, y \notin \min(\preccurlyeq_\Phi, [\![B]\!])$ and so, by ($\mathrm{IIAP}^*_{\preccurlyeq}$), we have: (2) $\rho_{\Theta * B}(x, y) = \rho_{\Theta * B}(x, y)$. Finally, from the fact that $x, y \notin \min(\preccurlyeq_\Phi, [\![A]\!]) \cup \min(\preccurlyeq_\Phi, [\![B]\!])$ and $\rho_A(x, y) = \rho_B(x, y)$, by ($\mathrm{IIAI}^*_{\preccurlyeq}$), we have: (3) $\rho_{\Phi * A}(x, y) = \rho_{\Phi * B}(x, y)$. The required result then follows from (1), (2) and (3). $\square$

**Proposition 4** ($\mathrm{IIAI}^*_{\preccurlyeq}$) *does not imply* ($\mathrm{IIAP}^*_{\preccurlyeq}$) *or vice versa, even in the presence of* ($\mathrm{KM}^*_{\preccurlyeq}$) *and* ($\mathrm{C1}, 2^*_{\preccurlyeq}$)-($\mathrm{C4}^*_{\preccurlyeq}$).

*Proof* For an operator satisfying (KM$^*_\preccurlyeq$), (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$), and (IIAI$^*_\preccurlyeq$) but not (IIAP$^*_\preccurlyeq$), we can consider the operator $*$ introduced in the proof of Proposition 2 above. It was defined as follows: for all states $\Psi$ and $A \in L$, $\Psi * A = \Psi *_{\mathrm{L}} A$ if $\preccurlyeq_\Psi$ is a chain (i.e. an antisymmetric TPO, such that, for all $x, y \in W$, if $x \sim_\Psi y$, then $x \neq y$), and $\Psi * A = \Psi *_{\mathrm{R}} A$ otherwise.

We've already noted above that it satisfies (KM$^*_\preccurlyeq$) and (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$). (IIAI$^*_\preccurlyeq$) is also satisfied, since, for any given state, the same method (i.e. $*_{\mathrm{L}}$ or $*_{\mathrm{R}}$) is used for all sentences and this method satisfies (IIAI$^*_\preccurlyeq$). But $*$ does not satisfy (IIAP$^*_\preccurlyeq$), as can be seen from the countermodel given in the proof of Proposition 2 above. For convenience, we repeat it here. Let $W = \{x, y, z, w\}$, $[\![A]\!] = \{z, x\}$, $z \prec_\Psi w \prec_\Psi x \prec_\Psi y$ and $\{z, w\} \prec_\Theta x \prec_\Theta y$, so that, notably, $\Psi * A = \Psi *_{\mathrm{L}} A$ but $\Theta * A = \Theta *_{\mathrm{R}} A$. Then $w, x \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![A]\!])$, $\rho_\Psi(w, x) = \rho_\Theta(w, x) = 1$, $\rho_A(w, x) - 1$, but $\rho_{\Psi * A}(w, x) = -1$ and $\rho_{\Theta * A}(w, x) = 1$.

For an operator that satisfies (KM$^*_\preccurlyeq$), (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$), and (IIAP$^*_\preccurlyeq$) but not (IIAI$^*_\preccurlyeq$), consider the operator $*$ defined as follows: For all states $\Psi$ and $A \in L$, $\Psi * A = \Psi *_{\mathrm{L}} A$ if $|[\![\neg A]\!]| = 1$, and $\Psi * A = \Psi *_{\mathrm{R}} A$ otherwise.

Again, as for the previous operator, it is easily verified that $*$ satisfies (KM$^*_\preccurlyeq$), and (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$). (IIAP$^*_\preccurlyeq$) is also satisfied, since, , for any given sentence, the same method (i.e. $*_{\mathrm{L}}$ or $*_{\mathrm{R}}$) is used for all states and this method satisfies (IIAI$^*_\preccurlyeq$). But $*$ does not satisfy (IIAI$^*_\preccurlyeq$), as can be seen from the following countermodel: Let $W = \{x, y, z, w\}$, $[\![A]\!] = \{x, z, w\}$, $[\![B]\!] = \{x, w\}$ and $w \prec_\Psi y \prec_\Psi x \prec_\Psi z$. Then $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Psi, [\![B]\!])$ and $\rho_A(x, y) = 1 = \rho_B(x, y)$, but $\rho_{\Psi * A}(x, y) = 1$, whereas $\rho_{\Psi * B}(x, y) = -1$. □

**Proposition 5** *Given* (KM$^{\dot{-}}_\preccurlyeq$), (IIAP$^*_\preccurlyeq$) *entails* (TPOR$^*_\preccurlyeq$)

*Proof* Assume that $\preccurlyeq_\Psi = \preccurlyeq_\Theta$. Consider arbitrary $x, y \in W$. If either $x$ or $y$ is in $\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![A]\!])$, then $\rho_{\preccurlyeq_{\Psi * A}}(x, y) = \rho_{\preccurlyeq_{\Theta * A}}(x, y)$ by virtue of (KM$^*_\preccurlyeq$). If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![A]\!])$, then $\rho_{\preccurlyeq_{\Psi * A}}(x, y) = \rho_{\preccurlyeq_{\Theta * A}}(x, y)$ by virtue of (IIAP$^*_\preccurlyeq$). Hence $\preccurlyeq_{\Psi * A} = \preccurlyeq_{\Theta * A}$, as required. □

**Proposition 6** *Given* ($KM^*_\preccurlyeq$), (C1, 2$^*_\preccurlyeq$)-(C4$^*_\preccurlyeq$), (IIAI$^*_\preccurlyeq$) *is equivalent to the conjunction of:*

($\beta 1^*_\preccurlyeq$) If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $\rho_A(x, y) = 1$ and $\rho_{\Psi * A}(x, y) \leq 0$, then $\rho_{\Psi * C}(x, y) \leq 0$

($\beta 2^*_\preccurlyeq$) If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $\rho_A(x, y) = 1$ and $\rho_{\Psi * A}(x, y) = -1$, then $\rho_{\Psi * C}(x, y) = -1$

*Proof* The proof of this claim closely resembles the proof of Proposition 3 of [8]. For ease of comparison, we use the $\preccurlyeq$-notation, rather than the $\rho$-notation, so that ($\beta 1^*_\preccurlyeq$) and ($\beta 2^*_\preccurlyeq$) are presented as follows:

($\beta 1^*_\preccurlyeq$) If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \prec_A y$, and $y \preccurlyeq_{\Psi * A} x$, then $y \preccurlyeq_{\Psi * C} x$

($\beta 2^*_\preccurlyeq$) If $x \notin \min(\preccurlyeq_\Psi, [\![C]\!])$, $x \prec_A y$, and $y \prec_{\Psi * A} x$, then $y \prec_{\Psi * C} x$ □

We first establish the following lemma:

**Lemma 4** *Given* $(\mathrm{C}1, 2^*_{\preccurlyeq})$–$(\mathrm{C}4^*_{\preccurlyeq})$,

*(a)* *If* $x \prec_A y$ *and* $\rho_A(x, y) \neq \rho_C(x, y)$*, then, if* $y \preccurlyeq_{\Psi * A} x$*, then* $y \preccurlyeq_{\Psi * C} x$.
*(b)* *If* $x \prec_A y$ *and* $\rho_A(x, y) \neq \rho_C(x, y)$*, then, if* $y \prec_{\Psi * A} x$*, then* $y \prec_{\Psi * C} x$.

*We simply derive (a), since the proof of (b) is analogous. Assume that* $x \prec_A y$ *and* $\rho_A(x, y) \neq \rho_C(x, y)$*. In other words:* $x \in [\![A]\!], y \in [\![\neg A]\!]$*, and either (i)* $x \in [\![C]\!], y \in [\![C]\!]$*, (ii)* $x \in [\![\neg C]\!], y \in [\![\neg C]\!]$ *or (iii)* $x \in [\![\neg C]\!], y \in [\![C]\!]$*. Assume that* $y \preccurlyeq_{\Psi * A} x$*. From this and* $x \in [\![A]\!], y \in [\![\neg A]\!]$*, it follows, by* $(\mathrm{C}3^*_{\preccurlyeq})$*, that* $y \preccurlyeq_{\Psi} x$*. From this, if either (i), (ii) or (iii) hold, then, by* $(\mathrm{C}1, 2^*_{\preccurlyeq})$, $(\mathrm{C}2^*_{\preccurlyeq})$*, and* $(\mathrm{C}4^*_{\preccurlyeq})$*, respectively, we have* $y \preccurlyeq_{\Psi * C} x$*, as required. This completes the proof of Lemma 4.*

*With this in hand, we can derive each direction of the equivalence:*

*(a)* *From* $(\mathrm{IIAI}^*_{\preccurlyeq})$ *to* $(\beta 1^*_{\preccurlyeq})$ *and* $(\beta 2^*_{\preccurlyeq})$*: Regarding* $(\beta 1^*_{\preccurlyeq})$*, assume* $x \notin \min(\preccurlyeq_{\Psi}, [\![C]\!])$*,* $x \prec_A y$*, and* $y \preccurlyeq_{\Psi * A} x$*. We need to show that* $y \preccurlyeq_{\Psi * C} x$*. If* $\rho_A(x, y) \neq \rho_C(x, y)$*, then the required result follows by principle (a) of Lemma 4. So assume* $\rho_A(x, y) = \rho_C(x, y)$*, and hence that* $x \prec_C y$*. We now establish that* $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Psi}, [\![C]\!])$*. We already have* $x \notin \min(\preccurlyeq_{\Psi}, [\![C]\!])$*. Since, by* $x \prec_C y$*, it follows that* $y \in [\![\neg C]\!]$*, we therefore have* $y \notin \min(\preccurlyeq_{\Psi}, [\![C]\!])$*. Furthermore, by* $x \prec_A y$*, it follows that* $y \in [\![\neg A]\!]$ *and so* $y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$*. Finally, assume for contradiction that* $x \in \min(\preccurlyeq_{\Psi}, [\![A]\!])$*. Then* $x \in \min(\preccurlyeq_{\Psi * A}, W)$*, by* $(\mathrm{KM}^*_{\preccurlyeq})$*. Since* $y \in [\![\neg A]\!]$*, by* $(\mathrm{KM}^*_{\preccurlyeq})$*,* $y \notin \min(\preccurlyeq_{\Psi * A}, W)$*. Hence* $x \prec_{\Psi * A} y$*, contradicting* $y \preccurlyeq_{\Psi * A} x$*. So we can infer that* $x \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$*. With this in hand, we can apply* $(\mathrm{IIAI}^*_{\preccurlyeq})$ *to derive* $y \preccurlyeq_{\Psi * C} x$*, as required. The derivation of* $(\beta 2^*_{\preccurlyeq})$ *is analogous, but using principle (b) of Lemma 4.*

*(b)* *From* $(\beta 1^*_{\preccurlyeq})$ *and* $(\beta 2^*_{\preccurlyeq})$ *to* $(\mathrm{IIAI}^*_{\preccurlyeq})$*: Assume that* $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Psi}, [\![C]\!])$ *and that* $\rho_A(x, y) = \rho_C(x, y)$*. We want to show that* $x \preccurlyeq_{\Psi * A} y$ *iff* $x \preccurlyeq_{\Psi * C} y$*. By symmetry, it suffices for this to show that* $x \preccurlyeq_{\Psi * A} y$ *implies* $x \preccurlyeq_{\Psi * C} y$*. So assume* $x \preccurlyeq_{\Psi * A} y$*. Since* $\rho_A(x, y) = \rho_C(x, y)$*, we have three cases to consider:*

  *(i)* $\rho_A(x, y) = \rho_C(x, y) = 1$*: Assume for contradiction that* $y \prec_{\Psi * C} x$*. From this,* $x \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$ *and* $x \prec_C y$*, it follows by* $(\beta 2^*_{\preccurlyeq})$ *that* $y \prec_{\Psi * A} x$*, contradicting* $x \preccurlyeq_{\Psi * A} y$*. Hence* $x \preccurlyeq_{\Psi * C} y$*, as required.*
  *(ii)* $\rho_A(x, y) = \rho_C(x, y) = 0$*: It follows from this, via* $(\mathrm{C}1, 2^*_{\preccurlyeq})$*, that* $x \preccurlyeq_{\Psi * A} y$ *iff* $x \preccurlyeq_{\Psi} y$ *iff* $x \preccurlyeq_{\Psi * C} y$*. Hence* $x \preccurlyeq_{\Psi * C} y$*, as required.*
  *(iii)* $\rho_A(x, y) = \rho_C(x, y) = -1$*: By* $(\beta 1^*_{\preccurlyeq})$*, it follows, from* $x \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$*,* $y \prec_A x$*, and* $x \preccurlyeq_{\Psi * A} y$*, that* $x \preccurlyeq_{\Psi * C} y$*, as required.*

**Lemma 2** *The conjunction of* $(\mathrm{KM}^*_{\preccurlyeq})$ *transitivity of* $\preccurlyeq_{\Psi * A}$, $(\mathrm{PI}^*_{\preccurlyeq})$ *and* $(\mathrm{IIA}^*_{\preccurlyeq})$ *implies the following principle of "Zero Symmetry":*

$(\mathrm{ZS}^*_{\preccurlyeq})$ If $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Theta}, [\![B]\!])$, $\rho_{\Psi}(x, y) = -\rho_{\Theta}(x, y)$ and $\rho_A(x, y) = -\rho_B(x, y)$ then, $\rho_{\Psi * A}(x, y) = -\rho_{\Psi * B}(x, y)$

*Proof* ($\mathrm{KM}^*_{\preccurlyeq}$) and ($\mathrm{IIA}^*_{\preccurlyeq}$) jointly tell us that that, for a pair of worlds $\{x, y\}$, the posterior relative rank $\rho_{\Psi * A}(x, y)$ is determined by the prior relative rank $\rho_{\Psi}(x, y)$ and input sentence relative rank $\rho_A(x, y)$ (although this mapping may be different for different pairs of worlds). With the case in which $x$ or $y \in \min(\preccurlyeq_{\Psi}, [\![A]\!])$ being taken care of by ($\mathrm{KM}^*_{\preccurlyeq}$), the behaviour of $*$ with respect to $\{x, y\}$ can therefore be represented in the form of a matrix giving us, for $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$, the values of $\rho_{\Psi * A}(x, y)$ as a function of those of $\rho_{\Psi}(x, y)$ and $\rho_A(x, y)$.

Since we assume $A$ to be consistent and hence that $\min(\preccurlyeq_{\Psi}, [\![A]\!])$ is non-empty, we must have strictly more than two, and therefore (since $|W| = 2^n$) at least four, worlds. Let these worlds be $x$, $y$, $z$ and $w$. We will consider the matrices for the pairs $\langle x, y\rangle$, $\langle y, z\rangle$ and $\langle z, x\rangle$ in Table 7.

($\mathrm{PI}^*_{\preccurlyeq}$) tells us that the value of the central cell in each matrix is 0. To establish ($\mathrm{ZS}^*_{\preccurlyeq}$), we then just need to show that $a_i = -b_i$, for $1 \leq i \leq 4$. We shall simply show that $a_1 = -b_1$, since the proof strategy is identical for other values of $i$.

Let $\Psi$ be such that $\rho_{\Psi}(x, z) = \rho_{\Psi}(x, y) = \rho_{\Psi}(y, z) = 0$. Let $A$ be such that $x, z \in [\![A]\!]$ but $y \notin [\![A]\!]$ (with $x, z \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$; this can be ensured by designating the fourth world $w$ to be the sole member of that set) and so $\rho_A(x, z) = 0$. Since $\rho_{\Psi}(x, z) = 0$, we therefore have $\rho_{\Psi * A}(x, z) = 0$. We also have $\rho_A(x, y) = 1$ and $\rho_A(y, z) = -1$. Therefore $\rho_{\Psi * A}(x, y) = a_1$ and $\rho_{\Psi * A}(y, z) = d_1$. Assume $a_1 \neq -d_1$ for reductio (so that $\langle a_1, d_1\rangle$ is equal to either $\langle 1, 0\rangle$, $\langle 0, 1\rangle$, $\langle -1, 0\rangle$, $\langle 0, -1\rangle$, $\langle 1, 1\rangle$, or $\langle -1, -1\rangle$). It then follows, by transitivity of $\preccurlyeq_{\Psi * A}$, that $\rho_{\Psi * A}(x, z) = 1$ or $-1$. But this contradicts our earlier finding that $\rho_{\Psi * A}(x, z) = 0$. Hence $a_1 = -d_1$. By similar reasoning, we can establish that $e_1 = -d_1$ (let $A$ be such that $y, x \in [\![A]\!]$ but $z \notin [\![A]\!]$, with $y, x \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$ and hence $\rho_A(y, x) = 0$) and $e_1 = -b_1$ (let $A$ be such that $z, y \in [\![A]\!]$ but $x \notin [\![A]\!]$, with $z, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$, and hence $\rho_A(z, y) = 0$) and hence that $a_1 = -b_1$, as required. □

**Table 7** Matrices for proof of Lemma 2

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | $a_4$ | $a_3$ | $a_2$ |
| **0** | $a_1$ | 0 | $b_1$ |
| **-1** | $b_2$ | $b_3$ | $b_4$ |

**(a)** Matrix for $\langle x, y\rangle$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | $c_4$ | $c_3$ | $c_2$ |
| **0** | $c_1$ | 0 | $d_1$ |
| **-1** | $d_2$ | $d_3$ | $d_4$ |

**(b)** Matrix for $\langle y, z\rangle$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | $e_4$ | $e_3$ | $e_2$ |
| **0** | $e_1$ | 0 | $f_1$ |
| **-1** | $f_2$ | $f_3$ | $f_4$ |

**(c)** Matrix for $\langle z, x\rangle$.

**Lemma 3** *The conjunction of* (KM$^*_{\preccurlyeq}$), *transitivity of* $\preccurlyeq_{\Psi * A}$, (PI$^*_{\preccurlyeq}$) *and* (IIA$^*_{\preccurlyeq}$) *implies the following "Representation Invariance" principle:*

(RI$^*_{\preccurlyeq}$) $\rho_{\Psi * A}(x, y) = \rho_{\Theta * \pi(A)}(\pi(x), \pi(y))$, for any order isomorphism $\pi$ from $\preccurlyeq_\Psi$ to $\preccurlyeq_\Theta$

*Proof* We consider 3 cases, depending on $x$ and $y$'s membership of $\min(\preccurlyeq_\Psi, [\![A]\!])$: (a) one of $x$ or $y$ is in the set, (b) both $x$ and $y$ are in the set and (c) neither $x$ nor $y$ are in the set.

For each case, we shall prove the identity of the matrices relating to the pair $\langle x, y\rangle$ and revision by $A$ on the one hand, and the pair $\langle \pi(x), \pi(y)\rangle$ and revision by $\pi(A)$, on the other. Since, if $\pi$ is an order isomorphism from $\preccurlyeq_\Psi$ to $\preccurlyeq_\Theta$, then $\rho_\Psi(x, y) = \rho_\Theta(\pi(x), \pi(y))$ and $\rho_A(x, y) = \rho_{\pi(A)}(\pi(x), \pi(y))$, it then follows that $\rho_{\Psi * A}(x, y) = \rho_{\Theta * \pi(A)}(\pi(x), \pi(y))$, as required.

Assume (a), so that, for example, $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$, $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ (the other case in analogous). Since we then have $\rho_A(x, y) \geq 0$, it must be the case that either (i) $\rho_A(x, y) = 1$, or (ii) $\rho_A(x, y) = 0$. Furthermore, if (ii) is the case, it must be the case that $\rho_\Psi(x, y) = 1$. The relevant matrix is then given in Table 8a, with impossible combinations of values indicated by "$\times$". Since, for all $x \in W$, $A \in L$ and order isomorphisms $\pi$ from $\preccurlyeq_\Psi$ to $\preccurlyeq_\Theta$, we have $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$ iff $\pi(x) \in \min(\preccurlyeq_\Psi, [\![\pi(A)]\!])$, the same matrix characterises the pair $\langle \pi(x), \pi(y)\rangle$ under revision by $\pi(A)$. Assume (b). The reasoning is analogous to the one provided in relation to (a) above, this time with reference to the matrix given in Table 8b. Assume (c). Again, as we have noted above, since we assume $A$ to be consistent and hence that $\min(\preccurlyeq_\Psi, [\![A]\!])$ is non-empty, we must have at least 4 worlds. Here we make use of the transitivity of $\preccurlyeq_{\Psi * A}$, much as we did in the proof of Lemma 2, and consider, in addition to $\langle x, y\rangle$, the pairs $\langle y, z\rangle$ and $\langle z, x\rangle$. The matrices for these pairs are given in Table 7 above.

Using the strategy applied in relation to the same case in the proof of Lemma 2, we can show that, for $1 \leq i \leq 4$, not only $a_i = -b_i$ and $c_i = -d_i$, but also $a_i = -d_i$ and so $a_i = c_i$. So the matrix for all sentences is the same for all pairs of worlds. In particular, the matrix relating to the pair $\langle x, y\rangle$ and revision by $A$ on the one hand, is identical to that relating the pair $\langle \pi(x), \pi(y)\rangle$ and revision by $\pi(A)$, on the other.

**Table 8** Matrices for cases (a) and (b) in proof of Lemma 2

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | 1 | 1 | $\times$ |
| **0** | 1 | $\times$ | $\times$ |
| **-1** | 1 | $\times$ | $\times$ |

**(a)** $x \in \min(\preccurlyeq_\Psi, [\![A]\!])$, $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$.

| $\rho_\Psi$ \ $\rho_A$ | **1** | **0** | **-1** |
|---|---|---|---|
| **1** | $\times$ | $\times$ | $\times$ |
| **0** | $\times$ | 0 | $\times$ |
| **-1** | $\times$ | $\times$ | $\times$ |

**(b)** $x, y \in \min(\preccurlyeq_\Psi, [\![A]\!])$.

**Proposition 7** *In the presence of* (UD$_{\preccurlyeq}$), (ZS$^*_{\preccurlyeq}$) *implies* (IIA$^*_{\preccurlyeq}$).

*Proof* Assume $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and $\rho_A(x, y) = \rho_B(x, y)$. We need to show that $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$. By (UD$_{\preccurlyeq}$), there will exist a state $\Xi$ and sentence $C$ such that $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Xi, [\![C]\!])$, $\rho_\Psi(x, y) = -\rho_\Xi(x, y)$ and $\rho_A(x, y) = -\rho_C(x, y)$. By (ZS$^*_{\preccurlyeq}$), we then have $\rho_{\Psi * A}(x, y) = -\rho_{\Xi * C}(x, y)$. But since $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and $\rho_A(x, y) = \rho_B(x, y)$, we also have $\rho_\Theta(x, y) = -\rho_\Xi(x, y)$ and $\rho_B(x, y) = -\rho_C(x, y)$. Since $x, y \notin \min(\preccurlyeq_\Theta, [\![B]\!]) \cup \min(\preccurlyeq_\Xi, [\![C]\!])$, we can reapply (ZS$^*_{\preccurlyeq}$), to obtain $\rho_{\Theta * B}(x, y) = -\rho_{\Xi * C}(x, y)$. Hence $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$, as required. □

**Proposition 8** (RI$^*_{\preccurlyeq}$) *does not imply either* (IIAI$^*_{\preccurlyeq}$) *or* (IIAP$^*_{\preccurlyeq}$) *even in the presence* (KM$^*_{\preccurlyeq}$) *and* (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$)

*Proof* We simply need to show that both the operators introduced in the proof of Proposition 4 satisfy (RI$^*_{\preccurlyeq}$). Indeed, both satisfy (KM$^*_{\preccurlyeq}$) and (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$), but the first does not satisfy (IIAP$^*_{\preccurlyeq}$) and the second does not satisfy (IIAI$^*_{\preccurlyeq}$).

Regarding the first operator: If $\pi$ is an order isomorphism between $\preccurlyeq_\Psi$ and $\preccurlyeq_\Theta$, then $\preccurlyeq_\Psi$ is a chain if and only if $\preccurlyeq_\Theta$ is. So we have $\rho_{\Psi * A}(x, y) = \rho_{\Psi *_{\mathrm{L}} A}(x, y)$ if and only if we have $\rho_{\Theta * \pi(A)}(\pi(x), \pi(y)) = \rho_{\Theta *_{\mathrm{L}} \pi(A)}(\pi(x), \pi(y))$ and similarly regarding $*_{\mathrm{R}}$. But we have also noted that $*_{\mathrm{L}}$ and $*_{\mathrm{R}}$ both satisfy (KM$^*_{\preccurlyeq}$) and (C1, 2$^*_{\preccurlyeq}$)-(C4$^*_{\preccurlyeq}$). Furthermore, since they also satisfy (PI$^*_{\preccurlyeq}$) (because they satisfy (C1, 2$^*_{\preccurlyeq}$); see Lemma 1) and (IIA$^*_{\preccurlyeq}$), we know from Lemma 3 that they also satisfy (RI$^*_{\preccurlyeq}$).

Regarding the second operator: If $\pi$ is an order isomorphism between $\preccurlyeq_\Psi$ and $\preccurlyeq_\Theta$, then $|[\![\neg A]\!]| = 1$ iff $|[\![\pi(\neg A)]\!]| = 1$. So we have $\rho_{\Psi * A}(x, y) = \rho_{\Psi *_{\mathrm{L}} A}(x, y)$ iff $\rho_{\Psi * \pi(A)}(\pi(x), \pi(y)) = \rho_{\Psi *_{\mathrm{L}} \pi(A)}(\pi(x), \pi(y))$ and similarly regarding $*_{\mathrm{R}}$. The reasoning to establish (RI$^*_{\preccurlyeq}$) then proceeds as above. □

**Proposition 9** (RI$^*_{\preccurlyeq}$) *entails* (TPOR$^*_{\preccurlyeq}$).

*Proof* Let $\preccurlyeq_\Psi = \preccurlyeq_\Theta$ and the order isomorphism $\pi$ be such that $\pi(x) = x$. It follows by (RI$^*_{\preccurlyeq}$) that $\rho_{\Psi * A}(x, y) = \rho_{\Theta * \pi(A)}(\pi(x), \pi(y)) = \rho_{\Theta * A}(x, y)$. □

**Proposition 10** *Given AGM,* (PI$^*_{\preccurlyeq}$) *is equivalent to:*

(PI$^*$) If $[\Psi * B] = Cn(B)$, then $[(\Psi * A) * A \wedge B] = [\Psi * A \wedge B]$ and $[(\Psi * A) * \neg A \wedge B] = [\Psi * \neg A \wedge B]$

*Proof* From (PI$^*_{\preccurlyeq}$) to (PI$^*$): Assume that $[\Psi * B] = \mathrm{Cn}(B)$ but, for reductio, $[(\Psi * A) * A \wedge B] \neq [\Psi * A \wedge B]$ (the case of $[(\Psi * A) * \neg A \wedge B] = [\Psi * \neg A \wedge B]$ is analogous). We consider two cases:

(i) $[\Psi * A \wedge B] \not\subseteq [(\Psi * A) * A \wedge B]$, so $[\![[(\Psi * A) * A \wedge B]]\!] \not\subseteq [\![[(\Psi * A) * A \wedge B]]\!]$: Then, since we assume $A \wedge B$ to be consistent, $\exists x \in$

$\min(\preccurlyeq_{\Psi * A}, [\![A \wedge B]\!]) \setminus \min(\preccurlyeq_{\Psi}, [\![A \wedge B]\!])$. Let $y \in \min(\preccurlyeq_{\Psi}, [\![A \wedge B]\!])$. On the one hand, since $x, y \in [\![B]\!]$ and $[\Psi * B] = \mathrm{Cn}(B)$, we have $x, y \in \min(\preccurlyeq_{\Psi}, [\![B]\!])$, by ($\mathrm{KM}^*_{\preccurlyeq}$), and so $x \sim_{\Psi} y$. On the other hand, since $y \in \min(\preccurlyeq_{\Psi}, [\![A \wedge B]\!])$, and $x \in [\![A \wedge B]\!]$ but $x \notin \min(\preccurlyeq_{\Psi}, [\![A \wedge B]\!])$, we have $y \prec_{\Psi} x$. Contradiction.

(ii) $[(\Psi * A) * A \wedge B] \nsubseteq [\Psi * A \wedge B]$ and so $[\![[\Psi * A \wedge B]]\!] \nsubseteq [\![[(\Psi * A) * A \wedge B]]\!]$: Then, since we assume $A \wedge B$ to be consistent, $\exists x \in \min(\preccurlyeq_{\Psi}, [\![A \wedge B]\!]) \setminus \min(\preccurlyeq_{\Psi * A}, [\![A \wedge B]\!])$. Let $y \in \min(\preccurlyeq_{\Psi * A}, [\![A \wedge B]\!])$. As above, since $x, y \in [\![B]\!]$ and $[\Psi * B] = \mathrm{Cn}(B)$, we have $x \sim_{\Psi} y$. Since $x, y \in [\![A]\!]$, it follows from this, by ($\mathrm{PI}^*_{\preccurlyeq}$), that $x \sim_{\Psi * A} y$. But since $y \in \min(\preccurlyeq_{\Psi * A}, [\![A \wedge B]\!])$, and $x \in [\![A \wedge B]\!]$ but $x \notin \min(\preccurlyeq_{\Psi * A}, [\![A \wedge B]\!])$, we also have $y \prec_{\Psi * A} x$. Contradiction.

From ($\mathrm{PI}^*$) to ($\mathrm{PI}^*_{\preccurlyeq}$): Assume $x \sim_{\Psi} y$. Then $\min(\preccurlyeq_{\Psi}, [\![x \vee y]\!]) = [\![x \vee y]\!]$ and so $[\Psi * x \vee y] = \mathrm{Cn}(x \vee y)$. By ($\mathrm{PI}^*$), letting $B = x \vee y$, we then have:

(a) $[(\Psi * A) * A \wedge (x \vee y)] = [\Psi * A \wedge (x \vee y)]$
(b) $[(\Psi * A) * \neg A \wedge (x \vee y)] = [\Psi * \neg A \wedge (x \vee y)]$

We consider two cases:

(i) Assume $x, y \in [\![A]\!]$. Then $A \wedge (x \vee y) \equiv x \vee y$. So by (a), $[(\Psi * A) * x \vee y] = [\Psi * x \vee y] = \mathrm{Cn}(x \vee y)$. Since $x \sim_{\Psi * A} y$ iff $[(\Psi * A) * x \vee y] = \mathrm{Cn}(x \vee y)$, it then follows that $x \sim_{\Psi * A} y$, as required.
(ii) Assume $x, y \in [\![\neg A]\!]$. Then we obtain $x \sim_{\Psi * A} y$ by the same reasoning as above, this time using (b). □

**Proposition 11** *Given AGM and* $(\mathrm{C1}, 2^*_{\preccurlyeq})$, ($\mathrm{IIA}^*_{\preccurlyeq}$),($\mathrm{IIAP}^*_{\preccurlyeq}$) *and* ($\mathrm{IIAI}^*_{\preccurlyeq}$) *are respectively equivalent to:*

($\mathrm{IIA}_{\preccurlyeq}$) *If* $\neg C \in [\Psi * A] \cap [\Theta * B]$, $A \equiv_C B$ *and* $\Psi$ *and* $\Theta$ *agree modulo* $C$, *then so do* $\Psi * A$ *and* $\Theta * B$

($\mathrm{IIAP}_{\preccurlyeq}$) *If* $\neg C \in [\Psi * A] \cap [\Theta * A]$, *then, if* $\Psi$ *and* $\Theta$ *agree modulo* $C$ , *so do* $\Psi * A$ *and* $\Theta * A$

($\mathrm{IIAI}_{\preccurlyeq}$) *If* $\neg C \in [\Psi * A] \cap [\Psi * B]$ *and* $A \equiv_C B$, *then* $\Psi * A$ *and* $\Psi * B$ *agree modulo* $C$

*Proof* We simply prove the equivalence of ($\mathrm{IIA}^*$) and ($\mathrm{IIA}^*_{\preccurlyeq}$), since the remaining equivalences are established in an analogous manner. We first note that $(\mathrm{C1}, 2^*_{\preccurlyeq})$ gives us the result that, if $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y)$ and $x, y \in [\![A \wedge \neg B]\!]$ or $x, y \in [\![\neg A \wedge B]\!]$, then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$. "Removing" these two cases from the condition $\rho_A(x, y) = \rho_B(x, y)$ leaves us with four cases: (i) $x \in [\![A \wedge B]\!]$ and $y \in [\![\neg A \wedge \neg B]\!]$, (ii) $y \in [\![A \wedge B]\!]$ and $x \in [\![\neg A \wedge \neg B]\!]$, (iii) $x, y \in [\![A \wedge B]\!]$, or (iv) $x, y \in [\![\neg A \wedge \neg B]\!]$. In view of this, ($\mathrm{IIA}^*_{\preccurlyeq}$) is equivalent to the following weaker principle in the presence of $(\mathrm{C1}, 2^*_{\preccurlyeq})$:

If $x, y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!]) \cup \min(\preccurlyeq_{\Theta}, [\![B]\!])$, then, if $\rho_{\Psi}(x, y) = \rho_{\Theta}(x, y)$ and (i) $x \in [\![A \wedge B]\!]$ and $y \in [\![\neg A \wedge \neg B]\!]$, (ii) $y \in [\![A \wedge B]\!]$ and $x \in [\![\neg A \wedge \neg B]\!]$, (iii) $x, y \in [\![A \wedge B]\!]$, or (iv) $x, y \in [\![\neg A \wedge \neg B]\!]$, then $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$

We first show that this weaker principle can alternatively be presented as follows:

> If $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])) = \varnothing$ and, for all $x, y \in [\![C]\!]$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and either (i) $x \in [\![A \wedge B]\!]$ and $y \in [\![\neg A \wedge \neg B]\!]$, (ii) $y \in [\![A \wedge B]\!]$ and $x \in [\![\neg A \wedge \neg B]\!]$, (iii) $x, y \in [\![A \wedge B]\!]$, or (iv) $x, y \in [\![\neg A \wedge \neg B]\!]$, then, for all $x, y \in [\![C]\!]$, $\rho_{\Psi * A}(x, y) = \rho_{\Theta * A}(x, y)$

Going from the second principle to the first, the entailment is obvious, since the latter is just the special case in which $[\![C]\!] = \{x, y\}$. Going the other way, assume that $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])) = \varnothing$ and, for all $x, y \in [\![C]\!]$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and (i) $x \in [\![A \wedge B]\!]$ and $y \in [\![\neg A \wedge \neg B]\!]$, (ii) $y \in [\![A \wedge B]\!]$ and $x \in [\![\neg A \wedge \neg B]\!]$, (iii) $x, y \in [\![A \wedge B]\!]$, or (iv) $x, y \in [\![\neg A \wedge \neg B]\!]$. Assume further that $x, y \in [\![C]\!]$, for arbitrary $x, y \in W$. From these assumptions, we have $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ and (i) $x \in [\![A \wedge B]\!]$ and $y \in [\![\neg A \wedge \neg B]\!]$, (ii) $y \in [\![A \wedge B]\!]$ and $x \in [\![\neg A \wedge \neg B]\!]$, (iii) $x, y \in [\![A \wedge B]\!]$, or (iv) $x, y \in [\![\neg A \wedge \neg B]\!]$. Given this, by (IIA$^*_\preccurlyeq$), it follows that $\rho_{\Psi * A}(x, y) = \rho_{\Theta * B}(x, y)$, as required.

Next we establish syntactic equivalents for the various semantic properties figuring in this principle.

Clearly $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])) = \varnothing$ iff $\neg C \in [\Psi * A] \cap [\Theta * B]$. Furthermore, it is easy to see that conditions (i) to (iv) hold for all $x, y \in [\![C]\!]$ iff $A \equiv_C B$.

Finally, we can show that $\rho_\Psi(x, y) = \rho_\Theta(x, y)$ for all $x, y \in [\![C]\!]$ iff $\Psi$ and $\Theta$ agree modulo $C$ (and obviously similarly regarding $\Psi * A$ and $\Theta * B$).

Indeed, assume that (1) $\forall B \in L$, $[\Psi * B \wedge C] = [\Theta * B \wedge C]$, but, for reductio, that (2) $\exists x, y \in [\![C]\!]$, such that $\rho_\Psi(x, y) \neq \rho_\Theta(x, y)$. Where $B \in L$ is such that $[\![B \wedge C]\!] = \{x, y\}$, it follows from (2) that $\min(\preccurlyeq_\Psi, [\![B \wedge C]\!]) \neq \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$. But from (1), we have $\min(\preccurlyeq_\Psi, [\![B \wedge C]\!]) = \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$. Contradiction.

Going the other way, assume that (1) $\forall x, y \in [\![C]\!]$, $\rho_\Psi(x, y) = \rho_\Theta(x, y)$, but, for reductio, that (2) $\exists B \in L$ such that $[\Psi * B \wedge C] \neq [\Theta * B \wedge C]$. From (2), $\min(\preccurlyeq_\Psi, [\![B \wedge C]\!]) \neq \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$. From this, either there exists $x \in W$ such that $x \in \min(\preccurlyeq_\Psi, [\![B \wedge C]\!])$ but $x \notin \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$, or exists $x \in W$ such that $x \in \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$ but $x \notin \min(\preccurlyeq_\Psi, [\![B \wedge C]\!])$. Assume the former (the other case is analogous). Let $y \in \min(\preccurlyeq_\Theta, [\![B \wedge C]\!])$. From this, we have $y \prec_\Theta x$ but $x \preccurlyeq_\Psi y$ and hence $\rho_\Psi(x, y) \neq \rho_\Theta(x, y)$. However, since $x, y \in [\![C]\!]$, this contradicts (1). $\square$

**Proposition 12** *Given AGM and* (C1, $2^*_\preccurlyeq$)*,* (ZS$^*_\preccurlyeq$) *is equivalent to:*

(ZS$^*$) *If* $\neg C \in [\Psi * A] \cap [\Theta * B]$, $A \equiv_C \neg B$ *and* $\Psi$ *and* $\Theta$ *are in opposition modulo* $C$*, then so are* $\Psi * A$ *and* $\Theta * B$

*Proof* The proof is somewhat similar to the one given in relation to Proposition 11. We first note that (C1, $2^*_\preccurlyeq$) gives us the result that, if $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and $x, y \in [\![A \wedge B]\!]$ or $x, y \in [\![\neg A \wedge \neg B]\!]$, then $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$. Note that in these two cases, we have $\rho_A(x, y) = -\rho_B(x, y) = 0$. “Removing” the cases from the condition $\rho_A(x, y) = -\rho_B(x, y)$ leaves us with two cases: (i) $x \in [\![A \wedge \neg B]\!]$ and $y \in [\![\neg A \wedge B]\!]$ and (ii) $y \in [\![A \wedge \neg B]\!]$ and $x \in [\![\neg A \wedge B]\!]$.

In view of this, ($ZS^*_{\preccurlyeq}$) is equivalent to the following weaker principle in the presence of ($C1, 2^*_{\preccurlyeq}$):

> If $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and (i) $x \in [\![A \wedge \neg B]\!]$ and $y \in [\![\neg A \wedge B]\!]$ or (ii) $y \in [\![A \wedge \neg B]\!]$ and $x \in [\![\neg A \wedge B]\!]$, then $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$

We first show that this weaker principle can alternatively be presented as follows:

> If $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])) = \varnothing$ and, for all $x, y \in [\![C]\!]$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and (i) $x \in [\![A \wedge \neg B]\!]$ and $y \in [\![\neg A \wedge B]\!]$ or (ii) $y \in [\![A \wedge \neg B]\!]$ and $x \in [\![\neg A \wedge B]\!]$, then, for all $x, y \in [\![C]\!]$, $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$

Going from the second principle to the first, the entailment is obvious, since the latter is just the special case in which $[\![C]\!] = \{x, y\}$. Going the other way, assume that $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!]) = \varnothing$ and, for all $x, y \in [\![C]\!]$, (i) $x \in [\![A \wedge \neg B]\!]$ and $y \in [\![\neg A \wedge B]\!]$ or (ii) $y \in [\![A \wedge \neg B]\!]$ and $x \in [\![\neg A \wedge B]\!]$, and $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$. From these assumptions, we have $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ and $\rho_A(x, y) = -\rho_B(x, y)$. Given this, by ($ZS^*_{\preccurlyeq}$), it follows that $\rho_{\Psi * A}(x, y) = -\rho_{\Theta * B}(x, y)$, as required.

Next we establish syntactic equivalents for the various semantic properties figuring in this principle.

As we have noted above, in the proof of Proposition 11, $[\![C]\!] \cap (\min(\preccurlyeq_\Psi, [\![A]\!]) \cup \min(\preccurlyeq_\Theta, [\![B]\!])) = \varnothing$ iff $\neg C \in [\Psi * A] \cap [\Theta * B]$. Furthermore, (i) and (ii) hold for all $x, y \in [\![C]\!]$ iff $A \equiv_C \neg B$.

Finally, we can show that $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ for all $x, y \in [\![C]\!]$ iff $\Psi$ and $\Theta$ are in opposition modulo $C$ (and obviously similarly regarding $\Psi * A$ and $\Theta * B$).

From left to right: Assume that (1) $\forall x, y \in [\![C]\!]$, $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$ but, for reductio, that $\exists B, D \in L$ such that $B$ is quasi-complete, $B \wedge C \nvdash D$, $D \in [\Psi * B \wedge C]$ but $\neg D \notin [\Theta * B \wedge C]$. Since $B \wedge C \nvdash D$, there exists $x \in [\![B \wedge C \wedge \neg D]\!]$. Let $y \in \min(\preccurlyeq_\Psi, [\![B \wedge C]\!])$ (we can assume that such a $y$ exists, since $B \wedge C \nvdash D$ and hence $[\![B \wedge C]\!] \neq \varnothing$). Since $D \in [\Psi * B \wedge C]$ and $x \in [\![B \wedge C \wedge \neg D]\!]$, we have $y \prec_{\Psi * B \wedge C} x$. Therefore, since $x, y \in [\![B \wedge C]\!]$, by ($C1, 2^*_{\preccurlyeq}$), it follows that $y \prec_\Psi x$, so that $\rho_\Psi(x, y) = -1$. Given our assumption that $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$, we then recover $\rho_\Theta(x, y) = 1$. From this, given that $x, y \in [\![B \wedge C]\!]$, we have $\rho_{\Theta * B \wedge C}(x, y) = 1$, by ($C1, 2^*_{\preccurlyeq}$). Since $B$ is quasi-complete and $x, y \in [\![B \wedge C]\!]$, we also have $[\![B \wedge C]\!] = \{x, y\}$. It then follows that $\neg D \in [\Theta * B \wedge C]$. Contradiction.

From right to left: Assume that (1) for all $B, D \in L$ such that $B$ is quasi-complete and $B \wedge C \nvdash D$, if $D \in [\Psi * B \wedge C]$, then $\neg D \in [\Theta * B \wedge C]$, but, for reductio, that (2) $\exists x, y \in [\![C]\!]$, such that $\rho_\Psi(x, y) \neq -\rho_\Theta(x, y)$. Where $B \in L$ is such that $[\![B]\!] = \{x, y\}$, and $D \in L$ is such that $x \in [\![D]\!]$ but $y \notin [\![D]\!]$, it follows from (1) that $x \prec_{\Psi * B \wedge C} y$ and $y \prec_{\Theta * B \wedge C} x$. By ($C1, 2^*_{\preccurlyeq}$), since $x, y \in [\![B \wedge C]\!]$, we then have $x \prec_\Psi y$ and $y \prec_\Theta x$ and hence $\rho_\Psi(x, y) = -\rho_\Theta(x, y)$. Contradiction. □

**Proposition 13** *Given AGM* ($\mathrm{RI}^*_{\preccurlyeq}$) *is equivalent to*

($\mathrm{RI}^*$) $B \in [(\Psi * A)*C]$ *iff* $\iota(B) \in [(\Theta * \iota(A))*\iota(C)]$, *for any c-belief isomorphism* $\iota$ *from* $\Psi$ *to* $\Theta$

*Proof* For convenience, we first recall the definition of ($\mathrm{RI}^*_{\preccurlyeq}$):

($\mathrm{RI}^*_{\preccurlyeq}$) $\rho_{\Psi*A}(x, y) = \rho_{\Theta*\pi(A)}(\pi(x), \pi(y))$, for any order isomorphism $\pi$ from $\preccurlyeq_\Psi$ to $\preccurlyeq_\Theta$

Let $I$ be the set of all c-belief isomorphisms between $\Psi$ and $\Theta$ and $\Pi$ the set of all order isomorphisms between the corresponding TPOs. We will show that there exists a bijection $f$ between $\Pi$ and $I$ such that if $\iota = f(\pi)$, then the biconditional of ($\mathrm{RI}^*$) holds for $\iota$ iff the equality of ($\mathrm{RI}^*_{\preccurlyeq}$) holds for $\pi$. It then follows from this that the biconditional holds for all $\iota$ in $I$ iff the equality holds for all $\pi$ in $\Pi$.

We divide the proof into two lemmas. The first establishes that a particular relation $f$ is a bijection. The second shows that $f$ has the required property stated above. The first lemma, then, is:

**Lemma 5** *The relation* $f \subseteq \Pi \times I$, *such that,* $\forall A \in L$, $\forall \pi \in \Pi$, $\forall \iota \in I$, $f(\pi, \iota)$ *iff* $[\![\iota(A)]\!] = \{x \in W \mid \exists y \in [\![A]\!],$ *such that* $x = \pi(y)\}$, *is a bijection from* $\Pi$ *to* $I$.

In other words $f$ maps $\pi$ onto the unique $\iota$ (modulo logical equivalence) such that the set of models of the image, under $\iota$, of a sentence $A$ is the set of images, under $\pi$, of the models of $A$. Since $f$ is a bijection, we write $f(\pi)$ for the unique $\iota$ (modulo logical equivalence) such that $f(\pi, \iota)$.

To establish Lemma 5, we first note that [18, Proposition 6.2] already prove the following first of two sublemmas, where $\Sigma \supseteq I$ is the set of all belief amount preserving symbol translations (i.e. permutations of $L$ satisfying properties (i)–(iii) of Definition 9) $\sigma$ on $L$ and $\Gamma \supseteq \Pi$ is the set of all permutations $\gamma$ of $W$:

**Sublemma 1** *The relation* $g \subseteq \Pi \times \Sigma$, *such that,* $\forall A \in L$, $\forall \gamma \in \Gamma$, $\forall \sigma \in \Sigma$, $f(\gamma, \sigma)$ *iff* $[\![\sigma(A)]\!] = \{x \in W \mid \exists y \in [\![A]\!],$ *such that* $x = \gamma(y)\}$, *is a bijection from* $\Gamma$ *to* $\Sigma$.

In view of this, it therefore simply remains to be shown that $\iota$ is in $I$ iff its preimage under $g$ is in $\Pi$, since it then follows from this, and the fact that $g$ is a bijection between $\Gamma$ and $\Sigma$, that $f$ is a bijection between $\Pi$ and $I$:

**Sublemma 2** *Where* $\iota = g(\pi)$, *the following are equivalent*

*(1)* *For all* $x, y \in W$, $\rho_\Psi(x, y) = \rho_\Theta(\pi(x), \pi(y))$
*(2)* *For all* $A, B \in L$, $B \in [\Psi * A]$ *iff* $\iota(B) \in [\Theta * \iota(A)]$

From (1) to (2): Assume $\iota = g(\pi)$ and (1). We need to establish (2), which, given ($\mathrm{KM}^*_{\preccurlyeq}$), we can reformulate in terms of minimal sets as: $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$ iff $\min(\preccurlyeq_\Theta, [\![\iota(A)]\!]) \subseteq [\![\iota(B)]\!]$. We simply derive the left to right direction of (2), since the other direction is established in an analogous manner. Assume that $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$ but, for reductio, that $\min(\preccurlyeq_\Theta, [\![\iota(A)]\!]) \nsubseteq [\![\iota(B)]\!]$, so that $\exists x \in \min(\preccurlyeq_\Theta, [\![\iota(A)]\!]) \cap [\![\neg\iota(B)]\!]$. By the definitions of $\iota$ and $g$, it follows that $\pi^{-1}(x) \in [\![A]\!] \cap [\![\neg B]\!]$. Let $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$. Since $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$, we then have $y \prec_\Psi \pi^{-1}(x)$. By (1), we then have $\pi(y) \prec_\Theta x$. Since $y \in [\![A]\!]$, it follows by the definition of $g$ that $\pi(y) \in [\![\iota(A)]\!]$ and so, given $\pi(y) \prec_\Theta x$, that $x \notin \min(\preccurlyeq_\Theta, [\![\iota(A)]\!])$ after all: contradiction.

From (2) to (1): Assume that $\iota = g(\pi)$ and that (2) holds. Given ($\mathrm{KM}^*_\preccurlyeq$), we can again reformulate (2) in terms of minimal sets as: $\min(\preccurlyeq_\Psi, [\![A]\!]) \subseteq [\![B]\!]$ iff $\min(\preccurlyeq_\Theta, [\![\iota(A)]\!]) \subseteq [\![\iota(B)]\!]$. Where $[\![A]\!] = \{x, y\}$, this gives us, in view of the definition of $g$:

$$\min(\preccurlyeq_\Psi, \{x, y\}) \subseteq [\![B]\!] \text{ iff } \min(\preccurlyeq_\Theta, \{\pi(x), \pi(y)\}) \subseteq [\![\iota(B)]\!]$$

By the definition of $g$, for all $x \in W$, $x \in [\![B]\!]$ iff $\pi(x) \in [\![\iota(B)]\!]$. Hence $\rho_\Psi(x, y) = \rho_\Theta(\pi(x), \pi(y))$, i.e. (1). This completes the proof of Sublemma 2 and hence of Lemma 5.

We finally derive our second lemma towards the proof of our main result, which tells us that $f$ has the required property, namely that, if $\iota = f(\pi)$, then the biconditional in ($\mathrm{RI}^*$) holds for $\iota$, iff the equality in ($\mathrm{RI}^*_\preccurlyeq$) holds for $\pi$:

**Lemma 6** *Where $\iota = f(\pi)$, the following are equivalent*

*(1) For all $x, y \in W$ and $A \in L$, $\rho_{\Psi * A}(x, y) = \rho_{\Theta * \pi(A)}(\pi(x), \pi(y))$*
*(2) For all $A, B, C \in L$, $B \in [(\Psi * A) * C]$ iff $\iota(B) \in [(\Theta * \iota(A)) * \iota(C)]$*

The proof of this is identical to that of Sublemma 2, save for the fact that we need to note that, as we defined the extension of $\pi$ to $L$, $\Theta * \pi(A) = \Theta * \iota(A)$.

The conjunction of Lemmas 5 and 6 then establishes the required result. □

**Proposition 15** *Lexicographic revision is the only elementary revision operator that satisfies* ($\mathrm{IIAP}+^*_\preccurlyeq$)

*Proof* We show that, given ($\mathrm{KM}^*_\preccurlyeq$) and ($\mathrm{UD}_\preccurlyeq$), ($\mathrm{IIAP}+^*_\preccurlyeq$) entails Recalcitrance (i.e. if $\rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$), which characterises lexicographic revision in the presence of ($\mathrm{C1}, 2^*_\preccurlyeq$). This was already established as Fact 2.2 (a) in [15]. Let $\rho_A(x, y) = 1$, so that $x \in [\![A]\!]$ and $y \in [\![\neg A]\!]$. Then, by ($\mathrm{UD}_\preccurlyeq$), for any state $\Psi$, there will exist a state $\Theta$ such that $\rho_\Theta(x, y) = \rho_\Psi(x, y)$ and $x \in \min(\preccurlyeq_\Theta, [\![A]\!])$ (and, since $y \in [\![\neg A]\!]$, $y \notin \min(\preccurlyeq_\Theta, [\![A]\!])$). But by ($\mathrm{KM}^*_\preccurlyeq$), if $x \in \min(\preccurlyeq_\Theta, [\![A]\!])$ but $y \notin \min(\preccurlyeq_\Theta, [\![A]\!])$, then $x \prec_{\Theta * A} y$. So, by ($\mathrm{IIAP}+^*_\preccurlyeq$), $x \prec_{\Psi * A} y$, as required. □

**Proposition 16** *Lexicographic revision is the only elementary revision operator that satisfies* ($\mathrm{IIAI}+^*_\preccurlyeq$)

*Proof* ($\mathrm{IIAI}+^*_\preccurlyeq$), in conjunction with ($\mathrm{C1}, 2^*_\preccurlyeq$)–($\mathrm{C4}^*_\preccurlyeq$), can be shown to entail a principle that we have called "($\beta 1+^*_\preccurlyeq$)" in previous work [6]. Indeed, in the proof of

Proposition 6 above, we established the equivalence between (IIAI$^*_{\preccurlyeq}$) and the conjunction of ($\beta 1^*_{\preccurlyeq}$) and ($\beta 2^*_{\preccurlyeq}$) using only (C1, $2^*_{\preccurlyeq}$)–(C4$^*_{\preccurlyeq}$). This proof can be adapted to establish a strengthening of Proposition 3 in [8], in which, unlike in the original, the principle of "Independence" (P$^*_{\preccurlyeq}$) is not appealed to. This yields the following: In the presence of (KM$^*_{\preccurlyeq}$) and (C1, $2^*_{\preccurlyeq}$)–(C4$^*_{\preccurlyeq}$), Booth & Meyer's strengthening of (IIAI$^*_{\preccurlyeq}$) is equivalent to the conjunction of what [6] call "($\beta 1+^*_{\preccurlyeq}$)" and "($\beta 2+^*_{\preccurlyeq}$)".

Furthermore, we also showed in [6] (see Corollary 1 there) that ($\beta 1+^*_{\preccurlyeq}$) characterises lexicographic revision, given (KM$^*_{\preccurlyeq}$) and (C1, $2^*_{\preccurlyeq}$)-(C2$^*_{\preccurlyeq}$). □

**Proposition 17** *Lexicographic revision is the only elementary revision operator that satisfies* (ZS+$^*_{\preccurlyeq}$)

*Proof* We show that, given (KM$^*_{\preccurlyeq}$) and (C3$^*_{\preccurlyeq}$), (ZS+$^*_{\preccurlyeq}$) entails Recalcitrance (i.e. if $\rho_A(x, y) = 1$, then $\rho_{\Psi * A}(x, y) = 1$), which characterises lexicographic revision in the presence of (C1, $2^*_{\preccurlyeq}$). Assume for reductio that Recalcitrance fails, so that there exists $\Psi, x, y \in W$, $A \in L$ such that

(1) $\rho_A(x, y) = 1$
(2) $\rho_{\Psi * A}(x, y) \neq 1$

From (1) and (2), by (C3$^*_{\preccurlyeq}$), we must have:

(3) $\rho_{\Psi}(x, y) \neq 1$

From (1), we have:

(4) $\rho_{\neg A}(x, y) = -1$

From (1), it follows that $y \in [\![\neg A]\!]$ and $x \neq y$, and so by (UD$_{\preccurlyeq}$) there exists a state $\Theta$ such that:

(5) $y \in \min(\preccurlyeq_{\Theta}, [\![\neg A]\!])$
(6) $\rho_{\Theta}(x, y) = -\rho_{\Psi}(x, y)$

From (1), (4), (3) and (6), by (ZS+$^*_{\preccurlyeq}$):

(7) $\rho_{\Theta * \neg A}(x, y) \neq -1$

However, by AGM, given (4) and (5), we have $\rho_{\Theta * \neg A}(x, y) = -1$, directly contradicting (7). □

**Proposition 18** *Given AGM* $\div_{\mathrm{R}}$ *is characterised by the following property:*

$$[\Psi \div_R A) \div_R B] = \begin{cases} [\Psi \div_R A] \cap [\Psi \div_R \neg A \vee B], \ \textit{if} A \vee B \in [\Psi \div_R B] \\ [\Psi \div_R A] \cap [\Psi \div_R A \vee B], \textit{otherwise} \end{cases}$$

*Proof* We first recall the definition of $\div_{\mathrm{R}}$:

If $x \in$ or $y \in \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A]\!])$, then:

$$\rho_{\Psi \div_{\mathrm{R}} A}(x, y) = \begin{cases} 1, \text{if } x \in \text{ and } y \notin \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A]\!]) \\ 0, \text{if } x \in \text{ and } y \in \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A]\!]) \\ -1, \text{if } x \notin \text{ and } y \in \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A]\!]) \end{cases}$$

If $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$, then:

$$\rho_{\Psi \div_{\mathrm{R}} A}(x, y) = \begin{cases} \rho_\Psi(x, y), \text{ if } \rho_\Psi(x, y) \neq 0 \\ \rho_{\neg A}(x, y), \text{ if } \rho_\Psi(x, y) = 0 \end{cases}$$

Next, with the help of ($\mathrm{KM}^{\div}_{\preccurlyeq}$), we translate the two conditionals on the right hand side of the equality of the characteristic syntactic property into the following statements about minimal sets:

(1) If $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \subseteq [\![A \vee B]\!]$, then $\min(\preccurlyeq_{(\Psi \div A) \div B}, W) = \min(\preccurlyeq_{\Psi \div A}, W) \cup \min(\preccurlyeq_{\Psi \div A}, [\![A \wedge \neg B]\!])$

(2) If $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \nsubseteq [\![A \vee B]\!]$, then $\min(\preccurlyeq_{(\Psi \div A) \div B}, W) = \min(\preccurlyeq_{\Psi \div A}, W) \cup \min(\preccurlyeq_{\Psi \div A}, [\![\neg A \wedge \neg B]\!])$

From the semantic characterisation to the syntactic one: We first note that, given ($\mathrm{KM}^{\div}_{\preccurlyeq}$), the consequent of (1) is equivalent to $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_\Psi, [\![A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W)$, while the consequent of (2) is equivalent to $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_\Psi, [\![\neg A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W)$. We then split the proof into two obvious parts:

(A) If $\div = \div_{\mathrm{R}}$, then (1): Assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \subseteq [\![A \vee B]\!]$, so that there does not exist $x \in \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \cap [\![\neg A]\!]$. Assume for reductio that $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) \nsubseteq [\![A]\!]$, so that there exists $x \in \min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) \cap [\![\neg A]\!]$ and $x \notin \min(\preccurlyeq_{\Psi \div A}, W)$. As we have noted, from our initial assumption, it must be the case that $x \notin \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \cap [\![\neg A]\!]$ and hence, since $x \in [\![A]\!]$, $x \notin \min(\preccurlyeq_\Psi, [\![\neg B]\!])$. So, since $x \in [\![\neg B]\!]$, there exists $y \in [\![\neg B]\!]$ such that $\rho_\Psi(y, x) = 1$. Given the semantic definition of $\div_{\mathrm{R}}$, the fact that $x \notin \min(\preccurlyeq_{\Psi \div A}, W)$ then suffices to ensure that $\rho_{\Psi \div A}(y, x) = 1$ (indeed, if $y \in \min(\preccurlyeq_{\Psi \div A}, W)$, then this follows from the first conditional of the definition, and if $y \notin \min(\preccurlyeq_{\Psi \div A}, W)$, then, since $\rho_\Psi(y, x) \neq 0$, the second conditional tells us that $\rho_{\Psi \div A}(y, x) = \rho_\Psi(y, x) = 1$), and hence, since $y \in [\![\neg B]\!]$, $x \notin \min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) \cap [\![\neg A]\!]$. Contradiction. We can therefore conclude that $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) \subseteq [\![A]\!]$. From this, it follows that $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_{\Psi \div A}, [\![A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W)$. Since $\div_{\mathrm{R}}$ satisfies ($\mathrm{C}1, 2^{\div}_{\preccurlyeq}$), it follows that $\min(\preccurlyeq_{\Psi \div A}, [\![A \wedge \neg B]\!]) = \min(\preccurlyeq_\Psi, [\![A \wedge \neg B]\!])$ and hence we finally have $\min(\preccurlyeq_{\Psi \div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W) = \min(\preccurlyeq_\Psi, [\![A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi \div A}, W)$, as required.

(B) If $\div = \div_{\mathrm{R}}$, then (2): We first note that, like $\div_{\mathrm{P}}$, $\div_{\mathrm{R}}$ satisfies the following property, which could be considered the analogue for contraction of the property ($\mathrm{P}^{*}_{\preccurlyeq}$), satisfied by $*_{\mathrm{L}}$ and $*_{\mathrm{R}}$:

($\mathrm{wP}^{\div}_{\preccurlyeq}$) If $y \notin \min(\preccurlyeq_\Psi, W)$, $\rho_{\neg A}(x, y) = 1$ and $\rho_\Psi(x, y) \geq 0$, then $\rho_{\Psi \div A}(x, y) = 1$[10]

[10]This principle is given as follows:

($\mathrm{P}^{*}_{\preccurlyeq}$) If $\rho_A(x, y) = 1$ and $\rho_\Psi(x, y) \geq 0$, then $\rho_{\Psi * A}(x, y) = 1$

Now assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg B]\!]) \nsubseteq [\![A \vee B]\!]$, so that there exists $x \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg B]\!])$, such that $x \in [\![\neg A \wedge \neg B]\!]$. If $x \in \min(\preccurlyeq_\Psi, W)$, since $x \in [\![\neg B]\!]$, we have $x \in \min(\preccurlyeq_\Psi, [\![\neg B]\!])$. So, either way, $x \in \min(\preccurlyeq_\Psi, [\![\neg B]\!])$. Assume for reductio that $\min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W) \nsubseteq [\![\neg A]\!]$, so that there exists $y \in \min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W)$, such that $y \in [\![A]\!]$ and therefore, since $x \in [\![\neg A]\!]$, $\rho_{\neg A}(x, y) = 1$ . Since $x \in \min(\preccurlyeq_\Psi, [\![\neg B]\!])$ and $y \in [\![\neg B]\!]$, we have $\rho_\Psi(x, y) \geq 0$. Since, by assumption, $y \notin \min(\preccurlyeq_{\Psi\div A}, W)$, it follows, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$), that $y \notin \min(\preccurlyeq_\Psi, W)$. Given, $y \notin \min(\preccurlyeq_\Psi, W)$, $\rho_{\neg A}(x, y) = 1$, and $\rho_\Psi(x, y) \geq 0$, we can now apply ($\mathrm{wP}^{\div}_{\preccurlyeq}$), to obtain $\rho_{\Psi\div A}(x, y) = 1$. Since $x, y \in [\![\neg B]\!]$, we then have $y \notin \min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!])$. Contradiction. We can therefore conclude that $\min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W) \subseteq [\![\neg A]\!]$. From this, it follows that $\min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W) = \min(\preccurlyeq_{\Psi\div A}, [\![\neg A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W)$. Since $\div_{\mathrm{R}}$ satisfies ($\mathrm{C1}, 2^{\div}_{\preccurlyeq}$), it follows that $\min(\preccurlyeq_{\Psi\div A}, [\![\neg A \wedge \neg B]\!]) = \min(\preccurlyeq_\Psi, [\![\neg A \wedge \neg B]\!])$ and hence we finally have $\min(\preccurlyeq_{\Psi\div A}, [\![\neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W) = \min(\preccurlyeq_\Psi, [\![\neg A \wedge \neg B]\!]) - \min(\preccurlyeq_{\Psi\div A}, W)$, as required.

From the syntactic characterisation to the semantic one: Assume $x \in$ or $y \in \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$. Then the required result follows simply by ($\mathrm{KM}^{\div}_{\preccurlyeq}$). So assume that $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A]\!])$. We divide the remainder of the proof into two obvious parts:

(A) Proof that, if $\rho_\Psi(x, y) \neq 0$, then $\rho_{\Psi\div A}(x, y) = \rho_\Psi(x, y)$: It suffices to show that, if $\rho_\Psi(x, y) = 1$, then $\rho_{\Psi\div A}(x, y) = 1$ (establishing that, if $\rho_\Psi(x, y) = -1$, then $\rho_{\Psi\div A}(x, y) = -1$, is analogous). So assume that $\rho_\Psi(x, y) = 1$. We first note that the required conclusion that $\rho_{\Psi\div A}(x, y) = 1$ holds iff $y \notin \min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W)$: Indeed, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$), we have $\min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W) = \min(\preccurlyeq_{\Psi\div A}, W) \cup \min(\preccurlyeq_{\Psi\div A}, \{x, y\})$. Since $\{x, y\} \subseteq W$, it then follows from this that $y \in \min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W)$ iff $y \in \min(\preccurlyeq_{\Psi\div A}, \{x, y\})$. Since, trivially, we have $\rho_{\Psi\div A}(x, y) = 1$ iff $y \notin \min(\preccurlyeq_{\Psi\div A}, \{x, y\})$, the required result then follows. We then split the proof into two cases:

(a) Assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, \{x, y\}) \subseteq [\![A \vee (\neg x \wedge \neg y)]\!]$. Then, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$) and the syntactic characteristic principle (setting $B = \neg x \wedge \neg y$) we have:

$$\begin{aligned}&\min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W)\\ &\qquad = \min(\preccurlyeq_{\Psi\div A}, W) \cup \min(\preccurlyeq_{\Psi\div\neg A\vee(\neg x\wedge\neg y)}, W)\end{aligned}$$

By assumption, $y \notin \min(\preccurlyeq_{\Psi\div A}, W)$. So it remains to be shown that $y \notin \min(\preccurlyeq_{\Psi\div\neg A\vee(\neg x\wedge\neg y)}, W)$. By ($\mathrm{KM}^{\div}_{\preccurlyeq}$), $\min(\preccurlyeq_{\Psi\div\neg A\vee(\neg x\wedge\neg y)}, W) = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y)]\!])$. Since we have already also assumed that $y \notin \min(\preccurlyeq_\Psi, W)$, we are just left with

It was noted, in [3], that, in its unqualified form (there called "($\mathrm{P}^{\div}_{\preccurlyeq}$)"), without the requirement that $y \notin \min(\preccurlyeq_\Psi, W)$, this principle would lead to trouble.

verifying that $y \notin \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y)]\!])$. So assume for reductio that $y \in \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y))]\!]$, so that $y \in [\![A]\!]$ and, if $x \in [\![A]\!]$, then $\rho_\Psi(x, y) \leq 0$. From $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, \{x, y\}) \subseteq [\![A \vee (\neg x \wedge \neg y)]\!]$ and $\rho_\Psi(x, y) = 1$, it follows that $x \in [\![A]\!]$. Hence $\rho_\Psi(x, y) \leq 0$. But this contradicts our assumption that $\rho_\Psi(x, y) = 1$. So $y \notin \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y))]\!]$ and we therefore have $y \notin \min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W)$, as required.

(b) Assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, \{x, y\}) \nsubseteq [\![A \vee (\neg x \wedge \neg y)]\!]$. Then the reasoning is similar to the above, except that we proceed by showing that $y \notin \min(\preccurlyeq_\Psi, [\![\neg A \wedge (x \vee y)]\!])$.

(B) Proof that, if $\rho_\Psi(x, y) = 0$, then $\rho_{\Psi \div A}(x, y) = \rho_{\neg A}(x, y)$: Assume that $\rho_\Psi(x, y) = 0$. We again split the proof into two cases:

(a) Assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, \{x, y\}) \subseteq [\![A \vee (\neg x \wedge \neg y)]\!]$. Then $x, y \in [\![A]\!]$ and so $\rho_{\neg A}(x, y) = 0$. So we need to establish that $\rho_{\Psi \div A}(x, y) = 0$, or equivalently $x, y \in \min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W)$. It also follows, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$) and the syntactic characteristic principle (setting $B = \neg x \wedge \neg y$), that:

$$\min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W) = \min(\preccurlyeq_{\Psi \div A}, W) \cup \min(\preccurlyeq_{\Psi \div (\neg A \vee (\neg x \wedge \neg y)}, W)$$

As we have already noted, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$), $\min(\preccurlyeq_{\Psi \div (\neg A \vee (\neg x \wedge \neg y)}, W) = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y))]\!]$. Furthermore, it follows from $x, y \in [\![A]\!]$ and $\rho_\Psi(x, y) = 0$ that $x, y \in \min(\preccurlyeq_\Psi, [\![A \wedge (x \vee y))]\!]$. Hence $x, y \in \min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W)$, as required.

(b) Assume $\min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, \{x, y\}) \nsubseteq [\![A \vee (\neg x \wedge \neg y)]\!]$. Then, by ($\mathrm{KM}^{\div}_{\preccurlyeq}$) and the syntactic characteristic principle (setting $B = \neg x \wedge \neg y$):

$$\min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W) = \min(\preccurlyeq_{\Psi \div A}, W) \cup \min(\preccurlyeq_{\Psi \div A \vee (\neg x \wedge \neg y)}, W)$$

From the fact that $\min(\preccurlyeq_\Psi, \{x, y\}) \nsubseteq [\![A \vee (\neg x \wedge \neg y)]\!]$ and hence, since $\rho_\Psi(x, y) = 0$, that $\{x, y\} \nsubseteq [\![A]\!]$, we are left with two possibilities to consider:

(i) Assume $x, y \in [\![\neg A]\!]$. Then $\rho_{\neg A}(x, y) = 0$. So we need to establish that $\rho_{\Psi \div A}(x, y) = 0$, or equivalently $x, y \in \min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W)$. By ($\mathrm{KM}^{\div}_{\preccurlyeq}$), it follows that we have $\min(\preccurlyeq_{\Psi \div A \vee (\neg x \wedge \neg y)}, W) = \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![\neg A \wedge (x \vee y)]\!])$. Since $x, y \in [\![\neg A]\!]$ and $\rho_\Psi(x, y) = 0$, we have $x, y \in \min(\preccurlyeq_\Psi, [\![\neg A \wedge (x \vee y)]\!])$ and so, finally, we recover $x, y \in \min(\preccurlyeq_{(\Psi \div A) \div \neg x \wedge \neg y}, W)$, as required.

(ii) Assume $x \in [\![A]\!]$, $y \in [\![\neg A]\!]$ (the other case, in which $y \in [\![A]\!]$, $x \in [\![\neg A]\!]$ is analogous). Then $\rho_{\neg A}(x, y) =$

$-1$. So we need to establish that $\rho_{\Psi\div A}(x, y) = -1$, or equivalently $x \notin \min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W)$. We already know that $x \notin \min(\preccurlyeq_{\Psi\div A}, W)$. So it remains to be established that $x \notin \min(\preccurlyeq_{\Psi\div A\vee(\neg x\wedge\neg y)}, W)$. By ($\mathrm{KM}^{\div}_{\preccurlyeq}$), $\min(\preccurlyeq_{\Psi\div(A\vee(\neg x\wedge\neg y)}, W) = \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi}, [\![\neg A \wedge (x \vee y)]\!])$. We have already assumed that $x \notin \min(\preccurlyeq_{\Psi}, W)$ and, since $x \in [\![A]\!]$, $x \notin \min(\preccurlyeq_{\Psi}, [\![\neg A \wedge (x \vee y)]\!])$. We therefore recover $x \notin \min(\preccurlyeq_{(\Psi\div A)\div\neg x\wedge\neg y}, W)$, as required. □

**Proposition 19** *Where $i \in \{ P, R, N \}$, it is not the case that for any state $\Psi$ and sentences $A, B \in L$ there exists $C \in L$ such that $[(\Psi \div_i A) \div_i B] = [\Psi \div_i C]$*

*Proof* We shall say that a *rank* associated with $\preccurlyeq_{\Psi}$ is an equivalence class generated by the indifference relation $\sim_{\Psi}$. All three operators are such that, if $A \in [\Psi]$, then the number of ranks associated with $\preccurlyeq_{\Psi\div A}$ is equal to $n - 1$, where $n$ is the number of ranks associated with $\preccurlyeq_{\Psi}$ (and equal to $n$ if $A \notin [\Psi]$). We can then find a countermodel by considering $\Psi$ and $A, B \in L$ such that $A \in [\Psi]$ and $B \in [\Psi \div A]$: we will then have $n - 2$ ranks associated with $\preccurlyeq_{(\Psi\div A)\div B}$ and therefore no $C \in L$ such that $\preccurlyeq_{\Psi\div C} = \preccurlyeq_{(\Psi\div A)\div B}$. Figure 6 depicts a countermodel relating to all three operators. □

**Proposition 21** *Let $\langle i, j\rangle \in \{\langle N, N\rangle, \langle R, R\rangle\}$. Then, if $*$ is defined from $\div_j$ using* ($\mathrm{iLIRC}_{\preccurlyeq}$), *then $* = *_i$.*

*Proof* Let $\langle i, j\rangle \in \{\langle N, N\rangle, \langle R, R\rangle\}$. We need to show that the following equality holds: $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y)$.

We first note that our operators satisfy certain preservation conditions for strict preference. In particular $*_{\mathrm{N}}$ and $*_{\mathrm{R}}$ satisfy:

($\mathrm{SPPres}^{*}_{\preccurlyeq}$) If $y \notin \min(\preccurlyeq_{\Psi}, [\![A]\!])$ and $x \prec_{\Psi} y$, then $x \prec_{\Psi * A} y$

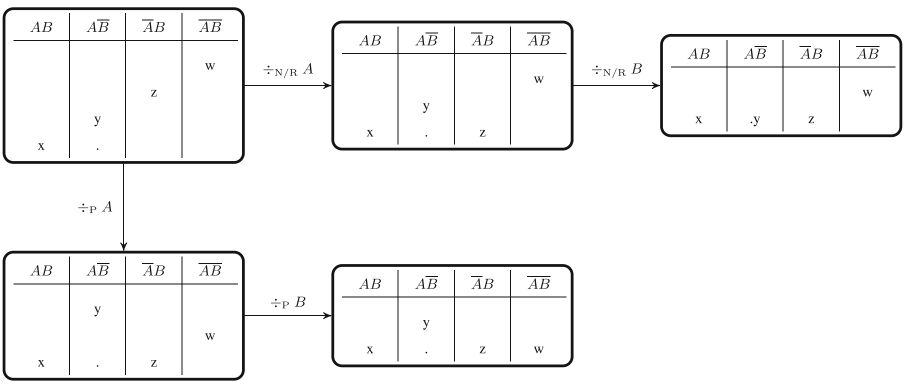


**Fig. 6** Countermodel establishing Proposition 19

whereas $\div_{\mathrm{N}}$ and $\div_{\mathrm{R}}$ satisfy:

(SPPres$^{\div}_{\preccurlyeq}$) If $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$ and $x \prec_\Psi y$, then $x \prec_{\Psi\div\neg A} y$

We divide the proof into two main cases:

(1) Assume $x \in$ or $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$. By (KM$^*_{\preccurlyeq}$), it follows that $\min(\preccurlyeq_{\Psi *_i A}, W) = \min(\preccurlyeq_\Psi, [\![A]\!])$. By (C1, 2$^{\div}_{\preccurlyeq}$), we have $\min(\preccurlyeq_\Psi, [\![A]\!]) = \min(\preccurlyeq_{\Psi\div_j\neg A}, [\![A]\!])$. Finally, again by (KM$^*_{\preccurlyeq}$), we recover the result that $\min(\preccurlyeq_{\Psi\div_j\neg A}, [\![A]\!]) = \min(\preccurlyeq_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}, W)$. These 3 equalities then yield $\min(\preccurlyeq_{\Psi *_i A}, W) = \min(\preccurlyeq_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}, W)$. By (KM$^*_{\preccurlyeq}$), again, we also have $\min(\preccurlyeq_\Psi, [\![A]\!]) = \min(\preccurlyeq_{\Psi *_i A}, W)$. Hence:

$$\min(\preccurlyeq_\Psi, [\![A]\!]) = \min(\preccurlyeq_{\Psi *_i A}, W) = \min(\preccurlyeq_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}, W)$$

With this in hand: $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = 1$, if $x \in$ and $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = 0$, if $x, y \in \min(\preccurlyeq_\Psi, [\![A]\!])$, and $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = -1$, if $x \notin$ and $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$.

(2) Assume $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$.

(a) Assume $x \in$ or $y \in \min(\preccurlyeq_\Psi, W)$.

(i) Assume $x, y \in \min(\preccurlyeq_\Psi, W)$. Then $\rho_\Psi(x, y) = 0$. Since $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, we must also have $x, y \in [\![\neg A]\!]$ and hence $\rho_A(x, y) = 0$. By (C1, 2$^*_{\preccurlyeq}$), it then follows that we have $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = 0$.

(ii) Assume that $x \in$ and $y \notin \min(\preccurlyeq_\Psi, W)$ (the remaining case is analogous). Then $\rho_\Psi(x, y) = 1$. Furthermore, since $y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$, by (SPPres$^*_{\preccurlyeq}$) and (SPPres$^{\div}_{\preccurlyeq}$), $\rho_{\Psi *_i A}(x, y) = \rho_{\Psi\div_j\neg A}(x, y) = 1$. By reapplying (SPPres$^*_{\preccurlyeq}$) again, we then obtain $\rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = 1$ and we are done.

(b) Assume $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W)$. We know from the definitions of these operators that, for $x, y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![A]\!])$, we have $\rho_{\Psi\div_j\neg A}(x, y) = \rho_{\Psi *_i A}(x, y)$. Therefore $\rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y) = \rho_{(\Psi *_i A)*_{\mathrm{N}}A}(x, y)$. But, since, by (KM$^*_{\preccurlyeq}$), $\rho_{(\Psi *_i A)*_{\mathrm{N}}A}(x, y) \subseteq [\![A]\!]$, it follows, by the definition of $*_{\mathrm{N}}$, that $\rho_{(\Psi *_i A)*_{\mathrm{N}}A}(x, y) = \rho_{\Psi *_i A}(x, y)$. We can therefore conclude that $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi\div_j\neg A)*_{\mathrm{N}}A}(x, y)$, as required. □

**Proposition 22** *If $*$ is defined from $\div_P$ via* (iLIRC$_{\preccurlyeq}$) *then $* \neq *_L$.*

*Proof* We provide a countermodel in Fig. 7. □

**Proposition 23** *If $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$ then $*_i$ and $\div_j$ jointly satisfy* (iLI$*_{\preccurlyeq}$).

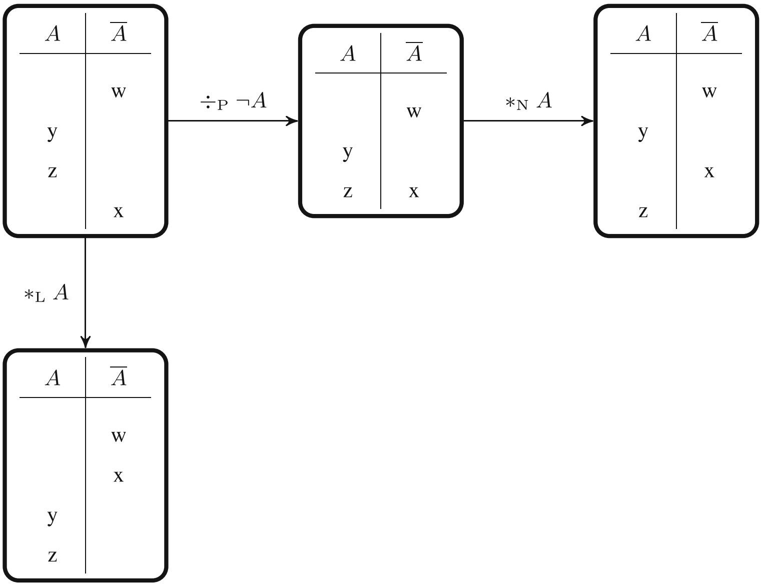


**Fig. 7** Countermodel establishing Proposition 22

*Proof* Let $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$. We need to show that $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi \div_j \neg A) *_i A}(x, y)$.

Regarding the case in which $\langle i, j\rangle = \langle L, P\rangle$: Assume $\rho_A(x, y) = 1$. Then, since it is a property of lexicographic revision that, if $\rho_A(x, y) = 1$, then $\rho_{\Psi *_{\mathrm{L}} A}(x, y) = 1$, it follows that $\rho_{\Psi *_{\mathrm{L}} A}(x, y) = \rho_{(\Psi \div_{\mathrm{P}} \neg A) *_{\mathrm{L}} A}(x, y) = 1$. Assume $\rho_A(x, y) = 0$. Then, by (C1, $2^*_{\preccurlyeq}$), $\rho_{\Psi *_{\mathrm{L}} A}(x, y) = \rho_{(\Psi \div_{\mathrm{P}} \neg A) *_{\mathrm{L}} A}(x, y) = \rho_{\Psi}(x, y)$.

Regarding the case in which $\langle i, j\rangle \in \{\langle N, N\rangle, \langle R, R\rangle\}$, we divide the proof into two main cases:

(1) Assume $x \in$ or $y \in \min(\preccurlyeq_\Psi, [\![A]\!])$. The proof proceeds as in the corresponding case in the proof of Proposition 21 above.

(2) Assume $x, y \notin \min(\preccurlyeq_\Psi, [\![A]\!])$.

   (a) Assume $x \in$ or $y \in \min(\preccurlyeq_\Psi, W)$. Again, the proof proceeds as in the corresponding case in the proof of Proposition 21.

   (b) Assume $x \notin$ and $y \notin \min(\preccurlyeq_\Psi, W)$. We know from the definitions of these operators that, for $x, y \notin \min(\preccurlyeq_\Psi, W) \cup \min(\preccurlyeq_\Psi, [\![A]\!])$, we have $\rho_{\Psi \div_j \neg A}(x, y) = \rho_{\Psi *_i A}(x, y)$. Since $*_{\mathrm{R}}$ and $*_{\mathrm{N}}$ both satisfy ($\mathrm{KM}^*_{\preccurlyeq}$) and ($\mathrm{IIA}^*_{\preccurlyeq}$) and the latter tell us that $\rho_\Psi(x, y)$ and $\rho_A(x, y)$ jointly determine $\rho_{\Psi * A}(x, y)$, it follows that $\rho_{(\Psi \div_j \neg A) *_i A}(x, y) = \rho_{(\Psi *_i A) *_i A}(x, y)$. For the final step, we note that $*_{\mathrm{R}}$ and $*_{\mathrm{N}}$ both satisfy:

($\mathrm{Idem}^*_{\preccurlyeq}$) $\quad \rho_{(\Psi * A) * A}(x, y) = \rho_{\Psi * A}(x, y)$

Indeed, in the presence of (KM$^*_{\preccurlyeq}$), (IIAP$^*_{\preccurlyeq}$) tells us that the posterior relative rank $\rho_{\Psi * A}(x, y)$ of a pair $\langle x, y\rangle$ of worlds is determined by the prior relative rank $\rho_{\Psi}(x, y)$, with the nature of this mapping depending, for non-minimal $A$-worlds, on the relevant proposition and pair of worlds. (Idem$^*_{\preccurlyeq}$) is then the requirement that, for any given value of $\rho_A(x, y)$, if the row for value $v$ points to value $w$, then the row for $w$ also points to $w$. That this condition holds for $*_{\mathrm{R}}$ and $*_{\mathrm{N}}$ can be seen from Table 2. (Note, in passing, that (Idem$^*_{\preccurlyeq}$) can more generally be shown to be derivable from (KM$^*_{\preccurlyeq}$), (IIAP$^*_{\preccurlyeq}$) and (C1, 2$^*_{\preccurlyeq}$)–(C4$^*_{\preccurlyeq}$) and is hence also satisfied by $*_{\mathrm{L}}$.)

Hence $\rho_{(\Psi *_i A) *_i A}(x, y) = \rho_{\Psi *_i A}(x, y)$ and we can conclude that $\rho_{\Psi *_i A}(x, y) = \rho_{(\Psi \div_j \neg A) *_{\mathrm{N}} A}(x, y)$, as required. □

**Proposition 24** *Let $\langle i, j\rangle \in \{\langle L, P\rangle, \langle N, N\rangle, \langle R, R\rangle\}$. Then if $\div$ defined from $*_i$ by $\preccurlyeq_{\Psi \div A} = \preccurlyeq_{\Psi} \oplus_{\mathrm{TQ2}} \preccurlyeq_{\Psi *_i \neg A}$, then $\div = \div_j$.*

*Proof* The result is obvious from the definitions of the various elementary contraction operators in Definition 10 and elementary revision operators in Definition 2. $\oplus_{\mathrm{TQ2}}$ combination of $\preccurlyeq_{\Psi}$ and $\preccurlyeq_{\Psi *_i \neg A}$ yields the result that the minimal equivalence class under $\preccurlyeq_{\Psi \div A}$, is given by $\min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi *_i \neg A}, W)$, which is equal to $\min(\preccurlyeq_{\Psi \div_j \neg A}, W)$. Regarding the subsequent equivalence classes, $\oplus_{\mathrm{TQ2}}$ combination gives us the result that, for $x \notin$ and $y \notin \min(\preccurlyeq_{\Psi}, W) \cup \min(\preccurlyeq_{\Psi *_i \neg A}, W)$, $\rho_{\Psi \div A}(x, y) = \rho_{\Psi *_i \neg A}(x, y)$. But as we noted in Section 5.1, for the same $x$ and $y$, we have $\rho_{\Psi \div_j A}(x, y) = \rho_{\Psi *_i \neg A}(x, y)$ and so $\rho_{\Psi \div A}(x, y) = \rho_{\Psi \div_j A}(x, y)$. □

**Funding** Open Access funding enabled and organized by CAUL and its Member Institutions.



## References

1. Alchourrón, C. E., Gärdenfors, P., & Makinson, D. (1985). On the logic of theory change: Partial meet contraction and revision functions. *Journal of Symbolic Logic*, *50*(02), 510–530.
2. Arrow, K. (1951). Social choice and individual values wiley: new york.
3. Booth, R., & Chandler, J. (2016). Extending the harper identity to iterated belief change. In *Proceedings of the twenty-fifth international joint conference on artificial intelligence, IJCAI 2016, New York, NY, USA, 9-15 July 2016. pp. 987–993.*
4. Booth, R., & Chandler, J. (2017). The irreducibility of iterated to single revision. *Journal of Philosophical Logic*, *46*(4), 405–418.
5. Booth, R., & Chandler, J. (2019). From iterated revision to iterated contraction: Extending the harper identity. Artificial Intelligence, **277**.
6. Booth, R., & Chandler, J. (2020). On strengthening the logic of iterated belief revision: Proper ordinal interval operators. Artificial Intelligence, **285**.

7. Booth, R., & Meyer, T. (2006). Admissible and restrained revision. *Journal of Artificial Intelligence Research*, *26*(1), 127–151.
8. Booth, R., & Meyer, T. (2011). How to revise a total preorder. Journal of Philosophical Logic, **1–46**.
9. Boutilier, C. (1996). Iterated revision and minimal change of conditional beliefs. *Journal of Philosophical Logic*, *25*(3), 263–305.
10. Caridroit, T., Konieczny, S., & Marquis, P. (2017). Contraction in propositional logic. *Int. J. Approx. Reasoning*, *80*(C), 428–442. https://doi.org/10.1016/j.ijar.2016.06.010.
11. Chandler, J., & Booth, R. (2019). Elementary iterated revision and the Levi Identity. In *Blackburn, p., lorini, e., guo, m. (eds.) logic, rationality, and interaction - 7th international workshop, LORI 2019, Chongqing, China, october 18-21, 2019, proceedings. Lecture notes in computer science, vol. 11813, pp. 15–28. Springer*.
12. Chopra, S., Ghose, A., Meyer, T., & Wong, K.S. (2008). Iterated belief change and the Recovery axiom. *Journal of Philosophical Logic*, *37*(5), 501–520.
13. Darwiche, A., & Pearl, J. (1997). On the logic of iterated belief revision. *Artificial Intelligence*, *89*(1), 1–29.
14. Fermė, E. L., & Hansson, S. O. (2011). AGM 25 Years - twenty-five years of research in belief change. *Journal of Philosophical Logic*, *40*(2), 295–331.
15. Glaister, S. M. (1998). Symmetry and belief revision. *Erkenntnis*, *49*(1), 21–56.
16. Katsuno, H., & Mendelzon, A. O. (1991). Propositional knowledge base revision and minimal change. *Artificial Intelligence*, *52*(3), 263–294.
17. Lehmann, D., & Magidor, M. (1992). What does a conditional knowledge base entail? *Artificial Intelligence*, *55*(1), 1–60.
18. Marquis, P., & Schwind, N. (2014). Lost in translation: Language independence in propositional logic - application to belief change. *Artificial Intelligence*, *206*, 1–24. https://doi.org/10.1016/j.artint.2013.09.005.
19. May, K. O. (1952). A set of independent necessary and sufficient conditions for simple majority decision. *Econometrica*, *20*(4), 680–684. http://www.jstor.org/stable/1907651.
20. Nayak, A. C., Goebel, R., & Orgun, M.A. (2007). Iterated belief contraction from first principles. In *Proceedings of the 20th international joint conference on artificial intelligence (IJCAI-07), Hyderabad, India, January 6–12 2007* (pp. 2568–2573).
21. Nayak, A. C., Goebel, R., Orgun, M. A., & Pham, T. (2005). Iterated belief change and the Levi Identity. In *Belief change in rational agents: perspectives from artificial intelligence, philosophy, and economics, 7.-12. August 2005*.
22. Nayak, A. C., Pagnucco, M., & Peppas, P. (2003). Dynamic belief revision operators. *Artificial Intelligence*, *146*(2), 193–228.
23. Ramachandran, R., Nayak, A. C., & Orgun, M.A. (2012). Three approaches to iterated belief contraction. *Journal of Philosophical Logic*, *41*(1), 115–142. http://www.jstor.org/stable/41427272.
24. Rott, H. (2009). Shifting priorities: Simple representations for twenty-seven iterated theory change operators. In *Malinowski, D.M.J., Wansing, H (eds.) Philosophy, Towards Mathematical* (pp. 269–296). Springer.
25. Segerberg, K. (1998). Irrevocable belief revision in dynamic doxastic logic. *Notre Dame Journal of Formal Logic*, *39*(3), 287–306.