Bosonic Ghostbusting

Vertex Algebras and Quantum Groups

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Summary

The aim of this project has been to investigate into the representation theory of the bosonic ghost vertex algebra and to construct a quantum group with an equivalent category of modules.

The bosonic ghost vertex algebra violates many properties of other vertex algebras which simplify the representation theory and this means one cannot directly apply many of the methods commonly used in these simpler cases. A key component in many of the methods employed is the screening operator. The vertex algebra itself is the kernel of the screening operator acting on a lattice vertex algebra. The module containing the screening operator characterises a Nichols algebra which is crucial in the construction of the corresponding quantum group.

An equivalence of representation theory is with respect to a certain level of structure. The consensus is that a sensible choice of module category for vertex algebras should admit a braided tensor structure. We realised that the notion of a dual module vertex algebras possess leads to a generalised duality called Grothendieck-Verdier structure on the module category. Therefore we construct the equivalence at the level of ribbon Grothendieck-Verdier structure, a braided monoidal category with a twist and Grothendieck-Verdier dual.

We develop a rigorous framework for constructing functors from a given category to a vertex algebra module category. We work through all the details of an equivalence for lattice vertex algebras and the bosonic ghost vertex algebra with their Hopf algebra counterparts.

Finally, we construct the candidate quantum group and prove the ribbon Grothendieck-Verdier equivalence to the bosonic ghost vertex algebra, up to one equation involving intertwining operators. A more general equivalence between vertex algebras which are the kernel of screenings and quantum groups constructed from the corresponding Nichols algebra is still conjectural but more evidence and tools are now available.

"I ain't afraid of no ghost"

— Ray Parker Jr., *Ghostbusters*

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— Chapter 1 —

Introduction

"Human sacrifice. Dogs and cats living together. Mass hysteria."

— Peter Venkman (Bill Murray), *Ghostbusters*

Context and History

Vertex algebras and quantum groups were both introduced in the 1980s, motivated by the study of the representation theory of infinite-dimensional Lie algebras. They have both mathematical and physical motivation, with deep connections to number theory, group theory, differential equations, topology, conformal field theory, integrable systems, statistical mechanics, string theory and quantum gravity. Many connections between vertex algebras and quantum groups have been explored, one direction of which is a connection between the representation theories of vertex algebras associated to affine Lie algebras, and quantum groups at roots of unity. This connection is known as Kazhdan-Lusztig correspondence and in this thesis we will investigate a similar equivalence between the bosonic ghost vertex algebra and a quantum group of $\mathfrak{gl}_2$, at the primitive fourth root of unity.

Representations and Category Theory

Many algebraic structures, including vertex algebras and quantum groups, possess a notion of action on a vector space, in which elements of the algebra act as linear maps. The study of these actions is called representation theory and the vector spaces themselves are referred to as modules. Instead of focussing on the details of the action on the modules, one can label each module and shift focus to maps between the modules which are compatible with the action, called morphisms. Packaging the representation theoretic data in this way yields a category, where the objects are modules and the arrows are morphisms.

Tensor Categories

Categories can possess additional structure such as a tensor product, which is a binary operation on the objects. Then they can be equipped with braiding and associativity morphisms which enforce a form of commutativity and associativity on the tensor product operation. A category equipped with these structures is called a braided monoidal category, as it parallels the axioms of a monoid. A rigid category is one which possesses a duality structure paralleling the one present in finite dimensional vector spaces. If the category also possesses a twist, then it is called ribbon.

Grothendieck-Verdier Categories

If the category lacks rigidity, it can still possess the more general notion of Grothendieck-Verdier structure. Grothendieck-Verdier structure depends on a specific choice of dualising object, and rigid categories are Grothendieck-Verdier with the tensor unit as the dualising object. If, in addition, the category is braided monoidal category with a twist, this leads to the notion of ribbon Grothendieck-Verdier categories. Categories can be mapped to other categories, and therefore compared, using functors. Two categories can be equivalent in a way which respects all, or none, of this additional structure.

Hopf Algebras

We will focus on two types of algebraic structures in this thesis. First we will introduce Hopf algebras, whose structure parallels that of a rigid monoidal category. Hopf algebras are constructed from a vector space and several linear maps. They act on vector spaces by multiplication, and their linear maps allow one to directly construct the tensor product and the other structure maps of a rigid monoidal category. Hopf algebras can further be equipped with a distinguished element that directly constructs the braiding, and another constructing the twist. This yields a ribbon category. Later in this thesis, we will encounter a specific type of Hopf algebra called a quantum group, whose structure parallels that of the universal enveloping algebra of a Lie algebra (in our case $\mathfrak{gl}_2$).

Vertex Algebras

By contrast to Hopf algebras, vertex algebras do not, in general, parallel a specific categorical structure. Instead they consist of power series whose coefficients are endomorphisms of some vector space. Sometimes, the commutation rules for these endomorphisms give known infinite dimensional Lie algebras, which makes the vertex algebras easier to study. These power series are called fields and obey a certain set of axioms, which can be motivated by conformal field theory.

Huang-Lepowsky-Zhang Tensor Categories

Vertex algebras can also be thought of as generalisations of commutative rings and in this sense they have a analogous notion of modules, or an action on a vector space. Vertex algebras act on vector spaces through power series whose coefficients are endomorphisms of the module, satisfying a modified version of the vertex algebra axioms, and under a certain set of sufficient conditions, they produce braided tensor categories. However, rigidity is significantly more rare for vertex algebras and instead they possess the more general structure of Grothendieck-Verdier duality. This leads to the notion of ribbon Grothendieck-Verdier categories as the natural representation theory of vertex algebras. Fortunately, we can equip Hopf algebras with ribbon Grothendieck-Verdier structure as well, and therefore we can compare vertex algebra and Hopf algebra module categories at this level.

Functors Involving Vertex Operator Algebra Module Categories

The goal of this thesis is to construct a quantum group with equivalent representation theory (as ribbon Grothendieck-Verdier categories) to the bosonic ghost vertex algebra. In order to do this, we develop a framework for studying functors from braided monoidal categories to vertex algebra module categories, with further conditions for when the functor is an equivalence and in particular when it is a ribbon Grothendieck-Verdier equivalence. We also prove that, under certain conditions, the functor only needs to be defined on a subcategory (projective modules) and can then be uniquely extended to the whole module category.

The Free Boson in Three Guises

We apply our framework for equivalences with vertex algebra module categories to the example of the free boson, or lattice vertex algebra. We study the lattice vertex algebra structure in detail, for any kind of lattice, and construct a 'lattice Hopf algebra' which we prove has ribbon Grothendieck-Verdier equivalent representation theory to the free boson. We also see that these categories are equivalent to a category of graded vector spaces.

Bosonic Ghosts

Most well-studied vertex algebras obey finiteness conditions, such as rationality and C_2-cofiniteness. The bosonic ghosts violate these conditions, but we show that it still exhibits many desirable features in its representation theory. We provide an extensive investigation into the representation theory of the bosonic ghost vertex algebra, classifying all the modules, compute the tensor products and prove that our choice of module category is in fact rigid braided monoidal. A major tool we use in this process is the free field representation, by which we embed the bosonic ghosts into a lattice vertex algebra. To recover the bosonic ghosts one takes the kernel of a screening operator acting on the lattice vertex algebra. Among other things, this screening operator can be used to twist the vertex algebra action on a sum of modules to yield a new module. This module is the projective cover of the vacuum module (the vertex algebra as a module over itself) and the construction from the free field realisation gives insight into the explicit structure of the module, which assists in proving facts about the module itself and aids in classifying the indecomposable modules.

Quantum Enveloping Algebras of $\mathfrak{gl}_2$

We construct the candidate quantum group for our equivalence with the bosonic ghosts. As quantum groups' structure parallels that of the universal enveloping algebra of Lie algebras, we begin by defining the quantum Cartan subalgebra, which is exactly the lattice Hopf algebra corresponding to the free field realisation of the bosonic ghosts. The screening operator is associated to a particular lattice vertex algebra module, which we use to generate a Hopf algebra that becomes the analogues of the positive and negative root spaces of the quantum group. We prove that the representation theory of the quantum group yields a category of Yetter-Drinfeld modules over graded vector spaces. These are modules over the quantum Cartan subalgebra, equipped with an action and coaction which satisfy a certain compatibility condition. This condition ensures that the category of Yetter-Drinfeld modules is equivalent to the categorical analogue of a center for modules over the Cartan subalgebra. It is also integral in providing the category of Yetter-Drinfeld modules with a braiding morphism.

The Equivalence

We use our framework to compare the representation theory of the quantum group to the bosonic ghosts. We classify the simple and projective modules of the quantum group, and their braiding, twist and associators. Using the restriction to projective modules, we prove the equivalence as abelian categories. We use the tensor product structure of both categories to construct a minimal set of data which need to agree for the categories to be ribbon Grothendieck-Verdier equivalent. Using this minimal set, we condense the conditions for the functor to be an equivalence down to one equation involving products and iterates of vertex algebraic intertwining operators.

Appendices

In Appendices A to C we give a brief overview of some of the mathematical definitions and results referred to throughout the thesis. In Appendix D we give the full proof of Theorem 6.3.2. In Appendix E we outline how we came to the construction of the quantum group which yields equivalent representation theory to the bosonic ghosts. In Appendix F we outline some potentially useful calculations for proving Conjecture 11.4.2.

Future Direction

The quantum groups we define depend on a parameter q which we take to be a $2p$th root of unity. The bosonic ghosts are also part of a family of vertex algebras called the $\mathcal{B}_p$ algebras. We consider the $p = 2$ case, but connections between these two families may be an interesting direction for exploration. Similarly, we investigate an equivalence fo the half-unrolled quantum group. We expect that the other stages of unrolling, yield equivalent representation theory to vertex algebras resembling the singlet and triplet vertex algebras. A natural extension of both these directions is the development of a general theory regarding equivalences between vertex algebras which are kernels of screening operators, and quantum groups at roots of unity constructed from Nichols algebras of screenings.

— Chapter 2 —

Tensor Categories

"Tenser, said the Tensor. Tension, apprehension, and dissension have begun."

— Alfred Bester, *The Demolished Man*

Recall that a monoid is a set with an associative multiplication and a multiplicative unit. Monoidal (or tensor) categories, are the categorification of this notion. That is, one replaces equalities and bijections between sets with natural isomorphisms and equivalences between categories. Monoidal categories provide a unifying framework which connects the representation theory of varied algebraic structures, as well as having numerous applications to fields such as knot theory and number theory. We will introduce additional structure in the form of braidings and twists to refine the notion of a monoidal category to encompass the features which arise in the categories of representations of vertex and Hopf algebras. For more in depth exposition on tensor categories see [1, 2]. For some basic definitions and results on category theory used in this thesis, see Appendix A.

2.1 Monoidal structure

Definition 2.1.1. A **monoidal category** $(\mathscr{C}, \otimes, A, 1, l, r)$ is a category $\mathscr{C}$, equipped with the following data and conditions.

- Tensor product bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$,
- Associativity isomorphisms $A_{\mathcal{M},\mathcal{N},\mathcal{P}} : \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P}) \to (\mathcal{M} \otimes \mathcal{N}) \otimes \mathcal{P}$,
- Unit object 1, and unit isomorphisms $l_{\mathcal{M}} : 1 \otimes \mathcal{M} \to \mathcal{M}$, $r_{\mathcal{M}} : \mathcal{M} \otimes 1 \to \mathcal{M}$,
- Each of the isomorphisms above are natural and the following diagrams must commute, for $\mathcal{M}, \mathcal{N}, \mathcal{P}, \mathcal{Q} \in \mathscr{C}$ (these commutative diagrams are known as the pentagon and triangle

axioms).

$$\begin{array}{ccccc}
\mathcal{M}\otimes(\mathcal{N}\otimes(\mathcal{P}\otimes\mathcal{Q})) & \xrightarrow{A_{\mathcal{M},\mathcal{N},\mathcal{P}\otimes\mathcal{Q}}} & (\mathcal{M}\otimes\mathcal{N})\otimes(\mathcal{P}\otimes\mathcal{Q}) & \xrightarrow{A_{\mathcal{M}\otimes\mathcal{N},\mathcal{P},\mathcal{Q}}} & ((\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})\otimes\mathcal{Q} \\
\big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{M}}\otimes A_{\mathcal{N},\mathcal{P},\mathcal{Q}}} & & & & \big\uparrow{\scriptstyle A_{\mathcal{M},\mathcal{N},\mathcal{P}}\otimes\mathrm{id}_{\mathcal{Q}}} \\
\mathcal{M}\otimes((\mathcal{N}\otimes\mathcal{P})\otimes\mathcal{Q}) & & \xrightarrow{\qquad A_{\mathcal{M},\mathcal{N}\otimes\mathcal{P},\mathcal{Q}} \qquad} & & (\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P}))\otimes\mathcal{Q}
\end{array} \tag{2.1.1}$$

$$\begin{array}{ccc}
\mathcal{M}\otimes(1\otimes\mathcal{N}) & \xrightarrow{A_{\mathcal{M},1,\mathcal{N}}} & (\mathcal{M}\otimes 1)\otimes\mathcal{N} \\
{\scriptstyle \mathrm{id}_{\mathcal{M}}\otimes l_{\mathcal{N}}}\searrow & & \swarrow{\scriptstyle r_{\mathcal{M}}\otimes\mathrm{id}_{\mathcal{N}}} \\
& \mathcal{M}\otimes\mathcal{N} &
\end{array} \tag{2.1.2}$$

The pentagon and triangle axioms have been shown to be sufficient requirements for any diagrams, that relate tensor products using associativity and unit isomorphisms, to commute (Mac Lane's coherence theorem) [3][VII.2]. That is, that any natural transformations constructed from A, l and r between any two functors constructed from $-\otimes-$ and 1 are equal.

Often, we will just write $\mathscr{C}$ to denote the monoidal category $(\mathscr{C},\otimes,A,1,l,r)$.

Now we define monoidal functors, which are the categorification of morphisms between monoids.

Definition 2.1.2. Let $\mathscr{C}$ and $\mathscr{D}$ be monoidal categories. Then a **monoidal functor** from $\mathscr{C}$ to $\mathscr{D}$ is defined by the following data and conditions.

- A functor $F:\mathscr{C}\to\mathscr{D}$,
- An isomorphism $F_0: 1_{\mathscr{D}}\to F(1_{\mathscr{C}})$,
- A natural isomorphism $F^T : \otimes_{\mathscr{D}}\circ(F\times F)\to F\circ\otimes_{\mathscr{C}}$.
- The following diagrams must commute, for $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}\in\mathscr{C}$.

$$\begin{array}{ccc}
1_{\mathscr{D}}\otimes F(\mathcal{M}) & \xrightarrow{l_{F(\mathcal{M})}} & F(\mathcal{M}) \\
\big\downarrow{\scriptstyle F_0\otimes\mathrm{id}_{F(\mathcal{M})}} & & \big\downarrow{\scriptstyle F(l_{\mathcal{M}})^{-1}} \\
F(1_{\mathscr{C}})\otimes F(\mathcal{M}) & \xrightarrow{F^T_{1_{\mathscr{C}},\mathcal{M}}} & F(1_{\mathscr{C}}\otimes\mathcal{M})
\end{array}
\qquad
\begin{array}{ccc}
F(\mathcal{M})\otimes 1_{\mathscr{D}} & \xrightarrow{r_{F(\mathcal{M})}} & F(\mathcal{M}) \\
\big\downarrow{\scriptstyle \mathrm{id}_{F(\mathcal{M})}\otimes F_0} & & \big\downarrow{\scriptstyle F(r_{\mathcal{M}})^{-1}} \\
F(\mathcal{M})\otimes F(1_{\mathscr{C}}) & \xrightarrow{F^T_{\mathcal{M},1_{\mathscr{C}}}} & F(\mathcal{M}\otimes 1_{\mathscr{C}})
\end{array} \tag{2.1.3}$$

$$\begin{array}{ccc}
F(\mathcal{M})\otimes(F(\mathcal{N})\otimes F(\mathcal{P})) & \xrightarrow{A_{F(\mathcal{M}),F(\mathcal{N}),F(\mathcal{P})}} & (F(\mathcal{M})\otimes F(\mathcal{N}))\otimes F(\mathcal{P}) \\
\Big\downarrow{\scriptstyle \mathrm{id}_{F(\mathcal{M})}\otimes F^T_{\mathcal{N},\mathcal{P}}} & & \Big\downarrow{\scriptstyle F^T_{\mathcal{M},\mathcal{N}}\otimes \mathrm{id}_{F(\mathcal{P})}} \\
F(\mathcal{M})\otimes F(\mathcal{N}\otimes\mathcal{P}) & & F(\mathcal{M}\otimes\mathcal{N})\otimes F(\mathcal{P}) \\
\Big\downarrow{\scriptstyle F^T_{\mathcal{M},\mathcal{N}\otimes\mathcal{P}}} & & \Big\downarrow{\scriptstyle F^T_{\mathcal{M}\otimes\mathcal{N},\mathcal{P}}} \\
F(\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P})) & \xrightarrow{F\left(A_{\mathcal{M},\mathcal{N},\mathcal{P}}\right)} & F((\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})
\end{array} \tag{2.1.4}$$

Definition 2.1.3. If the functors defining an equivalence of categories are monoidal, we call this a **monoidal equivalence**.

2.2 Braided monoidal structure

Similarly, we can categorify the definition of a commutative monoid, by replacing commutativity with a natural isomorphism called the braiding.

Definition 2.2.1. A **braided monoidal category** $(\mathscr{C},\otimes,A,1,l,r,c)$ is a monoidal category $\mathscr{C}$, equipped with the following data and conditions.

- Braiding isomorphisms $c_{\mathcal{M},\mathcal{N}}:\mathcal{M}\otimes\mathcal{N}\to\mathcal{N}\otimes\mathcal{M}$,
- The braiding isomomorphism is natural and the following diagrams commute, for $\mathcal{M},\mathcal{N},\mathcal{P}\in\mathscr{C}$ (Hexagon axioms).

$$\begin{array}{ccccc}
 & (\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P} & \xrightarrow{c_{\mathcal{M}\otimes\mathcal{N},\mathcal{P}}} & \mathcal{P}\otimes(\mathcal{M}\otimes\mathcal{N}) & \\
{\scriptstyle A_{\mathcal{M},\mathcal{N},\mathcal{P}}}\nearrow & & & & \searrow{\scriptstyle A_{\mathcal{P},\mathcal{M},\mathcal{N}}} \\
\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P}) & & & & (\mathcal{P}\otimes\mathcal{M})\otimes\mathcal{N} \\
{\scriptstyle \mathrm{id}_{\mathcal{M}}\otimes c_{\mathcal{N},\mathcal{P}}}\searrow & & & & \nearrow{\scriptstyle c_{\mathcal{M},\mathcal{P}}\otimes \mathrm{id}_{\mathcal{N}}} \\
 & \mathcal{M}\otimes(\mathcal{P}\otimes\mathcal{N}) & \xrightarrow{A_{\mathcal{M},\mathcal{P},\mathcal{N}}} & (\mathcal{M}\otimes\mathcal{P})\otimes\mathcal{N} &
\end{array} \tag{2.2.1}$$

$$\begin{array}{ccccc}
 & \mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P}) & \xrightarrow{c_{\mathcal{M},\mathcal{N}\otimes\mathcal{P}}} & (\mathcal{N}\otimes\mathcal{P})\otimes\mathcal{M} & \\
{\scriptstyle A^{-1}_{\mathcal{M},\mathcal{N},\mathcal{P}}}\nearrow & & & & \searrow{\scriptstyle A^{-1}_{\mathcal{N},\mathcal{P},\mathcal{M}}} \\
(\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P} & & & & \mathcal{N}\otimes(\mathcal{P}\otimes\mathcal{M}) \\
{\scriptstyle c_{\mathcal{M},\mathcal{N}}\otimes \mathrm{id}_{\mathcal{P}}}\searrow & & & & \nearrow{\scriptstyle \mathrm{id}_{\mathcal{N}}\otimes c_{\mathcal{M},\mathcal{P}}} \\
 & (\mathcal{N}\otimes\mathcal{M})\otimes\mathcal{P} & \xrightarrow{A^{-1}_{\mathcal{N},\mathcal{M},\mathcal{P}}} & \mathcal{N}\otimes(\mathcal{M}\otimes\mathcal{P}) &
\end{array} \tag{2.2.2}$$

Now we define braided monoidal functors, which are the categorification of morphisms between monoids that also respect the braided structure.

Definition 2.2.2. Let $\mathscr{C}$ and $\mathscr{D}$ be braided monoidal categories. Then a **braided monoidal functor** from $\mathscr{C}$ to $\mathscr{D}$ is a monoidal functor satisfying the following additional commutative diagram.

$$\begin{array}{ccc} F(\mathcal{M})\otimes F(\mathcal{N}) & \xrightarrow{c_{F(\mathcal{M}),F(\mathcal{N})}} & F(\mathcal{N})\otimes F(\mathcal{M}) \\ \downarrow F^T_{\mathcal{M},\mathcal{N}} & & \downarrow F^T_{\mathcal{N},\mathcal{M}} \\ F(\mathcal{M}\otimes\mathcal{N}) & \xrightarrow{F\left(c_{\mathcal{M},\mathcal{N}}\right)} & F(\mathcal{N}\otimes\mathcal{M}) \end{array} \tag{2.2.3}$$

Definition 2.2.3. If the functors defining an equivalence of categories are braided monoidal, we call this a **braided monoidal equivalence**.

2.3 Categories of vector spaces graded by abelian groups

Definition 2.3.1. Let Vect_G denote the category of finite dimensional complex vector spaces graded by an abelian group G, whose morphisms are all grade preserving linear maps. This category is semisimple with the isomorphism classes of simple objects represented by the one dimensional vector spaces $\mathbb{C}_g$ which are $\mathbb{C}$ at grade $g \in G$ and trivial at other grades.

Note that if G is not finite, then objects in Vect_G will have only finitely many non-trivial homogenous spaces. We define a tensor product bifunctor on Vect_G by asserting

$$(\mathcal{M}\otimes\mathcal{N})_g = \bigoplus_{h\in G} \mathcal{M}_{g-h}\otimes_{\mathbb{C}} \mathcal{N}_h, \qquad g\in G,\ \mathcal{M},\mathcal{N}\in \mathrm{Vect}_G, \tag{2.3.1}$$

where $\otimes_{\mathbb{C}}$ is the tensor product of complex vectors spaces and having the tensor product of morphisms be that of linear maps. Further the unit morphisms of vector spaces then also define unit morphisms for the tensor functor $\otimes$ on Vect_G. The associativity and braiding isomorphisms can then be defined on tensor products of the simple objects $\mathbb{C}_g$ to be scalar multiplies of the vector space associator and tensor flip respectively. We shall denote these scalar multiples as F and Ω, and they form a normalised abelian 3-cocycle which we shall define below.

Definition 2.3.2. Let G be an abelian group. A **normalised abelian 3-cocycle** (F,Ω) is a pair of maps $F: G\times G\times G \to \mathbb{C}^\times$ and $\Omega: G\times G\to\mathbb{C}^\times$ characterised by the following relations, for $g,h,k,l\in G$.

$$\begin{aligned} F(g+h,k,l)F(g,h,k+l) &= F(g,h,k)F(g,h+k,l)F(h,k,l), \\ F(h,k,g)^{-1}\Omega(g,h+k)F(g,h,k)^{-1} &= \Omega(g,k)F(h,g,k)^{-1}\Omega(g,h), \\ F(k,g,h)\Omega(g+h,k)F(g,h,k) &= \Omega(g,k)F(g,k,h)\Omega(h,k), \end{aligned} \tag{2.3.2}$$

and additionally requiring that both maps evaluate to $1 \in \mathbb{C}^\times$ if any argument is $0 \in G$. Inequivalent associativity and braiding structures are parametrised by the cohomology classes of the third abelian group cohomology $H^3_{\text{ab}}(G,\mathbb{C}^\times)$. The cohomology classes $\omega = [(F,\Omega)] \in H^3_{\text{ab}}(G,\mathbb{C}^\times)$ are uniquely characterised by their trace $(\text{tr}\,\omega)(g) = \Omega(g,g) = q(g)$, which yields a quadratic form $q : G \to \mathbb{C}^\times$.

Theorem 2.3.3 (Eilenberg and MacLane [4], Joyal and Street [2])**.** *Let G be an abelian group and Vect_G the category of finite dimensional G graded complex vector spaces with tensor functor and unit isomorphisms defined above. Then the braiding and associativity morphisms on Vect_G are in bijection with normalised abelian 3-cocycles (F,Ω).*

Due to the above theorem, we denote by Vect_G^q the equivalence class of braided tensor categories with structure characterised by q, and by $\text{Vect}_G^{(F,\Omega)}$ the specific representative category whose associativity and braiding structures corresponds to the abelian 3-cocycle (F,Ω). The only nontrivial datum for the equivalence functor is the natural isomorphism F^T.

Remark. Two elements (F,Ω) and (F',Ω') of the same cohomology class ω are multiples of each other, by a **coboundary** $df = (df_1, df_2)$. That is, $(F',\Omega') = (F df_1, \Omega df_2)$. Given a 2-cochain $f : G\times G \to \mathbb{C}^\times$, the corresponding coboundary is given by

$$df(g,h,k) = (f(h,k)f(h,g+k)^{-1}f(g,h+k)f(g,k)^{-1}, f(g,h)f(h,g)^{-1}). \tag{2.3.3}$$

2.4 Rigidity

Rigidity is a categorical generalisation of the duality structure of finite dimensional vector spaces. There are notions of left and right dual, but here we shall only consider the case where there is a braiding, and therefore the existence of the left dual guarantees the existence of the right dual.

Definition 2.4.1. An object $\mathcal{M}$ in a monoidal category is (left) **rigid** if there exists an object $\mathcal{M}^\vee$ (called a tensor dual of $\mathcal{M}$) and two morphisms $e_\mathcal{M} : \mathcal{M}^\vee \boxtimes \mathcal{M} \to 1$ and $i_\mathcal{M} : 1 \to \mathcal{M}\boxtimes\mathcal{M}^\vee$, respectively, called evaluation and coevaluation, such that the compositions

$$\mathcal{M} \cong 1\otimes\mathcal{M} \xrightarrow{i_\mathcal{M}\otimes 1} (\mathcal{M}\otimes\mathcal{M}^\vee)\otimes\mathcal{M} \xrightarrow{A^{-1}} \mathcal{M}\otimes(\mathcal{M}^\vee\otimes\mathcal{M}) \xrightarrow{1\otimes e_\mathcal{M}} \mathcal{M}\otimes 1 \cong \mathcal{M}, \tag{2.4.1a}$$

$$\mathcal{M}^\vee \cong \mathcal{M}^\vee\otimes 1 \xrightarrow{1\otimes i_\mathcal{M}} \mathcal{M}^\vee\otimes(\mathcal{M}\otimes\mathcal{M}^\vee) \xrightarrow{A} (\mathcal{M}^\vee\otimes\mathcal{M})\otimes\mathcal{M}^\vee \xrightarrow{e_\mathcal{M}\otimes 1} 1\otimes\mathcal{M}^\vee \cong \mathcal{M}^\vee, \tag{2.4.1b}$$

yield the identity maps $\text{id}_\mathcal{M}$ and $\text{id}_{\mathcal{M}^\vee}$, respectively. If it exists, the tensor dual is unique [1, Proposition 2.10.4]. Further, a category is called (left) rigid if every object is (left) rigid.

Lemma 2.4.2. *A left rigid category with a braiding is rigid [2, Propisition 7.2].*

Consider the case of a finite dimensional vector space V over a field $\mathbb{k}$, with basis $\{v_n\}$. Let the dual $V^\vee = V^*$ have basis $\{v^n\}$. Here, the tensor unit is the field $\mathbb{k}$ and the evaluation and coevaluation are given by

$$e : v^n \otimes v_m \longrightarrow v^n(v_m) = \delta^n_m, \qquad i(1) = \sum_n v_n \otimes v^n. \tag{2.4.2}$$

Note that the expression for $i(1)$ does not depend on the choice of basis.

Proposition 2.4.3. *Let $\mathscr{C}$ be a rigid abelian category. Then we have the following consequences.*

- *The rigid dual $(-)^\vee$ is an antiequivalence [5][Exercise 2.1.7].*
- *The tensor product bifunctor $\otimes : \mathscr{C} \times \mathscr{C} \longrightarrow \mathscr{C}$ is exact in both factors (biexact) [5][2.1.8].*
- *Projective modules in $\mathscr{C}$ form a tensor ideal (a subcategory which is closed under taking the tensor product with any object in $\mathscr{C}$) [1][4.2.12].*

2.5 Ribbon structure

Just as rigidity generalises the duality structure of finite dimensional vector spaces, we can take this a step further and include a functorial isomorphism

Definition 2.5.1. Let $\mathscr{C}$ be a braided monoidal category with braiding c, then the identity functor with monoidal structure given by the double braiding $J_{\mathscr{C}} = (\mathrm{id}_{\mathscr{C}}, \mathrm{id}_{\mathbf{1}}, c \circ c)$ is a braided monoidal auto-equivalence called the **Joyal-Street equivalence**. A **twist** on a braided monoidal category is a monoidal isomorphism $\theta : \mathrm{id}_{\mathscr{C}} \longrightarrow J_{\mathscr{C}}$. Explicitly, this means that the twist θ obeys

$$\theta_{\mathbf{1}} = \mathrm{id}_{\mathbf{1}}, \quad \text{and} \quad \theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y). \tag{2.5.1}$$

We refer to the choice of isomorphisms θ as a ribbon structure, and we call a rigid monoidal category with a family of twists a **ribbon** category if the twist satisfies

$$(\theta_X)^\vee = \theta_{X^\vee}. \tag{2.5.2}$$

— Chapter 3 —

Grothendieck-Verdier Categories

"One should never try to prove anything that is not almost obvious."

— Alexander Grothendieck

3.1 Definitions

Duality, in particular, in the form of rigidity, plays an important role in quantum topology, a subject intimately linked to vertex algebras and their representation categories. The tensor product of a rigid abelian tensor category is necessarily exact. In general, this can not be expected to be true for representation categories of vertex algebras, which are monoidal categories. Rigidity is a property; it is actually a special case of a more general duality *structure*. Categories with such a structure are called $*$-autonomous categories [6] or, more recently, Grothendieck-Verdier categories [7–9]. We will see that this duality structure is indeed very naturally realised on vertex operator algebra module categories to which the Huang-Lepowsky-Zhang tensor product theory applies (see Chapter 6). In fact, since these categories are naturally braided and have a canonical identification of the bidual with the original module, they admit a pivotal structure in the presence of rigidity. This pivotal structure is equivalent to the existence of a ribbon structure (which has a prominent manifestation in the context of vertex operator algebras). So in the absence of rigidity, we are naturally lead to study these module categories as ribbon Grothendieck-Verdier categories. This overview is based on [10], with an emphasis on the results necessary for later sections.

Definition 3.1.1. A **Grothendieck-Verdier category** is a monoidal category $\mathscr{C}$, together with a distinguished object $K \in \mathscr{C}$, called the **dualising object** satisfying the following conditions.

1. For any object $Y \in \mathscr{C}$, the contravariant functor $\mathrm{Hom}(- \otimes Y, K)$ is representable, that is,

there exists an object $DY \in \mathscr{C}$ such that there is a natural isomorphism

$$\mathrm{Hom}(-\otimes Y, K) \xrightarrow{\cong} \mathrm{Hom}(-, DY). \tag{3.1.1}$$

Therefore, by Yoneda's Lemma there exists a unique (up to natural isomorphism) contravariant functor D, called the **dualising functor**, which assigns to every $Y \in \mathscr{C}$ the representing object DY, that is $D(Y) = DY$.

2. The contravariant functor D above is an anti-equivalence.

If in addition the category $\mathscr{C}$ is braided, then it is called a **braided Grothendieck-Verdier category**.

Remark. An immediate consequence of the above definition of Grothendieck-Verdier categories is the existence of a natural isomorphism in two variables

$$\mathrm{Hom}(-_1 \otimes -_2, K) \xrightarrow{\cong} \mathrm{Hom}(-_1, D(-_2)), \tag{3.1.2}$$

of the contravariant functors $\mathrm{Hom}(-\otimes -, K)$ and $\mathrm{Hom}(-, D(-))$, where the subscripts indicate that the ordering of the variables is preserved.

Remark. An equivalent definition of a Grothendieck-Verdier category is a closed monoidal category $\mathscr{C}$ with a distinguished object K such that the internal Hom functor $D(-) = [-, K]$ is an antiequivalence of $\mathscr{C}$.

Note that the choice of a dualising object K is structure, rather than a property, as there can be many inequivalent choices.

Proposition 3.1.2 (Boyarchenko-Drinfeld [7, Proposition 1.3])**.** *Let $\mathscr{C}$ be a Grothendieck-Verdier category with dualising object K and corresponding dualising functor D.*

1. *For any invertible object U, the objects $D(U) \cong K \otimes U^{-1}$ and $D^{-1}(U) \cong U^{-1} \otimes K$ are again dualising objects in $\mathscr{C}$.*

2. *The functors $U \mapsto D(U) = K \otimes U^{-1}$ and $U \mapsto D^{-1}(U) \cong U^{-1} \otimes K$ are anti-equivalences between the full subcategory of invertible objects $U \in \mathscr{C}$ and the full subcategory of dualising objects.*

3. *If $U \in \mathscr{C}$ is invertible then so is $D^2 U$ and one has a canonical isomorphism $K \otimes U^{-1} \cong (D^2 U)^{-1} \otimes K$.*

Proposition 3.1.3. *Let $\mathscr{C}$ be a Grothendieck-Verdier category with dualising object $K_{\mathscr{C}}$ and let $\mathscr{D}$ be a monoidal category. For any monoidal equivalence $F:\mathscr{C}\longrightarrow\mathscr{D}$, the object $F(K_{\mathscr{C}})$ is a dualising object for $\mathscr{D}$. Thus $\mathscr{D}$ admits a Grothendieck-Verdier structure. In particular, if $\mathscr{D}$ has already been endowed with a dualising object $K_{\mathscr{D}}$, then $F(K_{\mathscr{C}})$ and $K_{\mathscr{D}}$ differ by tensoring with an invertible object.*

Proof. Let F^{-1} be a quasi-inverse of the equivalence F. For $X,Y\in\mathscr{D}$, consider

$$\begin{aligned}\mathrm{Hom}_{\mathscr{D}}(X\otimes Y,FK_{\mathscr{C}}) &\xrightarrow{\cong}\mathrm{Hom}_{\mathscr{D}}(FF^{-1}X\otimes FF^{-1}Y,FK_{\mathscr{C}})\xrightarrow{\cong}\mathrm{Hom}_{\mathscr{D}}(F(F^{-1}X\otimes F^{-1}Y),FK_{\mathscr{C}})\\ &\xrightarrow{\cong}\mathrm{Hom}_{\mathscr{C}}(F^{-1}X\otimes F^{-1}Y,K_{\mathscr{C}})\xrightarrow{\cong}\mathrm{Hom}_{\mathscr{C}}(F^{-1}X,D_{\mathscr{C}}F^{-1}Y)\\ &\xrightarrow{\cong}\mathrm{Hom}_{\mathscr{D}}(X,FD_{\mathscr{C}}F^{-1}Y),\end{aligned}\tag{3.1.3}$$

where the second bijection follows from the monoidal structure on F and the fourth uses the definition property of the dualising object $K_{\mathscr{C}}$. This implies that $FK_{\mathscr{C}}$ is a dualising object in $\mathscr{D}$ with corresponding dualising functor $F\circ D_{\mathscr{C}}\circ F^{-1}$. Finally, if $\mathscr{D}$ was already endowed with a dualising object $K_{\mathscr{D}}$, then an immediate consequence of Proposition 3.1.2.2 is that $FK_{\mathscr{C}}$ and $K_{\mathscr{D}}$ differ by tensoring with an invertible object. ■

Proposition 3.1.3 shows that monoidal equivalences transport dualising objects, thus allowing their comparison. In particular, in the notation of this proposition the pair of Grothendieck-Verdier categories $\mathscr{C},\mathscr{D}$ have equivalent Grothendieck-Verdier structures if and only if $FK_{\mathscr{C}}\cong K_{\mathscr{D}}$.

Braided Grothendieck-Verdier categories can admit ribbon structures that are compatible with the Grothendieck-Verdier structure. If θ is a twist on a Grothendieck-Verdier category $\mathscr{C}$, then

$$\theta^D_X = D^{-1}(\theta_{DX})$$

is also a twist on $\mathscr{C}$. By [7, Proposition 7.3], this is an involution on the set of twists. The fixed points under this involution are relevant for representation categories of conformal vertex operator algebras.

Definition 3.1.4. A **ribbon Grothendieck-Verdier category** is a braided Grothendieck-Verdier category $\mathscr{C}$ with a twist θ such that

$$D(\theta_X)=\theta_{DX},\quad \forall X\in\mathscr{C}.\tag{3.1.4}$$

Combining all of the notions above, we are lead to the following natural definition of an equivalence of ribbon Grothendieck-Verdier categories.

Definition 3.1.5. Let $\mathscr{C}$ and $\mathscr{D}$ be ribbon Grothendieck-Verdier categories with respective dualising objects $K_{\mathscr{C}}$ and $K_{\mathscr{D}}$, and twists $\theta_{\mathscr{C}}$ and $\theta_{\mathscr{D}}$. A **ribbon Grothendieck-Verdier equivalence** is a braided monoidal equivalence satisfying

- equivalence of dualising objects: $FK_{\mathscr{C}} \cong K_{\mathscr{D}}$,
- equivalence of twists: $F(\theta_{\mathscr{C}}) = \theta_{\mathscr{D}}$.

Lemma 3.1.6. *If two categories are equivalent as rigid categories, then they are also equivalent as Grothendieck-Verdier categories. That is, given a choice of dualising object for one of the categories, there exists a choice of dualising object for the other such that the categories are Grothendieck-Verdier equivalent. Similarly, if they are equivalent as ribbon categories, then they are also equivalent as ribbon Grothendieck-Verdier categories.*

Proof. Both categories have equivalent subcategories of invertible objects and therefore equivalent categories of dualising objects, by Proposition 3.1.2. ∎

3.2 Categories of vector spaces graded by abelian groups

Proposition 3.2.1. *Let (F, Ω) be an abelian 3-cocycle and consider the braided tensor category $\mathrm{Vect}_G^{(F,\Omega)}$.*

1. *For any $h \in G$, $K = \mathbb{C}_h$ is a dualising object and hence endows $\mathrm{Vect}_G^{(F,\Omega)}$ with the structure of a Grothendieck-Verdier category.*
2. *The dualising functor corresponding to the choice of dualising object $K = \mathbb{C}_h$, $h \in G$ is characterised by $D(\mathcal{M})_g \simeq (\mathcal{M}_{h-g})^*$, $g \in G$, $\mathcal{M} \in \mathrm{Vect}_G^{(F,\Omega)}$, where $*$ denotes the ordinary vector space dual.*
3. *Every dualising object of $\mathrm{Vect}_G^{(F,\Omega)}$ is isomorphic to one of the simple objects $\mathbb{C}_h$ for some $h \in G$.*

Denote by $\mathrm{Vect}_G^{(F,\Omega,h)}$ the Grothendieck-Verdier category constructed from $\mathrm{Vect}_G^{(F,\Omega)}$ with dualising object $K = \mathbb{C}_{2h}$.

4. *The Grothendieck-Verdier category $\mathrm{Vect}_G^{(F,\Omega,h)}$ admits a twist θ by defining*

$$\theta_Q|_{\mathcal{M}_g} = Q(g)\mathrm{id}_{\mathcal{M}_g}, \qquad Q(g) = \frac{q(g-h)}{q(h)}, \qquad \mathcal{M} \in \mathrm{Vect}_G^{(F,\Omega,h)},\ g \in G, \tag{3.2.1}$$

where q is the quadratic form which characterises the cohomology class of (F, Ω).

Proof. First we prove Part 1. The category $\mathrm{Vect}_G^{(F,\Omega,h)}$ is known to be rigid (the rigid dual of any object $\mathcal{M}$ is $\mathcal{M}_h^\vee \cong \mathcal{M}_{-h}$, for $h \in G$. and the evaluation and coevaluation maps are those of vector spaces rescaled by the 3-cocycle) and hence the unit object $\mathbb{C}_0$ is dualising. The simple modules $\mathbb{C}_h, h \in G$ are all invertible. Thus, by Proposition 3.1.2.2, $\mathbb{C}_0 \otimes (\mathbb{C}_h)^{-1} \cong \mathbb{C}_{-h}$ is also dualising. Proposition 3.1.2.2 also immediately implies Part 3. Next, Part 2 follows by noting that the proposed formula for D satisfies the defining relation (3.1.1) for the dualising object $K = \mathbb{C}_h$. Part 4 follows from the fact that the given formula for $Q(g)$ satisfies the relations implied by (2.5.1) and (3.1.4). This is most easily seen by using the bilinear form which gives the double braiding $\beta(g_1, g_2) = q(g_1 + g_2)/(q(g_1)q(g_2)) = \Omega(g_1, g_2)\Omega(g_2, g_1)$. Then we have $Q(g) = q(g)\beta(g, -h)$ and hence

$$\begin{aligned} Q(g_1+g_2) &= q(g_1+g_2)\beta(g_1+g_2,-h) = \beta(g_1,g_2)q(g_1)q(g_2)\beta(g_1,-h)\beta(g_2,-h) \\ &= \beta(g_1,g_2)Q(g_1)Q(g_2) = \Omega(g_2,g_1)\Omega(g_1,g_2)Q(g_1)Q(g_2), \\ Q(2h-g) &= \frac{q(h-g)}{q(-h)} = \frac{q(g-h)}{q(-h)} = Q(g). \end{aligned} \tag{3.2.2}$$

■

Remark. Observe that the choice of $\mathbb{C}_{2h}$ as dualising object in $\mathrm{Vect}_G^{(F,\Omega,h)}$ excludes those simple objects not labelled by the double of a group element in G. This is not an oversight; while every simple object in $\mathrm{Vect}_G^{(F,\Omega)}$ is a valid choice of dualising object (this follows from the category being rigid, hence the tensor unit is dualising, and by Proposition 3.1.2, we can shift by invertible objects), simple objects not labelled by the double of a group element need not admit a twist which satisfies $D(\theta) = \theta_{D(-)}$. Notice that $Q(g)$ in the example above is not well defined when $K = \mathbb{C}_{2h-1}$, for $h \in G$. Fortunately, the vertex operator algebraic constructions we consider will always yield dualising objects that admit twists and make a preferred choice of twist.

Functions of the form Q in Equation (3.2.1) are called weak quadratic forms centred at h. It is interesting to note that (at least in the special case of G being a finite group) the Grothendieck-Verdier ribbon twists on $\mathrm{Vect}_G^{(F,\Omega,h)}$ are in bijection with such weak quadratic forms centred at h, as was shown in Zetzsche's Masters thesis [11, Theorem 4.2.2]. This classification of ribbon Grothendieck-Verdier structures by weak quadratic forms is a generalisation of the classification of rigid braided tensor structures by quadratic forms, which corresponds to the special case $h = 0$ for the dualising object. Note also that for $h = 0$ the category is ribbon.

— Chapter 4 —

Hopf Algebras

"Often greater risk is involved in postponement than in making a wrong decision."

— Heinz Hopf

Hopf algebras are mathematical objects whose structure parallels that of a monoidal category with a notion of duals. The maps that consist their data endow their category of modules with a monoidal structure. For example, the comultiplication map gives the construction of tensor products. We require Hopf algebras for the free boson constructions in Chapter 8 and to provide a foundation for the quantum groups we define in Chapter 10.

4.1 Formal definition

Hopf algebras are vector spaces, endowed with an algebra structure and a coalgebra structure. We require that the coalgebra maps should be morphisms of unital algebras, yielding a bialgebra structure. Finally, we require the antipode, which provides a kind of dual.

Definition 4.1.1. A **Hopf algebra** $(H, M, \eta, \Delta, \varepsilon, s)$ consists of a vector space H over a field k equipped with linear maps

$$\begin{array}{ll} \text{Multiplication} & M : H \otimes H \to H, \\ \text{Unit} & \eta : k \to H, \\ \text{Comultiplication} & \Delta : H \to H \otimes H, \\ \text{Counit} & \varepsilon : H \to k, \\ \text{Antipode} & s : H \to H, \end{array} \tag{4.1.1}$$

satisfying the following conditions.

$$\begin{array}{ll} \text{Associative} & M \circ (\mathrm{id} \otimes M) = M \circ (M \otimes \mathrm{id}), \\ \text{Unital} & M \circ (\mathrm{id} \otimes \eta) = \mathrm{id} = M \circ (\eta \otimes \mathrm{id}), \\ \text{Coassociative} & (\mathrm{id} \otimes \Delta) \circ \Delta = (\Delta \otimes \mathrm{id}) \circ \Delta, \\ \text{Counital} & (\mathrm{id} \otimes \varepsilon) \circ \Delta = \mathrm{id} = (\varepsilon \otimes \mathrm{id}) \circ \Delta, \\ \text{Bialgebra} & \Delta \circ M = M \circ (\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta, \\ & \varepsilon \circ M = M \circ (\varepsilon \otimes \varepsilon), \quad \varepsilon \circ \eta = \eta, \\ \text{Antipode} & M \circ (\mathrm{id} \otimes s) \circ \Delta = \eta \circ \varepsilon = M \circ (s \otimes \mathrm{id}) \circ \Delta. \end{array} \tag{4.1.2}$$

In the unital condition, we implicitly use the natural scalar multiplication map $T : k \otimes H \longrightarrow H$ defined by $T(\lambda \otimes x) = \lambda x$ to ensure the domains acted on by each composition of maps are the same. We also identify the spaces $k \otimes H$ and $H \otimes k$. In the counital condition, we ensure the maps act on the same domains by implicitly using the natural inclusion map $I : H \longrightarrow k \otimes H$ defined by $I(x) = 1 \otimes x$, where 1 is the multiplicative unit of the field. To simplify the above conditions, we make use of some common notational conventions.

$$\eta(1) = e, \quad M(x \otimes y) = xy, \quad \Delta(x) = x_1 \otimes x_2 = \sum_i x_1^{(i)} \otimes x_2^{(i)}. \tag{4.1.3}$$

In the new notation, the conditions can be written in the following way. For all $x, y, z \in H$,

$$\begin{array}{ll} \text{Associative} & x(yz) = (xy)z, \\ \text{Unit} & xe = x = ex, \\ \text{Coassociative} & x_1 \otimes \Delta(x_2) = \Delta(x_1) \otimes x_2, \\ \text{Counit} & \varepsilon(x_1)x_2 = x = x_1\varepsilon(x_2), \\ \text{Bialgebra} & \Delta(xy) = \Delta(x)\Delta(y), \qquad \Delta(e) = e \otimes e, \\ & \varepsilon(xy) = \varepsilon(x)\varepsilon(y), \qquad \varepsilon(e) = 1, \\ \text{Antipode} & x_1 s(x_2) = \varepsilon(x)e = s(x_1)x_2. \end{array} \tag{4.1.4}$$

Proposition 4.1.2. *Let H be a Hopf algebra, then the antipode S is an anti-morphism of algebras and coalgebras (also called an antialgebra and anticoalgebra map). That is,*

$$S(xy) = S(y)S(x), \qquad S(e) = e, \tag{4.1.5}$$

$$S(x_1)S(x_2) = S(x)_2 \otimes S(x)_1, \qquad \varepsilon(S(x)) = \varepsilon(x). \tag{4.1.6}$$

For a more detailed introduction to Hopf algebras see [12–14].

4.2 Examples

Example. Let G be a group with identity element e and consider the group algebra $k[G]$. Then $k[G]$ is a Hopf algebra with

$$M(g,h) = gh, \quad \eta(1) = e, \quad \Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad s(g) = g^{-1}, \quad g,h \in G. \tag{4.2.1}$$

Example. Let $\mathfrak{g}$ be a Lie algebra and consider the universal enveloping algebra $\mathsf{U}(\mathfrak{g})$. Then $\mathsf{U}(\mathfrak{g})$ is a Hopf algebra with

$$M(x,y) = xy, \quad \eta(1) = 1, \quad \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad s(x) = -x, \quad x,y \in \mathfrak{g}, \tag{4.2.2}$$

and extended to relations on $\mathsf{U}(\mathfrak{g})$ using the properties of the algebra (anti)homomorphisms.

As a result of these two examples, we have the following definition.

Definition 4.2.1. Let H be a Hopf algebra. An element h is called

- **grouplike** if
$$\Delta(h) = h \otimes h, \qquad \varepsilon(h) = 1, \qquad s(h) = h^{-1}, \tag{4.2.3}$$
- **primitive** if
$$\Delta(h) = h \otimes 1 + 1 \otimes h, \qquad \varepsilon(h) = 0, \qquad s(h) = -h. \tag{4.2.4}$$

4.3 Modules and comodules

Definition 4.3.1. Let H be a Hopf algebra. Then a **left module** over H is a pair (V,ρ) consisting of a vector space V and an algebra morphism $\rho : H \otimes V \to V$, called the action.

Definition 4.3.2. Let H be a Hopf algebra. Then a **left comodule** over H is a pair (V,δ) consisting of a vector space V and a coalgebra morphism $\delta : V \to H \otimes V$, called the coaction.

Proposition 4.3.3. *Let H be a Hopf algebra. Then the categories of H-modules and H-comodules are monoidal categories, where the action and coaction on the tensor products are given by the following formulas, respectively.*

$$\rho_{V\otimes W} = (\rho_V \otimes \rho_W) \circ (\mathrm{id}_H \otimes P \otimes \mathrm{id}_W) \circ (\Delta \otimes \mathrm{id}_{V\otimes W}), \tag{4.3.1}$$
$$\delta_{V\otimes W} = (M \otimes \mathrm{id}_{V\otimes W}) \circ (\mathrm{id}_H \otimes P \otimes \mathrm{id}_W) \circ (\delta_V \otimes \delta_W), \tag{4.3.2}$$

where P is the tensor flip of vector spaces. The associativity and unit isomorphisms are the standard ones inherited from the underlying category of vector spaces.

Remark. In Sweedler notation, let the action and coaction above be given by $\rho_V(h \otimes v) = h.v$ and $\delta_V(v) = v_{-1} \otimes v_0$. Then the action and coaction on the tensor product of modules V and W is given by

$$\rho(h \otimes v \otimes w) = h_1.v \otimes h_2.w, \qquad \delta(v \otimes w) = v_{-1}w_{-1} \otimes v_0 \otimes w_0. \tag{4.3.3}$$

Proposition 4.3.4. *Let V be an H-module. Then the* ***dual module*** $V^* = \mathrm{Hom}(V, \Bbbk)$ *is also an H-module with action*

$$(hf)x = f(S(h)x). \tag{4.3.4}$$

Proof.

$$((hg)f)(x) = f(S(hg)x) = f(S(g)S(h)x) = (gf)(S(h)x) = (h(gf))(x). \tag{4.3.5}$$

■

4.4 Ribbon Grothendieck-Verdier structure

As we have seen, Hopf algebras naturally give rise to monoidal categories of modules and comodules. They can also be equipped with additional structure, in the form of distinguished invertible elements, which endow the module categories with braidings and twists.

Definition 4.4.1. A **quasi-triangular** Hopf algebra H is a Hopf algebra, equipped with an invertible element $R \in H \otimes H$, called the **R-matrix**, satisfying

$$R\Delta(x)R^{-1} = (P \circ \Delta)(x), \tag{4.4.1a}$$

$$(\Delta \otimes \mathrm{id})(R) = R_{13}R_{23}, \tag{4.4.1b}$$

$$(\mathrm{id} \otimes \Delta)(R) = R_{13}R_{12}, \tag{4.4.1c}$$

where P is the tensor flip of vector spaces and

$$R_{12} = R \otimes 1, \quad R_{23} = 1 \otimes R, \quad R_{13} = (P \otimes \mathrm{id}_H)(1 \otimes R). \tag{4.4.2}$$

Further, a **ribbon** Hopf algebra H is a quasi-triangular Hopf algebra, equipped with an invertible central element $r \in H$, called the **ribbon element**, satisfying

$$\Delta(r) = (R_{21}R)^{-1}(r \otimes r), \quad \varepsilon(r) = 1, \quad S(r) = r. \tag{4.4.3}$$

Proposition 4.4.2. *Let H be a quasi-triangular Hopf algebra, with invertible antipode, and let $\mathcal{M}, \mathcal{N}$ be H-modules. Then the ribbon element and R-matrix equip the category of H-modules with the structure of a ribbon category (rigid with twists) with twist θ and braiding c, respectively, given by*

$$\theta_{\mathcal{M}} = r^{-1}, \qquad c_{\mathcal{M},\mathcal{N}} = P \circ R. \tag{4.4.4}$$

Proof. This is easily verified by plugging in the appropriate expressions in pentagon, hexagon and balancing axioms. The first two relations in (4.4.3) ensure that the twist conditions (2.5.1) are satisfied. (4.4.1a) ensures that the braidings give homomorphisms of H-modules. (4.4.1b) and (4.4.1c) ensure that the hexagon axioms (11.3.7) and (11.3.8) are satisfied. S being invertible is required to define both left and right duals (using Proposition 4.3.4 with the pullback of S and of S^{-1}). ∎

Remark. Note that if we relax the condition $S(r) = r$, then Proposition 4.4.2 yields a ribbon Grothendieck-Verdier category as long as (3.1.4) holds. The Grothendieck-Verdier structure is inherited from the underlying vector space category, through Proposition 3.2.1, by selecting a dualising object.

Remark. A **quasi-Hopf** algebra H is a modification of a Hopf algebra, where the axioms include terms from a **coassociator** $\Phi \in H \otimes H \otimes H$ which endows the monoidal category of modules with a non-trivial associator. Similarly, a **quasi-triangular quasi-Hopf** algebra results from modifying the quasi-triangular conditions such that the hexagon axioms are satisfied. For an explicit definition, see [12, Chapter XV].

Remark. We also allow the ribbon element, R-matrix and coassociator to be linear maps rather than elements. That is, we write $r \in \mathrm{End}(H)$, $R \in \mathrm{End}(H^{\otimes 2})$ and $\Phi \in \mathrm{End}(H^{\otimes 3})$ for the actions of each map defined as multiplication by their respective elements. Then we generalise to allow them to be any linear maps.

4.5 Yetter-Drinfeld modules

The category of Yetter-Drinfeld modules of a Hopf algebra H is equivalent to the Drinfeld center of the monoidal category of left H-modules, where the Drinfeld center is the categorification of the notion of center of a monoid [1][7.13]. The enhanced structures such as braidings, twists and duals are inherited from the underlying category of H-modules.

Definition 4.5.1. Let H be a Hopf algebra. Then V is called a **Yetter-Drinfeld module** over H if it is equipped with the structure of a left H-module and a left H-comodule, satisfying the following compatibility condition.

$$\delta(h.v) = h_1 v_{-1} S(h_3) \otimes h_2.v_0, \tag{4.5.1}$$

where . denotes the left action and δ the left coaction.

We recall and prove the following results, which applies to categories such as in Theorem 10.2.6, and which is relevant for categories Yetter-Drinfeld modules in any category (not just vector spaces).

Proposition 4.5.2 (Yetter-Drinfeld braiding [13, Definition 3.4.4])**.** *Let $\mathscr{C}$ be a category of H-modules and H-comodules with braiding c. Then the braiding of the category ${}^H_H\mathcal{YD}(\mathscr{C})$ of Yetter-Drinfeld modules in $\mathscr{C}$ is given by*

$$c^{\mathcal{YD}} = (\rho\otimes \mathrm{id})\circ(\mathrm{id}\otimes c)\circ(\delta\otimes \mathrm{id}), \qquad c^{\mathcal{YD}}(v,w) = c(v_0,w)v_{-1}.w\otimes v_0. \tag{4.5.2}$$

An object is called **transparent** if it has trivial double braiding with every other object. Possessing no non-trivial transparent objects is a condition that modular tensor categories must satisfy. Modular tensor categories are intimately linked to conformal field theory, topological field theory and quantum groups at roots of unity [5][Chapter 3], [15]. Then we have the following result.

Proposition 4.5.3. *Let $\mathscr{C}$ be a braided tensor category and let $B\in\mathscr{C}$ be a Hopf algebra. Then the transparent objects in ${}^B_B\mathcal{YD}(\mathscr{C})$ are transparent objects in $\mathscr{C}$ with trivial B action and coaction. In particular if $\mathscr{C}$ has no non-trivial transparent objects, then ${}^B_B\mathcal{YD}(\mathscr{C})$ has none either.*

Proof. Let $b\otimes v\longrightarrow q(b,v)v\otimes b$ be the braiding, in $\mathscr{C}$, of an element $b\otimes v\in B\otimes V$. Then the double braiding of $b\otimes v$ is given by the following expression, for $V\in{}^B_B\mathcal{YD}(\mathscr{C})$ [16][p.12].

$$q(b_0,v)q((b_{-1}v)_0,b_0)(b_{-1}v)_{-1}b_0\otimes(b_{-1}v)_0. \tag{4.5.3}$$

Consider B as an object in $B-\mathrm{Mod}(\mathscr{C})$ via the product. Make it into an object in ${}^B_B\mathcal{YD}(\mathscr{C})$ with the trivial coaction $\delta(b)=1\otimes b$. Then the double braiding $B\otimes V\longrightarrow B\otimes V$ with any $V\in{}^B_B\mathcal{YD}(\mathscr{C})$ is equal to $\mathscr{C}$-braiding B-coaction $\mathscr{C}$-braiding B-product $(q(b,v)q(v_0,b)v_{-1}b\otimes v_0)$. In particular, applying to the unit in B (resp. precomposing with $\eta:1\longrightarrow B$) is the B-coaction on V ($\delta_B(v)=v_{-1}\otimes v_0$). So, if V is transparent, then B coacts trivially.

Consider B as an object in $B-\mathrm{Comod}(\mathscr{C})$ via the coproduct $\delta(b)=b_1\otimes b_2$. Make it into an object in ${}^B_B\mathcal{YD}(\mathscr{C})$ with the trivial action $a.b=b$ for $a,b\in B$. Then the double braiding $B\otimes V\longrightarrow B\otimes V$ with any $V\in{}^B_B\mathcal{YD}(\mathscr{C})$ is equal to B-coproduct $\mathbb{C}$-braiding B-action $\mathbb{C}$-braiding $(q(b_2,v)q((b_1v)_0,b_2)b_2\otimes(b_1v)_0)$. In particular applying the counit of B gives the B-action on V ($b.v$). So, if V is transparent, then B acts trivially.

Consider any object $X\in\mathscr{C}$ as a ${}^B_B\mathcal{YD}(\mathscr{C})$ module with trivial action and coaction. Then the double braiding $X\otimes V\longrightarrow X\otimes V$ with any $V\in{}^B_B\mathcal{YD}(\mathscr{C})$ is equal is the $\mathscr{C}$-double braiding. So, if V is transparent, then it is transparent as a $\mathscr{C}$-object.

■

4.6 Nichols algebras

Nichols algebras can be thought of as generalisations of Lie algebras, as they are endowed with generalised root systems. They will be important in our quantum group constructions of Chapter 10, where they play the roles of the quantum analogues of the positive and negative roots in the universal enveloping algebra of a Lie algebra.

Definition 4.6.1. Let $V \in {}^H_H\mathcal{YD}$ be a Yetter-Drinfeld module. Then the **Nichols algebra** $\mathfrak{B}(V)$ is the unique Hopf algebra in ${}^H_H\mathcal{YD}$ generated by V, such that $V \subset \mathfrak{B}(V)$ are the only primitive elements.

— Chapter 5 —

Vertex Algebras

"I am an old man, and I know that a definition cannot be so complicated."

— I.M. Gelfand

Vertex algebras describe chiral symmetries of two dimensional conformal field theories. They are generalisations of commutative rings, with an infinite number of products. These products can be written as coefficients of a formal power series, or formal distributions, called fields. In conformal field theory, the formal variables in the fields become insertion points in the complex plane, and the vertex algebra can be thought of as an algebra with a product parametrised by points in the complex plane. We cover some formal calculus basics in Appendix C but for a more comprehensive exposition on formal calculus and vertex algebras, see [17–19].

5.1 Formal definition

Definition 5.1.1. A **vertex algebra** consists of the following data.

- The **space of states**: a vector space V.
- The **vacuum vector**: a distinguished vector $|0\rangle \in V$.
- The **translation operator**: A linear operator $T : V \to V$.
- **Vertex operators**: A linear map $Y : V \to \mathrm{End}V[[x^{\pm 1}]]$ which takes each state $v \in V$ to a field (formal series) acting on V, written in the form below.

$$v \mapsto Y(v,x) = \sum_{n\in\mathbb{Z}} v_n x^{-n-h_v}, \tag{5.1.1}$$

where $h_v \in \mathbb{Z}$ is a choice determined by convention and $v_n \in \mathrm{End}V$ are the Fourier coefficients.

All the data above must also satisfy the following axioms, for all $u, v \in V$.

- **Field and vacuum axioms**: Fields must be lower truncated when acting on a state, and the field corresponding to the vacuum vector should be the identity map.

$$Y(u,x)v \in V((x)), \qquad Y(|0\rangle, x) = \mathrm{id}_V. \tag{5.1.2}$$

- **State-field correspondence**: A field acting on the vacuum is well defined as $x \to 0$ and vectors are recovered from their corresponding fields by taking this limit.

$$Y(v,x)|0\rangle \in V[[x]], \qquad \lim_{x\to 0} Y(v,x)|0\rangle = v. \tag{5.1.3}$$

- **Translation covariance**: The translation operator behaves as an infinitesimal generator of translations and the vacuum is translation invariant.

$$[T, Y(v,x)] = \partial_x Y(v,x), \qquad T|0\rangle = 0. \tag{5.1.4}$$

- **Locality**: All vertex operators satisfy the following relaxed version of a commutation relation, as formal power series in $\mathrm{End}V[[x_1{}^{\pm 1}, x_2{}^{\pm 1}]]$.

$$(x_1 - x_2)^N [Y(u,x_1), Y(v,x_2)] = 0 \;\text{ for some }\; N \in \mathbb{N}. \tag{5.1.5}$$

Remark. Equivalently, the locality axiom can be replaced by the **Jacobi identity**.

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right) Y(u,x_1)Y(v,x_2) - x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right) Y(v,x_2)Y(u,x_1) \\ = x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right) Y(Y(u,x_0)v, x_2), \end{aligned} \tag{5.1.6}$$

where δ denotes the algebraic delta distribution, that is the formal power series

$$\delta\left(\frac{z_2 - z_1}{z_0}\right) = \sum_{\substack{r\in\mathbb{Z} \\ s\geq 0}} \binom{r}{s} (-1)^s z_1^s z_2^{r-s} z_0^{-r}. \tag{5.1.7}$$

The Virasoro algebra plays a crucial role in two-dimensional conformal field theory where it generates holomorphic conformal transformations.

Definition 5.1.2. A **vertex operator algebra** is a vertex algebra with an additional distinguished vector ω satisfying the following conditions, where $Y(\omega, x) = \sum_{n\in\mathbb{Z}} L_n x^{-n-2}$.

- L_n endow V with an action of the Virasoro algebra with central charge c.

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m\left(m^2 - 1\right)\delta_{m+n,0}, \qquad c \in \mathbb{C}. \tag{5.1.8}$$

- L_0 grades the space of states by weights $n \in \mathbb{Z}$. That is,

$$V = \bigoplus_{n\in\mathbb{Z}} V_{(n)}, \qquad L_0 v = nv \;\text{ for }\; v \in V_{(n)}. \tag{5.1.9}$$

  We call $h_v = n$ the conformal weight of the vector v.

- The vacuum and conformal vectors have appropriate conformal weights $|0\rangle \in V_{(0)}$ and $\omega \in V_{(2)}$.

- L_{-1} acts as the translation operator $L_{-1} = T$.

Remark. In general, there are many choices for the conformal vector ω, or equivalently the Virasoro field $Y(\omega, x)$, and we refer to this as a choice of **conformal structure**.

5.2 Operator product expansion

Definition 5.2.1. The **operator product expansion** (OPE) of two fields is given by the expansion of the product of the two fields as a power series in the difference of the insertion points, with vertex operator coefficients.

$$Y(u,z)Y(v,w) = \sum_{n\in\mathbb{Z}} \frac{Y(u_n v, w)}{(z-w)^{n+h_u}}, \tag{5.2.1}$$

where the left hand side is expanded as a power series in positive powers of $(z-w)/w$, by sending z to $w+(z-w)$ and z^{-1} to $(w+(z-w))^{-1}$ [17].

Only the singular part of the OPE contributes to the commutator of two Fourier coefficients

$$[u_n, v_m] = \sum_{k\geq 0} \binom{n}{k} (u_k v)_{n+m-k}. \tag{5.2.2}$$

Therefore, the operator product expansion is usually written without the regular part of the expansion (the normally ordered product).

$$: Y(u,z)Y(v,w) :\, = \sum_{n\leq -h_u} \frac{Y(u_n v, w)}{(z-w)^{n+h_u}}. \tag{5.2.3}$$

The fact that the fields are lower truncated when acting on a state ensures the existence of the normally ordered product, which provides a way of multiplying fields together at the same point, as it doesn't contain any singular terms which diverge as $z \to w$.

5.3 Affine Kac-Moody algebras at level k

Affine Kac-Moody algebras are a class of infinite-dimensional Lie algebras which are the result of allowing the Cartan matix to not be positive definite. They are of particular interest in conformal field theory and we include this example of their associated vertex algebra as it is a commonly studied example which explicitly demonstrates a lot of the key features of vertex algebras.

Let $\mathfrak{g}$ be a finite-dimensional simple complex Lie algebra. The corresponding loop algebra is defined as

$$\mathcal{L}\mathfrak{g} = \mathfrak{g}((t)) = \mathfrak{g} \otimes \mathbb{C}((t)), \qquad [A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t). \tag{5.3.1}$$

Let $\langle -, - \rangle$ be the Killing form, renormalised by a factor of $1/2h^\vee$, where $h^\vee$ is the dual coxeter number of $\mathfrak{g}$. The affine Kac-Moody algebra $\hat{\mathfrak{g}}$ is a central extension

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}} \to \mathcal{L}\mathfrak{g} \to 0. \tag{5.3.2}$$

As a vector space $\hat{\mathfrak{g}} \simeq \mathcal{L}\mathfrak{g} \oplus \mathbb{C}K$ with

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\mathrm{Res}_{t=0} f dg)(A, B)K, \tag{5.3.3}$$

where $\mathrm{Res}_{t=0}$ corresponds to taking the coefficient of t^{-1}. Let $J^a_n = J^a \otimes t^n$ be a basis for $\hat{\mathfrak{g}}$, then we have the commutation relations

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n\left\langle J^a, J^b \right\rangle \delta_{n,-m} K. \tag{5.3.4}$$

This becomes a vertex operator algebra by assigning the vertex operators as follows.

$$Y(J^a, z) = J^a(z) = \sum_{n \in \mathbb{Z}} J^a_n z^{-n-1}, \qquad Y(\omega, z) = \frac{1}{2(k + h^\vee)} \sum_a : J^a(z) J^a(z) :, \tag{5.3.5}$$

where $h^\vee$ is the dual coxeter number of $\mathfrak{g}$.

When $\mathfrak{g}$ is abelian, this is called the free boson or Heisenberg Lie algebra, which we will explore more in Chapter 8.

— Chapter 6 —

Huang-Lepowsky-Zhang Tensor Categories

"You're right. No human being would stack books like this."
— Peter Venkman (Bill Murray), *Ghostbusters*

As with associative unital algebras, vertex algebras possess notions of actions on a vector space, producing a module. These modules form categories which can be equipped with a tensor product and further structure. The complete reference for tensor structures arising from vertex operator algebras and intertwining operators is provided in the series of papers [20] by Huang, Lepowsky and Zhang. Due to the series admirably operating in great generality, while also providing many technical details, it can be perceived as intimidatingly long. There are therefore a number of articles in the literature, which include helpful reviews highlighting different aspects of the series relevant to different types of vertex operator algebras and choices of module category [21–23]. This overview is based on [10], with an emphasis on the results necessary for later sections.

6.1 Definition

There will be three types of grading appearing below, whose relative importance might not be immediately clear for readers unfamiliar with the theory. There is the conformal grading by generalised eigenvalues of the Virasoro L_0 operator and an additional grading by two abelian groups $A \leq B$, with A grading the vertex operator algebra and B its modules. The latter two gradings have an analogy in the setting of a simple finite dimensional Lie algebra, where A is the root lattice (which grades the Lie algebra) and B is the dual of the Cartan subalgebra (which grades general weight modules).

Definition 6.1.1. Let V be a vertex operator algebra and B an abelian group with subgroup A.

- The vertex operator algebra V is said to be A-graded, if V admits a decomposition into homogenous spaces $V^{(\gamma)}$, $\gamma \in A$ such that
  1. $V = \bigoplus_{\gamma\in A} V^{(\gamma)}$.
  2. For any $\alpha, \beta \in A$ and any $v \in V^{(\alpha)}$
  $$Y(v,z)V^{(\beta)} \subset V^{(\alpha+\beta)}[\![z,z^{-1}]\!]. \tag{6.1.1}$$
- A **weak V-module** is a vector space M together with a field map
  $$\begin{aligned} Y_M : V &\to (\mathrm{End} M)[\![z,z^{-1}]\!] \\ v &\mapsto Y_M(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1} \end{aligned}$$
  satisfying
  1. **Lower truncation**: For all $v \in V$ and $m \in M$, $v_n m = 0$ for sufficiently large $n \in \mathbb{Z}$.
  2. **Vacuum property**: $Y_M(\mathbf{1}, z) = \mathrm{id}_M$, where $\mathbf{1} \in V$ is the vacuum vector.
  3. **$\mathbf{L_{-1}}$ derivation**: For any $v \in V$
  $$Y_M(L_{-1}v,z) = \frac{\mathrm{d}}{\mathrm{d}z} Y_M(v,z). \tag{6.1.2}$$
  4. **Jacobi identity**: For any $u, v \in V$,
  $$\begin{aligned} z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right) Y_M(u,z_1)Y_M(v,z_2) =& z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right) Y_M(v,z_2)Y_M(u,z_1) \\ &+ z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right) Y_M(Y(u,z_0)v,z_2), \end{aligned} \tag{6.1.3}$$

  If in addition there is a B-grading on the weak module $M = \bigoplus_{\beta\in B} M^{(\beta)}$, then M is a **B-graded weak V-module**, if the following condition is satisfied.

  5. **Grading compatibility**: For all $\alpha \in A$, $v \in V^{(\alpha)}$, $\beta \in B$,
  $$Y_M(v,z)M^{(\beta)} \subset M^{(\alpha+\beta)}[\![z,z^{-1}]\!]. \tag{6.1.4}$$
- A **B-graded generalised V-module** M is a B-graded weak V-module that is graded by generalised L_0 eigenvalues, that is, $M = \bigoplus_{h\in\mathbb{C},\beta\in B} M_h^{(\beta)}$, where
  $$M_h^{(\beta)} = \left\{ m \in M^{(\beta)} \ : \ \exists n \in \mathbb{N},\ (L_0 - h)^n m = 0 \right\}. \tag{6.1.5}$$

  The elements of $M_h^{(\beta)}$ are called **doubly homogeneous vectors**. Note that B -graded generalised V-modules together with module homomorphisms form an abelian category.

- A B-graded generalised V-module M is called **lower bounded** if for each $\beta \in B$, $M_h^{(\beta)} = 0$ for $\operatorname{Re} h$ sufficiently negative.

- A **strongly B-graded generalised V-module M** is a B-graded generalised V-module whose simultaneous homogeneous spaces $M_h^{(\beta)}$ are all finite dimensional and for fixed $h \in \mathbb{C}$ and $\beta \in B$, $M_{h+k}^{(\beta)} = 0$ for sufficiently negative $k \in \mathbb{Z}$. Such a module is called **discretely strongly graded** if all non-zero homogeneous spaces have real conformal weight and for any $h \in \mathbb{R}$, $\beta \in B$, the space $\bigoplus_{\tilde{h} \in \mathbb{R}, \tilde{h} \le h} M_h^{(\beta)}$ is finite dimensional.

- A strongly B-graded generalised V-module M is called **graded C_1-cofinite** if for any $\beta \in B$ the space

$$C_1(M)^{(\beta)} = \operatorname{span}_{\mathbb{C}}\Big\{v_{-1}m \in M^{(\beta)} \ : \ v \in V_h, h > 0, m \in M\Big\} \tag{6.1.6}$$

has finite codimension in $M^{(\beta)}$.

Remark. We abbreviate B-graded generalised V-module as B-graded V-module, or when the abelian group B is obvious from context as V-module. For the specific vertex operator algebras to be considered in later chapters, we will mainly be interested in discretely strongly graded modules which are also graded C_1-cofinite with respect to a suitable choice of vertex operator subalgebra.

Proposition 6.1.2 (Huang-Lepowsky-Zhang [20, Part I, Theorem 2.34])**.** *Let $A \le B$ be abelian groups, V an A-graded vertex operator algebra, let M be a B-graded weak V-module and define the vector spaces*

$$M' = \bigoplus_{b \in B, h \in \mathbb{C}} \left(M_h^{(\beta)}\right)^*, \qquad \left(M_h^{(\beta)}\right)^* = \operatorname{Hom}_{\mathbb{C}}\left(M_h^{(\beta)}, \mathbb{C}\right). \tag{6.1.7}$$

If M is strongly B-graded, then the canonical linear isomorphisms, identifying a finite dimensional vector space with its double dual, extend to a canonical linear isomorphism $M \cong M''$ of bigraded vector spaces. If, in addition, M is discretely strongly B-graded, then M' is also a discretely strongly B-graded with field map $Y_{M'}$ uniquely characterised by

$$\langle Y_{M'}(v,z)\phi, m\rangle = \left\langle \phi, Y_M^{\mathrm{opp}}(v,z)m\right\rangle, \qquad v \in V,\ \phi \in M', m \in M, \tag{6.1.8}$$

where Y_M^{opp} is the ***opposed field map***

$$Y_M^{\mathrm{opp}}(v,z) = Y_M\left(\mathrm{e}^{zL_1}\left(-z^{-2}\right)^{L_0} v, z^{-1}\right). \tag{6.1.9}$$

The module M' is called the ***contragredient*** *of M. Opposing the field map is involutive, that is, $Y_M^{\text{opp opp}} = Y_M$, hence the canonical linear isomorphism $M \cong M''$ of bigraded vector spaces is an isomorphism of V-modules.*

Note that by (6.1.9) the opposed field map depends on the conformal (or at least the Möbius) structure on the vertex operator algebra, that is, the actions of the Virasoro L_0 and L_1 operators enter explicitly. Note further that the opposed field map can be used to define an action of V on M' (or even the full vector space dual M^*) for any weak module M. However, in general the lower truncation axiom need not hold and thus the terms in the Jacobi identity need not converge. There are numerous boundedness conditions on conformal weights which are sufficient for module structures on M'. We shall only consider discrete strong gradation, as this is also a sufficient condition for tensor product structures in the module categories we will study.

Definition 6.1.3. Let $A \leq B$ be abelian groups, V an A-graded vertex operator algebra and let M_1, M_2, M_3 be B-graded weak V-modules. Denote by $M_3\{z\}[\log z]$ the space of formal power series in z and $\log z$ with coefficients in M_3, where the exponents of z can be arbitrary complex numbers and with only finitely many $\log z$ terms for any fixed exponent of z. A **logarithmic intertwining operator of type $\binom{M_3}{M_1,M_2}$** is a linear map

$$\begin{aligned} \mathcal{Y} : M_1 \otimes M_2 &\to M_3\{z\}[\log z], \\ m_1 \otimes m_2 &\mapsto \mathcal{Y}(m_1,z)m_2 = \sum_{\substack{s\geq 0\\ t\in\mathbb{C}}} (m_1)_{t,s} m_2 z^{-t-1} (\log z)^s \end{aligned} \tag{6.1.10}$$

where $(m_1)_{t,s} \in \mathrm{Hom}_{\mathbb{C}}(M_2, M_3)$, satisfying the following properties.

1. **Lower truncation**: For any $m_i \in M_i$, $i = 1,2$, and $s \geq 0$

$$(m_1)_{t+k,s} m_2 = 0 \tag{6.1.11}$$

   for sufficiently large $k \in \mathbb{Z}$.

2. **$\mathbf{L_{-1}}$ derivation**: For any $m_i \in M_i$, $i = 1,2$,

$$\mathcal{Y}(L_{-1}m_1, z)m_2 = \frac{\mathrm{d}}{\mathrm{d}z}\mathcal{Y}(m_1,z)m_2. \tag{6.1.12}$$

3. **Jacobi identity**: For any $v \in V$, $m_i \in M_i$, $i = 1,2$,

$$z_0^{-1}\delta\left(\frac{z_1 - z_2}{z_0}\right) Y_{M_3}(v,z_1)\mathcal{Y}(m_1,z_2)m_2 = z_0^{-1}\delta\left(\frac{-z_2 + z_1}{z_0}\right)\mathcal{Y}(m_1,z_2)Y_{M_2}(v,z_1)m_2$$

$$+z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}\big(Y_{M_1}(v,z_0)m_1,z_2\big)m_2. \tag{6.1.13}$$

The intertwining operator $\mathcal{Y}$ is called **grading compatible** if addition to the conditions above it also satisfies the following condition.

4. **Grading compatibility**: For any $\beta_1,\beta_2 \in B$, $m_1 \in M_1^{(\beta_1)}$

$$\mathcal{Y}(m_1,z)M_2^{(\beta_2)} \subset M_3^{(\beta_1+\beta_2)}\{z\}[\log z]. \tag{6.1.14}$$

The conditions above are all linear and so we denote by

$$\mathrm{I}\binom{M_3}{M_1,M_2}, \qquad \mathrm{Gr}\binom{M_3}{M_1,M_2}, \tag{6.1.15}$$

respectively, the vector space of all logarithmic intertwining operators of type $\binom{M_3}{M_1,M_2}$ and the subspace of all grading compatible ones.

Note that if, as will be the case in Section 8.3, the B-grading corresponds to eigenvalues of zero modes of certain vectors in V of conformal weight 1, then the Jacobi identity implies that all logarithmic intertwining operators are grading compatible. Intertwining operators admit a dualisation analogous to the opposed field map in Proposition 6.1.2 which will prove crucial to showing the existence of Grothendieck-Verdier structures on vertex operator algebra module categories.

Theorem 6.1.4 (Huang-Lepowsky-Zhang [20, Part II Proposition 3.46])**.** *Let M_1,M_2,M_3 be strongly graded generalised modules over some vertex operator algebra V. Then there exists a natural linear isomorphism $A : \mathrm{I}\binom{M_3}{M_1,M_2} \to \mathrm{I}\binom{M_2'}{M_1,M_3'}$, which on any intertwining operator $\mathcal{Y} \in \mathrm{I}\binom{M_3}{M_1,M_2}$ evaluates as*

$$\big\langle A(\mathcal{Y})(m_1,x)m_3',m_2\big\rangle_{M_2} = \left\langle m_3', \mathcal{Y}\left(\mathrm{e}^{xL_1}\mathrm{e}^{\mathrm{i}\pi L_0}\left(x^{-L_0}\right)^2 m_1, x^{-1}\right)m_2\right\rangle_{M_3}, \tag{6.1.16}$$

for $m_1 \in M_1$, $m_2 \in M_2$, $m_3' \in M_3'$, where the subscript indicates which module the pairings are evaluated in. The isomorphism A preserves grading and hence restricts to a natural isomorphism $\mathrm{Gr}\binom{M_3}{M_1,M_2} \to \mathrm{Gr}\binom{M_2'}{M_1,M_3'}$.

The Jacobi identity for intertwining operators implies that intertwining operators are essentially maps from a pair M_1,M_2 of modules to a third module M_3, which are bilinear in the action of the vertex operator algebra V. It therefore makes sense to ask if there exists some universal tensor product module through which all intertwining operators factor. That is, given some choice of V module category $\mathscr{C}$ and two modules $M_1, M_2 \in \mathscr{C}$, does there exist a module $M_1 \boxtimes M_2 \in \mathscr{C}$ together with an intertwining operator $\mathcal{Y}_{M_1,M_2} \in V\binom{M_1 \boxtimes M_2}{M_1, M_2}$ such that for any $R \in \mathscr{C}$ and intertwining operator $\mathcal{Y} \in V\binom{R}{M_1, M_2}$ there exists a unique module map $f \in \mathrm{Hom}_{\mathscr{C}}(M_1 \boxtimes M_2, R)$ such that $\mathcal{Y} = f \circ \mathcal{Y}_{M_1,M_2}$? That is, such that the diagram

$$\begin{array}{ccc} M_1 \otimes M_2 & \xrightarrow{\mathcal{Y}(M_1,M_2)} & M_1 \boxtimes M_2\{z\}[\log z] \\ & \searrow^{\mathcal{Y}} & \Big\downarrow {\scriptstyle \exists! f} \\ & & R\{z\}[\log z] \end{array} \tag{6.1.17}$$

commutes. Assuming that $M_1 \boxtimes M_2 \in \mathscr{C}$ exists for all pairs of modules in $\mathscr{C}$, $- \boxtimes -$ becomes a bifunctor after defining the following evaluation on pairs of morphisms. For $M_1, N_1, M_2, N_2 \in \mathscr{C}$ and morphisms $f_1 : M_1 \to N_1$, $f_2 : M_2 \to N_2$, the tensor product morphism $f_1 \boxtimes f_2 : M_1 \boxtimes M_2 \to N_1 \boxtimes N_2$ is the unique morphism, characterised by the universal property (6.1.17), such that $\mathcal{Y}(N_1, N_2) \circ (f_1 \otimes f_2) = (f_1 \boxtimes f_2) \circ \mathcal{Y}(M_1, M_2)$. This characterisation of tensor products (also called fusion products) of vertex operator algebra modules, via a universal property, is conceptually very clear and allows us to construct maps out of a tensor product in terms of vertex operators. However, it does not provide an actual construction, nor does it guarantee existence. We sketch some of the ideas of the construction of $M_1 \boxtimes M_2$ here but refer to the original source [20, Part IV] and the review [22] for details. The fusion product of two modules M_1, M_2 can be constructed inside $(M_1 \otimes M_2)^*$, the full vector space dual of the complex tensor product. While $(M_1 \otimes M_2)^*$ is not a V-module, it is possible to move the action of fields in V from either of the tensor factors M_1, M_2 to $(M_1 \otimes M_2)^*$ using a generalisation of the opposed field map. This leads to the consideration of the subspace $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2) \subset (M_1 \otimes M_2)^*$ consisting of all linear functionals on which the evaluation of a field has only finitely many singular terms and on which the transported action of fields from either tensor factor agrees. That is, it consists of all linear functionals compatible with the lower truncation property of Definition 6.1.1 and the vertex operator algebra version of bilinearity. The name COMP refers to the compatibility of the actions V on the two tensor factors transported to $(M_1 \otimes M_2)^*$. It was shown in [20, Part IV, Theorem 5.48] that $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ is a weak V module and should morally be thought of as the contragredient of the fusion product $M_1 \boxtimes M_2$. The subspace $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ is however usually too large to be contained in the category $\mathscr{C}$ one is considering. For example, it generally contains vectors which are not finite sums of homogeneous vectors. Under suitable conditions on $\mathscr{C}$ (such as those in Theorem 6.3.1) one can construct the subspace $M_1 \boxslash M_2 \subset \mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ consisting of the sum of all images of module maps from objects in $\mathscr{C}$ into $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$. The contragredient $(M_1 \boxslash M_2)'$ is then the fusion product module satisfying the universal property (6.1.17). For more explanation of COMP($\mathcal{M}_1, \mathcal{M}_2$) and the construction of $(M_1 \boxtimes M_2)$, see Section 9.7 in which we use the explicit details to prove that a module category of the bosonic ghost vertex algebra is rigid.

Proposition 6.1.5 (Huang-Lepowsky-Zhang [20, Part VIII, Section 12]). *Let $A \le B$ be abelian groups, let V be an A-graded vertex operator algebra and $\mathscr{C}$ a choice of category of V-modules (that is a subcategory of the category of all B-graded V-modules) containing V as an object such that the following conditions hold.*

1. *For any $M_1, M_2 \in \mathscr{C}$ there exist $M_1 \boxtimes M_2 \in \mathscr{C}$ and $\mathcal{Y}_{M_1,M_2} \in \mathrm{Gr}\binom{M_1 \boxtimes M_2}{M_1,M_2}$ such that the universal property (6.1.17) holds.*
2. *For any $M_1, M_2, M_3 \in \mathscr{C}$, there is a family of isomorphisms $A^{x_1,x_2}_{M_1,M_2,M_3} : (M_1 \boxtimes M_2) \boxtimes M_3 \to M_1 \boxtimes (M_2 \boxtimes M_3)$ depending on complex variables x_1, x_2 that is functorial in M_1, M_2, M_3. Further, for $m_i \in M_i$, $x_1, x_2 \in \mathbb{C}$, $|x_1| > |x_2| > |x_1 - x_2| > 0$, the expressions*
$$\mathcal{Y}_{M_1,M_2\boxtimes M_3}(m_1,x_1)\mathcal{Y}_{M_2,M_3}(m_2,x_2)m_3, \qquad \mathcal{Y}_{M_1\boxtimes M_2,M_3}(\mathcal{Y}_{M_1,M_2}(m_1,x_1-x_2)m_2,x_2)m_3, \tag{6.1.18}$$
converge absolutely for any choice of branch of logarithm for x_1, x_2 in the algebraic completions of $M_1 \boxtimes (M_2 \boxtimes M_3)$ and $(M_1 \boxtimes M_2) \boxtimes M_3$, respectively. Finally,
$$\begin{aligned}\overline{A}^{x_1,x_2}_{M_1,M_2,M_3}&\big(\mathcal{Y}_{M_1,M_2\boxtimes M_3}(m_1,x_1)\mathcal{Y}_{M_2,M_3}(m_2,x_2)m_3\big)\\ &= \mathcal{Y}_{M_1\boxtimes M_2,M_3}(\mathcal{Y}_{M_1,M_2}(m_1,x_1-x_2)m_2,x_2)m_3,\end{aligned} \tag{6.1.19}$$
where $\overline{A}^{x_1,x_2}_{M_1,M_2,M_3}$ is the natural extension of $A^{x_1,x_2}_{M_1,M_2,M_3}$ to algebraic completions.

Then $\mathscr{C}$ is a braided monoidal category with the vertex operator algebra V as the monoidal unit, whose structure isomorphisms are uniquely characterised by the following.

- *For $M \in \mathscr{C}$ the unit morphisms are uniquely characterised by*
$$\begin{aligned}\ell_M\big(\mathcal{Y}_{V,M}(v,z)m\big) &= Y_M(v,z)m\\ r_M\big(\mathcal{Y}_{M,V}(m,z)v\big) &= \mathrm{e}^{zL_{-1}}Y_M(v,-z)m,\end{aligned} \tag{6.1.20}$$
where Y_M is the field map of V acting on the module M, $v \in V$ and $m \in M$.
- *For $M_1, M_2 \in \mathscr{C}$ the braiding isomorphism $c_{M_1,M_2} : M_1 \boxtimes M_2 \to M_2 \boxtimes M_1$ is uniquely characterised by*
$$c_{M_1,M_2}\big(\mathcal{Y}_{M_1,M_2}(m_1,z)m_2\big) = \mathrm{e}^{zL_{-1}}\mathcal{Y}_{M_2,M_1}(m_2,\mathrm{e}^{\mathrm{i}\pi}z)m_1, \tag{6.1.21}$$
where $m_1 \in M_1$ and $m_2 \in M_2$.

- *There is a twist morphism* $\theta_{M_1} = \mathrm{e}^{2\pi \mathrm{i} L_0}|_{M_1}$*,* $M_1 \in \mathscr{C}$*, which satisfies* $\theta_V = \mathrm{id}_V$ *and for any* $M_2 \in \mathscr{C}$ *also satisfies the balancing equation*

$$\theta_{M_1 \boxtimes M_2} = c_{M_2,M_1} \circ c_{M_1,M_2} \circ \left(\theta_{M_1} \boxtimes \theta_{M_2}\right). \tag{6.1.22}$$

- *For* $i = 1,2,3$*,* $M_i \in \mathscr{C}$*,* $m_i \in M_i$*,* $x_1, x_2 \in \mathbb{R}$*,* $x_1 > x_2 > x_1 - x_2 > 0$*, the associativity isomorphism* $A_{M_1,M_2,M_3} : M_1 \boxtimes (M_2 \boxtimes M_3) \to (M_1 \boxtimes M_2) \boxtimes M_3$ *is given by the isomorphism of Part 2 above, where the branches of any logarithms are chosen such that the arguments for* $x_1, x_2, x_1 - x_2$ *are all* 0*.*

6.2 Grothendieck-Verdier structure

With the braided monoidal properties described in Proposition 6.1.5 and the sufficient conditions of Theorem 6.3.1 in hand, we can now connect these structures with the Grothendieck-Verdier structures introduced in Chapter 3.

Theorem 6.2.1. *Let* V *be a vertex operator algebra and* $\mathscr{C}$ *a choice of category of* V*-modules which contains* V *as an object, which is closed under taking contragredients and which satisfies the two conditions (and hence also the conclusions) of Proposition 6.1.5. Then* $\mathscr{C}$ *is a ribbon Grothendieck-Verdier category, with dualising object* V' *(the contragredient of the vertex operator algebra as a module over itself), with dualising functor given by the taking of contragredients, and with twist* $\theta = \mathrm{e}^{2\pi \mathrm{i} L_0}$*.*

Proof. Let $X, Y, Z \in \mathscr{C}$ and recall the linear isomorphism $A : \mathrm{I}\binom{Z}{X,Y} \to \mathrm{I}\binom{Y'}{X,Z'}$ of Theorem 6.1.4. When working with a strong grading consider instead the restriction $A : \mathrm{Gr}\binom{Z}{X,Y} \to \mathrm{Gr}\binom{Y'}{X,Z'}$. Since category $\mathscr{C}$ is closed under contragredients we therefore also have a natural isomorphism $\mathrm{Hom}(X \boxtimes Y, Z) \xrightarrow{\cong} \mathrm{Hom}(X \boxtimes Z', Y')$. Setting $Z = V'$, we have

$$\mathrm{Hom}(X \boxtimes Y, V') \xrightarrow{\cong} \mathrm{Hom}(X \boxtimes V'', Y') \xrightarrow{\cong} \mathrm{Hom}(X \boxtimes V, Y') \xrightarrow{\cong} \mathrm{Hom}(X, Y') \tag{6.2.1}$$

where we have made use of the canonical isomorphism $V'' \simeq V$ of Proposition 6.1.2 and the left unit isomorphism $V \boxtimes Y \simeq Y$. This proves that $\mathscr{C}$ is a braided Grothendieck-Verdier category with dualising object V'. Next we show that the twist $\theta_M = \mathrm{e}^{2\pi \mathrm{i} L_0}|_M$ and the contragredient functor satisfy the compatibility condition (3.1.4). From formula (6.1.9) for the opposed field map one sees that $L_0^{\mathrm{opp}} = L_0$ and hence for any module $M \in \mathscr{C}$, $(\theta_M)' = \theta_{M'}$. Thus $\mathscr{C}$ is ribbon Grothendieck-Verdier. ∎

Remark. Note that in Theorem 6.2.1 we do not require V' and V to be isomorphic as V-modules. Indeed, V' plays the important structural role of a dualising object.

6.3 Sufficient conditions

Proving that a choice of vertex operator algebra module category admits the braided tensor structure of Proposition 6.1.5 is a highly non-trivial task. Fortunately a number of sufficient conditions were identified in [20], which we quote and summarise in the following theorem.

Theorem 6.3.1 (Huang-Lepowsky-Zhang)**.** *Let $A \le B$ be abelian groups and let V be an A-graded vertex operator algebra. Then the following conditions on a choice of B-graded module category $\mathscr{C}$ are sufficient for $\mathscr{C}$ to have the braided monoidal structure induced from intertwining operators described in Proposition 6.1.5.*

1. *The vertex operator algebra V is an object in $\mathscr{C}$ and all objects of $\mathscr{C}$ are strongly B-graded. For any $M_1, M_2 \in \mathscr{C}$ the logarithmic intertwining operator $\mathcal{Y}_{M_1,M_2}$ satisfying the universal property in the definition of the tensor product of M_1 and M_2 is grading compatible (hence all logarithmic intertwining operators are grading compatible) [20, Part III, Assumption 4.1].*

2. *$\mathscr{C}$ is a full subcategory of the category of strongly B-graded modules and is closed under the contragredient functor and under taking finite direct sums [20, Part IV, Assumption 5.30].*

3. *For any object in $\mathscr{C}$ all conformal weights are real and the non-semisimple part of L_0 acts nilpotently, that is, there is a uniform bound on the size of Jordan blocks for any given module though there need not be global bound for the entire category [20, Part V, Assumption 7.11].*

4. *$\mathscr{C}$ is closed under images of module homomorphisms [20, Part VI, Assumption 10.1.7].*

5. *The convergence and extension properties for either products or iterates holds [20, Part VII, Theorem 11.4].*

6. *For any objects $M_1, M_2 \in \mathscr{C}$, let M_v be the V-module generated by a B-homogeneous generalised L_0 eigenvector $v \in \mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$. If M_v is lower bounded then M_v is strongly graded and an object in $\mathscr{C}$ [24, Theorem 3.1].*

Remark. The above sufficient conditions are in a sense the weakest known conditions for a vertex operator algebra module category admitting a braided monoidal structure. However, they can in practice be difficult to verify (especially Conditions 5 and 6). Other more restrictive and hence more tractable sets of conditions are therefore also commonly considered in the literature. The most famous set arguably being:

- The vertex operator algebra V is C_2-cofinite, all L_0 eigenspaces are finite dimensional, the only non-zero eigenspaces have non-negative integral L_0 eigenvalue, $\dim V_0 = 1$ and $V \cong V'$.
- The category $\mathscr{C}$ of all admissible (also known as $\mathbb{N}$ gradable) modules is semisimple.

If the above conditions hold, then $\mathscr{C}$ is a modular tensor category and the much celebrated Verlinde formula holds [25]. A weaker set of sufficient conditions only requires the vertex operator algebra to be C_2-cofinite and the category to be the category of admissible modules without any assumptions on semisimplicity. Comparatively few general results are known when the C_2-cofiniteness condition is not satisfied, however, a recent flurry of new insights appears to be changing this at last [23, 26–29].

The convergence and extension property of Theorem 6.3.1.5 is a technical condition on the analytic properties of intertwining operators, whose details we shall not state here. Instead we give sufficient conditions for the convergence and extension property to hold.

Theorem 6.3.2 (Allen-Wood [29, Theorem 5.7])**.** *Let $A \le B$ be abelian groups, let V be an A-graded vertex operator algebra and let $\overline{V}$ be a vertex subalgebra of $V^{(0)}$. Further, let M_i, $i = 0, 1, 2, 3, 4$ be B-graded V-modules. Finally let $\mathcal{Y}_1$, $\mathcal{Y}_2$, $\mathcal{Y}_3$ and $\mathcal{Y}_4$ be logarithmic grading compatible intertwining operators of types $\binom{M_0}{M_1,\ M_4}$, $\binom{M_4}{M_2,\ M_3}$, $\binom{M_0}{M_4,\ M_3}$ and $\binom{M_4}{M_1,\ M_2}$ respectively. If the modules M_i, $i = 0, 1, 2, 3$ (note $i = 4$ is excluded) are discretely strongly graded, and graded C_1-cofinite as $\overline{V}$-modules, then $\mathcal{Y}_1$, $\mathcal{Y}_2$ satisfy the convergence and extension property for products and $\mathcal{Y}_3$, $\mathcal{Y}_4$ satisfy the convergence and extension property for iterates.*

Proof. The above theorem follows from the proof of [30, Theorem 7.2], however, in [30] some assumptions are made on the category of strongly graded modules (see [30, Assumption 7.1, Part 3]) which do not hold in general. Fortunately, the proof of Theorem 6.3.2 does not depend at all on any categorical considerations or even on the details of the intertwining operators $\mathcal{Y}_i$ beyond their types. It merely depends on certain finiteness properties of the modules $\mathcal{W}_i$. We reproduce the proof of Yang in Appendix D, with some minor tweaks to the arguments, to show that the conclusion of Theorem 6.3.2 holds, without making any assumptions on the category of all strongly graded modules. ■

Choosing the module category to be abelian and combining Theorems 6.3.1 and 6.3.2 we obtain the following simplified sufficient conditions.

Corollary 6.3.3. *Let $A \leq B$ be abelian groups, let V be an A-graded vertex operator algebra and let $\mathscr{C}$ be an abelian full subcategory of the category of all B-graded V-modules, which contains V. Then the following conditions are sufficient for $\mathscr{C}$ to have the ribbon Grothendieck-Verdier category structures induced from intertwining operators described in Proposition 6.1.5 and Theorem 6.2.1.*

1. *All objects in $\mathscr{C}$ are discretely strongly graded and $\mathscr{C}$ is closed under taking contragredients.*
2. *The non-semisimple part of L_0 acts nilpotently on any object in $\mathscr{C}$.*
3. *There exists a vertex operator subalgebra $\overline{V} \subset V^{(0)}$ such that all objects in $\mathscr{C}$ are graded C_1-cofinite as $\overline{V}$ modules.*
4. *For any objects $M_1, M_2 \in \mathscr{C}$, every lower bounded submodule of* $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ *that is finitely generated by doubly homogeneous vectors is an object in $\mathscr{C}$.*

We have ordered the conditions in Corollary 6.3.3 by how difficult they are to verify in practice. Note in particular that Conditions 1 – 3 are merely properties of the types of modules one wishes to consider and make no reference to tensor products.

— Chapter 7 —

Functors Involving Vertex Operator Algebra Module Categories

"Don't cross the streams."
— Dr. Egon Spengler (Harold Ramis), *Ghostbusters*

The monoidal structures of vertex operator algebra module categories are a consequence of the properties of intertwining operators. This chapter presents results from [10] on how monoidal functors, from linear braided monoidal categories to vertex operator algebra module categories, interact with intertwining operators, conditions for when the functor is an isomorphism and when it is ribbon Grothendieck-Verdier are also presented. Finally we prove a proposition which describes conditions under which one can extend the functor from projectives to the full category.

7.1 Functors and equivalences

Lemma 7.1.1. *Let V be a vertex algebra with choice of module category $(\mathscr{C}, \boxtimes, l, r, \alpha, c)$, admitting the braided monoidal structure induced from intertwining operators described in Proposition 6.1.5. Let $(\mathscr{D}, \otimes, l, r, \alpha, c)$ be a linear braided monoidal category, $G : \mathscr{D} \to \mathscr{C}$ a $\mathbb{C}$-linear abelian functor and φ_0 a choice of morphism $\varphi_0 : V \to G(1_{\mathscr{D}})$. Then the following are equivalent.*

1. *There exists a natural transformation $\varphi_2 : G(-) \boxtimes G(-) \to G(- \otimes -)$ such that $(G, \varphi_0, \varphi_2)$ is a braided lax monoidal functor (lax here means that φ_0 and φ_2 are not required to be isomorphisms).*

2. *There exists a family of linear maps*

$$G^T : \mathrm{Hom}_{\mathscr{D}}(\mathcal{M}\otimes\mathcal{N},\mathcal{P}) \to \mathrm{I}\binom{G(\mathcal{P})}{G(\mathcal{M}),G(\mathcal{N})},$$
$$f \mapsto G^T_f(z), \tag{7.1.1}$$

for all $\mathcal{M},\mathcal{N},\mathcal{P}\in\mathscr{D}$*, satisfying the following conditions.*

- *Functoriality: For any* $\mathcal{M},\mathcal{M}',\mathcal{N},\mathcal{N}',\mathcal{P},\mathcal{P}'\in\mathscr{D}$ *and any* $g:\mathcal{M}'\to\mathcal{M}$*,* $h:\mathcal{N}'\to\mathcal{N}$*,* $k:\mathcal{P}\to\mathcal{P}'$*, we have*

$$G^T_{k\circ f\circ(g\otimes h)}(z) = G(k)\circ G^T_f(z)\circ(G(g)\otimes_{\mathbb{C}} G(h)), \tag{7.1.2}$$

where $\otimes_{\mathbb{C}}$ *denotes the tensor product of complex vector spaces and linear maps.*

- *Unitality: For any* $\mathcal{N}\in\mathscr{D}$*,*

$$G^T_{l_{\mathcal{N}}}(\varphi_0(v),z)n = Y_{G(\mathcal{N})}(v,z)n, \tag{7.1.3}$$

where $Y_{G(\mathcal{N})}$ *is the vertex operator algebra field map on the module* $G(\mathcal{N})$*.*

- *Skew symmetry: For any* $\mathcal{M}$*,* $\mathcal{N}\in\mathscr{D}$ *and* $m\in G(\mathcal{M})$*,* $n\in G(\mathcal{N})$*,*

$$G^T_{c_{\mathcal{N},\mathcal{M}}}(n,z)m = \mathrm{e}^{zL_{-1}}G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}\left(m,\mathrm{e}^{\mathrm{i}\pi}z\right)n. \tag{7.1.4}$$

- *Associativity: For any* $\mathcal{M}$*,* $\mathcal{N}$*,* $\mathcal{P}\in\mathscr{D}$*,* $m\in G(\mathcal{M})$*,* $n\in G(\mathcal{N})$*,* $p\in G(\mathcal{P})$ *and* $x_1,x_2\in\mathbb{C}$ *such that* $|x_1|>|x_2|>0$ *and* $|x_2|>|x_1-x_2|>0$*,*

$$G^T_{\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}}(m,x_1)G^T_{\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}}}(n,x_2)p = G^T_{\mathrm{id}_{(\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P}}}\left(G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(m,x_1-x_2)n,x_2\right)p, \tag{7.1.5}$$

where both sides of the equality are to be seen as elements of the algebraic completion of $G((\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})$ *and the associativity map* $\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}$ *is to be seen as an element of* $\mathrm{Hom}(\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P}),(\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})$ *so that* $G^T_{\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}}(z)$ *is an intertwiner of type* $\binom{G((\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})}{G(\mathcal{M}),G(\mathcal{N}\otimes\mathcal{P})}$*.*

The linear maps G^T *and the natural transformation* φ_2 *uniquely characterise each other through the equality* $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z) = \varphi_2(\mathcal{M},\mathcal{N})\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(-,z)$*, where* $\mathcal{M},\mathcal{N}\in\mathscr{D}$ *and where* $\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}$ *is the intertwining operator of the universal property* (6.1.17) *characterising* $G(\mathcal{M})\boxtimes G(\mathcal{N})$*.*

Remark. By the functoriality condition above, the linear maps G^T are completely determined by their values on $\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}} \in \mathrm{Hom}_{\mathscr{D}}(\mathcal{M}\otimes\mathcal{N},\mathcal{M}\otimes\mathcal{N})$. If $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}} \in \mathrm{Gr}\binom{G(\mathcal{M}\otimes\mathcal{N})}{G(\mathcal{M}),G(\mathcal{N})}$ for all $\mathcal{M},\mathcal{N} \in \mathscr{D}$, then all G^T_f will be graded intertwining operators, since all morphisms in $\mathscr{C}$ preserve the grading. Further, for each of the equations (7.1.3), (7.1.4), (7.1.5), the left-hand sides and right-hand sides are respectively in the same space of intertwining operators. If these spaces of intertwining operators are finite dimensional, then it is sufficient to verify the equation for only a finite number of coefficients. In particular, if the intertwining operator space is one dimensional then it is sufficient to compare the leading coefficients.

Proof. Note that if the family of maps G^T in (7.1.1) exists, then since $\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}} \in \mathrm{End}_{\mathscr{D}}(\mathcal{M}\otimes\mathcal{N})$, it follows that $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}$ is an intertwining operator of type $\binom{G(\mathcal{M}\otimes\mathcal{N})}{G(\mathcal{M}),G(\mathcal{N})}$. Let $\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}$ be the universal intertwining operator of type $\binom{G(\mathcal{M})\boxtimes G(\mathcal{N})}{G(\mathcal{M}),G(\mathcal{N})}$ coming from universal property (6.1.17) characterising $G(\mathcal{M})\boxtimes G(\mathcal{N})$. This universal property further implies the existence and uniqueness of a family of morphisms $\varphi_2(\mathcal{M},\mathcal{N}) \in \mathrm{Hom}_{\mathscr{C}}(G(\mathcal{M})\boxtimes G(\mathcal{N}), G(\mathcal{M}\otimes\mathcal{N}))$ satisfying $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z) = \varphi_2(\mathcal{M},\mathcal{N})\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(-,z)$. Conversely given a family of morphisms $\varphi_2(\mathcal{M},\mathcal{N}) : G(\mathcal{M})\boxtimes G(\mathcal{N}) \to G(\mathcal{M}\otimes\mathcal{N})$, we can define $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}$ via $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z) = \varphi_2(\mathcal{M},\mathcal{N})\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}$.

We show the logical equivalence of Assertions 1 and 2 by respectively showing the equivalence of naturality of φ_2 and functoriality of G^T; the left unit square constraint for φ_2 commuting and the unitality of G^T; the braiding square constraint for φ_2 commuting and the skew symmetry of G^T; and the associativity hexagon constraint for φ_2 commuting and the associativity of G^T. Note that the right unit square constraint does not need to be verified, since it is implied by the left unit and braiding.

Assume G^T is functorial, and $g : \mathcal{M}' \to \mathcal{M}$, $h : \mathcal{N}' \to \mathcal{N}$, then

$$\begin{aligned}(G(g\otimes h))\circ\varphi_2(\mathcal{M}',\mathcal{N}')\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})} &= G(g\otimes h)\circ G^T_{\mathrm{id}_{\mathcal{M}'\otimes\mathcal{N}'}} = G^T_{g\otimes h} = G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}\circ G(g)\otimes_{\mathbb{C}} G(h)\\ &= \varphi_2(\mathcal{M},\mathcal{N})\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}\circ G(g)\otimes_{\mathbb{C}} G(h) = \varphi_2(\mathcal{M},\mathcal{N})\circ(G(g)\boxtimes G(h))\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}. \quad (7.1.6)\end{aligned}$$

Thus $(G(g\otimes h))\circ\varphi_2(\mathcal{M}',\mathcal{N}') = \varphi_2(\mathcal{M},\mathcal{N})\circ(G(g)\boxtimes G(h))$ and hence φ_2 is natural.

Conversely, assume φ_2 is natural. As noted above, We first define G^T on identity morphisms $\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}$ by $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}} = \varphi_2(\mathcal{M},\mathcal{N})\circ\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}$ and extend functorially, that is for $f \in \mathrm{Hom}_{\mathscr{D}}(\mathcal{M}\otimes\mathcal{N},\mathcal{P})$, $g : \mathcal{M}' \to \mathcal{M}$, $h : \mathcal{N}' \to \mathcal{N}$, $k : \mathcal{P} \to \mathcal{P}'$,

$$G^T_{k\circ f\circ(g\otimes h)}(z) = G(k)\circ G(f)\circ G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z)\circ(G(g)\otimes_{\mathbb{C}} G(h)). \quad (7.1.7)$$

This is well defined if and only if $G(g\otimes h)\circ G^T_{\mathrm{id}_{\mathcal{M}'\otimes\mathcal{N}'}}(z) = G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z)\circ G(g)\otimes_{\mathbb{C}} G(h)$. Consider

$$G(g\otimes h)\circ G^T_{\mathrm{id}_{\mathcal{M}'\otimes\mathcal{N}'}}(z) = G(g\otimes h)\circ\varphi_2(\mathcal{M}',\mathcal{N}')\circ\mathcal{Y}_{G(\mathcal{M}'),G(\mathcal{N}')}(-,z)$$

$$= \varphi_2(\mathcal{M},\mathcal{N}) \circ (G(g) \boxtimes G(h)) \circ \mathcal{Y}_{G(\mathcal{M}'),G(\mathcal{N}')}(-,z)$$
$$= \varphi_2(\mathcal{M},\mathcal{N}) \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(-,z) \circ G(g) \otimes_{\mathbb{C}} G(h), \tag{7.1.8}$$

where the second equality uses the naturality of φ_2 is natural and the third uses the definition of the tensor product of morphisms in $\mathscr{C}$. Hence the formula (7.1.7) is well defined and G^T is functorial. For the remainder of the proof we will assume that φ_2 is natural and hence also that G^T is functorial.

We next show the logical equivalence of the left unit constraint for (G,φ_0,φ_2) and the unitality of G^T. Consider the following squares.

$$\begin{array}{ccccc} V \otimes_{\mathbb{C}} G(\mathcal{M}) & \xrightarrow{\mathcal{Y}_{V,G(M)}} & V \boxtimes G(\mathcal{M}) & \xrightarrow{l_{G(\mathcal{M})}} & G(\mathcal{M}) \\ \downarrow{\scriptstyle \varphi_0 \otimes_{\mathbb{C}} \mathrm{id}_{G(M)}} & & \downarrow{\scriptstyle \varphi_0 \boxtimes \mathrm{id}_{G(M)}} & & \uparrow{\scriptstyle G(l_{\mathcal{M}})} \\ G(1_{\mathscr{D}}) \otimes_{\mathbb{C}} G(\mathcal{M}) & \xrightarrow{\mathcal{Y}_{G(1_{\mathscr{D}}),G(M)}} & G(1_{\mathscr{D}}) \boxtimes G(\mathcal{M}) & \xrightarrow{\varphi_2(1_{\mathscr{D}},\mathcal{M})} & G(1_{\mathscr{D}} \otimes \mathcal{M}) \end{array} \tag{7.1.9}$$

Note that we have suppressed formal variables in the images of intertwining operators for visual clarity. The left square commutes by the definition (see Theorem 6.2.1) of how the functor $\boxtimes$ is evaluated on pairs of morphisms in $\mathscr{C}$. Consider the two sequences of equalities

$$G(l_{\mathcal{M}}) \circ \varphi_2(1_{\mathscr{D}},\mathcal{M}) \circ \big(\varphi_0 \boxtimes \mathrm{id}_{G(\mathcal{M})}\big)\big(\mathcal{Y}_{V,G(\mathcal{M})}(v,z)m\big)$$
$$= G(l_{\mathcal{M}}) \circ \varphi_2(1_{\mathscr{D}},\mathcal{M})\big(\mathcal{Y}_{G(1_{\mathscr{D}}),G(\mathcal{M})}(\varphi_0(v),z)m\big)$$
$$= G(l_{\mathcal{M}}) \circ G^T_{\mathrm{id}_{1_{\mathscr{D}}\otimes\mathcal{M}}}(\varphi_0(v),z)m = G^T_{l_{\mathcal{M}}}(\varphi_0(v),z)m, \tag{7.1.10a}$$
$$l_{G(M)}\big(\mathcal{Y}_{V,G(M)}(v,z)m\big) = Y_{G(\mathcal{M})}(v,z)m. \tag{7.1.10b}$$

The first equality of (7.1.10a) follows from the definition of $\boxtimes$ evaluated on a pair of morphisms, the second from the identity relating G^T and φ_2, the third from the functoriality of G^T, while (7.1.10b) is the defining property of left unit morphisms in $\mathscr{C}$. If we assume that G^T is unital, then the last expressions of (7.1.10a) and (7.1.10b) are equal and hence the first terms must also be equal, implying the commutativity of the right square in (7.1.9). Conversely, if we assume that the right square in (7.1.9) commutes, then the first expressions of (7.1.10a) and (7.1.10b) are equal and hence the last terms must also be equal. Thus G^T is unital.

We next show the logical equivalence of the braiding constraint for (G,φ_0,φ_2) and the skew symmetry of G^T. Consider the following squares.

$$\begin{array}{ccccc} G(\mathcal{M}) \otimes_{\mathbb{C}} G(\mathcal{N}) & \xrightarrow{\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}} & G(\mathcal{M}) \boxtimes G(\mathcal{N}) & \xrightarrow{\varphi_2(\mathcal{M},\mathcal{N})} & G(\mathcal{M} \otimes \mathcal{N}) \\ \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle c_{G(\mathcal{M}),G(\mathcal{N})}} & & \downarrow{\scriptstyle G(c_{\mathcal{M},\mathcal{N}})} \\ G(\mathcal{N}) \otimes_{\mathbb{C}} G(\mathcal{M}) & \xrightarrow{\mathcal{Y}_{G(\mathcal{N}),G(\mathcal{M})}} & G(\mathcal{N}) \boxtimes G(\mathcal{M}) & \xrightarrow{\varphi_2(\mathcal{N},\mathcal{M})} & G(\mathcal{N} \otimes \mathcal{M}) \end{array} \tag{7.1.11}$$

where P is the tensor flip. We have again suppressed formal variables. The left square commutes by the definition (see Theorem 6.2.1) of braiding for intertwining operators. Consider the two sequences of equalities

$$\varphi_2(\mathcal{N},\mathcal{M}) \circ c_{G(\mathcal{M}),G(\mathcal{N})}\big(\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(m,z,n)\big) = \varphi_2(\mathcal{N},\mathcal{M})\Big(\mathrm{e}^{zL_{-1}}\mathcal{Y}_{G(\mathcal{N}),G(\mathcal{M})}(n,\mathrm{e}^{\mathrm{i}\pi}z,m)\Big)$$
$$= \mathrm{e}^{zL_{-1}}\varphi_2(\mathcal{N},\mathcal{M})\Big(\mathcal{Y}_{G(\mathcal{N}),G(\mathcal{M})}(n,\mathrm{e}^{\mathrm{i}\pi}z,m)\Big)\mathrm{e}^{zL_{-1}}G^T_{\mathrm{id}_{\mathcal{N}\otimes\mathcal{M}}}\Big(n,\mathrm{e}^{\mathrm{i}\pi}z\Big)m, \tag{7.1.12a}$$
$$G(c_{\mathcal{M},\mathcal{N}}) \circ \varphi_2(\mathcal{M},\mathcal{N}) \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(m,z)n = G(c_{\mathcal{M},\mathcal{N}}) \circ G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(m,z)n = G^T_{c_{\mathcal{M},\mathcal{N}}}(m,z)n. \tag{7.1.12b}$$

As for the unitality argument above, the equalities follow from the defining properties of the tensor structures in $\mathscr{C}$ and the functoriality of G^T or naturality of φ_2. Note that $\varphi_2(\mathcal{N},\mathcal{M})$ is a module map and hence commutes with L_{-1}. If we assume that G^T is skew symmetric, then the last expressions of (7.1.12a) and (7.1.12b) are equal and hence the first are also equal. Thus the right square in (7.1.11) commutes. Conversely, if the right square in (7.1.11) commutes, then the first expressions of (7.1.12a) and (7.1.12b) are equal and hence the last are also equal, implying the skew symmetry of G^T.

Finally we show the equivalence of the associativity hexagon condition for φ_2, and the associativity of G^T. Consider the following triangle and hexagon.

$$\begin{array}{ccccc}
 & & G(\mathcal{M})\otimes_{\mathbb{C}} G(\mathcal{N})\otimes_{\mathbb{C}} G(\mathcal{P}) & & \\
 & \swarrow^{\mathcal{Y}(\,)\mathcal{Y}(\,)} & & \searrow^{\mathcal{Y}(\mathcal{Y}(\,))} & \\
G(\mathcal{M})\boxtimes(G(\mathcal{N})\boxtimes G(\mathcal{P})) & & \xrightarrow{\ \alpha_{\mathscr{C}}\ } & & (G(\mathcal{M})\boxtimes G(\mathcal{N}))\boxtimes G(\mathcal{P}) \\
\downarrow{\scriptstyle \mathrm{id}\boxtimes\varphi_2} & & & & \downarrow{\scriptstyle \varphi_2\boxtimes\mathrm{id}} \\
G(\mathcal{M})\boxtimes G(\mathcal{N}\otimes\mathcal{P}) & & & & G(\mathcal{M}\otimes\mathcal{N})\boxtimes G(\mathcal{P}) \\
\downarrow{\scriptstyle \varphi_2} & & & & \downarrow{\scriptstyle \varphi_2} \\
G(\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P})) & & \xrightarrow{\ G(\alpha_{\mathscr{D}})\ } & & G((\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P})
\end{array} \tag{7.1.13}$$

Here $\mathcal{Y}(\,)\mathcal{Y}(\,)$ and $\mathcal{Y}(\mathcal{Y}(\,))$ denote the obvious product and iterate of intertwining operators and we have suppressed the objects labelling the natural transformations $\varphi_2, \alpha_{\mathscr{C}}, \alpha_{\mathscr{D}}$. The left triangle commutes by the definition (see Theorem 6.2.1) of associativity for intertwining operators. Let $m \in G(\mathcal{M})$, $n \in G(\mathcal{N})$, $p \in G(\mathcal{P})$, $x_1, x_2 \in \mathbb{C}$, $|x_1| > |x_2| > 0$, $|x_2| > |x_1 - x_2| > 0$ and consider the two sequences of equalities

$$G(\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}) \circ \varphi_2(\mathcal{M},\mathcal{N}\otimes\mathcal{P}) \circ \big(\mathrm{id}_{G(M)} \boxtimes \varphi_2(\mathcal{N},\mathcal{P})\big) \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})\boxtimes G(\mathcal{P})}(m,x_1)\mathcal{Y}_{G(\mathcal{N}),G(\mathcal{P})}(n,x_2)p$$
$$= G(\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}) \circ \varphi_2(\mathcal{M},\mathcal{N}\otimes\mathcal{P}) \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N}\otimes\mathcal{P})}(m,x_1)\varphi_2(\mathcal{N},\mathcal{P}) \circ \mathcal{Y}_{G(\mathcal{N}),G(\mathcal{P})}(n,x_2)p$$

$$= G(\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}) \circ G^T_{\mathrm{id}_{\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P})}}(m,x_1) G^T_{\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}}}(n,x_2)p = G^T_{\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}}(m,x_1) G^T_{\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}}}(n,x_2)p, \tag{7.1.14a}$$

$$\begin{aligned}
&\varphi_2(\mathcal{M}\otimes\mathcal{N},\mathcal{P}) \circ \big(\varphi_2(\mathcal{M},\mathcal{N}) \boxtimes \mathrm{id}_{G(\mathcal{P})}\big) \circ \alpha_{G(\mathcal{M}),G(\mathcal{N}),G(\mathcal{P})} \\
&\quad \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})\boxtimes G(\mathcal{P})}(m,x_1)\mathcal{Y}_{G(\mathcal{N}),G(\mathcal{P})}(n,x_2)p \\
&\quad = \varphi_2(\mathcal{M}\otimes\mathcal{N},\mathcal{P}) \circ \big(\varphi_2(\mathcal{M},\mathcal{N}) \boxtimes \mathrm{id}_{G(\mathcal{P})}\big) \circ \mathcal{Y}_{G(\mathcal{M}\otimes\mathcal{N}),G(\mathcal{P})}(\mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(m,x_1-x_2)n,x_2)p \\
&\quad = \varphi_2(\mathcal{M}\otimes\mathcal{N},\mathcal{P}) \circ \mathcal{Y}_{G(\mathcal{M}\otimes\mathcal{N}),G(\mathcal{P})}(\varphi_2(\mathcal{M},\mathcal{N}) \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(m,x_1-x_2)n,x_2)p \\
&\quad = G^T_{\mathrm{id}_{(\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P}}}\Big(G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(m,x_1-x_2)n,x_2\Big)p.
\end{aligned} \tag{7.1.14b}$$

As with the arguments for the previous commutative diagrams, the equivalence of G^T being associative and the hexagon in (7.1.13) commuting follows by recognising the equality of either the first or last terms of (7.1.14a) and (7.1.14b).

Assertions 1 and 2 are therefore equivalent. ∎

Corollary 7.1.2. *Let $\mathcal{C}$, $\mathcal{D}$, G and φ_0 be as in Lemma 7.1.1. Further, assume φ_0 is an isomorphism and that there exists a natural transformation $\varphi_2 : G(\mathcal{M}) \boxtimes G(\mathcal{N}) \to G(\mathcal{M} \otimes \mathcal{N})$ such that (G,φ_0,φ_2) is a braided monoidal functor. Then φ_2 is a natural isomorphism (equivalently (G,φ_0,φ_2) is a strong braided monoidal functor) if either of the following sets of sufficient conditions are satisfied.*

1. *The unique morphism $f_{\mathcal{M},\mathcal{N}} \in \mathrm{Hom}_{\mathcal{C}}(G(\mathcal{M})\boxtimes G(\mathcal{N}), G(\mathcal{M}\otimes\mathcal{N}))$ satisfying $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z) = f_{\mathcal{M},\mathcal{N}} \circ \mathcal{Y}_{G(\mathcal{M}),G(\mathcal{N})}(-,z)$ is an isomorphism for all $\mathcal{M},\mathcal{N} \in \mathcal{D}$.*

2. *For all $\mathcal{M},\mathcal{N} \in \mathcal{D}$, the objects $G(\mathcal{M})\boxtimes G(\mathcal{N})$ and $G(\mathcal{M}\otimes\mathcal{N})$ are isomorphic, and $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z)$ is a surjective intertwining operator.*

If in addition $\mathcal{D}$ and $\mathcal{C}$ are ribbon Grothendieck-Verdier with dualising objects $K_{\mathcal{D}}, K_{\mathcal{C}}$ and twists $\theta_{\mathcal{D}}, \theta_{\mathcal{C}}$, respectively. Then a braided monoidal equivalence (G,φ_0,φ_2) is a ribbon Grothendieck-Verdier equivalence, if and only if

$$G(K_{\mathcal{D}}) \cong K_{\mathcal{C}} \qquad \text{and} \qquad G(\theta_{\mathcal{D}}) = \theta_{\mathcal{C}}|_{G(-)}. \tag{7.1.15}$$

7.2 Restriction to projectives

Due to the categories and functors above being abelian, the functor G and its monoidal structure morphisms, or equivalently the family of linear maps G^T, distribute over direct sums. It is therefore sufficient to only consider indecomposable modules $\mathcal{M}, \mathcal{N}$ and $\mathcal{P}$ when verifying the properties of G^T. In practice general indecomposable vertex operator algebra modules can still be intractably complicated and so it would be convenient to not have to consider all modules. For sufficiently well behaved functors and categories one can, for example, restrict one's attention to projective modules only (and if projective modules are sufficient, then indecomposable projectives are too), as we show in the next proposition. This result will be required in Chapter 11 where we apply it to the bosonic ghost vertex algebra.

Proposition 7.2.1 ([10][Proposition 2.17]). *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian braided monoidal categories both satisfying that there are sufficiently many projectives, that the tensor products are biexact and that projectives form a tensor ideal. Let $\mathscr{C}^p$ and $\mathscr{D}^p$, respectively, be the full subcategories of projective objects of $\mathscr{C}$ and $\mathscr{D}$. If we have*

- *an exact functor $F : \mathscr{C} \to \mathscr{D}$, satisfying that the image of any projective object is projective,*
- *an isomorphism $\varphi_0 : 1_{\mathscr{D}} \to F(1_{\mathscr{C}})$,*
- *a natural isomorphism $\varphi_2^p : F(-|_p) \otimes_{\mathscr{D}} F(-|_p) \to F \circ \left(-|_p \otimes_{\mathscr{C}} -|_p\right)$, where $-|_p$ denotes the restriction to $\mathscr{C}^p$,*
- *a projective cover of the unit object $\mathcal{Q} \xrightarrow{\pi} 1_{\mathscr{C}}$,*

satisfying that, for all $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P} \in \mathscr{C}^p$, the four diagrams below commute, then φ_2^p admits a unique extension φ_2 to $\mathscr{C}$ and $(F, \varphi_0, \varphi_2)$ is a braided lax tensor functor. If, in addition, φ_2^p is a natural isomorphism, then so is φ_2 and therefore $(F, \varphi_0, \varphi_2)$ is a strong braided tensor functor.

$$\begin{array}{ccccc}
1_{\mathscr{D}} \otimes_{\mathscr{D}} F(\mathcal{M}) & \xleftarrow[\varphi_0^{-1} \otimes_{\mathscr{D}} \mathrm{id}_{F(\mathcal{M})}]{} & F(1_{\mathscr{C}}) \otimes_{\mathscr{D}} F(\mathcal{M}) & \xleftarrow[F(\pi_1) \otimes_{\mathscr{D}} \mathrm{id}_{F(\mathcal{M})}]{} & F(\mathcal{Q}) \otimes_{\mathscr{D}} F(\mathcal{M}) \\
\big\downarrow {\scriptstyle l_{F(\mathcal{M})}} & & & & \big\downarrow {\scriptstyle \varphi_2^p(\mathcal{Q},\mathcal{M})} \\
F(\mathcal{M}) & \xleftarrow[F(l_{\mathcal{M}})]{} & F(1_{\mathscr{C}} \otimes_{\mathscr{C}} \mathcal{M}) & \xleftarrow[F\left(\pi_1 \otimes_{\mathscr{C}} \mathrm{id}_{F(\mathcal{M})}\right)]{} & F(\mathcal{Q} \otimes_{\mathscr{C}} \mathcal{M})
\end{array} \tag{7.2.1}$$

$$\begin{array}{ccccc}
F(\mathcal{M}) \otimes_{\mathscr{D}} 1_{\mathscr{D}} & \xleftarrow[\mathrm{id}_{F(\mathcal{M})} \otimes_{\mathscr{D}} \varphi_0^{-1}]{} & F(\mathcal{M}) \otimes_{\mathscr{D}} F(1_{\mathscr{C}}) & \xleftarrow[\mathrm{id}_{F(\mathcal{M})} \otimes_{\mathscr{D}} F(\pi_1)]{} & F(\mathcal{M}) \otimes_{\mathscr{D}} F(\mathcal{Q}) \\
\big\downarrow {\scriptstyle r_{F(\mathcal{M})}} & & & & \big\downarrow {\scriptstyle \varphi_2^p(\mathcal{M},\mathcal{Q})} \\
F(\mathcal{M}) & \xleftarrow[F(r_{\mathcal{M}})]{} & F(\mathcal{M} \otimes_{\mathscr{C}} 1_{\mathscr{C}}) & \xleftarrow[F\left(\mathrm{id}_{F(\mathcal{M})} \otimes_{\mathscr{C}} \pi_1\right)]{} & F(\mathcal{M} \otimes_{\mathscr{C}} \mathcal{Q})
\end{array} \tag{7.2.2}$$

$$\begin{array}{ccc}
F(\mathcal{M}) \otimes_{\mathscr{D}} F(\mathcal{N}) & \xrightarrow{c_{F(\mathcal{M}),F(\mathcal{N})}} & F(\mathcal{N}) \otimes_{\mathscr{D}} F(\mathcal{M}) \\
\big\downarrow {\scriptstyle \varphi_2^p(\mathcal{M},\mathcal{N})} & & \big\downarrow {\scriptstyle \varphi_2^p(\mathcal{N},\mathcal{M})} \\
F(\mathcal{M} \otimes_{\mathscr{C}} \mathcal{N}) & \xrightarrow{F\left(c_{\mathcal{M},\mathcal{N}}\right)} & F(\mathcal{N} \otimes_{\mathscr{C}} \mathcal{M})
\end{array} \tag{7.2.3}$$

$$\begin{array}{ccc} F(\mathcal{M})\otimes_{\mathcal{D}}(F(\mathcal{N})\otimes_{\mathcal{D}}F(\mathcal{P})) & \xrightarrow{\alpha_{F(\mathcal{M}),F(\mathcal{N}),F(\mathcal{P})}} & (F(\mathcal{M})\otimes_{\mathcal{D}}F(\mathcal{N}))\otimes_{\mathcal{D}}F(\mathcal{P}) \\ \downarrow{\scriptstyle \mathrm{id}_{F(\mathcal{M})}\otimes_{\mathcal{D}}\varphi_2^P(\mathcal{N},\mathcal{P})} & & \downarrow{\scriptstyle \varphi_2^P(\mathcal{M},\mathcal{N})\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{P})}} \\ F(\mathcal{M})\otimes_{\mathcal{D}}F(\mathcal{N}\otimes_{\mathcal{C}}\mathcal{P}) & & F(\mathcal{M}\otimes_{\mathcal{C}}\mathcal{N})\otimes_{\mathcal{D}}F(\mathcal{P}) \\ \downarrow{\scriptstyle \varphi_2^P(\mathcal{M},\mathcal{N}\otimes_{\mathcal{C}}\mathcal{P})} & & \downarrow{\scriptstyle \varphi_2^P(\mathcal{M}\otimes_{\mathcal{C}}\mathcal{N},\mathcal{P})} \\ F(\mathcal{M}\otimes_{\mathcal{C}}(\mathcal{N}\otimes_{\mathcal{C}}\mathcal{P})) & \xrightarrow{F(\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}})} & F((\mathcal{M}\otimes_{\mathcal{C}}\mathcal{N})\otimes_{\mathcal{C}}\mathcal{P}) \end{array} \tag{7.2.4}$$

Proof. We apply Lemma B.2.4, with functors $F(-)\otimes_{\mathcal{D}}F(-)$ and $F(-\otimes_{\mathcal{C}}-)$ from $\mathcal{C}\times\mathcal{C}\to\mathcal{D}$. By biexactness of the tensor products and projectives forming a tensor ideal we know that the images of projective objects under these functors are again projective. We use the projective resolution of $\mathcal{M}\times\mathcal{N}$ given by the total complex $\mathrm{Tot}^{\oplus}(\mathcal{M}_\bullet\times\mathcal{N}_\bullet)$ of their individual projective resolutions (see Definition B.5.1). Hence φ_2^P admits a unique extension φ_2. It remains to show that (F,φ_0,φ_2) satisfies the constraints of a braided lax tensor functor. First, we show that the first two commutative diagrams lead to the unital constraints for projective objects. Consider the commuting square (7.2.1), where we add the morphism $\varphi_2(1_{\mathcal{C}},\mathcal{M})$ to the middle column.

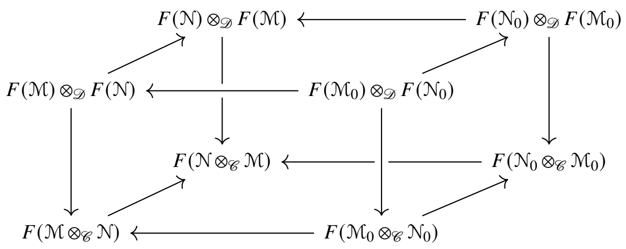
$$\begin{array}{ccccc} 1_{\mathcal{D}}\otimes_{\mathcal{D}}F(\mathcal{M}) & \xleftarrow{\varphi_0^{-1}\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{M})}} & F(1_{\mathcal{C}})\otimes_{\mathcal{D}}F(\mathcal{M}) & \xleftarrow{F(\pi)\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{M})}} & F(\mathcal{Q})\otimes_{\mathcal{D}}F(\mathcal{M}) \\ \downarrow{\scriptstyle l_{F(\mathcal{M})}} & & \downarrow{\scriptstyle \varphi_2(1_{\mathcal{C}},\mathcal{M})} & & \downarrow{\scriptstyle \varphi_2^P(\mathcal{Q},\mathcal{M})} \\ F(\mathcal{M}) & \xleftarrow{F(l_{\mathcal{M}})} & F(1_{\mathcal{C}}\otimes_{\mathcal{C}}\mathcal{M}) & \xleftarrow{F(\pi_1\otimes_{\mathcal{C}}\mathrm{id}_{F(\mathcal{M})})} & F(\mathcal{Q}\otimes_{\mathcal{C}}\mathcal{M}) \end{array} \tag{7.2.5}$$

The right square commutes by the naturality of φ_2 and the outer rectangle commutes by assumption. Therefore

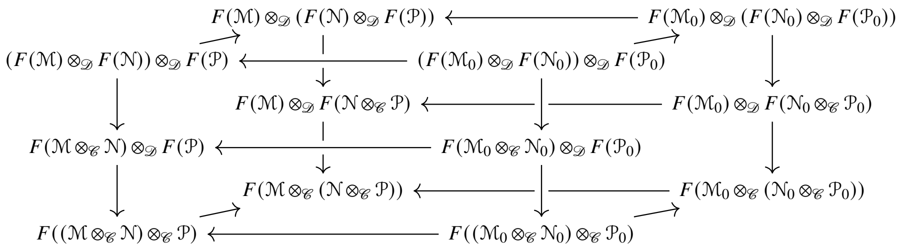
$$\begin{aligned} F(l_{\mathcal{M}})\circ\varphi_2(1_{\mathcal{C}},\mathcal{M})\circ\big(F(\pi)\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{M})}\big) &= F(l_{\mathcal{M}})\circ F(\pi\otimes_{\mathcal{C}}\mathrm{id}_{F(\mathcal{M})})\circ\varphi_2(Q,\mathcal{M}) \\ &= l_{F(\mathcal{M})}\circ\varphi_0^{-1}\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{M})}\circ\big(F(\pi)\otimes_{\mathcal{D}}\mathrm{id}_{F(M)}\big). \end{aligned} \tag{7.2.6}$$

Since $F(\pi)\otimes_{\mathcal{D}}\mathrm{id}_{F(M)}$ is an epimorphism $F(l_{\mathcal{M}})\circ\varphi_2(1_{\mathcal{C}},\mathcal{M}) = l_{F(\mathcal{M})}\circ\varphi_0^{-1}\otimes_{\mathcal{D}}\mathrm{id}_{F(\mathcal{M})}$ and it follows that the left square commutes. A similar argument holds for the right unit.

Next we show that φ_2 satisfies the constraints of the braiding square and the associativity hexagon. For any $\mathcal{M},\mathcal{N},\mathcal{P}\in\mathcal{C}$ and choices of projective resolutions for these objects consider the following diagrams.

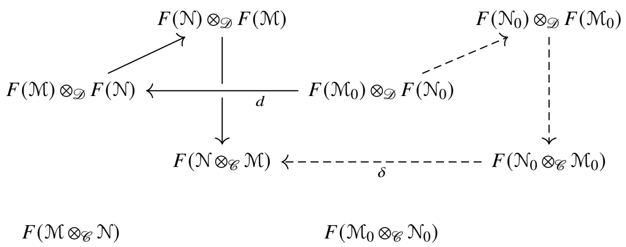


(7.2.7)

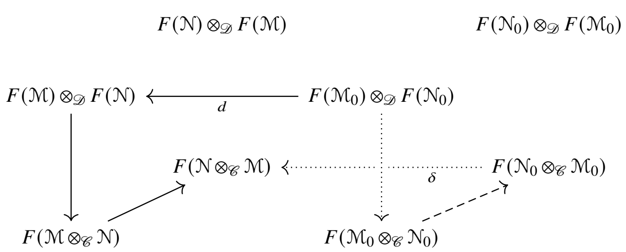


(7.2.8)

The left faces are, respectively, the braiding and associativity constraints whose commutativity we need to show. The right faces are the same diagrams evaluated on the first projective coefficients of the appropriate projective resolutions formed by taking total complexes, while the horizontal arrows are from these resolutions. Note that the right faces commute by (7.2.3) and (7.2.4), as they are evaluated on projective objects, while the front and back commute by the naturality of φ_2. We present the detailed argument for the braiding square. The argument for the associativity hexagon is similar. Consider the following paths through the braiding diagram.



(7.2.9)



(7.2.10)

The solid paths in the two diagrams denote the compositions of maps $\varphi_2(\mathcal{N},\mathcal{M}) \circ c_{F(\mathcal{M}),F(\mathcal{N})} \circ d$ and $F(c_{\mathcal{M},\mathcal{N}}) \circ \varphi_2(\mathcal{M},\mathcal{N}) \circ d$, respectively. The dashed paths denote analogous morphisms on the projective modules, that is, $\varphi_2(\mathcal{N}_0, \mathcal{M}_0) \circ c_{F(\mathcal{M}_0),F(\mathcal{N}_0)}$ and $F(c_{\mathcal{M}_0,\mathcal{N}_0}) \circ \varphi_2^p(\mathcal{M}_0, \mathcal{N}_0)$ in the top and bottom diagrams, respectively. By construction, all four morphisms $F(\mathcal{M}_0) \otimes_{\mathcal{D}} F(\mathcal{N}_0) \to F(\mathcal{N} \otimes_{\mathcal{C}} \mathcal{M})$ in the two diagrams are equal, in particular the morphisms consisting of compositions of solid arrows. Thus, since d is an epimorphism, the left face commutes. ∎

— Chapter 8 —

The Free Boson in Three Guises

"Triples makes it safe. Triples is best."

— Man in Diner (Bob Odenkirk), *I Think You Should Leave with Tim Robinson*

While the previous chapters have been very general, we now change gears and consider a specific family of monoidal categories, typically called free bosons in the context of vertex operator algebras. We will, however, consider these free bosons in slightly greater generality than is usually done in the literature (by allowing for different choices of conformal structures) and we will show that they admit ribbon Grothendieck-Verdier structures. This section presents the bulk of the content from [10].

8.1 Lattice data for free bosons

Throughout this section, we will make frequent use of certain linear algebraic and lattice data, whose structure we record here.

Definition 8.1.1. A set of **bosonic lattice data** is a quadruple $(\mathfrak{h}, \langle -, - \rangle, \Lambda, \xi)$, where

- $\mathfrak{h}$ is a finite dimensional real vector space,
- $\langle -, - \rangle$ is a non-degenerate symmetric real-valued bilinear form on $\mathfrak{h}$,
- $\Lambda \subset \mathfrak{h}$ is a lattice (that is, a discrete subgroup of $\mathfrak{h}$), which is even and integral with respect to $\langle -, - \rangle$,
- $\xi \in \Lambda^*/\Lambda$ is a distinguished element called the **Feigin-Fuchs boson**, where $\Lambda^* = \{\mu \in \mathfrak{h} \mid \langle \mu, \Lambda \rangle \subset \mathbb{Z}\}$.

Note also that we do not assume that $\langle -, - \rangle$ is positive definite. Nor do we assume that Λ is non-trivial, or that $\langle -, - \rangle$ restricted to Λ is non-degenerate. Further, if Λ is not full rank, then Λ^* will not be discrete. For any set of bosonic lattice data we can always choose a section $s : \Lambda^*/\Lambda \to \Lambda^*$, that is $\forall \rho \in \Lambda^*/\Lambda$, $s(\rho) \in \rho$ or in other words a map which chooses a representative for each coset. Note that s will generally only be a set theoretic section and not a group homomorphism. Additionally, we will always assume that $s(\Lambda) = 0 \in \Lambda^*$. From the section s we construct the associated 2-cocycle $k : \Lambda^*/\Lambda \times \Lambda^*/\Lambda \to \Lambda$.

$$k(\mu,\nu) = s(\mu+\nu) - s(\mu) - s(\nu), \quad \mu,\nu \in \Lambda^*. \tag{8.1.1}$$

The 2-cocycle k is completely determined by s and encodes the failure of s to be a group homomorphism. Finally, let $\varepsilon : \Lambda \times \Lambda \to \mathbb{C}^\times$ be a normalised 2-cocycle with commutator function $C(\alpha,\beta) = (-1)^{\langle \alpha,\beta \rangle}$, that is, ε satisfies the following conditions, for $\alpha,\beta,\gamma \in \Lambda$.

$$\begin{aligned} &\varepsilon(\alpha,0) = \varepsilon(0,\alpha) = 1, \quad \varepsilon(\alpha,\beta)\varepsilon(\beta,\alpha)^{-1} = (-1)^{\langle \alpha,\beta\rangle}, \\ &\varepsilon(\beta,\gamma)\varepsilon(\alpha+\beta,\gamma)^{-1}\varepsilon(\alpha,\beta+\gamma)\varepsilon(\alpha,\beta)^{-1} = 1. \end{aligned} \tag{8.1.2}$$

An example of such a 2-cocycle can be constructed from any ordered $\mathbb{Z}$-basis $\{\alpha_i\}$ of Λ by defining ε to be the group homomorphism uniquely characterised by

$$\varepsilon\big(\alpha_i,\alpha_j\big) = \begin{cases} (-1)^{\langle \alpha_i,\alpha_j \rangle} & \text{if } i < j \\ 1 & \text{if } i \geq j \end{cases}. \tag{8.1.3}$$

Note that in general ε need not be a homomorphism, however, since Λ is an abelian group, all choices of 2-cocycle are cohomologous. The section and 2-cocycles will always be denoted s, k and ε, respectively, for any set of bosonic lattice data. They will be required for giving explicit formulae for certain structures such as braiding and associativity isomorphisms.

Each set of bosonic lattice data will allow us to define a category of graded vectors spaces, a vertex operator algebra module category and a quasi-Hopf algebra module category all with a natural choice of ribbon Grothendieck-Verdier structure determined by these bosonic lattice data. Any such triple of categories will be shown to be ribbon Grothendieck-Verdier equivalent provided their bosonic lattice data are equal. Different choices of section s and 2-cocycle ε will yield equivalent categories, hence s and ε are not data.

Lemma 8.1.2. *Let* $(\mathfrak{h}, \langle -,- \rangle, \Lambda, \xi)$ *be a set of bosonic lattice data.*

1. *Let* $\Lambda^\perp = \{\mu \in \mathfrak{h} \ : \ \langle \mu,\Lambda \rangle = 0\}$. *There exists a finitely generated free abelian subgroup* $\Gamma \subset \Lambda^*$ *such that* $\Lambda^* = \Lambda^\perp \oplus \Gamma$, *where* $\oplus$ *is the (internal) direct sum of* $\mathbb{Z}$*-modules.*

2. *Let* $\Lambda^\circ = \{v \in \Lambda \ : \ \langle v,w \rangle = 0, \forall w \in \Lambda\}$. *The subgroup* Γ *from above can be chosen in such a way that there exists a vector subspace* $V \in \Lambda^\perp$ *and free finitely generated groups* $F, T \subset \Gamma$ *such that all of the following hold.*

- *As an abelian group,* Λ^* *admits a direct sum decomposition*

$$\Lambda^* = V \oplus \mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F \oplus T. \tag{8.1.4}$$

- *The three subgroups* V*,* $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F$ *and* T *are mutually orthogonal.*
- *The restriction of* $\langle -,-\rangle$ *to each of the three subgroups* V*,* $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \oplus F$ *and* T *individually is non-degenerate.*
- *The restriction of* $\langle -,-\rangle$ *to* $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\}$ *and* F *individually is trivial.*

Proof. 1. If Γ exists, then it must be isomorphic to the quotient $\Lambda^*/\Lambda^\perp$, we therefore first need to show that $\Lambda^*/\Lambda^\perp$ is freely finitely generated. By definition $\Lambda^\perp$ is the kernel of the surjective group homomorphism

$$\begin{aligned} \psi : \Lambda^* &\to \mathrm{Hom}(\Lambda, \mathbb{Z}), \\ \kappa &\mapsto \langle \kappa, -\rangle|_{\Lambda}. \end{aligned} \tag{8.1.5}$$

Hence $\Lambda^*/\Lambda^\perp \cong \mathrm{Hom}(\Lambda,\mathbb{Z}) \cong \Lambda$ which is freely finitely generated. Thus $\Lambda^* \to \Lambda^*/\Lambda^\perp$ is a surjective homomorphism onto a free finitely generated $\mathbb{Z}$ module with $\Lambda^\perp$ as its kernel, hence $\Lambda^\perp$ admits a free finitely generated direct sum complement in Λ^*.

2. Note that Λ/Λ° is torsion free. This can be seen by contradiction. If there was an element $t \in \Lambda \setminus \Lambda^\circ$ such that $kt \in \Lambda^\circ$ for some non-zero $k \in \mathbb{Z}$, then $0 = \langle kt, w\rangle = k\langle t, w\rangle$ for all $w \in \Lambda$, but this would imply $t \in \Lambda^\circ$. Since Λ/Λ° is torsion free, Λ° admits a direct sum complement Λ^c in Λ. The directness of the sum $\Lambda = \Lambda^\circ \oplus \Lambda^c$ implies that $\langle -,-\rangle$ restricted to Λ^c is non-degenerate. Note that the subgroup Γ from above can be chosen such that $\Lambda^c \subset \Gamma$. Define $T = \Gamma \cap \mathrm{span}_{\mathbb{R}}\{\Lambda^c\}$, then the restriction of $\langle -,-\rangle$ to T is non-degenerate, because it is non-degenerate on Λ^c. Next define $F = \{t \in \Gamma \ :\ \langle t, f\rangle = 0, \forall f \in T\}$, $V = \{v \in \Lambda^\perp \ :\ \langle v, t\rangle = 0, \forall t \in F\}$ and $W = \{w \in \Lambda^\perp \ :\ \langle w, v\rangle = 0, \forall v \in V\}$.

We show that $\Gamma = F \oplus T$. Let $\{f_i\}_{i=1}^{\mathrm{rk}T}$ be a $\mathbb{Z}$-basis of T. Since $\langle -,-\rangle$ is non-degenerate on T, there exists an $\mathbb{R}$-basis $\{f^i\}_{i=1}^{\mathrm{rk}T}$ of $\mathrm{span}_{\mathbb{R}}\{T\}$, which is dual to $\{f_i\}_{i=1}^{\mathrm{rk}T}$, that is $\left\langle f_i, f^j\right\rangle = \delta_{i,j}$. Note that this implies that the f^i basis elements pair integrally with any element in T. Consider $v \in \Gamma$, then

$$\tilde{v} = \sum_{i=1}^{\mathrm{rk}T} \left\langle v, f^i\right\rangle f_i \in T \tag{8.1.6}$$

and for any f^j we have

$$\left\langle v - \tilde{v}, f^j\right\rangle = \left\langle v, f^j\right\rangle - \left\langle v, f^j\right\rangle = 0. \tag{8.1.7}$$

Since all elements of T are $\mathbb{R}$-linear combinations of the $\mathbb{R}$-basis elements f^i, this implies $v - \tilde{v} \in F$ and hence $v \in F + T$. Next consider $v \in F \cap T$, then $\langle v, f\rangle = 0$ for all $f \in T$, but $\langle -, -\rangle$ is non-degenerate on T, hence $v = 0$ and $\Gamma = F \oplus T$.

Note that T is orthogonal to F by construction and to $\Lambda^\perp$ since $T \subset \mathrm{span}_{\mathbb{R}}\{\Lambda\}$. Thus V is orthogonal to T, F and W and so $\langle -, -\rangle$ must be non-degenerate on V in order to be non-degenerate on $\mathfrak{h}$. By a similar argument as for T and F, we have that $\Lambda^\perp = V \oplus W$.

By construction Λ° is orthogonal to V (because it is orthogonal to $\Lambda^\perp$) and also $\Lambda^\circ \subset \Lambda^\perp$. Hence $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} \subset W$. A brief counting of dimensions and ranks reveals $\mathrm{rk}\Lambda^\circ = \mathrm{rk}\Lambda - \mathrm{rk}T = \dim W = \mathrm{rk}F$, implying that $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\} = W$. Finally, by construction F is orthogonal to V and Γ hence, by the non-degeneracy of $\langle -, -\rangle$ on $\mathfrak{h}$, F must pair non-trivially with W.

■

Remark. The quotient group Λ^*/Λ will feature prominently below. The decomposition in Lemma 8.1.2 Part 2, after observing that $\Lambda = \Lambda^\circ \oplus T \cap \Lambda$, implies the decomposition

$$\Lambda^*/\Lambda = V \oplus \frac{\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\}}{\Lambda^\circ} \oplus F \oplus \frac{T}{T \cap \Lambda}. \tag{8.1.8}$$

Thus Λ^*/Λ decomposes into an abelian Lie group with a vector space part V and a compact part $\mathrm{span}_{\mathbb{R}}\{\Lambda^\circ\}/\Lambda^\circ$, and a finitely generated group with a free part F and a finite part $T/T \cap \Lambda$.

Example. We have the following natural examples to consider.

1. An empty lattice: $\mathfrak{h} = \mathbb{R}^n$ and $\Lambda = \{0\}$. In the decomposition of Lemma 8.1.2, we have $\Lambda^* = \mathfrak{h} = V$, $\Gamma = \Lambda^\circ = F = T = \{0\}$. Hence $\Lambda^*/\Lambda \cong \mathfrak{h}$ and ξ can be any element in $\mathfrak{h}$. In this case there is only one choice of section s, the canonical identification of $\mathfrak{h}/\{0\}$ with $\mathfrak{h}$, and $k = 0$.

2. A full rank lattice: $\mathfrak{h} = \mathbb{R}^n$ and Λ a rank n even integral lattice. In the decomposition of Lemma 8.1.2, $\Lambda^\perp = \{0\}$ and so $\Lambda^* = \Gamma = T$ is finitely generated. Further, Λ^*/Λ is a finite group whose order is equal to the determinant (up to a sign) of the Gram matrix of the pairing in any choice of $\mathbb{Z}$-basis of Λ. We can construct a section s by fixing a $\mathbb{Z}$-basis $\{e_i\}$ of Λ^*. The image of this basis in Λ^* will be a set of generators $\{\overline{e_i}\}$ and each $\mu \in \Lambda^*/\Lambda$ has a unique expansion $\mu = \sum_i a_i \overline{e_i}$ such that the coefficients a_i are minimal non-negative integers. Then $s(\mu) = \sum_i a_i e_i$ is a choice of section.

3. Half rank indefinite lattice: $\mathfrak{h} = \mathbb{R}^2$ with pairing $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1 y_2 + x_2 y_1$ and lattice $\Lambda = \{(0, m) \ : \ m \in \mathbb{Z}\}$. Then, in the decomposition of Lemma 8.1.2, $\Lambda^\circ = \Lambda$, $\Lambda^\perp =$

$\operatorname{span}_{\mathbb{R}}\{\Lambda^\circ\} = \{(0,x) \ : \ x \in \mathbb{R}\}$, $\Lambda^* = \{(m,x) \ : \ m \in \mathbb{Z},\ x \in \mathbb{R}\} \cong \mathbb{Z}\times\mathbb{R}$, $\Lambda^*/\Lambda \cong \mathbb{Z}\times\mathbb{R}/\mathbb{Z}$, $V = T = \{0\}$ and $F = \{(0,m) \ : \ m \in \mathbb{Z}\}$. Since the pairing is trivial when restricted to Λ, we can choose the 2-cocycle to be trivial, that is, $\varepsilon = 1$. We choose $\xi = (1, 0+\mathbb{Z})$ for the Feigin-Fuchs boson, as this a convenient choice for the free field realisations of bosonic ghost systems. See [31] for an example. Finally, we can define a choice of section s by $s(x, y+\mathbb{Z}) = (x, \tilde{y})$, where $\tilde{y}$ is the unique representative of the coset $y+\mathbb{Z}$ in the interval $[0,1)$.

8.2 Categories of vector spaces graded by abelian groups

Let $\Psi = (\mathfrak{h}, \langle -,-\rangle, \Lambda, \xi)$ be a set of bosonic lattice data and recall the decomposition $\Lambda^* = \Lambda^\perp \oplus \Gamma$ of Lemma 8.1.2.1. We specialise the results of Theorem 2.3.3 and Proposition 3.2.1 using Ψ. We choose the abelian group to be $G = \Lambda^*/\Lambda$ and the quadratic form to be

$$q(\alpha) = \mathsf{e}^{\mathsf{i}\pi\langle s(\alpha), s(\alpha)\rangle}, \quad \alpha \in \Lambda^*/\Lambda, \tag{8.2.1}$$

which defines the equivalence class of braided monoidal categories $\mathrm{Vect}^q_{\Lambda^*/\Lambda}$. Note that this choice of quadratic form is independent of the choice of section s due to Λ being even. Note further that $\langle s(\alpha), s(\alpha)\rangle$ need not be integral and so we have chosen $\mathsf{e}^{\mathsf{i}\pi}$ as a specific branch of logarithm for -1. The section s then allows us to realise a representative $\mathrm{Vect}^{(F,\Omega)}_{\Lambda^*/\Lambda}$ of $\mathrm{Vect}^q_{\Lambda^*/\Lambda}$, by defining the abelian 3-cocycle, for $\alpha, \beta, \gamma \in \Lambda^*/\Lambda$, to be

$$\Omega(\alpha,\beta) = \mathsf{e}^{\mathsf{i}\pi\langle s(\alpha), s(\beta)\rangle}, \qquad F(\alpha,\beta,\gamma) = (-1)^{\langle s(\alpha), k(\beta,\gamma)\rangle}\frac{\varepsilon(k(\alpha,\beta), k(\alpha+\beta,\gamma))}{\varepsilon(k(\beta,\gamma), k(\alpha,\beta+\gamma))}. \tag{8.2.2}$$

It is tedious but straightforward to verify this by substituting the expressions above into the equations which define an abelian 3-cocycle. Note that the abelian 3-cocycle does depend on the choice of section s, however, all choices of s yield the same trace and hence yield equivalent braided monoidal structures. Similarly, different choices of the 2-cocycle ε will yield equivalent associators. Finally, every $\xi \in \Lambda^*/\Lambda$ yields a ribbon Grothendieck-Verdier category $\mathrm{Vect}^{(F,\Omega,\xi)}_{\Lambda^*/\Lambda}$ with dualising object $\mathbb{C}_{2\xi}$ and with ribbon twist $\theta|_{\mathcal{M}_\alpha} = Q(\alpha)\mathrm{id}_{\mathcal{M}_\alpha}$, $\mathcal{M} \in \mathrm{Vect}^{(F,\Omega,\xi)}_{\Lambda^*/\Lambda}$, $\alpha \in \Lambda^*/\Lambda$, given by

$$\begin{aligned} Q(\alpha) &= \mathsf{e}^{\mathsf{i}\pi[\langle s(\alpha-\xi), s(\alpha-\xi)\rangle - \langle s(-\xi), s(-\xi)\rangle]} = \mathsf{e}^{\mathsf{i}\pi[\langle s(\alpha)-s(\xi), s(\alpha)-s(\xi)\rangle - \langle s(-\xi), s(-\xi)\rangle]} \\ &= \mathsf{e}^{\mathsf{i}\pi\langle s(\alpha), s(\alpha)+2s(-\xi)\rangle} = \mathsf{e}^{\mathsf{i}\pi\langle s(\alpha), s(\alpha)-2s(\xi)\rangle}, \end{aligned} \tag{8.2.3}$$

where we have used the fact that the lattice Λ is even. As with the quadratic form, the weak quadratic form Q, which characterises the twist, is independent of the choice of section due to Λ being even. We denote the ribbon Grothendieck-Verdier category constructed above by $\mathrm{Vect}(\Psi)$.

Example. Recall the half rank indefinite lattice example on the previous page. In the notation and conventions introduced there, we have the ribbon Grothendieck-Verdier structure defined by the abelian 3-cocycle, trace and twist

$$\begin{aligned}
&F((x_1,x_2+\mathbb{Z}),(y_1,y_2+\mathbb{Z}),(z_1,z_2+\mathbb{Z})) = (-1)^{x_1\left(\widetilde{y_2+z_2}-\tilde{y}_2-\tilde{z}_2\right)},\\
&\Omega((x_1,x_2+\mathbb{Z}),(y_1,y_2+\mathbb{Z})) = \mathsf{e}^{\mathsf{i}\pi(x_1\tilde{y}_2+\tilde{x}_2y_1)},\\
&q(x_1,x_2+\mathbb{Z}) = \mathsf{e}^{\mathsf{i}2\pi x_1\tilde{x}_2}, \qquad Q(x_1,x_2+\mathbb{Z}) = \mathsf{e}^{\mathsf{i}2\pi(x_1-1)\tilde{x}_2}.
\end{aligned} \tag{8.2.4}$$

We will return to this example again in the next section, as it will be relevant in later chapters, associated to a free field realisation of the bosonic ghosts (Section 9.4).

8.3 Categories of Heisenberg and lattice vertex operator algebra modules

Let $\Psi = (\mathfrak{h}, \langle -,-\rangle, \Lambda, \xi)$ be a set of bosonic lattice data. Treating $\mathfrak{h}$ as a real abelian Lie algebra, let $\widehat{\mathfrak{h}} = \mathfrak{h}_{\mathbb{C}} \otimes \mathbb{C}\left[t,t^{-1}\right] \oplus \mathbb{C}\mathbf{1}$ be the affinisation of $\mathfrak{h}_{\mathbb{C}}$ (the complexification of $\mathfrak{h}$ with the bilinear form extended in the obvious way) with respect to the bilinear form $\langle -,-\rangle$. This is called the Heisenberg Lie algebra (at level 1). For $\alpha \in \mathfrak{h}_{\mathbb{C}}$ and $n \in \mathbb{Z}$ denote $\alpha_n = \alpha \otimes t^n$, then we have

$$[\alpha_n, \beta_m] = n\langle \alpha,\beta\rangle \delta_{n,-m}\mathbf{1}, \quad \alpha_n,\beta_m \in \widehat{\mathfrak{h}}, \tag{8.3.1}$$

with $\mathbf{1}$ central and always taken to act as scalar multiplication by 1 in modules. We choose the triangular decomposition $\widehat{\mathfrak{h}} = \widehat{\mathfrak{h}}_- \oplus \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+$ with $\widehat{\mathfrak{h}}_0 = \mathfrak{h}_{\mathbb{C}} \otimes 1 \oplus \mathbb{C}\mathbf{1}$ and $\widehat{\mathfrak{h}}_\pm = \operatorname{span}_{\mathbb{C}}\{\alpha_n \;:\; \alpha \in \mathfrak{h}_{\mathbb{C}},\ \pm n > 0\}$. The highest weight modules with respect to this decomposition ($\widehat{\mathfrak{h}}_-$ acting freely, $\widehat{\mathfrak{h}}_+$ nilpotently and $\widehat{\mathfrak{h}}_0$ semisimply) are called Fock spaces

$$\mathcal{F}_\lambda = \operatorname{Ind}_{\widehat{\mathfrak{h}}_+ \oplus \widehat{\mathfrak{h}}_0}^{\widehat{\mathfrak{h}}} \mathbb{C}|\lambda\rangle, \quad \lambda \in \mathfrak{h}_{\mathbb{C}}, \tag{8.3.2}$$

where

$$\widehat{\mathfrak{h}}_+|\lambda\rangle = 0, \qquad \mathbf{1}|\lambda\rangle = |\lambda\rangle, \qquad \alpha_0|\lambda\rangle = \langle\alpha,\lambda\rangle|\lambda\rangle, \quad \alpha \in \mathfrak{h}_{\mathbb{C}}, \tag{8.3.3}$$

and $\widehat{\mathfrak{h}}_-$ acts freely. In sequel, any reference to a Fock space $\mathcal{F}_\lambda$ will assume the explicit choice of highest weight vector $|\lambda\rangle$ given in (8.3.2). This explicit choice of highest weight vector will be required for giving explicit normalisations of intertwining operators. For the lattice vertex operator algebras and modules to be considered in this section, we shall mostly focus on real weights, that is, $\lambda \in \mathfrak{h}$. For any coset $\mu \in \Lambda^*/\Lambda$ we define the lattice Fock space

$$\mathbb{F}_\mu = \bigoplus_{\nu \in \mu} \mathcal{F}_\nu. \tag{8.3.4}$$

Proposition 8.3.1 (Free boson vertex operators [32, Section 6.3.2]). *The Fock space $\mathcal{F}_0$ admits the structure of a vertex operator algebra uniquely characterised by the choice of field map*

$$Y(\alpha_{-1}|0\rangle, z) = \alpha(z) = \sum_{n=0}^{\infty} \alpha_n z^{-n-1}, \quad \alpha \in \mathfrak{h}_{\mathbb{C}}, \tag{8.3.5}$$

and choice of conformal vector

$$\omega_\gamma = \frac{1}{2} \sum_i \alpha^i_{-1} \alpha^{i*}_{-1} |0\rangle + \gamma_{-2}|0\rangle, \quad \gamma \in \mathfrak{h}_{\mathbb{C}}, \tag{8.3.6}$$

where $\{\alpha^i\}_{i=1}^{\dim \mathfrak{h}}$ and $\{\alpha^{j*}\}_{j=1}^{\dim \mathfrak{h}}$ are any dual choices of basis of $\mathfrak{h}_{\mathbb{C}}$. We denote this vertex operator algebra by $\mathsf{V}(\gamma)$. For any $\alpha, \beta \in \mathfrak{h}_{\mathbb{C}}$, the operator product expansions of the corresponding fields $\alpha(z), \beta(z)$ with each other and with the conformal field $T_\gamma(z) = Y(\omega_\gamma, z)$ are

$$\alpha(z)\beta(w) \sim \frac{\langle \alpha, \beta \rangle}{(z-w)^2}, \qquad T_\gamma(z)\alpha(w) \sim \frac{-2\langle \gamma, \alpha \rangle}{(z-w)^3} + \frac{\alpha(w)}{(z-w)^2} + \frac{\partial \alpha(w)}{z-w}, \tag{8.3.7}$$

and the central charge determined by ω_γ is

$$c_\gamma = \dim \mathfrak{h} - 12\langle \gamma, \gamma \rangle. \tag{8.3.8}$$

Any choice of basis of $\mathfrak{h}_{\mathbb{C}}$ is a set of strong generators of $\mathsf{V}(\gamma)$. For any $\alpha \in \mathfrak{h}_{\mathbb{C}}$, the Fock space $\mathcal{F}_\alpha$ is a module over $\mathsf{V}(\gamma)$ with field map Y_α characterised by the same formula (8.3.5) as the field map of $\mathsf{V}(\gamma)$ acting on itself.

Let $\mathbb{C}[\mathfrak{h}_{\mathbb{C}}]$ be the group algebra of $\mathfrak{h}_{\mathbb{C}}$ seen as an abelian group under addition and denote the basis element corresponding to any group element $\alpha \in \mathfrak{h}_{\mathbb{C}}$ by e^α. To each such basis vector we assign a linear map e^α, called a shift operator,

$$\begin{aligned} \mathsf{e}^\alpha : \mathcal{F}_\gamma &\to \mathcal{F}_{\alpha+\gamma}, \\ p|\gamma\rangle &\mapsto p|\alpha+\gamma\rangle, \end{aligned} \tag{8.3.9}$$

where $p \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$. Further let

$$E^\pm(\alpha, x) = \exp\left(\mp \sum_{n=1}^{\infty} \frac{\alpha_{\pm n}}{n} x^{\mp n}\right), \quad U(p, \alpha, x) = E^-(\alpha, x) Y(p, x) E^+(\alpha, x), \qquad \alpha \in \mathfrak{h}_{\mathbb{C}},\ p \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right). \tag{8.3.10}$$

Then we define linear maps $I_{\mu,\nu} : \mathcal{F}_\mu \otimes \mathcal{F}_\nu \to \mathcal{F}_{\mu+\nu}[\![z, z^{-1}]\!] z^{\langle \mu, \nu \rangle}$, for $\mu, \nu \in \mathfrak{h}_{\mathbb{C}}$ by

$$\begin{aligned} I_{\mu,\nu}(p|\mu\rangle, z) q|\nu\rangle &= z^{\langle \mu,\nu \rangle} \mathsf{e}^\mu U(p, \alpha, z) q|\nu\rangle \\ &= z^{\langle \mu,\nu \rangle} \mathsf{e}^\mu E^-(\mu, z) Y(p, z) E^+(\mu, z) q|\nu\rangle, \qquad p, q \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right), \end{aligned} \tag{8.3.11}$$

where $Y(p,z)$ is the series of Heisenberg algebra valued coefficients obtained by expanding the field map $Y(p|0\rangle,z)$ in the vertex operator algebra $\mathsf{V}(\gamma)$. The linear maps $I_{\mu,\nu}$ are generally known as (chiral) vertex operators in theoretical physics literature and are called untwisted vertex operators in [19].

Proposition 8.3.2 (Abelian intertwining algebras [19, Propositions 12.5, 12.9])**.** *Let $\mu,\nu,\rho,\in \mathfrak{h}_{\mathbb{C}}$, then*

$$\dim \mathrm{I}\begin{pmatrix} \mathcal{F}_\rho \\ \mathcal{F}_\mu, \mathcal{F}_\nu \end{pmatrix} = \begin{cases} 1 & \rho = \mu+\nu \\ 0 & \rho \neq \mu+\nu \end{cases} \tag{8.3.12}$$

and $I_{\mu,\nu}$ is an intertwining operator of type $\binom{\mathcal{F}_{\mu+\nu}}{\mathcal{F}_\mu,\mathcal{F}_\nu}$.

Lattice vertex operator algebras are constructed from Heisenberg vertex operator algebras by taking the underlying vector space to be a sum over Fock spaces whose weights lie in a lattice. The field maps for vectors lying in Fock spaces with non-zero weight are then constructed from the untwisted intertwining operators $I_{\mu,\nu}$ above. As can be seen from the definitions of modules and intertwining operators, and the unit isomorphism conditions (6.1.20), the field maps encoding the action of a vertex operator algebra on its modules are a special case of an intertwining operator with a canonical choice of normalisation. General intertwining operators, however, have no obvious choice of normalisation. So in order to extend a Heisenberg vertex operator algebra to a lattice vertex operator algebra, one needs to specify normalisations. These normalisations need to be compatible with the vacuum, skew-symmetry and associativity properties of vertex operator algebras, which implies that they satisfy the defining properties of the 2-cocycles ε in (8.1.2). As previously noted all choices of 2-cocycle are cohomologous and hence give rise to isomorphic lattice vertex operator algebras [33, Chapter 5].

Proposition 8.3.3. *Let $\widetilde{\xi}$ be a choice of representative of ξ.*

1. *The lattice Fock space $\mathbb{F}_\Lambda = \bigoplus_{\alpha\in\Lambda} \mathcal{F}_\alpha$ admits the structure of a vertex operator algebra, uniquely characterised by the choice of field map*

$$Y|_{\mathcal{F}_\mu\otimes\mathcal{F}_\nu} = \varepsilon(\mu,\nu) I_{\mu,\nu}, \quad \mu,\nu\in\Lambda, \tag{8.3.13}$$

(note that on $\mathcal{F}_0$ this specialises to the field map of the Heisenberg vertex operator algebra) and choice of conformal vector

$$\omega_{\widetilde{\xi}} = \frac{1}{2}\sum_i \alpha^i_{-1}\alpha^{i*}_{-1}|0\rangle + \widetilde{\xi}_{-2}|0\rangle, \quad \widetilde{\xi}\in\Lambda^*, \tag{8.3.14}$$

where $\{\alpha^i\}_{i=1}^{\dim\mathfrak{h}}$ *and* $\{\alpha^{j*}\}_{j=1}^{\dim\mathfrak{h}}$ *are any dual choices of basis of* $\mathfrak{h}_{\mathbb{C}}$. *We denote this vertex operator algebra by* $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$. *The central charge determined by* $\omega_{\widetilde{\xi}}$ *is*

$$c_{\widetilde{\xi}} = \dim\mathfrak{h} - 12\left\langle\widetilde{\xi},\widetilde{\xi}\right\rangle. \tag{8.3.15}$$

2. *The zero modes of* $Y(\alpha_{-1}|0\rangle,z)$, $\alpha\in\Lambda$ *furnish* $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ *with a* Λ*-grading.*

3. *For any* $\rho\in\Lambda^*/\Lambda$, *the lattice Fock space* $\mathbb{F}_\rho$ *equipped with the field map*

$$Y_{\mathbb{F}_\rho}\Big|_{\mathcal{F}_\mu\otimes\mathcal{F}_{s(\rho)+\nu}} = \varepsilon(\mu,\nu)I_{\mu,s(\rho)+\nu},\quad \mu,\nu\in\Lambda, \tag{8.3.16}$$

   is a simple discretely strongly Λ^**-graded generalised* $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ *module. The conformal weight of the highest weight vector* $|\mu\rangle$ *of a Fock space direct summand* $\mathcal{F}_\mu$, $\mu\in\rho$ *is*

$$h_\mu = \frac{1}{2}\left\langle\mu,\mu-2\widetilde{\xi}\right\rangle. \tag{8.3.17}$$

4. *Every lattice Fock space* $\mathbb{F}_\rho$, $\rho\in\Lambda^*/\Lambda$, *is graded* C_1*-cofinite as a module over the Heisenberg vertex operator algebra* $\mathsf{V}(\beta)$.

Proof. 1. The existence of the vertex algebra structure on $\mathbb{F}_\Lambda$ was shown in [34, Theorem 3.6, Remark 3.7]. Note that this vertex algebra structure is also unique in the sense that all choices of normalised 2-cocycles are cohomologous and yield isomorphic vertex algebras. The restriction of $\widetilde{\xi}$ to Λ^* is equivalent to requiring that the grading of $\mathbb{F}_\Lambda$ be integral.

2. This follows by construction.

3. That the lattice Fock space $\mathbb{F}_\rho$ is a module follows from [34, Theorem 3.6]. Each doubly homogeneous space of $\mathbb{F}_\rho$ is just an L_0 eigenspace of one of the underlying Fock spaces $\mathcal{F}_\mu$, $\mu\in\rho$. Since these eigenspaces are all finite dimensional, the doubly homogeneous spaces are too. Formula (8.3.17) follows by direct computation and implies that all conformal weights are real and that the Fock spaces $\mathcal{F}_\mu$ are discretely strongly graded. Hence the $\mathbb{F}_\rho$ are also discretely strongly graded.

4. The Λ^* homogeneous spaces of lattice Fock spaces are just the ordinary Fock spaces. These are all C_1-cofinite over $\mathsf{V}\left(\widetilde{\xi}\right)$ because the C_1 subspace has codimension 1.

∎

Remark. Note that the conformal structure of $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ genuinely depends on the choice of vector $\widetilde{\xi}\in\Lambda^*$ rather than its coset $\xi=\widetilde{\xi}+\Lambda\in\Lambda^*/\Lambda$. For example, shifting $\widetilde{\xi}$ by some $\alpha\in\Lambda$ will generally give a different central charge. It will also shift the conformal weight of any lattice module by some integer. However, the ribbon Grothendieck-Verdier structure of the module category to be defined below will only depend on the coset ξ (specifically, the dualising object and the twist depend on ξ, the associativity and braiding isomorphisms do not), rather than a choice of representative of this coset.

Definition 8.3.4. For any set of bosonic lattice data $\Psi=(\mathfrak{h},\langle -,-\rangle,\Lambda,\xi)$ and a representative $\widetilde{\xi}\in\xi$, let $\mathsf{VM}(\Psi)$ be the full subcategory of generalised Λ^*-graded $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$-modules whose objects are finitely generated, with $\widehat{\mathfrak{h}}_+$ acting locally nilpotently and $\mathfrak{h}$ acting semisimply with real eigenvalues.

Proposition 8.3.5. *The category $\mathsf{VM}(\Psi)$ is linear, abelian and semisimple. The lattice Fock spaces $\mathbb{F}_\mu$, $\mu\in\Lambda^*/\Lambda$ form a complete set of mutually inequivalent representatives of isomorphism classes of simple objects. Further, the category $\mathsf{VM}(\Psi)$ satisfies all of the conditions of Corollary 6.3.3, and therefore admits the braided monoidal structure of Proposition 6.1.5 and the ribbon Grothendieck-Verdier structure of Theorem 6.2.1.*

Proof. The category $\mathsf{VM}(\Psi)$ is clearly linear and abelian by construction. We first show semisimplicity. Let $M\in\mathsf{VM}(\Psi)$ be indecomposable. Since $\mathfrak{h}$ is required to act semisimply and real, M must be $\mathfrak{h}$ graded. Further, in order for M to be a $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$-module all fields in $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ must have integral exponents when expanded on M. Hence M is Λ^* graded and its Λ^* homogeneous spaces are modules over the Heisenberg vertex algebra $\mathsf{V}\left(\widetilde{\xi}\right)$, by restriction. Since $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ is Λ-graded, homogenous spaces of M corresponding to elements in Λ^*, which are in different cosets of Λ, cannot mix under the action of $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$. Since M is indecomposable the weights of non-zero Λ^* homogeneous spaces of M must all lie in the same Λ coset. Local nilpotence of $\widehat{\mathfrak{h}}_+$ and semisimple action of $\mathfrak{h}$ then implies, by an algebraic version of the Stone-von Neumann theorem [35, Prop 3.6], that each Λ^* homogeneous space of M is a semisimple $\mathsf{V}\left(\widetilde{\xi}\right)$ module and a possibly infinite direct sum of Fock spaces. So assume there exists a direct sum decomposition $M^{(\mu)}=A\oplus B$ of the homogeneous space of weight $\mu\in\Lambda^*$ into non-zero but not necessarily simple $\mathsf{V}\left(\widetilde{\xi}\right)$ modules A,B. Then the $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ submodules of M generated by A and B would intersect trivially and hence provide a direct sum decomposition of M, contradicting indecomposability. Thus every non-trivial homogeneous space of M is isomorphic to a single Fock space of the same weight. The module M is therefore isomorphic to a lattice Fock space and hence simple. Further, lattice Fock spaces form a complete set of mutually inequivalent simple objects. Here we implicitly use the uniqueness of module structures on lattice Fock spaces which was shown in [34, Proposition 4.2].

The first three conditions of Corollary 6.3.3 clearly hold and so we only need to verify the fourth. Consider two lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu$, $\mu, \nu \in \Lambda^*/\Lambda$ and let M be a finitely generated lower bounded submodule of $\mathrm{COMP}(\mathbb{F}_\mu, \mathbb{F}_\nu)$. We need to verify that M is an object in $\mathsf{VM}(\Psi)$. Since $\mathsf{VM}(\Psi)$ is closed under contragredients, this is equivalent to M' being in $\mathsf{VM}(\Psi)$. By [20, Part IV, Proposition 5.24], $M \subset \mathrm{COMP}(\mathbb{F}_\mu, \mathbb{F}_\nu)$ implies the existence of a surjective intertwining operator $\mathcal{Y}$ of type $\binom{M'}{\mathbb{F}_\mu, \mathbb{F}_\nu}$, we show that the image of any such intertwining operator must be an object in $\mathsf{VM}(\Psi)$. By assumption M' is finitely generated and hence we need only verify that $\mathfrak{h}$ acts semisimply and $\widehat{\mathfrak{h}}_+$ locally nilpotently. Assume $m_\mu \in \mathbb{F}_\mu, m_\nu \in \mathbb{F}_\nu$, the Jacobi identity for intertwining operators implies for any $\alpha \in \mathfrak{h}_\mathbb{C}$ and $n \geq 1$

$$\alpha_0 \mathcal{Y}(m_\mu, x) m_\nu = \mathcal{Y}(m_\mu, x)\alpha_0 m_\nu + \mathcal{Y}(\alpha_0 m_\mu, x) m_\nu,$$

$$(\alpha_n - x\alpha_{n-1})\mathcal{Y}(m_\mu, x) m_\nu = \mathcal{Y}(m_\mu, x)(\alpha_n - x\alpha_{n-1}) m_\nu + \sum_{t=0}^{n} \binom{t-n}{t} (-1)^t x^{n-t-1} \mathcal{Y}(\alpha_{t+1} m_\mu, x) m_\nu.$$

The first equality shows that the semisimplicity of α_0 on m_μ and m_ν implies the semisimplicity of α_0 on the image of $\mathcal{Y}$. The second equality shows that the nilpotency of $\widehat{\mathfrak{h}}_+$ on m_μ and m_ν implies the local nilpotency of $\widehat{\mathfrak{h}}_+$ on the image of $\mathcal{Y}$. Thus all conditions of Corollary 6.3.3 are satisfied, hence intertwining operators equip $\mathsf{VM}(\Psi)$ with the braided monoidal structures of Proposition 6.1.5.

Finally, the contragredient of a lattice Fock space is again a lattice Fock space (though generally of different weight). Hence $\mathsf{VM}(\Psi)$ is closed under taking contragredients and thus admits a ribbon Grothendieck-Verdier structure. ∎

Recall again that we are not assuming the lattice Λ to be non-zero and so the above considerations capture the ordinary free boson without a lattice by setting $\Lambda = \{0\}$. Henceforth all references to $\mathsf{VM}(\Psi)$ are to be understood as including the braided monoidal and ribbon Grothendieck-Verdier structures provided in Proposition 8.3.5.

Proposition 8.3.6. *[10][Proposition 3.11] Let Ψ be a set of bosonic lattice data and let (Ω, F) be the abelian 3-cocycle constructed from Ψ by the formulae (8.2.2). Since $\mathsf{VM}(\Psi)$ is semisimple its structure isomorphisms are uniquely determined by their values on simple modules. Consider the lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu, \mathbb{F}_\rho$, $\mu, \nu, \rho \in \Lambda^*/\Lambda$.*

1. *For any two lattice Fock spaces $\mathbb{F}_\mu, \mathbb{F}_\nu$ a choice of fusion product is given by*

$$\mathbb{F}_\mu \boxtimes \mathbb{F}_\nu = \mathbb{F}_{\mu+\nu}, \tag{8.3.18}$$

with corresponding universal intertwining operator

$$\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_\nu}\Big|_{\mathcal{F}_{s(\mu)+\alpha_1}\otimes\mathcal{F}_{s(\nu)+\alpha_2}} = (-1)^{\langle s(\mu),\alpha_2\rangle}\varepsilon(\alpha_1,\alpha_2)\varepsilon(\alpha_1+\alpha_2,k(\mu,\nu))I_{s(\mu)+\alpha_1,s(\nu)+\alpha_2}, \tag{8.3.19}$$

for $\alpha_1,\alpha_2 \in \Lambda$.

2. *The braiding isomorphism* $c_{\mu,\nu} : \mathbb{F}_\mu \boxtimes \mathbb{F}_\nu \to \mathbb{F}_\nu \boxtimes \mathbb{F}_\mu$ *is given by*

$$c_{\mu,\nu} = \mathsf{e}^{\mathsf{i}\pi\langle s(\mu),s(\nu)\rangle}\mathrm{id}_{\mathbb{F}_{\mu+\nu}} = \Omega(\mu,\nu)\mathrm{id}_{\mathbb{F}_{\mu+\nu}}. \tag{8.3.20}$$

3. *The associativity isomorphism* $A_{\mu,\nu,\rho} : \mathbb{F}_\mu \boxtimes (\mathbb{F}_\nu \boxtimes \mathbb{F}_\rho) \to (\mathbb{F}_\mu \boxtimes \mathbb{F}_\nu) \boxtimes \mathbb{F}_\rho$ *is given by*

$$A_{\mu,\nu,\rho} = (-1)^{\langle s(\mu),k(\nu,\rho)\rangle}\frac{\varepsilon(k(\mu,\nu),k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho),k(\mu,\nu+\rho))}\mathrm{id}_{\mathbb{F}_{\mu+\nu+\rho}} = F(\mu,\nu,\rho)\mathrm{id}_{\mathbb{F}_{\mu+\nu+\rho}}. \tag{8.3.21}$$

4. *The contragredient of a lattice Fock space is*

$$\mathbb{F}_\rho' = \mathbb{F}_{2\xi-\rho}, \qquad \rho \in \Lambda^*/\Lambda, \tag{8.3.22}$$

and hence the dualising object is $\mathbb{F}_{2\xi+\Lambda}$.

5. *The twist isomorphism is given by*

$$\theta_{\mathbb{F}_\rho} = \mathsf{e}^{\pi\mathsf{i}\langle s(\rho),s(\rho)-2\widetilde{\xi}\rangle}\mathrm{id}_{\mathbb{F}_\rho}, \qquad \rho \in \Lambda^*/\Lambda. \tag{8.3.23}$$

Note that Λ *being even guarantees that the above twist formula is independent of the choice of section* s.

Proof. Parts 2 – 5 follow by simple computations from the explicit formulae for intertwining operators in Part 1.

1. The lattice intertwining operator formulae (8.3.19) were proved in [19] in the context of full rank even lattices, however, the arguments showing that these formulae satisfy the intertwining operator axioms, such as the Jacobi identity, do not depend on the lattice being full rank. See also, [36] for detailed descriptions on how to compute with Heisenberg intertwining operators.

2. Since the lattice Fock spaces are simple modules, the braiding isomorphism is determined by comparing the leading terms of $\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_\nu}(|s(\mu)+\alpha_1\rangle,z)|s(\nu)+\alpha_2\rangle$ and $\mathsf{e}^{zL_{-1}}\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_\nu}(|s(\nu)+\alpha_2\rangle,\mathsf{e}^{\mathsf{i}\pi}z)|s(\mu)+\alpha_1\rangle$, where $\mu,\nu \in \Lambda^*/\Lambda$ and $\alpha_1,\alpha_2 \in \Lambda$. These are

$$\mathsf{e}^{zL_{-1}}\mathcal{Y}_{\mathbb{F}_\nu,\mathbb{F}_\mu}(|s(\nu)+\alpha_2\rangle,\mathsf{e}^{\mathsf{i}\pi}z)|s(\mu)+\alpha_1\rangle \tag{8.3.24}$$

$$= (\mathrm{e}^{\mathrm{i}\pi}z)^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle}\varepsilon(\alpha_2,\alpha_1)(-1)^{\langle s(\nu),\alpha_1\rangle}(|s(\mu)+\alpha_1+s(\nu)+\alpha_2\rangle+\mathcal{O}(z)),$$

$$\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_\nu}(|s(\mu)+\alpha_1\rangle, z)|s(\nu)+\alpha_2\rangle \tag{8.3.25}$$

$$= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle}\varepsilon(\alpha_1,\alpha_2)(-1)^{\langle s(\mu),\alpha_2\rangle}(|s(\mu)+\alpha_1+s(\nu)+\alpha_2\rangle+\mathcal{O}(z)). \tag{8.3.26}$$

Comparing the leading terms we obtain

$$\frac{\varepsilon(\alpha_2,\alpha_1)}{\varepsilon(\alpha_1,\alpha_2)}\mathrm{e}^{\mathrm{i}\pi\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle}(-1)^{\langle s(\nu),\alpha_1\rangle}(-1)^{-\langle s(\mu),\alpha_2\rangle} = \mathrm{e}^{\mathrm{i}\pi\langle s(\mu),s(\nu)\rangle}, \tag{8.3.27}$$

and hence $c_{\mu,\nu} = \mathrm{e}^{\mathrm{i}\pi\langle s(\mu),s(\nu)\rangle}\mathrm{id}_{\mathbb{F}_{\mu+\nu}}$.

3. As with the braiding isomorphisms, since the lattice Fock spaces are simple modules, thus the associativity isomorphisms are determined by comparing the leading terms. Let $\mu,\nu,\rho \in \Lambda^*/\Lambda$ and $\alpha_1,\alpha_2,\alpha_3 \in \Lambda$ and consider

$$\begin{aligned}
&\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_{\nu+\rho}}(|s(\mu)+\alpha_1\rangle, x_1)\mathcal{Y}_{\mathbb{F}_\nu,\mathbb{F}_\rho}(|s(\nu)+\alpha_2\rangle, x_2)|s(\rho)+\alpha_3\rangle\\
&\quad= (-1)^{\langle s(\mu),k(\nu,\rho)+\alpha_2+\alpha_3\rangle}\varepsilon(\alpha_1,\alpha_2+\alpha_3+k(\nu,\rho))\varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\nu,\rho),k(\mu,\nu+\rho))\\
&\qquad(-1)^{\langle s(\nu),\alpha_3\rangle}\varepsilon(\alpha_2,\alpha_3)\varepsilon(\alpha_2+\alpha_3,k(\nu,\rho))\\
&\qquad(x_1-x_2)^{\langle s(\mu)+\alpha_1,s(\nu)+\alpha_2\rangle}x_1^{\langle s(\mu)+\alpha_1,s(\rho)+\alpha_3\rangle}x_2^{\langle s(\nu)+\alpha_2,s(\rho)+\alpha_3\rangle}\\
&\qquad\big(|s(\mu)+s(\nu)+s(\rho)+\alpha_1+\alpha_2+\alpha_3\rangle+\mathcal{O}(z)\big)\\
&\mathcal{Y}_{\mathbb{F}_{\mu+\nu},\mathbb{F}_\rho}(\mathcal{Y}_{\mathbb{F}_\mu,\mathbb{F}_\nu}(|s(\mu)+\alpha_1\rangle, x_1-x_2)|s(\nu)+\alpha_2\rangle, x_2)|s(\rho)+\alpha_3\rangle\\
&\quad= (-1)^{\langle s(\mu),\alpha_2\rangle}\varepsilon(\alpha_1,\alpha_2)\varepsilon(\alpha_1+\alpha_2,k(\mu,\nu))\\
&\qquad(-1)^{\langle s(\mu+\nu),\alpha_3\rangle}\varepsilon(\alpha_1+\alpha_2+k(\mu,\nu),\alpha_3)\varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\mu,\nu),k(\mu+\nu,\rho))\\
&\qquad(x_1-x_2)^{\langle s(\mu)+\alpha_1,s(\nu+\alpha_2)\rangle}x_2^{\langle s(\mu)+s(\nu)+\alpha_1+\alpha_2,s(\rho)+\alpha_3\rangle}\left(\frac{x_1}{x_2}\right)^{\langle s(\mu)+\alpha_1,s(\rho)+\alpha_3\rangle}\\
&\qquad\big(|s(\mu)+s(\nu)+s(\rho)+\alpha_1+\alpha_2+\alpha_3\rangle+\mathcal{O}(z)\big).
\end{aligned} \tag{8.3.28}$$

The ratio of the x_1,x_2 dependent factors is

$$\frac{(x_1-x_2)^{\langle s(\mu),s(\nu)\rangle}x_2^{\langle s(\mu)+s(\nu),s(\rho)\rangle}\left(\frac{x_1}{x_2}\right)^{\langle s(\mu),s(\rho)\rangle}}{(x_1-x_2)^{\langle s(\mu),s(\nu)\rangle}x_1^{\langle s(\mu),s(\rho)\rangle}x_2^{\langle s(\nu),s(\rho)\rangle}} = 1 \tag{8.3.29}$$

and the associativity isomorphism is scalar multiplication by

$$\begin{aligned}
&\frac{(-1)^{\langle s(\mu),\alpha_2\rangle}\varepsilon(\alpha_1,\alpha_2)\varepsilon(\alpha_1+\alpha_2,k(\mu,\nu))(-1)^{\langle s(\mu+\nu),\alpha_3\rangle}\varepsilon(\alpha_1+\alpha_2+k(\mu,\nu),\alpha_3)}{(-1)^{\langle s(\mu),k(\nu,\rho)+\alpha_2+\alpha_3\rangle}\varepsilon(\alpha_1,\alpha_2+\alpha_3+k(\nu,\rho))\varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\nu,\rho),k(\mu,\nu+\rho))}\\
&\quad\frac{\varepsilon(\alpha_1+\alpha_2+\alpha_3+k(\mu,\nu),k(\mu+\nu,\rho))}{(-1)^{\langle s(\nu),\alpha_3\rangle}\varepsilon(\alpha_2,\alpha_3)\varepsilon(\alpha_2+\alpha_3,k(\nu,\rho))} = (-1)^{\langle s(\mu),k(\nu,\rho)\rangle}\frac{\varepsilon(k(\mu,\nu),k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho),k(\mu,\nu+\rho))}
\end{aligned} \tag{8.3.30}$$

4. The Heisenberg weight of $\mathbb{F}'_\rho$ is determined by computing the opposed field map of $\alpha_{-1}|0\rangle$, $\alpha \in \mathfrak{h}_\mathbb{C}$. This is given by

$$Y(\alpha_{-1}|0\rangle, z)^{\text{opp}} = Y(e^{zL_1}(-z^{-2})^{L_0}\alpha_{-1}|0\rangle, z^{-1}) = -z^{-2}Y(\alpha_{-1}|0\rangle, z^{-1}) + z^{-1}2\left\langle\widetilde{\xi},\alpha\right\rangle Y(|0\rangle, z^{-1})$$
$$= -z^{-2}Y(\alpha_{-1}|0\rangle, z^{-1}) + z^{-1}2\left\langle\widetilde{\xi},\alpha\right\rangle\text{id}, \tag{8.3.31}$$

   for any $\alpha \in \mathfrak{h}_\mathbb{C}$. This implies that the Heisenberg weight of $\mathbb{F}'_\rho$ is $2\widetilde{\xi} - \rho$.

5. The formula for the twist isomorphism follows immediately from the conformal weight of Fock space highest weight vectors.

■

We prepare some notation in order to use Lemma 7.1.1 to show that Vect(Ψ) and VM(Ψ) are equivalent as ribbon Grothendieck-Verdier categories. To any object in Vect(Ψ) we can associate an object in VM(Ψ) by the following induction construction. Let $M = \bigoplus_{\alpha\in\Lambda^*/\Lambda} M_\alpha$ be a decomposition into homogeneous spaces and consider the vector space $M \otimes \mathbb{C}[\Lambda]$ and endow it with the structure of an $\widehat{\mathfrak{h}}_\geq = \widehat{\mathfrak{h}}_0 \oplus \widehat{\mathfrak{h}}_+$ module by defining

$$\widehat{\mathfrak{h}}_+ \cdot M\otimes\mathbb{C}[\Lambda] = 0, \qquad \alpha\cdot m_\rho \otimes \beta = \langle\alpha, s(\rho)+\beta\rangle\text{id}, \quad \alpha\in\mathfrak{h}_\mathbb{C},\ \rho\in\Lambda^*/\Lambda,\ \beta\in\Lambda. \tag{8.3.32}$$

Further induce $M \otimes \mathbb{C}[\Lambda]$ to a module over $\widehat{\mathfrak{h}}$ by defining

$$\mathbb{F}[M] = \text{Ind}_{\widehat{\mathfrak{h}}_\geq}^{\widehat{\mathfrak{h}}} M\otimes\mathbb{C}[\Lambda]. \tag{8.3.33}$$

Next, define the action of the shift operators e^γ, $\gamma\in\Lambda$ on $M\otimes\mathbb{C}[\Lambda]$ to be

$$\mathsf{e}^\gamma m\otimes\mathsf{e}^\delta = m\otimes\mathsf{e}^{\gamma+\delta}, \quad m\in M,\ \gamma,\delta\in\Lambda \tag{8.3.34}$$

and extend to all of $\mathbb{F}[M]$ to obtain a well defined action of the obvious analogue of untwisted vertex operators (8.3.11) (with the first of the two indices parametrising weights in Λ) and hence also the field map (8.3.16) by defining

$$Y_{\mathbb{F}[M]}(u|\alpha_1\rangle, z)v\cdot m\otimes\mathsf{e}^{\alpha_2} = \varepsilon(\alpha_1,\alpha_2)(-1)^{\langle s(\Lambda),\alpha_2\rangle} z^{\langle\alpha_1, s(\mu)+\alpha_2\rangle}\mathsf{e}^{\alpha_1}U(\alpha_1,u,z)\,v\cdot m\otimes\mathsf{e}^{\alpha_2}, \tag{8.3.35}$$

for $\alpha_1,\alpha_2\in\Lambda$, $\mu\in\Lambda^*/\Lambda$, $m\in M_\mu$, $u,v\in\mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ and where $U(\alpha_1,u,z)$ is the Heisenberg algebra valued series (8.3.10). Thus $\mathbb{F}[M]$ has the structure of a $\mathsf{V}\left(\widetilde{\xi},\Lambda\right)$ module, with decomposition into lattice Fock spaces given by

$$\mathbb{F}[M] \cong \bigoplus_{\rho\in\Lambda^*/\Lambda}\dim(M_\rho)\mathbb{F}_\rho. \tag{8.3.36}$$

To define intertwining operators for the modules constructed above, we shall need the following auxiliary linear maps which for any $M, N, P \in \mathrm{Vect}(\Psi)$ and $f \in \mathrm{Hom}(M \otimes N, P)$ are defined to be

$$\begin{aligned} f_m : \mathbb{F}[N] &\to \mathbb{F}[P], \qquad\qquad m \in M,\ n \in N,\ u \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right),\ \gamma \in \Lambda. \\ u \cdot n \otimes \mathsf{e}^{\gamma} &\mapsto u \cdot f(m \otimes n) \otimes \mathsf{e}^{\gamma} \end{aligned} \tag{8.3.37}$$

Theorem 8.3.7. *Let Ψ be a set of bosonic lattice data,* $\mathrm{Vect}(\Psi)$ *be the associated ribbon Grothendieck-Verdier category from the previous section and* $\mathsf{VM}(\Psi)$ *the module category of the lattice vertex operator algebra described above. Further, let $G : \mathrm{Vect}(\Psi) \to \mathsf{VM}(\Psi)$ be the functor which assigns to any $M \in \mathrm{Vect}(\Psi)$ the object $G(M) = \mathbb{F}[M]$ from (8.3.33) with the obvious extension to morphisms. Consider the following maps.*

- *Let $\varphi_0 : \mathsf{V}\left(\widetilde{\xi}, \Lambda\right) \mapsto G(\mathbb{C}_0)$ be the module map uniquely characterised by $\varphi_0(|0\rangle) = 1_0 \otimes \mathsf{e}^0$, where $1_0 \in \mathbb{C}_0$.*
- *For $M, N, P \in \mathrm{Vect}(\Psi)$, $f \in \mathrm{Hom}(M \otimes N, P)$, $\mu, \nu \in \Lambda^*/\Lambda$, $m \in M_\mu$, $n \in N_\nu$, $\alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ define G^T by*

$$\begin{aligned} G^T_f(u \cdot m \otimes \mathsf{e}^{\alpha_1}, z) v \cdot n \otimes \mathsf{e}^{\alpha_2} &= (-1)^{\langle s(\mu), \alpha_2 \rangle} \varepsilon(\alpha_1, \alpha_2) \varepsilon(\alpha_1 + \alpha_2, k(\mu, \nu)) \\ &\quad z^{\langle s(\mu) + \alpha_1, s(\nu) + \alpha_2 \rangle} f_m \mathsf{e}^{\alpha_1} U(u, s(\mu) + \alpha_1, z) v \cdot n \otimes \mathsf{e}^{\alpha_2}. \end{aligned} \tag{8.3.38}$$

Then φ_0 and G^T satisfy the conditions of Lemma 7.1.1 and hence endow G with the structure of a braided monoidal functor. The functor G with this choice of monoidal structure is an equivalence of ribbon Grothendieck-Verdier categories. In particular, for $\xi = 0$, the functor G is a ribbon equivalence.

The equivalence of $\mathsf{VM}(\Psi)$ and $\mathrm{Vect}(\Psi)$ as braided tensor categories is well known [19] in the special case of positive definite even full rank lattices. Here we use the opportunity to illustrate the application of Lemma 7.1.1 and to show the equivalence of the ribbon Grothendieck-Verdier structures as well.

Proof. We prove the theorem by showing that φ_0 and the family of linear maps G^T of (8.3.38) satisfy the conditions of Lemma 7.1.1.2 and Corollary 7.1.2.2. We first show the functoriality of G^T. For any $\mathcal{M}, \mathcal{M}', \mathcal{N}, \mathcal{N}', \mathcal{P}, \mathcal{P}' \in \mathrm{Vect}(\Psi)$, $\mu, \nu \in \Lambda^*/\Lambda$, $m \in \mathcal{M}'_\mu$, $n \in \mathcal{N}'_\nu$, $\alpha_1, \alpha_2 \in \Lambda$ and $u, v \in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ consider

$$G^T_{k \circ f \circ (g \otimes h)}(u \cdot m \otimes \mathsf{e}^{\alpha_1}; z) v \cdot n \otimes \mathsf{e}^{\alpha_2}$$

$$= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle}(k\circ f\circ(g\otimes h))_m \mathrm{e}^{\alpha_1} U(s(\mu)+\alpha_1,u,z))\, v\cdot n\otimes \mathrm{e}^{\alpha_2}$$
$$= G(k) z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} f_{g(m)} \mathrm{e}^{\alpha_1} U(s(\mu)+\alpha_1,u,z) v\cdot h(n)\otimes \mathrm{e}^{\alpha_2}$$
$$= G(k)\circ G^T_f(G(g)u\cdot m\otimes \mathrm{e}^{\alpha_1},z)G(h)v\cdot n\otimes \mathrm{e}^{\alpha_2}, \tag{8.3.39}$$

where the second and third equalities follow from the definition of the f_m notation in (8.3.37). Thus G^T is functorial.

Next we show the unitality of G^T. For any $\mathcal{N}\in \mathrm{Vect}(\Psi)$, $\nu\in\Lambda^*/\Lambda$, $n\in\mathcal{N}_\nu$, $\alpha_1,\alpha_2\in\Lambda$ and $u,v\in \mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ consider

$$G^T_{l_{\mathcal{N}}}(\varphi_0(u|\alpha_1\rangle),z)v\cdot n\otimes \mathrm{e}^{\alpha_2} = G^T_{l_{\mathcal{N}}}(u\cdot 1_0\otimes \mathrm{e}^{\alpha_1},z)v\cdot n\otimes \mathrm{e}^{\alpha_2}$$
$$= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle}(l_{\mathcal{N}})_{1_0} \mathrm{e}^{\alpha_1} U(s(\mu)+\alpha_1,u,z)\, v\cdot n\otimes \mathrm{e}^{\alpha_2}$$
$$= z^{\langle s(\mu)+\alpha_1, s(\nu)+\alpha_2\rangle} \mathrm{e}^{\alpha_1} U(s(\mu)+\alpha_1,u,z)\, v\cdot n\otimes \mathrm{e}^{\alpha_2}$$
$$= Y_{G(\mathcal{N})}(u|\alpha_1\rangle,z)v\cdot n\otimes \mathrm{e}^{\alpha_2}, \tag{8.3.40}$$

where in the third identity we have used that $l_{\mathcal{N}}(1_0\otimes n)=n$. Thus G^T is unital.

Next we show the skew symmetry of G^T. For any $\mathcal{M},\mathcal{N}\in\mathrm{Vect}(\Psi)$, $\mu,\nu\in\Lambda^*/\Lambda$, $m\in\mathcal{M}_\mu$, $n\in\mathcal{N}_\nu$, $\alpha_1,\alpha_2\in\Lambda$ and $u,v\in\mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ consider

$$G^T_{c_{\mathcal{N},\mathcal{M}}}(v\cdot n\otimes \mathrm{e}^{\alpha_2},z)u\cdot m\otimes \mathrm{e}^{\alpha_1}$$
$$= z^{\langle s(\nu)+\alpha_2, s(\mu)+\alpha_1\rangle}\left(c_{\mathcal{N},\mathcal{M}}\right)_n \mathrm{e}^{\alpha_2} U(s(\nu)+\alpha_2,u,z)\, v\cdot n\otimes \mathrm{e}^{\alpha_1}$$
$$= \mathrm{e}^{\mathrm{i}\pi\langle s(\nu),s(\mu)\rangle} z^{\langle s(\nu)+\alpha_2, s(\mu)+\alpha_1\rangle}(P_{\mathcal{N}\otimes\mathcal{M}})_n \mathrm{e}^{\alpha_2} U(s(\nu)+\alpha_2,u,z)\, v\cdot n\otimes \mathrm{e}^{\alpha_1}$$
$$= \mathrm{e}^{\mathrm{i}\pi\langle s(\nu),s(\mu)\rangle} G^T_{P_{\mathcal{N}\otimes\mathcal{M}}}(v\cdot n\otimes \mathrm{e}^{\alpha_2},z)u\cdot m\otimes \mathrm{e}^{\alpha_1}$$
$$= \mathrm{e}^{zL_{-1}} G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}\left(u\cdot m\otimes \mathrm{e}^{\alpha_1}, \mathrm{e}^{\mathrm{i}\pi}z\right)v\cdot n\otimes \mathrm{e}^{\alpha_2}, \tag{8.3.41}$$

where $P_{\mathcal{N}\otimes\mathcal{M}}:\mathcal{N}\otimes\mathcal{M}\to\mathcal{M}\otimes\mathcal{N}$ is the standard tensor flip of (graded) vector spaces and where in the fourth identity we have used the well known behaviour of untwisted vertex operators (8.3.11) with respect to L_{-1}. Thus G^T is skew symmetric.

Next we show G^T is associative. For any $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P}\in\mathrm{Vect}(\Psi)$, $m\in\mathcal{M}$, $n\in\mathcal{N}$, $p\in\mathcal{P}$, $\mu,\nu,\rho\in\Lambda^*/\Lambda$, $\alpha_1,\alpha_2,\alpha_3\in\Lambda$, $u,v,w\in\mathsf{U}\left(\widehat{\mathfrak{h}}_-\right)$ and $x_1,x_2\in\mathbb{C}$ such that $|x_1|>|x_2|>0$ and $|x_2|>|x_1-x_2|>0$, consider

$$G^T_{\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}}(u\cdot m\otimes \mathrm{e}^{\alpha_1},x_1)G^T_{\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}}}(v\cdot n\otimes \mathrm{e}^{\alpha_2},x_2)w\cdot p\otimes \mathrm{e}^{\alpha_3}$$
$$= x_1^{\langle s(\mu)+\alpha_1, s(\nu+\rho)-k(\nu,\rho)+\alpha_2+\alpha_3\rangle} x_2^{\langle s(\nu)+\alpha_2, s(\rho)+\alpha_3\rangle}$$
$$\left(\alpha_{\mathcal{M},\mathcal{N},\mathcal{P}}\right)_m \mathrm{e}^{\alpha_1} U(s(\nu)+\alpha_1,u,x_1)(\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}})_n \mathrm{e}^{\alpha_2} U(s(\nu)+\alpha_2,v,x_2)\, w\cdot p\otimes \mathrm{e}^{\alpha_3}$$
$$= (-1)^{\langle s(\mu),k(\nu,\rho)\rangle}\frac{\varepsilon(k(\mu,\nu),k(\mu+\nu,\rho))}{\varepsilon(k(\nu,\rho),k(\mu,\nu+\rho))} x_1^{\langle s(\mu)+\alpha_1, s(\nu+\rho)-k(\nu,\rho)+\alpha_2+\alpha_3\rangle} x_2^{\langle s(\nu)+\alpha_2, s(\rho)+\alpha_3\rangle}$$

$$\begin{aligned}&\left(\mathrm{id}_{\mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P})}\right)_m \mathrm{e}^{\alpha_1} U(s(\mu)+\alpha_1,u,x_1)(\mathrm{id}_{\mathcal{N}\otimes\mathcal{P}})_n \mathrm{e}^{\alpha_2} U(s(\nu)+\alpha_2,v,x_2)\, w\cdot p\otimes \mathrm{e}^{\alpha_3}\\ &= G^T_{\mathrm{id}_{(\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P}}}\left(G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(u\cdot m\otimes \mathrm{e}^{\alpha_1},x_1-x_2)v\cdot n\otimes \mathrm{e}^{\alpha_2},x_2\right) w\cdot p\otimes \mathrm{e}^{\alpha_3},\end{aligned} \tag{8.3.42}$$

where in the third equality we have used the well known behaviour of untwisted vertex operators (8.3.11), see for example [19, Section 12] or [36]. Thus G^T is associative.

The intertwining operators $G^T_{\mathrm{id}_{\mathcal{M}\otimes\mathcal{N}}}(z)$ are surjective by construction for any $\mathcal{M},\mathcal{N}\in \mathrm{Vect}(\Psi)$. Hence, by Corollary 7.1.2, the functor G, with the monoidal structure constructed from G^T, is a braided monoidal equivalence.

The equivalence of the ribbon Grothendieck-Verdier structures then follows from noting that the dualising objects are isomorphic, that is,

$$G(\mathbb{C}_{2\xi}) \cong \dim(\mathbb{C}_{2\xi})\, \mathbb{F}_{2\xi} = \mathbb{F}_{2\xi}, \tag{8.3.43}$$

and that the twists are equivalent, that is, for any $\mu\in\Lambda^*/\Lambda$

$$G(\theta_{\mathbb{C}_\mu}) = \mathrm{e}^{\pi\mathrm{i}\langle s(\mu),s(\mu)-2s(\xi)\rangle}\mathrm{id}_{G(\mathbb{C}_\mu)} = \mathrm{e}^{\pi\mathrm{i}\langle s(\mu),s(\mu)-2\tilde{\xi}\rangle}\mathrm{id}_{G(\mathbb{C}_\mu)} = \mathrm{e}^{2\pi\mathrm{i}L_0}|_{G(\mathbb{C}_\mu)} = \theta_{G(\mathbb{C}_\mu)}, \tag{8.3.44}$$

where in the second equality we have used that $s(\xi)$ and $\tilde{\xi}$ differ at most by an element in Λ. ■

Example. Recall the half rank lattice example at the end of Section 8.1. In the notation and conventions introduced there, we choose $\tilde{\xi}=(1,0)$ as a representative of ξ. Further let $\alpha=(1,0)\in\mathbb{R}^2$ and $\beta=(0,1)\in\mathbb{R}^2$, then vertex operator algebra structure on $\mathcal{F}_0$ is strongly generated by the fields corresponding to α,β, whose defining operator product expansions are

$$\alpha(z)\alpha(w)\sim 0\sim\beta(z)\beta(w),\qquad \alpha(z)\beta(w)\sim\frac{1}{(z-w)^2}. \tag{8.3.45}$$

The choice of element $\tilde{\xi}$ defines the conformal vector and central charge

$$\omega_{\tilde{\xi}}=\alpha_{-1}\beta_{-1}|0\rangle+\tilde{\xi}_{-2}|0\rangle,\qquad c_{\tilde{\xi}}=2. \tag{8.3.46}$$

Further, β generates the lattice Λ and the Fock spaces with weights in Λ have generating highest weight vectors of the following conformal weights, by (8.3.17).

$$h_{n\beta}=\frac{1}{2}\langle(0,n),(-2,n)\rangle=-n. \tag{8.3.47}$$

In the basis $\psi=\frac{1}{\sqrt{2}}(1,1)\in\mathbb{R}^2$ and $\theta=\frac{1}{\sqrt{2}}(-1,1)\in\mathbb{R}^2$, the operator product expansions become

$$\psi(z)\psi(w)\sim\frac{1}{(z-w)^2}\sim-\theta(z)\theta(w),\qquad \psi(z)\theta(w)\sim 0. \tag{8.3.48}$$

These OPEs will arise again in (9.4.1) as a lattice vertex algebra into which the bosonic ghost vertex algebra embeds.

8.4 Categories of Hopf algebra modules

For any set of bosonic lattice data $\Psi = (\mathfrak{h}, \langle -,-\rangle, \Lambda, \xi)$, let $\mathsf{U}(\Lambda^\perp)$ denote the universal enveloping algebra (or symmetric algebra) of the complexification $\Lambda^\perp_{\mathbb{C}}$ of the vector space $\Lambda^\perp$ seen as an abelian Lie algebra and $\mathbb{C}[\Lambda^*/\Lambda^\perp]$ the group algebra of the abelian group $\Lambda^*/\Lambda^\perp$. These associative algebras both admit well known Hopf algebra structures by defining the elements of $\Lambda^\perp_{\mathbb{C}}$ to be primitive and those of $\Lambda^*/\Lambda^\perp$ to be group like, that is

$$\begin{aligned} \Delta(\mu) &= \mu\otimes 1 + 1\otimes\mu, & \epsilon(\mu) &= 0, & s(\mu) &= -\mu, & \mu &\in \Lambda^\perp_{\mathbb{C}}, \\ \Delta(K_\nu) &= K_\nu\otimes K_\nu, & \epsilon(K_\nu) &= 1, & s(K_\nu) &= K_{-\nu} = K_\nu^{-1}, & \nu &\in \Lambda^*/\Lambda^\perp, \end{aligned} \tag{8.4.1}$$

where K_ν is the basis element of $\mathbb{C}[\Lambda^*/\Lambda^\perp]$ corresponding to $\nu \in \Lambda^*/\Lambda^\perp$. We call

$$H_\Lambda = \mathsf{U}\big(\Lambda^\perp\big)\otimes\mathbb{C}[\Lambda^*/\Lambda^\perp] \tag{8.4.2}$$

the **lattice Hopf algebra** of Λ, where the Hopf algebra structures are those inherited from the two tensor factors. Note that we omit the subscript $\mathbb{C}$ from $\Lambda^\perp_{\mathbb{C}}$ in $\mathsf{U}(\Lambda^\perp)$.

Every object $\mathcal{M}$ in $\mathrm{Vect}(\Psi)$ can be given the structure of an H_Λ module by defining the representation $\rho_{\mathcal{M}} : H_\Lambda \to \mathrm{End}\mathcal{M}$ on homogeneous spaces by

$$\rho_{\mathcal{M}}(\mu)|_{\mathcal{M}_\alpha} = \langle \mu, s(\alpha)\rangle \mathrm{id}_{\mathcal{M}_\alpha}, \quad \rho_{\mathcal{M}}(K_\nu)|_{\mathcal{M}_\alpha} = \mathrm{e}^{2\pi i\langle \nu, s(\alpha)\rangle}\mathrm{id}_{\mathcal{M}_\alpha}, \tag{8.4.3}$$

for $\alpha \in \Lambda^*/\Lambda$, $\mu \in \Lambda^\perp_{\mathbb{C}}$, $\nu \in \Lambda^*/\Lambda^\perp$. Note that the above formulae do not depend on the choice of section s. We can therefore interpret $\mathrm{Vect}(\Psi)$ as a category of representations of the group Λ^*. Further, for $\mu \in \Lambda^\perp_{\mathbb{C}} \cap \Lambda$, $\rho_{\mathcal{M}}(\mu)|_{\mathcal{M}_\alpha} = 0$ and for $\nu \in \Lambda^*/\Lambda^\perp$, $\nu \cap \Lambda \neq 0$, $\rho_{\mathcal{M}}(K_\nu)|_{\mathcal{M}_\alpha} = \mathrm{id}_{\mathcal{M}_\alpha}$. Hence the objects of $\mathrm{Vect}(\Psi)$ also can be interpreted as representations of the quotient group Λ^*/Λ. Since H_Λ is a Hopf algebra, there is of course a natural representation on tensor products of objects $\mathcal{M}, \mathcal{N} \in \mathrm{Vect}(\Psi)$ given by $\rho_{\mathcal{M}\otimes\mathcal{N}} = (\rho_{\mathcal{M}}\otimes\rho_{\mathcal{N}})\circ\Delta$. Now that we have recast $\mathrm{Vect}(\Psi)$, as an abelian category, as a category of modules over H_Λ, it is interesting to see if we can capture the braided monoidal, Grothendieck-Verdier and ribbon structures of $\mathrm{Vect}(\Psi)$ in Hopf algebraic terms by specifying an R-matrix, coassociator and ribbon element. To do so, we recall the decomposition $\Lambda^* = \Lambda^\perp \oplus \Gamma$ of Λ^* in Lemma 8.1.2.1. We define formal operators in terms of their action on the objects of $\mathrm{Vect}(\Psi)$ (though they could also be thought of as lying in suitable completions of tensor powers of H_Λ). Let $\{\mu_i\}_{i=1}^{\dim\Lambda^\perp}$ be an $\mathbb{R}$-basis of $\Lambda^\perp$ and let $\{\nu_j\}_{j=1}^{\mathrm{rk}\Lambda}$ be a $\mathbb{Z}$-basis of Γ. Since the real span of Λ^* is $\mathfrak{h}$, $\{\mu_i, \nu_j\}$ is an $\mathbb{R}$ basis of $\mathfrak{h}$. Hence there exists a dual basis $\{\mu^i, \nu^j\}$. Let $\log_s K_\nu$, $\nu \in \Lambda^*/\Lambda^\perp$ be the formal operator, depending on the section s, defined on the homogeneous spaces of an object $\mathcal{M} \in \mathrm{Vect}(\Psi)$ to act as

$$\log_s(K_\nu)|_{\mathcal{M}_\alpha} = \langle \nu, s(\alpha)\rangle \mathrm{id}_{\mathcal{M}_\alpha}, \qquad \alpha \in \Lambda^*/\Lambda. \tag{8.4.4}$$

Further, consider the $\mathfrak{h}$ valued operators

$$X = \sum_{i=1}^{\dim\Lambda^\perp} \mu^i \otimes \mu_i, \qquad \log_s K = \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \otimes \log_s K_{\nu_j}, \tag{8.4.5}$$

which define maps $\mathcal{M} \to \mathfrak{h}_{\mathbb{C}} \otimes \mathcal{M}$ by the action

$$X|_{\mathcal{M}_\alpha} = \sum_{i=1}^{\dim\Lambda^\perp} \mu^i \langle \mu_i, s(\alpha)\rangle \otimes \mathrm{id}_{\mathcal{M}_\alpha}, \qquad \log_s K|_{\mathcal{M}_\alpha} = \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \big\langle \nu_j, s(\alpha)\big\rangle \otimes \mathrm{id}_{\mathcal{M}_\alpha}. \tag{8.4.6}$$

So for any function $f : (\Lambda^*)^n \to \mathbb{C}^\times$, $n \in \mathbb{N}$ and $\alpha_1, \ldots, \alpha_n \in \Lambda^*$, we define the linear operator

$$f(X_1 + \log_s K_1, \ldots, X_n + \log_s K_n)|_{\mathcal{M}_{\alpha_1} \otimes \cdots \otimes \mathcal{M}_{\alpha_n}} = f(\alpha_1, \ldots, \alpha_n)\mathrm{id}_{\mathcal{M}_{\alpha_1} \otimes \cdots \otimes \mathcal{M}_{\alpha_n}}, \tag{8.4.7}$$

Then we define the following ribbon element $r : \mathcal{M} \to \mathcal{M}$, R-matrix $R : \mathcal{M} \otimes \mathcal{N} \to \mathcal{M} \otimes \mathcal{N}$ and coassociator $\Phi : \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P}) \to \mathcal{M} \otimes (\mathcal{N} \otimes \mathcal{P})$, whose names will be justified by Theorem 8.4.1.

$$\begin{aligned} r &= \exp\big[-\pi\mathrm{i}\big\langle X_1 + \log_s K_1, X_1 + \log_s K_1 - 2s(\xi)\big\rangle\big], \qquad R = \exp\big[\mathrm{i}\pi\big\langle X_1 + \log_s K_1, X_2 + \log_s K_2\big\rangle\big], \\ \Phi &= \exp\big[\mathrm{i}\pi\big\langle X_1 + \log_s K_1, \log_s K_2 + \log_s K_3 - \log_s K_{2\otimes 3}\big\rangle\big] \\ &\quad \cdot \frac{\varepsilon\big(k(X_1 + \log_s K_1, X_2 + \log_s K_2), k(X_1 + \log_s K_1 + X_2 + \log_s K_2, X_3 + \log_s K_3)\big)}{\varepsilon\big(k(X_2 + \log_s K_2, X_3 + \log_s K_3), k(X_1 + \log_s K_1, X_2 + \log_s K_2 + X_3 + \log_s K_3)\big)}, \end{aligned} \tag{8.4.8}$$

where $\log_s(K)_{2\otimes 3}$ is to be evaluated after the $\mathcal{N} \otimes \mathcal{P}$ tensor product has been evaluated, that is, $\log_s(K_\nu)|_{\mathcal{N}_\alpha \otimes \mathcal{P}_\gamma} = \langle \nu, s(\alpha + \gamma)\rangle \mathrm{id}_{\mathcal{N}_\alpha \otimes \mathcal{P}_\gamma}$.

Theorem 8.4.1. *Let* $\mathsf{HM}(\Psi)$ *be the category of* H_Λ *modules constructed from* $\mathrm{Vect}(\Psi)$*, that is, the objects are the pairs* $(\mathcal{M}, \rho_{\mathcal{M}})$*,* $\mathcal{M} \in \mathrm{Vect}(\Psi)$ *and the morphism are* H_Λ *module homomorphisms (these are precisely the morphisms of* $\mathrm{Vect}(\Psi)$*). Define a tensor functor on* $\mathsf{HM}(\Psi)$ *by*

$$(\mathcal{M}, \rho_{\mathcal{M}}) \otimes (\mathcal{N}, \rho_{\mathcal{N}}) = (\mathcal{M} \otimes \mathcal{N}, (\rho_{\mathcal{M}} \otimes \rho_{\mathcal{N}}) \circ \Delta), \qquad \mathcal{M}, \mathcal{N} \in \mathrm{Vect}(\Psi), \tag{8.4.9}$$

where the tensor product of morphisms is the standard tensor product of linear maps.

1. *The ribbon element, R-matrix and coassociator given in* (8.4.8) *equip* $\mathsf{HM}(\Psi)$ *with the structure of a ribbon Grothendieck-Verdier category with twist* θ*, braiding* c *and associator* α *respectively given by*

$$\theta_{\mathcal{M}} = r^{-1}, \qquad c_{\mathcal{M},\mathcal{N}} = P \circ R, \qquad \alpha_{\mathcal{M},\mathcal{N},\mathcal{P}} = \alpha^{vec} \circ \Phi, \tag{8.4.10}$$

where P *is the tensor flip of vector spaces and* α^{vec} *is the standard associator of vector spaces. All future references to* $\mathsf{HM}(\Psi)$ *will include the ribbon Grothendieck-Verdier structure given here.*

2. *Let* $F : \mathrm{Vect}(\Psi) \to \mathsf{HM}(\Psi)$ *be the functor which equips the vector space* $\mathcal{M} \in \mathrm{Vect}(\Psi)$ *with the* H_Λ *action defined by the representation* $\rho_{\mathcal{M}}$*, that is,*

$$F : \mathcal{M} \mapsto (\mathcal{M}, \rho_{\mathcal{M}}), \tag{8.4.11}$$

and which is the identity on morphisms. Let the isomorphism $\varphi_0 : (\mathbb{C}_0, \rho_{\mathbb{C}_0}) \to F(\mathbb{C}_0) = \mathbb{C}_0$ *be the identity map* $\mathrm{id}_{\mathbb{C}_0}$ *on the tensor unit* $\mathbb{C}_0$*. Let* $\varphi_2 : F(-) \otimes F(-) \to F(- \otimes -)$ *be the natural transformation given by*

$$\varphi_2((\mathcal{M}, \rho_{\mathcal{M}}), (\mathcal{N}, \rho_{\mathcal{N}})) = \mathrm{id}_{\mathcal{M} \otimes \mathcal{N}}. \tag{8.4.12}$$

Then $(F, \varphi_0, \varphi_2)$ *is a ribbon Grothendieck-Verdier equivalence.*

Proof. The proposed tensor product functor $\otimes$ is well defined, because H_Λ is a Hopf algebra. We can therefore use the proposed tensor functor F to map the twist, braiding and associativity isomorphisms from $\mathrm{Vect}(\Psi)$ to $\mathsf{HM}(\Psi)$. If the images of these structure morphisms match the evaluations of the formal operators (8.4.8), it then automatically follows that these operators satisfy the defining properties of ribbon elements, R-matrices and coassociators and that $(F, \varphi_0, \varphi_2)$ is an equivalence of ribbon Grothendieck-Verdier categories. Let $\eta, \kappa, \tau \in \Lambda^*/\Lambda$ and $\mathcal{M}, \mathcal{N}, \mathcal{P} \in \mathsf{HM}(\Psi)$, then

$$\begin{aligned} r|_{\mathcal{M}_\eta} &= \exp\left[-\pi \mathrm{i}\left\langle \sum_{i=1}^{\dim \Lambda^\perp} \mu^i \langle \mu_i, s(\eta)\rangle + \sum_{j=1}^{\mathrm{rk}\Lambda} \nu^j \langle \nu_j, s(\eta)\rangle, \sum_{k=1}^{\dim \Lambda} \mu^k \langle \mu_k, s(\eta)\rangle + \sum_{l=1}^{\mathrm{rk}\Lambda} \nu^l \langle \nu_l, s(\eta)\rangle - 2s(\xi)\right\rangle\right] \\ &= \mathrm{e}^{-\pi \mathrm{i}\langle s(\eta), s(\eta) - 2s(\xi)\rangle} \mathrm{id}_{\mathcal{M}_\eta} \end{aligned} \tag{8.4.13}$$

and similarly,

$$\begin{aligned} R|_{\mathcal{M}_\eta \otimes \mathcal{N}_\kappa} &= \mathrm{e}^{\mathrm{i}\pi\langle s(\eta), s(\kappa)\rangle} \mathrm{id}_{\mathcal{M}_\eta \otimes \mathcal{M}_\kappa}, \\ \Phi|_{\mathcal{M}_\eta \otimes \mathcal{N}_\kappa \otimes \mathcal{P}_\tau} &= (-1)^{\langle s(\eta), k(\kappa,\tau)\rangle} \frac{\varepsilon(k(\eta,\kappa), k(\eta+\kappa,\tau))}{\varepsilon(k(\kappa,\tau), k(\eta,\kappa+\tau))} \mathrm{id}_{\mathcal{M}_\eta \otimes \mathcal{N}_\kappa \otimes \mathcal{P}_\tau}. \end{aligned} \tag{8.4.14}$$

Therefore the ribbon element, R matrix and coassociator evaluate exactly as the twist, braiding isomorphisms and associativity isomorphism in $\mathrm{Vect}(\Psi)$ do and the theorem follows. The equivalence of the Grothendieck-Verdier structures then follows by noting that $\mathbb{C}_{2\xi}$ is the dualising object for both categories. ∎

Example. Recall the example from the end of Section 8.1.

1. If the lattice Λ is full rank, then $\Lambda^\perp$ is trivial and Λ^*/Λ is a finite group . In this case the lattice Hopf algebra is just the group algebra $\mathbb{C}[\Lambda^*]$.

2. If Λ is the trivial lattice, then $\Lambda^\perp = \Lambda^* = \mathfrak{h}$ and in this case the lattice Hopf algebra is the universal enveloping algebra $\mathsf{U}(\Lambda^\perp)$ of the complexification $\Lambda^\perp_{\mathbb{C}}$.

3. Finally, in the half rank example $\Lambda^* \cong \mathbb{Z}\times\mathbb{R}$ and so the lattice Hopf algebra is a tensor product of the $\mathbb{Z}$-group algebra and the universal enveloping algebra of the abelian one-dimensional Lie algebra $\mathfrak{gl}(1)$. Further, the modules defined by the action (8.4.3) descend to modules over the group $\mathbb{Z}\times U(1)$. Explicitly we can give the lattice Hopf algebra as

$$\begin{aligned} H_\Lambda &= \mathbb{C}[X,K,K^{-1}], & K^{\pm 1}K^{\mp 1} &= 1, \\ \Delta(X) &= X\otimes 1 + 1\otimes X, & S(X) &= -X, \\ \Delta(K^{\pm 1}) &= K^{\pm 1}\otimes K^{\pm 1}, & S(K^{\pm 1}) &= K^{\mp 1}. \end{aligned} \tag{8.4.15}$$

The action on the module $\mathbb{C}_{x_1,x_2+\mathbb{Z}}$ is then given by

$$\rho_{\mathbb{C}_{x_1,x_2+\mathbb{Z}}}(X) = x_1 \mathrm{id}_{\mathbb{C}_{x_1,x_2+\mathbb{Z}}}, \qquad \rho_{\mathbb{C}_{x_1,x_2+\mathbb{Z}}}(K) = \mathsf{e}^{2\pi\mathsf{i}\tilde{x}_2}\mathrm{id}_{\mathbb{C}_{x_1,x_2+\mathbb{Z}}}. \tag{8.4.16}$$

The ribbon element, R-matrix and coassociator for this choice of data are then

$$\begin{aligned} &r = \exp\big(-2\mathsf{i}\pi\big(X_1 \log_s K_1 - \log_s K_1\big)\big), \qquad R = \exp[\mathsf{i}\pi(\log K_1\otimes X_2 + X_1\otimes \log K_2)], \\ &\Phi = \exp[\mathsf{i}\pi(X_1\otimes\log K_2\otimes \mathrm{id} + X_1\otimes\mathrm{id}\otimes\log K_3 - X_1\otimes \log K_{2\otimes 3})], \end{aligned} \tag{8.4.17}$$

where $\log K$ acts as $\tilde{x}_2$ on $\mathbb{C}_{x_1,x_2+\mathbb{Z}}$.

8.5 Simple Current Extensions

The process of extending a vertex operator algebra by (tensor powers of) modules whose tensor product is invertible (such extentions are called **simple current extentions**) has a long history in the conformal field theory and vertex operator algebra literature for both finite order extensions [37] and more recently infinite ones [23, 38]. At a categorical level, extensions (not necessarily the simple current type) correspond to algebra objects in a braided monoidal category [21, 39, 40]. In particular, algebra objects in categories of graded vectors spaces and their connections to vertex operator algebras and conformal field theory have been studied in [41]. Let $\Psi_i = (\mathfrak{h}, \langle -,-\rangle, \Lambda_i, \xi_i)$ for $i = 1,2$ be two sets of bosonic lattice data. Then by Theorems 8.3.7 and 8.4.1 we have two triples of ribbon Grothendieck-Verdier equivalent categories

$$\mathrm{Vect}(\Psi_i) \;\cong\; \mathsf{VM}(\Psi_i) \;\cong\; \mathsf{HM}(\Psi_i), \quad i = 1,2. \tag{8.5.1}$$

We will show that if $\Lambda_1 \subset \Lambda_2$ and $\xi_1 \subset \xi_2$, we can find an algebra object $\mathcal{A}$ in the direct sum completion $\mathrm{Vect}(\Psi_1)_\oplus$ such that the module category for $\mathcal{A}$ is equivalent to $\mathrm{Vect}(\Psi_2)$.

Transferring the algebra object $\mathcal{A}$ to $\mathsf{VM}(\Psi_1)_\oplus$ then yields the simple current extension of $\mathsf{V}\left(\widetilde{\xi},\Lambda_1\right)$ to $\mathsf{V}\left(\widetilde{\xi},\Lambda_2\right)$, if we choose the same representative $\widetilde{\xi}$ for both ξ_1 and ξ_2. Finally, we will pose the problem of constructing H_{Λ_2} from H_{Λ_1}.

Proposition 8.5.1. *Let Ψ_1, Ψ_2 be two sets of bosonic lattice data, satisfying $\Lambda_1 \subset \Lambda_2$ and $\xi_1 \subset \xi_2$. Let $\sigma : \Lambda_2/\Lambda_1 \otimes \Lambda_2/\Lambda_1 \to \mathbb{C}^\times$ satisfy*

$$\begin{aligned}&\sigma(\lambda,\Lambda_1) = \sigma(\Lambda_1,\lambda) = 1, \qquad \sigma(\lambda_1,\lambda_2)\sigma(\lambda_2,\lambda_1)^{-1} = \Omega(\lambda_1,\lambda_2),\\ &\sigma(\lambda_2,\lambda_3)\sigma(\lambda_1+\lambda_2,\lambda_3)^{-1}\sigma(\lambda_1,\lambda_2+\lambda_3)\sigma(\lambda_1,\lambda_2)^{-1} = F(\lambda_1,\lambda_2,\lambda_3),\end{aligned} \tag{8.5.2}$$

where (F,Ω) is the abelian 3-cocycle associated to Ψ_1. Taking $\Lambda_2/\Lambda_1 \subset \Lambda_1^*/\Lambda_1$ as a subgroup, we define the triple $(\mathcal{A},\ \mu : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A},\ \eta : \mathbb{C}_{\Lambda_1} \to \mathcal{A})$ by

$$\mathcal{A} = \bigoplus_{\lambda\in\Lambda_2/\Lambda_1} \mathbb{C}_\lambda, \qquad \mu|_{\mathbb{C}_{\lambda_1}\otimes\mathbb{C}_{\lambda_2}} = \sigma(\lambda_1,\lambda_2)J_{\lambda_1,\lambda_2}, \qquad \eta = \mathrm{id}_{\mathbb{C}_{\Lambda_1}}, \tag{8.5.3}$$

where J_{λ_1,λ_2} is the canonical identification $\mathbb{C}_{\lambda_1}\otimes\mathbb{C}_{\lambda_2} \cong \mathbb{C}_{\lambda_1+\lambda_2}$. Then

1. *$(\mathcal{A},\mu,\eta)$ defines an associative commutative algebra with trivial twist and a unique unit (that is,* $\dim\mathrm{Hom}(\mathbb{C}_0,\mathcal{A}) = 1$*, also called the haploid condition), in* $\mathrm{Vect}(\Psi_1)_\oplus$.

2. *The category of local $\mathcal{A}$-modules* $\mathcal{A}\text{-Mod}^{\mathrm{loc}}(\mathrm{Vect}(\Psi_1)_\oplus)$ *(also called dyslectic modules) is a ribbon Grothendieck-Verdier category and is equivalent to* $\mathrm{Vect}(\Psi_2)$. *Thus the images of $\mathcal{A}$ under the functors G and F in Theorems 8.3.7 and 8.4.1 define equivalent algebras $\mathcal{A}_{\mathsf{V}} = G(\mathcal{A})$ and $\mathcal{A}_{\mathsf{H}} = F(\mathcal{A})$ in $\mathsf{VM}(\Psi_1)_\oplus$ and $\mathsf{HM}(\Psi_1)_\oplus$, respectively. Hence we have the sequence*

$$\mathrm{Vect}(\Psi_2) \cong \mathcal{A}_{\mathsf{V}}\text{-Mod}^{\mathrm{loc}}\big(\mathsf{VM}(\Psi_1)_\oplus\big) \cong \mathsf{VM}(\Psi_2) \cong \mathcal{A}_{\mathsf{H}}\text{-Mod}^{\mathrm{loc}}\big(\mathsf{HM}(\Psi_1)_\oplus\big) \cong \mathsf{HM}(\Psi_2). \tag{8.5.4}$$

   of ribbon Grothendieck-Verdier equivalences.

3. *Let $\widetilde{\xi} \in \xi_1 \subset \xi_2$ be a choice of representative for both ξ_1 and ξ_2. The algebra object $\mathcal{A}_{\mathsf{V}} = G(\mathcal{A})$ admits the structure of a vertex operator algebra via the field map $Y = G^T_\mu$, with vacuum and conformal vectors given by the images of the vacuum and conformal vectors in $\mathsf{V}\left(\widetilde{\xi},\Lambda_1\right)$ under the tensor structure map φ_0. Further, this vertex operator algebra is isomorphic to $\mathsf{V}\left(\widetilde{\xi},\Lambda_2\right)$.*

Proof. Denote by s_i the respective sections of the bosonic lattice data Ψ_i. The 2-cocycles k and ε shall only be needed for Ψ_1 and will hence not be given an index, to reduce notational clutter.

1. The conditions (8.5.2) are equivalent to the constraints imposed on μ and η by the definition of an associative unital commutative algebra [1][Definition 7.8.1]. Unitality is implied by the first relation, commutativity by the second and associativity by the third.

   The haploid or uniqueness of the unit property follows from $\mathcal{A}$ containing $\mathbb{C}_{\Lambda_1}$ only once as a direct summand and $\dim \mathrm{Hom}\left(\mathbb{C}_{\Lambda_1}, \mathbb{C}_{\Lambda_1}\right) = 1$.

   The algebra having trivial twist follows by direct computation. On each summand of $\mathcal{A}$, the twist evaluates to $\theta(\lambda) = \mathrm{e}^{\mathrm{i}\pi\langle s_1(\lambda), s_1(\lambda) - 2s_1(\xi_1)\rangle}$, $\lambda_2 \in \Lambda_2/\Lambda_1$. Since Λ_2 is even, $s_1(\lambda) \in \Lambda_2$ and $s_1(\xi_1) \in \xi_2 \subset \Lambda_2^*$, we have $\langle s_1(\lambda), s_1(\lambda)\rangle$, $2\langle s_1(\lambda), s_1(\xi_2)\rangle \in 2\mathbb{Z}$ and hence the twist is trivial.

2. Let $\mathcal{A}$-Mod(Vect$(\Psi_1)_\oplus$) be the category of all $\mathcal{A}$-modules in Vect$(\Psi_1)_\oplus$. Combining [39, Theorem 1.6], which asserts that induction and restriction are adjoint, exact and injective on morphisms, and that induction is a tensor functor with the semisimplicity of Vect$(\Psi_1)_\oplus$, we can quickly deduce that $\mathcal{A}$-Mod(Vect$(\Psi_1)_\oplus$) is also semisimple and that every simple object in $\mathcal{A}$-Mod(Vect$(\Psi_1)_\oplus$) is the induction of a simple object in Vect(Ψ_1). We denote the simple modules induced from the $\mathbb{C}_\alpha$, $\alpha \in \Lambda_1^*/\Lambda_1$ by

$$\mathcal{N}_\alpha = \mathcal{A} \otimes \mathbb{C}_\alpha \cong \bigoplus_{\lambda \in \Lambda_2/\Lambda_1} \mathbb{C}_{\lambda+\alpha}. \tag{8.5.5}$$

   Let $\mathcal{A}$-Mod$^{\mathrm{loc}}$(Vect(Ψ_1)) be the full subcategory of local modules, that is, all objects which have trivial double braiding with the algebra $\mathcal{A}$. For one of the $\mathcal{N}_\alpha$ above this means that for all $\lambda \in \Lambda_2/\Lambda_1$, we require that

$$\Omega(\lambda,\alpha)\Omega(\alpha,\lambda) = \mathrm{e}^{2\pi\mathrm{i}\langle s_1(\lambda), s_1(\alpha)\rangle} = 1, \quad \text{or equivalently } \langle s_1(\lambda), s_1(\alpha)\rangle \in \mathbb{Z}. \tag{8.5.6}$$

   By assumption $s_1(\alpha) \in \Lambda_1^*$. If $s_1(\alpha) \in \Lambda_2^*$, then the above condition is satisfied for all $\lambda \in \Lambda_2/\Lambda_1$. Conversely, if $s_1(\alpha) \notin \Lambda_2^*$ then there exits a $\mu \in \Lambda_2$ such that $\langle \mu, s_1(\alpha)\rangle \notin \mathbb{Z}$. But then $s_1(\alpha)$ would pair non-integrally with every representative of the Λ_1 coset of μ and hence the above condition cannot be satisfied. Therefore $\alpha \in \Lambda_2^*/\Lambda_1$ exhausts all labels for simple objects in $\mathcal{A}$-Mod$^{\mathrm{loc}}$(Vect(Ψ_1)). Two induced simple modules $\mathcal{N}_\alpha$, $\mathcal{N}_\beta$ are isomorphic if and only if their labels differ by a coset in Λ_2/Λ_1. Therefore the isomorphism classes of simple modules are labelled by the elements of the quotient group $\left(\Lambda_2^*/\Lambda_1\right)/\left(\Lambda_2/\Lambda_1\right) \cong \Lambda_2^*/\Lambda_2$. This implies that $\mathcal{A}$-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) and Vect(Ψ_2) are equivalent as abelian categories. By [39, Theorem 1.10] or [42, Theorem 2.5], $\mathcal{A}$-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) is braided monoidal with the braiding descending from Vect(Ψ_1). Further, from [39, Theorem 1.6] one can deduce that $\mathcal{N}_\alpha \otimes_{\mathcal{A}} \mathcal{N}_\beta \cong \mathcal{N}_{\alpha+\beta}$. Thus $\mathcal{A}$-Mod$^{\mathrm{loc}}$(Vect$(\Psi_1)_\oplus$) also has the same tensor product as Vect(Ψ_2), hence the braiding and associativity isomorphisms are characterised by abelian 3-cocycles for the group Λ_2^*/Λ_2. To conclude equivalence as braided monoidal categories it is therefore sufficient for the trace of the abelian 3-cocycles of $\mathcal{A}\text{-Mod}^{\text{loc}}(\text{Vect}(\Psi_1)_\oplus)$ and $\text{Vect}(\Psi_2)$ to be equal. Let Ω_i, $i = 1,2$ be the respective braidings associated to Ψ_i, then for $\alpha \in \Lambda_2^*/\Lambda_1$ we need to compare $\Omega_1(\alpha,\alpha)$ and $\Omega_2(\alpha+\Lambda_2,\alpha+\Lambda_2)$. Recall that $s_1(\alpha) \in \Lambda_2^*$ and hence $s_2(\alpha+\Lambda_2) - s_1(\alpha) = \kappa \in \Lambda_2$, so

$$\Omega_2(\alpha+\Lambda_2,\alpha+\Lambda_2) = \mathsf{e}^{\mathsf{i}\pi\langle s_2(\alpha+\Lambda_2),s_2(\alpha+\Lambda_2)\rangle} = \mathsf{e}^{\mathsf{i}\pi\langle s_1(\alpha)+\kappa,s_1(\alpha)+\kappa\rangle} = \mathsf{e}^{\mathsf{i}\pi\langle s_1(\alpha),s_1(\alpha)\rangle} = \Omega_1(\alpha,\alpha), \tag{8.5.7}$$

where the third equality follows from Λ_2 being even. Thus $\mathcal{A}\text{-Mod}^{\text{loc}}(\text{Vect}(\Psi_1)_\oplus)$ and $\text{Vect}(\Psi_2)$ are equivalent as braided monoidal categories. Grothendieck-Verdier equivalence follows by noting that the induction of the dualising object $\mathcal{N}_{\xi_1}$ has $\xi_1 + \Lambda_2 = \xi_2$ as its label and is hence equivalent to the dualising object of $\text{Vect}(\Psi_2)$. Finally, ribbon equivalence follows by comparing the twist scalars θ_1,θ_2 in both categories. We denote $s_2(\xi_2) - s_1(\xi_1) = \tau \in \Lambda_2$ and consider for any $\alpha \in \Lambda_2^*/\Lambda_1$

$$\begin{aligned}\theta_2(\alpha+\Lambda_2) &= \mathsf{e}^{\mathsf{i}\pi\langle s_2(\alpha+\Lambda_2),s_2(\alpha+\Lambda_2)-2s_2(\xi_2)\rangle} = \mathsf{e}^{\mathsf{i}\pi\langle s_1(\alpha)+\kappa,s_1(\alpha)+\kappa-2s_1(\xi_1)-2\tau\rangle}\\ &= \mathsf{e}^{\mathsf{i}\pi\langle s_1(\alpha),s_1(\alpha)-2s_1(\xi_1)\rangle} = \theta_1(\alpha),\end{aligned} \tag{8.5.8}$$

where we have again used the Λ_2 is even. Thus $\mathcal{A}\text{-Mod}^{\text{loc}}(\text{Vect}(\Psi_1)_\oplus)$ and $\text{Vect}(\Psi_2)$ are ribbon Grothendieck-Verdier equivalent.

3. As a module over the Heisenberg algebra $\mathcal{A}_{\mathsf{V}}$ decomposes as follows.

$$\mathcal{A}_{\mathsf{V}} = G(\mathcal{A}) = \bigoplus_{\lambda\in\Lambda_2/\Lambda_1} G(\mathcal{M}_\lambda) \cong \bigoplus_{\lambda\in\Lambda_2/\Lambda_1} \mathbb{F}_\lambda \cong \bigoplus_{\lambda\in\Lambda_2/\Lambda_1}\bigoplus_{\alpha\in\Lambda_1} \mathcal{F}_{s_1(\lambda)+\alpha} = \bigoplus_{\lambda\in\Lambda_2} \mathcal{F}_\lambda, \tag{8.5.9}$$

which is isomorphic to the vector space on which $\mathsf{V}\left(\widetilde{\xi},\Lambda_2\right)$ is defined. We need to verify that $Y = G_\mu^T$ is indeed a field map. We show this by comparing G_μ^T to the field map of $\mathsf{V}\left(\widetilde{\xi},\Lambda_2\right)$. Consider $\lambda_1,\lambda_2 \in \Lambda_2/\Lambda_1$, $\alpha_1,\alpha_2 \in \Lambda_1$, then $G_\mu^T|_{(G(\mathbb{C}_{\lambda_1})\otimes \mathsf{e}^{\alpha_1})\otimes(G(\mathbb{C}_{\lambda_2})\otimes\mathsf{e}^{\alpha_2})}$ is essentially an untwisted vertex operator of the form (8.3.11) up to a scaling factor of

$$(-1)^{\langle s_1(\lambda_1),\alpha_2\rangle}\varepsilon(\alpha_1,\alpha_2)\varepsilon(\alpha_1+\alpha_2,k(\lambda_1,\lambda_2))\sigma(\lambda_1,\lambda_2). \tag{8.5.10}$$

Therefore G_μ^T defines a vertex operator algebra structure if and only if

$$\begin{aligned}\tau(\gamma,\delta) = (-1)^{\langle s_1(\gamma+\Lambda_1),\delta-s_1(\delta+\Lambda_1)\rangle}&\varepsilon(\gamma-s_1(\gamma+\Lambda_1),\delta-s_1(\delta+\Lambda_1))\\ &\varepsilon(\gamma-s_1(\gamma+\Lambda_1)+\delta-s_1(\delta+\Lambda_1),k(\gamma+\Lambda_1,\delta+\Lambda_1))\sigma(\gamma+\Lambda_1,\delta+\Lambda_1),\end{aligned} \tag{8.5.11}$$

for $\gamma, \delta \in \Lambda_2$, satisfies the 2-cocycle conditions of (8.1.2) for Λ_2. Since all 2-cocycles for Λ_2 are cohomologous, the vertex operator algebra structure defined by G_μ^T is isomorphic to that of $\mathsf{V}\left(\widetilde{\xi}, \Lambda_2\right)$.

∎

— Chapter 9 —

Bosonic Ghosts

"Back off, man! I'm a scientist!"

— Peter Venkman (Bill Murray), *Ghostbusters*

The bosonic ghost vertex algebra is a simple example of a conformal field theory which is neither rational, nor C_2-cofinite. We identify a natural module choice of module category, which we prove is closed under fusion and rigid. Further, we classify all indecomposable modules and compute all fusion products. This chapter presents the bulk of the content from [29].

9.1 Bosonic ghost vertex algebra

In this section we introduce the bosonic ghost vertex algebra, along with its gradings and automorphisms. We will go on to define the module category which will be the focus of this chapter, and introduce useful tools for the classification of modules and calculation of fusion products, including two free field realisations. Note that we will make specific choices of conformal structure for all vertex algebras considered in this chapter and so will not distinguish between vertex algebras and vertex operator algebras.

The bosonic ghost vertex algebra (also called $\beta\gamma$ ghosts) is closely related to the Weyl algebra. Their defining relations resemble each other and the Zhu algebra of the bosonic ghosts is isomorphic to the Weyl algebra. The bosonic ghosts are therefore also often referred to as the Weyl vertex algebra. Due to these connections, we first introduce the Weyl algebra and its modules before going on to consider the bosonic ghosts.

Definition 9.1.1. The (rank 1) **Weyl** algebra $\mathfrak{A}$ is the unique unital associative algebra with two generators p, q, subject to the relations

$$[p, q] = 1, \tag{9.1.1}$$

and no additional relations beyond those required by the axioms of an associative algebra. The grading operator is the element $N = qp$.

Definition 9.1.2. We define the following indecomposable $\mathfrak{A}$-modules:

1. $\mathbb{C}[x]$, where p acts as $\partial/\partial x$ and q acts as x. Denote this module by $\overline{\mathcal{V}}$.

2. $\mathbb{C}[x]$, where p acts as x and q acts as $-\partial/\partial x$. Denote this module by $\mathsf{c}\overline{\mathcal{V}}$.

3. $\mathbb{C}[x,x^{-1}]x^{\lambda}$, $\lambda \in \mathbb{C} \setminus \mathbb{Z}$, where p acts as $\partial/\partial x$ and q acts as x. Note that shifting λ by an integer yields an isomorphic module. Denote the mutually inequivalent isomorphism classes of these modules by $\overline{\mathcal{W}_{\mu}}$, where $\mu \in \mathbb{C}/\mathbb{Z}$, $\mu \neq \mathbb{Z}$ and $\lambda \in \mu$.

4. $\mathbb{C}[x,x^{-1}]$, where p acts as $\partial/\partial x$ and q acts as x. Denote this module by $\overline{\mathcal{W}_0^+}$. This module is uniquely characterised by the non-split exact sequence
$$0 \longrightarrow \overline{\mathcal{V}} \longrightarrow \overline{\mathcal{W}_0^+} \longrightarrow \mathsf{c}\overline{\mathcal{V}} \longrightarrow 0. \tag{9.1.2}$$

5. $\mathbb{C}[x,x^{-1}]$, where p acts as x and q acts as $-\partial/\partial x$. Denote this module by $\overline{\mathcal{W}_0^-}$. This module is uniquely characterised by the non-split exact sequence
$$0 \longrightarrow \mathsf{c}\overline{\mathcal{V}} \longrightarrow \overline{\mathcal{W}_0^-} \longrightarrow \overline{\mathcal{V}} \longrightarrow 0. \tag{9.1.3}$$

A module on which $N = qp$ acts semisimply is called a **weight module**. Note that N acts semisimply on all modules above.

Proposition 9.1.3 (Block [43])**.** *Any simple $\mathfrak{A}$-module on which N acts semisimply is isomorphic to one of those listed in Definition 9.1.2, Parts 1 – 3.*

Definition 9.1.4. The **bosonic ghost vertex algebra** G is the unique vertex algebra strongly generated by two fields β, γ, subject to the defining operator product expansions
$$\gamma(z)\beta(w) \sim \frac{1}{z-w}, \qquad \beta(z)\beta(w) \sim \gamma(z)\gamma(w) \sim 0, \tag{9.1.4}$$
and no additional relations beyond those required by vertex algebra axioms.

The bosonic ghost vertex algebra admits a one-parameter family of conformal structures. Here we choose the Virasoro field (or energy momentum tensor) to be

$$T(z) = -{:}\,\beta(z)\partial\gamma(z){:}, \tag{9.1.5}$$

thus determining the central charge to be $c = 2$ and the conformal weights of β and γ to be 1 and 0, respectively. The bosonic ghost fields can thus be expanded as formal power series with the mode indexing chosen to reflect the conformal weights.

$$\beta(z) = \sum_{n\in\mathbb{Z}} \beta_n z^{-n-1}, \qquad \gamma(z) = \sum_{n\in\mathbb{Z}} \gamma_n z^{-n}. \tag{9.1.6}$$

The operator product expansions of β and γ fields imply that their modes generate the **bosonic ghost Lie algebra** $\mathfrak{G}$ satisfying the Lie brackets

$$[\gamma_m, \beta_n] = \delta_{m+n,0}\mathbf{1}, \qquad [\beta_m, \beta_n] = [\gamma_m, \gamma_n] = 0, \qquad m, n \in \mathbb{Z}, \tag{9.1.7}$$

where $\mathbf{1}$ is central and acts as the identity on any G-module, since it corresponds to the identity (or vacuum) field.

Within G there is a rank 1 Heisenberg vertex algebra generated by the field

$$J(z) = {:}\,\beta(z)\gamma(z){:}. \tag{9.1.8}$$

A quick calculation reveals that J is a free boson of Lorentzian signature, not a conformal primary, and that J defines a grading on β and γ called ghost weight (or ghost number), that is,

$$\begin{aligned} J(z)J(w) &\sim \frac{-1}{(z-w)^2}, & T(z)J(w) &\sim \frac{-1}{(z-w)^3} + \frac{J(w)}{(z-w)^2} + \frac{\partial J(w)}{z-w}, \\ J(z)\beta(w) &\sim \frac{\beta(w)}{z-w}, & J(z)\gamma(w) &\sim \frac{-\gamma(w)}{z-w}. \end{aligned} \tag{9.1.9}$$

Note that for the distinguished elements β, γ, J, and T we suppress the field map symbol $Y : \mathsf{G} \to \mathsf{G}[\![z, z^{-1}]\!]$. For generic elements $A \in \mathsf{G}$ we will use both $Y(A, z)$ and $A(z)$ to denote the field corresponding to A, depending on what is easier to read in the given context.

We make frequent use of two automorphisms of $\mathfrak{G}$. The first is spectral flow, which acts on the $\mathfrak{G}$ modes as

$$\sigma^\ell \beta_n = \beta_{n-\ell}, \qquad \sigma^\ell \gamma_n = \gamma_{n+\ell}, \qquad \sigma^\ell \mathbf{1} = \mathbf{1}. \tag{9.1.10}$$

The second is conjugation which is given by

$$\mathsf{c}\beta_n = \gamma_n, \qquad \mathsf{c}\gamma_n = -\beta_n, \qquad \mathsf{c}\mathbf{1} = \mathbf{1}. \tag{9.1.11}$$

These automorphisms satisfy the relation

$$\mathsf{c}\sigma^{\ell} = \sigma^{-\ell}\mathsf{c}. \tag{9.1.12}$$

At the level of fields, these automorphisms act as

$$\begin{aligned}
&\sigma^{\ell}\beta(z) = \beta(z)z^{-\ell}, \qquad \sigma^{\ell}\gamma(z) = \gamma(z)z^{\ell}, \qquad \sigma^{\ell}J(z) = J(z) + \ell\mathbf{1}z^{-1},\\
&\sigma^{\ell}T(z) = T(z) - \ell J(z)z^{-1} - \tfrac{1}{2}\ell(\ell-1)\mathbf{1}z^{-2},\\
&\mathsf{c}\beta(z) = \gamma(z), \qquad \mathsf{c}\gamma(z) = -\beta(z), \qquad \mathsf{c}J(z) = -J(z) + \mathbf{1}z^{-1},\\
&\mathsf{c}T(z) = T(z) + \partial J(z) + J(z)z^{-1}.
\end{aligned} \tag{9.1.13}$$

The primary utility of the conjugation and spectral flow automorphisms lies in constructing new modules from known ones through twisting.

Definition 9.1.5. Let $\mathcal{M}$ be a G-module and α an automorphism of $\mathfrak{G}$. The α-twisted module $\alpha\mathcal{M}$ is defined to be $\mathcal{M}$ as a vector space, but with the action of G redefined to be

$$A(z)\cdot_{\alpha} m = \alpha^{-1}(A(z))m, \qquad A \in \mathsf{G},\ m \in \mathcal{M}, \tag{9.1.14}$$

where the action of G on the right-hand side is the original untwisted action of G on $\mathcal{M}$.

Remark. Due to being algebra automorphisms, spectral flow and conjugation twists both define exact covariant functors $\mathcal{M} \mapsto \alpha\mathcal{M}$, on the category of G modules. These α-twisted modules are well defined as vertex algebra modules, as the actions of the automorphisms preserve the integral grading of the modes. Further, the respective ghost and conformal weights $[j,h]$ of a vector m in a G-Module $\mathcal{M}$ transform as follows under conjugation and spectral flow.

$$\begin{aligned}
\sigma^{\ell} &: [j,\ h] \mapsto \left[j-\ell,\ h+\ell j - \tfrac{1}{2}\ell(\ell+1)\right],\\
\mathsf{c} &: [j,\ h] \mapsto [1-j,\ h].
\end{aligned} \tag{9.1.15}$$

Since $\mathsf{c}^2\beta_n = -\beta_n$ and $\mathsf{c}^2\gamma_n = -\gamma_n$, we have $\mathsf{c}^2\mathcal{M} \cong \mathcal{M}$, for any G-module $\mathcal{M}$. We shall later see that spectral flow has infinite order and thus the relations (9.1.12) imply that at the level of the module category spectral flow and conjugation generate the infinite dihedral group.

Theorem 9.1.6. *For any G-modules $\mathcal{M}$ and $\mathcal{N}$, conjugation and spectral flow are compatible with fusion products in the following sense.*

$$\begin{aligned}
\sigma^{\ell}\mathcal{M} \boxtimes \sigma^{m}\mathcal{N} &\cong \sigma^{\ell+m}(\mathcal{M}\boxtimes\mathcal{N}),\\
\mathsf{c}\mathcal{M}\boxtimes\mathsf{c}\mathcal{N} &\cong \mathsf{c}\sigma(\mathcal{M}\boxtimes\mathcal{N}).
\end{aligned} \tag{9.1.16}$$

The behaviour of spectral flow under fusion was proven for vertex algebras with finite dimensional conformal weight spaces in [44, Proposition 2.4]. However, the proof does not rely on this fact, and so we can apply the result to G-modules, as in [45, Proposition 3.1]. The behaviour of conjugation under fusion was proven in [45, Proposition 2.1], where conjugation was denoted by σ and spectral flow by ρ_ℓ. There the automorphism g corresponds to $\sigma^{-1}\mathsf{c} = \mathsf{c}\sigma$ here. These formulae mean that the fusion of modules twisted by spectral flow is determined by the fusion of untwisted modules, a simplification we shall make frequent use of.

9.2 Module category

Every G-module is a $\mathfrak{G}$-module, however, the converse is not true (consider for example the adjoint $\mathfrak{G}$-module, where $\mathfrak{G}$ acts on itself via the Lie bracket). The category of smooth $\mathfrak{G}$-modules consists of precisely those modules which are also G-modules. Such modules are also commonly called weak G-modules and we shall use these terms interchangeably. Unfortunately the category of all smooth modules is at present too unwieldy to analyse and so we must invariably consider some subcategory.

In this section we define the module category, which we believe to be the natural one from the perspective of conformal field theory, because it is compatible with the following two necessary conformal field theoretic conditions.

1. Non-degeneracy of n-point conformal blocks (chiral correlation functions) on the sphere.

2. Well-definedness of conformal blocks at higher genera.

Condition 1 can be reduced to the non-degeneracy of two and three-point conformal blocks. The non-degeneracy of two-point conformal blocks requires the module category to be closed under taking contragredients, while non-degeneracy of three-point conformal blocks requires the module category to be closed under fusion . Conformal blocks at higher genera can be constructed from those on the sphere provided there is a well-defined action of the modular group on characters. Thus Condition 2 requires characters to be well-defined, that is, for all modules to decompose into direct sums of finite dimensional simultaneous generalised J_0 and L_0 eigenspaces. On any simple such module both L_0 and J_0 will act semisimply. Further, the action of J_0 is semisimple on a fusion product if J_0 acts semisimply on both factors of the product. We can therefore restrict ourselves to a category of J_0-semisimple modules without endangering closure under fusion. We cannot, however, assume that L_0 will act semisimply in general.

The main tool for identifying and classifying vertex operator algebra modules is Zhu's algebra. Sadly Zhu's algebra is only sensitive to modules containing vectors annihilated by all positive modes. Any simple such module is a simple module in the category called $\mathscr{R}$ below. We will see that $\mathscr{R}$ is closed under taking the contragredient dual, however, as can be seen later in Section 9.8, category $\mathscr{R}$ is not closed under fusion. Further, it was shown in [46] that the action of the modular group does not close on its characters. Thus a larger category is needed, which will be denoted $\mathscr{F}$ below. It was shown in [46] that the action of the modular group closes on the characters of $\mathscr{F}$ and strong evidence was presented that fusion does as well. We will see in Section 9.8 that category $\mathscr{F}$ is indeed closed under fusion and that it satisfies numerous other nice properties.

The definition of the module categories mentioned above requires the following choice of parabolic triangular decomposition of $\mathfrak{G}$.

$$\mathfrak{G}^{\pm} = \operatorname{span}\{\beta_{\pm n}, \gamma_{\pm n} \ : \ n \geq 1\}, \qquad \mathfrak{G}^0 = \operatorname{span}\{\mathbf{1}, \beta_0, \gamma_0\}. \tag{9.2.1}$$

This decomposition is parabolic, because $\mathfrak{G}^0$ is not abelian and thus not a choice of Cartan subalgebra. The role of the Cartan subalgebra will instead be played by $\operatorname{span}\{\mathbf{1}, J_0\}$.

Definition 9.2.1.

1. Let G-WMod be the category of smooth weight $\mathfrak{G}$-modules, that is the category whose objects are all smooth (or weak) G-modules $\mathcal{M}$ (we follow the conventions of [17] regarding smooth modules) which in addition satisfy that J_0 acts semisimply and whose arrows are all $\mathfrak{G}$-module homomorphisms.

2. Let $\mathscr{R}$ be the full subcategory of G-WMod consisting of those modules $\mathcal{M} \in$ G-WMod satisfying

   - $\mathcal{M}$ is finitely generated,
   - $\mathfrak{G}^+$ acts locally nilpotently, that is, for all $m \in \mathcal{M}$, $\mathsf{U}(\mathfrak{G}^+)m$ is finite dimensional.

3. Let $\mathscr{F}$ be the full subcategory of G-WMod consisting all finite length extensions of arbitrary spectral flows of modules in $\mathscr{R}$ with real J_0 weights.

The $\mathfrak{A}$-modules of Definition 9.1.2 induce to modules in category $\mathscr{R}$. See [47, Section 3.4] for a detailed explanation of this procedure for a general Lie superalgebra.

Definition 9.2.2. Let $\mathcal{M}$ be a $\mathfrak{A}$-module, then we induce $\mathcal{M}$ to a G-module $\operatorname{Ind}\mathcal{M}$ in $\mathscr{R}$ by having $\mathfrak{G}^+$ act trivially on $\mathcal{M}$, β_0 and γ_0 act as $-p$ and q, respectively, and $\mathfrak{G}^-$ act freely. We denote

1. $\mathcal{V} \cong \operatorname{Ind} \overline{\mathcal{V}}$, the vacuum module or bosonic ghost vertex algebra as a module over itself.

2. $\mathsf{c}\mathcal{V} \cong \sigma^{-1}\mathcal{V} \cong \operatorname{Ind} \mathsf{c}\overline{\mathcal{V}}$, the conjugation twist of the vacuum module.

3. $\mathcal{W}_\lambda \cong \operatorname{Ind} \overline{\mathcal{W}_\lambda}$ with $\lambda \in \mathbb{C}/\mathbb{Z}$, $\lambda \neq \mathbb{Z}$.

4. $\mathcal{W}_0^\pm \cong \operatorname{Ind} \overline{\mathcal{W}_0^\pm}$.

Note that due to the simple nature of the $\mathfrak{G}$ commutation relations (9.1.7) $\operatorname{Ind} \mathcal{M}$ is simple whenever $\mathcal{M}$ is, that is, the modules listed in parts 1 – 3 are simple.

Proposition 9.2.3.

1. *Any simple module in $\mathscr{R}$ is isomorphic to one of those listed in Parts 1 – 3 of Definition 9.2.2.*

2. *Any simple module in $\mathscr{F}$ is isomorphic to one of the following mutually inequivalent modules.*
$$\sigma^\ell \mathcal{V}, \qquad \sigma^\ell \mathcal{W}_\lambda, \qquad \ell \in \mathbb{Z},\ \lambda \in \mathbb{R}/\mathbb{Z},\ \lambda \neq \mathbb{Z}. \tag{9.2.2}$$

3. *The conjugation twists of simple modules in $\mathscr{F}$ satisfy*
$$\mathsf{c}\sigma^\ell \mathcal{V} \cong \sigma^{-1-\ell}\mathcal{V}, \qquad \mathsf{c}\sigma^\ell \mathcal{W}_\lambda \cong \sigma^{-\ell}\mathcal{W}_{-\lambda}, \qquad \ell \in \mathbb{Z},\ \lambda \in \mathbb{R}/\mathbb{Z},\ \lambda \neq \mathbb{Z}. \tag{9.2.3}$$

4. *The indecomposable modules $\mathcal{W}_0^\pm$ satisfy the non-split exact sequences*
$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{W}_0^+ \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow 0, \tag{9.2.4a}$$
$$0 \longrightarrow \sigma^{-1}\mathcal{V} \longrightarrow \mathcal{W}_0^- \longrightarrow \mathcal{V} \longrightarrow 0. \tag{9.2.4b}$$

This proposition was originally given in [46, Proposition 1], however, Part 1 is an immediate consequence of Block's classification of simple Weyl modules [43]. We shall show in Proposition 9.5.2 that, up to spectral flow twists, the indecomposable modules $\mathcal{W}_0^\pm$ are the only indecomposable length 2 extensions of spectral flows of the vacuum module. In Section 9.6 we extend the indecomposable modules $\mathcal{W}_0^\pm$ to infinite families of indecomposable modules.

9.3 Contragredient duals

As mentioned above, conformal field theories require their representation categories to be closed under taking contragredient duals. They are also an essential tool for the computation of fusion products using the Huang-Lepowsky-Zhang (HLZ) double dual construction [20, Part IV]. Recall the definition from Proposition 6.1.2.

Definition 9.3.1. Let $\mathcal{M}$ be a weight G-module. The contragredient (or restricted dual) module is defined to be

$$\mathcal{M}' = \bigoplus_{h,j\in\mathbb{C}} \mathrm{Hom}\Big(\mathcal{M}_{[h]}^{(j)}, \mathbb{C}\Big), \quad \mathcal{M}_{[h]}^{(j)} = \{m \in \mathcal{M} \;:\; (J_0 - j)m = 0,\ (L_0 - h)^n m = 0,\ n \gg 0\}, \tag{9.3.1}$$

where the action of G is characterised by

$$\langle Y(A,z)\psi, m\rangle = \langle \psi, Y(A,z)^{\mathrm{opp}} m\rangle, \qquad A \in \mathsf{G},\ \psi \in \mathcal{M}',\ m \in \mathcal{M}, \tag{9.3.2}$$

and where $Y(A,z)^{\mathrm{opp}}$ is given by the formula

$$Y(A,z)^{\mathrm{opp}} = Y\Big(\mathsf{e}^{zL_1}\big(-z^{-2}\big)^{L_0} A, z^{-1}\Big). \tag{9.3.3}$$

Proposition 9.3.2. *The vertex algebra G and its modules have the following properties.*

1. *The modes of the generating fields and the Heisenberg field satisfy*

$$\beta_n^{\mathrm{opp}} = -\beta_{-n}, \qquad \gamma_n^{\mathrm{opp}} = \gamma_{-n}, \qquad J_n^{\mathrm{opp}} = \delta_{n,0} - J_{-n}. \tag{9.3.4}$$

2. *The contragredient duals of spectral flows of the indecomposable modules in Definition 9.2.2 can be identified as*

$$\big(\sigma^\ell \mathcal{V}\big)' \cong \sigma^{-1-\ell}\mathcal{V}, \qquad \big(\sigma^\ell \mathcal{W}_\lambda\big)' \cong \sigma^{-\ell}\mathcal{W}_{-\lambda}, \qquad \big(\sigma^\ell \mathcal{W}_0^\pm\big)' \cong \sigma^{-\ell}\mathcal{W}_0^\pm. \tag{9.3.5}$$

3. *Denote by * the composition of twisting by c and taking the contragredient, then*

$$\big(\sigma^\ell \mathcal{V}\big)^* \cong \sigma^{\ell}\mathcal{V}, \qquad \big(\sigma^\ell \mathcal{W}_\lambda\big)^* \cong \sigma^{\ell}\mathcal{W}_{\lambda}, \qquad \big(\sigma^\ell \mathcal{W}_0^\pm\big)^* \cong \sigma^{\ell}\mathcal{W}_0^\mp. \tag{9.3.6}$$

4. *Let $\mathcal{A}, \mathcal{B} \in \mathscr{F}$ and $\ell \in \mathbb{Z}$, then the homomorphism and first extension groups satisfy*

$$\begin{aligned}\mathrm{Hom}(\mathcal{A},\mathcal{B}) &= \mathrm{Hom}(\mathsf{c}\mathcal{A},\mathsf{c}\mathcal{B}) = \mathrm{Hom}\Big(\sigma^\ell\mathcal{A},\sigma^\ell\mathcal{B}\Big) = \mathrm{Hom}(\mathcal{B}',\mathcal{A}') = \mathrm{Hom}(\mathcal{B}^*,\mathcal{A}^*),\\ \mathrm{Ext}(\mathcal{A},\mathcal{B}) &= \mathrm{Ext}(\mathsf{c}\mathcal{A},\mathsf{c}\mathcal{B}) = \mathrm{Ext}\Big(\sigma^\ell\mathcal{A},\sigma^\ell\mathcal{B}\Big) = \mathrm{Ext}(\mathcal{B}',\mathcal{A}') = \mathrm{Ext}(\mathcal{B}^*,\mathcal{A}^*).\end{aligned} \tag{9.3.7}$$

Proof. Part 1 follows immediately from Definition 9.3.1.

Part 2: Since $\sigma^{\ell}\mathcal{V}$ is simple, $(\sigma^{\ell}\mathcal{V})'$ is too, due to taking duals being an invertible exact contravariant functor. Further, by the action given in Definition 9.3.1 it is easy to see that β_n, $n \geq \ell+1$ and γ_m, $m \geq -\ell$ act locally nilpotently and therefore $(\sigma^{\ell}\mathcal{V})'$ is an object of both $\sigma^{-\ell}\mathcal{R}$ and $\sigma^{-1-\ell}\mathcal{R}$. Thus, $(\sigma^{\ell}\mathcal{V})' \cong \sigma^{-1-\ell}\mathcal{V}$.

Similarly, since $\sigma^{\ell}\mathcal{W}_{\lambda}$ is simple, $(\sigma^{\ell}\mathcal{W}_{\lambda})'$ is too. The modes β_n, $n \geq \ell+1$ and γ_m, $m \geq 1-\ell$ act locally nilpotently and therefore $(\sigma^{\ell}\mathcal{W}_{\lambda})'$ is an object of $\sigma^{-\ell}\mathcal{R}$. Further, for J_0 homogeneous $m \in \sigma^{\ell}\mathcal{W}_{\lambda}$ and $\psi \in (\sigma^{\ell}\mathcal{W}_{\lambda})'$, consider

$$\langle J_0\psi, m\rangle = \langle \psi, (1-J_0)m\rangle. \tag{9.3.8}$$

Thus, $(\sigma^{\ell}\mathcal{W}_{\lambda})' \cong \sigma^{-\ell}\mathcal{W}_{-\lambda}$.

Finally, the duals of $\sigma^{\ell}\mathcal{W}_0^{\pm}$ follow from that fact that the duality functor is exact and contravariant, and by applying it to the exact sequences (9.2.4).

Part 3 follows from composing the formulae of Part 2 with the conjugation twist formulae of Proposition 9.2.3.

Part 4 follows from c, σ and $'$ being exact invertible functors, the first two covariant and the last contravariant. ∎

9.4 Free field realisation

We present two embeddings of G into a rank 1 lattice algebra constructed from a rank 2 Heisenberg vertex algebra (recall the definition of the Heisenberg and lattice vertex algebras in Section 8.3).

Let ψ and θ to denote a basis of a rank 2 lattice $L_{\mathbb{Z}} = \mathrm{span}_{\mathbb{Z}}\{\psi,\theta\}$ with symmetric bilinear lattice form, that is $\langle\psi,\psi\rangle = -\langle\theta,\theta\rangle = 1$ and $(\psi,\theta) = 0$. Let $L = \mathrm{span}_{\mathbb{R}}\{\psi,\theta\}$ be the extension of scalars of $L_{\mathbb{Z}}$ by $\mathbb{R}$, $K = \mathrm{span}_{\mathbb{Z}}\{\psi+\theta\}$ the indefinite rank 1 lattice generated by $\psi+\theta$ and $K^* = \{\lambda \in L \ : \ \langle\lambda,\kappa\rangle \in \mathbb{Z}, \ \forall\kappa \in K\}$.

Let V be the rank 2 Heisenberg vertex algebra with choice of generating fields ψ,θ equipped with defining operator product expansions which correspond to the bilinear form above

$$\psi(z)\psi(w) \sim \frac{1}{(z-w)^2}, \qquad \theta(z)\theta(w) \sim \frac{-1}{(z-w)^2}, \qquad \psi(z)\theta(w) \sim 0. \tag{9.4.1}$$

Recall that the Fock spaces of V are denoted by $\mathcal{F}_{\lambda}$, $\lambda \in L$, where the zero mode of a Heisenberg vertex algebra field $a(z)$, $a \in L$ acts as scalar multiplication by $\langle a,\lambda\rangle$. Let V_K be the lattice vertex algebra extension of V along K. The lattice modules

$$\mathbb{F}_{\Lambda} = \bigoplus_{\lambda\in\Lambda} \mathcal{F}_{\lambda}, \qquad \Lambda \in K^*/K, \tag{9.4.2}$$

are simple modules for V_K. It will occasionally be convenient to label the lattice modules by a representative $\lambda \in \Lambda$ rather than the coset itself, that is $\mathbb{F}_\lambda = \mathbb{F}_\Lambda$. Recall we have the (chiral) vertex operator given by (8.3.11)

$$I_{\lambda,\mu} : \mathcal{F}_\lambda \otimes \mathcal{F}_\mu \to \mathcal{F}_{\lambda+\mu}[\![z, z^{-1}]\!] z^{\langle \lambda,\mu \rangle}, \tag{9.4.3}$$

and let $Y(-,z)$ denote the result of reinterpreting these vertex operators as maps $\mathcal{F}_\mu \to \mathcal{F}_{\mu+\lambda}[\![z,z^{-1}]\!] z^{\langle \lambda,\mu\rangle}$. That is,

$$\mathrm{V}(p|\lambda\rangle, z) = \mathsf{e}^\lambda z^{\lambda_0} \prod_{m\geq 1} \exp\left(\frac{\lambda_{-m}}{m} z^m\right) Y(p|0\rangle, z) \prod_{n \geq 1} \exp\left(-\frac{\lambda_n}{n} z^{-n}\right), \tag{9.4.4}$$

where $Y(p|0\rangle,z)$ is the field associated to $p|0\rangle \in \mathsf{V}$, and where $\mathsf{e}^\lambda \in \mathbb{C}[L]$ is the basis element in the group algebra of L corresponding to $\lambda \in L$ and satisfies the relations

$$[b_n, \mathsf{e}^\lambda] = \delta_{n,0} \langle b, \lambda \rangle \mathsf{e}^\lambda, \qquad \mathsf{e}^\lambda |\mu\rangle = |\lambda + \mu\rangle. \tag{9.4.5}$$

We abbreviate the vertex operator $Y(|\lambda\rangle, z)$ as $Y(\lambda,z)$. Note that our notation differs from conventions common in theoretical physics literature. There, for $a \in L$, $Y(a,z)$ would be denoted by $:\mathsf{e}^{a(z)}:$ and $a(z)$ by $\partial a(z)$, or both by $a(z)$.

Recall that this is exactly the case of the indefinite half rank lattice example at the end of Section 8.1, Section 8.2 and Section 8.3. By Proposition 8.3.6, we have the following braiding and associativity structures for the category of V_K lattice modules.

$$\Omega(\alpha,\beta) = \mathsf{e}^{\mathrm{i}\pi\langle s(\alpha), s(\beta)\rangle}, \qquad F(\alpha,\beta,\gamma) = \mathsf{e}^{\mathrm{i}\pi\langle s(\alpha), s(\beta+\gamma) - s(\beta) - s(\gamma)\rangle}, \qquad \alpha,\beta,\gamma \in K^*/K. \tag{9.4.6}$$

Here $K^* \cong \mathbb{Z} \times \mathbb{R}$, where $\mathbb{Z}$ corresponds to the integer span of $\psi - \theta$ and $\mathbb{R}$ to the real span of K. Hence $K^*/K \cong \mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. If we denote by $\overline{x} \in [0,1)$ the unique representative of $x \in \mathbb{R}/\mathbb{Z}$ in the half open unit interval, then a choice of section s is given by the formula

$$s(\alpha_1,\alpha_2) = (\alpha_1, \overline{\alpha_2}), \quad (\alpha_1,\alpha_2) \in \mathbb{Z} \times \mathbb{R}/\mathbb{Z}. \tag{9.4.7}$$

Resulting in

$$\begin{aligned} &\Omega(\alpha,\beta) = \mathsf{e}^{\mathrm{i}\pi\left(\alpha_1 \overline{\beta_2} + \beta_1 \overline{\alpha_2}\right)}, \qquad F(\alpha,\beta,\gamma) = \mathsf{e}^{\mathrm{i}\pi \alpha_1 \left(\overline{\beta_2+\gamma_2} - \overline{\beta_2} - \overline{\gamma_2}\right)}, \\ &\alpha = (\alpha_1,\alpha_2),\ \beta = (\beta_1,\beta_2),\ \gamma = (\gamma_1,\gamma_2)\ \in \mathbb{Z} \times \mathbb{R}/\mathbb{Z}. \end{aligned} \tag{9.4.8}$$

Redefining (Ω, F) by rescaling by the coboundary

$$dk(\alpha,\beta,\gamma) = (\mathsf{e}^{\mathrm{i}\pi\beta_1\overline{\alpha_2} - \mathrm{i}\pi\alpha_1\overline{\beta_2}}, \mathsf{e}^{-\mathrm{i}\pi\alpha_1\left(\overline{\beta_2+\gamma_2} - \overline{\beta_2} - \overline{\gamma_2}\right)}) \tag{9.4.9}$$

of the 2-cochain $k(\alpha,\beta) = \mathsf{e}^{-\mathsf{i}\pi\alpha_1\overline{\beta_2}}$ (see Section 2.3) yields

$$\Omega(\alpha,\beta) = \mathsf{e}^{2\mathsf{i}\pi\beta_1\overline{\alpha_2}}, \qquad F(\alpha,\beta,\gamma) = 1, \tag{9.4.10}$$

and we see that the intertwining operators of V_K lattice modules can be normalised in such a way as to have trivial associators.

Proposition 9.4.1.

1. *The assignment*

$$\beta(z) \mapsto Y(\theta+\psi,z), \qquad \gamma(z) \mapsto :\psi(z)Y(-\theta-\psi,z): \tag{9.4.11}$$

induces an embedding $\phi_1 : \mathsf{G} \to \mathsf{V}_K$. Restricting to the image of this embedding, V_K-modules can be identified with G-modules as

$$\mathbb{F}_{\ell\psi} \cong \sigma^{\ell+1}\mathcal{W}_0^-, \qquad \mathbb{F}_\Lambda \cong \sigma^{\langle\Lambda,\psi+\theta\rangle+1}\mathcal{W}_{\langle\Lambda,\psi\rangle}, \quad \Lambda \in L/K,\ \langle\Lambda,\psi+\theta\rangle \in \mathbb{Z} \text{ and } \langle\Lambda,\psi\rangle \neq \mathbb{Z}, \tag{9.4.12}$$

where $\langle\Lambda,\psi\rangle$ is the coset in $\mathbb{R}/\mathbb{Z}$ formed by pairing all representatives of Λ with ψ.

2. *The assignment*

$$\beta(z) \mapsto :\psi(z)Y(\theta+\psi,z):, \qquad \gamma(z) \mapsto Y(-\theta-\psi,z) \tag{9.4.13}$$

induces an embedding $\phi_2 : \mathsf{G} \to \mathsf{V}_K$. Restricting to the image of this embedding, V_K-modules can be identified with G-modules as

$$\mathbb{F}_{\ell\psi} \cong \sigma^{\ell}\mathcal{W}_0^+, \qquad \mathbb{F}_\Lambda \cong \sigma^{\langle\Lambda,\psi+\theta\rangle}\mathcal{W}_{\langle\Lambda,\psi\rangle}, \quad \Lambda \in L/K,\ \langle\Lambda,\psi+\theta\rangle \in \mathbb{Z} \text{ and } \langle\Lambda,\psi\rangle \neq \mathbb{Z}, \tag{9.4.14}$$

where $\langle\Lambda,\psi\rangle$ is the coset in $\mathbb{R}/\mathbb{Z}$ formed by pairing all representatives of Λ with ψ.

The embeddings are well known and the identifications of V_K-modules with G-modules follow by comparing characters and was shown in [31, Proposition 4.7] and [45, Proposition 4.1]. To understand why $\langle\Lambda,\psi+\theta\rangle$ is an integer, rather than a coset, let $n \in \mathbb{Z}$. Then note that for a coset $\Lambda = n\psi + \mathbb{Z}(\psi+\theta) \in L/K$ we have $\langle\Lambda,\psi+\theta\rangle = \langle n\psi+\mathbb{Z}(\psi+\theta),\psi+\theta\rangle = n \in \mathbb{Z}$.

Theorem 9.4.2.

1. *Let $\mathcal{S}_1 = \operatorname{Res} Y(\psi,z)$, then $\ker(\mathcal{S}_1 : \mathsf{V}_K \to \mathbb{F}_\psi) = \operatorname{im}\phi_1$, where $\phi_1 : \mathsf{G} \to \mathsf{V}_K$ is the embedding of Proposition 9.4.1.1, that is, $\mathcal{S}_1$ is a screening operator for the free field realisation ϕ_1 of G. Further the sequence*

$$\cdots \xrightarrow{\mathcal{S}_1} \mathbb{F}_{-\psi} \xrightarrow{\mathcal{S}_1} \mathbb{F}_0 \xrightarrow{\mathcal{S}_1} \mathbb{F}_\psi \xrightarrow{\mathcal{S}_1} \cdots \tag{9.4.15}$$

is exact and is therefore a Felder complex.

2. *Let* $\mathcal{S}_2 = \operatorname{Res} Y(-\psi, z)$*, then* $\ker(\mathcal{S}_2 : \mathsf{V}_K \to \mathbb{F}_{-\psi}) = \operatorname{im}\phi_2$*, where* $\phi_2 : \mathsf{G} \to \mathsf{V}_K$ *is the embedding of Proposition 9.4.1.2, that is,* $\mathcal{S}_2$ *is a screening operator for the free field realisation* ϕ_2 *of* G*. Further the sequence*

$$\cdots \xrightarrow{\mathcal{S}_2} \mathbb{F}_\psi \xrightarrow{\mathcal{S}_2} \mathbb{F}_0 \xrightarrow{\mathcal{S}_2} \mathbb{F}_{-\psi} \xrightarrow{\mathcal{S}_2} \cdots \tag{9.4.16}$$

is exact and is therefore a Felder complex.

Proof. We prove part 1 only, as part 2 follows analogously. The operator product expansion of $Y(\psi, z)$ with the images of β and γ in V_K are

$$Y(\psi, z)\beta(w) \sim 0, \qquad Y(\psi, z)\gamma(w) \sim -\frac{Y(-\theta, w)}{(z-w)^2}, \tag{9.4.17}$$

which are total derivatives in z implying that $\mathcal{S}_1 = \operatorname{Res} Y(\psi, z)$ is a screening operator and that $\operatorname{im}\phi_1 \subset \ker \mathcal{S}_1$. Therefore, $\mathcal{S}_1$ commutes with G and hence defines a G-module map $\mathbb{F}_0 \to \mathbb{F}_\psi$. The identification (9.4.12) implies $\mathbb{F}_0 \cong \sigma \mathcal{W}_0^-$ and $\mathbb{F}_\psi \cong \sigma^2 \mathcal{W}_0^-$. By comparing composition factors we see that the kernel must be either $\operatorname{im}\phi_1 \cong \mathcal{V}$ or all of $\mathbb{F}_0$, so it is sufficient to show that the map $\mathcal{S}_1 : \mathbb{F}_0 \to \mathbb{F}_\psi$ is non-trivial. A quick calculation reveals that

$$\mathcal{S}_1 \left|-\psi - \theta\right\rangle = \left|-\theta\right\rangle, \tag{9.4.18}$$

and thus $\mathcal{S}_1$ is not trivial. By comparing the composition factors of the sequence (9.4.15) we also see that the sequence is an exact complex if each arrow is non-zero. Finally, the arrows are non-zero because

$$\mathcal{S}_1 \left|-\psi + m\theta\right\rangle = \left|m\theta\right\rangle, \qquad \forall m \in \mathbb{Z}. \tag{9.4.19}$$

■

Remark. The existence of Felder complexes will not specifically be needed for any of the results that follow, however, it is interesting to note that the bosonic ghosts admit such complexes. These complexes were crucial in [46] for computing the character formulae needed for the standard module formalism via resolutions of simple modules.

9.5 Projective modules

In this section we construct reducible yet indecomposable modules $\mathcal{P}$ on which the L_0 operator has rank 2 Jordan blocks. We further prove that the modules $\sigma^\ell \mathcal{P}$ and $\sigma^\ell \mathcal{W}_\lambda$ are both projective and injective, and that in particular the $\sigma^\ell \mathcal{P}$ are projective covers and injective hulls of $\sigma^\ell \mathcal{V}$ for any $\ell \in \mathbb{Z}$. We refer readers unfamiliar with homological algebra concepts such as injective and projective modules or extension groups to the book [48] and recall the following result for later use.

Proposition 9.5.1. *For a module $\mathcal{R}$ which is both projective and injective, the Hom-Ext sequences terminate. That is, if we have the short exact sequence*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{R} \longrightarrow \mathcal{B} \longrightarrow 0, \tag{9.5.1}$$

for modules $\mathcal{A}$, $\mathcal{B}$. then this implies that the following two sequences are exact, for any module $\mathcal{M}$.

$$0 \longrightarrow \operatorname{Hom}(\mathcal{M},\mathcal{A}) \longrightarrow \operatorname{Hom}(\mathcal{M},\mathcal{R}) \longrightarrow \operatorname{Hom}(\mathcal{M},\mathcal{B}) \longrightarrow \operatorname{Ext}(\mathcal{M},\mathcal{A}) \longrightarrow 0, \tag{9.5.2}$$

$$0 \longrightarrow \operatorname{Hom}(\mathcal{B},\mathcal{M}) \longrightarrow \operatorname{Hom}(\mathcal{R},\mathcal{M}) \longrightarrow \operatorname{Hom}(\mathcal{A},\mathcal{M}) \longrightarrow \operatorname{Ext}(\mathcal{B},\mathcal{M}) \longrightarrow 0. \tag{9.5.3}$$

Furthermore, $\operatorname{Hom}(\mathcal{R},-)$ *and* $\operatorname{Hom}(-,\mathcal{R})$ *are exact covariant and exact contravariant functors respectively.*

This proposition assists with the calculation of Hom and Ext groups, when all but one of the dimensions in the sequence are known. Using the fact that the Euler characteristic (the alternating sum of the dimensions of the coefficients) of an exact sequence vanishes, there is only one possibility for the remaining group.

Proposition 9.5.2. *The first extension groups of simple modules in $\mathscr{F}$ satisfy*

$$\dim \operatorname{Ext}\left(\sigma^k \mathcal{V}, \sigma^\ell \mathcal{V}\right) = \begin{cases} 1, & |k-\ell| = 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \dim \operatorname{Ext}\left(\sigma^k \mathcal{W}_\lambda, \mathcal{M}\right) = \dim \operatorname{Ext}\left(\mathcal{M}, \sigma^k \mathcal{W}_\lambda\right) = 0, \tag{9.5.4}$$

where $\lambda \in \mathbb{R}/\mathbb{Z}$, $\lambda \neq \mathbb{Z}$, $k,\ell \in \mathbb{Z}$ and $\mathcal{M}$ is any module in $\mathscr{F}$. In particular the simple modules $\sigma^k \mathcal{W}_\lambda$ are both projective and injective in $\mathscr{F}$.

Proof. To conclude that $\sigma^k \mathcal{W}_\lambda$ is projective in $\mathscr{F}$ it is sufficient to show that $\dim \operatorname{Ext}(\mathcal{W}_\lambda, \mathcal{M}) = 0$ for all simple objects $\mathcal{M} \in \mathscr{F}$. Injectivity in $\mathscr{F}$ then follows by applying the * functor and noting that $\mathcal{W}_\lambda{}^* \cong \mathcal{W}_\lambda$. Let $\mathcal{M} \in \mathscr{F}$ be simple, then a necessary condition for the short exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{W}_\lambda \longrightarrow 0, \qquad \mathcal{M} \in \mathscr{F} \tag{9.5.5}$$

being non-split is that the respective ghost and conformal weights of $\mathcal{W}_\lambda$ and $\mathcal{M}$ differ only by integers. For simple $\mathcal{M}$ this rules out $\mathcal{M} = \sigma^\ell \mathcal{V}$ or $\mathcal{M} = \sigma^\ell \mathcal{W}_\mu$, $\mu \neq \lambda$. So we consider $\mathcal{M} = \sigma^\ell \mathcal{W}_\lambda$. Assume $\ell = 0$, let $j \in \lambda$ and let v be a non-zero vector in the ghost and conformal weight $[j,0]$ space of the submodule $\mathcal{M} = \mathcal{W}_\lambda \subset \mathcal{N}$ and let $w \in \mathcal{N}$ be a representative of a non-zero coset in the $[j,0]$ weight space of the quotient $\mathcal{N}/\mathcal{W}_\lambda$. Without loss of generality, we can assume that w is a J_0-eigenvector and a generalised L_0-eigenvector. A necessary condition for the indecomposability of $\mathcal{N}$, is the existence of an element U in the universal enveloping algebra $\mathsf{U}(\mathfrak{G})$ such that $Uv = w$. Since v has minimal generalised conformal weight all positive modes annihilate v, thus Uv can be expanded as a sum of products of β_0 and γ_0 with each summand containing as many β_0 as γ_0 factors, that is, $Uv = f(J_0)v$ can be expanded as a polynomial in J_0 acting on v. Since $\mathcal{N} \in \mathscr{F}$, J_0 acts semisimply, hence $f(J_0)v \propto v$. Since v is not a scalar multiple of w, this contradicts the indecomposability of $\mathcal{N}$. Thus the exact sequence splits or, equivalently, the corresponding extension group vanishes.

Assume $\mathcal{M} = \sigma^\ell \mathcal{W}_\lambda$ with $\ell \neq 0$, then by applying the * and σ functors, we have $\operatorname{Ext}(\mathcal{W}_\lambda, \sigma^\ell \mathcal{W}_\lambda) = \operatorname{Ext}(\sigma^\ell \mathcal{W}_\lambda, \mathcal{W}_\lambda) = \operatorname{Ext}(\mathcal{W}_\lambda, \sigma^{-\ell} \mathcal{W}_\lambda)$. Thus the sign of ℓ can be chosen at will and we can assume without loss of generality that $\ell \geq 1$. Further, from the formulae for the conformal weights of spectral flow twisted modules (9.1.15), the conformal weights of $\mathcal{W}_\lambda$ and $\sigma^\ell \mathcal{W}_\lambda$ differ by integers if and only if $\ell \cdot \lambda = \mathbb{Z}$. Let $j \in \lambda$ be the minimal representative satisfying that the space of ghost weight j in $\sigma^\ell \mathcal{W}_\lambda$ has positive least conformal weight. The least conformal weight of the ghost weight $j-1$ space is a negative integer, which we denote by $-k$. See Figure 9.1 for an illustration of how the weight spaces are arranged. Let $v \in \mathcal{N}$ be a non-zero vector of ghost weight j and generalised L_0 eigenvalue 0, and hence a representative of a non-trivial coset of ghost and conformal weight $[j, 0]$ in $\mathcal{W}_\lambda \cong \mathcal{N}/\sigma^\ell \mathcal{W}_\lambda$. Further let $w \in \sigma^\ell \mathcal{W}_\lambda \subset \mathcal{N}$ be a non-zero vector of ghost and conformal weight $[j-1, -k]$. Both v and w lie in one-dimensional weight spaces and hence span them. If $\mathcal{N}$ is indecomposable, then there must exist an element U of ghost and conformal weight $[-1, -k]$ in $\mathsf{U}(\mathfrak{G})$, such that $Uv = w$. We pick a Poincaré-Birkhoff-Witt ordering such that generators with larger mode index are placed to the right of those with lesser index and γ_n is placed to the right of β_n for any $n \in \mathbb{Z}$. Thus $Uv = \sum_{i=1}^k U^{(i)} \gamma_i v$, where $U^{(i)}$ is an element of $\mathsf{U}(\mathfrak{G})$ of ghost and conformal weight $[0, i-k]$. In $\mathcal{W}_\lambda$, γ_0 acts bijectively on the space of conformal weight 0 vectors, hence there exists a $\tilde{v} \in \mathcal{N}$ such that $\gamma_0 \tilde{v} = v$. Since at ghost weight j the conformal weights of $\mathcal{N}$ are non-negative, we have $\gamma_n \tilde{v} = 0$, $n \geq 1$ and thus $Uv = \sum_{i=1}^k U^{(i)} \gamma_i \gamma_0 \tilde{v} = \sum_{i=1}^k U^{(i)} \gamma_0 \gamma_i \tilde{v} = 0$, contradicting the indecomposability of $\mathcal{N}$.

Next we consider the extensions of spectral flows of the vacuum module. By judicious application of the * and σ functors, we can identify $\operatorname{Ext}(\sigma^k \mathcal{V}, \sigma^\ell \mathcal{V}) = \operatorname{Ext}(\mathcal{V}, \sigma^{k-\ell} \mathcal{V}) = \operatorname{Ext}(\mathcal{V}, \sigma^{\ell-k} \mathcal{V})$. So without loss of generality, it is sufficient to consider the extension groups $\operatorname{Ext}(\mathcal{V}, \sigma^\ell \mathcal{V})$ or equivalently short exact sequences of the form

$$0 \longrightarrow \sigma^\ell \mathcal{V} \longrightarrow \mathcal{M} \longrightarrow \mathcal{V} \longrightarrow 0, \qquad \ell \in \mathbb{Z}_{\geq 0},\ \mathcal{M} \in \mathscr{F}. \tag{9.5.6}$$

Let $\sigma^\ell |0\rangle \in \sigma^\ell \mathcal{V} \subset \mathcal{M}$ denote the the spectral flow image of the highest weight vector of $\mathcal{V}$ and let $\omega \in \mathcal{M}$ be a J_0-eigenvector and a choice of representative of the highest weight vector in $\mathcal{V} \cong \mathcal{M}/\sigma^\ell \mathcal{V}$. We first show that these sequences necessarily split if $\ell \neq 1$. Assume $\ell = 0$, then the exact sequence can only be non-split if there exists a ghost and conformal weight $[0, 0]$

element U in $\mathsf{U}(\mathfrak{G})$ such that $U\omega = a\sigma^\ell|0\rangle - b\omega$, $a, b \in \mathbb{C}$, $a \neq 0$. Without loss of generality we can replace U by $\tilde{U} = U - b\mathbf{1}$ to obtain $\tilde{U}\omega = a\sigma^\ell|0\rangle$. Since the conformal weights of $\mathcal{V}$ are bounded below by 0, they satisfy the same bound in $\mathcal{M}$ and $\beta_n\omega = \gamma_n\omega = 0$, $n \geq 1$, so $\tilde{U}\omega$ can be expanded as a sum of products of β_0 and γ_0 acting on ω, with each summand containing the same number of β_0 and γ_0 factors. Equivalently, $\tilde{U}\omega$ can be expanded as a polynomial in J_0 acting on ω. Since ω is a J_0-eigenvector $\tilde{U}\omega \propto \omega$. Since ω is not a scalar multiple of $\sigma^\ell|0\rangle$, $\tilde{U}\omega = 0$ contradicting indecomposability, and the exact sequence splits.

Assume $\ell \geq 2$. The ghost and conformal weights of $\sigma^\ell|0\rangle$ are $[-\ell, -\frac{\ell(\ell+1)}{2}]$. Further, from the spectral flow formulae (9.1.15), one can see that the weight spaces of ghost and conformal weight $[-1, h]$ of $\sigma^\ell\mathcal{V}$ vanish for $h < \frac{(\ell+1)(\ell-2)}{2}$ and similarly the $[1, h]$ weight spaces of $\sigma^\ell\mathcal{V}$ vanish for $h < \frac{(\ell+1)(\ell+2)}{2}$. Since we are assuming $\ell \geq 2$, $\frac{(\ell+1)(\ell\pm2)}{2} \geq 0$. Thus $\gamma_n\omega = \beta_n\omega = 0$, $n \geq 1$. If $\mathcal{M}$ is indecomposable, there must exist a ghost and conformal weight $[-\ell, -\frac{\ell(\ell+1)}{2}]$ element U in $\mathsf{U}(\mathfrak{G})$ such that $U\omega = \sigma^\ell|0\rangle$. Since the conformal weight of U is $-\ell$, every summand of the expansion of $U\omega$ into β and γ modes must contain factors of γ_n or β_n with $n \geq 1$ and we can choose a Poincaré-Birkhoff-Witt ordering where these modes are placed to the right. Thus $U\omega = 0$, contradicting indecomposability and the exact sequence splits.

Assume $\ell = 1$, then $\sigma\mathcal{W}_0^+$ provides an example for which the exact sequence does not split and the dimension of the corresponding extension group is at least 1. We show that it is also at most 1. Let ω and $\sigma|0\rangle$ be defined as for $\ell \geq 2$. By arguments analogous to those for $\ell \geq 2$, it follows that the $[1, h]$ weight space vanishes for $h < 0$ and the $[-1, h]$ weight space vanishes for $h < -1$. Thus $\beta_n\omega = \gamma_{n+1}\omega = 0$, $n \geq 1$. The $[-1, -1]$ weight space of $\sigma\mathcal{V}$ is one-dimensional and is hence spanned by $\sigma|0\rangle$. If $\mathcal{M}$ is indecomposable, there must exist a ghost and conformal weight $[-1, -1]$ element U in $\mathsf{U}(\mathfrak{G})$ such that $U\omega = \sigma|0\rangle$. Thus, $U\omega$ can be expanded as $f(J_0)\gamma_1\omega = f(0)\gamma_1\omega = a|0\rangle$, where $f(J_0)$ is a polynomial. Hence the isomorphism class of $\mathcal{M}$ is determined by the value of $\gamma_1\omega$ in the one-dimensional $[-1, -1]$ weight space and $\dim \mathrm{Ext}(\mathcal{V}, \sigma\mathcal{V}) = 1$.

■

Now we introduce a general procedure for twisting actions by screening operators which we apply to direct sums of simple modules to create new modules.

Theorem 9.5.3 ([49, Theorem 2.1], proof in [44, 50], see also [51])**.** *Let V be a vertex algebra and $v \in V$ an even vector satisfying*

$$[v_n, v_m] = 0 \;\forall n, m \in \mathbb{Z}, \qquad L_n v = \delta_{n,0} \;\forall n \in \mathbb{Z}_{\geq 0}, \tag{9.5.7}$$

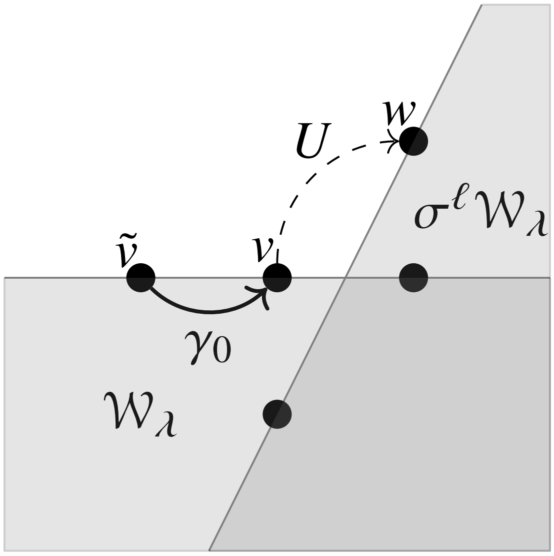


Figure 9.1: This diagram is a visual aid for the proof of the inextensibility of the simple module $\mathcal{W}_\lambda \in \mathscr{F}$, $\lambda \in \mathbb{R}/\mathbb{Z}, \lambda \neq \mathbb{Z}$. Here $\ell \geq 1$, $\ell \cdot \lambda = \mathbb{Z}$. The nodes represent the (spectral flows of) relaxed highest weight vectors of each module. Weight spaces are filled in grey. Conformal weight increases from top to bottom and ghost weight increases from right to left.

and let $\overline{V}$ be the vertex subalgebra of V such that $\overline{V} \subseteq \ker_V v_0$. Let $\langle M, Y_M \rangle$ be a V-module. Then define $\widetilde{M} = M$ as a vector space and define

$$Y_{\widetilde{M}}(a,x) = Y_M(\Delta(v,x)a,x), \qquad \Delta(v,x) = x^{v_0} \exp\left(\sum_{n=1}^{\infty} \frac{v_n}{-n}(-x)^{-n}\right). \tag{9.5.8}$$

Then $\left\langle \widetilde{M}, Y_{\widetilde{M}} \right\rangle$ is a $\overline{V}$-module.

Remark. Note that $Y(v_n a, w)$ is exactly the coefficient of $(z-w)^{-n}$ in the following OPE.

$$Y(v,z)Y(a,w) \sim \sum_{n\geq 1} \frac{Y(v_n a, w)}{(z-w)^n}. \tag{9.5.9}$$

Therefore, the modified fields $Y_{\widetilde{M}}(a,z)$ in Theorem 9.5.3 can be determined from the OPEs of $Y(v,z)$ with the original fields $Y_M(a,z)$.

Armed with the above results on extension groups and the twisting procedure, we can construct indecomposable modules $\sigma^\ell \mathcal{P} \in \mathscr{F}$, which will turn out to be projective covers and injective hulls of $\sigma^\ell \mathcal{V}$.

Proposition 9.5.4. *Recall that by the first free field realisation ϕ_1 of Proposition 9.4.1, we can identify $\mathbb{F}_{\ell\psi} \cong \sigma^{\ell+1}\mathcal{W}_0^-$. Define the $\mathcal{S}_1$-twisted action of* G *on $\mathbb{F}_{-\psi} \oplus \mathbb{F}_0$ by assigning*

$$\begin{aligned}
\beta(z) &\mapsto \phi_1(\beta(z)) = Y(\psi+\theta, z), \\
\gamma(z) &\mapsto \phi_1(\gamma(z)) - \frac{Y(-\theta,z)}{z} = :\psi(z)Y(-\psi-\theta,z): - \frac{Y(-\theta,z)}{z},
\end{aligned} \tag{9.5.10}$$

and determining the action of all other fields in G *through normal ordering and taking derivatives, where any vertex operator $Y(\lambda,z)$ whose Heisenberg weight λ is in the coset $[\psi] = [-\theta]$ is defined to act as* 0 *on $\mathbb{F}_0$ and as usual on $\mathbb{F}_{-\psi}$.*

1. *The assignment is well-defined, that is, it represents the operator product expansions of* G*, and hence defines an action of* G *on $\mathbb{F}_{-\psi} \oplus \mathbb{F}_0$, where $\oplus$ is meant as a direct sum of vector spaces without considering the module structure. Denote the module with this $\mathcal{S}_1$-twisted action by $\mathcal{P}$.*

2. *The composite fields $J(z) = :\beta(z)\gamma(z):$, $T(z) = -:\beta(z)\partial\gamma(z):$ act as*

$$\begin{aligned}
J(z) &\mapsto \phi_1(J(z)) = -\theta(z), \\
T(z) &\mapsto \phi_1(T(z)) + \frac{Y(\psi,z)}{z} = \frac{:\psi(z)^2: - :\theta(z)^2:}{2} - \partial\frac{\psi(z)-\theta(z)}{2} + \frac{Y(\psi,z)}{z}.
\end{aligned} \tag{9.5.11}$$

   The zero mode J_0 therefore acts semisimply and L_0 has rank 2 Jordan blocks. The vectors $|-\psi\rangle, |-\psi-\theta\rangle, |\theta\rangle, |0\rangle \in \mathcal{P}$ satisfy the relations

$$\beta_0|-\psi\rangle = |\theta\rangle, \quad \gamma_1|-\psi\rangle = -|-\psi-\theta\rangle, \quad \gamma_0|\theta\rangle = -|0\rangle, \quad \beta_{-1}|-\psi-\theta\rangle = |0\rangle, \quad L_0|-\psi\rangle = |0\rangle. \tag{9.5.12}$$

3. *The module $\mathcal{P}$ is indecomposable and satisfies the non-split exact sequences*

$$0 \longrightarrow \sigma\mathcal{W}_0^- \longrightarrow \mathcal{P} \longrightarrow \mathcal{W}_0^- \longrightarrow 0, \tag{9.5.13a}$$

$$0 \longrightarrow \mathcal{W}_0^+ \longrightarrow \mathcal{P} \longrightarrow \sigma\mathcal{W}_0^+ \longrightarrow 0, \tag{9.5.13b}$$

   which implies that its composition factors are $\sigma^{\pm1}\mathcal{V}$ and $\mathcal{V}$ with multiplicities 1 and 2, respectively.

4. *$\mathcal{P}$ is an object in $\mathscr{F}$.*

See Figure 9.2 for an illustration of how the composition factors of $\mathcal{P}$ are linked by the action of G.

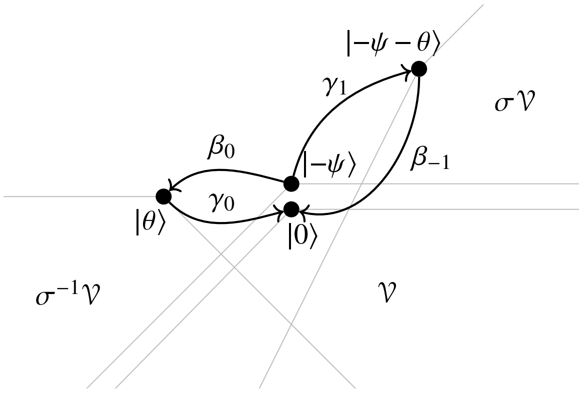


Figure 9.2: The composition factors of $\mathcal{P}$ with the nodes representing the spectral flows of the highest weight vectors of $\sigma^{\ell}\mathcal{V}$ for $-1 \le \ell \le 1$. The arrows give the action of $\mathfrak{G}$ modes on the highest-weight vectors of each factor. In this diagram, ghost weight increases to the left and conformal weight increases downwards. Note that there are two copies of $\mathcal{V}$, illustrated by a small vertical shift in their weights.

Proof. Part 1 follows from Theorem 9.5.3. The field identifications (9.5.11) of Part 2 follow by evaluating the definition for the deformed fields, with $v = |\psi\rangle$, while the relations (9.5.12) follow by applying the field identifications. The results follow directly from the following OPEs, as well as (9.4.17).

$$Y(\psi, z)T(w) \sim \frac{Y(\psi, w)}{(z-w)^2}, \qquad Y(\psi, z)J(w) \sim 0. \tag{9.5.14}$$

To conclude the first exact sequence of Part 3 note that the action of β and γ closes on $\mathbb{F}_0 \cong \sigma\mathcal{W}_0^-$, because $Y(-\theta, z)$ acts trivially and quotienting by $\mathbb{F}_0$ leaves only $\mathbb{F}_{-\psi} \cong \mathcal{W}_0^-$.

To conclude the second exact sequence, let $|0\rangle$ be the highest weight vector of $\mathcal{V}$ and let $\sigma^{\ell}|0\rangle$ be the spectral flow images of $|0\rangle$. Then $|0\rangle \in \mathbb{F}_0 \cong \sigma^{-1}\mathcal{W}_0^-$ can be identified with $|0\rangle$ in the $\mathcal{V}$ composition factor of $\sigma^{-1}\mathcal{W}_0^-$ and $|-\psi-\theta\rangle$ can be identified with $\sigma|0\rangle$ in the $\sigma\mathcal{V}$ composition factor. Further, $|-\psi\rangle \in \mathbb{F}_{-\psi} \cong \mathcal{W}_0^-$ can be identified with $|0\rangle$ in the $\mathcal{V}$ composition factor and $|\theta\rangle$ can be identified with $\sigma^{-1}|0\rangle$ in the $\sigma^{-1}\mathcal{V}$ composition factor. See Figure 9.2 for a diagram of the action of β and γ modes on $\mathcal{P}$ and how they connect the different composition factors. It therefore follows that $|0\rangle$ generates an indecomposable module whose composition factors are $\sigma^{-1}\mathcal{V}$ and $\mathcal{V}$, with $\mathcal{V}$ as a submodule and $\sigma^{-1}\mathcal{V}$ as a quotient. The module therefore satisfies the same non-split exact sequence (9.2.4) as $\mathcal{W}_0^+$ does and since the extension groups in (9.5.4) are one-dimensional, this submodule is isomorphic to $\mathcal{W}_0^+$. After quotienting by the submodule generated by $|\theta\rangle$, the formulae above imply that the quotient is isomorphic to $\sigma\mathcal{W}_0^+$ and the second exact sequence of Part 3 follows.

Part 4 follows because J_0 acts diagonalisably on $\mathcal{P}$ and because $\mathcal{P}$ has only finitely many composition factors all of which lie in $\mathscr{R}$ or $\sigma\mathscr{R}$. ■

Theorem 9.5.5. *For every $\ell \in \mathbb{Z}$ the indecomposable module $\sigma^\ell\mathcal{P}$ is projective and injective in $\mathscr{F}$, and hence is a projective cover and an injective hull of the simple module $\sigma^\ell\mathcal{V}$.*

Proof. Since spectral flow is an exact invertible functor, it is sufficient to prove projectivity and injectivity of $\mathcal{P}$, rather than all spectral flow twists of $\mathcal{P}$. We first show that $\mathcal{P}$ is injective by showing that $\dim \operatorname{Ext}(\mathcal{W},\mathcal{P}) = 0$ for any simple module $\mathcal{W} \in \mathscr{F}$. Following that we will show $\mathcal{P}^* = \mathcal{P}$, which, since * is an exact invertible contravariant functor, implies $\mathcal{P}$ is also projective.

A necessary condition for the non-triviality of such an extension is ghost weights differing only by integers. We therefore need not consider extensions by $\sigma^\ell\mathcal{W}_\lambda$, $\lambda \neq \mathbb{Z}$, so we restrict our attention to short exact sequences of the form

$$0 \longrightarrow \mathcal{P} \longrightarrow \mathcal{M} \longrightarrow \sigma^\ell\mathcal{V} \longrightarrow 0. \tag{9.5.15}$$

If the above extension is non-split, then there must exist a subquotient of $\mathcal{M}$ which is a non-trivial extension of $\sigma^\ell\mathcal{V}$ by one of the composition factors of $\mathcal{P}$. By Proposition 9.5.2 the above sequence must split if $|\ell| \geq 3$ and we therefore only consider $|\ell| \leq 2$.

If $\ell = 2$, then the composition factor of $\mathcal{P}$ non-trivially extending $\sigma^2\mathcal{V}$ must be $\sigma\mathcal{V}$. If the extension is non-trivial, then this subquotient must be isomorphic to $\sigma^2\mathcal{W}_0^-$. Further, if $\sigma^2|0\rangle$ is the spectrally flowed highest weight vector of $\sigma^2\mathcal{V}$ and $|-\psi-\theta\rangle \in \mathcal{P}$ (see Figure 9.2) is the spectrally flowed highest weight vector of the $\sigma\mathcal{V}$ composition factor of $\mathcal{P}$, then $\beta_{-2}\sigma^2|0\rangle = a|-\psi-\theta\rangle$, $a \in \mathbb{C}\setminus\{0\}$. The relations (9.5.12) thus imply

$$a|0\rangle = a\beta_{-1}|-\psi-\theta\rangle = a\beta_{-1}\beta_{-2}\sigma^2|0\rangle = a\beta_{-2}\beta_{-1}\sigma^2|0\rangle. \tag{9.5.16}$$

However, $\beta_{-1}\sigma^2|0\rangle$ has conformal and ghost weight $[-1,-2]$ and this weight space vanishes for both $\mathcal{P}$ and $\sigma^2\mathcal{V}$. Thus $\beta_{-1}\sigma^2|0\rangle$ and hence $a = 0$, which is a contradiction.

If $\ell = 1$, then the composition factor of $\mathcal{P}$ non-trivially extending $\sigma\mathcal{V}$ must be $\mathcal{V}$. There are two such composition factors in $\mathcal{P}$. Any such non-trivial extension must be isomorphic to $\sigma\mathcal{W}_0^-$. If the non-trivial extension involves the composition factor whose spectrally flowed highest weight vector is represented by $|-\psi\rangle$, then $\beta_{-1}\sigma|0\rangle = a|-\psi\rangle$, $a \in \mathbb{C}\setminus\{0\}$. The relations (9.5.12) thus imply

$$a|\theta\rangle = a\beta_0|-\psi-\rangle = a\beta_0\beta_{-1}\sigma|0\rangle = a\beta_{-1}\beta_0\sigma|0\rangle. \tag{9.5.17}$$

However, $\beta_0\sigma|0\rangle = 0$, so $a = 0$, which is a contradiction. If the non-trivial extension involves the composition factor whose spectrally flowed highest weight vector is represented by $|0\rangle$, then there would exist $a \in \mathbb{C} \setminus \{0\}$ such that $\beta_{-1}\sigma|0\rangle = a|0\rangle$. But then, by the relations (9.5.12), $\beta_{-1}(\sigma|0\rangle - a)|-\psi-\theta\rangle = 0$. Hence $(\sigma|0\rangle - a)|-\psi-\theta\rangle$ generates a direct summand isomorphic to $\sigma\mathcal{V}$, making the extension trivial.

If $\ell = 0$, then the composition factor of $\mathcal{P}$ non-trivially extending $\mathcal{V}$ must be $\sigma\mathcal{V}$ or $\sigma^{-1}\mathcal{V}$. If there is a subquotient isomorphic to a non-trivial extension of $\mathcal{V}$ by $\sigma^{-1}\mathcal{V}$, that is, isomorphic to $\mathcal{W}_0^-$, then there exists $a \in \mathbb{C} \setminus \{0\}$ such that $\beta_0|0\rangle = a|\theta\rangle$. But then, by the relations (9.5.12), $\beta_0(|0\rangle - a)|\theta\rangle = 0$. Hence $(|0\rangle - a)|\theta\rangle$ generates a direct summand isomorphic to $\mathcal{V}$, making the extension trivial. An analogous argument rules out the existence of subquotient isomorphic a non-trivial extension of $\mathcal{V}$ by $\sigma^{-1}\mathcal{V}$.

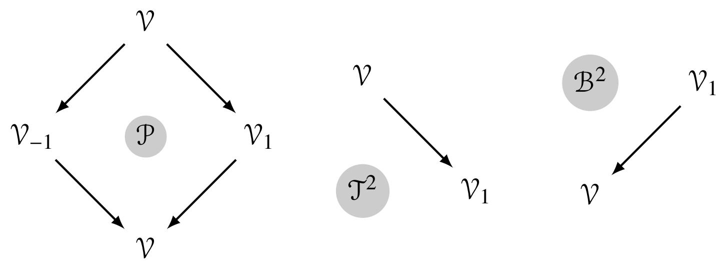
The cases $\ell = -2$ and $\ell = -1$ follow the same reasoning as $\ell = 2$ and $\ell = 1$, respectively.

Now that we have established that $\mathcal{P}$ is injective, we can apply the functors $\operatorname{Hom}(\mathcal{W}_0^-, -)$ and $\operatorname{Hom}(\sigma\mathcal{W}_0^+, -)$ to the short exact sequences (9.5.13a) and (9.5.13b), respectively, to deduce $\dim\operatorname{Ext}(\mathcal{W}_0^-, \sigma\mathcal{W}_0^-) = 1 = \dim\operatorname{Ext}(\sigma\mathcal{W}_0^+, \mathcal{W}_0^+)$. The indecomposable module $\mathcal{P}$ is therefore the unique module making the short exact sequences (9.5.13a) and (9.5.13b) non-split. By applying the functor * to these exact sequences, we see that $\mathcal{P}^*$ also satisfies these same sequences and hence $\mathcal{P} \cong \mathcal{P}^*$. This in turn implies $\operatorname{Ext}(\mathcal{P}, -) = 0$ and hence that $\sigma^\ell\mathcal{P}$ is projective for all $\ell \in \mathbb{Z}$. ■

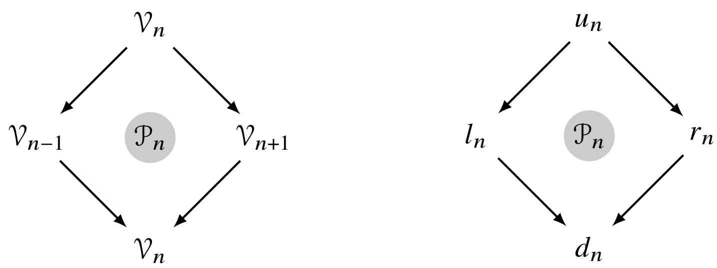
9.6 Classification of indecomposables

In this section, we give a classification of all indecomposable modules in category $\mathscr{F}$. We already know any simple module is isomorphic to either $\sigma^m\mathcal{W}_\lambda$ or $\sigma^m\mathcal{V}$, and we also know that the $\sigma^m\mathcal{W}_\lambda$ are inextensible due to being injective and projective in $\mathscr{F}$. We now complete the classification by finding all the reducible indecomposables which can be built as finite length extensions with composition factors isomorphic to spectral flows of $\mathcal{V}$. To unclutter formulae, we use the notation $\mathcal{M}_n = \sigma^n\mathcal{M}$ for any module $\mathcal{M}$. The classification of indecomposable modules in $\mathscr{F}$ closely resembles the classification of indecomposable modules over the Temperley-Lieb algebra with parameter at roots of unity given in [52] and also the classification of indecomposable modules over the $(1, p)$ triplet model given in [53].

The reducible yet indecomposable modules constituting the classification are the spectral flows of the projective module $\mathcal{P}$, and two infinite families. These two families, denoted $\mathcal{B}_n^m$ and $\mathcal{T}_n^m$, $m, n \in \mathbb{Z}$, $n \geq 1$, are dual to each other with respect to *, meaning $(\mathcal{B}_n^m)^* = \mathcal{T}_n^m$, and further satisfy the identifications $\mathcal{B}^1 = \mathcal{T}^1 = \mathcal{V}$, $\mathcal{B}^2 = \sigma\mathcal{W}_0^-$ and $\mathcal{T}^2 = \sigma\mathcal{W}_0^+$, where an absence of subscript indicates $n = 0$. The superscript m is the number of composition factors or length of the module. As a visual aid, we represent these indecomposable modules using Loewy diagrams.



Here the edges indicate the action of G and the vertices represent the composition factors. Recall the generating vectors $u = |{-\psi}\rangle$, $l = |\theta\rangle$, $r = |{-\psi - \theta}\rangle$, $d = |0\rangle$ for the composition factors of $\mathcal{P}$ in Figure 9.2 constructed from the first free field realisation Proposition 9.4.1.1. We denote their spectral flow images by $u_n = \sigma^n u$, $l_n = \sigma^n l$, $r_n = \sigma^n r$, $d_n = \sigma^n d$. The letters labelling these vectors have been chosen according to the position of their corresponding composition factor in the Loewy diagram below.



Since $\mathscr{F}$ has sufficiently many projectives and injectives, every indecomposable module is isomorphic to some quotient of a finite sum of indecomposable projective modules and to a submodule of a finite sum of indecomposable injective modules. We therefore define the two families of reducible yet indecomposable modules, $\mathcal{B}_m^n$ and $\mathcal{T}_m^n$ as certain images or coimages of homomorphisms from projective to injective modules. We prepare the necessary notation, For $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, let

$$\begin{aligned}
\mathsf{P}[\mathcal{T}_m^{2k}] &= \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i}, \quad \mathsf{J}[\mathcal{T}_m^{2k}] = \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i+1}, \quad \mathsf{P}[\mathcal{T}_m^{2k+1}] = \bigoplus_{i=0}^{k} \mathcal{P}_{m+2i}, \quad \mathsf{J}[\mathcal{T}_m^{2k+1}] = \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i+1},\\
\mathsf{P}[\mathcal{B}_m^{2k}] &= \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i+1}, \quad \mathsf{J}[\mathcal{B}_m^{2k}] = \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i}, \quad \mathsf{P}[\mathcal{B}_m^{2k+1}] = \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i+1}, \quad \mathsf{J}[\mathcal{B}_m^{2k+1}] = \bigoplus_{i=0}^{k} \mathcal{P}_{m+2i}.
\end{aligned} \tag{9.6.1}$$

As the $\mathsf{P}[\,]$, $\mathsf{J}[\,]$ notation suggests, the above modules will be shown to be projective covers and injective hulls of the $\mathcal{T}$ and $\mathcal{B}$ modules. Further, for $n \in \mathbb{Z}$ let $\psi_n^{\pm} : \mathcal{P}_n \to \mathcal{P}_{n\pm 1}$ be the module homomorphisms uniquely characterised by $\psi_n^+(u_n) = l_{n+1}$ and $\psi_n^-(u_n) = r_{n-1}$. Note that these homomorphisms satisfy the relations

$$\psi_{m\pm1}^{\pm} \circ \psi_m^{\pm} = 0, \qquad \psi_{m+1}^{-} \circ \psi_m^{+} = \psi_{m-1}^{+} \circ \psi_m^{-}, \tag{9.6.2}$$

in fact $\ker \psi_{m\pm1}^{\pm} = \operatorname{im} \psi_m^{\pm}$. Finally, consider the module homomorphisms $\psi[\mathcal{M}] : \mathsf{P}[\mathcal{M}] \to \mathsf{J}[\mathcal{M}]$, which, for $\mathcal{M}$ being any of the indecomposables above, are given by the formulae

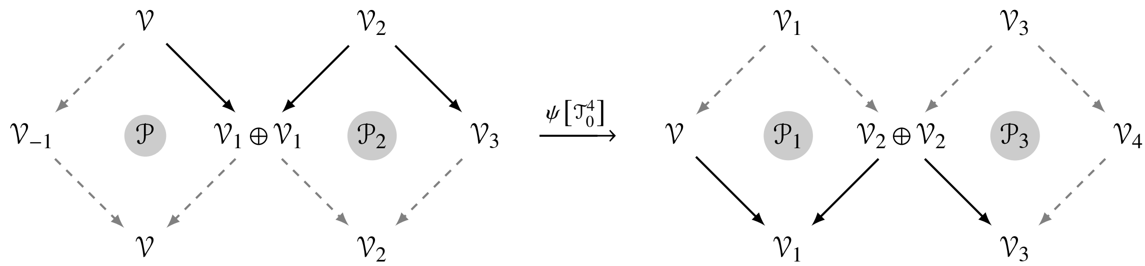
$$\psi\left[\mathcal{T}_m^{2k}\right] = \psi_m^+ + \sum_{i=1}^{k-1} \psi_{m+2i}^- + \psi_{m+2i}^+, \qquad \psi\left[\mathcal{T}_m^{2k+1}\right] = \psi_m^+ + \psi_{m+2k}^- + \sum_{i=1}^{k-1} \psi_{m+2i}^- + \psi_{m+2i}^+,$$
$$\psi\left[\mathcal{B}_m^{2k}\right] = \psi_{m+1}^- + \sum_{i=1}^{k-1} \psi_{m+2i-1}^+ + \psi_{m+2i+1}^-, \qquad \psi\left[\mathcal{B}_m^{2k+1}\right] = \sum_{i=0}^{k-1} \psi_{m+2i+1}^- + \psi_{m+2i+1}^+. \tag{9.6.3}$$

Definition 9.6.1. For $m \in \mathbb{Z}$ and $k \in \mathbb{Z}_{>0}$, we define the following indecomposable modules

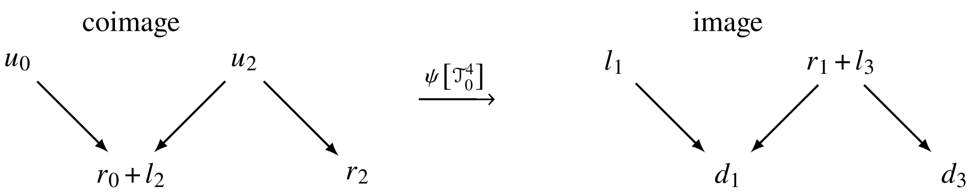
$$\mathcal{T}_m^k = \operatorname{im} \psi\left[\mathcal{T}_m^k\right] \cong \operatorname{coim} \psi\left[\mathcal{T}_m^k\right], \qquad \mathcal{B}_m^k = \operatorname{im} \psi\left[\mathcal{B}_m^k\right] \cong \operatorname{coim} \psi\left[\mathcal{B}_m^k\right], \tag{9.6.4}$$

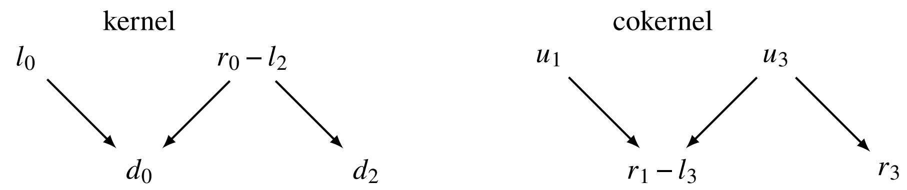
where the isomorphisms follow by the first isomorphism theorem.

Example. Consider the example of $\mathcal{T}_0^4$.



The thick arrows above indicate the edges of the Loewy diagram of the coimage and image. The repeated composition factors $\mathcal{V}_1$ and $\mathcal{V}_2$ in the domain and codomain, respectively, each contribute one factor to the image and coimage and one to the kernel and cokernel as can be seen in the diagrams below.





In the cases of the coimage and the cokernel above, the indicated vectors are to be thought of as representatives of their equivalences classes in their respective quotients.

Theorem 9.6.2. *Any reducible indecomposable module in $\mathscr{F}$ is isomorphic to one of the following.*

$$\mathcal{P}_m, \qquad \mathcal{B}^n_m, \qquad \mathcal{T}^n_m, \qquad m,n \in \mathbb{Z},\ n \geq 2. \tag{9.6.5}$$

The remainder of this section will be dedicated to proving the above theorem. For any module $\mathcal{M}$, we recall the following two well known substructures. The first is the maximal semisimple submodule of $\mathcal{M}$, called the socle and which we denote $\operatorname{soc}\mathcal{M}$. The second, called the head, is the maximal semisimple quotient of $\mathcal{M}$, defined to be the quotient of $\mathcal{M}$ by its radical (the intersection of its maximal proper submodules), which we denote $\operatorname{hd}\mathcal{M}$. We also let $\mathsf{J}[\mathcal{M}]$ and $\mathsf{P}[\mathcal{M}]$ denote the injective hull and the projective cover of $\mathcal{M}$ respectively. We then have the following standard homological algebra result.

Proposition 9.6.3. *For any module $\mathcal{M} \in \mathscr{F}$, we have*

$$\operatorname{Hom}(\mathcal{V}_n, \mathcal{M}) \cong \operatorname{Hom}(\mathcal{V}_n, \operatorname{soc}\mathcal{M}), \qquad \operatorname{Hom}(\mathcal{M}, \mathcal{V}_n) \cong \operatorname{Hom}(\operatorname{hd}\mathcal{M}, \mathcal{V}_n), \tag{9.6.6}$$

and

$$\mathsf{J}[\mathcal{M}] \cong \mathsf{J}[\operatorname{soc}\mathcal{M}], \qquad \mathsf{P}[\mathcal{M}] \cong \mathsf{P}[\operatorname{hd}\mathcal{M}]. \tag{9.6.7}$$

Corollary 9.6.4. *The heads and socles of the $\mathcal{T}^k_m$ and $\mathcal{B}^k_m$ are given by*

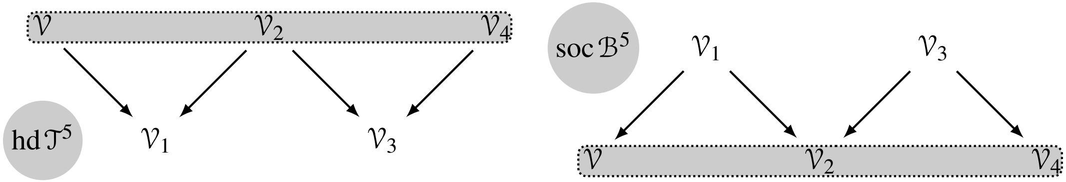
$$\begin{aligned}
&\operatorname{hd}\mathcal{T}^{2k}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i}, \quad \operatorname{soc}\mathcal{T}^{2k}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i+1}, \quad \operatorname{hd}\mathcal{T}^{2k+1}_m \cong \bigoplus_{i=0}^{k} \mathcal{V}_{m+2i}, \quad \operatorname{soc}\mathcal{T}^{2k+1}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i+1},\\
&\operatorname{hd}\mathcal{B}^{2k}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i+1}, \quad \operatorname{soc}\mathcal{B}^{2k}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i}, \quad \operatorname{hd}\mathcal{B}^{2k+1}_m \cong \bigoplus_{i=0}^{k-1} \mathcal{V}_{m+2i+1}, \quad \operatorname{soc}\mathcal{B}^{2k+1}_m \cong \bigoplus_{i=0}^{k} \mathcal{V}_{m+2i},
\end{aligned} \tag{9.6.8}$$

and the dimensions of Hom groups involving $\mathcal{V}_n$ by the following table.

	$\mathcal{B}^{2k+1}_m$	$\mathcal{B}^{2k}_m$	$\mathcal{T}^{2k+1}_m$	$\mathcal{T}^{2k}_m$
$\dim \mathrm{Hom}(\mathcal{V}_n, -)$	$\sum_{i=0}^{k} \delta_{n,m+2i}$	$\sum_{i=0}^{k-1} \delta_{n,m+2i}$	$\sum_{i=0}^{k-1} \delta_{n,m+2i+1}$	$\sum_{i=0}^{k-1} \delta_{n,m+2i+1}$
$\dim \mathrm{Hom}(-, \mathcal{V}_n)$	$\sum_{i=0}^{k-1} \delta_{n,m+2i+1}$	$\sum_{i=0}^{k-1} \delta_{n,m+2i+1}$	$\sum_{i=0}^{k} \delta_{n,m+2i}$	$\sum_{i=0}^{k-1} \delta_{n,m+2i}$

Thus, for $\mathcal{M}$ being any of the $\mathcal{T}^k_m$ and $\mathcal{B}^k_m$ in Definition 9.6.1, $\mathsf{P}[\mathcal{M}]$ and $\mathsf{J}[\mathcal{M}]$ are the projective cover and injective hull of the indecomposable module $\mathcal{M}$, respectively.

Proof. The Loewy diagrams for the $\mathcal{T}^k_m$ and $\mathcal{B}^k_m$ immediately suggest the heads and socles. For example for $\mathcal{T}^5_0$ and $\mathcal{B}^5_0$ we have the following.



They can, of course, also be easily determined from the calculations similar to those in the example above Theorem 9.6.2. The dimensions of the Hom groups in the table above, and the projective cover and injective hull formulae then immediately follow by Proposition 9.6.3. ■

Lemma 9.6.5. *For $\mathcal{M}$ being any of the $\mathcal{T}^k_m$ and $\mathcal{B}^k_m$ in Definition 9.6.1, the projective and injective presentations of $\mathcal{M}$ are characterised by the following.*

$\mathcal{M}$	$\mathcal{B}^{2k+1}_m$	$\mathcal{B}^{2k}_m$	$\mathcal{T}^{2k+1}_m$	$\mathcal{T}^{2k}_m$
$\ker(\mathsf{P}[\mathcal{M}] \longrightarrow \mathcal{M})$	$\mathcal{B}^{2k-1}_{m+1}$	$\mathcal{B}^{2k}_{m+1}$	$\mathcal{T}^{2k+3}_{m-1}$	$\mathcal{T}^{2k}_{m-1}$
$\mathrm{coker}(\mathcal{M} \longrightarrow \mathsf{J}[\mathcal{M}])$	$\mathcal{B}^{2k+3}_{m-1}$	$\mathcal{B}^{2k}_{m-1}$	$\mathcal{T}^{2k-1}_{m+1}$	$\mathcal{T}^{2k}_{m+1}$

Proof. Observe that we can precompose any of the homomorphisms $\psi[\mathcal{M}]$ by any automorphism A of the domain $\mathsf{P}[\mathcal{M}]$ that consists of a non-zero rescaling chosen independently on each direct summand. Since this is a precomposition by an automorphism, the kernels of $\psi[\mathcal{M}]$ and $\psi[\mathcal{M}] \circ A$ are isomorphic. Similarly we can postcompose by any automorphism B of the codomain $\mathsf{J}[\mathcal{M}]$ that consists of a non-zero rescaling, chosen independently on each direct summand. In summary, the homomorphisms $\psi[\mathcal{M}]$ and $B \circ \psi[\mathcal{M}] \circ A$ have isomorphic images and isomorphic kernels. Appropriately chosen pre- and postcompositions hence allow us to freely rescale each summand in the formulae (9.6.3). Hence we can define the morphisms

$$\overline{\psi}[\mathcal{T}^{2k}_m] = \psi^+_m + \sum_{i=1}^{k-1} -\psi^-_{m+2i} + \psi^+_{m+2i}, \qquad \overline{\psi}[\mathcal{T}^{2k+1}_m] = \psi^+_m - \psi^-_{m+2k} + \sum_{m+i=1}^{k-1} -\psi^-_{m+2i} + \psi^+_{2i},$$

$$\overline{\psi}\big[\mathcal{B}_m^{2k}\big] = -\psi_{m+1}^- + \sum_{i=1}^{k-1} \psi_{m+2i-1}^+ - \psi_{m+2i+1}^-, \quad \overline{\psi}\big[\mathcal{B}_m^{2k+1}\big] = \sum_{i=0}^{k-1} -\psi_{m+2i+1}^- + \psi_{m+2i+1}^+. \tag{9.6.9}$$

We prove the $\mathcal{T}_m^{2k}$ column of the lemma above, the remaining columns follow by analogous arguments. First the kernel of $\psi\big[\mathcal{T}_m^{2k}\big]$. Note that $\mathsf{P}[\mathcal{T}_m^{2k}] = \mathsf{J}[\mathcal{T}_{m-1}^{2k}]$ and hence the domain of $\psi\big[\mathcal{T}_m^{2k}\big]$ is equal to the codomain of $\overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big]$ and hence these two homomorphisms can be composed. A direct computation using the composition relations (9.6.2) then shows that $\psi\big[\mathcal{T}_m^{2k}\big] \circ \overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big] = 0$ and hence $\ker \psi\big[\mathcal{T}_m^{2k}\big] \supset \operatorname{im} \overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big]$. Finally, $\operatorname{im} \overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big] \cong \operatorname{im} \psi\big[\mathcal{T}_{m-1}^{2k}\big] = \mathcal{T}_{m-1}^{2k}$. By inspection we also see that both $\ker \psi\big[\mathcal{T}_m^{2k}\big]$ and $\operatorname{im} \overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big]$ have $2k$ composition factors and hence $\ker \psi\big[\mathcal{T}_m^{2k}\big] = \operatorname{im} \overline{\psi}\big[\mathcal{T}_{m-1}^{2k}\big] \cong \mathcal{T}_{m-1}^{2k}$. To compute the cokernel of $\psi\big[\mathcal{T}_m^{2k}\big]$, note that $\mathsf{J}[\mathcal{T}_m^{2k}] = \mathsf{P}[\mathcal{T}_{m+1}^{2k}]$. An analogous argument to the above then proves $\operatorname{coker} \psi\big[\mathcal{T}_m^{2k}\big] \cong \mathcal{T}_{m+1}^{2k}$. ■

Combining all of the results above we can now prove Theorem 9.6.2.

Proof of Theorem 9.6.2. The idea is to show that any extension of the $\mathcal{T}_m^k$ and $\mathcal{B}_m^k$ decomposes into a direct sum of the modules listed in (9.6.5). In particular it is sufficient to only consider extensions by $\mathcal{V}_n$ (as either a submodule or a quotient). We classify the $\mathcal{T}_m^k$ modules and the classification of the $\mathcal{B}_m^k$ follows from an analogous argument.

We first consider $\mathcal{T}_m^3$ as this module constitutes a special case that need to be considered separately form the other $\mathcal{T}_m^k$. Recall that we have the respective injective and projective presentations

$$0 \longrightarrow \mathcal{T}_m^3 \longrightarrow \mathcal{P}_{m+1} \longrightarrow \mathcal{V}_{m+1} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}_{m-1}^5 \longrightarrow \mathcal{P}_m \oplus \mathcal{P}_{m+2} \longrightarrow \mathcal{T}_m^3 \longrightarrow 0. \tag{9.6.10}$$

Applying the $\operatorname{Hom}(\mathcal{V}_n, -)$ functor to the injective presentation yields (recall Proposition 9.5.1) the long exact sequence

$$0 \longrightarrow \operatorname{Hom}\Big(\mathcal{V}_n, \mathcal{T}_m^3\Big) \longrightarrow \operatorname{Hom}(\mathcal{V}_n, \mathcal{P}_{m+1}) \longrightarrow \operatorname{Hom}(\mathcal{V}_m, \mathcal{V}_{m+1}) \longrightarrow \operatorname{Ext}\Big(\mathcal{V}_n, \mathcal{T}_m^3\Big) \longrightarrow 0, \tag{9.6.11}$$

which implies $\dim \operatorname{Ext}\big(\mathcal{V}_n, \mathcal{T}_m^3\big) = \dim \operatorname{Hom}(\mathcal{V}_m, \mathcal{V}_{m+1}) - \dim \operatorname{Hom}(\mathcal{V}_n, \mathcal{P}_{m+1}) + \dim \operatorname{Hom}\big(\mathcal{V}_n, \mathcal{T}_m^3\big) = \delta_{n,m+1}$. Up to isomorphism there therefore exists exactly one indecomposable module with $\mathcal{V}_{m+1}$ as a quotient and $\mathcal{T}_m^3$ as a submodule. The projective module $\mathcal{P}_{m+1}$ an example of this extension and hence no new indecomposable has been constructed. Similarly, applying the $\operatorname{Hom}(-, \mathcal{V}_n)$ functor to the projective presentation above and computing dimensions yields $\dim \operatorname{Ext}\big(\mathcal{T}_m^3, \mathcal{V}_n\big) = \delta_{n,m-1} + \delta_{n,m+1} + \delta_{n,m+3}$. The cases $n = m-1$ and $n = m+3$ respectively correspond to $\mathcal{B}_{m-1}^4$ and $\mathcal{T}_m^4$, however, $n = m+1$ requires closer inspection. This final case corresponds to the extension constructed from the non-split exact sequence

$$0 \longrightarrow \mathcal{V}_{m+1} \longrightarrow \mathcal{T}_m^2 \oplus \mathcal{B}_{m+1}^3 \longrightarrow \mathcal{T}_m^3 \longrightarrow 0, \tag{9.6.12}$$

where the second arrow is characterised by the generating vector (the $m+1$-fold spectral flow of the highest weight vector of $\mathcal{V}$) to any linear combination with non-zero coefficients of the two corresponding generating vectors of the two $\mathcal{V}_{m+1}$ summands of the socles of $\mathcal{T}^3_m$ and $\mathcal{B}^3_{m+1}$. A brief calculation reveals that the quotient of $\mathcal{T}^2_m \oplus \mathcal{B}^3_{m+1}$ by the image of $\mathcal{V}_{m+1}$ is isomorphic to $\mathcal{T}^3_m$ and it is the projection to this quotient that makes up the third arrow in the exact sequence above. Hence no new indecomposables have been constructed.

Next we consider $\mathcal{T}^{2k+1}_m$, $k \geq 2$. The injective and projective presentations are given by

$$0 \longrightarrow \mathcal{T}^{2k+1}_m \longrightarrow \bigoplus_{i=0}^{k-1} \mathcal{P}_{m+2i+1} \longrightarrow \mathcal{T}^{2k-1}_{m+1} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}^{2k+3}_{m-1} \longrightarrow \bigoplus_{i=0}^{k} \mathcal{P}_{2i} \longrightarrow \mathcal{T}^{2k+1}_m \longrightarrow 0. \tag{9.6.13}$$

Again we can compute the dimensions of Ext groups by applying the $\operatorname{Hom}(\mathcal{V}_n, -)$ and $\operatorname{Hom}(-, \mathcal{V}_n)$ and obtain

$$\dim \operatorname{Ext}\left(\mathcal{V}_n, \mathcal{T}^{2k+1}_m\right) = \sum_{i=0}^{k-1} \delta_{n,m+2i}, \qquad \dim \operatorname{Ext}\left(\mathcal{T}^{2k+1}_m, \mathcal{V}_n\right) = \sum_{i=0}^{k+1} \delta_{n,m+2i-1}. \tag{9.6.14}$$

The non-vanishing Ext groups correspond to the respective non-split exact sequences

$$\begin{aligned} &0 \longrightarrow \mathcal{T}^{2k+1}_m \longrightarrow \mathcal{T}^{2i+1}_m \oplus \mathcal{T}^{2(k-i)+1}_{m+2i} \longrightarrow \mathcal{V}_{m+2i} \longrightarrow 0, && i = 0, \dots, k-1, \\ &0 \longrightarrow \mathcal{V}_{m+2i-1} \longrightarrow \mathcal{T}^{2i}_m \oplus \mathcal{B}^{2(k-i)+2}_{m+2i-1} \longrightarrow \mathcal{T}^{2k+1}_m \longrightarrow 0, && i = 0, \dots, k+1, \end{aligned} \tag{9.6.15}$$

where $\mathcal{T}^0$ and $\mathcal{B}^0$ are to be interpreted as the 0 module.

Similar computations for $\mathcal{T}^{2k}_m$, $k \geq 1$ yield the Ext group dimensions

$$\dim \operatorname{Ext}\left(\mathcal{V}_n, \mathcal{T}^{2k}_m\right) = \sum_{i=1}^{k} \delta_{n,m+2i}, \qquad \dim \operatorname{Ext}\left(\mathcal{T}^{2k}_m, \mathcal{V}_n\right) = \sum_{i=0}^{k-1} \delta_{n,m+2i-1}, \tag{9.6.16}$$

which correspond to the non-split exact sequences

$$\begin{aligned} &0 \longrightarrow \mathcal{T}^{2k}_m \longrightarrow \mathcal{T}^{2i+1}_m \oplus \mathcal{T}^{2(k-i)}_{m+2i} \longrightarrow \mathcal{V}_{m+2i} \longrightarrow 0, && i = 1, \dots, k, \\ &0 \longrightarrow \mathcal{V}_{m+2i-1} \longrightarrow \mathcal{T}^{2i}_m \oplus \mathcal{B}^{2(k-i)+1}_{m+2i-1} \longrightarrow \mathcal{T}^{2k}_m \longrightarrow 0, && i = 0, \dots, k+1. \end{aligned} \tag{9.6.17}$$

Extensions of the $\mathcal{T}^k_m$ modules therefore only yield indecomposable already accounted for in the list (9.6.5). The extensions of the $\mathcal{B}$ modules are computed analogously ■

We end this section with some characterisations of the classified indecomposable modules which will prove helpful in later sections.

Corollary 9.6.6. *The $\mathcal{B}$, $\mathcal{T}$ and $\mathcal{P}$ indecomposable modules are uniquely characterised by the following non-split exact sequences.*

$$0 \longrightarrow \mathcal{V} \longrightarrow \mathcal{B}^n \longrightarrow \mathcal{T}_1^{n-1} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{B}_1^{n-1} \longrightarrow \mathcal{T}^n \longrightarrow \mathcal{V} \longrightarrow 0, \tag{9.6.18a}$$

$$0 \longrightarrow \mathcal{V}_{2n} \longrightarrow \mathcal{B}^{2n+1} \longrightarrow \mathcal{B}^{2n} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}^{2n} \longrightarrow \mathcal{T}^{2n+1} \longrightarrow \mathcal{V}_{2n} \longrightarrow 0, \tag{9.6.18b}$$

$$0 \longrightarrow \mathcal{B}^{2n-1} \longrightarrow \mathcal{B}^{2n} \longrightarrow \mathcal{V}_{2n-1} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{V}_{2n-1} \longrightarrow \mathcal{T}^{2n} \longrightarrow \mathcal{T}^{2n-1} \longrightarrow 0. \tag{9.6.18c}$$

$$0 \longrightarrow \mathcal{B}_2^{n-2} \longrightarrow \mathcal{B}^n \longrightarrow \mathcal{B}^2 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}^2 \longrightarrow \mathcal{T}^n \longrightarrow \mathcal{T}_2^{n-2} \longrightarrow 0, \tag{9.6.18d}$$

$$0 \longrightarrow \mathcal{B}^{2n-1} \longrightarrow \mathcal{B}^{2n+1} \longrightarrow \mathcal{T}_{2n-1}^2 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{B}_{2n-1}^2 \longrightarrow \mathcal{T}^{2n+1} \longrightarrow \mathcal{T}^{2n-1} \longrightarrow 0, \tag{9.6.18e}$$

$$0 \longrightarrow \mathcal{B}_{2n-2}^2 \longrightarrow \mathcal{B}^{2n} \longrightarrow \mathcal{B}^{2n-2} \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}^{2n-2} \longrightarrow \mathcal{T}^{2n} \longrightarrow \mathcal{T}_{2n-2}^2 \longrightarrow 0, \tag{9.6.18f}$$

$$0 \longrightarrow \mathcal{B}^2 \longrightarrow \mathcal{P} \longrightarrow \mathcal{B}_{-1}^2 \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{T}_{-1}^2 \longrightarrow \mathcal{P} \longrightarrow \mathcal{T}^2 \longrightarrow 0. \tag{9.6.18g}$$

Proof. Sequences (9.6.18a), (9.6.18b) and (9.6.18c) follow from the one-dimensionality of the Ext groups in the proof of Theorem 9.6.2. We illustrate the procedure by which the remainder of the exact sequences can be seen to be non-split, by focussing on the first sequence of (9.6.18d), starting with the projective presentation of $\mathcal{B}^2$.

$$0 \longrightarrow \mathcal{B}_1^2 \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{B}^2 \longrightarrow 0 \tag{9.6.19}$$

We apply the contravariant functor $\operatorname{Hom}(-,\mathcal{B}_2^{n-2})$ to obtain the following long exact sequence.

$$0 \to \operatorname{Hom}\left(\mathcal{B}^2,\mathcal{B}_2^{n-2}\right) \to \operatorname{Hom}\left(\mathcal{P}_1,\mathcal{B}_2^{n-2}\right) \to \operatorname{Hom}\left(\mathcal{B}_{-1}^2,\mathcal{B}_2^{n-2}\right) \to \operatorname{Ext}\left(\mathcal{B}^2,\mathcal{B}_2^{n-2}\right) \to 0, \tag{9.6.20}$$

which evaluates to

$$0 \to 0 \to 0 \to \mathbb{C} \to \operatorname{Ext}\left(\mathcal{B}^2,\mathcal{B}_2^{n-2}\right) \to 0. \tag{9.6.21}$$

Therefore $\dim \operatorname{Ext}(\mathcal{B}^2,\mathcal{B}_2^{n-2}) = 1$, with the nontrivial extension given by (9.6.18d). The uniqueness of the remaining non-split exact sequences follows analogously, where we determine $\operatorname{Ext}(\mathcal{M},\mathcal{N})$ by applying the functor $\operatorname{Hom}(-,\mathcal{N})$ to the projective presentation for the module $\mathcal{M}$ or by applying $\operatorname{Hom}(\mathcal{M},-)$ to the injective presentation of the module $\mathcal{N}$. ∎

Proposition 9.6.7. *The evaluation of the * functor of Proposition 9.3.2 on reducible indecomposable modules is given by*

$$(\mathcal{P}_n)^* \cong \mathcal{P}_n, \qquad \left(\mathcal{B}_n^m\right)^* \cong \mathcal{T}_n^m, \qquad \left(\mathcal{T}_n^m\right)^* \cong \mathcal{B}_n^m. \tag{9.6.22}$$

Proof. The evaluation of the * functor on the $\mathcal{B}$ and $\mathcal{T}$ modules follows inductively from the characterising sequences (9.6.18a), with the base step given by ${\mathcal{V}_m}^* \cong \mathcal{V}_m$. The self duality of $\mathcal{P}$ is a consequence of Proposition 9.5.4.3 and the fact that $\left(\mathcal{W}_0^{\pm}\right)^* = \mathcal{W}_0^{\mp}$, as argued above. ∎

9.7 Rigid tensor category

In this section we prove that fusion furnishes category $\mathscr{F}$ with the structure of a rigid tensor category and define evaluation and coevaluation maps for the simple projective modules to verify that these modules and maps satisfy the conditions required for rigidity.

Theorem 9.7.1. *Category $\mathscr{F}$ with the tensor structures defined by fusion is a braided tensor category.*

This theorem follows by verifying certain conditions which were proved to be sufficient in [20], and [24]. Recall the definitions in Section 6.1. Here we will go into more depth on the construction of the HLZ tensor product spaces in order to prove the above theorem.

Definition 9.7.2. Let $A \le B$ be abelian groups. Let V be a vertex algebra graded by A and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be modules over V, graded by B. We define the following properties for functionals $\psi \in \mathrm{Hom}(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathbb{C})$.

1. **P(w)-compatibility**:

   a) **Lower truncation**: For any $v \in \mathsf{V}$, $v_n\psi = 0$, for any sufficiently large $n \in \mathbb{Z}$.

   b) For any $v \in \mathsf{V}$ and $f \in \mathbb{C}[t, t^{-1}, (t^{-1} - w)^{-1}]$ the identity

   $$v f(t)\psi = v\iota_+(f(t))\psi \tag{9.7.1}$$

   holds. Here ι_+ means expanding about $t = 0$ such that the exponents of t are bounded below and the action of $\mathsf{V} \otimes \mathbb{C}[t, t^{-1}, (t^{-1} - w)^{-1}]$ or $\mathsf{V} \otimes \mathbb{C}((t))$ on ψ is characterised by

   $$\langle vg(t)\psi, m_1 \otimes m_2\rangle = \langle \psi, \iota_+ \circ T_w \left(v^{\mathrm{opp}} g(t^{-1})\right) m_1 \otimes m_2\rangle + \langle \psi, m_1 \otimes \iota_+\left(v^{\mathrm{opp}} g(t^{-1})\right) m_2\rangle, \tag{9.7.2}$$

   where $m_i \in \mathcal{M}_i$, $v \in \mathsf{V}$, $g \in \mathbb{C}[t, t^{-1}, (t^{-1} - w)^{-1}]$, T_w replaces t by $t + w$, $v^{\mathrm{opp}} = \mathsf{e}^{t^{-1}L_1}(-t^2)^{L_0} v t^{-2}$, and (assuming v has conformal weight h) $vt^n m_i = v_{n-h+1} m_i$.

   Denote by $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ the vector space of all $P(w)$-compatible functionals.

2. **P(w)-local grading restriction**:

   a) The functional ψ is a finite sum of vectors that are both B-homogeneous and L_0 generalised eigenvectors.

b) Denote the smallest subspace of $\mathrm{Hom}(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathbb{C})$ containing ψ and stable under $V \otimes \mathbb{C}[t, t^{-1}]$ by $\mathcal{M}_\psi$. Then $\mathcal{M}_\psi$ must satisfy for any $r \in \mathbb{C}, b \in B$

$$\dim\left(\mathcal{M}_\psi{}^{(b)}_{[r]}\right) < \infty, \qquad \text{and} \qquad \dim\left(\mathcal{M}_\psi{}^{(b)}_{[r+k]}\right) = 0, \tag{9.7.3}$$

for sufficiently large $k \in \mathbb{Z}$.

Denote by $\mathrm{LGR}(\mathcal{M}_1, \mathcal{M}_2)$ the vector space of all $P(w)$-local grading restricted functionals.

Define $\mathcal{M}_1 \boxslash \mathcal{M}_2 = \mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2) \cap \mathrm{LGR}(\mathcal{M}_1, \mathcal{M}_2)$.

Remark. The variable w in $P(w)$ denotes the insertion point of the tensor product constructed in [20], where it is usually denoted z and hence the tensor product is referred to as the $P(z)$-tensor product.

Theorem 9.7.3 (Huang-Lepowsky-Zhang [20, Part IV, Theorem 5.44, 5.45, 5.50])**.** *Let $A \leq B$ be abelian groups. Let V be a vertex algebra graded by A with a choice of module category $\mathscr{C}$ which is closed under contragredients and let $\mathcal{M}_1, \mathcal{M}_2 \in \mathscr{C}$ be graded by B. Then* $\mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ *and $\mathcal{M}_1 \boxslash \mathcal{M}_2$ are modules over V. Further, if $\mathcal{M}_1 \boxslash \mathcal{M}_2 \in \mathscr{C}$, then $\mathcal{M}_1 \boxtimes \mathcal{M}_2 \cong (\mathcal{M}_1 \boxslash \mathcal{M}_2)'$.*

In [20] $\mathcal{M}_1 \boxslash \mathcal{M}_2$ is originally defined as the image of all intertwining operators with $\mathcal{M}_1$ and $\mathcal{M}_2$ as factors, but it is then shown that this is equivalent to the definition given above. The construction of fusion products through Definition 9.7.2 is sometimes called the HLZ double dual construction. In addition to the primary reference [20], the survey [22] relates this construction of fusion to others in the literature.

Now we apply Theorem 6.3.1. Conditions 1 – 4 of Theorem 6.3.1 hold by construction for category $\mathscr{F}$, so all that remains is verifying Conditions 5 and 6.

Lemma 9.7.4. *The convergence and extension properties for products and iterates holds for $\mathscr{F}$.*

Proof. If, in the assumptions of Theorem 6.3.2, we set $\mathsf{V} = \mathsf{G}$ and grade by ghost weight, so that $A = \mathbb{Z}$, then the modules of $\mathscr{F}$ are graded by $B = \mathbb{R}$. We further choose $\overline{\mathsf{V}} = \mathsf{G}^{(0)}$, that is, the vertex subalgebra given by the ghost weight 0 subspace of G. The lemma then follows by verifying that all modules in $\mathscr{F}$ are discretely strongly graded and graded C_1-cofinite as modules over $\overline{\mathsf{V}}$.

All modules in $\mathscr{F}$ are discretely strongly graded by ghost weight $j \in \mathbb{R}$. To prove this, we need to check that the simultaneous ghost and conformal weight spaces are finite dimensional and that every ghost weight homogeneous space has lower bounded conformal weights. The simultaneous ghost and conformal weight spaces of objects in $\mathscr{R}$ and therefore also those of $\sigma^\ell\mathscr{R}$ are finite dimensional by construction. Thus, since the objects of $\mathscr{F}$ are finite length extensions of those in $\sigma^\ell\mathscr{R}$, the objects of $\mathscr{F}$ also have finite dimensional simultaneous ghost and conformal weight spaces. Similarly we have that the objects in $\mathscr{F}$ are graded lower bounded and therefore discretely strongly graded.

Next we need to decompose objects of $\mathscr{F}$ as $\overline{\mathsf{V}}$-modules,. It is known that $\overline{\mathsf{V}}$ is generated by $\{:\beta(z)(\partial^n\gamma(z)):, n\geq 0\}$ and is isomorphic to $W_{1+\infty}\cong W_{3,-2}\otimes\mathcal{H}$ where $W_{3,-2}$ is the singlet algebra at $c=-2$ and $\mathcal{H}$ is a rank 1 Heisenberg algebra [54, 55]. Note that the conformal vector of $W_{1+\infty}$ is usually chosen so as to have a central charge of 1. Since we require $\overline{\mathsf{V}}$ to embed conformally into G, that is, to have the same conformal vector as G and the central charge of G is 2, we choose conformal vector of our Heisenberg algebra $\mathcal{H}$ so that its central charge is 4 (the conformal structure of $W_{3,-2}$ is unique). Fortunately, this does not complicate matters, as the simple modules over $\mathcal{H}$ are just Fock spaces regardless of the central charge or conformal vector. The tensor factors of $W_{1+\infty}$ decompose nicely with respect to the free field realisation of Proposition 9.4.1.2. The Heisenberg algebra $\mathcal{H}$ is generated by $\theta(z)$ and the singlet algebra $W_{3,-2}$ is a vertex subalgebra of the Heisenberg algebra generated by $\psi(z)$.

We denote Fock spaces over the rank 1 Heisenberg algebras generated by ψ and θ, respectively, by the same symbol $\mathcal{F}_\mu$, where, the index $\mu\in\mathbb{C}$ indicates the respective eigenvalues of the zero modes ψ_0 and θ_0. All simple $\overline{\mathsf{V}}\cong W_{1+\infty}$ modules can be constructed via its free field realisation as $\mathcal{V}_{\langle\lambda,\psi\rangle}\otimes\mathcal{F}_{\langle\lambda,\theta\rangle}$ [56, Corollary 6.1], where $\mathcal{V}_{\langle\lambda,\psi\rangle}$, as a $W_{3,-2}$-module, is the simple quotient of the submodule of $\mathcal{F}_{\langle\lambda,\psi\rangle}$ generated by the highest weight vector. The homogeneous space $(\sigma^\ell\mathcal{V})^{(j)}$ is simple, as a $\overline{\mathsf{V}}$-module [54, Lemma 4.1], see also [57, 58]. Recall from Proposition 9.4.1.2 that with $K=\mathrm{span}_{\mathbb{Z}}\{\psi,\theta\}$ and $\Lambda\in L/K$, we can construct the simple projective G-modules as $\sigma^{\langle\Lambda,\psi+\theta\rangle}\mathcal{W}_{\langle\Lambda,\psi\rangle}\cong\mathbb{F}_\Lambda$. To identify the homogeneous space $\left(\sigma^{\langle\Lambda,\psi+\theta\rangle}\mathcal{W}_{\langle\Lambda,\psi\rangle}\right)^{(j)}$ as a $\overline{\mathsf{V}}$-module, we use the fact that $J(z)=-\theta(z)$, thus the $\mathbb{R}$-grading on $\mathbb{F}_\Lambda$ is given by the eigenvalue of $-\theta_0$. Therefore, for $j\in\mathbb{R}$,

$$\left(\sigma^{\langle\Lambda,\psi+\theta\rangle}\mathcal{W}_{\langle\Lambda,\psi\rangle}\right)^{(j)}\cong\mathbb{F}_\Lambda^{(j)}\cong\left(\bigoplus_{\lambda\in\Lambda}\mathcal{F}_{\langle\lambda,\psi\rangle}\otimes\mathcal{F}_{\langle\lambda,\theta\rangle}\right)^{(j)}=\begin{cases}\mathcal{F}_{\langle\Lambda,\psi+\theta\rangle+j}\otimes\mathcal{F}_{-j}, & j\in\langle\Lambda,\psi\rangle,\\ 0, & j\notin\langle\Lambda,\psi\rangle.\end{cases}\tag{9.7.4}$$

For $\langle\Lambda,\psi+\theta\rangle+j\notin\mathbb{Z}$, $\mathcal{F}_{\langle\Lambda,\psi+\theta\rangle+j}$ is irreducible as a $\mathcal{W}_{3,-2}$ module, by [59, Section 3.2], see also [60, Section 5]. Thus, $\left(\sigma^{\langle\Lambda,\psi+\theta\rangle}\mathcal{W}_{\langle\Lambda,\psi\rangle}\right)^{(j)}\cong\mathcal{F}_{\langle\Lambda,\psi+\theta\rangle+j}\otimes\mathcal{F}_{-j}=\mathcal{V}_{\langle\Lambda,\psi+\theta\rangle+j}\otimes\mathcal{F}_{-j}$. The finite length modules of $W_{3,-2}$ are all C_1-cofinite [61, Corollary 14], as are $\mathcal{H}$-modules, since Fock spaces have C_1-codimension 1. Therefore all $\overline{\mathsf{V}}$-modules appearing as the homogeneous spaces of modules in $\mathscr{F}$ are C_1-cofinite and the lemma follows. ∎

Lemma 9.7.5. *Let $\mathcal{M}_1, \mathcal{M}_2$ be modules in $\mathscr{F}$, let $\mathcal{W}$ be an indecomposable smooth (or weak)* G *module and let $\mathcal{Y}$ be a surjective logarithmic intertwining operator of type $\binom{\mathcal{W}}{\mathcal{M}_1,\ \mathcal{M}_2}$.*

1. *The logarithmic intertwining operator $\mathcal{Y}$ is grading compatible and the module $\mathcal{W}$ is graded by ghost weight.*

2. *If $\mathcal{M}_1 \in \sigma^k\mathscr{R}, \mathcal{M}_2 \in \sigma^\ell\mathscr{R}$, then $\mathcal{W} \in \mathscr{F}$ and $\mathcal{W}$ has composition factors only in $\sigma^{k+\ell}\mathscr{R}$ and $\sigma^{k+\ell-1}\mathscr{R}$.*

3. *If $\mathcal{M}_1$ has composition factors only in $\sigma^k\mathscr{R}$ and $\sigma^{k-1}\mathscr{R}$, and $\mathcal{M}_2$ has composition factors only in $\sigma^\ell\mathscr{R}$ and $\sigma^{\ell-1}\mathscr{R}$, $\mathcal{W} \in \mathscr{F}$ and $\mathcal{W}$ has composition factors only in $\sigma^{k+\ell+i}\mathscr{R}, -3 \le i \le 0$.*

Proof. Due to the compatibility of fusion with spectral flow, see Theorem 9.1.6, it is sufficient to only consider $k = \ell = 0$. We prove Part 1 first. Let $\mathcal{M}_1, \mathcal{M}_2$, be modules in $\mathscr{F}$. Let $v \in \mathsf{G}$ be the vector corresponding to the field $J(z)$ and take the residue with respect to x_0 and x_1 in the Jacobi identity (6.1.13). This yields

$$J_0\mathcal{Y}(m_1,x_2)m_2 = \mathcal{Y}(m_1,x_2)J_0m_2 + \mathcal{Y}(J_0m_1,x_2)m_2. \tag{9.7.5}$$

Hence, since the fusion factors $\mathcal{M}_i$ are graded by ghost weight, the fusion product will be too. This means that the intertwining operator will be grading compatible and $\mathcal{W}$ must be graded by ghost weight.

Next we prove Part 3. Assume that $\mathcal{M}_1, \mathcal{M}_2$ have composition factors only in $\mathscr{R}$ and $\sigma^{-1}\mathscr{R}$. Note that $J_n,\ n \ge 1$ acts locally nilpotently on any object in $\mathscr{F}$ and that $\beta_{n-\ell},\ \gamma_{n+\ell},\ n \ge 1$ act locally nilpotently on any object in $\sigma^\ell\mathscr{R}$ (recall that local nilpotence is one of the defining properties of $\sigma^\ell\mathscr{R}$). We first show that $J_n,\ \beta_{n+1},\ \gamma_n,\ n \ge 1$ acting locally nilpotently on $\mathcal{M}_1, \mathcal{M}_2$ implies that $J_n,\ \beta_{n+3},\ \gamma_n,\ n \ge 1$ act locally nilpotently on $\mathcal{W}$. Let h be the conformal weight of $v = \beta, \gamma$ or J, multiply both sides of the Jacobi identity (6.1.13) by $x_0^k x_1^{n+h-1},\ n,k \in \mathbb{Z}$ and take residues with respect to x_0 and x_1. This yields

$$\begin{aligned}\sum_{s\ge0}\binom{k}{s}(-1)^s x_2^s v_{n-s}\mathcal{Y}(m_1,x_2)m_2 &= \sum_{s\ge0}\binom{k}{s}(-1)^s x_2^{k-s}\mathcal{Y}(m_1,x_2)v_{n-k+s}m_2\\ &\quad+\sum_{s\ge0}\binom{s-n+k-h}{s}(-1)^s x_2^{n-k+h-s-1}\mathcal{Y}(v_{s-h+k+1}m_1,x_2)m_2.\end{aligned} \tag{9.7.6}$$

Set $v = \gamma$ (and thus $h = 0$) and $k = 0$ in (9.7.6) to obtain

$$\gamma_n\mathcal{Y}(m_1,x_2)m_2 = \mathcal{Y}(m_1,x_2)\gamma_n m_2 + \sum_{s=0}^{n}\binom{s-n}{s}(-1)^s x_2^{n-s-1}\mathcal{Y}(\gamma_{s+1}m_1,x_2)m_2. \tag{9.7.7}$$

This implies the local nilpotence of $\gamma_n,\ n \geq 1$ on $\mathcal{Y}(m_1,x_2)m_2$ from its local nilpotence on m_1 and m_2. Next consider $v = J$ (and thus $h = 1$) and $k = 1$ in (9.7.6) to obtain

$$(J_n - x_2 J_{n-1})\mathcal{Y}(m_1,x_2)m_2 = \mathcal{Y}(m_1,x_2)(J_n - x_2 J_{n-1})m_2 + \sum_{s=0}^{n} \binom{s-n}{s}(-1)^s x_2^{n-s-1}\mathcal{Y}(J_{s+1}m_1,x_2)m_2. \tag{9.7.8}$$

Since $J_k,\ k \geq 1$ is nilpotent on both m_1 and m_2, we see that $J_n - x_2 J_{n-1}$ is nilpotent for $n \geq 2$. Recall that the series expansion of the intertwining operator

$$\mathcal{Y}(m_1,x_2)m_2 = \sum_{\substack{t\in\mathbb{C}\\ s\geq 0}} (m_1)_{(t,s)} m_2 x_2^{-t-1}(\log x_2)^s \tag{9.7.9}$$

satisfies a lower truncation condition, that is, for fixed s, if there exists a $u \in \mathbb{C}$ satisfying $m_{(u,s)} \neq 0$, then there exists a minimal representative $t \in u+\mathbb{Z}$ such that $m_{(t,s)} \neq 0$ and $m_{(t',s)} = 0$ for all $t' < t$. Since $J_n - x_2 J_{n-1}$ is nilpotent on $\mathcal{Y}(m_1,x_2)m_2$ it is also nilpotent on the leading term $m_{(t,s)}$. By comparing coefficients of x_2 and $\log x_2$ it then follows that $J_n,\ n \geq 2$ acts nilpotently on $m_{(t,s)}$ and by induction also on all coefficients of higher powers of x_2. To show that J_1 acts locally nilpotently, assume that m_1 has J_0-eigenvalue j and set $n = 1,\ k = 0$ in (9.7.6) to obtain

$$J_1\mathcal{Y}(m_1,x_2)m_2 = \mathcal{Y}(m_1,x_2)J_1 m_2 + x_2 j\mathcal{Y}(m_1,x_2)m_2 + \sum_{s\geq 1}(-1)^s\binom{s-2}{s}x_2^{1-s}\mathcal{Y}(J_s m_1,x_2)m_2. \tag{9.7.10}$$

Thus $J_1 - x_2 j$ is nilpotent, which by the previous leading term argument implies that J_1 is too. Finally, consider $v = \beta$ (and thus $h = 1$) and $k = 2$ in (9.7.6) to obtain

$$\begin{aligned}(\beta_n - 2x_2\beta_{n-1} + x_2^2\beta_{n-2})\mathcal{Y}(m_1,x_2)m_2 &= \mathcal{Y}(m_1,x_2)(\beta_n - 2x_2\beta_{n-1} + x_2^2\beta_{n-2})m_2 \\ &\quad + \sum_{s\geq 0}\binom{s-n+1}{s}(-1)^s x_2^{n-s-2}\mathcal{Y}(\beta_{s+2}m_1,x_2)m_2.\end{aligned} \tag{9.7.11}$$

By leading term arguments analogous to those used for J_n, this implies that β_n acts locally nilpotently for $n \geq 4$.

Consider the subspace $V \subset \mathcal{W}$ annihilated by $\beta_{n+3}, \gamma_n, n \geq 1$. Then V is a module over four commuting copies of the Weyl algebra respectively generated by the pairs (β_0,γ_0), (β_1,γ_{-1}), (β_2,γ_{-2}), (β_3,γ_{-3}). Further, V is closed under the action of $J_n,\ n \geq 1$ and restricted to acting on V, the first few J_n modes expand as

$$J_3 = \beta_3\gamma_0, \qquad J_2 = \beta_2\gamma_0 + \beta_3\gamma_{-1}, \qquad J_1 = \beta_1\gamma_0 + \beta_2\gamma_{-1} + \beta_3\gamma_{-2}. \tag{9.7.12}$$

We show that on any composition factor of V at least three of the four Weyl algebras have a generator acting nilpotently and that thus the induction of such a composition factor is an object in one of the categories $\sigma^i\mathscr{R}$, $-3 \le i \le 0$. Let $C_0 \otimes C_1 \otimes C_2 \otimes C_3$ be isomorphic to a composition factor of V, where C_i is a simple module over the Heisenberg algebra generated by the pair (β_i, γ_{-i}). Since J_1, J_2, J_3 act locally nilpotently on V they must also do so on $C_0 \otimes C_1 \otimes C_2 \otimes C_3$ using the expansions (9.7.12). If we assume that neither β_3 nor γ_0 act locally nilpotently on C_3 and C_0, respectively, that is there exist $c_3 \in C_3$ and $c_0 \in C_0$ such that $\mathsf{U}(\beta_3)c_3$ and $\mathsf{U}(\gamma_0)c_0$ are both infinite dimensional, and choose c_1, c_2, to be non-zero vectors in C_1 and C_2, respectively. Then $\mathsf{U}(J_3)(c_0 \otimes c_1 \otimes c_2 \otimes c_3)$ will be infinite dimensional contradicting the local nilpotence of J_3. So assume β_3 acts locally nilpotently but γ_0 does not, and let $c_3 \in C_3$ be annihilated by β_3 and c_0, c_1, c_2 be non-zero vectors in C_0, C_2, C_3, respectively. On this vector J_2 evaluates to

$$J_2(c_0 \otimes c_1 \otimes c_2 \otimes c_3) = \gamma_0 c_0 \otimes c_1 \otimes \beta_2 c_2 \otimes c_3. \tag{9.7.13}$$

By the same reasoning as before, unless either β_2 or γ_0 act nilpotently, we have a contradiction to the nilpotence of J_2, so β_2 must act nilpotently on c_2. Repeating this argument for J_1 and assuming $\beta_2 c_2 = 0$ we have a contradiction to the nilpotence of J_1 unless β_1 acts nilpotently. The composition factor isomorphic to $C_0 \otimes C_1 \otimes C_2 \otimes C_3$ thus induces to an object in $\mathscr{R}$. Repeating the previous arguments, assuming that γ_0 acts locally nilpotently but β_3 does not, implies that γ_{-1} and γ_{-2} must act locally nilpotently to avoid contradictions to the local nilpotence of J_1, J_2, J_3. Such a composition factor would induce to a module in $\sigma^{-3}\mathscr{R}$. Finally assume both β_3 and γ_0 act locally nilpotently, then analogous arguments to those used above applied to the action of J_1 imply that at least one of β_2 or γ_{-1} act locally nilpotently. Such a composition factor would induce to an object in $\sigma^{-2}\mathscr{R}$ or $\sigma^{-1}\mathscr{R}$, respectively.

The final potential obstruction to $\mathcal{W}$ lying in $\mathscr{F}$ is that such a submodule might not be finite length. However, if $\mathcal{W}$ had infinite length, it would have to admit indecomposable subquotients of arbitrary finite length, yet by the classification of indecomposable modules in Theorem 9.6.2, a finite length indecomposable module with composition factors only in $\sigma^i\mathscr{R}$, $-3 \le i \le 0$ has length at most 5. Therefore $\mathcal{W} \in \mathscr{F}$.

Part 2 follows by a similar but simplified version of the above arguments. J_n and γ_n continue to satisfy the same nilpotence conditions as above, however for β one needs to reconsider (9.7.6) with $k = 1$ to conclude that β_n, $n \ge 2$ is nilpotent. The remainder of the argument follows analogously. ■

Proof of Theorem 9.7.1. We verify that the assumptions of Theorem 6.3.1 hold, in numerical order. Theorem 6.3.1 thus implies that category $\mathscr{F}$ is an additive braided tensor category. Additionally, since category $\mathscr{F}$ is abelian, it is a braided tensor category.

1. All modules in category $\mathscr{F}$ are strongly graded by ghost weight $j \in \mathbb{R}$. Further, by Lemma 9.7.5.1, all logarithmic intertwining operators are grading compatible.

2. By Proposition 9.3.2, category $\mathscr{F}$ is closed under taking contragredients. Closure under finite direct sums holds by construction, since category $\mathscr{F}$ is abelian.

3. The modules in $\mathscr{F}$ have real conformal weights by definition. The only modules on which the non semi-simple part of L_0 acts non-trivially are $\sigma^m \mathcal{P}_n$, for which it squares to zero.

4. Closure under images of module homomorphisms holds by construction, since category $\mathscr{F}$ is abelian.

5. The convergence and extension properties hold by Lemma 9.7.4.

6. Since the $P(w)$-tensor product is right exact, by [20, Part IV, Proposition 4.26], and since category $\mathscr{F}$ has sufficiently many projectives, that is, every module can be realised as a quotient of a direct sum of indecomposable projectives, we can without loss of generality assume $\mathcal{M}_1$ and $\mathcal{M}_2$ are indecomposable projective modules, as Condition 6 holding for projective modules implies that it also holds for their quotients. Further, due to the compatibility of fusion with spectral flow, we can pick $\mathcal{M}_1$ and $\mathcal{M}_2$ to be isomorphic to $\mathcal{W}_\lambda$ or $\mathcal{P}$. Let $\nu \in \mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$ be doubly homogenous and assume that the module $\mathcal{M}_\nu$ generated by ν is lower bounded. By assumption, the functional ν therefore satisfies all the properties of $P(w)$-local grading restriction except for the finite dimensionality of the doubly homogeneous spaces of $\mathcal{M}_\nu$. We need to show the finite dimensionality of these doubly homogeneous spaces and that $\mathcal{M}_\nu$ is an object in $\mathscr{F}$. Since $\mathcal{M}_\nu$ is finitely generated (cyclic even) it is at most a finite direct sum. To see this, assume the module admits an infinite direct sum. Then the partial sums define an ascending filtration whose union is the entire module. Hence after some finite number of steps all generators must appear within this filtration, but if this finite sum contains all generators, it must be equal to the entire module and hence all later direct summands must be zero. Denote the direct summands by $\mathcal{M}_{\nu,i}$, $i \in I$, where I is some finite index set. By [20, Part IV, Proposition 5.24] there exists a smooth G module $\mathcal{W}_{\nu,i}$ such that $\mathcal{W}'_{\nu,i} \cong \mathcal{M}_{\nu,i}$ and a surjective intertwiner of type $\binom{\mathcal{W}_{\nu,i}}{\mathcal{M}_1,\ \mathcal{M}_2}$. Hence, by Lemma 9.7.5, $\mathcal{W}_{\nu,i} \in \mathscr{F}$. In particular, since category $\mathscr{F}$ is closed under taking contragredients and all its objects have finite dimensional doubly homogeneous spaces, we have $\mathcal{M}_{\nu,i} \in \mathscr{F}$ and $\mathcal{M}_\nu \in \mathscr{F}$.

∎

Remark. Note that the above proof did not make any use of $\mathcal{M}_\nu$ being lower bounded to conclude that $\mathcal{M}_\nu \in \mathscr{F}$ and that membership of category $\mathscr{F}$ implies lower boundedness.

Lemma 9.7.6. *Let $\lambda \in \mathbb{R}/\mathbb{Z}$, $\lambda \neq \mathbb{Z}$,. Then exactly one direct summand of the fusion product $\sigma^\ell \mathcal{W}_\lambda \boxtimes \sigma^k \mathcal{W}_{-\lambda}$ is isomorphic to $\sigma^{\ell+k-1}\mathcal{P}$.*

We will prove the above lemma by showing that $\mathcal{W}_\lambda \boxbslash \mathcal{W}_{-\lambda}$ has exactly one submodule isomorphic to $\mathcal{P}$. This requires finding linear functionals which satisfy $P(w)$ compatibility. This is very difficult to do in practice, since (9.7.1) needs to be checked for every vector $v \in V$. Fortunately there is a result by Zhang which cuts this down to generators. Zhang originally formulated the theorem below for a related type of fusion product called the $Q(z)$-tensor product, so we have translated his result to the $P(w)$-tensor product, which we use here.

Theorem 9.7.7 (Zhang [62, Theorem 4.7])**.** *Let $A \leq B$ be abelian groups. Let V be a vertex algebra graded by A with a set of strong generators S and let $\mathcal{M}_1$ and $\mathcal{M}_2$ be modules over V, graded by B. A functional $\psi \in \mathrm{Hom}(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathbb{C})$ is said to satisfy the **strong lower truncation condition** for a vector $v \in \mathsf{V}$, if there exists an $N \in \mathbb{N}$ such that for all $n, m \in \mathbb{Z}$, with $m \geq N$, we have*

$$vt^{m+n}(t^{-1} - w)^n \psi = 0. \tag{9.7.14}$$

Then $\psi \in \mathrm{Hom}(\mathcal{M}_1 \otimes \mathcal{M}_2, \mathbb{C})$ satisfies the $P(w)$-compatibility condition if and only if it satisfies the strong lower truncation condition for all elements of S.

We further prepare some helpful identities.

Lemma 9.7.8. *Let $\mathcal{M}_1, \mathcal{M}_2 \in \mathscr{F}$, $m_i \in \mathcal{M}_i$, $i = 1, 2$, and $\psi \in \mathrm{COMP}(\mathcal{M}_1, \mathcal{M}_2)$, then we have the identities, for $k, n \in \mathbb{Z}$.*

$$\langle J_n\psi, m_1 \otimes m_2\rangle = \delta_{n,0}\langle \psi, m_1 \otimes m_2\rangle - \sum_{i \geq 0} \binom{-n}{i} w^{-n-i} \langle \psi, J_i m_1 \otimes m_2\rangle - \langle \psi, m_1 \otimes J_{-n} m_2\rangle, \tag{9.7.15}$$

$$\langle L_0\psi, m_1 \otimes m_2\rangle = \langle \psi, L_0 m_1 \otimes m_2\rangle + w\langle \psi, L_{-1} m_1 \otimes m_2\rangle + \langle \psi, m_1 \otimes L_0 m_2\rangle, \tag{9.7.16}$$

$$\begin{aligned} \left\langle \beta t^{k+n}(t^{-1} - w)^n \psi, m_1 \otimes m_2 \right\rangle &= -\sum_{i \geq 0} \binom{-k-n}{i} w^{-k-n-i} \langle \psi, \beta_{n+i} m_1 \otimes m_2\rangle \\ &\quad - \sum_{i \geq 0} \binom{n}{i} (-w)^{n-i} \langle \psi, m_1 \otimes \beta_{i-k-n} m_2\rangle, \end{aligned} \tag{9.7.17}$$

$$\left\langle \gamma t^{k+n}(t^{-1} - w)^n \psi, m_1 \otimes m_2 \right\rangle = \sum_{i \geq 0} \binom{-k-n-2}{i} w^{-k-n-2-i} \langle \psi, \gamma_{n+i+1} m_1 \otimes m_2\rangle$$

$$+\sum_{i\geq 0}\binom{n}{i}(-w)^{n-i}\langle\psi, m_1\otimes\gamma_{i-k-n-1}m_2\rangle. \tag{9.7.18}$$

Proof. These identities follow by evaluating (9.7.2) for the fields β, γ, J and T. ∎

Proof of Lemma 9.7.6. We shall use the details of the HLZ double dual construction covered in Definition 9.7.2. By the compatibility of fusion with spectral flow, Theorem 9.1.6, it is sufficient to consider the case $\ell = k = 0$. Note since $\sigma^{-1}\mathcal{P}$ is both projective and injective, it must be a direct summand if it appears as either a quotient or a subspace. Further, by Lemma 9.7.5, all composition factors must lie in categories $\sigma^i\mathscr{R}$, $i = -1, 0$. This implies that the composition factors of $\mathcal{W}_\lambda\boxtimes\mathcal{W}_{-\lambda}$ must all lie in $\sigma^i\mathscr{R}$, $i = 0, 1$. Note further, that $(\sigma^{-1}\mathcal{P})' \cong \mathcal{P}$ and so we seek to find a copy of $\mathcal{P}$ within $\mathcal{W}_\lambda\boxtimes\mathcal{W}_{-\lambda}$. We do so by considering a certain characterising two dimensional subspace of $\mathcal{P}$. For a G-module $\mathcal{M}$ consider the subspace

$$K(\mathcal{M}) = \{m \in \mathcal{M} \ :\ \beta_n m = \gamma_{n+1} m = J_1 m = J_0 m = 0,\ n \geq 1\}. \tag{9.7.19}$$

From the expansions of $T(z)$ and $J(z)$ in terms of the fields β and γ, it follows that for any $m \in K(\mathcal{M})$, $L_0^2 m = L_n m = J_n m = 0$, $n \geq 1$. In particular, in the notation of Figure 9.2, $K(\mathcal{P}) = \mathrm{span}_{\mathbb{C}}\{|0\rangle, |-\psi\rangle\}$ and thus $K(\mathcal{P})$ is two dimensional and L_0 has a rank 2 Jordan block of generalised eigenvalue 0 on this space. Further, $\mathcal{P}$ is the only indecomposable module with composition factors in categories $\sigma^i\mathscr{R}$, $i = 0, 1$ admitting L_0 Jordan blocks. The remaining indecomposable modules with composition factors in categories $\sigma^i\mathscr{R}$, $i = 0, 1$ all have $K(\mathcal{M})$ subspaces of dimension zero or one.

Let $\psi \in \mathrm{Hom}(\mathcal{W}_\lambda\otimes\mathcal{W}_{-\lambda}, \mathbb{C})$ satisfy $\beta t^{k+n}(t^{-1}-w)^n\psi = \gamma t^{k+n}(t^{-1}-w)^n\psi = 0$ for all $m \geq 1$. Thus by Theorem 9.7.7, ψ satisfies the $P(w)$-compatibility property and $\beta_m\psi = \gamma_{m+1}\psi = 0$, $m \geq 1$. If in addition ψ is doubly homogeneous, then ψ lies in $\mathcal{W}_\lambda\boxtimes\mathcal{W}_{-\lambda}$. By assumption the left-hand sides of (9.7.17) and (9.7.18) vanish for $k \geq 1$. These relations imply that the value of ψ on any vector in $\mathcal{W}_\lambda\otimes\mathcal{W}_{-\lambda}$ is determined by its value on tensor products of relaxed highest weight vectors, because negative modes on one factor can be traded for less negative modes on the other factor. For example, for $k = 1$, $n = 0$ in (9.7.17), we have the relation

$$\langle\psi, m_1\otimes\beta_{-1}m_2\rangle = -\sum_{i\geq 0}\binom{-1}{i}w^{-1-i}\langle\psi, \beta_i m_1\otimes m_2\rangle. \tag{9.7.20}$$

Let $u_{\pm j} \in \mathcal{W}_{\pm\lambda}$, $j \in \pm\lambda$ be a choice of normalisation of relaxed highest weight vectors satisfying $u_{\pm j-1} = \gamma_0 u_{\pm j}$. This implies $\beta_0 u_{\pm j} = \pm j u_{1\pm j}$. Since the negative β and γ modes act freely on the simple projective modules $\mathcal{W}_\lambda$ and $\mathcal{W}_{-\lambda}$, there are no relations in addition to those coming from $\beta t^{k+n}(t^{-1}-w)^n\psi = \gamma t^{k+n}(t^{-1}-w)^n\psi = 0$ for all $m \geq 1$. Thus there is a linear isomorphism

$$\{\psi \in \mathcal{W}_\lambda\boxtimes\mathcal{W}_{-\lambda} \ :\ \beta_n\psi = \gamma_{n+1}\psi = 0,\ n \geq 1\} \xrightarrow{\cong} \mathrm{Hom}\big(\mathrm{span}_{\mathbb{C}}\{u_j\otimes u_{-i}\}, \mathbb{C}\big). \tag{9.7.21}$$

Clearly, $K(\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda})$ is a subspace of $\{\psi \in \mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda} \;:\; \beta_n\psi = \gamma_{n+1}\psi = 0,\ n \geq 1\}$ and so we impose the remaining two relations, the vanishing of J_0 and J_1, via (9.7.15). The vanishing of $J_0\psi$ implies

$$0 = \langle J_0\psi, u_j \otimes u_{-i}\rangle = \langle \psi, u_j \otimes u_{-i}\rangle - \langle \psi, J_0 u_j \otimes u_{-i}\rangle - \langle \psi, u_j \otimes J_0 u_{-i}\rangle = (1-j+i)\langle \psi, u_j \otimes u_{-i}\rangle. \tag{9.7.22}$$

Thus ψ vanishes on $u_j \otimes u_{-i}$ unless $i = j-1$. The vanishing of $J_1\psi$ implies

$$(2j-1)\langle \psi, u_j \otimes u_{1-j}\rangle - j\langle \psi, u_{j+1} \otimes u_{-j}\rangle + (1-j)\langle \psi, u_{j-1} \otimes u_{2-j}\rangle = 0, \tag{9.7.23}$$

where we have used $J_{-1}u_{1-j} = (\gamma_{-1}\beta_0 + \beta_{-1}\gamma_0)u_{1-j}$. Thus ψ is completely characterised by its value on a two pairs of relaxed highest weight vectors, say $u_j \otimes u_{1-j}$ and $u_{j+1} \otimes u_{-j}$. Therefore, the subspace $K(\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda})$ is two dimensional. Next we show that that L_0 has a rank two Jordan block on it when acting on this space. Let $\psi \in K(\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda})$. If $\psi \neq 0$, then there exist $a, b \in \mathbb{C}$, not both zero, such that

$$\langle \psi, u_j \otimes u_{1-j}\rangle = a, \qquad \langle \psi, u_{j+1} \otimes u_{-j}\rangle = b. \tag{9.7.24}$$

The evaluation of $L_0\psi$ on $u_j \otimes u_{1-j}$ and $u_{j+1} \otimes u_{-j}$ is then

$$\langle L_0\psi, u_j \otimes u_{1-j}\rangle = j(a-b), \qquad \langle L_0\psi, u_{j+1} \otimes u_{-j}\rangle = -j(a-b). \tag{9.7.25}$$

Therefore if $a \neq b$ (a choice which we can make as $K(\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda})$ is two dimensional), the vectors ψ and $L_0\psi$ are linearly independent and span $K(\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda})$, which also shows that L_0 has a rank two Jordan block. ■

Remark. In [46, Section 7] the above fusion product was computed using the NGK algorithm up to certain conjectured additional conditions. In light of the survey [22] explaining the equivalence of the HLZ double dual construction and the NGK algorithm, it seemed appropriate to supplement the NGK calculation of [46] with an HLZ double dual calculation here.

Proposition 9.7.9. *For all $\ell \in \mathbb{Z}$ and $\lambda \in \mathbb{R}/\mathbb{Z}$, $\lambda \neq \mathbb{Z}$, the simple module $\sigma^\ell \mathcal{W}_\lambda$ is rigid in category $\mathscr{F}$, with tensor dual given by $(\sigma^\ell \mathcal{W}_\lambda)^\vee = \sigma^{1-\ell}\mathcal{W}_{-\lambda}$.*

Proof. Recall that a module $\mathcal{M}$ is rigid if there exists an object $\mathcal{M}^\vee$ (called a tensor dual of $\mathcal{M}$) and two morphisms $e_\mathcal{M} : \mathcal{M}^\vee \boxtimes \mathcal{M} \to \mathcal{V}$ and $i_\mathcal{M} : \mathcal{V} \to \mathcal{M} \boxtimes \mathcal{M}^\vee$, respectively, called evaluation and coevaluation, such that the compositions

$$\mathcal{M} \cong \mathcal{V} \boxtimes \mathcal{M} \xrightarrow{i_\mathcal{M} \otimes 1} (\mathcal{M} \boxtimes_{w_2} \mathcal{M}^\vee) \boxtimes_{w_1} \mathcal{M} \xrightarrow{\mathcal{A}^{-1}} \mathcal{M} \boxtimes_{w_2} (\mathcal{M}^\vee \boxtimes_{w_1} \mathcal{M}) \xrightarrow{1 \otimes e_\mathcal{M}} \mathcal{M} \boxtimes \mathcal{V} \cong \mathcal{M}, \tag{9.7.26a}$$

$$\mathcal{M}^\vee \cong \mathcal{M}^\vee \boxtimes \mathcal{V} \xrightarrow{1\otimes i_{\mathcal{M}}} \mathcal{M}^\vee \boxtimes_{w_2} \left(\mathcal{M} \boxtimes_{w_1} \mathcal{M}^\vee\right) \xrightarrow{\mathcal{A}} \left(\mathcal{M}^\vee \boxtimes_{w_2} \mathcal{M}\right) \boxtimes_{w_1} \mathcal{M}^\vee \xrightarrow{e_{\mathcal{M}}\otimes 1} \mathcal{V} \boxtimes \mathcal{M}^\vee \cong \mathcal{M}^\vee, \tag{9.7.26b}$$

yield the identity maps $1_{\mathcal{M}}$ and $1_{\mathcal{M}^\vee}$, respectively. Here w_1, w_2 are distinct non-zero complex numbers satisfying $|w_2| > |w_1|$ and $|w_2| > |w_2 - w_1|$; $\boxtimes_w$ indicates the relative positioning of insertion points of fusion factors, that is, the right most factor will be inserted at 0, the middle factor at w_1 and the left most at w_2; Technically there exist distinct notions of left and right duals and the above properties are those for left duals. We prove below that $\mathcal{M} = \sigma^\ell \mathcal{W}_\lambda$ is left rigid. Right rigidity follows from left rigidity due to category $\mathscr{F}$ being braided.

For $\mathcal{M} = \sigma^\ell \mathcal{W}_\lambda$ we take the tensor dual to be $\mathcal{M}^\vee = \sigma^{1-\ell}\mathcal{W}_{-\lambda}$ and we will construct the evaluation and coevaluation morphisms using the first free field realisation (9.4.11) given in Proposition 9.4.1.1. In particular, we have

$$\sigma^\ell \mathcal{W}_\lambda \cong \mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi}, \qquad \sigma^{1-\ell}\mathcal{W}_{-\lambda} \cong \mathbb{F}_{-\lambda(\theta+\psi)-\ell\psi}, \qquad \ell \in \mathbb{Z},\ \lambda \in \mathbb{R}/\mathbb{Z},\ \lambda \neq \mathbb{Z}. \tag{9.7.27}$$

We denote fusion over the lattice vertex algebra V_K of the free field realisation by $\boxtimes^{\mathrm{ff}}$ to distinguish it from fusion over G. Recall that the fusion product of Fock spaces over the lattice vertex algebra V_K of the free field realisation just adds Fock space weights. Thus the fusion product over V_K of the modules corresponding to $\sigma^\ell \mathcal{W}_\lambda$ and $\sigma^{1-\ell}\mathcal{W}_{-\lambda}$ is given by

$$\mathbb{F}_{-\lambda(\theta+\psi)-\ell\psi} \boxtimes^{\mathrm{ff}} \mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi} \cong \mathbb{F}_{-\psi} \cong \mathcal{W}_0^-. \tag{9.7.28}$$

Therefore we have the V_K-module map $\mathcal{Y} : \mathbb{F}_{-\lambda(\theta+\psi)-\ell\psi} \boxtimes^{\mathrm{ff}} \mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi} \to \mathbb{F}_{-\psi}$ given by the intertwining operator that maps the kets in the Fock space $\mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi}$ to vertex operators, that is, operators of the form (9.4.4). Since V_K-module maps are also G-module maps by restriction and since the fusion product of two modules over a vertex subalgebra is a quotient of the fusion product over the larger vertex algebra, $\mathcal{Y}$ also defines a G-module map $\mathbb{F}_{-\lambda(\theta+\psi)-\ell\psi} \boxtimes \mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi} \to \mathbb{F}_{-\psi} \cong \mathcal{W}_0^-$. Furthermore, the screening operator $\mathcal{S}_1 = \oint Y(\psi, z)\mathrm{d}z$ defines a G-module map $\mathcal{S}_1 : \mathbb{F}_{-\psi} \to \mathbb{F}_0$ with the image being the bosonic ghost vertex algebra G. Up to a normalisation factor, to be determined later, we define the evaluation map for $\mathcal{M} = \sigma^\ell \mathcal{W}_\lambda$ to be the composition of $\mathcal{Y}$ and the screening operator $\mathcal{S}_1$.

$$e_{\mathcal{M}} = \mathcal{S}_1 \circ \mathcal{Y} : \mathcal{M}^\vee \boxtimes \mathcal{M} \to \mathcal{V}. \tag{9.7.29}$$

To define the coevaluation we need to identify a submodule of $\mathcal{M} \boxtimes \mathcal{M}^\vee$ isomorphic to $\mathcal{V}$. By Lemma 9.7.6, we know that $\mathcal{M} \boxtimes \mathcal{M}^\vee$ has a unique direct summand isomorphic to $\mathcal{P}$, which by Proposition 9.5.4 we know has a unique submodule isomorphic to $\mathcal{V}$. It is this copy of $\mathcal{V}$ which the coevaluation shall map to. Since $\mathcal{V}$ is the vector space underlying the vertex algebra G and any vertex algebra is generated from its vacuum vector, we characterise the coevaluation map by the image of the vacuum vector.

$$\begin{aligned} i_M : |0\rangle &\longrightarrow |0\rangle \xrightarrow{\mathcal{S}_1^{-1}} |-\psi\rangle \longrightarrow Y((j-1)\psi + (j-\ell)\theta, w)|-j\psi - (j-\ell)\theta\rangle \\ &\xrightarrow{\mathcal{S}_1} \oint_w \mathcal{S}_1(z) Y((j-1)\psi + (j-\ell)\theta, w)|-j\psi - (j-\ell)\theta\rangle \mathrm{d}z, \end{aligned} \tag{9.7.30}$$

where the first arrow is the inclusion of $\mathcal{V}$ into $\mathbb{F}_0 \cong \mathcal{W}_0^- \subset \mathcal{P}$, $\mathcal{S}_1^{-1}$ denotes picking preimages of $\mathcal{S}_1$ and j the unique representative of the coset λ satisfying $0 < j < 1$. The third arrow maps the highest weight vector $|-\psi\rangle$ to its corresponding expression in the fusion product $\mathbb{F}_{-\lambda(\theta+\psi)-\ell\psi} \boxtimes^{\mathrm{ff}} \mathbb{F}_{\lambda(\theta+\psi)+(\ell-1)\psi}$. Note that the ambiguity of picking preimages of $\mathcal{S}_1$ in the second arrow is undone by reapplying $\mathcal{S}_1$ in the fourth arrow and hence the map is well-defined. This map maps to $\mathbb{F}_0$, which is a submodule of $\mathcal{P}$ as shown in Proposition 9.5.4.

Note that since the modules $\mathcal{M}$ and $\mathcal{M}^\vee$ considered here are simple, the compositions of coevaluations and evaluations (9.7.26) are proportional to the identity by Schur's lemma. Rigidity therefore follows, if we can show that the proportionality factors for (9.7.26a) and (9.7.26b) are equal and non-zero.

We determine the proportionality factor for (9.7.26a) by applying the map to the ket $|(j-1)\psi + (j-\ell)\theta\rangle \in \mathbb{F}_{\lambda(\psi+\theta)+(\ell-1)\theta} \cong \sigma^\ell \mathcal{W}_\lambda$. Following the sequence of maps in (9.7.26a) we get

$$\begin{aligned} &|(j-1)\psi + (j-\ell)\theta\rangle \to |0\rangle \boxtimes |(j-1)\psi + (j-\ell)\theta\rangle \\ &\to \oint_{w_1,w_2} \mathcal{S}_1(z) Y((j-1)\psi - (j-\ell)\theta, w_2) Y(-j\psi - (j-\ell)\theta, w_1)|(j-1)\psi - (j-\ell)\theta\rangle \mathrm{d}z \\ &\to \oint_{0,w_1} \oint_{w_1,w_2} \mathcal{S}_1(z_2)\mathcal{S}_1(z_1) Y((j-1)\psi + (j-\ell)\theta, w_2) \\ &\qquad\qquad Y(-j\psi - (j-\ell)\theta, w_1)|(j-1)\psi + (j-\ell)\theta\rangle \mathrm{d}z_1 \mathrm{d}z_2, \end{aligned} \tag{9.7.31}$$

where $\oint_{0,w_2}$ denotes a contour about 0 and w_2 but not w_1, $\oint_{w_1,w_2}$ denotes a contour about w_1 and w_2 but not 0, and where we have used the fact that the integration contours lie in domains in which the product and iterate of the vertex operators are equal. The proportionality factor is obtained by pairing the above with the dual of the Fock space highest weight vector, which we denote by an empty bra $\langle|$. Evaluating these matrix elements requires the associativity isomorphisms for replacing products of intertwining operators with their iterates. This is a characterising property of associativity isomorphisms for vertex operator algebra module categories, see [20, Part VII, Theorem 10.6]. The intertwining operators we are considering here are all lattice intertwining operators with lattice modules as codomains. Further, all lattice module endomorphisms are scalar multiples of the identity, therefore the associativity isomorphisms are also scalars. By the remark after (9.4.10) , these intertwining operators can be normalised such that the associativity scalars are 1. Note that this does not imply that the associativity isomorphisms are trivial on all of category $\mathscr{F}$, just that they can be scaled away when only considering lattice modules that are isomorphic to simple G modules in the free field realisation. The proportionality factor from (9.7.26a) is therefore given by the matrix element

$$\begin{aligned}
&I(w_1,w_2)\\
&=\oint_{0,w_1}\oint_{w_1,w_2}\langle|\mathcal{S}_1(z_2)\mathcal{S}_1(z_1)Y((j-1)\psi+(j-\ell)\theta,w_2)Y(-j\psi-(j-\ell)\theta,w_1)|(j-1)\psi+(j-\ell)\theta\rangle \mathrm{d}z_1\mathrm{d}z_2\\
&=f(w_1,w_2)\oint_{0,w_1}\oint_{w_1,w_2}(z_2-z_1)z_2^{j-1}(z_2-w_2)^{j-1}(z_2-w_1)^{-j}z_1^{j-1}(z_1-w_2)^{j-1}(z_1-w_1)^{-j}\mathrm{d}z_1\mathrm{d}z_2\\
&=f(w_1,w_2)\bigg(\oint_{0,w_1}z^j(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\oint_{w_1,w_2}z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\\
&\qquad-\oint_{0,w_1}z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\oint_{w_1,w_2}z^{j}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\bigg),
\end{aligned}\tag{9.7.32}$$

where

$$f(w_1,w_2)=(w_2-w_1)^{\ell^2+j(1-2\ell)}w_2^{(j-1)(2j-\ell-1)}w_1^{\ell^2+j(1-2\ell)}.\tag{9.7.33}$$

Note that the second equality of (9.7.32) is where the associativity isomorphisms are used to pass from compositions (or products) of vertex operators to their operator product expansions (also called iterates). For intertwining operators, associativity amounts to the analytic continuation of their series expansions and then reexpanding in a different domain. On the left-hand side of the second equality the intertwining operators (or here specifically vertex operators) are in radial ordering, while on the right-hand side they have been analytically continued and then reexpanded as operator product expansions. By an analogous argument the proportionality factor produced by the sequence of maps (9.7.26b) is the matrix element

$$\begin{aligned}
&\tilde{I}(w_1,w_2)\\
&=\oint_{0,w_1}\oint_{w_1,w_2}\langle|\mathcal{S}_1(z_2)\mathcal{S}_1(z_1)Y(-j\psi-(j-\ell)\theta,w_2)Y((j-1)\psi+(j-\ell)\theta,w_1)|-j\psi-(j-\ell)\theta\rangle \mathrm{d}z_1\mathrm{d}z_2\\
&=f(w_1,w_2)\bigg(\oint_{0,w_1}z^j(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\oint_{w_1,w_2}z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\\
&\qquad-\oint_{0,w_1}z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\oint_{w_1,w_2}z^{j}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z\bigg).
\end{aligned}\tag{9.7.34}$$

Since both matrix elements are equal, $I(w_1,w_2)=\tilde{I}(w_1,w_2)$, rigidity follows by showing that they are non-zero.

We evaluate the four integrals appearing in $I(w_1,w_2)$. We simplify the first integral using the substitution $z=w_1x$.

$$\begin{aligned}\oint_{0,w_1} z^j(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z &= -w_2^{j-1}w_1\oint_{0,1} x^j(1-x)^{-j}\left(1-\frac{w_1}{w_2}x\right)^{j-1}\mathrm{d}x\\ &= -\left(e^{2\pi ij}-1\right)w_2^{j-1}w_1\int_0^1 x^j(1-x)^{-j}\left(1-\frac{w_1}{w_2}x\right)^{j-1}\mathrm{d}x\\ &= -\left(e^{2\pi ij}-1\right)w_2^{j-1}w_1\,B(1+j,1-j)\,{}_2F_1\left(1-j,1+j;2;\frac{w_1}{w_2}\right),\end{aligned} \tag{9.7.35}$$

where the second equality follows by deforming the contour about 0 and 1 to a dumbbell or dog bone contour, whose end points vanish because the contributions from the end points are $O(\varepsilon^{1+j})$ and $O(\varepsilon^{1-j})$ respectively, and $0<j<1$; and the third equality is the integral representation of the hypergeometric function and B is the beta function. Similarly,

$$\oint_{0,w_1} z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z = -\left(e^{2\pi ij}-1\right)w_2^{j-1}\,\mathrm{B}(j,1-j)\,{}_2F_1\left(1-j,j;1;\frac{w_1}{w_2}\right). \tag{9.7.36}$$

For the integrals with contours about w_1 and w_2 we use the substitution $z=w_2-(w_2-w_1)x$ and then again obtain integral representations of the hypergeometric function.

$$\begin{aligned}\oint_{w_1,w_2} z^{j-1}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z &= (-1)^j\left(e^{2\pi ij}-1\right)w_2^{j-1}B(j,1-j){}_2F_1\left(1-j,j;1;\frac{w_2-w_1}{w_2}\right),\\ \oint_{w_1,w_2} z^{j}(z-w_2)^{j-1}(z-w_1)^{-j}\mathrm{d}z &= (-1)^j\left(e^{2\pi ij}-1\right)w_2^{j}B(j,1-j){}_2F_1\left(-j,j;1;\frac{w_2-w_1}{w_2}\right).\end{aligned} \tag{9.7.37}$$

Note that for the three integrals above, the end point contributions of the contour also vanish due to being $O(\varepsilon^j)$ and $O(\varepsilon^{1-j})$ for 0 and 1 respectively.

Making use of the hypergeometric and beta function identities

$$\begin{aligned}{}_2F_1\left(1-\mu,1+\mu;2;\frac{w_2}{w_1}\right) &= \frac{w_1}{w_2}\,{}_2F_1\left(-\mu,\mu;1;1-\frac{w_2}{w_1}\right),\\ {}_2F_1\left(1-\mu,\mu;1;1-\frac{w_2}{w_1}\right) &= {}_2F_1\left(1-\mu,\mu;1;\frac{w_2}{w_1}\right),\\ \mathrm{B}(1+\mu,1-\mu) = \mu\,\mathrm{B}(\mu,1-\mu) &= \frac{\pi\mu}{\sin(\pi\mu)},\end{aligned} \tag{9.7.38}$$

the proportionality factor $I(w_1,w_2)$ simplifies to

$$I(w_1,w_2) = (-1)^j f(w_1,w_2)\left(e^{2\pi ij}-1\right)^2 w_2^{2j-1}\frac{\pi^2(j-1)}{\sin(\pi j)^2}\,{}_2F_1\left(-j,j;1;\frac{w_2-w_1}{w_2}\right){}_2F_1\left(1-j,j;1;\frac{w_2}{w_1}\right). \tag{9.7.39}$$

Since $j \notin \mathbb{Z}$, $I(w_1, w_2)$ can only vanish, if one of the hypergeometric factors does. We specialise the complex numbers w_1, w_2, such that $w_2 = 2w_1$, as if $I(w_1, w_2)$ vanishes for some choice of w_1, w_2 then it vanishes for all of them. Then,

$$ {}_2F_1\left(1-j, j; 1; \frac{w_1}{w_2}\right) = {}_2F_1\left(1-j, j; 1; \tfrac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1)}{\Gamma\left(1-\frac{j}{2}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2}\right)} \neq 0, \tag{9.7.40} $$

and the relationship between contiguous hypergeometric functions implies

$$ {}_2F_1\left(-j, j; 1; \frac{w_2 - w_1}{w_2}\right) = \tfrac{1}{2}\left({}_2F_1\left(1-j, j; 1; \tfrac{1}{2}\right) + {}_2F_1\left(-j, 1+j; 1; \tfrac{1}{2}\right)\right) \tag{9.7.41} $$

$$ = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma(1)}{\Gamma\left(1-\frac{j}{2}\right)\Gamma\left(\frac{1}{2}+\frac{j}{2}\right)} \neq 0. \tag{9.7.42} $$

Thus $I(w_1, w_2) \neq 0$ and we can rescale the evaluation map by $I(w_1, w_2)^{-1}$ so that the sequences of maps (9.7.26) are equal to the identity maps on $\mathcal{M}$ and $\mathcal{M}^\vee$. Thus $\sigma^\ell \mathcal{W}_\lambda$ is rigid. ■

9.8 Fusion product formulae

In this section we determine the decomposition of all fusion products in category $\mathscr{F}$. A complete list of fusion products among representatives of each spectral flow orbit is collected in Theorem 9.8.1, while the proofs of these decomposition formulae have been split into the dedicated Subsections 9.9 and 9.10. To simplify some of the decomposition formulae we introduce dedicated notation for certain sums of spectral flows of the projective module $\mathcal{P}$. Consider the polynomial of spectral flows

$$ f_n(\sigma) = \sum_{k=1}^{n} \sigma^{2k-1}, \qquad n \in \mathbb{N}, \tag{9.8.1} $$

and let

$$ \mathcal{Q}^n = f_n(\sigma)\mathcal{P} = \bigoplus_{k=1}^{n} \mathcal{P}_{2k-1}, \qquad n \in \mathbb{N}. \tag{9.8.2} $$

Further, let

$$ \mathcal{Q}^n_k = \sigma^k \mathcal{Q}^n, \qquad \mathcal{Q}^{m,n}_k = \sigma^{k-1} f_m(\sigma)\mathcal{Q}^n = \bigoplus_{r=1}^{m+n-1} N_r\, \mathcal{P}_{k+2r-1}, \quad N_r = \min\{r, m, n, m+n-r\}, \tag{9.8.3} $$

where $m, n \in \mathbb{N}$, $k \in \mathbb{Z}$.

Theorem 9.8.1.

1. *Category $\mathscr{F}$ under fusion is a rigid braided tensor category.*

2. *The following is a list of all non-trivial fusion products, those not involving the fusion unit (the vacuum module $\mathcal{V}$), in category $\mathscr{F}$ among representatives for each spectral flow orbit. All other fusion products are determined from these through spectral flow and the compatibility of spectral flow with fusion as given in Theorem 9.1.6.*

   Since $\mathscr{F}$ is rigid, the fusion product of a projective module $\mathcal{R}$ with any indecomposable module $\mathcal{M}$ is given by

$$\mathcal{R} \boxtimes \mathcal{M} \cong \bigoplus_{\mathcal{S}} [\mathcal{M} : \mathcal{S}]\, \mathcal{R} \boxtimes \mathcal{S}, \tag{9.8.4}$$

   where the summation index runs over all isomorphism classes of composition factors of $\mathcal{M}$ and $[\mathcal{M} : \mathcal{S}]$ is the multiplicity of the composition factor $\mathcal{S}$ in $\mathcal{M}$.

   For all $\lambda, \mu \in \mathbb{R}/\mathbb{Z}$, $\lambda, \mu, \lambda+\mu \neq \mathbb{Z}$,

$$\begin{aligned} \mathcal{W}_\lambda \boxtimes \mathcal{W}_\mu &\cong \mathcal{W}_{\lambda+\mu} \oplus \sigma^{-1}\mathcal{W}_{\lambda+\mu}, \\ \mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda} &\cong \sigma^{-1}\mathcal{P}. \end{aligned} \tag{9.8.5}$$

   For $m, n \in \mathbb{Z}$, $m \geq n$, such that the lengths of indecomposables below are positive, we have the following fusion product formulae.

$$\begin{aligned} \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{B}^{2m+2n+1} \oplus \mathcal{Q}_1^{m,n} & \mathcal{T}^{2m+1} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{T}^{2m+2n+1} \oplus \mathcal{Q}_1^{m,n} \\ \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{B}^{2n} \oplus \mathcal{Q}_1^{m,n} & \mathcal{T}^{2m+1} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{T}^{2n} \oplus \mathcal{Q}_1^{m,n} \\ \mathcal{B}^{2m} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{B}^{2n}_{2m-1} \oplus \mathcal{B}^{2n} \oplus \mathcal{Q}_1^{m-1,n} & \mathcal{T}^{2m} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{T}^{2n}_{2m-1} \oplus \mathcal{T}^{2n} \oplus \mathcal{Q}_1^{m-1,n} \end{aligned} \tag{9.8.6a}$$

$$\begin{aligned} \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{T}^{2m-2n+1}_{2n} \oplus \mathcal{Q}^{m+1,n} & \mathcal{B}^{2m+1} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{B}^{2m-2n+1}_{2n} \oplus \mathcal{Q}^{m+1,n} \\ \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{T}^{2m}_{2n} \oplus \mathcal{Q}^{m,n} & \mathcal{B}^{2m} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{B}^{2m}_{2n} \oplus \mathcal{Q}^{m,n} \\ \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{Q}^{m,n} & \mathcal{B}^{2m} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{Q}^{m,n} \end{aligned} \tag{9.8.6b}$$

We split the proof of Theorem 9.8.1 into multiple parts. Theorem 9.8.1.1 is shown in Proposition 9.9.3. The fusion formulae (9.8.5),(9.8.6a),(9.8.6b) are determined in Propositions 9.9.1, 9.10.4 and 9.10.5 and Lemma 9.7.6, respectively.

Remark. The fusion product formulae of Theorem 9.8.1 projected onto the Grothendieck group match the conjectured Verlinde formula of [46, Corollaries 7 and 10], thereby proving that category $\mathscr{F}$ satisfies the standard module formalism version of the Verlinde formula. It will be an interesting future problem to find a more conceptual and direct proof for the validity of the Verlinde formula, rather than a proof by inspection.

9.9 Fusion products of simple projective modules

In this section we determine the fusion products of the simple projective modules.

Proposition 9.9.1. *For $\lambda, \mu \in \mathbb{R}/\mathbb{Z}$, $\lambda, \mu, \lambda+\mu \notin \mathbb{Z}$, we have*

$$\mathcal{W}_\lambda \boxtimes \mathcal{W}_\mu \cong \mathcal{W}_{\lambda+\mu} \oplus \sigma^{-1}\mathcal{W}_{\lambda+\mu}. \tag{9.9.1}$$

Proof. Since $\mathcal{W}_\lambda$ and $\mathcal{W}_\mu$ both lie in category $\mathscr{R}$, we know, by Lemma 9.7.5, that the composition factors of the fusion product lie in categories $\mathscr{R}$ or $\sigma^{-1}\mathscr{R}$. Further, since $J(z)$ is a conformal weight 1 field, its corresponding weight, the ghost weight, adds under fusion. Therefore the only possible composition factors are $\mathcal{W}_{\lambda+\mu}$ and $\sigma^{-1}\mathcal{W}_{\lambda+\mu}$. Since these composition factors are both projective and injective, they can only appear as direct summands and all that remains is to determine their multiplicity. In [45] Adamović and Pedić computed dimensions of spaces of intertwining operators for fusion products of the simple projective modules. In particular, [45, Corollary 6.1] states that

$$\dim \begin{pmatrix} \mathcal{M} \\ \mathcal{W}_\lambda, \ \mathcal{W}_\mu \end{pmatrix} = 1, \tag{9.9.2}$$

if $\mathcal{M}$ is isomorphic to $\sigma^\ell \mathcal{W}_{\lambda+\mu}$, $\ell = 0, -1$. Thus the proposition follows. ∎

Remark. To prove the above proposition directly without citing the literature, we could have used the two free field realisations in Section 9.4 to construct intertwining operators of the type appearing in equation (9.9.2), thereby showing that the dimension of the corresponding space of intertwining operators is at least 1. This was also done in [45]. An upper bound of 1 can then easily be determined by calculations involving either the HLZ double dual construction (similar to the calculations done in Lemma 9.7.6) or the NGK algorithm.

Proposition 9.9.2. *For $\lambda \in \mathbb{R}/\mathbb{Z}$, $\lambda \neq \mathbb{Z}$, we have*

$$\mathcal{W}_\lambda \boxtimes \mathcal{W}_{-\lambda} \cong \sigma^{-1}\mathcal{P}. \tag{9.9.3}$$

Proof. By Proposition 9.7.9, $\mathcal{W}_\lambda$ is rigid and hence its fusion product with a projective module must again be projective. Further, by Lemma 9.7.5, all composition factors must lie in categories $\sigma^\ell\mathscr{R}$, $\ell = -1, 0$. Finally, since ghost weights add under fusion, the ghost weights of the fusion product must lie in $\mathbb{Z}$. Thus the fusion product must be isomorphic to a direct sum of some number of copies of $\sigma^{-1}\mathcal{P}$. By Lemma 9.7.6, we know there is exactly one such summand. ∎

Proposition 9.9.3. *Category $\mathscr{F}$ is rigid.*

Proof. Category $\mathscr{F}$ has sufficiently many injective and projective modules, that is, all simple modules have projective covers and injective hulls, and all projectives are injective and vice-versa. Further, the simple projective modules $\sigma^\ell \mathcal{W}_\lambda$ are rigid and generate the non-simple projective modules under fusion, so all projective modules are rigid. Category $\mathscr{F}$ is therefore a Frobenius category (it has enough projectives and enough injectives, where the classes of projectives and injectives coincide) and hence any for short exact sequence with two rigid terms (whose duals are also rigid) the third term is also rigid. This implies that all modules are rigid and hence so is category $\mathscr{F}$. ∎

Corollary 9.9.4. *Let* $\mathcal{M}, \mathcal{N} \in \mathscr{F}$, *then*

$$\mathcal{M}^* \boxtimes \mathcal{N}^* \cong (\mathcal{M} \boxtimes \mathcal{N})^*. \tag{9.9.4}$$

Proof. Due to rigidity, the tensor duality functor $^\vee$ defines an equivalence of categories and is therefore exact. Further, the tensor duality functor satisfies

$$\mathcal{M}^\vee \boxtimes \mathcal{N}^\vee \cong (\mathcal{M} \boxtimes \mathcal{N})^\vee. \tag{9.9.5}$$

This also implies that ${\mathcal{V}_k}^\vee = \mathcal{V}_{-k}$. We see that the tensor dual $\mathcal{M}^\vee$ agrees with $\sigma(\mathcal{M}')$ on all simple modules in $\mathscr{F}$. As both $(-)^\vee$ and $\sigma(-)'$ are exact contravariant invertible functors and all reducible indecomposable objects are uniquely characterised by the non-split exact sequences (9.6.18) with 1-dimensional corresponding extension groups, it follows by induction in module length that $\mathcal{M}^\vee \cong \sigma(\mathcal{M}')$ for any module in $\mathscr{F}$. Recalling $(-)^* = c(-)'$, we further have $M^* \cong \sigma \mathsf{c} M^\vee$. Theorem 9.1.6 then implies

$$\mathcal{M}^* \boxtimes \mathcal{N}^* \cong \left(\sigma \mathsf{c} \mathcal{M}^\vee\right) \boxtimes \left(\sigma \mathsf{c} \mathcal{N}^\vee\right) \cong \sigma \mathsf{c} (\mathcal{M} \boxtimes \mathcal{N})^\vee \cong (\mathcal{M} \boxtimes \mathcal{N})^*. \tag{9.9.6}$$

∎

9.10 Fusion products of reducible indecomposable modules

In this section we calculate the remaining fusion product formulae involving indecomposable modules in $\mathscr{F}$. The main tool for determining these fusion products is that category $\mathscr{F}$ is rigid by Proposition 9.9.3. Hence fusion is biexact and projective modules form a tensor ideal (Proposition 2.4.3). We begin by calculating certain basic fusion products from which the remainder can be determined inductively.

Lemma 9.10.1.

$$\mathcal{T}^2 \boxtimes \mathcal{B}^2 \cong \mathcal{P}_1,$$

$$\mathcal{B}^2 \boxtimes \mathcal{B}^2 \cong \mathcal{B}^2 \oplus \mathcal{B}^2_1,$$
$$\mathcal{T}^2 \boxtimes \mathcal{T}^2 \cong \mathcal{T}^2 \oplus \mathcal{T}^2_1. \tag{9.10.1}$$

Proof. Taking the short exact sequence (9.2.4a) for $\mathcal{W}_0^+ = \mathcal{T}^2_{-1}$ and fusing it with $\mathcal{W}_0^- = \mathcal{B}^2_{-1}$ yields the short exact sequence

$$0 \longrightarrow \mathcal{W}_0^- \longrightarrow \mathcal{W}_0^+ \boxtimes \mathcal{W}_0^- \longrightarrow \sigma^{-1}\mathcal{W}_0^- \longrightarrow 0. \tag{9.10.2}$$

Similarly, fusing the short exact sequence (9.2.4b) for $\mathcal{W}_0^-$ with $\mathcal{W}_0^+$ yields

$$0 \longrightarrow \sigma^{-1}\mathcal{W}_0^+ \longrightarrow \mathcal{W}_0^- \boxtimes \mathcal{W}_0^+ \longrightarrow \mathcal{W}_0^+ \longrightarrow 0. \tag{9.10.3}$$

If either of the above exact sequences splits there is a contradiction, because if $\sigma^{-1}\mathcal{W}_0^+$ and $\mathcal{W}_0^+$ are direct summands of $\mathcal{W}_0^+ \boxtimes \mathcal{W}_0^-$, (9.10.2) is not exact, and if $\mathcal{W}_0^-$ and $\sigma^{-1}\mathcal{W}_0^-$ are direct summands, (9.10.3) is not exact. Hence both sequences must be non-split. As can be read off from the tables in Corollary 9.6.4, $\dim \operatorname{Ext}(\sigma^{-1}\mathcal{W}_0^-, \mathcal{W}_0^-) = \dim \operatorname{Ext}(\mathcal{W}_0^+, \sigma^{-1}\mathcal{W}_0^+) = 1$. There is only one candidate for the middle coefficient of these exact sequences, namely $\sigma^{-1}\mathcal{P}$. Thus the first fusion rule follows. The other two fusion products by are determined by fusing $\mathcal{W}_0^\pm$ with the short exact sequences for $\mathcal{W}_0^\pm$. The extension groups corresponding to these fused exact sequences are zero-dimensional and hence the sequences split and the lemma follows. ∎

We further prepare the following Ext group dimensions for later use.

Lemma 9.10.2. *The indecomposable modules $\mathcal{T}^{2n+1}$, $\mathcal{B}^m_{2n+1}$, $\mathcal{B}^{2n}$ and $\mathcal{B}^m_{2n}$ satisfy*

$$\dim \operatorname{Ext}\left(\mathcal{T}^{2n+1}, \mathcal{B}^m_{2n+1}\right) = \dim \operatorname{Ext}\left(\mathcal{B}^{2n}, \mathcal{B}^m_{2n}\right) = 1. \tag{9.10.4}$$

The corresponding extensions are given by $\mathcal{T}^{2n+m+1}$ and $\mathcal{B}^{2n+m}$ respectively.

Proof. We start with the following presentation of $\mathcal{T}^{2n+1}$

$$0 \longrightarrow \mathcal{T}^{2n+2} \longrightarrow \mathsf{P}[\mathcal{T}^{2n+1}] \longrightarrow \mathcal{T}^{2n+1} \longrightarrow 0. \tag{9.10.5}$$

Applying the functor $\operatorname{Hom}(-, \mathcal{B}^m_{2n+1})$ yields

$$0 \to \operatorname{Hom}\left(\mathcal{T}^{2n+1}, \mathcal{B}^m_{2n+1}\right) \to \operatorname{Hom}\left(\mathsf{P}[\mathcal{T}^{2n+1}], \mathcal{B}^m_{2n+1}\right) \to \operatorname{Hom}\left(\mathcal{T}^{2n+2}, \mathcal{B}^m_{2n+1}\right) \to \operatorname{Ext}\left(\mathcal{T}^{2n+1}, \mathcal{B}^m_{2n+1}\right) \to 0. \tag{9.10.6}$$

The first coefficient vanishes due to $\mathcal{T}^{2n+1}$ and $\mathcal{B}^m_{2n+1}$ having no common composition factors. The second coefficient can be shown to vanish using the projective cover formulae in Lemma 9.6.5 and reading off Hom group dimensions from the Loewy diagrams. For the third coefficient, the only composition factor common to both $\mathcal{T}^{2n+2}$ and $\mathcal{B}^m_{2n+1}$ is $\mathcal{V}_{2n+1}$, which occurs as a quotient for $\mathcal{T}^{2n+2}$ and a submodule for $\mathcal{B}^m_{2n+1}$, so this gives rise to a one dimensional Hom group. The vanishing Euler characteristic then implies that $\dim \mathrm{Ext}(\mathcal{T}^{2n+1}, \mathcal{B}^m_{2n+1}) = 1$ as expected. Furthermore, we can examine $\mathcal{T}^{2n+m+1}$ to see that it has a $\mathcal{B}^m_{2n}$ submodule which yields $\mathcal{T}^{2n+1}$ when quotiented out, therefore this is the unique extension characterised by $\mathrm{Ext}(\mathcal{T}^{2n+1}, \mathcal{B}^m_{2n+1})$.

We can follow the same procedure starting with the projective presentation of $\mathcal{B}^{2n}$ to obtain the following exact sequence

$$0 \longrightarrow \mathrm{Hom}\Big(\mathcal{B}^{2n}, \mathcal{B}^m_{2n}\Big) \longrightarrow \mathrm{Hom}\Big(\mathsf{P}[\mathcal{B}^{2n}], \mathcal{B}^m_{2n}\Big) \longrightarrow \mathrm{Hom}\Big(\mathcal{B}^{2n}_1, \mathcal{B}^m_{2n}\Big) \longrightarrow \mathrm{Ext}\Big(\mathcal{B}^{2n+1}, \mathcal{B}^m_{2n}\Big) \longrightarrow 0. \tag{9.10.7}$$

By the same argument as above we can calculate the Hom groups, and vanishing Euler characteristic implies $\dim \mathrm{Ext}(\mathcal{B}^{2n}, \mathcal{B}^m_{2n}) = 1$. Similarly we see that $\mathcal{B}^{2n+m}$ provides an extension of $\mathcal{B}^{2n}$ by $\mathcal{B}^m_{2n}$ and must therefore be the unique one. ■

We can now determine fusion products when one factor has length 2 and the other has arbitrary length.

Lemma 9.10.3. *The fusion products of length 2 indecomposables with any indecomposable of types $\mathcal{B}$ or $\mathcal{T}$ satisfy the following decomposition formulae.*

$$\begin{aligned}
\mathcal{B}^{2n+1} \boxtimes \mathcal{B}^2 &\cong \mathcal{B}^2 \oplus \mathcal{Q}^n_1 & \qquad \mathcal{T}^{2n+1} \boxtimes \mathcal{T}^2 &\cong \mathcal{T}^2 \oplus \mathcal{Q}^n_1 \\
\mathcal{B}^{2n+2} \boxtimes \mathcal{B}^2 &\cong \mathcal{B}^2_{2n+1} \oplus \mathcal{B}^2 \oplus \mathcal{Q}^n_1 & \qquad \mathcal{T}^{2n+2} \boxtimes \mathcal{T}^2 &\cong \mathcal{T}^2_{2n+1} \oplus \mathcal{T}^2 \oplus \mathcal{Q}^n_1 \\
\mathcal{B}^{2n+1} \boxtimes \mathcal{T}^2 &\cong \mathcal{T}^2_{2n} \oplus \mathcal{Q}^n & \qquad \mathcal{T}^{2n+1} \boxtimes \mathcal{B}^2 &\cong \mathcal{B}^2_{2n} \oplus \mathcal{Q}^n \\
\mathcal{B}^{2n} \boxtimes \mathcal{T}^2 &\cong \mathcal{Q}^n & \qquad \mathcal{T}^{2n} \boxtimes \mathcal{B}^2 &\cong \mathcal{Q}^n
\end{aligned} \tag{9.10.8}$$

Proof. We prove the left column of identities. The right column then follows from Corollary 9.9.4 and applying the * functor. We start with the short exact sequence (9.6.18e) satisfied by $\mathcal{B}^{2n+1}$,

$$0 \longrightarrow \mathcal{B}^{2n-1} \longrightarrow \mathcal{B}^{2n+1} \longrightarrow \mathcal{T}^2_{2n-1} \longrightarrow 0. \tag{9.10.9}$$

We then take the fusion product with $\mathcal{B}^2$,

$$0 \longrightarrow \mathcal{B}^{2n-1} \boxtimes \mathcal{B}^2 \longrightarrow \mathcal{B}^{2n+1} \boxtimes \mathcal{B}^2 \longrightarrow \mathcal{P}_{2n} \longrightarrow 0. \tag{9.10.10}$$

Because $\mathcal{P}_{2n}$ is projective, the sequence splits and we have the recurrence relation

$$\mathcal{B}^{2n+1} \boxtimes \mathcal{B}^2 \cong (\mathcal{B}^{2n-1} \boxtimes \mathcal{B}^2) \oplus \mathcal{P}_{2n}. \tag{9.10.11}$$

Then, the first fusion product formula of the lemma follows by induction with $\mathcal{B}^1 = \mathcal{V}$ as the base case.

We next consider the short exact sequence (9.6.18c) and fuse it with $\mathcal{B}^2$ to obtain

$$0 \longrightarrow \mathcal{B}^{2n+1} \boxtimes \mathcal{B}^2 \longrightarrow \mathcal{B}^{2n+2} \boxtimes \mathcal{B}^2 \longrightarrow \mathcal{B}^2_{2n+1} \longrightarrow 0. \tag{9.10.12}$$

Since $\mathrm{Ext}\big(\mathcal{B}^2_{2n+1}, \mathcal{B}^2\big) = 0$, by the tables in Corollary 9.6.4, this sequence splits and we obtain the second fusion product of the lemma.

For the final two fusion products, we perform the same exercises with different exact sequences. For the third and fourth fusion products we use (9.6.18d), with odd and even length respectively. Fusing with $\mathcal{T}^2$ gives the short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathcal{B}^{2n-1}_2 \boxtimes \mathcal{T}^2 \longrightarrow \mathcal{B}^{2n+1} \boxtimes \mathcal{T}^2 \longrightarrow \mathcal{P}_1 \longrightarrow 0,\\ 0 \longrightarrow \mathcal{B}^{2n}_2 \boxtimes \mathcal{T}^2 \longrightarrow \mathcal{B}^{2n+2} \boxtimes \mathcal{T}^2 \longrightarrow \mathcal{P}_1 \longrightarrow 0. \end{aligned} \tag{9.10.13}$$

In both cases, the sequences split because $\mathcal{P}_1$ is projective. ■

We now use Lemma 9.10.3 to prove the fusion product formulae (9.8.6a) of Theorem 9.8.1.

Proposition 9.10.4. *The fusion products of indecomposable modules of types $\mathcal{B}$ and $\mathcal{T}$ with themselves satisfy the decomposition formulae below, for $m \geq n$.*

$$\begin{aligned} \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{B}^{2m+2n+1} \oplus \mathcal{Q}^{m,n}_1 & \mathcal{T}^{2m+1} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{T}^{2m+2n+1} \oplus \mathcal{Q}^{m,n}_1\\ \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{B}^{2n} \oplus \mathcal{Q}^{m,n}_1 & \mathcal{T}^{2m+1} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{T}^{2n} \oplus \mathcal{Q}^{m,n}_1\\ \mathcal{B}^{2m} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{B}^{2n}_{2m-1} \oplus \mathcal{B}^{2n} \oplus \mathcal{Q}^{m-1,n}_1 & \mathcal{T}^{2m} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{T}^{2n}_{2m-1} \oplus \mathcal{T}^{2n} \oplus \mathcal{Q}^{m-1,n}_1 \end{aligned} \tag{9.10.14}$$

Proof. We prove the left column of identities. The right column then follows from Corollary 9.9.4 and applying the * functor. First, for both superscripts odd, we take two short exact sequences (9.6.18d) and (9.6.18e) for $\mathcal{B}^{2n+1}$ and fuse with $\mathcal{B}^{2m+1}$ to find

$$\begin{aligned} 0 \longrightarrow \mathcal{B}^{2n-1}_2 \boxtimes \mathcal{B}^{2m+1} \longrightarrow \mathcal{B}^{2n+1} \boxtimes \mathcal{B}^{2m+1} \longrightarrow \mathcal{B}^2 \oplus \mathcal{Q}^m_1 \longrightarrow 0,\\ 0 \longrightarrow \mathcal{B}^{2n-1} \boxtimes \mathcal{B}^{2m+1} \longrightarrow \mathcal{B}^{2n+1} \boxtimes \mathcal{B}^{2m+1} \longrightarrow \mathcal{T}^2_{2n+2m-1} \oplus \mathcal{Q}^m_{2n-1} \longrightarrow 0. \end{aligned} \tag{9.10.15}$$

Now comparing these exact sequences, and using the fact that $\mathcal{P}$ is projective, we find that the sequences cannot both split, as they would give different direct sums. For the first short exact sequence, we use Lemma 9.10.2, to find $\dim \mathrm{Ext}\big(\mathcal{B}^2, \mathcal{B}^{2m+2n-1}_2\big) = 1$, with the extension being given by $\mathcal{B}^{2m+2n+1}$ so we can determine the fusion product formulae inductively to get

$$\mathcal{B}^{2m+1} \boxtimes \mathcal{B}^3 \cong \mathcal{B}^{2m+3} \oplus \mathcal{Q}^m_1,$$

$$\mathcal{B}^{2m+1} \boxtimes \mathcal{B}^5 \cong \mathcal{B}^{2m+5} \oplus \left(1+\sigma^2\right)\mathcal{Q}_1^m,$$

$$\mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n+1} \cong \mathcal{B}^{2m+2n+1} \oplus \bigoplus_{k=1}^{m} \mathcal{Q}_{2k-1}^n = \mathcal{B}^{2m+2n+1} \oplus \mathcal{Q}_1^{m,n}. \tag{9.10.16}$$

We can deduce the remaining rules from short exact sequences that relate even and odd $\mathcal{B}$s. Firstly, we take the two short exact sequences (9.6.18d) and (9.6.18f), and fuse them with $\mathcal{B}^{2m+1}$ to get

$$0 \longrightarrow \mathcal{B}_2^{2m+1} \boxtimes \mathcal{B}^{2n} \longrightarrow \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n+2} \longrightarrow \mathcal{B}^2 \oplus \mathcal{Q}_1^m \longrightarrow 0,$$

$$0 \longrightarrow \mathcal{B}_{2n}^2 \oplus \mathcal{Q}_{2n+1}^m \longrightarrow \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n+2} \longrightarrow \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n} \longrightarrow 0. \tag{9.10.17}$$

Either of these exact sequences splitting would lead to a contradiction, hence both must be non-split. Further, by Lemma 9.10.2 we find $\dim \operatorname{Ext}\left(\mathcal{B}^2, \mathcal{B}_2^{2n}\right) = \dim \operatorname{Ext}\left(\mathcal{B}^{2n}, \mathcal{B}_{2n}^2\right) = 1$, with the corresponding non-split extension given by $\mathcal{B}^{2n+2}$. Therefore

$$\mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n} \cong \mathcal{B}^{2n} \oplus \mathcal{Q}_1^{m,n}. \tag{9.10.18}$$

Finally we fuse (9.6.18c) with $\mathcal{B}^{2n}$ to find

$$0 \longrightarrow \mathcal{B}^{2m+1} \boxtimes \mathcal{B}^{2n} \longrightarrow \mathcal{B}^{2m+2} \boxtimes \mathcal{B}^{2n} \longrightarrow \mathcal{B}_{2m+1}^{2n} \longrightarrow 0. \tag{9.10.19}$$

For $m \geq n$, $\dim \operatorname{Ext}\left(\mathcal{B}_{2m+1}^{2n}, \mathcal{B}^{2n}\right) = 0$, which follows because the composition factors are separated by at least two units of spectral flow and $\operatorname{Ext}(\mathcal{V}_n, \mathcal{V}_m) = 0$ for $|n-m| > 1$, the above sequence splits. In the case when $m = n-1$, we have that $\operatorname{Ext}\left(\mathcal{B}_{2n-1}^{2n}, \mathcal{V}_k\right) = 0$ for all the composition factors of $\mathcal{B}^{2n}$, that is, $(0 \leq k \leq 2n-1)$. Hence $\dim \operatorname{Ext}\left(\mathcal{B}_{2n-1}^{2n}, \mathcal{B}^{2n}\right) = 0$ and the above sequence again splits. Thus,

$$\mathcal{B}^{2m+2} \boxtimes \mathcal{B}^{2n} \cong \mathcal{B}_{2m+1}^{2n} \oplus \mathcal{B}^{2n} \oplus \mathcal{Q}_1^{m,n}, \qquad m \geq n-1. \tag{9.10.20}$$

∎

Proposition 9.10.5. *The fusion products of indecomposable modules of types $\mathcal{B}$ and $\mathcal{T}$ with each other satisfy the decomposition formulae below, for $m \geq n$.*

$$\begin{aligned} \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{T}_{2n}^{2m-2n+1} \oplus \mathcal{Q}^{m+1,n} & \qquad \mathcal{B}^{2m+1} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{B}_{2n}^{2m-2n+1} \oplus \mathcal{Q}^{m+1,n} \\ \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{T}_{2n}^{2m} \oplus \mathcal{Q}^{m,n} & \mathcal{B}^{2m} \boxtimes \mathcal{T}^{2n+1} &\cong \mathcal{B}_{2n}^{2m} \oplus \mathcal{Q}^{m,n} \\ \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n} &\cong \mathcal{Q}^{m,n} & \mathcal{B}^{2m} \boxtimes \mathcal{T}^{2n} &\cong \mathcal{Q}^{m,n} \end{aligned} \tag{9.10.21}$$

Proof. We prove the left column of identities. The right column then follows from Corollary 9.9.4 and applying the * functor to each module. We start with sequences (9.6.18e) and (9.6.18d) for odd length $\mathcal{B}$, and fuse them with $\mathcal{T}^{2m+1}$ to find

$$0 \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n-1} \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n+1} \longrightarrow \mathcal{T}^{2}_{2n-1} \oplus \mathcal{Q}^{m}_{2n} \longrightarrow 0, \tag{9.10.22}$$

$$0 \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n-1}_{2} \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n+1} \longrightarrow \mathcal{B}^{2}_{2m} \oplus \mathcal{Q}^{m} \longrightarrow 0. \tag{9.10.23}$$

Specialising to n=1 we have

$$0 \longrightarrow \mathcal{T}^{2m+1} \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{3} \longrightarrow \mathcal{T}^{2}_{1} \oplus \bigoplus_{k=1}^{m} \sigma^{2k+1}\mathcal{P} \longrightarrow 0, \tag{9.10.24}$$

$$0 \longrightarrow \mathcal{T}^{2m+1}_{2} \longrightarrow \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{3} \longrightarrow \mathcal{B}^{2}_{2m} \oplus \bigoplus_{k=1}^{m} \sigma^{2k-1}\mathcal{P} \longrightarrow 0. \tag{9.10.25}$$

Since $\mathcal{P}$ is projective, its spectral flows must appear as direct summands in the middle coefficient of the above exact sequences. Thus,

$$\mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{3} \cong \mathcal{A} \oplus \bigoplus_{k=1}^{m+1} \sigma^{2k-1}\mathcal{P} = \mathcal{A} \oplus \mathcal{Q}^{m+1}. \tag{9.10.26}$$

Therefore the module $\mathcal{A}$ satisfies the exact sequences

$$\begin{aligned} &0 \longrightarrow \mathcal{T}^{2m+1} \longrightarrow \mathcal{A} \oplus \mathcal{P}_{1} \longrightarrow \mathcal{T}^{2}_{1} \longrightarrow 0, \\ &0 \longrightarrow \mathcal{T}^{2m+1}_{2} \longrightarrow \mathcal{A} \oplus \mathcal{P}_{2m+1} \longrightarrow \mathcal{B}^{2}_{2m} \longrightarrow 0. \end{aligned} \tag{9.10.27}$$

Because either of these sequences splitting would lead to a contradiction and the corresponding extension groups are one-dimensional, the sequences uniquely characterise the fusion product. Proceeding by induction, we obtain

$$\begin{aligned} \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{3} &\cong \mathcal{T}^{2m-1}_{2} \oplus \mathcal{Q}^{m+1}, \\ \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{5} &\cong \mathcal{T}^{2m-3}_{4} \oplus \left(1+\sigma^{2}\right)\mathcal{Q}^{m+1}, \\ \mathcal{T}^{2m+1} \boxtimes \mathcal{B}^{2n+1} &\cong \mathcal{T}^{2m-2n+1}_{2n} \oplus \mathcal{Q}^{m+1,n}. \end{aligned} \tag{9.10.28}$$

Next we take two short exact sequences (9.6.18c) and (9.6.18a), for $\mathcal{T}^{2m}$ and fuse them with $\mathcal{B}^{2n+1}$ to get

$$\begin{aligned} &0 \longrightarrow \mathcal{B}^{2n+1}_{2m-1} \longrightarrow \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n+1} \longrightarrow \mathcal{T}^{2m-2n-1}_{2n} \oplus \mathcal{Q}^{m,n} \longrightarrow 0, \\ &0 \longrightarrow \mathcal{B}^{2m+2n-1}_{1} \oplus \mathcal{Q}^{m-1,n}_{2} \longrightarrow \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n+1} \longrightarrow \mathcal{B}^{2n+1} \longrightarrow 0. \end{aligned} \tag{9.10.29}$$

Again either of these sequences splitting would lead to a contradiction, and by Lemma 9.10.2, $\dim \operatorname{Ext}\big(\mathcal{T}_{2n}^{2m-2n-1}, \mathcal{B}_{2m-1}^{2n+1}\big) = 1$, with the extension being given by $\mathcal{T}_{2n}^{2m}$, so the second fusion rule follows. Finally, fusing (9.6.18f) with $\mathcal{B}^{2n}$, we have

$$0 \longrightarrow \mathcal{T}^{2m-2} \boxtimes \mathcal{B}^{2n} \longrightarrow \mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n} \longrightarrow \mathcal{Q}_{2m-2}^{n} \longrightarrow 0, \tag{9.10.30}$$

$$\mathcal{T}^{2m} \boxtimes \mathcal{B}^{2n} \cong \bigoplus_{k=1}^{m} \mathcal{Q}_{2k-2}^{n} = \mathcal{Q}^{m,n}. \tag{9.10.31}$$

∎

— Chapter 10 —

Quantum Enveloping Algebras of $\mathfrak{gl}_2$

"I would have the judicious reader pause before accusing such asseverations of an undue quantum of absurdity."

— Edgar Allan Poe, *Loss of Breath*

Quantum groups are quasitriangular Hopf algebras which are deformations of the universal enveloping algebras of certain Lie algebras. Here we study a one parameter deformation of the universal enveloping algebra of $\mathfrak{gl}_2$, whose representation theory will yield an equivalent category to the bosonic ghosts. We will prove this equivalence at the level of ribbon Grothendieck-Verdier structure, up to one condition which is an equation satisfied by a subset of intertwining operators.

10.1 Definition

Let $p \geq 2$ be a positive integer and $q = e^{\pi i/p}$ a primitive $2p$-th root of unity. Unless otherwise stated, subscript indices run over all possible values. We start by defining three algebras which will be the quantum analogues of the Cartan subalgebras for our quantum groups.

Definition 10.1.1. Let the associative algebra H_0 be the quotient of the polynomial ring $\mathbb{C}[K_1, K_1{}^{-1}, K_2, K_2{}^{-1}, X_1, X_2]$ by the relations $K_i{}^{-1}K_i = \mathbf{1}$.

Definition 10.1.2. Let the associative algebra H_1 be the quotient of the polynomial ring $\mathbb{C}[K_1, K_1{}^{-1}, K_2, K_2{}^{-1}, X_1]$ by the relations $K_i{}^{-1}K_i = \mathbf{1},\ K_1{}^p - \mathbf{1} = 0$.

Definition 10.1.3. Let the associative algebra H_2 be the quotient of the polynomial ring $\mathbb{C}[K_1, K_1{}^{-1}, K_2, K_2{}^{-1}]$ by the relations $K_i{}^{-1}K_i = \mathbf{1},\ K_i{}^p - \mathbf{1}$.

In contrast to Section 8.4, here we include more grouplike elements K_i but we impose the condition $K_i^p = 1$, which corresponds to imposing $K_\Lambda = 1$. Therefore, the inclusion of this extra generator yields the same module category, as we shall see in the next section.

Proposition 10.1.4. *H_n for $n = 0,1,2$ can be endowed with Hopf algebra structure given by counit $\varepsilon : H_n \to \mathbb{C}$, coproduct $\Delta : H_n \to H_n \otimes H_n$ and antipode $S : H_n \to H_n$ defined by their actions on generators below, where i takes values in the appropriate index set of generators for each H_n.*

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i{}^{-1}, \\
&\Delta(K_i{}^{-1}) = K_i{}^{-1} \otimes K_i{}^{-1}, \quad \varepsilon(K_i{}^{-1}) = 1, \quad S(K_i{}^{-1}) = K_i, \\
&\Delta(X_i) = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i, \quad \varepsilon(X_i) = 0, \quad S(X_i) = -X_i.
\end{aligned} \tag{10.1.1}$$

Proof. We take the generators $K_i, K_i{}^{-1}$ to be grouplike and X_i to be primitive (Definition 4.2.1). Both of the consequently generated algebras are well known to be Hopf algebras individually, and it is easy to check the axioms are satisfied. The Hopf algebra structure on arbitrary elements in H_n is then evaluated through linearity and by Δ and ε acting as algebra homomorphisms and S as an algebra antihomomorphism (See Chapter 4). ∎

Definition 10.1.5. The quantum group associated to $\mathfrak{gl}_2$, denoted $U_q(\mathfrak{gl}_2)$, is the associative algebra generated by $E, F, K_1, K_1{}^{-1}, K_2, K_2{}^{-1}$ and the following relations, for $i, j = 1, 2$.

$$\begin{aligned}
&[K_i, K_j] = [K_i{}^{-1}, K_j{}^{-1}] = [K_i, K_j{}^{-1}] = 0, \quad K_i{}^{-1}K_i = K_iK_i{}^{-1} = \mathbf{1}, \\
&K_iEK_i{}^{-1} = q^2E, \quad K_iFK_i{}^{-1} = q^{-2}F, \quad [E,F] = \frac{K_1 - K_2{}^{-1}}{q - q^{-1}}.
\end{aligned} \tag{10.1.2}$$

Proposition 10.1.6. *$U_q(\mathfrak{gl}_2)$ can be endowed with Hopf algebra structure given by counit $\varepsilon : U_q(\mathfrak{gl}_2) \to \mathbb{C}$, coproduct $\Delta : U_q(\mathfrak{gl}_2) \to U_q(\mathfrak{gl}_2) \otimes U_q(\mathfrak{gl}_2)$ and antipode $S : U_q(\mathfrak{gl}_2) \to U_q(\mathfrak{gl}_2)$, defined by their actions on generators below. Note that Δ and ε are defined on products of these generators as they extend to algebra homomorphisms and S to an algebra antihomomorphism.*

$$\begin{aligned}
&\Delta(K_i) = K_i \otimes K_i, \quad \varepsilon(K_i) = 1, \quad S(K_i) = K_i{}^{-1}, \\
&\Delta(K_i{}^{-1}) = K_i{}^{-1} \otimes K_i{}^{-1}, \quad \varepsilon(K_i{}^{-1}) = 1, \quad S(K_i{}^{-1}) = K_i, \\
&\Delta(E) = E \otimes K_1 + \mathbf{1} \otimes E, \quad \varepsilon(E) = 0, \quad S(E) = -EK_1^{-1}, \\
&\Delta(F) = K_2{}^{-1} \otimes F + F \otimes \mathbf{1}, \quad \varepsilon(F) = 0, \quad S(F) = -K_2F.
\end{aligned} \tag{10.1.3}$$

Proof. For $i = 1, 2$, K_i and $K_i{}^{-1}$ generate the group algebra $\mathbb{C}[\mathbb{Z}^2]$, which is a Hopf algebra with all grouplike elements. It remains to check all axioms for a Hopf algebra which involve the new generators E and F. Below we present the non-trivial calculations involving E, as the calculations for F follow similarly. Firstly, we verify coassociativity and counit respectively.

$$\begin{aligned}(\Delta\otimes\mathrm{id})\Delta(E) &= \Delta(E)\otimes K_1+\Delta(\mathbf{1})\otimes E = E\otimes K_1\otimes K_1+\mathbf{1}\otimes E\otimes K_1+\mathbf{1}\otimes\mathbf{1}\otimes E\\ &= E\otimes\Delta(K_1)+\mathbf{1}\otimes\Delta(E) = (\mathrm{id}\otimes\Delta)\Delta(E), \qquad (10.1.4)\\ (\varepsilon\otimes\mathrm{id})\Delta(E) &= \varepsilon(E)\otimes K_1+\varepsilon(\mathbf{1})\otimes E = \mathbf{1}\otimes E\simeq E\simeq E\otimes\mathbf{1} = E\otimes\varepsilon(K_1)+\mathbf{1}\otimes\varepsilon(E) = (\mathrm{id}\otimes\varepsilon)\Delta(E).\end{aligned}$$

Then we verify that comultiplication and counit extend to algebra homomorphisms, by showing they are compatible with the $U_q(\mathfrak{gl}_2)$ relations. The counit of the quantities concerned vanishes so we omit it here.

$$\begin{aligned}\Delta\left(K_iEK_i{}^{-1}\right) &= \Delta(K_i)\Delta(E)\Delta\left(K_i{}^{-1}\right) = (K_i\otimes K_i)(E\otimes K_1+\mathbf{1}\otimes E)\left(K_i{}^{-1}\otimes K_i{}^{-1}\right)\\ &= K_iEK_i{}^{-1}\otimes K_iK_1K_i{}^{-1}+K_iK_i{}^{-1}\otimes K_iEK_i{}^{-1} = q^2E\otimes K_1+\mathbf{1}\otimes q^2E = q^2\Delta(E),\\ \Delta([E,F]) &= [\Delta(E),\Delta(F)] = [E\otimes K_1+\mathbf{1}\otimes E, K_2{}^{-1}\otimes F+F\otimes\mathbf{1}]\\ &= EK_2{}^{-1}\otimes K_1F-K_2{}^{-1}E\otimes FK_1+[E,F]\otimes K_1+K_2{}^{-1}\otimes[E,F]\\ &= q^2K_2{}^{-1}E\otimes q^{-2}FK_1-K_2{}^{-1}E\otimes FK_1+\frac{K_1\otimes K_1-K_2{}^{-1}\otimes K_1}{q-q^{-1}}+\frac{K_2{}^{-1}\otimes K_1-K_2{}^{-1}\otimes K_2{}^{-1}}{q-q^{-1}}\\ &= \frac{K_1\otimes K_1-K_2{}^{-1}\otimes K_2{}^{-1}}{q-q^{-1}} = \Delta\left(\frac{K_1-K_2{}^{-1}}{q-q^{-1}}\right). \qquad (10.1.5)\end{aligned}$$

We verify the antipode extends to an algebra antihomomorphism, by showing it is compatible with the $U_q(\mathfrak{gl}_2)$ relations.

$$\begin{aligned}S\left(K_iEK_i{}^{-1}\right) &= S(K_i{}^{-1})S(E)S(K_i) = -K_iEK_1^{-1}K_i{}^{-1} = -q^2EK_1^{-1} = q^2S(E), \qquad (10.1.6)\\ S([E,F]) &= [S(F),S(E)] = K_2FEK_1{}^{-1}-EK_1{}^{-1}K_2F\\ &= -K_2[E,F]K_1{}^{-1} = \frac{K_1{}^{-1}-K_2}{q-q^{-1}} = S\left(\frac{K_1-K_2{}^{-1}}{q-q^{-1}}\right).\end{aligned}$$

Finally, we verify the defining property of the antipode, on generators.

$$\begin{aligned}\mu\circ(\mathrm{id}\otimes S)\Delta(E) &= ES(K_1)+\mathbf{1}S(E) = EK_1^{-1}-EK_1^{-1} = 0 = \eta(\varepsilon(E)),\\ \mu\circ(S\otimes\mathrm{id})\Delta(E) &= S(E)K_1+S(\mathbf{1})E = -EK_1^{-1}K_1+\mathbf{1}E = 0 = \eta(\varepsilon(E)), \qquad (10.1.7)\end{aligned}$$

where we use the notation μ for multiplication and η for the unit map. ∎

Definition 10.1.7. The half-unrolled quantum group of $\mathfrak{gl}_2$, denoted $U_q^{X_1}(\mathfrak{gl}_2)$, is the associative algebra obtained by extending $U_q(\mathfrak{gl}_2)$ through the addition of a generator X_1 and taking the quotient by the following relations.

$$[X_1,K_i] = [X_1,K_i{}^{-1}] = 0, \quad [X_1,E] = E, \quad [X_1,F] = -F. \qquad (10.1.8)$$

Definition 10.1.8. The unrolled quantum group of $\mathfrak{gl}_2$, denoted $U_q^{X_1,X_2}(\mathfrak{gl}_2)$, is the associative algebra obtained by extending $U_q(\mathfrak{gl}_2)$ through the addition of two generators X_1 and X_2 and taking the quotient by the following relations.

$$[X_1,X_2] = [X_i,K_j] = [X_i,{K_j}^{-1}] = 0, \quad [X_i,E] = E, \quad [X_i,F] = -F. \tag{10.1.9}$$

Proposition 10.1.9. *$U_q^{X_1}(\mathfrak{gl}_2)$ and $U_q^{X_1,X_2}(\mathfrak{gl}_2)$ can be endowed with Hopf algebra structure given by the counit, coproduct and antipode for $U_q(\mathfrak{gl}_2)$, extended by the following definition of their action on the new generators.*

$$\Delta(X_i) = X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i, \quad \varepsilon(X_i) = 0, \quad S(X_i) = -X_i. \tag{10.1.10}$$

Proof. The new generators X_i are primitive elements, so the Hopf algebra structure is already known. Therefore, the result follows from Proposition 10.1.6, by checking that comultiplication and counit are still algebra homomorphisms and the antipode is an algebra antihomomorphism. The counit of the quantities concerned vanishes so we omit it here.

$$\begin{aligned}
\Delta([X_i,E]) &= [\Delta(X_i),\Delta(E)] = [X_i \otimes \mathbf{1} + \mathbf{1} \otimes X_i, E \otimes K_1 + \mathbf{1} \otimes E] \\
&= [X_i,E] \otimes K_1 + E \otimes [X_i,K_1] + \mathbf{1} \otimes [X_i,E] = E \otimes K_1 + \mathbf{1} \otimes E = \Delta(E), \\
S([X_i,E]) &= [S(E),S(X_i)] = [E{K_1}^{-1},X_i] = E{K_1}^{-1}X_i - X_iE{K_1}^{-1} \\
&= -[X_i,E]{K_1}^{-1} = -E{K_1}^{-1} = S(E).
\end{aligned} \tag{10.1.11}$$

■

We now introduce the "restricted" versions of $U_q(\mathfrak{gl}_2)$ at various stages of unrolling, by imposing certain relations on the powers of the generators. In the fully rolled case, this is known as the small quantum group.

Definition 10.1.10. The restricted quantum group of $\mathfrak{gl}_2$, denoted $\overline{U}_q(\mathfrak{gl}_2)$, is the associative algebra obtained by taking the quotient of $U_q(\mathfrak{gl}_2)$ by the two-sided ideal generated by the relations $E^p = F^p = {K_1}^p - \mathbf{1} = {K_2}^p - \mathbf{1} = 0$.

Definition 10.1.11. The half-unrolled restricted quantum group of $\mathfrak{gl}_2$, denoted $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$, is the associative algebra obtained by taking the quotient of $U_q^{X_1}(\mathfrak{gl}_2)$ by the two-sided ideal generated by the relations $E^p = F^p = {K_1}^p - \mathbf{1} = 0$.

Definition 10.1.12. The unrolled restricted quantum group of $\mathfrak{gl}_2$, denoted $\overline{U}_q^{X_1,X_2}(\mathfrak{gl}_2)$, is the associative algebra obtained by taking the quotient of $U_q^{X_1,X_2}(\mathfrak{gl}_2)$ by the two-sided ideal generated by the relations $E^p = F^p = 0$.

Proposition 10.1.13. *The restricted quantum groups of $\mathfrak{gl}_2$ can be endowed with the same Hopf algebra structure as their unrestricted counterparts.*

Proof. This follows from Proposition 10.1.6 and Proposition 10.1.9, by checking that the comultiplication and counit are algebra homomorphisms and the antipode is an algebra anti-homomorphism, in the sense that they are compatible with the new relations. The counit of E vanishes so we omit that calculation.

$$\begin{aligned}
&\Delta(E^p) = \Delta(E)^p = (E \otimes K_1 + \mathbf{1} \otimes E)^p = \sum_{n=0}^{p} q^{n(p-n)} \begin{bmatrix} p \\ n \end{bmatrix} E^{p-n} \otimes E^n {K_1}^{p-n} = 0, \\
&S(E^p) = S(E)^p = (-1)^p E^p = 0, \\
&\Delta({K_1}^p) = \Delta(K_1)^p = {K_1}^p \otimes {K_1}^p = \mathbf{1} \otimes \mathbf{1} = \Delta(\mathbf{1}), \\
&S({K_1}^p) = S(K_1)^p = {K_1}^{-p} = \mathbf{1}, \qquad \varepsilon({K_1}^p) = 1 = \varepsilon(\mathbf{1}).
\end{aligned} \tag{10.1.12}$$

The terms with $n = 0$ and $n = p$ vanish by imposing $E^p = 0$ and the rest vanish because $[p] = \frac{q^p - q^{-p}}{q - q^{-1}} = 0$ when $q^{2p} = 1$, which implies $[p]! = 0$ and therefore $\begin{bmatrix} p \\ n \end{bmatrix} = 0$. For more information on this notation, refer to [12][Section VI.1]. ■

Definition 10.1.14. Let U be one of the restricted quantum groups for $\mathfrak{gl}_2$. The corresponding **quantum Cartan subalgebra** for each of these quantum groups, is the subalgebra generated by K_i and X_i, which we list below.

$$H_0 \subset U_0 = \overline{U}_q^{X_1,X_2}(\mathfrak{gl}_2), \quad H_1 \subset U_1 = \overline{U}_q^{X_1}(\mathfrak{gl}_2), \quad H_2 \subset U_2 = \overline{U}_q(\mathfrak{gl}_2). \tag{10.1.13}$$

10.2 Modules

We begin by introducing the categories of vector spaces which are equivalent to the module categories of the quantum Cartan subalgebras and endow them with braided tensor structure. Then we prove the equivalence and build up to the categories for the full quantum groups.

Definition 10.2.1. Let the following denote the categories of finite dimensional modules, graded by their respective groups.

$$\mathcal{C}_0 = \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{R}^2}, \qquad \mathcal{C}_1 = \mathrm{Vect}^{\mathrm{fd}}_{(\mathbb{R}/p\mathbb{Z})\times\mathbb{Z}}, \qquad \mathcal{C}_2 = \mathrm{Vect}^{\mathrm{fd}}_{\mathbb{Z}_p\times\mathbb{Z}_p}. \tag{10.2.1}$$

Denote their simple objects by $\mathbb{C}_{(x,y)}$, where $(x,y) \in (\mathbb{R},\mathbb{R})$, $(\mathbb{Z},\mathbb{R}/p\mathbb{Z})$ or $(\mathbb{Z}_p,\mathbb{Z}_p)$, respectively. We endow them with the structure of ribbon Grothendieck-Verdier categories in the following way, where P is the tensor flip of vector spaces.

- Tensor product $\mathbb{C}_{(x,y)} \otimes \mathbb{C}_{(x',y')} \cong \mathbb{C}_{(x+x',y+y')}$.
- Braiding $\sigma(\mathbb{C}_{(x,y)}, \mathbb{C}_{(x',y')}) = q^{2xy'}P$ and a trivial associator.
- Dualising object $\mathbb{C}_{(2\xi_1,2\xi_2)}$ and twist $\theta(\mathbb{C}_{(x,y)}) = q^{2(xy-\xi_1 y-\xi_2 x)}\mathrm{id}_{\mathbb{C}_{(x,y)}}$.

Note that the braiding and the twist above do not depend on the choice of representatives, so we omit any choices of sections $\mathbb{R}/p\mathbb{Z} \to \mathbb{R}$ and $\mathbb{Z}_p \to \mathbb{Z}$.

Remark. We can apply Proposition 8.5.1 to see how each of these categories are related by simple current extensions. $\mathcal{C}_1$ can be obtained from $\mathcal{C}_0$ by taking the category of local (trivial double braiding) modules over the commutative ring $A_1 = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_{(0,pk)}$ for $\mathbb{C}_{(0,pk)} \in \mathcal{C}_0$. The corresponding modules in $\mathcal{C}_1$ are given by $A_1 \otimes \mathbb{C}_{(x,y)}$ for $\mathbb{C}_{(x,y)} \in \mathcal{C}_0$. Similarly, $\mathcal{C}_2$ can be obtained from $\mathcal{C}_1$ by taking the category of local modules over the commutative ring $A_2 = \bigoplus_{k\in\mathbb{Z}} \mathbb{C}_{(pk,p\mathbb{Z})}$ for $\mathbb{C}_{(pk,p\mathbb{Z})} \in \mathcal{C}_1$. The corresponding modules in $\mathcal{C}_2$ are given by $A_2 \otimes_{\mathcal{C}_0} \mathbb{C}_{(x,y)}$, for $\mathbb{C}_{(x,y)} \in \mathcal{C}_0$.

Definition 10.2.2. Let H be a Hopf algebra with a subalgebra isomorphic to an H_n from Definition 10.1.14, and let V be a finite dimensional H-module. Then V is a **weight module** if it splits into a direct sum of K_n and X_n eigenspaces and $K_n = q^{2X_n} = e^{\frac{2\pi I}{p} X_n}$ as operators on V, whenever K_n and X_n are both in the set of generators for H, with n fixed. We also require that the eigenvalues (weights) of X_n are real and the eigenvalues of K_n have modulus 1. We denote the category of weight modules of H by $H\text{-Mod}^{\mathrm{wt}}$.

Proposition 10.2.3. *Let H be as in Definition 10.2.2. Then H-Mod$^{\mathrm{wt}}$ is a monoidal category. If H is also equipped with a ribbon element and an R-matrix, then H-Mod$^{\mathrm{wt}}$ is a ribbon category.*

Proof. That the categories are monoidal follows from Proposition 4.3.3, after checking that the "weight" conditions are preserved by the tensor product prescribed by the coproduct of the Hopf algebra. As the coproduct of the X elements are primitive and the coproduct of the K elements are grouplike, their eigenvalues on a tensor product module are the sums and products of the individual weights, respectively. The second statement follows immediately from Proposition 4.4.2. ■

Proposition 10.2.4. *H_n-Mod$^{\mathrm{wt}}$ are equivalent to $\mathcal{C}_n$, as ribbon Grothendieck-Verdier categories, for $n = 0,1,2$. The categories are rigid when the dualising object is the tensor unit.*

Proof. Working through the procedure in the case of H_1, we start with the bosonic lattice data $\Psi = (\eta, \langle -,- \rangle, \Lambda, \xi)$ where $\eta = \mathbb{R}^2$, $\langle x,y \rangle = (x_1y_2 + x_2y_1)/p$, $\Lambda = p\mathbb{Z}e_2$. We choose the section

$s : (x_1, x_2) \mapsto (x_1, \overline{x_2})$ where $\overline{z}$ is the unique representative of $z \in \mathbb{R}/p\mathbb{Z}$ in the interval $[0, p)$. Then the construction in Section 8.4 gives us the following Hopf algebra.

$$H_\Lambda = \mathsf{U}\big(\Lambda^\perp\big) \otimes \mathbb{C}[\Lambda^*/\Lambda^\perp] = \mathbb{C}[X_{(0,1)}, K_{(1,0)}, K^{-1}_{(1,0)}], \tag{10.2.2}$$

where the X elements are primitive and the K elements are grouplike and satisfy $K^{-1}K = 1$. H_Λ acts on modules in $\mathrm{Vect}_{\Lambda^*/\Lambda} = \mathrm{Vect}_{\mathbb{Z}\times(\mathbb{R}/p\mathbb{Z})}$ by

$$X_{(0,1)}|_{\mathbb{C}_{(x_1,x_2)}} = x_1 \mathrm{id}_{\mathbb{C}_{(x_1,x_2)}}, \qquad K_{(1,0)}|_{\mathbb{C}_{(x_1,x_2)}} = q^{2x_2}\mathrm{id}_{\mathbb{C}_{(x_1,x_2)}}, \qquad x_1 \in \mathbb{Z},\ x_2 \in \mathbb{R}/p\mathbb{Z}. \tag{10.2.3}$$

We let $X_1 = X_{(0,1)}$ and $K_2 = K_{(1,0)}$. Then we can include a generator $K_1 = q^{2X_1}$ which acts as

$$K_1|_{\mathbb{C}_{(x_1,x_2)}} = q^{2x_1}\mathrm{id}_{\mathbb{C}_{(x_1,x_2)}}, \qquad x_1 \in \mathbb{Z}. \tag{10.2.4}$$

As $q^{2p} = 1$, this generator must satisfy $K_1^p = 1$ and we recover the form of H_1 stated in Definition 10.1.2. As a result of this correspondence, we know that the category H_1-$\mathrm{Mod}^{\mathrm{wt}}$ is ribbon Grothendieck-Verdier equivalent to $\mathrm{Vect}_{\mathbb{Z}\times(\mathbb{R}/p\mathbb{Z})}$ with the following structure morphisms. First, the braiding and associator

$$\Omega(x,y) = q^{x_1\overline{y_2}+y_1\overline{x_2}}, \quad F(x,y,z) = q^{x_1(\overline{y_2+z_2}-\overline{y_2}-\overline{z_2})}. \tag{10.2.5}$$

We can shift (Ω, F) by a coboundary $k(x,y) = q^{-x_1\overline{y_2}}$ to yield $(\sigma, 1)$, where

$$\sigma(x,y) = q^{2x_1\overline{y_2}}. \tag{10.2.6}$$

We also have the twist corresponding to dualising object $\mathbb{C}_{2\xi}$, giving a rigid twist for $\xi = (0, p\mathbb{Z})$.

$$\theta(x) = q^{2(x_1\overline{x_2}-x_1\overline{\xi_2}-\xi_1\overline{x_2})}. \tag{10.2.7}$$

Correspondingly, we have the following R-matrix and ribbon element.

$$R_1 = \exp\left(\frac{2\pi i}{p} X_1 \otimes \log K_2\right), \qquad r_1 = \exp\left(-\frac{2\pi i}{p}(X_1 \otimes \log K_2 - X_1 \otimes \overline{\xi_2} - \xi_1 \otimes \log K_2)\right). \tag{10.2.8}$$

We can repeat this process for the case of the trivial and the full rank lattices. For $\Lambda = 0$ we have

$$\Lambda^*/\Lambda = \mathbb{R}^2, \qquad H_\Lambda = \mathbb{C}[X_{(1,0)}, X_{(0,1)}]. \tag{10.2.9}$$

We obtain H_0 by including $K_{(1,0)}$ and $K_{(0,1)}$, which are not constrained. For $\Lambda = p\mathbb{Z}e_1 + p\mathbb{Z}e_2$ we have

$$\Lambda^*/\Lambda = \mathbb{Z}_p \times \mathbb{Z}_p, \qquad H_\Lambda = \mathbb{C}[K_{(1,0)}, K_{(0,1)}]. \tag{10.2.10}$$

with the following R-matrix and ribbon element.

$$R_0 = \exp\left(\frac{2\pi i}{p} X_1 \otimes X_2\right), \qquad r_0 = \exp\left(-\frac{2\pi i}{p}(X_1 \otimes X_2 - X_1 \otimes \overline{\xi_2} - \xi_1 \otimes X_2)\right). \tag{10.2.11}$$

We obtain H_2 by imposing the constraints $K_{(1,0)}^p = K_{(p,0)} = 1$ and $K_{(0,1)}^p = K_{(0,p)} = 1$. Each of the structure morphisms do not depend on the choices of sections so the same formulae hold. For the R-matrix and ribbon element we can replace X_i with $\log K_i$ when H_n does not contain the relevant X_i and these evaluate in the same way.

$$R_2 = \exp\left(\frac{2\pi i}{p} \log K_1 \otimes \log K_2\right), \qquad r_2 = \exp\left(-\frac{2\pi i}{p}(\log K_1 \otimes \log K_2 - \log K_1 \otimes \overline{\xi_2} - \xi_1 \otimes \log K_2)\right). \tag{10.2.12}$$

Technically these R-matrices are not in the tensor square of the quantum group, but the action is a well defined linear map on such a tensor product, so it still gives rise to a braiding in the expected way. ■

Remark. We can check directly that H_n-Mod$^{\mathrm{wt}}$ are equivalent to $\mathcal{C}_n$, as abelian categories, in the following way. For each H_n, let $\mathbb{C}_{(x_1,x_2)}$ denote the module satisfying $K_i|_{\mathbb{C}_{(x_1,x_2)}} = q^{2x_i}\mathrm{id}_{\mathbb{C}_{(x_1,x_2)}}$. For H_0, the X_i eigenvalues determine the K_i eigenvalues as $K_i = q^{2X_i}$. Therefore the simple modules are graded by $x, y \in \mathbb{R}$ which are unconstrained. For H_1, we have the constraint $1 = K_1{}^p = q^{2pX_1} = e^{2\pi i X_1}$ which implies that $x_1 \in \mathbb{Z}$. As the generator X_2 is not in H_1, we find that K_2 acts the same way if x_2 is shifted by any multiple of p. Therefore, $x_2 \in \mathbb{R}/p\mathbb{Z}$. For H_2, $K_1{}^p = K_2{}^p = 1$ implies that $x_1, x_2 \in \mathbb{Z}$. We see that the actions of K_1 and K_2 are invariant under shifting x_1, x_2 by multiples of p and therefore we have $x_1, x_2 \in \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$.

Remark. Note that in the fully rolled case we can rewrite the R-matrix as

$$R_2 = \frac{1}{p} \sum_{n,m \in \mathbb{Z}_p} q^{-2nm} K_1{}^n \otimes K_2{}^m \tag{10.2.13}$$

Definition 10.2.5. For any choice of category $\mathcal{C} \in \{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2\}$, let $M \in \mathcal{C}$ denote the simple objects $\mathbb{C}_{(1,1)}$, where each label denotes the appropriate coset. Let $B = \mathfrak{B}(M)$ be the Nichols algebra of M. Then M has self braiding $Q(\mathcal{M}) = e^{2\pi i/p} = q^2$, so we find that $B \cong \mathbb{C}[x]/x^p$ as a vector space. B is endowed with the following Hopf algebra structure, defined by its action on the generator x.

$$\Delta(x) = x \otimes \mathbf{1} + \mathbf{1} \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x. \tag{10.2.14}$$

Similarly, we define $B^* = \mathfrak{B}(M^*)$ where $M^* \in \mathcal{C}$ is the dual module to M, given by $\mathbb{C}_{(-1,-1)}$.

These choices of modules M and M^* are significant as they contain the screening operators for the two free field realisations of the bosonic ghost vertex algebra, see Proposition 9.4.1.

It is expected that quantum group modules form Yetter-Drinfeld categories (see Section 4.5) over the Nichols algebra, in the category of modules over the quantum Cartan subalgebra. The Nichols algebra structure yields the Serre relations, the action and coaction yield the coproduct and commutation relations, and the Yetter-Drinfeld condition yields the commutator $[E,F]$. There has been related work by Laugwitz [63], and similar statements involving Yetter-Drinfeld modules (or crossed bi-modules) in connection with quantum groups are well known [12][IX.5]. Here we calculate and verify the specific relations for $\mathfrak{gl}_2$ and prove the equivalence in the language of Yetter-Drinfeld categories.

Theorem 10.2.6. *U_n-Mod$^{\mathrm{wt}}$ are equivalent to ${}^B_B\mathcal{YD}(H_n\text{-Mod}^{\mathrm{wt}})$ as ribbon Grothendieck-Verdier categories. The categories are rigid when the dualising object is the tensor unit. That is,*

- $\overline{U}_q^{X_1,X_2}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ *is equivalent to* ${}^B_B\mathcal{YD}\left(\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{C}^2}\right)$ *as braided tensor categories.*
- $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ *is equivalent to* ${}^B_B\mathcal{YD}\left(\mathrm{Vect}^{\mathrm{fd}}_{(\mathbb{R}/p\mathbb{Z})\times\mathbb{Z}}\right)$ *as braided tensor categories.*
- $\overline{U}_q(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ *is equivalent to* ${}^B_B\mathcal{YD}\left(\mathrm{Vect}^{\mathrm{fd}}_{\mathbb{Z}_p\times\mathbb{Z}_p}\right)$ *as braided tensor categories.*

Proof. We start with a general set up for a quantum group, then we construct a Yetter-Drinfeld module from a given quantum group module. We specialise to our choices of quantum groups and show that the commutator of E and F in the quantum group relations is equivalent to the Yetter-Drinfeld condition, and that the coproduct on the Hopf algebra side corresponds to a specific action and coaction on the Yetter-Drinfeld side. The category of Yetter-Drinfeld modules comes equipped with a braiding, and we construct the corresponding R-matrix.

Let U be a Hopf algebra. We proceed as generally as possible, specialising to our case (U is the restricted quantum group of $\mathfrak{gl}_2$) when necessary. Suppose we have a Hopf subalgebra $H \subset U$ and two subalgebras $B, B' \subset U$. Suppose B has an H-action such that U has a relation of the form $h.b = h_1 b S(h_2) \in B$. Suppose B has a coproduct Δ_B and a left H-coaction $\delta_B(b) = k_b \otimes b$, for grouplike $k_b \in H$ such that $\Delta_U(b) = (\mu_U \otimes \mathrm{id})(\mathrm{id} \otimes \delta_B)(\Delta_B b)$ for $b \in B$. Suppose the same for B', and let $\langle , \rangle : B' \otimes B \to \mathbb{C}$ be a dual pairing compatible with all structure on B, B'. We see that these satisfy the conditions required of an action and a coaction below.

$$
\begin{aligned}
&(hk).b = (hk)_1 b S((hk)_2) = h_1 k_1 b S(h_2 k_2) = h_1(k_1 b S(k_2))S(h_2) = h.(k.b),\\
&1.b = 1bS(1) = b,\\
&(\Delta \otimes 1)\delta_B(b) = \Delta(k_b) \otimes b = k_b \otimes k_b \otimes b = k_b \otimes \Delta_B(b) = (1 \otimes \delta_B)\delta_B(b),\\
&(\varepsilon \otimes 1)\delta_B(b) = \varepsilon(k_b) \otimes b = 1 \otimes b \simeq b.
\end{aligned}
\tag{10.2.15}
$$

Let V be a U module. We define $\psi(V) \in {}^B_B\mathcal{YD}(H\text{-Mod}^{\text{wt}})$. First, we restrict V to $H \subset U$ which gives an H-module.

Restricting V to B gives a B-module. The algebra relation between B and H proves in general that this is a B-module inside H-mod. That is, the B-action is an H-module morphism

$$(h_1.b).(h_2.v) = (h_1 bS(h_2)).(h_3.v) = (h_1 bS(h_2)h_3).v = (\varepsilon(h_2)h_1 b).v = (hb).v = h.(b.v). \tag{10.2.16}$$

Restricting V to B' gives a B' module. From the pairing, we obtain a B-comodule structure via $v_{-1} \otimes v_0 = \delta(v) \in B \otimes V$ with the property that, for all $b' \in B'$, we have $\langle b', v_{-1}\rangle v_0 = b'.v$. More explicitly, if b_i, b'_i are a dual basis $\langle b'_i, b_j\rangle = \delta_{ij}$ then $v_{-1} \otimes v_0 = \sum_i b_i \otimes (b'_i.v)$. Again, we see that this is a B-coaction inside H-mod. This follows directly from the B'-action being an H-module morphism, as above for B.

We check the Yetter-Drinfeld condition for diagonal braiding on H. That is, $v \otimes w \mapsto q(v,w) w \otimes v$.

$$\begin{aligned} LHS &= q(b_2, v)q(b_{12}, v_{-1})q(b_{12}.v_0, S(b_2))b_{11}v_{-1}S(b_2) \otimes b_{12}.v_0, \\ RHS &= (b.v)_{-1} \otimes (b.v)_0. \end{aligned} \tag{10.2.17}$$

Insert coaction

$$\begin{aligned} LHS &= \sum_i q(b_2, v)q(b_{12}, b_i)q(b_{12}.b'_i.v, S(b_2))b_{11}b_iS(b_2) \otimes b_{12}.b'_i.v, \\ RHS &= \sum_i b_i \otimes b'_i.(b.v). \end{aligned} \tag{10.2.18}$$

Let b be a primitive element in B, then $b_{11} \otimes b_{12} \otimes b_2 = b \otimes 1 \otimes 1 + 1 \otimes b \otimes 1 + 1 \otimes 1 \otimes b$.

$$LHS = \sum_i bb_i \otimes b'_i.v + \sum_j q(b, b_j)b_j \otimes bb'_j.v + \sum_i q(b, v)q(b'_i.v, S(b))b_i(-b) \otimes b'_i.v, \tag{10.2.19}$$

$$RHS = \sum_j b_j \otimes b'_j b.v. \tag{10.2.20}$$

Further, let $\lambda(i)b_i \otimes b'_i = F^i \otimes G^i$ and test the relation on $b = F$, using the braidings from H_0, H_1, H_2. Let $\mathbb{C}G = M$ be the module generating the Nichols algebra B' and let $\mathbb{C}F = M^*$ be the dual module to M, which generates the Nichols algebra B. Then from the braiding on H, we have $q(G,G) = q(F,F) = q^2$, $q(G,F) = q(F,G) = q^{-2}$, $q(v,F) = q^{-2x}$ and $q(F,v) = q^{-2y}$, where $K_1 v = q^{2x}v$ and $K_2 v = q^{2y}v$, for $v \in M$.

$$LHS = \sum_i \lambda(i)^{-1}FF^i \otimes G^i.v + \sum_j \lambda(j)^{-1}q^{2j}F^j \otimes FG^j.v - \sum_i \lambda(i)^{-1}q^{-2y}q^{-2x-2i}F^iF \otimes G^i.v,$$

$$RHS = \sum_j \lambda(j)^{-1} F^j \otimes G^j F.v. \tag{10.2.21}$$

Compare the terms with F^{i+1} resp $F^j, j = i+1$.

$$\lambda(i)^{-1} G^i.v + \lambda(i+1)^{-1} q^{2i+2} F G^{i+1}.v - \lambda(i)^{-1} q^{-2x-2y-2i} G^i.v = \lambda(i+1)^{-1} G^{i+1} F.v,$$
$$\lambda(i+1)^{-1}\left(q^{2i+2} F G^{i+1} - G^{i+1} F\right) = \lambda(i)^{-1}(q^{-2x-2y-2i} - 1) G^i. \tag{10.2.22}$$

eg. $i = 0$ gives $q^2 FG - GF = \frac{\lambda(1)}{\lambda(0)}(q^{-2x-2y} - 1)$. Substituting the relation $G = (q - q^{-1}) {K_1}^{-1} E$ gives $[E, F] = \frac{\lambda(1)}{\lambda(0)}\left(\frac{K_1 - {K_2}^{-1}}{q - q^{-1}}\right)$.

Now we can work out $\lambda(i)$ to check this is really equivalent to our quantum group commutator. For $\langle -, - \rangle$ to be a dual pairing, it must satisfy the following conditions (using Sweedler notation to suppress the sum).

$$\langle ab, c \rangle = \langle a, c_2 \rangle \langle b, c_1 \rangle, \quad \langle a, cd \rangle = \langle a_2, c \rangle \langle a_1, d \rangle, \quad \langle a, 1 \rangle = \varepsilon(a), \quad \langle 1, c \rangle = \varepsilon(c). \tag{10.2.23}$$

Once we have defined its value on our generators F and G, we can work out the pairing on F^i and G^j as follows, where we apply the above relations with $a = G^i$, $c = F^n$, $d = F^{i-n}$ for some choice of n. We work out the coproduct $\Delta(G^i)$ using the braided multiplication of the tensor product given by $(a \otimes b)(c \otimes d) = q(b, c) ac \otimes bd$.

$$\langle G, F \rangle = 1, \quad \langle G^2, F^2 \rangle = (1 + q^2), \quad \langle G^i, F^j \rangle = \lambda(i)\delta_{ij},$$
$$\langle G^i, F^i \rangle = \sum_{m=0}^{i} q^{m(i-m)} \begin{bmatrix} i \\ m \end{bmatrix} \langle G^{i-m}, F^{i-n} \rangle \langle G^m, F^n \rangle = q^{n(i-n)} \begin{bmatrix} i \\ n \end{bmatrix} \lambda(i-n)\lambda(n). \tag{10.2.24}$$

We make the choice $n = 1$ to find the following recurrence relation for $\lambda(i)$.

$$\lambda(0) = \lambda(1) = 1, \quad \lambda(2) = (1 + q^2) = q[2], \quad \lambda(i) = q^{i-1}[i]\lambda(i-1). \tag{10.2.25}$$

Therefore we find the following expression for $\lambda(i)$.

$$\lambda(i) = q^{i(i-1)/2}[i]! \tag{10.2.26}$$

Note that this pairing is compatible with the relations $F^p = G^p = 0$ as $[p] = 0 \implies \lambda(p) = 0$. The pairing also implies the relations because non-degeneracy of the pairing means that $\langle G^p, F^i \rangle = \langle G^i, F^p \rangle = 0 \implies G^p = F^p = 0$. We can also rewrite the Yetter-Drinfeld condition in terms of E and F.

$$[E^{i+1}, F] = \frac{\lambda(i+1)}{\lambda(i)}\left(\frac{q^{2x} - q^{-2y-2i}}{q - q^{-1}}\right) E^i = \frac{\lambda(i+1)}{\lambda(i)}\left(\frac{E^i K_1 - {K_2}^{-1} E^i}{q - q^{-1}}\right). \tag{10.2.27}$$

To ensure that this also holds for $i+1$ and higher, we can rewrite the condition in the following way.

$$[E^{i+2},F]=E[E^{i+1},F]+[E,F]E^{i+1}=\left(\frac{\lambda(i+1)}{\lambda(i)}+q^{2i+2}\right)\frac{E^{i+1}K_1}{q-q^{-1}}-\left(\frac{\lambda(i+1)}{\lambda(i)}q^2+1\right)\frac{K_2{}^{-1}E^{i+1}}{q-q^{-1}}. \tag{10.2.28}$$

Both coefficients must be equal to $\frac{\lambda(i+2)}{\lambda(i+1)}$ and therefore also to each other, so we find that

$$\frac{\lambda(i+1)}{\lambda(i)}=\frac{q^{2i+2}-1}{q^2-1}=q^i\frac{q^{i+1}-q^{-i-1}}{q-q^{-1}}=q^i[i+1]. \tag{10.2.29}$$

This is exactly the condition on $\lambda(i)$ which we require.

We check the tensor product (comultiplication), starting with the following coactions.

$$\delta_B(F)=K_2{}^{-1}\otimes F,\qquad \delta_{B'}(G)=K_1{}^{-1}\otimes G. \tag{10.2.30}$$

F and G are primitive in B and B'. With the coactions above, the coproducts in U take the following forms.

$$\Delta_U(F)=K_2{}^{-1}\otimes F+F\otimes\mathbf{1},\qquad \Delta_U(G)=K_1{}^{-1}\otimes G+G\otimes\mathbf{1}, \tag{10.2.31}$$

$$\Delta_U(E)=\frac{\Delta_U(K_1)\Delta_U(G)}{q-q^{-1}}=K_1K_1{}^{-1}\otimes\left(\frac{K_1G}{q-q^{-1}}\right)+\left(\frac{K_1G}{q-q^{-1}}\right)\otimes K_1=\mathbf{1}\otimes E+E\otimes K_1.$$

To demonstrate that these act the same way we evaluate the coproducts from the subalgebra B with braided multiplication and from the quantum group U on the module v such that $K_1v=q^{2x}v$, $K_2v=q^{2y}v$.

$$\begin{aligned}\Delta_B(F).(v\otimes w)&=F.v\otimes w+q(F,v)v\otimes Fw,\\ \Delta_U(F).(v\otimes w)&=F.v\otimes w+K_2{}^{-1}.v\otimes Fw.\end{aligned} \tag{10.2.32}$$

These are equal because the braiding on H gives $q(F,v)v=q^{-2y}v=(K_2)^{-1}v$. Similarly for G, we find

$$\begin{aligned}\Delta_{B'}(G).(v\otimes w)&=G.v\otimes w+q'(G,v)v\otimes Gw,\\ \Delta_U(G).(v\otimes w)&=G.v\otimes w+K_1{}^{-1}.v\otimes Gw.\end{aligned} \tag{10.2.33}$$

These agree as long as the braiding q' on G as a B'-module is the inverse braiding $q^{-1}\left(\mathbb{C}_{(x,y)},\mathbb{C}_{(x',y')}\right)=q^{-2yx'}\implies q'(G,v)v=q^{-2x}v=(K_1)^{-1}v$. This follows from the conversion process between B-comodules and B'-modules. For $b_1=b=F$, $b'=G$ we can see this from the following expressions for the coaction of B on the tensor product. The first comes from the action of B' and the second from the coaction of B. For them to be equal we must have $q'(G,v)=q(v,F)=q^{-2x}$.

$$\delta(v\otimes w)=\sum_i b_i\otimes b'_i.(v\otimes w) \tag{10.2.34}$$

$$= \sum_i q'((b_i')_2, v) b_i \otimes (b_i')_1.v \otimes (b_i')_2.w = 1 \otimes v \otimes w + b \otimes b'.v \otimes w + q'(b', v) b \otimes v \otimes b'.w + \dots,$$

$$\delta(v \otimes w) = \sum_{i,j} q(b_i'.v, b_j) b_i b_j \otimes b_i'.v \otimes b_j'.w = 1 \otimes v \otimes w + b \otimes b'.v \otimes w + q(v, b) b \otimes v \otimes b'.w + \dots.$$

We find the R-Matrix from the braiding in ${}^B_B\mathcal{YD}(H\text{-Mod}^{\mathrm{wt}})$.

$$\sigma(v, w) = q(v_0, w) v_{-1}.w \otimes v_0 = \sum_i q(b_i'.v, w) b_i.w \otimes b_i'.v = \sum_i q(b_i', w)\big(b_i \otimes b_i'\big)(R_0)_{21}.(w \otimes v),$$

$$R = \sum_i (b_i' \otimes b_i)\big(1 \otimes {K_2}^i\big) R_0, \qquad R_{21} R = \left(\sum_{i,j} b_i {K_2}^i b_j' \otimes b_i' b_j {K_2}^j\right) R_0 R_0. \tag{10.2.35}$$

Therefore, R is factorisable. Explicitly we have

$$\begin{aligned} R &= \left(\sum_{k=0}^{p-1} \lambda(k)^{-1} (q - q^{-1})^k \left({K_1}^{-1} E\right)^k \otimes F^k {K_2}^k\right) R_0 \\ &= \left(\sum_{k=0}^{p-1} \frac{(q - q^{-1})^k}{[k]!} q^{-k(3k+1)/2} E^k {K_1}^{-k} \otimes F^k K_2^k\right) R_0. \end{aligned} \tag{10.2.36}$$

The ribbon Grothendieck-Verdier equivalence follows from braided monoidal equivalence, by including the rigid twist defined from the Hopf algebra antipode and applying Lemma 3.1.6. ■

Using the expression (10.2.13) for the R-matrix of the fully rolled quantum Cartan subalgebra, we have the following expression for the R-matrix.

$$\begin{aligned} R &= \left(\sum_{k=0}^{p-1} \frac{(q - q^{-1})^k}{[k]!} q^{-k(3k+1)/2} E^k {K_1}^{-k} \otimes F^k K_2^k\right)\left(\frac{1}{p} \sum_{n,m=0}^{p-1} q^{-2nm} {K_1}^n \otimes {K_2}^m\right) \\ &= \frac{1}{p} \sum_{k=0}^{p-1} \sum_{n,m} \frac{(q - q^{-1})^k}{[k]!} q^{-3k(k+1)/2} q^{-2mn} E^k {K_1}^{n-k} \otimes F^k {K_2}^{m+k} \\ &= \frac{1}{p} \sum_{k=0}^{p-1} \sum_{n,m} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2} q^{-2(n+k)(m-k)} E^k {K_1}^n \otimes F^k {K_2}^m \\ &= \frac{1}{p} \sum_{k=0}^{p-1} \sum_{n,m} \frac{(q - q^{-1})^k}{[k]!} q^{k(k-1)/2} q^{-2mn} {K_1}^n E^k \otimes {K_2}^m F^k \\ &= R_0 \left(\sum_{k=0}^{p-1} \frac{q^{k(k-1)/2}}{[k]!} (q - q^{-1})^k E^k \otimes F^k\right) = R_0 \Theta. \end{aligned} \tag{10.2.37}$$

The quasi R-matrix part Θ is the same as for $U_q(\mathfrak{sl}_2)$, see [64][(4.43)] and the form of the R-matrix resembles the R-matrices given in [65]. Our expression also agrees with the R-matrix developed for quantum $\mathfrak{gl}_2$ in [66][(41), (42)], where our elements $(q, 2X_1, 2X_2, K_1, K_2, E, F)$ correspond to $(e^{h/2}, H, G, K, L, X, Y)$ in [66] and, in addition, we impose $E^p = F^p = 0$. This correspondence requires the following identity.

$$\prod_{k\geq 0}(1-(1-q^2)q^{2k})^{-1} = \sum_{k\geq 0}\frac{(1-q^2)^k}{(1-q^2)\dots(1-q^{2k})} = \sum_{k\geq 0}\frac{q^{k(k-1)/2}}{[k]!}. \tag{10.2.38}$$

— Chapter 11 —

The Equivalence

"It's always the same with these bogus equivalences"

— Christopher Hitchens

In this section we will work towards proving the following theorem.

Theorem 11.0.1. *Subject to Conjecture 11.4.2, category $\mathscr{F}$ is equivalent to $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ for $p = 2$, as a Grothendieck-Verdier category. Depending on the choice of dualising object $K \in \{\mathcal{V}_k \ : \ k \in \mathbb{Z}\} \subset \mathscr{F}$, they are also equivalent as*

- *Rigid categories, for $K = \mathcal{V}_0$,*
- *Ribbon Grothendieck-Verdier categories, for $K = \mathcal{V}_{2k-1}$.*

Recall that the bosonic ghost vertex algebra module category $\mathscr{F}$ from Definition 9.2.1 is equipped with ribbon Grothendieck-Verdier structure given by Theorem 6.2.1, and recall $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ for $p = 2$, from Definition 10.2.2.

Conjecture 11.4.2 is that there exists a solution to an equation which relates products and iterates of intertwining operators for a triple of modules $\mathcal{W}_{\lambda_i}^{\ell_i}$ where $\lambda_1 + \lambda_2 + \lambda_3 \in \mathbb{Z}$ but $\lambda_i + \lambda_j \notin \mathbb{Z}$ for any distinct pairs $i, j \in 1, 2, 3$. Using the free field realisation, one can construct intertwining operators where the products and iterates agree on the quotients of the modules seen by each realisation. The conjecture is required in order to ensure that the products and iterates agree on the full tensor product.

Remark. By Theorem 10.2.6, the two categories in Theorem 11.0.1 are also equivalent to ${}^{B}_{B}\mathcal{YD}\left(\mathrm{Vect}^{\mathrm{fd}}_{(\mathbb{R}/p\mathbb{Z})\times\mathbb{Z}}\right)$, as ribbon Grothendieck-Verdier categories, where $B = \mathfrak{B}(M)$ is the Nichols algebra of a module M with self braiding $Q(M) = e^{2\pi i/p} = q^2 = -1$, given in Definition 10.2.5.

We start with the equivalence as abelian categories.

11.1 Abelian equivalence

We begin by classifying the simple and projective modules in $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$, for $p=2$.

Proposition 11.1.1. *Consider the following three infinite dimensional families of modules in $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$, for $p=2$. These modules have dimensions one, two and four, respectively, with the generators represented by the following matrices.*

$$\mathcal{V}_n: \quad X_1 = n, \quad K_2 = (-1)^n e^{2\pi i\lambda}, \qquad F = E = 0. \tag{11.1.1}$$

$$\begin{aligned} \mathcal{W}_\lambda^n: \quad & X_1 = \mathrm{diag}(n, n-1), && K_2 = (-1)^n e^{2\pi i\lambda}\mathrm{diag}(1,-1), \\ & F = e_{21}, && E = \frac{1}{2i}(-1)^n\left(1-e^{-2\pi i\lambda}\right)e_{12}. \end{aligned} \tag{11.1.2}$$

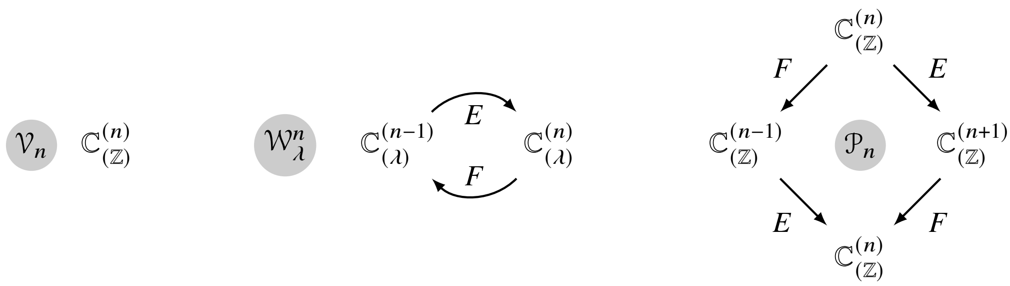
$$\begin{aligned} \mathcal{P}_n: \quad & X_1 = \mathrm{diag}(n, n-1, n+1, n), && K_2 = \mathrm{diag}(1,-1,-1,1) \\ & F = e_{21}+e_{43}, && E = e_{31}+e_{42}, \end{aligned} \tag{11.1.3}$$

where e_{ij} is the matrix which satisfies $\left(e_{ij}\right)_{k\ell} = \delta_{ik}\delta_{j\ell}$.

We claim the following.

- *The modules $\mathcal{V}_n$ and $\mathcal{W}_\lambda^n$ are simple.*
- *The modules $\mathcal{W}_\lambda^n$ and $\mathcal{P}_n$ are projective and injective.*
- *These modules form a complete list of the indecomposable modules in $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$, for $p=2$, which are simple, projective or injective, up to isomorphism.*

Figure 11.1: Diagrams visualising the structure of the modules classified in Proposition 11.1.1. The arrows denote the action of the generators E and F between the one dimensional spaces acted on by X_1 as $n \in \mathbb{Z}$ and by K_2 as $(-1)^n e^{2\pi i\lambda}$, $\lambda \in \mathbb{R}/\mathbb{Z}$, which we denote $\mathbb{C}_{(\lambda)}^{(n)}$.



Proof. We begin by considering weight modules, which split into direct sums of eigenspaces of the quantum Cartan subalgebra H_1. Recall that by Proposition 10.2.4, the category of H_1-modules is semisimple, and the simple modules are one dimensional vector spaces $\mathbb{C}_{(x_1,x_2)}$, for $x_1 \in \mathbb{Z}$, $x_2 \in \mathbb{R}/2\mathbb{Z}$. For convenience, we change variables to $n = x \in \mathbb{Z}$, $\lambda = (x+y)/2 \in \mathbb{R}/\mathbb{Z}$ and denote the module by $\mathbb{C}^{(n)}_{(\lambda)}$. Recall the defining relations of $\overline{U}^{X_1}_q$, for $p = 2$ $(q = e^{i\pi/2})$. By the commutator relation in (10.1.5), we find the following condition which relates the actions of E and F.

$$[E,F]\big|_{\mathbb{C}^{(n)}_{(\lambda)}} = \frac{1}{2i}(-1)^n\Big(1-e^{-2\pi i\lambda}\Big)\mathrm{id}_{\mathbb{C}^{(n)}_{(\lambda)}} = f_{n,\lambda}\mathrm{id}_{\mathbb{C}^{(n)}_{(\lambda)}}. \tag{11.1.4}$$

By (10.1.8) we see that E and F raise and lower the weight n by 1, respectively, and leave the weight λ unchanged. By (10.1.11), we have $E^2 = F^2 = 0$.

We will classify the simple modules. Let S be a simple finite dimensional weight module. Then all non-zero vectors are cyclic. Let v be such a vector, satisfying $Ev = 0$. The non-existence of such a vector would violate either finite dimensionality or the relations $E^2 = 0$. Consider the subspace $\mathrm{span}_{\mathbb{C}}\{v, Fv\}$ which is closed under the actions of E and F. This subspace is therefore a submodule of S and hence equal to S. Let $v \in \mathbb{C}^{(n)}_{(\lambda)}$. If $Fv = 0$ then S is one dimensional and must satisfy $[E,F]v = f_{n,\lambda}v = 0$ which implies that $\lambda = \mathbb{Z}$ and yields the simple module $\mathcal{V}_n$. If $Fv \neq 0$ then S is two dimensional. As simple modules are cyclic, we require that Fv also generates the module. The only way this can happen is if $EFv = [E,F]v = f_{n,\lambda}v$ is a non-zero scalar multiple of v. This implies $\lambda \neq \mathbb{Z}$ and yields the simple module $\mathcal{W}^n_\lambda$.

We wish to prove that $\mathcal{W}^n_\lambda$ and $\mathcal{P}_n$ are both projective and injective. To do this, it suffices to consider extensions with simple modules. As E and F leave the weight λ unchanged, the Ext groups we need to calculate are $\mathrm{Ext}\big(\mathcal{W}^m_\lambda, \mathcal{W}^n_\lambda\big)$, $\mathrm{Ext}(\mathcal{V}_m, \mathcal{P}_n)$ and $\mathrm{Ext}(\mathcal{P}_n, \mathcal{V}_m)$.

First, we calculate $\mathrm{Ext}\big(\mathcal{W}^m_\lambda, \mathcal{W}^n_\lambda\big)$. Consider such an extension and let v_n be the generating vector annihilated by E in the submodule $\mathcal{W}^n_\lambda$. Let w_m denote a representative of the coset corresponding to the equivalent vector in the quotient $\mathcal{W}^m_\lambda$. Then if the extension is indecomposable, there must exist an element U of the universal enveloping algebra of $U^{X_1}_q(\mathfrak{gl}_2)$ such that $Uw_m = v_n$. Recall that E and F raise and lower the weights and the quantum Cartan subalgebra H_1 acts semisimply. If $m = n$, then the most general form is $Uw_m = (a + bEF)w_m$, for some $a, b \in \mathbb{C}$, where a is the constant that comes from evaluating sums and products of H_1 on w_m. Then we have

$$v_n = Uw_m = (a+bEF)w_m = (a+b[E,F])w_m = (a+bf_{\lambda,m})w_m. \tag{11.1.5}$$

But w_m cannot be a scalar multiple of v_n as this contradicts indecomposability, and the extension group vanishes. If $m = n+1$ then $U = aF$, so $0 = aF^2w_m = FUw_m = Fv_n = v_{n-1}$, so no such U exists. If $m = n-1$ then $U = aE$, so $v_n = aEw_m = aE^2 f_{m,\lambda}w_{m-1} = 0$, so no such U exists. The existence of such a U for other choices of m is ruled out by $F^2 = E^2 = 0$. Therefore $\mathcal{W}_\lambda^n$ is both injective and projective.

Now we need to calculate $\mathrm{Ext}(\mathcal{V}_m, \mathcal{P}_n)$. The only extensions of this form that aren't ruled out immediately by $E^2 = F^2 = 0$ are from $\mathcal{V}_n$ into the vector subspace $\mathbb{C}_{(\mathbb{Z})}^{(n\pm1)} \subset \mathcal{P}_n$ and from $\mathcal{V}_{n\pm1}$ into the submodule $\mathbb{C}_{(\mathbb{Z})}^{(n)} \subset \mathcal{P}_n$. In each of these cases, a change of basis reveals that the extension decomposes into a direct sum. In the first case let $w \in \mathcal{V}_m$ be the generating vector. As $EF = FE$, if one of E or F act trivially on this vector they both must, so we let $Ew = aEv$ and $Fw = bFv$. Applying F to the first equation and E to the second, we see that $a = b$. Then E and F act the same way on both w and av. The vector $w - av$ has vanishing image under E and F, so the extension decomposes into a direct sum. Similar reasoning applies to the other cases, except that either E or F act trivially on the new vectors, so we can skip the $a = b$ step. Therefore $\mathcal{P}_n$ is injective. By Proposition 4.3.4, we can use the exact contravariant functor given by the rigid dual given by the antipode to find $\mathrm{Ext}(\mathcal{P}_n, \mathcal{V}_m) = \mathrm{Ext}(\mathcal{V}_m{}^*, \mathcal{P}_n{}^*) = \mathrm{Ext}(\mathcal{V}_{-m}, \mathcal{P}_{-n}) = 0$ so $\mathcal{P}_n$ is also projective.

■

Theorem 11.1.2. *Category $\mathscr{F}$ is equivalent to $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-$\mathrm{Mod}^{\mathrm{wt}}$ for $p = 2$, as abelian categories.*

Proof. We show that the full subcategories of projective modules are equivalent, and then use the fact that both categories are abelian, with enough projectives, to determine the functor on the remaining modules and their Hom spaces via projective resolutions. The functor on projectives is given by

$$\begin{aligned} F^p : \mathscr{F}^p &\to \left(\overline{U}_q^{X_1}(\mathfrak{gl}_2)\text{-}\mathrm{Mod}^{\mathrm{wt}}\right)^p \\ \sigma^n \mathcal{W}_\lambda &\mapsto \mathcal{W}_\lambda^n \\ \mathcal{P}_n &\mapsto \mathcal{P}_n. \end{aligned} \tag{11.1.6}$$

From now on, we denote this module by $\mathcal{W}_\lambda^n$ in both categories. Recall that $\mathcal{W}_\lambda^n$ are simple in both categories, so

$$\mathrm{Hom}\left(\mathcal{W}_\lambda^n, \mathcal{W}_\mu^m\right) = \delta_{\lambda,\mu}\delta_{n,m}\mathbb{C}\mathrm{id}_{\mathcal{W}_\lambda^n}. \tag{11.1.7}$$

Also, the indecomposable projective Hom groups in both categories are given by

$$\mathrm{Hom}(\mathcal{P}_n, \mathcal{P}_m) = \delta_{n,m}\left(\mathbb{C}\mathrm{id}_{\mathcal{P}_n} + \mathbb{C}\psi_n^0\right) + \delta_{n,m-1}\mathbb{C}\psi_n^+ + \delta_{n,m+1}\mathbb{C}\psi_n^-, \tag{11.1.8}$$

where the homomorphisms $\psi_n^\pm : \mathcal{P}_n \to \mathcal{P}_{n\pm1}$ and $\psi_n^0 : \mathcal{P}_n \to \mathcal{P}_n$ are uniquely characterised on the vertex algebra side by their actions on the generating vectors introduced in Section 9.6, and continued as module homomorphisms. On the quantum groups side, they are characterised by their actions on the basis of column vectors $u_n = e_1,\ l_n = e_2,\ r_n = e_3,\ d_n = e_4$ in Proposition 11.1.1, where $(e_i)_j = \delta_{ij}$, and they extend linearly.

$$\psi_n^+(u_n) = l_{n+1}, \qquad \psi_n^-(u_n) = r_{n-1}, \qquad \psi_n^0(u_n) = d_n. \tag{11.1.9}$$

Along with the fact that $\dim \operatorname{Hom}(\mathcal{P}_n, \mathcal{W}_\lambda^m) = \dim \operatorname{Hom}(\mathcal{W}_\lambda^n, \mathcal{P}_m) = 0$ due to the weight λ in $\overline{U}_q^{X_1}(\mathfrak{gl}_2)\text{-Mod}^{\mathrm{wt}}$ and the ghost weight in $\mathscr{F}$, this gives a clear way to map the morphisms as well as the objects between the categories. Therefore, we have an equivalence between the categories of projective modules.

Now we show how to extend the functor F^p on projectives to a functor F on the whole category. Given a morphism $f : \mathcal{M} \to \mathcal{N}$ between modules $\mathcal{M}$ and $\mathcal{N}$ with projective covers $0 \leftarrow \mathcal{M} \leftarrow \mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \cdots$ and $0 \leftarrow \mathcal{N} \leftarrow \mathcal{N}_0 \leftarrow \mathcal{N}_1 \leftarrow \cdots$, respectively, we can construct the following diagram.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{M} & \xleftarrow[d_0]{} & \mathcal{M}_0 & \xleftarrow[d_1]{} & \mathcal{M}_1 & \longleftarrow & \cdots \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \\ 0 & \longleftarrow & \mathcal{N} & \xleftarrow[\delta_0]{} & \mathcal{N}_0 & \xleftarrow[\delta_1]{} & \mathcal{N}_1 & \longleftarrow & \cdots \end{array} \tag{11.1.10}$$

First we lift the morphism $f : \mathcal{M} \to \mathcal{N}$ to the projective resolutions of $\mathcal{M}$ and $\mathcal{N}$. This lifting $f_\bullet : \mathcal{M}_\bullet \to \mathcal{N}_\bullet$ is unique up to chain homotopy. Now we construct a new diagram by applying F to the diagram above. F is already defined as F^p on the projective modules and we can use the projective resolutions to define $F(\mathcal{M})$, $F(\mathcal{N})$, $F(d_0)$, $F(\delta_0)$ and $F(f)$.

$$\begin{array}{ccccccccc} 0 & \longleftarrow & F(\mathcal{M}) & \dashleftarrow & F(\mathcal{M}_0) & \xleftarrow[F(d_1)]{} & F(\mathcal{M}_1) & \longleftarrow & \cdots \\ & & \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle F(f_0)} & & \downarrow{\scriptstyle F(f_1)} & & \\ 0 & \longleftarrow & F(\mathcal{N}) & \dashleftarrow & F(\mathcal{N}_0) & \xleftarrow[F(\delta_1)]{} & F(\mathcal{N}_1) & \longleftarrow & \cdots \end{array} \tag{11.1.11}$$

The functor extends to objects by requiring that the rows of the diagram are exact.

$$F(\mathcal{M}) = F(\mathcal{M}_0)/\operatorname{Im} F(d_1), \qquad F(\mathcal{N}) = F(\mathcal{N}_0)/\operatorname{Im} F(\delta_1). \tag{11.1.12}$$

Then $F(f)$ is the unique morphism factorising $F(\delta_0) \circ F(f_0)$ over $F(d_0)$, by Lemma B.2.3. Different choices of resolution give isomorphic functors, as projective resolutions are unique up to chain homotopy.

■

11.2 Hopf algebra categorical structure

Proposition 11.2.1. *The Hopf algebra maps of* $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$ *endow* $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-$\mathrm{Mod}^{\mathrm{wt}}$ *with the structure of a monoidal category with the following products of simple projective modules,*

$$\mathcal{W}_\lambda^n \otimes \mathcal{W}_\mu^m \cong \mathcal{W}_{\lambda+\mu}^{n+m} \oplus \mathcal{W}_{\lambda+\mu}^{m+n-1}, \qquad \mathcal{W}_\lambda^n \otimes \mathcal{W}_{-\lambda}^m \cong \mathcal{P}_{n+m-1}. \tag{11.2.1}$$

Further, $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-$\mathrm{Mod}^{\mathrm{wt}}$ *is a ribbon Grothendieck-Verdier category with the standard associators and unitors of vector spaces, and the following braidings, twists and duality functor on simple projective modules.*

$$c_{\sigma^n\mathcal{W}_\lambda\otimes\sigma^m\mathcal{W}_\mu} = (-1)^{nm}e^{2\pi i n\mu}\left(\mathrm{id}_{\sigma^{n+m}\mathcal{W}_{\lambda+\mu}} \oplus (-1)^{n+m+1}e^{-2\pi i\mu}\mathrm{id}_{\sigma^{n+m-1}\mathcal{W}_{\lambda+\mu}}\right), \tag{11.2.2}$$

$$\theta_{\mathcal{W}_\lambda^n} = e^{2\pi i(n-k)\lambda}\mathrm{id}_{\mathcal{W}_\lambda^n}, \qquad D(\mathcal{W}_\lambda^n) = \mathcal{W}_{-\lambda}^{2k-n}, \qquad k \in \mathbb{Z}. \tag{11.2.3}$$

The dualising object is given by $K = \mathcal{V}_{2k-1}$. *Any other choice of dualising object does not admit a twist and therefore yields a Grothendieck-Verdier category. If instead* $K = \mathcal{V}$, *then the category is rigid.*

Proof. $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-$\mathrm{Mod}^{\mathrm{wt}}$ is a monoidal category, by Proposition 10.2.3. We make use of the equivalence to ${}^B_B\mathcal{YD}\left(\mathrm{Vect}^{\mathrm{fd}}_{(\mathbb{R}/p\mathbb{Z})\boxtimes\mathbb{Z}}\right)$ in order to calculate the relevant morphisms (see Theorem 10.2.6). This category is naturally equipped with trivial associators and the following R matrix

$$R = (1 - 2i(EK_1^{-1} \otimes FK_2))R_0, \tag{11.2.4}$$

where R_0 is the R-matrix which gives the braiding $e^{i\pi xy'} = e^{i\pi n(m+2\mu)}$, between modules $\mathbb{C}_{x,y} \otimes \mathbb{C}_{x',y'}$ or $\mathbb{C}_{(\lambda)}^{(n)} \otimes \mathbb{C}_{(\mu)}^{(m)}$, from (10.2.8). Let v_λ and v_μ denote a choice of basis vectors annihilated by E, in $\mathcal{W}_\lambda$ and $\mathcal{W}_\mu$, respectively. Then we can evaluate R on each pair of vectors in the fusion product $\sigma^n\mathcal{W}_\lambda \otimes \sigma^m\mathcal{W}_\mu$ as follows, using the fact that

$$\begin{gathered}
R_0\left(\mathbb{C}_{(\lambda)}^{(n)} \otimes \mathbb{C}_{(\mu)}^{(m)}\right) = (-1)^{nm}e^{2\pi i n\mu}\mathrm{id}_{\mathbb{C}_{(\lambda+\mu)}^{(n+m)}}, \qquad K_1\mathbb{C}_{(\lambda)}^{(n)} = (-1)^n\mathrm{id}_{\mathbb{C}_{(\lambda)}^{(n)}}, \qquad K_2\mathbb{C}_{(\lambda)}^{(n)} = (-1)^n e^{2\pi i\lambda}\mathrm{id}_{\mathbb{C}_{(\lambda)}^{(n)}}, \\
\left(K_1^{-1} \otimes K_2\right)\left(\mathbb{C}_{(\lambda)}^{(n)} \otimes \mathbb{C}_{(\mu)}^{(m)}\right) = (-1)^{m+n}e^{2\pi i\mu}\mathrm{id}_{\mathbb{C}_{(\lambda+\mu)}^{(n+m)}}, \qquad 2i[E,F]\mathbb{C}_{(\lambda)}^{(n)} = (-1)^n(1 - e^{-2\pi i\lambda})\mathrm{id}_{\mathbb{C}_{(\lambda)}^{(n)}},
\end{gathered} \tag{11.2.5}$$

we have

$$\begin{aligned}
&R(v_\lambda \otimes v_\mu) = (-1)^{nm}e^{2\pi i n\mu}v_\lambda \otimes v_\mu, \qquad R(v_\lambda \otimes Fv_\mu) = (-1)^{n(m-1)}e^{2\pi i n\mu}v_\lambda \otimes Fv_\mu, \\
&R(Fv_\lambda \otimes Fv_\mu) = (-1)^{(n-1)(m-1)}e^{2\pi i(n-1)\mu}Fv_\lambda \otimes Fv_\mu, \\
&R(Fv_\lambda \otimes v_\mu) = (-1)^{(n-1)m}e^{2\pi i(n-1)\mu}Fv_\lambda \otimes v_\mu + (-1)^{nm}e^{2\pi i n\mu}(1 - e^{-2\pi i\lambda})v_\lambda \otimes Fv_\mu.
\end{aligned} \tag{11.2.6}$$

The tensor structure is given by the following equations.

$$
\begin{aligned}
&\Delta(E)v_\lambda\otimes v_\mu=0, \qquad \Delta(F)v_\lambda\otimes v_\mu=(-1)^n e^{-2\pi i\lambda}v_\lambda\otimes Fv_\mu+Fv_\lambda\otimes v_\mu,\\
&\Delta(E)Fv_\lambda\otimes v_\mu=\frac{1}{2i}(-1)^{n+m}(1-e^{-2\pi i\lambda})v_\lambda\otimes v_\mu, \qquad \Delta(F)Fv_\lambda\otimes v_\mu=-(-1)^n e^{-2\pi i\lambda}Fv_\lambda\otimes Fv_\mu,\\
&\Delta(E)v_\lambda\otimes Fv_\mu=\frac{1}{2i}(-1)^{m}(1-e^{-2\pi i\mu})v_\lambda\otimes v_\mu, \qquad \Delta(F)v_\lambda\otimes Fv_\mu=Fv_\lambda\otimes Fv_\mu,\\
&\Delta(E)Fv_\lambda\otimes Fv_\mu=\frac{1}{2i}\Big((-1)^m\Big(1-e^{-2\pi i\mu}\Big)Fv_\lambda\otimes v_\mu-(-1)^{n+m}\Big(1-e^{-2\pi i\lambda}\Big)v_\lambda\otimes Fv_\mu\Big),\\
&\Delta(F)Fv_\lambda\otimes Fv_\mu=0.
\end{aligned}
\tag{11.2.7}
$$

In the case $\lambda+\mu\notin\mathbb{Z}$, we can identify the tensor product module as $\sigma^{n+m}\mathcal{W}_{\lambda+\mu}\oplus\sigma^{n+m-1}\mathcal{W}_{\lambda+\mu}$, where we denote the highest weight vectors by $v_{\lambda+\mu}$ and $w_{\lambda+\mu}$, respectively. Then we have the following identifications.

$$
\begin{aligned}
v_{\lambda+\mu}&=v_\lambda\otimes v_\mu,\\
w_{\lambda+\mu}&=\frac{1}{2i}\Big((-1)^m\Big(1-e^{-2\pi i\mu}\Big)Fv_\lambda\otimes v_\mu-(-1)^{n+m}\Big(1-e^{-2\pi i\lambda}\Big)v_\lambda\otimes Fv_\mu\Big),\\
Fv_{\lambda+\mu}&=(-1)^n e^{-2\pi i\lambda}v_\lambda\otimes Fv_\mu+Fv_\lambda\otimes v_\mu,\\
Fw_{\lambda+\mu}&=-\frac{1}{2i}(-1)^{n+m}\Big(1-e^{-2\pi i(\lambda+\mu)}\Big)Fv_\lambda\otimes Fv_\mu.
\end{aligned}
\tag{11.2.8}
$$

The braiding is determined by our R-matrix above, by $\sigma=\tau\circ R$ where τ is the vector space flip.

$$
\begin{aligned}
&\sigma(v_{\lambda+\mu})=(-1)^{nm}e^{2\pi i n\mu}v_{\mu+\lambda}, \qquad \sigma(w_{\lambda+\mu})=(-1)^{(n-1)(m-1)}e^{2\pi i(n-1)\mu}w_{\mu+\lambda}\\
&\sigma(Fv_{\lambda+\mu})=(-1)^{nm}e^{2\pi i n\mu}Fv_{\mu+\lambda} \qquad \sigma(Fw_{\lambda+\mu})=(-1)^{(n-1)(m-1)}e^{2\pi i(n-1)\mu}Fw_{\mu+\lambda}.
\end{aligned}
\tag{11.2.9}
$$

Therefore, we obtain the braiding

$$
c_{\sigma^n\mathcal{W}_\lambda\otimes\sigma^m\mathcal{W}_\mu}=(-1)^{nm}e^{2\pi i n\mu}\Big(\mathrm{id}_{\sigma^{n+m}\mathcal{W}_{\lambda+\mu}}\oplus(-1)^{n+m+1}e^{-2\pi i\mu}\mathrm{id}_{\sigma^{n+m-1}\mathcal{W}_{\lambda+\mu}}\Big), \tag{11.2.10}
$$

In the case where $\lambda+\mu\in\mathbb{Z}$, $w_{\lambda+\mu}$ and $Fv_{\lambda+\mu}$ become proportional, and we can identify the tensor product module as $\mathcal{P}_{n+m-1}$ in the following way,

$$
e_1=Fv_\lambda\otimes v_\mu, \qquad e_2=Fv_\lambda\otimes Fv_\nu, \qquad e_3=v_\lambda\otimes v_\mu, \qquad e_4=w_{\lambda+\mu}, \tag{11.2.11}
$$

where $(e_i)_j=\delta_{ij}$ are a basis of column vectors in Proposition 11.1.1.

Finally, as the category is rigid ($U_q^{X_1}(\mathfrak{gl}_2)$ has an invertible antipode), we can use Proposition 4.3.4 to find the following rigid duals of the simple and projective modules.

$$
\mathcal{P}_n{}^*\cong\mathcal{P}_{-n}, \qquad \big(\mathcal{W}_\lambda^n\big)^*\cong\mathcal{W}_{-\lambda}^{1-n}, \qquad (\mathcal{V}_n)^*\cong\mathcal{V}_{-n}, \tag{11.2.12}
$$

which match the rigid tensor dual of the bosonic ghosts, from Section 9.7. This corresponds to the choice of dualising object $K = \mathcal{V}$, and we can tensor with any invertible object to yield the full set of dualising objects $\{\mathcal{V}_n \,:\, n \in \mathbb{Z}\}$. Take the dualising object to be $K = \mathcal{V}_{k-1}$. Then the dual is given by $D(\mathcal{W}^n_\lambda) = \mathcal{W}^{k-n}_{-\lambda}$, as

$$\mathrm{Hom}\Big(\mathcal{W}^{k-n}_{-\lambda}, D\big(\mathcal{W}^n_\lambda\big)\Big) \cong \mathrm{Hom}\Big(\mathcal{W}^{k-n}_{-\lambda} \otimes \mathcal{W}^n_\lambda, \mathcal{V}_{k-1}\Big) \cong \mathrm{Hom}(\mathcal{P}_{k-1}, \mathcal{V}_{k-1}) \cong \mathbb{C}. \tag{11.2.13}$$

The modules $\mathcal{W}^n_\lambda$ are simple, so let $\theta_{\mathcal{W}^n_\lambda} = Q(n,\lambda)\mathrm{id}_{\mathcal{W}^n_\lambda}$. Then the conditions for θ to be a ribbon Grothendieck-Verdier twist become

$$Q(n,\lambda) = Q(k-n,\lambda), \qquad Q(n+m,\lambda+\mu) = e^{2\pi i(n\mu+n\lambda)}Q(n,\lambda)Q(m,\mu). \tag{11.2.14}$$

$Q(n,\lambda) = e^{2\pi i(n-k/2)\lambda}$ satisfies these equations, but this is only well defined when $\lambda \in 2\mathbb{Z}$, so we have compatibility with a twist only when our choice of dualising object is $\mathcal{V}_{2k-1}$ for $k \in \mathbb{Z}$. ■

11.3 Minimal data

Lemma 11.3.1. *Let $\mathscr{C}$ be an abelian braided tensor category satisfying that there are enough projectives, that the tensor products are biexact and that projectives form a tensor ideal. Let $\mathscr{C}^p$ be the full subcategory of projective objects of $\mathscr{C}$. Then any choice of natural isomorphisms θ^p, c^p and α^p obeying the balancing, hexagon and pentagon conditions on $\mathscr{C}^p$, admit unique extensions to $\mathscr{C}$ which obey the same conditions.*

Proof. We proceed in a similar way to Proposition 7.2.1. For $\mathcal{M}$, $\mathcal{N}$, $\mathcal{P} \in \mathscr{C}$, we define the families of morphisms θ, c and α from the definition of θ^p, c^p and α^p on projective objects. Choose projective resolutions $0 \leftarrow \mathcal{M} \leftarrow \mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \cdots$, $0 \leftarrow \mathcal{N} \leftarrow \mathcal{N}_0 \leftarrow \mathcal{N}_1 \leftarrow \cdots$ and $0 \leftarrow \mathcal{P} \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \cdots$. The biexactness of the tensor product and projectives forming a tensor ideal implies that the $\mathcal{M}_i \otimes_{\mathscr{C}} \mathcal{N}_j$ are projective, and that the total complexes form projective resolutions. Therefore by Lemma B.2.4 we can construct the following diagrams which illustrate how the structure morphisms are extended to the whole category $\mathscr{C}$.

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{M} & \xleftarrow{\;d_0\;} & \mathcal{M}_0 & \xleftarrow{\;d_1\;} & \mathcal{M}_1 & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle \theta(\mathcal{M})} & & \downarrow{\scriptstyle \theta^p(\mathcal{M}_0)} & & \downarrow{\scriptstyle \theta^p(\mathcal{M}_1)} & & \\
0 & \longleftarrow & \mathcal{M} & \xleftarrow{\;\delta_0\;} & \mathcal{M}_0 & \xleftarrow{\;\delta_1\;} & \mathcal{M}_1 & \longleftarrow & \cdots
\end{array} \tag{11.3.1}$$

$$\begin{array}{ccccccccc}
0 & \longleftarrow & \mathcal{M}\otimes\mathcal{N} & \xleftarrow{\;d_0\;} & \mathcal{M}_0\otimes\mathcal{N}_0 & \xleftarrow{\;d_1\;} & (\mathcal{M}_1\otimes\mathcal{N}_0)\oplus(\mathcal{M}_0\otimes\mathcal{N}_1) & \longleftarrow & \cdots \\
 & & \downarrow{\scriptstyle c(\mathcal{M},\mathcal{N})} & & \downarrow{\scriptstyle c^p(\mathcal{M}_0,\mathcal{N}_0)} & & \downarrow{\scriptstyle c^p(\mathcal{M}_1,\mathcal{N}_0)\oplus c^p(\mathcal{M}_0,\mathcal{N}_1)} & & \\
0 & \longleftarrow & \mathcal{N}\otimes\mathcal{M} & \xleftarrow{\;\delta_0\;} & \mathcal{N}_0\otimes\mathcal{M}_0 & \xleftarrow{\;\delta_1\;} & (\mathcal{N}_0\otimes\mathcal{M}_1)\oplus(\mathcal{N}_1\otimes\mathcal{M}_0) & \longleftarrow & \cdots
\end{array} \tag{11.3.2}$$

$$\begin{array}{ccccccccc} 0 & \longleftarrow & \mathcal{M}\otimes(\mathcal{N}\otimes\mathcal{P}) & \xleftarrow[d_0]{} & \mathcal{M}_0\otimes(\mathcal{N}_0\otimes\mathcal{P}_0) & \xleftarrow[d_1]{} & \mathcal{Q} & \longleftarrow & \cdots \\ & & \downarrow{\scriptstyle a(\mathcal{M},\mathcal{N},\mathcal{P})} & & \downarrow{\scriptstyle a^P(\mathcal{M}_0,\mathcal{N}_0,\mathcal{P}_0)} & & \downarrow{\scriptstyle a^P(\mathcal{Q})} & & \\ 0 & \longleftarrow & (\mathcal{M}\otimes\mathcal{N})\otimes\mathcal{P} & \xleftarrow[\delta_0]{} & (\mathcal{M}_0\otimes\mathcal{N}_0)\otimes\mathcal{P}_0 & \xleftarrow[\delta_1]{} & \mathcal{Q}' & \longleftarrow & \cdots \end{array} \tag{11.3.3}$$

where

$$\begin{aligned} \mathcal{Q} &= (\mathcal{M}_1\otimes(\mathcal{N}_0\otimes\mathcal{P}_0))\oplus(\mathcal{M}_0\otimes(\mathcal{N}_1\otimes\mathcal{P}_0))\oplus(\mathcal{M}_0\otimes(\mathcal{N}_0\otimes\mathcal{P}_1)),\\ \mathcal{Q}' &= ((\mathcal{M}_1\otimes\mathcal{N}_0)\otimes\mathcal{P}_0)\oplus((\mathcal{M}_0\otimes\mathcal{N}_1)\otimes\mathcal{P}_0)\oplus((\mathcal{M}_0\otimes\mathcal{N}_0)\otimes\mathcal{P}_1),\\ a^P(\mathcal{Q}) &= a^P(\mathcal{M}_1,\mathcal{N}_0,\mathcal{P}_0)\oplus a^P(\mathcal{M}_0,\mathcal{N}_1,\mathcal{P}_0)\oplus a^P(\mathcal{M}_0,\mathcal{N}_0,\mathcal{P}_1). \end{aligned} \tag{11.3.4}$$

The proof follows the same idea as Proposition 7.2.1 where the constraints (balancing, hexagon and pentagon axioms) which we know are satisfied for $\mathcal{M}_0$ make up the right face of a wire prism and the constraint for $\mathcal{M}$ makes up the left face. Then we have arrows running right to left which connect the two faces that come from the resolutions. The right faces commute as they are evaluated on projective objects and the front and back commute by naturality of the structure maps. ■

The following proposition introduces a set of structure morphisms which are sufficient to determine the braided tensor structure on the full category. This greatly reduces the amount of computation required to characterise the braided tensor structure. The proposition depends only on the fusion rules of the category and not the vertex algebra or Hopf algebra specific structure. In the Hopf algebra case, the braiding is given for all the modules by the R-matrix, but this is not true for the vertex algebra case. Similarly, the twist is specified by $e^{2\pi i L_0}$ in the vertex algebra case, but not in the Hopf algebra case.

Proposition 11.3.2. *Let $\mathscr{C}$ denote any category which is equivalent to $\mathscr{F}$ (and therefore also $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$) as an abelian category. Equip $\mathscr{C}$ with a tensor product bifunctor which acts on the isomorphism classes of objects in the same way as that of $\mathscr{F}$. Then the ribbon Grothendieck-Verdier structure on $\mathscr{C}$ is determined by the following set of structure morphisms, a choice of dualising object and the left and right unit isomorphisms (which for $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$-Mod$^{\mathrm{wt}}$ and $\mathscr{F}$ are determined for all modules by the scalar multiplication and the field map, respectively). Note that we allow the case $\lambda+\mu+\rho=\mathbb{Z}$.*

$$\{c_{\mathcal{W}_\lambda^n,\mathcal{W}_\mu^m},\qquad a_{\mathcal{W}_\lambda^n,\mathcal{W}_\mu^m,\mathcal{W}_\rho^\ell},\qquad \theta_{\mathcal{W}_\lambda^n},\qquad \lambda+\mu,\ \lambda+\rho,\ \mu+\rho\neq\mathbb{Z}.\} \tag{11.3.5}$$

Proof. We make use of the axioms of a braided monoidal category to prove that the rest of the structure morphisms for projective modules are fully determined. Recall the pentagon, hexagon and balancing identities are given by

$$a_{A\otimes B,C,D}\circ a_{A,B,C\otimes D}=\big(a_{A,B,C}\otimes \mathrm{id}_D\big)\circ a_{A,B\otimes C,D}\circ\big(\mathrm{id}_A\otimes a_{B,C,D}\big), \tag{11.3.6}$$

$$a_{C,A,B}\circ c_{A,B\otimes C}\circ a_{A,B,C}=\big(\mathrm{id}_B\otimes c_{A,C}\big)\circ a_{B,A,C}\circ\big(c_{A,B}\otimes \mathrm{id}_C\big), \tag{11.3.7}$$

$$a^{-1}_{C,A,B}\circ c_{A\otimes B,C}\circ a^{-1}_{A,B,C}=\big(c_{A,C}\otimes \mathrm{id}_B\big)\circ a^{-1}_{A,C,B}\circ\big(\mathrm{id}_A\otimes c_{B,C}\big), \tag{11.3.8}$$

$$\theta_{A\otimes B}=c_{B,A}\circ c_{A,B}\circ(\theta_A\otimes\theta_B). \tag{11.3.9}$$

Let $A=\sigma^{n_1}\mathcal{W}_{\lambda_1}$, $B=\sigma^{n_2}\mathcal{W}_{\lambda_2}$, $C=\sigma^{n_3}\mathcal{W}_{\lambda_3}$, $D=\sigma^{n_4}\mathcal{W}_{\lambda_4}$. Recall that the tensor products are given by

$$\begin{aligned}&\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_\mu\cong\mathcal{W}^{n+m}_{\lambda+\mu}\oplus\mathcal{W}^{m+n-1}_{\lambda+\mu}, &&\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_{-\lambda}\cong\mathcal{P}_{n+m-1},\\ &\mathcal{W}^n_\lambda\otimes\mathcal{P}_m\cong\mathcal{W}^{-1}_\lambda\oplus 2\mathcal{W}_\lambda\oplus\mathcal{W}^1_\lambda, &&\mathcal{P}_n\otimes\mathcal{P}_m\cong\mathcal{P}_{n+m-1}\oplus 2\mathcal{P}_{n+m}\oplus\mathcal{P}_{n+m+1}.\end{aligned} \tag{11.3.10}$$

We will also repeatedly make use of the fact that $a_{A\oplus B,C,D}=a_{A,C,D}\oplus a_{B,C,D}$ and the fact that each step can use the newly determined data of the previous steps. Now if we take

- $\lambda_1+\lambda_3=\mathbb{Z}$ in (11.3.6) gives $a_{A,B,C}$ as the only unknown, hence we determine $a_{\mathcal{W}^n_\lambda,\mathcal{W}^m_\mu,\mathcal{W}^\ell_\rho}$ for $\lambda+\rho=\mathbb{Z}$,

$$\begin{aligned}a_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3}_{\lambda_3}}\otimes\mathrm{id}_{\mathcal{W}^{n_4}_{\lambda_4}}&=a_{\mathcal{W}^{n_1}_{\lambda_1}\otimes\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\circ a_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3}_{\lambda_3}\otimes\mathcal{W}^{n_4}_{\lambda_4}}\\ &\quad\circ\left(\mathrm{id}_{\mathcal{W}^{n_1}_{\lambda_1}}\otimes a^{-1}_{\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\right)\circ a^{-1}_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2}_{\lambda_2}\otimes\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\\ &=\left(a_{\mathcal{W}^{n_1+n_2}_{\lambda_1+\lambda_2},\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\oplus a_{\mathcal{W}^{n_1+n_2-1}_{\lambda_1+\lambda_2},\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\right)\\ &\quad\circ\left(a_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3+n_4}_{\lambda_3+\lambda_4}}\oplus a_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3+n_4-1}_{\lambda_3+\lambda_4}}\right)\\ &\quad\circ\left(\mathrm{id}_{\mathcal{W}^{n_1}_{\lambda_1}}\otimes a^{-1}_{\mathcal{W}^{n_2}_{\lambda_2},\mathcal{W}^{n_3}_{\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\right)\circ\left(a^{-1}_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2+n_3}_{\lambda_2+\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\oplus a^{-1}_{\mathcal{W}^{n_1}_{\lambda_1},\mathcal{W}^{n_2+n_3-1}_{\lambda_2+\lambda_3},\mathcal{W}^{n_4}_{\lambda_4}}\right),\end{aligned} \tag{11.3.11}$$

- $\lambda_1+\lambda_2+\lambda_3,\ \lambda_1+\lambda_4=\mathbb{Z}$ in (11.3.6) gives $a_{A\otimes B,C,D}$ as the only unknown, hence we determine $a_{\sigma^n\mathcal{W}_\lambda,\sigma^m\mathcal{W}_\mu,\sigma^\ell\mathcal{W}_\rho}$ for $\lambda+\mu=\mathbb{Z}$,
- $\lambda_2+\lambda_3+\lambda_4,\ \lambda_1+\lambda_4=\mathbb{Z}$ in (11.3.6) gives $a_{A,B,C\otimes D}$ as the only unknown, hence we determine $a_{\sigma^n\mathcal{W}_\lambda,\sigma^m\mathcal{W}_\mu,\sigma^\ell\mathcal{W}_\rho}$ for $\mu+\rho=\mathbb{Z}$,
- either $\lambda_1+\lambda_2$, $\lambda_2+\lambda_3$, or $\lambda_3+\lambda_4=\mathbb{Z}$ in (11.3.6) determine the associators where one of the modules is $\sigma^\ell\mathcal{P}$. Taking combinations of these then determine the associators where multiple modules are of the form $\sigma^\ell\mathcal{P}$.

- $\lambda_1+\lambda_2+\lambda_3=\mathbb{Z}$ in (11.3.7) gives $c_{A,B\otimes C}$ as the only unknown, hence we determine $c_{\sigma^n\mathcal{W}_\lambda,\sigma^m\mathcal{W}_\mu}$ for $\lambda+\mu=\mathbb{Z}$,
- $\lambda_2+\lambda_3=\mathbb{Z}$ in (11.3.7) gives $c_{A,B\otimes C}$ as the only unknown, hence we determine $c_{\sigma^n\mathcal{W}_\lambda,\sigma^\ell\mathcal{P}}$,
- $\lambda_1+\lambda_2=\mathbb{Z}$ in (11.3.8) gives $c_{A\otimes B,C}$ as the only unknown, hence we determine $c_{\sigma^\ell\mathcal{P},\sigma^n\mathcal{W}_\lambda}$,
- $\lambda_1+\lambda_2=\mathbb{Z}$, $C=\sigma^s\mathcal{P}$ in (11.3.8) gives $c_{A\otimes B,C}$ as the only unknown, hence we determine $c_{\sigma^\ell\mathcal{P},\sigma^s\mathcal{P}}$.
- $\lambda_1+\lambda_2=\mathbb{Z}$ in (11.3.9) gives $\theta_{A\otimes B}$ as the only unknown, hence we determine $\theta_{\sigma^\ell\mathcal{P}}$.

Now we have the braided monoidal structure on projective modules, and by Lemma 11.3.1 we have the braided monoidal structure on the whole category. ■

11.4 Ribbon Grothendieck-Verdier equivalence

We would like to apply Lemma 7.1.1, and therefore we must make a choice of intertwining operators G^T_f (which are determined by choice of $G^T_{\mathrm{id}_{\mathcal{M}\boxtimes\mathcal{N}}}$ for each pair of modules) such that the constraints (7.1.3) - (7.1.5) are satisfied. Thanks to Proposition 7.2.1 we only need to make this choice for projective modules, and Proposition 11.3.2 lets us reduce the list further to only certain simple projectives (11.3.5), for each of these constraints, and we shall see that the obstacle to completing the proof of Theorem 11.0.1 is the associativity constraint (7.1.5), when the sum of the three weights is integral.

Proposition 11.4.1. *The choice of intertwining operators given by the free field realisation of Proposition 9.4.1 and the proof of Proposition 9.7.9 satisfies the constraints (7.1.3) - (7.1.5) for simple projective modules such that no sums of the weights are integral. That is, recall that* $\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_\mu\cong\mathcal{W}^{n+m}_{\lambda+\mu}\oplus\mathcal{W}^{m+n-1}_{\lambda+\mu}$, *for* $\lambda+\mu\neq\mathbb{Z}$. *Let* $\mathcal{Y}=G^T_{\mathrm{id}_{\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_\mu}}$ *be our choice of intertwining operator, defined as the sum of the following two maps*

$$\mathcal{Y}_1:\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_\mu\xrightarrow{\cong_1}\mathbb{F}_{\lambda(\psi+\theta)-(n+1)\psi}\otimes\mathbb{F}_{\mu(\psi+\theta)-(m+1)\psi}\xrightarrow{I}\mathbb{F}_{(\lambda+\mu)(\psi+\theta)-(n+m+2)\psi}\xrightarrow{\cong_1}\mathcal{W}^{n+m-1}_{\lambda+\mu},$$
$$\mathcal{Y}_2:\mathcal{W}^n_\lambda\otimes\mathcal{W}^m_\mu\xrightarrow{\cong_2}\mathbb{F}_{\lambda(\psi+\theta)-n\psi}\otimes\mathbb{F}_{\mu(\psi+\theta)-m\psi}\xrightarrow{I}\mathbb{F}_{(\lambda+\mu)(\psi+\theta)-(n+m)\psi}\xrightarrow{\cong_2}\mathcal{W}^{n+m}_{\lambda+\mu},\qquad(11.4.1)$$

where I *represents the free field intertwining operators given in (8.3.11),* $\cong_n$ *denotes the identification of modules using the nth free field realisation, for* $n=1,2$.

Proof. The unit constraint (7.1.3) is automatically satisfied by the left unit isomorphism of the Hopf algebra being given by scalar multiplication, and intertwining operators specialising to the field map when the vertex algebra is one of the arguments. We can take the choice of braiding and associator given in (9.4.10) to see that (7.1.5) is automatically satisfied, as the intertwining operators inherit the same associator as the lattice vertex algebra. Rewriting $\mathbb{F}_{\lambda(\psi+\theta)+n\psi}$ as $\mathbb{F}_{(\lambda+n/2)(\psi+\theta)+n(\psi-\theta)/2}$, see that this module has the weights $(n, \lambda+n/2) \in (\mathbb{Z}, \mathbb{R}/\mathbb{Z})$ and each summand will obtain the following contribution due to the lattice vertex algebra braiding.

$$c_{\mathbb{F}_{\lambda(\psi+\theta)+n\psi},\mathbb{F}_{\mu(\psi+\theta)+m\psi}} = e^{i\pi n(m+2\mu)}. \tag{11.4.2}$$

These summands match the desired braiding $G(c)$ from the Hopf algebra, so Equation (7.1.4) is also satisfied. ■

Remark. The case where $\lambda+\mu = \mathbb{Z}$ is more complicated because the images of these intertwining operators give the modules $(\mathcal{W}_0^+)^{n+m}$ and $(\mathcal{W}_0^-)^{n+m-1}$ which are quotients of $\mathcal{W}_\lambda^n \otimes \mathcal{W}_{-\lambda}^m \cong \mathcal{P}_{n+m-1}$, but are not direct summands like in the case $\lambda+\mu \neq \mathbb{Z}$.

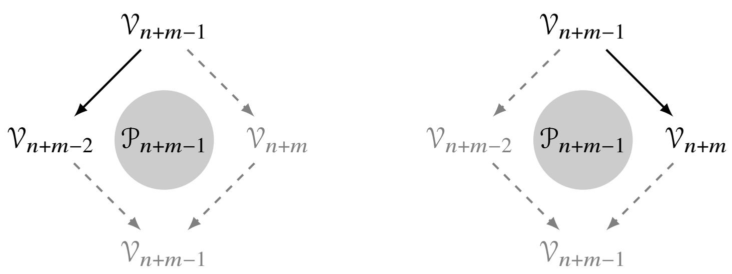


Figure 11.2: Quotients of the full module $\mathcal{P}_{n+m-1}$ seen by first free and second field realisations respectively.

By Proposition 11.3.2, the only constraint left to consider is the associativity, for triples of modules $\mathcal{W}_{\lambda_i}^{\ell_i}$ where $\lambda_1+\lambda_2+\lambda_3 \in \mathbb{Z}$ but $\lambda_i+\lambda_j \notin \mathbb{Z}$ for any distinct pairs $i, j \in 1,2,3$. The full tensor product on both sides is isomorphic to $\mathcal{W}_{\lambda_1}^{\ell_1} \otimes \mathcal{W}_{\lambda_2}^{\ell_2} \otimes \mathcal{W}_{\lambda_3}^{\ell_3} \cong \mathcal{P}_{\ell_1+\ell_2+\ell_3} \oplus \mathcal{P}_{\ell_1+\ell_2+\ell_3-1}$. Using the free field realisation, one can construct intertwining operators where the left and right hand sides agree on the quotients of the modules seen by each realisation (see Figure 11.2). This leaves a submodule of the full tensor product on which the conjecture is required. This conjecture is hard to prove because one needs a relation involving the composition factors which are not seen by the free field realisations. For some ideas about what form these relations might take and the difficulty involved in explicit intertwining operator calculations, see Appendix F.

Conjecture 11.4.2. *There exists a solution to (7.1.5) for the triple of modules $\mathcal{W}_{\lambda_i}^{\ell_i}$, with $i = 1,2,3$ and $\lambda_i+\lambda_j \notin \mathbb{Z}$, $\lambda_1+\lambda_2+\lambda_3 \in \mathbb{Z}$. That is, intetwining operators $\mathcal{Y}$ can be chosen such* *that the following equation is satisfied.*

$$\left(\mathcal{Y}_{\mathcal{W}^{\ell_1}_{\lambda_1},\mathcal{W}^{\ell_2+\ell_3}_{\lambda_2+\lambda_3}}(m_1,x_1)+\mathcal{Y}_{\mathcal{W}^{\ell_1}_{\lambda_1},\mathcal{W}^{\ell_2+\ell_3-1}_{\lambda_2+\lambda_3}}(m_1,x_1)\right)\mathcal{Y}_{\mathcal{W}^{\ell_2}_{\lambda_2},\mathcal{W}^{\ell_3}_{\lambda_3}}(m_2,x_2)m_3$$
$$=\left(\mathcal{Y}_{\mathcal{W}^{\ell_2+\ell_3}_{\lambda_2+\lambda_3},\mathcal{W}^{\ell_3}_{\lambda_3}}+\mathcal{Y}_{\mathcal{W}^{\ell_2+\ell_3-1}_{\lambda_2+\lambda_3},\mathcal{W}^{\ell_3}_{\lambda_3}}\right)\left(\mathcal{Y}_{\mathcal{W}^{\ell_1}_{\lambda_1},\mathcal{W}^{\ell_2}_{\lambda_2}}(m_1,x_1-x_2)m_2,x_2\right)m_3, \tag{11.4.3}$$

where equality here means the convergence of both sides as power series when $x_1,x_2\in\mathbb{R}$ *and* $|x_1|>|x_2|>|x_1-x_2|>0$.

Proof of Theorem 11.0.1. By Theorem 11.1.2, we already know that the categories are equivalent as abelian categories. We can use the set of minimal data for determining the structure morphisms on the whole category (Proposition 11.3.2) to compare the structure morphisms on the Hopf algebra side (Proposition 11.2.1) with the vertex algebra side, using our framework for constructing functors from vertex algebra categories (Lemma 7.1.1 and its refinements).

We know that $\mathscr{F}$ is equipped with ribbon Grothendieck-Verdier structure by Theorem 6.2.1, with the twist $\theta=e^{2\pi i L_0}$ which on the simple projectives gives $\theta_{\mathcal{W}^n_\lambda}=e^{2\pi i n\lambda}$. This matches the data for the twist on the quantum group side, when the choice of dualising object is $\mathcal{V}'=\mathcal{V}_{-1}$. The set of all possible dualising objects are precisely the invertible objects, which are the same as for the quantum group, so the same twists are admitted (see the proof of Proposition 11.2.1). We also know by Theorem 9.8.1.1 that $\mathscr{F}$ is rigid, which corresponds to the dualising object being the tensor unit. Therefore, this structure on the vertex algebra side agrees with the quantum group data.

By Proposition 11.4.1 and Conjecture 11.4.2 we know that the desired conditions of Lemma 7.1.1 are satisfied and therefore that the tensor functor φ_2^P, which is determined by the choice of intertwiners G^T for projective modules, extends to a tensor functor on the whole category, by Proposition 7.2.1. ∎

— Appendix A —

Category Theory

Category theory was introduced by Eilenberg and Maclane in the 1940s. Categories are abstractions of other mathematical concepts and provide a framework which includes many areas of mathematics. This appendix summarises the necessary definitions and results used throughout this thesis.

A.1 Categories and functors

Definition A.1.1. A **category** $\mathscr{C}$ consists of

- a collection of **objects** $\mathrm{Ob}(\mathscr{C})$,
- a collection of **morphisms** $\mathrm{Hom}(A,B)$, for every $A,B \in \mathrm{Ob}(\mathscr{C})$,
- a binary operation called the composition of morphisms, for every $A,B,C \in \mathrm{Ob}(\mathscr{C})$,

$$\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C) \tag{A.1.1}$$

$$f \times g \mapsto g \circ f, \tag{A.1.2}$$

  which is associative and has an identity morphism.

Further, a **functor** is a morphism of categories. A **natural transformation** η between two functors $F,G \in \mathrm{Hom}(\mathscr{C},\mathscr{D})$ is a family of morphisms η_C for every $C \in \mathrm{Ob}(\mathscr{C})$, satisfying the following requirement, for every morphism $f \in \mathrm{Hom}(A,B)$, where $G(A) = X$ and $F(B) = Y$.

$$\eta_Y \circ F(f) = G(f) \circ \eta_X. \tag{A.1.3}$$

Example. **Set** is the category whose objects are sets and morphisms are functions.
Ab is the category whose objects are abelian groups and morphisms are group homomorphisms.

A.2 Covariance and contravariance

Definition A.2.1. Given a category $\mathscr{C}$, the **opposite** category $\mathscr{C}^{\mathrm{op}}$ is formed by reversing the morphisms. A **contravariant** functor between two categories $\mathscr{C}$ and $\mathscr{D}$ is a functor from $\mathscr{C} \to \mathscr{D}^{\mathrm{op}}$ and therefore reverses the direction of morphisms. In contrast a **covariant** functor is one which is not contravariant.

Definition A.2.2. An **equivalence** between two categories $\mathscr{C}$ and $\mathscr{D}$ is consists of the following data .

- Two functors $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$,
- Two natural isomorphisms $\varepsilon : FG \to \mathrm{id}_{\mathscr{D}}$ and $\eta : \mathrm{id}_{\mathscr{C}} \to GF$,

An **anti-equivalence** between two categories $\mathscr{C}$ and $\mathscr{D}$ is an equivalence between $\mathscr{C}$ and $\mathscr{D}^{\mathrm{op}}$.

A.3 Yoneda's lemma

Yoneda's lemma is considered by many to be the most important theorem in category theory. Let $A, X, Y \in \mathrm{Ob}(\mathscr{C})$. The Hom functor $\mathrm{Hom}(A, -)$ sends X to the set of morphisms $\mathrm{Hom}(A, X)$, and sends a morphism $f : X \to Y$ to the morphism $f \circ -$, which sends $g \in \mathrm{Hom}(A, X)$ to the morphism $f \circ g \in \mathrm{Hom}(A, Y)$. For any functor F from $\mathscr{C}$ to **Set**, Yoneda's lemma provides a surprisingly simple characterisation of the family of morphisms from $\mathrm{Hom}(A, X)$ to $F(X)$, for any fixed $X \in \mathscr{C}$.

Theorem A.3.1. *For every $A, \in \mathrm{Ob}(\mathscr{C})$ and functor $F : \mathscr{C} \to$ **Set**, the natural transformations between the functors $\mathrm{Hom}(A, -)$ and F are in bijection with elements of $F(A)$.*

$$\mathrm{Hom}(\mathrm{Hom}(A, -), F) \cong F(A). \tag{A.3.1}$$

Explicitly, the natural transformation corresponding to the element $u \in F(A)$ is

$$\begin{aligned} \Phi_X : \mathrm{Hom}(A, X) &\to F(X) \\ f &\mapsto (Ff)u. \end{aligned} \tag{A.3.2}$$

— Appendix B —

Homological Algebra

Homological algebra is the study of sequences of algebraic objects, in particular with regard to topological properties. This appendix summarises the necessary definitions and results used throughout this thesis.

B.1 Exact sequences

Definition B.1.1. Some important definitions

- **Proper submodule** = not the full module
- **Reducible** = contains non-trivial proper submodule
- **Irreducible/Simple** = no non-trivial proper submodules
- **Decomposable** = direct sum of two non-trivial submodules
- **Semisimple** = direct sum of irreducible modules
- **Socle** = maximal semisimple submodule
- **Head** = maximal semisimple quotient

Exact sequences provide a graphical way to express and manipulate concepts such as submodules, quotients and homomorphisms to prove statements about modules. There are many rich concepts related to exact sequences and their properties, but here we only cover what will be directly relevant to the thesis.

Definition B.1.2. An **exact sequence** is a sequence of modules, connected by module homomorphisms,

$$\cdots \xrightarrow{f_{-2}} \mathcal{M}_{-1} \xrightarrow{f_{-1}} \mathcal{M}_0 \xrightarrow{f_0} \mathcal{M}_1 \xrightarrow{f_1} \mathcal{M}_2 \xrightarrow{f_2} \cdots \tag{B.1.1}$$

which satisfies the exactness condition $\ker f_n = \operatorname{im} f_{n-1}$. A **short exact sequence** is an exact sequence of the form

$$0 \longrightarrow \mathcal{M} \xrightarrow{\iota} \mathcal{N} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0. \tag{B.1.2}$$

Note that ι is an injection and π is a surjection. A short exact sequence is called **split** if $\mathcal{N} \cong \mathcal{M} \oplus \mathcal{Q}$.

Definition B.1.3. A functor F is called an **exact functor** if it preserves short exact sequences. That is, given any short exact sequence

$$0 \longrightarrow \mathcal{M} \xrightarrow{\iota} \mathcal{N} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0, \tag{B.1.3}$$

for $\mathcal{M}, \mathcal{N}, \mathcal{Q} \in \mathscr{C}$, the functor $F : \mathscr{C} \to \mathscr{D}$ is exact if the following is also a short exact sequence.

$$0 \longrightarrow F(\mathcal{M}) \xrightarrow{F(\iota)} F(\mathcal{N}) \xrightarrow{F(\pi)} F(\mathcal{Q}) \longrightarrow 0. \tag{B.1.4}$$

B.2 Projectivity and injectivity

Projective and injective modules cannot occur as quotients or submodules, respectively, of an indecomposable module. In other words, they can't form a non-trivial extension with another module and must yield a direct sum. We are interested in categories with enough projectives/injectives, which means that they contain all the other modules as submodules and quotients. Therefore classifying the projectives/injectives is an important first step towards a full module classification.

Definition B.2.1. A module $\mathcal{P}$ is **projective** if and only if every short exact sequence of the following form splits.

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{P} \longrightarrow 0. \tag{B.2.1}$$

A module $\mathcal{J}$ is **injective** if and only if every short exact sequence of the following form splits.

$$0 \longrightarrow \mathcal{J} \longrightarrow \mathcal{N} \longrightarrow \mathcal{Q} \longrightarrow 0. \tag{B.2.2}$$

Consequently, $\operatorname{Ext}(\mathcal{P}, \mathcal{M}) = 0 = \operatorname{Ext}(\mathcal{M}, \mathcal{J})$ for projective $\mathcal{P}$, injective $\mathcal{J}$ and any module $\mathcal{M}$.

Definition B.2.2. A **projective resolution** of $\mathcal{M}$ is an exact sequence

$$0 \longleftarrow \mathcal{M} \longleftarrow \mathcal{M}_0 \longleftarrow \mathcal{M}_1 \longleftarrow \cdots \tag{B.2.3}$$

where each of the $\mathcal{M}_i$ are projective. Similarly, an **injective resolution** of $\mathcal{M}$ is an exact sequence

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}_0 \longrightarrow \mathcal{M}_1 \longrightarrow \cdots \tag{B.2.4}$$

where each of the $\mathcal{M}_i$ are injective.

Lemma B.2.3. *For any exact sequence*

$$\cdots \longrightarrow \mathcal{M} \xrightarrow{\iota} \mathcal{N} \xrightarrow{\pi} \mathcal{Q} \longrightarrow 0, \tag{B.2.5}$$

any morphism $\pi' : \mathcal{N} \to \mathcal{Q}'$ satisfying $\pi' \circ \iota = 0$ factorises uniquely over π. That is, there exists a unique morphism $\kappa : \mathcal{Q} \to \mathcal{Q}'$ such that $\pi' = \kappa \circ \pi$.

Proof. $\pi' \circ \iota = 0$ implies that $\ker \pi' \subset \ker \pi$ and therefore we can use that $\mathcal{Q} \cong \mathcal{N}/\ker \pi$ to define

$$\begin{aligned} \kappa : \mathcal{Q} &\to \mathcal{Q}' \\ x + \ker \pi &\mapsto \pi'(x). \end{aligned} \tag{B.2.6}$$

■

Lemma B.2.4. *Let $\mathscr{C}$ and $\mathscr{D}$ be abelian categories with enough projectives, and take their full subcategories of projectives, denoted $\mathscr{C}^p$ and $\mathscr{D}^p$, respectively. Let $F, G : \mathscr{C} \to \mathscr{D}$ be exact functors satisfying that the images of any projective objects are projective. If there exists a natural isomorphism $\eta^p : F(-|_p) \to G(-|_p)$, where $-|_p$ denotes the restriction to $\mathscr{C}^p$, then η^p admits a unique extension η to $\mathscr{C}$.*

Proof. For $\mathcal{M} \in \mathscr{C}$, we define a family of morphisms $\eta(\mathcal{M}) : F(\mathcal{M}) \to G(\mathcal{M})$ from the definition of η^p on projective objects. Choose a projective resolution $0 \leftarrow \mathcal{M} \leftarrow \mathcal{M}_0 \leftarrow \mathcal{M}_1 \leftarrow \cdots$. By assumption, F and G are exact and take projectives to projectives, so the images of these projective resolutions under F and G are again projective resolutions for $F(\mathcal{M})$, $G(\mathcal{M})$ respectively. We can therefore consider the following diagram, where the dashed arrow is the morphism we are seeking.

$$\begin{array}{ccccccc} 0 \leftarrow & F(\mathcal{M}) & \xleftarrow[d_0]{} & F(\mathcal{M}_0) & \xleftarrow[d_1]{} & F(\mathcal{M}_1) & \leftarrow \cdots \\ & \downarrow{\scriptstyle \eta(\mathcal{M})} & & \downarrow{\scriptstyle \eta^p(\mathcal{M}_0)} & & \downarrow{\scriptstyle \eta^p(\mathcal{M}_1)} & \\ 0 \leftarrow & G(\mathcal{M}) & \xleftarrow[\delta_0]{} & G(\mathcal{M}_0) & \xleftarrow[\delta_1]{} & G(\mathcal{M}_1) & \leftarrow \cdots \end{array} \tag{B.2.7}$$

By the assumed naturality of η^p, the squares with solid edges of the diagram commute. Then we have

$$\delta_0 \circ \eta^p(\mathcal{M}_0) \circ d_1 = \delta_0 \circ \delta_1 \circ \eta^p(\mathcal{M}_1), \tag{B.2.8}$$

so by Lemma B.2.3 $\eta(\mathcal{M})$ is the unique morphism factorising $\delta_0 \circ \eta^p(\mathcal{M}_0)$ over d_0.

Next we show that this characterisation of η is independent of the projective resolutions. Suppose $0 \leftarrow \mathcal{M} \leftarrow \mathcal{M}'_0 \leftarrow \mathcal{M}'_1 \leftarrow \cdots$ is also a projective resolution of $\mathcal{M}$, then by [48][Theorem 4.1] there exist chain maps $f_\bullet : \mathcal{M}_\bullet \to \mathcal{M}'_\bullet$ and $f'_\bullet : \mathcal{M}'_\bullet \to \mathcal{M}_\bullet$ lifting the identity on $\mathcal{M}$, and which are unique up to chain homotopy and inverse to each other up to chain homotopy. To show that this resolution of $\mathcal{M}$ gives the same morphism $\eta(\mathcal{M})$, we need to verify that

$$\eta^p(\mathcal{M}_0') \circ F(f_0) = G(f_0) \circ \eta^p(\mathcal{M}_0). \tag{B.2.9}$$

This equality holds due to the assumed naturality of $\eta^p(\mathcal{M}_0)$ on projectives. The independence of $\eta(\mathcal{M})$ from choice of projective resolution of $\mathcal{N}$ follows in the same way. ■

Remark. When the category $\mathscr{D}$ in Lemma B.2.4 is a concrete category where the objects are at the very least abelian groups and the morphisms group homomorphisms, for example, a module category for a vertex operator algebra, it may desirable to have an actual formula for the natural transformation η on non-projective modules. This can be done by considering the diagram (B.2.7) and picking a set theoretic right inverse d_0^{-1} to the morphism d_0, that is, $d_0 \circ d_0^{-1} = \mathrm{id}_{F(\mathcal{M})}$. Note that in general d_0^{-1} cannot be chosen to be a morphism of $\mathscr{D}$. One can then define $\eta(\mathcal{M}) = \delta_0 \circ \eta(\mathcal{M}_0) \circ d_0^{-1}$. To show that this formula gives a morphism in $\mathscr{D}$ and that it does not depend on the choice of right inverse d_0^{-1}, let $d_0^{-1}(x) = d_1(x_1)$. Then

$$\begin{aligned}\eta(\mathcal{M})(x) &= \delta_0 \circ \eta(\mathcal{M}_0) \circ d_0^{-1}(x) = \delta_0 \circ \eta(\mathcal{M}_0) \circ d_1(x_1)\\ &= \delta_0 \circ \delta_1 \circ \eta(\mathcal{M}_1)(x_1) = 0.\end{aligned} \tag{B.2.10}$$

B.3 Hom-Ext exact sequences

Proposition B.3.1. *If the short sequence*

$$0 \longrightarrow \mathcal{M} \longrightarrow \mathcal{N} \longrightarrow \mathcal{Q} \longrightarrow 0 \tag{B.3.1}$$

is exact, then the following sequences are exact.

$$\begin{gathered}0 \to \mathrm{Hom}(\mathcal{A},\mathcal{M}) \to \mathrm{Hom}(\mathcal{A},\mathcal{N}) \to \mathrm{Hom}(\mathcal{A},\mathcal{Q}) \to \mathrm{Ext}(\mathcal{A},\mathcal{M}) \to \mathrm{Ext}(\mathcal{A},\mathcal{N}) \to \mathrm{Ext}(\mathcal{A},\mathcal{Q}) \to \cdots\\ 0 \to \mathrm{Hom}(\mathcal{Q},\mathcal{A}) \to \mathrm{Hom}(\mathcal{N},\mathcal{A}) \to \mathrm{Hom}(\mathcal{M},\mathcal{A}) \to \mathrm{Ext}(\mathcal{Q},\mathcal{A}) \to \mathrm{Ext}(\mathcal{N},\mathcal{A}) \to \mathrm{Ext}(\mathcal{M},\mathcal{A}) \to \cdots\end{gathered} \tag{B.3.2}$$

Proposition B.3.2. *The Euler characteristic of an exact sequence vanishes. That is, any exact sequence*

$$\cdots \xrightarrow{f_{-2}} \mathcal{M}_{-1} \xrightarrow{f_{-1}} \mathcal{M}_0 \xrightarrow{f_0} \mathcal{M}_1 \xrightarrow{f_1} \mathcal{M}_2 \xrightarrow{f_2} \cdots \tag{B.3.3}$$

satisfies

$$\sum_{n\in\mathbb{Z}} (-1)^n \dim \mathcal{M}_n = 0. \tag{B.3.4}$$

Remark. Note that if any of the modules involved are projective or injective, then the sequences (B.3.2) terminate. If all but one terms in the sequence are known then the remaining term can be determined by using the fact that the Euler characteristic of an exact sequence vanishes.

B.4 Loewy diagrams

Definition B.4.1. The following is a filtration of a module $\mathcal{M}$ by submodules,

$$0 = \mathcal{M}_0 \subset \mathcal{M}_1 \subset \cdots \subset \mathcal{M}_{\ell-1} \subset \mathcal{M}_\ell = \mathcal{M}. \tag{B.4.1}$$

The subquotients are given by $\mathcal{C}_i = \mathcal{M}_i/\mathcal{M}_{i-1}$.

- If we demand that the subquotients are irreducible, then they are called **composition factors** and the filtration is called a **composition series**.
- If we demand that the subquotients are instead the socles of $\mathcal{M}/\mathcal{M}_{i-1}$ then the filtration is called a **socle series**.

Definition B.4.2. A **Loewy diagram** is constructed by horizontal layers, where the i-th layer corresponds to the subquotients of the socle series. We annotate Loewy diagrams with arrows that go from composition factors (direct summands) of each socle in the i-th layer to composition factors in the $(i-1)$th. These arrows roughly represent the direction of the extensions.

For some examples of Loewy diagrams, see Section 9.6 and Section 11.1.

B.5 Complexes and total complexes

Definition B.5.1. A **chain complex** $\mathcal{M}_\bullet$ is a sequence of modules, connected by module homomorphisms

$$\cdots \xleftarrow{d_{-1}} \mathcal{M}_{-1} \xleftarrow{d_0} \mathcal{M}_0 \xleftarrow{d_1} \mathcal{M}_1 \xleftarrow{d_2} \mathcal{M}_2 \xleftarrow{d_3} \cdots \tag{B.5.1}$$

such that $d_n \circ d_{n+1} = 0$. Note that exact sequences are a special case of chain complexes.

One can construct the **total complex** of the tensor product of two chain complexes $\mathcal{M}_\bullet$ and $\mathcal{N}_\bullet$ by

$$\mathrm{Tot}^{\oplus}(\mathcal{M}_\bullet \otimes_{\mathscr{C}} \mathcal{N}_\bullet)_n = \bigoplus_{k+\ell=n} \mathcal{M}_k \otimes \mathcal{N}_\ell. \tag{B.5.2}$$

— Appendix C —

Formal calculus

Formal calculus is the manipulation of various kinds of formal distributions, which are series in formal variables. Vertex algebras are formulated in this framework and we use much of this notation.

C.1 Formal distributions

Definition C.1.1. Important spaces:

- **formal Laurent series**

$$V[[z^{\pm}]] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \ : \ a_n \in V \right\} \tag{C.1.1}$$

- **polynomials**

$$V[z] = \left\{ \sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n \ : \ a_n \in V \text{ with only finitely many non-zero} \right\} \tag{C.1.2}$$

- **formal Laurent polynomials**

$$V[z^{\pm}] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n \ : \ a_n \in V \text{ with only finitely many non-zero} \right\} \tag{C.1.3}$$

- **formal power series**

$$V[[z]] = \left\{ \sum_{n\in\mathbb{Z}_{\geq 0}} a_n z^n \ : \ a_n \in V \right\} \tag{C.1.4}$$

- **truncated formal Laurent series**

$$V((z)) = \left\{ \sum_{n\in\mathbb{Z}_{\geq k}} a_n z^n \ : \ a_n \in V,\ k \in \mathbb{Z} \right\} \tag{C.1.5}$$

Now we introduce the notion of residue, which is taken in analogy with complex analysis, the delta distribution and the binomial expansion which we expand in the second variables by convention.

Definition C.1.2. Important definitions:

- **residue**

$$\mathrm{Res}_z\left(\sum_{n\in\mathbb{Z}} a_n z^n\right) = a_{-1} \tag{C.1.6}$$

- **delta distribution**

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} w^n z^{-n-1} \tag{C.1.7}$$

- **binomial expansion**

$$(z-w)^k = \sum_{n=0} \binom{k}{j} (-w)^j z^{k-j} \tag{C.1.8}$$

Remark. The delta distribution satisfies the following equations, for $a(z) \in V[[z^{\pm 1}]]$.

$$a(z)\delta(z-w) = a(w)\delta(z-w), \qquad \mathrm{Res}_z a(z)\delta(z-w) = a(w). \tag{C.1.9}$$

— Appendix D —

Sufficient conditions for convergence and extension – Proof of Theorem 6.3.2

In this section we give a proof of Theorem 6.3.2 by reviewing reasoning presented by Yang in [30] and showing that certain assumptions on the category of strongly graded modules (see [30, Assumption 7.1, Part 3]) are not required, if one only wishes to conclude that convergence and extension properties hold. Instead all that is required is that the modules considered satisfy suitable finiteness conditions. This appendix closely follows the logic of [30, Sections 5 & 6] and also [67, Section 2].

Throughout this section let $A \leq B$ be abelian groups. Further, let V be an A-graded vertex algebra with a vertex subalgebra $\overline{\mathsf{V}} \subset \mathsf{V}^{(0)}$. In this section only, all mode expansions of fields from a vertex operator algebra V will be of the form $Y(v,z) = \sum_{n\in\mathbb{Z}} v_n z^{-n-1}$ regardless of the conformal weight of $v \in \mathsf{V}$, that is, v_n refers to the coefficient of z^{-n-1} rather than the one which shifts conformal weight by $-n$.

Definition D.0.1. Let $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3, \mathcal{W}_4$ be B-graded V-modules.

1. We say that two B-graded logarithmic intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ of respective types $\binom{\mathcal{W}_0}{\mathcal{W}_1,\ \mathcal{W}_4}$, $\binom{\mathcal{W}_4}{\mathcal{W}_2,\ \mathcal{W}_3}$ satisfy the *convergence and extension property for products* if for any $a_1, a_2, \in B$ and any doubly homogeneous elements $w_0' \in \mathcal{W}_0'$, $w_3 \in \mathcal{W}_3$, $w_i \in \mathcal{W}_i^{(a_i)}$, $i = 1,2$, there exist $M \in \mathbb{Z}_{\geq 0}$, $r_1, \ldots, r_M, s_1, \ldots s_M \in \mathbb{R}$, $u_1, \ldots u_M, v_1, \ldots v_M \in \mathbb{Z}_{\geq 0}$ and analytic functions $f_1(z), \ldots, f_M(z)$ on the disc $|z| < 1$ satisfying
$$\mathrm{wt}\, w_1 + \mathrm{wt}\, w_2 + s_k > N, \quad \text{for each } k = 1, \ldots, M, \tag{D.0.1}$$
where $N \in \mathbb{Z}$ depends only on the intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $a_1 + a_2$, such that as a formal power series the matrix element
$$\langle w_0', \mathcal{Y}_1(w_1, z_1)\mathcal{Y}_2(w_2, z_2)w_3\rangle \tag{D.0.2}$$

converges absolutely in the region $|z_1| > |z_2| > 0$ and may be analytically continued to the multivalued analytic function

$$\sum_{k=1}^{M} z_2^{r_k}(z_1-z_2)^{s_k}(\log z_2)^{u_k}(\log(z_1-z_2))^{v_k} f_k\left(\frac{z_1-z_2}{z_2}\right) \tag{D.0.3}$$

in the region $|z_2| > |z_1 - z_2| > 0$.

2. We say that two B-graded logarithmic intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ of respective types $\binom{\mathcal{W}_0}{\mathcal{W}_4,\ \mathcal{W}_3}$, $\binom{\mathcal{W}_4}{\mathcal{W}_1,\ \mathcal{W}_2}$ satisfy the *convergence and extension property for iterates* if for any $a_2, a_3, \in B$ and any doubly homogeneous elements $w_0' \in \mathcal{W}_0'$, $w_1 \in \mathcal{W}_3$, $w_i \in \mathcal{W}_i^{(a_i)}$, $i = 2,3$, there exist $M \in \mathbb{Z}_{\geq 0}$, $r_1, \dots, r_M, s_1, \dots s_M \in \mathbb{R}$, $u_1, \dots u_M, v_1, \dots v_M \in \mathbb{Z}_{\geq 0}$ and analytic functions $f_1(z), \dots f_M(z)$ on the disc $|z| < 1$ satisfying

$$\mathrm{wt}\, w_2 + \mathrm{wt}\, w_3 + s_k > N, \quad \text{for each } k = 1, \dots, M, \tag{D.0.4}$$

where $N \in \mathbb{Z}$ depends only on the intertwining operators $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $a_2 + a_3$, such that as a formal power series the matrix element

$$\langle w_0', \mathcal{Y}_1(\mathcal{Y}_2(w_1, z_1 - z_2)w_2, z_2)w_3\rangle \tag{D.0.5}$$

converges absolutely in the region $|z_2| > |z_1 - z_2| > 0$ and may be analytically continued to the multivalued analytic function

$$\sum_{k=1}^{M} z_1^{r_k} z_2^{s_k}(\log z_1)^{u_k}(\log z_2)^{v_k} f_k\left(\frac{z_2}{z_1}\right) \tag{D.0.6}$$

in the region $|z_1| > |z_2| > 0$.

Consider the Noetherian ring $R = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}, (z_1 - z_2)^{-1}]$. Then for any quadruple of B-graded V-modules $\mathcal{W}_0, \mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$, and any triple $(a_1, a_2, a_3) \in B^3$, we define the R-module

$$T^{(a_1,a_2,a_3)} = R \otimes \left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)} \otimes \mathcal{W}_1^{(a_1)} \otimes \mathcal{W}_2^{(a_2)} \otimes \mathcal{W}_3^{(a_3)}, \tag{D.0.7}$$

where all the tensor product symbols denote complex tensor products. We will generally omit the tensor product symbol separating R from the V-modules. The motivation for considering this module is that for any B-graded module $\mathcal{W}_4$ and any pair of grading compatible logarithmic intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of respective types $\binom{\mathcal{W}_0}{\mathcal{W}_1,\ \mathcal{W}_4}$ and $\binom{\mathcal{W}_4}{\mathcal{W}_2,\ \mathcal{W}_3}$ it produces matrix elements via the map $\phi_{\mathcal{Y}_1,\mathcal{Y}_2} : T^{(a_1,a_2,a_3)} \to z_1^h \mathbb{C}(\{z_2/z_1\})[z_1^{\pm 1}, z_2^{\pm 1}]$, where h is the combined conformal weight of w_0', w_1, w_2, w_3 and $\mathbb{C}(\{x\})$ is the space of all power series in x with bounded below real exponents (the modules $\mathcal{W}_i$, $i = 0, 1, 2, 3$ will always have real conformal weights below), defined by

$$\phi_{\mathcal{Y}_1,\mathcal{Y}_2}(f(z_1,z_2)w_0' \otimes w_1 \otimes w_2 \otimes w_3) = \iota(f(z_1,z_2))\langle w_0', \mathcal{Y}_1(w_1,z_1)\mathcal{Y}_2(w_2,z_2)w_3\rangle, \tag{D.0.8}$$

where $\iota : R \to \mathbb{C}[\![z_2/z_1]\!][z_1^{\pm 1}, z_2^{\pm 1}]$ is the map expanding elements of R such that the powers of z_2 are bounded below. This in turn justifies considering the submodule

$$\begin{aligned} J^{(a_1,a_2,a_3)} = \operatorname{span}_R \Big\{ &\mathcal{A}\big(v, w_0', w_1, w_2, w_3\big), \mathcal{B}\big(v, w_0', w_1, w_2, w_3\big), \\ &\mathcal{C}\big(v, w_0', w_1, w_2, w_3\big), \mathcal{D}\big(v, w_0', w_1, w_2, w_3\big) \in T^{(a_1,a_2,a_3)} \ : \\ &v \in \overline{\mathbb{V}},\ w_0' \in (\mathcal{W}_0')^{(a_1+a_2+a_3)},\ w_i \in W^{(a_i)},\ i = 1,2,3 \Big\}, \end{aligned} \tag{D.0.9}$$

where the generators

$$\begin{aligned} \mathcal{A}\big(v, w_0', w_1, w_2, w_3\big) &= -w_0' \otimes v_{-1}w_1 \otimes w_2 \otimes w_3 + \sum_{k\ge 0} \binom{-1}{k} (-z_1)^k v^*_{-1-k} w_0' \otimes w_1 \otimes w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0' \otimes w_1 \otimes v_k w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} (-z_1)^{-1-k} w_0' \otimes w_1 \otimes w_2 \otimes v_k w_3, \\ \mathcal{B}\big(v, w_0', w_1, w_2, w_3\big) &= -w_0' \otimes w_1 \otimes v_{-1}w_2 \otimes w_3 + \sum_{k\ge 0} \binom{-1}{k} (-z_2)^k v^*_{-1-k} w_0' \otimes w_1 \otimes w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} (-(z_1 - z_2))^{-1-k} w_0' \otimes v_k w_1 \otimes w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} (-z_2)^{-1-k} w_0' \otimes w_1 \otimes w_2 \otimes v_k w_3, \\ \mathcal{C}\big(v, w_0', w_1, w_2, w_3\big) &= v^*_{-1} w_0' \otimes w_1 \otimes v_{-1} w_2 \otimes w_3 - \sum_{k\ge 0} \binom{-1}{k} z_1^{-1-k} w_0' \otimes v_k w_1 \otimes w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} z_2^{-1-k} w_0' \otimes w_1 \otimes v_k w_2 \otimes w_3 - w_0' \otimes w_1 \otimes w_2 \otimes v_{-1} w_3, \\ \mathcal{D}\big(v, w_0', w_1, w_2, w_3\big) &= v_{-1} w_0' \otimes w_1 \otimes v_{-1} w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} z_1^{k+1} w_0' \otimes \mathrm{e}^{z_1^{-1} L_1} \left(-z_1^2\right)^{L_0} v_k \left(-z_1^{-2}\right)^{L_0} \mathrm{e}^{-z_1^{-1} L_1} w_1 \otimes w_2 \otimes w_3 \\ &\quad - \sum_{k\ge 0} \binom{-1}{k} z_2^{-1-k} w_0' \otimes w_1 \otimes \mathrm{e}^{z_2^{-1} L_1} \left(-z_2^2\right)^{L_0} v_k \left(-z_2^{-2}\right)^{L_0} \mathrm{e}^{-z_2^{-1} L_1} w_2 \otimes w_3 \\ &\quad - w_0' \otimes w_1 \otimes w_2 \otimes v^*_{-1} w_3, \end{aligned} \tag{D.0.10}$$

are preimages of the relations coming from residues of the Jacobi identity for intertwining operators and where $v_k^* : \mathcal{W}_i' \to \mathcal{W}_i'$ denotes the adjoint of $v_k : \mathcal{W}_i \to \mathcal{W}_i$. Hence $J^{(a_1,a_2,a_3)}$ lies in the kernel of $\phi_{\mathcal{Y}_1,\mathcal{Y}_2}$ for any choice of intertwining operators $\mathcal{Y}_1, \mathcal{Y}_2$ of the correct types.

Next consider the doubly homogeneous space

$$T_{[r]}^{(a_1,a_2,a_3)} = \prod_{\substack{r_0,r_1,r_2,r_3\in\mathbb{R}\\ r_0+r_1+r_2+r_3=r}} R\otimes\left(\mathcal{W}_0'\right)_{[r_0]}^{(a_1+a_2+a_3)}\otimes(\mathcal{W}_1)_{[r_1]}^{(a_1)}\otimes(\mathcal{W}_2)_{[r_2]}^{(a_2)}\otimes(\mathcal{W}_3)_{[r_3]}^{(a_3)} \tag{D.0.11}$$

to construct the subspaces

$$\begin{aligned} F_r(T^{(a_1,a_2,a_3)}) &= \prod_{s\le r} T_{[s]}^{(a_1,a_2,a_3)},\\ F_r(J^{(a_1,a_2,a_3)}) &= J^{(a_1,a_2,a_3)}\cap F_r(T^{(a_1,a_2,a_3)}). \end{aligned} \tag{D.0.12}$$

These define filtrations on $T^{(a_1,a_2,a_3)}$ and $J^{(a_1,a_2,a_3)}$, respectively, since $F_s(T^{(a_1,a_2,a_3)}) \subset F_r(T^{(a_1,a_2,a_3)})$ and $F_s(J^{(a_1,a_2,a_3)}) \subset F_r(J^{(a_1,a_2,a_3)})$, if $s \le r$, and $\bigcup_{r\in\mathbb{R}} F_r(T^{(a_1,a_2,a_3)}) = T^{(a_1,a_2,a_3)}$ and $\bigcup_{r\in\mathbb{R}} F_r(J^{(a_1,a_2,a_3)}) = J^{(a_1,a_2,a_3)}$. Note that if the $\mathcal{W}_i, i = 0,1,2,3$ are discretely strongly graded, then $T_{[r]}^{(a_1,a_2,a_3)}$ is a finite sum of finite dimensional doubly homogeneous spaces tensored with R. Hence $T_{[r]}^{(a_1,a_2,a_3)}$ is a finitely generated free R-module. Further, $F_r(T^{(a_1,a_2,a_3)})$ is also a finite sum and hence also a finitely generated free R-module. Finally, the ring R is Noetherian and so the submodule $F_r(J^{(a_1,a_2,a_3)})$ is also finitely generated.

Proposition D.0.2. *Let the V-modules $\mathcal{W}_i,\ i = 0,1,2,3$ be discretely strongly B-graded and B-graded C_1-cofinite as $\overline{\mathsf{V}}$-modules, then for any $a_1,a_2,a_3 \in B$ there exists $M \in \mathbb{Z}$ such that for any $r \in \mathbb{R}$*

$$F_r(T^{(a_1,a_2,a_3)}) \subset F_r(J^{(a_1,a_2,a_3)}) + F_M(T^{(a_1,a_2,a_3)}) \quad \text{and} \quad T^{(a_1,a_2,a_3)} \subset J^{(a_1,a_2,a_3)} + F_M(T^{(a_1,a_2,a_3)}). \tag{D.0.13}$$

Proof. By assumption the modules $\mathcal{W}_i,\ i = 0,1,2,3$ are B-graded C_1-cofinite as $\overline{V}$-modules, that is, the spaces

$$C_1(\mathcal{M})^{(a)} = \operatorname{span}_{\mathbb{C}}\left\{v_{-h}w \in M^{(a)}\ :\ v\in\overline{\mathsf{V}}_{[h]} h>0, w\in\mathcal{M}\right\} \tag{D.0.14}$$

have finite codimension in $\mathcal{M}^{(a)}$ for $\mathcal{M} = \mathcal{W}_i,\ i = 0,1,2,3$. Thus $\mathcal{M}_{[h]}^{(a)} \subset C_1(\mathcal{M})^{(a)}$ for sufficiently large conformal weight $h \in \mathbb{R}$ and hence there exists $M \in \mathbb{Z}$ such that

$$\begin{aligned} \bigoplus_{n>M} T_{[n]}^{(a_1,a_2,a_3)} \subset\ & C_1\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}\otimes\mathcal{W}_1^{(a_1)}\otimes\mathcal{W}_2^{(a_2)}\otimes\mathcal{W}_3^{(a_3)}\\ &+\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}\otimes C_1(\mathcal{W}_1)^{(a_1)}\otimes\mathcal{W}_2^{(a_2)}\otimes\mathcal{W}_3^{(a_3)} \end{aligned}$$

$$+\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}\otimes\mathcal{W}_1^{(a_1)}\otimes C_1(\mathcal{W}_2)^{(a_2)}\otimes\mathcal{W}_3^{(a_3)}+\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}\otimes\mathcal{W}_1^{(a_1)}\otimes\mathcal{W}_2^{(a_2)}\otimes C_1(\mathcal{W}_3)^{(a_3)}. \tag{D.0.15}$$

We prove the first inclusion of the proposition by induction on $r\in\mathbb{R}$. If $r\le M$, then the inclusion is true by $F_r(T^{(a_1,a_2,a_3)})$ defining a filtration. Next assume that $F_r(T^{(a_1,a_2,a_3)})\subset F_r(J^{(a_1,a_2,a_3)})+F_M(T^{(a_1,a_2,a_3)})$ is true for all $r<s\in\mathbb{R}$ for some $s>M$. We will show that any element of the homogeneous space $T_{[s]}^{(a_1,a_2,a_3)}$ can be written as a sum of elements in $F_s(J^{(a_1,a_2,a_3)})$ and $F_M(T^{(a_1,a_2,a_3)})$. Since $s>M$, this homogeneous element is an element of the right-hand side of (D.0.15). We shall only consider the case of this element lying in the second summand of the right-hand side, as the other cases follow analogously. Without loss of generality we can assume the element has the form $w_0'\otimes v_{-1}w_1\otimes w_2\otimes w_3\in T_{[s]}^{(a_1,a_2,a_3)}$, where $w_0'\in\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}$, $w_i\in\mathcal{W}_i^{(a_i)}, i=1,2,3$, $v\in\overline{V}_{[h]}, h>0$. By computing the degrees of the summands making up $\mathcal{A}(v,w_0',w_1,w_2,w_3)$ in (D.0.10) we see that the three sums over k all lie in $F_{s-1}(T^{(a_1,a_2,a_3)})\subset F_{s-1}(J^{(a_1,a_2,a_3)})+F_M(T^{(a_1,a_2,a_3)})$ and that $\mathcal{A}(v,w_0',w_1,w_2,w_3)\in F_s(J^{(a_1,a_2,a_3)})$. Further,

$$\begin{aligned}w_0'\otimes v_{-1}w_1\otimes w_2\otimes w_3 &= -\mathcal{A}(v,w_0',w_1,w_2,w_3)+\sum_{k\ge0}\binom{-1}{k}(-z_1)^k v_k^* w_0'\otimes w_1\otimes w_2\otimes w_3\\ -\sum_{k\ge0}\binom{-1}{k}(-(z_1-z_2))^{-1-k}&w_0'\otimes w_1\otimes v_k w_2\otimes w_3-\sum_{k\ge0}\binom{-1}{k}(-z_1)^{-1-k}w_0'\otimes w_1\otimes w_2\otimes v_k w_3.\end{aligned} \tag{D.0.16}$$

Thus $w_0'\otimes v_{-1}w_1\otimes w_2\otimes w_3$ lies in the sum $F_s(J^{(a_1,a_2,a_3)})+F_M(T^{(a_1,a_2,a_3)})$ and the first inclusion of the proposition follows. The second inclusion follows from $F_r(T^{(a_1,a_2,a_3)})$ and $F_r(J^{(a_1,a_2,a_3)})$ defining filtrations.

$$\begin{aligned}T^{(a_1,a_2,a_3)} &= \bigcup_{r\in\mathbb{R}}F_r(T^{(a_1,a_2,a_3)})\subset\bigcup_{r\in\mathbb{R}}\Big(F_r(J^{(a_1,a_2,a_3)})+F_M(T^{(a_1,a_2,a_3)})\Big)\\ &= \left(\bigcup_{r\in\mathbb{R}}F_r(J^{(a_1,a_2,a_3)})\right)+F_M(T^{(a_1,a_2,a_3)})=J^{(a_1,a_2,a_3)}+F_M(T^{(a_1,a_2,a_3)}).\end{aligned} \tag{D.0.17}$$

■

Corollary D.0.3. *Let the* V*-modules* $\mathcal{W}_i$, $i=0,1,2,3$ *be discretely strongly* B*-graded and* B*-graded* C_1*-cofinite as* $\overline{\mathsf{V}}$*-modules.*

1. *The quotient* R*-module* $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ *is finitely generated.*
2. *For any representative* $w\in T^{(a_1,a_2,a_3)}$*, we denote its coset in* $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ *by* $[w]$*. Let* $w_0'\in\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}$ *and* $w_i\in\mathcal{W}_i^{(a_i)}$ $i=1,2,3$*, and consider the submodules of*

$T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ given by

$$M_1 = \mathrm{span}_R\Big\{\Big[w_0 \otimes L_{-1}^j w_1 \otimes w_2 \otimes w_3\Big] \ : \ j \in \mathbb{Z}_{\geq 0}\Big\},$$
$$M_2 = \mathrm{span}_R\Big\{\Big[w_0 \otimes w_1 \otimes L_{-1}^j w_2 \otimes w_3\Big] \ : \ j \in \mathbb{Z}_{\geq 0}\Big\}. \tag{D.0.18}$$

Then M_1 and M_2 are finitely generated, in particular, there exist $m,n \in \mathbb{Z}_{\geq 0}$ and $a_k(z_1,z_2), b_\ell(z_1,z_2) \in R$, $1 \leq k \leq m$, $1 \leq \ell \leq n$ such that

$$\begin{aligned}
&\big[w_0 \otimes L_{-1}^m w_1 \otimes w_2 \otimes w_3\big] + a_1(z_1,z_2)\big[w_0 \otimes L_{-1}^{m-1} w_1 \otimes w_2 \otimes w_3\big] + \cdots \\
&\qquad + a_m(z_1,z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0, \\
&\big[w_0 \otimes w_1 \otimes L_{-1}^n w_2 \otimes w_3\big] + b_1(z_1,z_2)\big[w_0 \otimes w_1 \otimes L_{-1}^{n-1} w_2 \otimes w_3\big] + \cdots \\
&\qquad + b_n(z_1,z_2)[w_0 \otimes w_1 \otimes w_2 \otimes w_3] = 0.
\end{aligned} \tag{D.0.19}$$

Proof. Since R is a Noetherian ring, Part 1 holds if $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ is isomorphic to a subquotient of a finitely generated module over R. By Proposition D.0.2 we have the inclusion and identification

$$\begin{aligned}
T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)} &\subset \Big(J^{(a_1,a_2,a_3)} + F_M(T^{(a_1,a_2,a_3)})\Big)/J^{(a_1,a_2,a_3)} \\
&\cong F_M(T^{(a_1,a_2,a_3)})/\Big(F_M(T^{(a_1,a_2,a_3)}) \cap J^{(a_1,a_2,a_3)}\Big).
\end{aligned} \tag{D.0.20}$$

Thus $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ is isomorphic to a subquotient of the finitely generated module $F_M(T^{(a_1,a_2,a_3)})$ and Part 1 follows. Part 2 is an immediate consequence of Part 1 and the fact that a submodule of a finitely generated module over a Noetherian ring is again finitely generated. ∎

Theorem D.0.4. *Let the V-modules $\mathcal{W}_i$, $i = 0,1,2,3$ be discretely strongly B-graded and B-graded C_1-cofinite as $\overline{\mathsf{V}}$-modules, let $\mathcal{W}_4$ be a B-graded V-module and let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be logarithmic grading compatible intertwining operators of types $\binom{\mathcal{W}_0}{\mathcal{W}_1\ \mathcal{W}_4}$, $\binom{\mathcal{W}_4}{\mathcal{W}_2\ \mathcal{W}_3}$, respectively. Then for any homogeneous elements $w_0' \in \mathcal{W}_0'$, $w_i \in \mathcal{W}_i$, $i = 1,2,3$, there exist $m,n \in \mathbb{Z}_{\geq 0}$ and $a_k(z_1,z_2), b_\ell(z_1,z_2) \in R$, $1 \leq k \leq m$, $1 \leq \ell \leq n$ such that the power series expansion of the matrix element*

$$\langle w_0', \mathcal{Y}_1(w_1,z_1)\mathcal{Y}_2(w_2,z_2)w_3\rangle \tag{D.0.21}$$

is a solution to the power series expansion of the system of differential equations

$$\frac{\partial^m \phi}{\partial z_1^m} + a_1(z_1,z_2)\frac{\partial^{m-1}\phi}{\partial z_1^{m-1}} + \cdots + a_m(z_1,z_2)\phi = 0, \quad \frac{\partial^n \phi}{\partial z_2^n} + b_1(z_1,z_2)\frac{\partial^{n-1}\phi}{\partial z_1^{n-1}} + \cdots + b_n(z_1,z_2)\phi = 0, \tag{D.0.22}$$

in the region $|z_1| > |z_2| > 0$.

Proof. Let a_1, a_2, a_3 be the respective B-grades of w_1, w_2, w_3, then we can assume that the B-grade of w_0' is $a_1 + a_2 + a_3$, because otherwise the matrix element vanishes and the theorem follows trivially. Recall the map $\phi_{y_1,y_2} : T^{(a_1,a_2,a_3)} \to z_1^h \mathbb{C}(\{z_2/z_1\})\big[z_1^{\pm 1}, z_2^{\pm 2}\big]$, defined by the formula (D.0.8). Since $J^{(a_1,a_2,a_3)}$ lies in the kernel of ϕ_{y_1,y_2}, we have an induced map

$$\overline{\phi}_{y_1,y_2} : T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)} \to z_1^h \mathbb{C}(\{z_2/z_1\})\big[z_1^{\pm 1}, z_2^{\pm 2}\big]. \tag{D.0.23}$$

The theorem then follows by applying $\overline{\phi}_{y_1,y_2}$ to the relations (D.0.19) of Corollary D.0.3.2, using the L_{-1} derivative property of intertwining operators and expanding in the region $|z_1| > |z_2| > 0$. ∎

Systems of differential equations of the form (D.0.22) have solutions very close to the expansion required if their singular points are regular, see for example [68, Appendix B]. A sufficient condition, whose validity we shall verify shortly, for regularity at a given singular point is that the coefficients a_i, b_j in the system (D.0.22) have poles of degree at most $m - i$ and $n - j$ respectively. Such singular points are called simple (see [68, Appendix B] for the general definition). The singular points relevant for the convergence and extension property for products are $z_1 = z_2$ and $(z_1 - z_2)/z_2 = 0$.

We need to consider new filtrations in addition to those considered previously. Let $\overline{R} = \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 2}]$, then $R_n = (z_1 - z_2)^{-n}\overline{R}$, $n \in \mathbb{Z}$ equips R with the structure of a filtered ring in the sense that $R_n \subset R_m$, if $n \le m$, $R = \bigcup_{n\in\mathbb{Z}} R_n$ and $R_n \cdot R_m \subset R_{m+n}$. The R-module $T^{(a_1,a_2,a_3)}$ can then also be equipped with a compatible filtration

$$R_r(T^{(a_1,a_2,a_3)}) = \prod_{\substack{n+h_0+h_1+h_2+h_3 \le r\\ h_i \in \mathbb{R}}} R_n \otimes (\mathcal{W}_0')^{(a_1+a_2+a_3)}_{[h_0]} \otimes (\mathcal{W}_1)^{(a_1)}_{[h_1]} \otimes (\mathcal{W}_2)^{(a_2)}_{[h_2]} \otimes (\mathcal{W}_3)^{(a_3)}_{[h_3]}, \quad r \in \mathbb{R}, \tag{D.0.24}$$

in the sense that $R_r(T^{(a_1,a_2,a_3)}) \subset R_s(T^{(a_1,a_2,a_3)})$, if $r \le s$, $T^{(a_1,a_2,a_3)} = \bigcup_{r\in\mathbb{R}} R_r(T^{(a_1,a_2,a_3)})$ and $R_n \cdot R_r(T^{(a_1,a_2,a_3)}) \subset R_{n+r}(T^{(a_1,a_2,a_3)})$. Further, let $R_r(J^{(a_1,a_2,a_3)}) = R_r(T^{(a_1,a_2,a_3)}) \cap J^{(a_1,a_2,a_3)}$.

Proposition D.0.5. *Let the* V*-modules* $\mathcal{W}_i$, $i = 0,1,2,3$ *be discretely strongly* B*-graded and* B*-graded* C_1*-cofinite as* $\overline{\mathsf{V}}$*-modules. Then for any* $a_1, a_2, a_3 \in B$ *there exists* $M \in \mathbb{Z}$ *such that for any* $r \in \mathbb{R}$

$$R_r(T^{(a_1,a_2,a_3)}) \subset R_r(J^{(a_1,a_2,a_3)}) + F_M(T^{(a_1,a_2,a_3)}), \quad T^{(a_1,a_2,a_3)} = J^{(a_1,a_2,a_3)} + F_M(T^{(a_1,a_2,a_3)}). \tag{D.0.25}$$

Further, $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$ *is finitely generated.*

Proof. The proof of this proposition mimics the proof of Proposition D.0.2 once one has verified that the elements $\mathcal{A}(u,w_0',w_1,w_2,w_3)$, $\mathcal{B}(u,w_0',w_1,w_2,w_3)$, $\mathcal{C}(u,w_0',w_1,w_2,w_3)$ and $\mathcal{D}(u,w_0',w_1,w_2,w_3)$ lie in $R_h(J)$, where h is the sum of the conformal weights of u,w_0',w_1,w_2,w_3. ■

We also need to consider the $\overline{R}$-module $U^{(a_1,a_2,a_3)} = \overline{R}\otimes\left(\mathcal{W}_0'\right)^{(a_1+a_2+a_3)}\otimes\mathcal{W}_1^{(a_1)}\otimes\mathcal{W}_2^{(a_2)}\otimes\mathcal{W}_3^{(a_3)}$ and denote by $U_{[r]}^{(a_1,a_2,a_3)}$ the subspace of conformal weight $r\in\mathbb{R}$. Thus $U^{(a_1,a_2,a_3)} = \prod_{r\in\mathbb{R}} U_{[r]}^{(a_1,a_2,a_3)}$.

Lemma D.0.6. *Let the V-modules $\mathcal{W}_i$, $i=0,1,2,3$ be discretely strongly B-graded and B-graded C_1-cofinite as $\overline{\mathsf{V}}$-modules. For any $a_1,a_2,a_3\in B$ and any doubly homogeneous vectors $w_0'\in\left(\mathcal{W}_0'\right)_{[h_0]}^{(a_1+a_2+a_3)}$, $w_i\in\left(\mathcal{W}_i\right)_{[h_i]}^{(a_i)}$, let $h=\sum_i h_i$, let $\overline{h}$ be the smallest non-negative representative of the coset $h+\mathbb{Z}$ and let $m_J\in R_h(J^{(a_1,a_2,a_3)})$, $m_T\in F_M(T^{(a_1,a_2,a_3)})$ be vectors satisfying*

$$w_0'\otimes w_1\otimes w_2\otimes w_3 = m_J + m_T. \tag{D.0.26}$$

Then there exists $S\in\mathbb{R}$ such that $\overline{h}+S\in\mathbb{Z}_{\geq 0}$ and $(z_1-z_2)^{h+S}m_T\in U^{(a_1,a_2,a_3)}$.

Proof. Note that the existence of the vectors m_J, m_T is guaranteed by Proposition D.0.5. Choose $S\in\mathbb{R}$ such that $\overline{h}+S\in\mathbb{Z}_{\geq 0}$ and such that for any $r\leq -S$, $T_{[r]}^{(a_1,a_2,a_3)}=0$. Such an S must exist, since the conformal weights of $T^{(a_1,a_2,a_3)}$ are bounded below by assumption. By definition, $R_r(T^{(a_1,a_2,a_3)})$ is spanned by elements of the form $(z_1-z_2)^{-n}f(z_1,z_2)\widetilde{w}_0\otimes\widetilde{w}_1\otimes\widetilde{w}_2\otimes\widetilde{w}_3$, where $f\in\overline{R}$ and $n+\sum_i \mathrm{wt}\,\widetilde{w}_i\leq r$. The number S was therefore chosen such that $(z_1-z_2)^{r+S}R_r(T^{(a_1,a_2,a_3)})\subset U^{(a_1,a_2,a_3)}$ whenever $r+S\in\mathbb{Z}$. Now, by assumption,

$$m_T = w_0'\otimes w_1\otimes w_2\otimes w_3 - m_J. \tag{D.0.27}$$

The right-hand side of this equality lies in $R_h(T^{(a_1,a_2,a_3)})$ by construction and therefore so does the left-hand side. Hence $(z_1-z_2)^{h+S}m_T\in U^{(a_1,a_2,a_3)}$. ■

Theorem D.0.7. *Let the V-modules $\mathcal{W}_i$, $i=0,1,2,3$ be discretely strongly B-graded and B-graded C_1-cofinite as $\overline{\mathsf{V}}$-modules, let $\mathcal{W}_4$ be a B-graded V-module and let $\mathcal{Y}_1$, $\mathcal{Y}_2$ be logarithmic grading compatible intertwining operators of types $\binom{\mathcal{W}_0}{\mathcal{W}_1\ \mathcal{W}_4}$, $\binom{\mathcal{W}_4}{\mathcal{W}_2\ \mathcal{W}_3}$, respectively and consider the system of differential equations of Theorem D.0.4. For the singular points $z_1=z_2$ and $(z_1-z_2)/z_2=0$ there exist coefficients $a_k(z_1,z_2)$, $b_l(z_1,z_2)\in R$ such that these singular points of the system of differential equations (D.0.22) satisfied by the matrix elements (D.0.21) are regular.*

Proof. We consider first the singular point $z_1=z_2$. By Proposition D.0.5 and Lemma D.0.6, for any $k\in\mathbb{Z}_{\geq 0}$ together with a vector $w_0'\otimes L_{-1}^k w_1\otimes w_2\otimes w_3\in T^{(a_1,a_2,a_3)}$, where the w_i are doubly homogeneous vectors of total conformal weight $h \in \mathbb{R}$, there exist $m_J^{(k)} \in R_{h+k}(J^{(a_1,a_2,a_3)})$ and $m_T^{(k)} \in F_M(T^{(a_1,a_2,a_3)})$ such that

$$w_0' \otimes L_{-1}^k w_1 \otimes w_2 \otimes w_3 = m_J^{(k)} + m_T^{(k)}. \tag{D.0.28}$$

Let $\overline{h}$ be the smallest non-negative representative of the coset $h + \mathbb{Z}$. Then, by Lemma D.0.6, there exists $S \in \mathbb{R}$ such that $\overline{h} + S \in \mathbb{Z}_{\geq 0}$ and $(z_1 - z_2)^{h+k+S} m_T^{(k)} \in U^{(a_1,a_2,a_3)}$ and thus $(z_1 - z_2)^{h+k+S} m_T^{(k)} \in \bigcup_{r \leq M} U_{[r]}^{(a_1,a_2,a_3)}$. Since the V-modules $\mathcal{W}_i$ are discretely strongly B-graded and B-graded C_1-cofinite, $\prod_{r \leq M} U_{[r]}^{(a_1,a_2,a_3)}$ is a finite sum of finitely generated $\overline{R}$-modules and hence also finitely generated. Thus, since $\overline{R}$ is Noetherian, the submodule generated by the $(z_1 - z_2)^{h+k+S} m_T^{(k)}$, $k \in \mathbb{Z}_{\geq 0}$ is also finitely generated. Hence there exists an $m \in \mathbb{Z}_{\geq 0}$ such that $\{(z_1 - z_2)^{h+k+S} m_T^{(k)} \ : \ 0 \leq k \leq m-1\}$ is a finite generating set for this submodule and subsequently there exist $c_k(z_1, z_2) \in \overline{R}$ such that

$$(z_1 - z_2)^{h+m+S} m_T^{(m)} + \sum_{k=0}^{m-1} c_k(z_1,z_2)(z_1 - z_2)^{h+k+S} m_T^{(k)} = 0. \tag{D.0.29}$$

Therefore,

$$w_0' \otimes L_{-1}^m w_1 \otimes w_2 \otimes w_3 + \sum_{k=0}^{m-1} c_k(z_1,z_2)(z_1-z_2)^{k-m} w_0' \otimes L_{-1}^k w_1 \otimes w_2 \otimes w_3 = m_J^{(m)} + \sum_{k=0}^{m-1} c_k(z_1,z_2) m_J^{(k)}. \tag{D.0.30}$$

Thus in the quotient module $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$, we obtain (where we again use square brackets to denote cosets)

$$\left[w_0' \otimes L_{-1}^m w_1 \otimes w_2 \otimes w_3\right] + \sum_{k=0}^{m-1} c_k(z_1,z_2)(z_1 - z_2)^{k-m} \left[w_0' \otimes L_{-1}^k w_1 \otimes w_2 \otimes w_3\right] = 0, \tag{D.0.31}$$

since $m_J^{(k)} \in J^{(a_1,a_2,a_3)}$. By a similar line of reasoning there exists an $n \in \mathbb{Z}_{\geq 0}$ and $d_\ell(z_1,z_2) \in \overline{R}$ such that

$$\left[w_0' \otimes w_1 \otimes L_{-1}^n w_2 \otimes w_3\right] + \sum_{\ell=0}^{m-1} d_\ell(z_1,z_2)(z_1 - z_2)^{\ell-n} \left[w_0' \otimes w_1 \otimes L_{-1}^k w_2 \otimes w_3\right] = 0. \tag{D.0.32}$$

Applying the map ϕ_{y_1,y_2} defined by (D.0.8) and using the L_{-1} property for intertwining operators will then result in a system of differential equations for which $z_1 = z_2$ is a simple, and hence regular, singular point.

To show the regularity of the singular point $(z_1 - z_2)/z_2 = 0$, we introduce new gradings on R and $T^{(a_1,a_2,a_3)}$. We assign degree -1 to the variables z_1, z_2, thus giving R a $\mathbb{Z}$ grading and then grade $T^{(a_1,a_2,a_3)}$ by adding R-degrees and conformal weights. This implies that the elements

$\mathcal{A}(v,w_0',w_1,w_2,w_3)$, $\mathcal{B}(v,w_0',w_1,w_2,w_3)$, $\mathcal{C}(v,w_0',w_1,w_2,w_3)$ and $\mathcal{D}(v,w_0',w_1,w_2,w_3)$ are homogeneous with respect to this new grading if their arguments are doubly homogeneous. The new grading therefore descends to $T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}$. Further, for doubly homogeneous elements w_0',w_1,w_2,w_3, the elements

$$\left[w_0'\otimes L_{-1}^k w_1\otimes w_2\otimes w_3\right],\quad \left[w_0'\otimes w_1\otimes L_{-1}^\ell w_2\otimes w_3\right]\in T^{(a_1,a_2,a_3)}/J^{(a_1,a_2,a_3)}, \tag{D.0.33}$$

are also homogeneous. Thus the coefficients $c_k(z_1,z_2), d_\ell(z_1,z_2)$ of equations (D.0.31) and (D.0.32) are elements of degree 0 in R and can therefore be written as Laurent polynomials in $(z_1-z_2)/z_2$. It then follows that the singular point $(z_1-z_2)/z_2=0$ is regular. ■

The fact that the matrix element (D.0.2) satisfies an expansion of the form (D.0.3) now follows by the reasoning of [67, Theorem 3.5]. A little care is needed when following the reasoning of [67], since there only modules with a diagonalisable action of L_0 are considered. However, as noted in [20, Part VII, Proof of Theorem 11.8 and Remark 11.9] the argument extends easily to modules where L_0 has Jordan blocks. The basic idea is that one can use the L_0 conjugation property of intertwining operators (recall that L_0 is the generator of dilations) to rescale the variables in the matrix element (D.0.2) by z_2 so that it becomes a function in $z_3=(z_1-z_2)/z_2$ only and the system of differential equations (D.0.31) and (D.0.32) then becomes an ordinary differential equation for z_3 with a regular singularity at $z_3=0$. Similar reasoning for the matrix element (D.0.5) leads one to conclude that it satisfies the expansion (D.0.6). Hence Theorem 6.3.2 follows.

— Appendix E —

Construction of the quantum group

E.1 Construction Method

Let Ψ be a set of bosonic lattice data and let H_Λ be the associated lattice Hopf algebra. Further, let H be the result of including all grouplike elements K_μ corresponding to exponentials of the primitive elements X_μ, and enforcing that $K_\Lambda = 1$. In the language of Section 8.4, we have

$$H_\Lambda = \mathbb{C}[X_{\Lambda^\perp}, K_{\Lambda^*/\Lambda^\perp}], \qquad H = \mathbb{C}[X_{\Lambda_\perp}, K_{\Lambda^*}]/(K_\Lambda = 1). \tag{E.1.1}$$

Note that the category of weight modules is the same for both Hopf algebras, as the condition $K_\Lambda = 1$ implies that $e^{2\pi i X_\Lambda} = 1$ which is true because $X_{\Lambda^\perp \cap \Lambda}$ acts trivially on modules in VectΨ.

Let H modules be equipped with the self braiding $Q(x) = e^{i\pi\langle x,x\rangle}$ and double braiding $B(x,y) = e^{2\pi i\langle x,y\rangle}$ (specialised to $\langle (x,y),(x',y')\rangle = xy' + yx'$ in Definition 10.2.1). Then the braiding is given by $R(\alpha_i,\alpha_j) = \epsilon(\alpha_i,\alpha_j)e^{i\pi\langle\alpha_i,\alpha_j\rangle}$ for a choice of cocycle ϵ, satisfying $\epsilon\langle x,y\rangle\epsilon\langle y,x\rangle = 1$.

We can equip the modules containing the screening operators, $\mathbb{C}_{\alpha_i^\ominus+\Lambda} \in \mathrm{Vect}_{\Lambda^*/\Lambda}$, for distinguished $\alpha_i^\ominus$, with an action from the H-module structure, and a co-action constructed from the action and the R-matrix, such that they are compatible with the Yetter-Drinfeld condition. That is, $\mathbb{C}_{\alpha_i^\ominus+\Lambda} \in {}^H_H\mathcal{YD}$. The action and co-action respectively are given by

$$H_{\alpha_j} \cdot \mathbf{1}_{\alpha_i^\ominus} = \langle \alpha_j, \alpha_i^\ominus\rangle \mathbf{1}_{\alpha_i^\ominus}, \qquad \mathbf{1}_{\alpha_i^\ominus} \mapsto K_{\lambda_i} \otimes \mathbf{1}_{\alpha_i^\ominus}, \tag{E.1.2}$$

where we find a condition on λ_i from enforcing that the Yetter-Drinfeld braiding coincides with the expected braiding from the R-matrix, when applied to $x = \mathbf{1}_{\alpha_i^\ominus} \otimes \mathbf{1}_\mu$, where x^τ denotes the tensor flip of x.

$$c_{\mathcal{YD}}(x) = (K_{\lambda_i} \cdot \mathbf{1}_\mu) \otimes \mathbf{1}_{\alpha_i^\ominus} = e^{i\pi\langle\lambda_i,\mu\rangle} x^\tau, \qquad c_R(x) = \epsilon\langle\alpha_i^\ominus,\mu\rangle e^{i\pi\langle\alpha_i^\ominus,\mu\rangle} x^\tau. \tag{E.1.3}$$

Therefore we find that

$$\langle\lambda_i,\mu\rangle = \langle\alpha_i^\ominus,\mu\rangle + \langle\alpha_i^\ominus,\mu\rangle_\varepsilon \mod 2\mathbb{Z}, \quad \text{where } \varepsilon\langle\alpha_i^\ominus,\mu\rangle = e^{i\pi\langle\alpha_i^\ominus,\mu\rangle_\varepsilon}. \tag{E.1.4}$$

Now for the Nichols algebra B generated by the screening operators $E_{\alpha_i^\ominus}$ with the braiding above, we can construct the quantum group $B \rtimes H \ltimes B^*$ with module category ${}^B_B\mathcal{YD}(H-\mathrm{Mod}) = {}^B_B\mathcal{YD}({}^H_H\mathcal{YD}) = {}^{B\rtimes H}_{B\rtimes H}\mathcal{YD}$.

The Radford biproduct relations for $B \rtimes H$ and $H \ltimes B^*$ imply

$$[H_{\alpha_j}, E_{\alpha_i^\ominus}] = H_{\alpha_j} \cdot E_{\alpha_i^\ominus} = \langle \alpha_j, \alpha_i^\ominus \rangle E_{\alpha_i^\ominus}, \qquad K_{\lambda_i} E_{\alpha_i^\ominus} K_{\lambda_i}^{-1} = K_{\lambda_i} \cdot E_{\alpha_i^\ominus} = e^{i\pi\langle \lambda_i, \alpha_i^\ominus \rangle} E_{\alpha_i^\ominus}, \tag{E.1.5}$$

$$[H_{\alpha_j}, E^*_{\alpha_i^\ominus}] = H_{\alpha_j} \cdot E^*_{\alpha_i^\ominus} = -\langle \alpha_j, \alpha_i^\ominus \rangle E^*_{\alpha_i^\ominus}, \qquad K_{\lambda_i} E^*_{\alpha_i^\ominus} K_{\lambda_i}^{-1} = K_{\lambda_i} \cdot E^*_{\alpha_i^\ominus} = e^{-i\pi\langle \lambda_i, \alpha_i^\ominus \rangle} E^*_{\alpha_i^\ominus}. \tag{E.1.6}$$

We use the Yetter-Drinfeld condition (see Figure E.1) to find the commutator

$$E^*_{\alpha_i^\ominus} E_{\alpha_j^\ominus} - \epsilon\left\langle \alpha_i^\ominus, \alpha_j^\ominus \right\rangle e^{i\pi\left\langle \alpha_i^\ominus, \alpha_j^\ominus \right\rangle} E_{\alpha_j^\ominus} E^*_{\alpha_i^\ominus} = \delta_{ij}\left(1 - K^2_{\alpha_i^\ominus}\right). \tag{E.1.7}$$

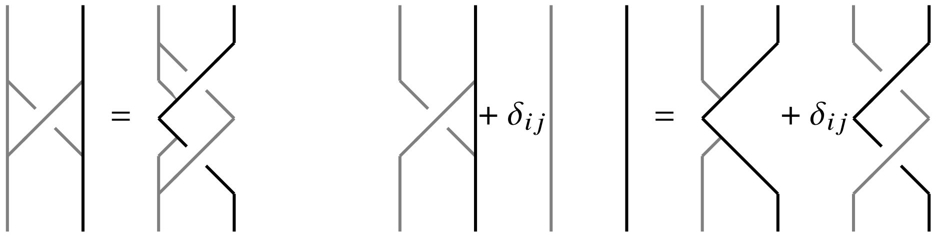


Figure E.1: A diagrammatic interpretation of the Yetter-Drinfeld condition. The grey lines represent the generators $E_{\alpha_j^\ominus}$ inserted at the bottom and $E^*_{\alpha_i^\ominus}$ at the top. The black lines represent the vector space V as a Yetter-Drinfeld module. The diagram should be read vertically upwards and encodes the product, coproduct, action and coaction. The equation on the right is expressed in terms of individual channels, where the δ_{ij} indicates that the diagram is valid as long as $i = j$. For more explanation see [13].

We also have the coproducts

$$\Delta E_{\alpha_i^\ominus} = K_{\mu_i} \otimes E_{\alpha_i^\ominus} + E_{\alpha_i^\ominus} \otimes \mathbf{1}, \qquad \Delta E^*_{\alpha_i^\ominus} = K_{\nu_i} \otimes E^*_{\alpha_i^\ominus} + E^*_{\alpha_i^\ominus} \otimes \mathbf{1}, \tag{E.1.8}$$

where $E^*_{\alpha_i^\ominus}$ is the dual of $E_{\alpha_i^\ominus}$ and both μ_i and ν_i are determined by the choice of cocycle ϵ in the following way. We fix a choice of μ_i and take the coproduct of the Yetter-Drinfeld condition, which implies that the only compatible choice is $\nu_i = 2\alpha_i^\ominus - \mu_i$. We also find that both $\lambda_i = \mu_i$ and $\lambda_i = \nu_i$ must satisfy E.1.4. Now, let $F_{\alpha_i^\ominus} = K_{\nu_i}^{-1} E^*_{\alpha_i^\ominus}$. Then $F_{\alpha_i^\ominus}$ satisfies the product and coproduct relations below.

$$[H_{\alpha_j}, F_{\alpha_i^\ominus}] = H_{\alpha_j} \cdot F_{\alpha_i^\ominus} = -\langle \alpha_j, \alpha_i^\ominus \rangle F_{\alpha_i^\ominus}, \qquad K_\lambda F_{\alpha_i^\ominus} K_\lambda^{-1} = K_\lambda \cdot F_{\alpha_i^\ominus} = e^{-i\pi\langle \lambda, \alpha_i^\ominus \rangle} F_{\alpha_i^\ominus}, \tag{E.1.9}$$

$$\Delta F_{\alpha_i^\ominus} = \Delta\left(K_{\nu_i}^{-1} E^*_{\alpha_i^\ominus}\right) = \Delta K_{\nu_i}^{-1} \Delta E^*_{\alpha_i^\ominus} = \mathbf{1} \otimes F_{\alpha_i^\ominus} + F_{\alpha_i^\ominus} \otimes K_{\nu_i}^{-1}. \tag{E.1.10}$$

Substituting $F_{\alpha_i^\ominus}$ into E.1.7 yields the last of the defining relations of our quantum group.

$$[E_{\alpha_j^\ominus}, F_{\alpha_i^\ominus}] = \delta_{ij}\left(K_{\mu_i} - K_{\nu_i}^{-1}\right). \tag{E.1.11}$$

E.2 Application to specific case

In our case, we have the following data.

$$\mathfrak{h} = \mathbb{R}^2, \quad \langle x, y\rangle = x^T \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) y, \quad \Lambda = (1,1)\mathbb{Z}, \tag{E.2.1}$$

$$\alpha_i^\ominus = (1,0), \quad \epsilon(x,y) = e^{i\pi x^T \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) y}. \tag{E.2.2}$$

From these, we can deduce

$$\Lambda^* = \{(a,b) : a - b \in \mathbb{Z}\} = (1,0)\mathbb{Z} + (1,1)\mathbb{R}, \quad \Lambda^*/\Lambda = (\mathbb{R}/\mathbb{Z}) \boxtimes \mathbb{Z}. \tag{E.2.3}$$

Therefore, using the construction method above, we find the following Hopf algebra.

$$H = \left\langle X_{(1,1)},\ K_{(1,1)},\ K_{(1,-1)},\ E_{(1,0)},\ F_{(1,0)} \right\rangle / (\star), \tag{E.2.4}$$

where $\star$ are the relations below.

$$E_{(1,0)}^2 = F_{(1,0)}^2 = K_{(1,1)}^2 - \mathbf{1} = 0, \tag{E.2.5}$$

$$[X_{(1,1)}, E_{(1,0)}] = E_{(1,0)}, \quad [X_{(1,1)}, F_{(1,0)}] = -F_{(1,0)}, \tag{E.2.6}$$

$$K_{(1,-1)} E_{(1,0)} K_{(1,-1)}^{-1} = -E_{(1,0)}, \quad K_{(1,-1)} F_{(1,0)} K_{(1,-1)}^{-1} = -F_{(1,0)}, \tag{E.2.7}$$

$$K_{(1,1)} E_{(1,0)} K_{(1,1)}^{-1} = -E_{(1,0)}, \quad K_{(1,1)} F_{(1,0)} K_{(1,1)}^{-1} = -F_{(1,0)}, \tag{E.2.8}$$

$$[E_{(1,0)}, F_{(1,0)}] = K_{(1,-1)} - K_{(1,1)}^{-1}, \tag{E.2.9}$$

$$\Delta\left(X_{(1,1)}\right) = X_{(1,1)} \otimes \mathbf{1} + \mathbf{1} \otimes X_{(1,1)}, \tag{E.2.10}$$

$$\Delta\left(K_{(1,1)}\right) = K_{(1,1)} \otimes K_{(1,1)}, \quad \Delta\left(K_{(1,-1)}\right) = K_{(1,-1)} \otimes K_{(1,-1)}, \tag{E.2.11}$$

$$\Delta\left(E_{(1,0)}\right) = K_{(1,-1)} \otimes E_{(1,0)} + E_{(1,0)} \otimes \mathbf{1}, \quad \Delta\left(F_{(1,0)}\right) = F_{(1,0)} \otimes K_{(1,1)}^{-1} + \mathbf{1} \otimes F_{(1,0)}, \tag{E.2.12}$$

$$S(X_{(1,1)}) = -X_{(1,1)}, \quad S(K_{(1,1)}) = K_{(1,1)}^{-1}, \quad S(K_{(1,-1)}) = K_{(1,-1)}^{-1}, \tag{E.2.13}$$

$$S(E_{(1,0)}) = -K_{(1,-1)}^{-1} E_{(1,0)}, \quad S(F_{(1,0)}) = -F_{(1,0)} K_{(1,1)}, \tag{E.2.14}$$

$$\varepsilon(K_{(1,1)}) = \varepsilon(K_{(1,-1)}) = 0, \quad \varepsilon(X_{(1,1)}) = \varepsilon(E_{(1,0)}) = \varepsilon(F_{(1,0)}) = 0. \tag{E.2.15}$$

This can be identified with $\overline{U}_q^{X_1}(\mathfrak{gl}_2)$ with $q = e^{i\pi/2}$ and appropriate rescaling of the generators.

— Appendix F —

Ideas for proving Conjecture 11.4.2

F.1 Logarithmic intertwining operator construction

Proposition F.1.1. *Let $\mathcal{Y}$ be a logarithmic intertwining operator of type $\binom{\sigma^{n+m-1}\mathcal{P}}{\sigma^n\mathcal{W}_\lambda,\,\sigma^m\mathcal{W}_{-\lambda}}$. Let $j = 1-n-m$, $h = -\frac{1}{2}(n+m-1)(n+m)$. Then*

$$\mathcal{Y}(m_1,z)m_2 = (u+d\log(z))z^k + R(z), \qquad k = h-h_1-h_2, \tag{F.1.1}$$

where $m_1 \in \mathcal{W}_\lambda^n$ and $m_2 \in \mathcal{W}_{-\lambda}^m$ are homogeneous vectors with conformal weights h_1 and h_2 respectively, and ghost weights j_1 and j_2 such that $j_1+j_2 = j$. $u,d \in \sigma^{n+m-1}\mathcal{P}$ are a choice of generating vector u and its Jordan partner d satisfying $L_0u = hu+d$. $R(z)$ denotes the terms which are proportional to descendent vectors of u and d.

Proof. The respective ghost and conformal weights of $u,d \in \mathcal{P}^{n+m-1}$ are given by $[j,h] = \left[1-n-m,\ -\frac{1}{2}(n+m-1)(n+m)\right]$. Therefore, by grading compatibility,

$$\mathcal{Y}(m_1,z)m_2 \subset \left(\sigma^{n+m-1}\mathcal{P}\right)^{(j)}\{z\}[\log z], \tag{F.1.2}$$

which is the space containing the vectors v, w and their descendants. In order to determine the coefficients of the vectors v and w in the intertwining operator expansion, we start with the Jacobi identity for $\omega \in V$ the conformal vector, and take residues with respect to z_0 and z_1. This yields

$$\begin{aligned} L_0\mathcal{Y}(m_1,z)m_2 &= \mathcal{Y}(m_1,z)L_0m_2 + \sum_{s=0}^{1}\binom{s-2}{s}(-1)^s z^{1-s}\mathcal{Y}(L_{s-1}m_1,z)m_2 \\ &= \mathcal{Y}(m_1,z)L_0m_2 + \mathcal{Y}(L_0m_1,z)m_2 + z\mathcal{Y}(L_{-1}m_1,z)m_2 \\ &= \left(h_1+h_2+z\frac{\mathrm{d}}{\mathrm{d}z}\right)\mathcal{Y}(m_1,z)m_2. \end{aligned} \tag{F.1.3}$$

We let

$$\mathcal{Y}(m_1,z)m_2 = a(z)u + b(z)d + R(z), \tag{F.1.4}$$

where $R(z)$ contains the terms proportional to descendent vectors.

$$L_0\mathcal{Y}(m_1,z)m_2 = ha(z)u + (a(z)+hb(z))d + L_0R(z). \tag{F.1.5}$$

Therefore, we have the following differential equations

$$z\frac{\mathrm{d}}{\mathrm{d}z}a(z) = ka(z), \qquad z\frac{\mathrm{d}}{\mathrm{d}z}b(z) = a(z)+kb(z), \qquad k = h-h_1-h_2. \tag{F.1.6}$$

Which admit the following unique solution

$$a(z) = az^k, \qquad b(z) = (b+a\log z)z^k. \tag{F.1.7}$$

Therefore we have the following solution for the logarithmic intertwining operator.

$$\mathcal{Y}(m_1,z)m_2 = (au+bd+ad\log z)z^k + R(z). \tag{F.1.8}$$

We obtain the desired expression by renormalising our intertwining operator so that $a = 1$, and by redefining our generating vector in order to set $b = 0$. This is possible because any vector of the form $v+cw$ is a generating vector for $\sigma^{n+m-1}\mathcal{P}$, and our choice corresponds to $c = -b/a$. ■

Remark. Note that one can only fix the normalisation above for one pair m_1, m_2 of vectors, and then the logarithmic intertwining operator on the other vectors are all determined by this.

F.2 Comparison with free field intertwining operators

Free field intertwining operators give the following coefficients

$$\begin{aligned} &\mathbb{F}_{\lambda(\psi+\theta)+n\psi} \otimes \mathbb{F}_{\mu(\psi+\theta)+m\psi} \to \mathbb{F}_{(\lambda+\mu)(\psi+\theta)+(n+m)\psi}\{z\} \\ |i(\psi+\theta)+n\psi\rangle \otimes |j(\psi+\theta)+m\psi\rangle &\mapsto z^{nm+nj+mi}|(i+j)(\psi+\theta)+(n+m)\psi\rangle + \dots \end{aligned} \tag{F.2.1}$$

Let $|j\rangle \in \mathcal{W}^n_\lambda$ be shorthand for $|j(\theta+\psi)+n\psi\rangle$ or $|j(\theta+\psi)+(n-1)\psi\rangle$, in the first and second free field realisations, respectively. Then the two free field intertwining operators give the following coefficients in front of u.

$$z^{nj_2+mj_1+nm}, \quad z^{(n-1)j_2+(m-1)j_1+(n-1)(m-1)} = z^{nj_2+mj_1+nm}z^{1-n-m-j_1-j_2} = z^{nj_2+mj_1+nm}. \tag{F.2.2}$$

This differs from our logarithmic intertwining operator construction coefficient by a factor of

$$\begin{aligned} z^{k-nj_2-mj_1-nm} &= z^{-\frac{1}{2}(n+m-1)(n+m)-nj_1-\frac{1}{2}n(n-1)-mj_2-\frac{1}{2}m(m-1)} = z^{-n^2-m^2+n+m-2nm-(n+m)(j_1+j_2)} \\ &= z^{-n^2-m^2+n+m-2nm-(n+m)(1-n-m)} = 1. \end{aligned} \tag{F.2.3}$$

Therefore the free field intertwining operators are already appropriately normalised.

F.3 Twisted action and the Jacobi identity

Let $\mathcal{Y}(m_1,z)m_2 = \mathcal{Y}^{(0)}(m_1,z)m_2 + \mathcal{Y}^{(1)}(m_1,z)m_2$ where $\mathcal{Y}^{(1)} : \mathcal{W}_\lambda \otimes \mathcal{W}_{-\lambda} \to \mathcal{W}_0^-\{z\}[\log z]$ is a linear map and $\mathcal{Y}^{(0)} \in \binom{\sigma^{-1}\mathcal{W}_0^-}{\mathcal{W}_\lambda,\mathcal{W}_{-\lambda}}$ is an intertwining operator determined by the free field intertwining operator of type $\binom{\mathbb{F}_{-2\psi}}{\mathbb{F}_{\lambda(\psi+\theta)-\psi},\mathbb{F}_{-\lambda(\psi+\theta)-\psi}}$. Further, let $\widetilde{v}(z) = v(z) + f(z)$, where $f(z_1)\mathcal{Y}^{(1)}(m_1,z_2)m_2 = 0$. Then the Jacobi identity for $\mathcal{Y}$ yields

$$\begin{aligned}
0 &= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\widetilde{v}(z_1)\mathcal{Y}(m_1,z_2)m_2 - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{Y}(m_1,z_2)v(z_1)m_2\\
&- z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}(v(z_0)m_1,z_2)m_2\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)(v(z_1)+f(z_1))\Big(\mathcal{Y}^{(0)}(m_1,z_2)+\mathcal{Y}^{(1)}(m_1,z_2)\Big)m_2\\
&- z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\Big(\mathcal{Y}^{(0)}(m_1,z_2)+\mathcal{Y}^{(1)}(m_1,z_2)\Big)v(z_1)m_2\\
&- z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\Big(\mathcal{Y}^{(0)}(v(z_0)m_1,z_2)+\mathcal{Y}^{(1)}(v(z_0)m_1,z_2)\Big)m_2\\
&= z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)v(z_1)\mathcal{Y}^{(0)}(m_1,z_2)m_2 - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{Y}^{(0)}(m_1,z_2)v(z_1)m_2\\
&- z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}^{(0)}(v(z_0)m_1,z_2)m_2\\
&\quad + z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)v(z_1)\mathcal{Y}^{(1)}(m_1,z_2)m_2 - z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{Y}^{(1)}(m_1,z_2)v(z_1)m_2\\
&- z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}^{(1)}(v(z_0)m_1,z_2)m_2\\
&\quad + z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)f(z_1)\mathcal{Y}^{(0)}(m_1,z_2)m_2. \qquad \text{(F.3.1)}
\end{aligned}$$

The first three terms of the last equation is the Jacobi identity for $\mathcal{Y}^{(0)}$ so it vanishes, leaving us with the following equation.

$$\begin{aligned}
&z_0^{-1}\delta\left(\frac{z_1-z_2}{z_0}\right)\Big(v(z_1)\mathcal{Y}^{(1)}(m_1,z_2)+f(z_1)\mathcal{Y}^{(0)}(m_1,z_2)\Big)m_2\\
&= z_0^{-1}\delta\left(\frac{-z_2+z_1}{z_0}\right)\mathcal{Y}^{(1)}(m_1,z_2)v(z_1)m_2 + z_2^{-1}\delta\left(\frac{z_1-z_0}{z_2}\right)\mathcal{Y}^{(1)}(v(z_0)m_1,z_2)m_2. \qquad \text{(F.3.2)}
\end{aligned}$$

By expanding the delta functions as formal power series, we can write this equation as

$$\sum_{\substack{r,t\in\mathbb{Z}\\ s\geq 0}} \binom{r}{s}(-1)^s z_2^s z_1^{r-s-t-h} z_0^{-r-1}\Big(v_t\mathcal{Y}^{(1)}(m_1,z_2)+f_{t+h-h'}\mathcal{Y}^{(0)}(m_1,z_2)\Big)m_2$$

$$
\begin{aligned}
&= \sum_{\substack{r,t\in\mathbb{Z}\\ s\geq 0}} \binom{r}{s}(-1)^{r-s} z_1^{s-t-h} z_2^{r-s} z_0^{-r-1} \mathcal{Y}^{(1)}(m_1,z_2)v_t m_2 \\
&+ \sum_{\substack{r,t\in\mathbb{Z}\\ s\geq 0}} \binom{r}{s}(-1)^{s} z_0^{s-t-h} z_1^{r-s} z_2^{-r-1} \mathcal{Y}^{(1)}(v_t m_1,z_2)m_2,
\end{aligned} \tag{F.3.3}
$$

where h and h' denote the conformal weights of v and f, respectively. Then multiplying both sides by $z_0^k z_1^{n+h-1}$, $n,k\in\mathbb{Z}$, taking residues with respect to z_0 and z_1, and relabelling z_2 as z yields

$$
\begin{aligned}
&\sum_{s\geq 0}\binom{k}{s}(-1)^s z^s\Big(v_{n-s}\mathcal{Y}^{(1)}(m_1,z)+f_{n-s+h-h'}\mathcal{Y}^{(0)}(m_1,z)\Big)m_2 \\
&=\sum_{s\geq 0}\binom{k}{s}(-1)^{k-s}z^{k-s}\mathcal{Y}^{(1)}(m_1,z)v_{n+s-k}m_2+\sum_{s\geq 0}\binom{s-n+k-h}{s}(-1)^s z^{n-k+h-s-1}\mathcal{Y}^{(1)}(v_{s-h+k+1}m_1,z)m_2.
\end{aligned} \tag{F.3.4}
$$

We also use of the free field realisation for the vectors $\{u=|-\psi+\theta\rangle,\ l=|2\theta\rangle\}\in\mathbb{F}_{-2\psi}\cong\operatorname{Im}\mathcal{Y}^{(0)}$, $\{r=|-\psi\rangle, d=|\theta\rangle\}\in\mathbb{F}_{-\psi}\cong\operatorname{Im}\mathcal{Y}^{(1)}$. Their ghost and conformal weights respectively are $[u]=[d]=[1,0]$, $[l]=[2,-1]$, $[r]=[0,0]$. If we only keep track of the terms involving d, we can let $\mathcal{Y}^{(1)}(u_i,z)u_j=a(z)d+\ldots$ for $i+j=1$, and take $v=T$, $k=n=0$ to find

$$
\begin{aligned}
z\frac{\mathrm{d}}{\mathrm{d}z}a(z)d &= \tfrac{1}{w}Y(\psi,w)\mathcal{Y}^{(0)}(u_i,z)u_j|_{w^{-2}\text{coeff}} = \tfrac{1}{w}Y(\psi,w)Y(i(\psi+\theta)-\psi,z)|j(\psi+\theta)-\psi\rangle|_{w^{-2}\text{coeff}} \\
&= w^{j-2}z^{1-i-j}(w-z)^{i-1}|(i+j)(\psi+\theta)-\psi\rangle|_{w^{-2}\text{coeff}} = w^{-2}(1-\tfrac{z}{w})^{i-1}|\theta\rangle|_{w^{-2}\text{coeff}} \\
&= \sum_{n\in\mathbb{Z}}\binom{i-1}{n}z^n w^{-2-n}d|_{w^{-2}\text{coeff}} = d
\end{aligned} \tag{F.3.5}
$$

So $a'(z)=1/z$ and hence $a(z)=a+\log(z)$ for some a. We know that $\mathcal{Y}^{(0)}(u_i,z)u_j=u+\ldots$ so we reproduce that $\mathcal{Y}(u_i,z)u_j=u+ad+d\log z+\ldots$, which we can choose so that $\mathcal{Y}(u_i,z)u_j=u+d\log z+\ldots$, as before.

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