FURTHER INSIGHTS INTO THE TIMOSHENKO-EHRENFEST BEAM THEORY

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In this paper, the theory of a Timoshenko-Ehrenfest beam is revisited and given a new perspective with particular emphasis on the relative significances of the parameters underlying the theory. The investigation is intended to broaden the scope and applicability of the theory. It has been shown that the two parameters that characterise the Timoshenko-Ehrenfest beam theory, namely the rotary inertia and the shear deformation, can be related and hence they can be combined into one parameter when predicting the beam's free vibration behaviour. It is explained why the effect of the shear deformation on the free vibration behaviour of a Timoshenko-Ehrenfest beam for any boundary condition will be always more pronounced than that of the rotary inertia. The range of applicability of the Timoshenko-Ehrenfest beam theory for realistic problems is demonstrated by a set of new curves, which provide considerable insights into the theory.

Key words: Timoshenko-Ehrenfest beam, vibration, shear deformation, rotary inertia

1. INTRODUCTION

The beam theory given by Euler and Bernoulli in the eighteenth century [1, 2] has magnificently survived and endured the passage of time and has continued to be successfully used by scientists and engineers even to this day. The theory gives sufficiently accurate results, particularly for slender beams with cross-sectional dimensions much smaller than the length of the beam. From a historical perspective, attempts to improve the Bernoulli-Euler beam theory for free vibration analysis was made around a century later since its inception due to the ingenuity of Bresse [3] and Lord Rayleigh [4] and who included the effect of rotary inertia arising from the rotation of the beam cross-section. However, it is often overlooked that following the work of Bresse [3] and Rayleigh [4], Searle [5] investigated the effect of rotary inertia on the free vibration behaviour of beams for various boundary conditions which was without doubt a note-worthy contribution at the time. The beam theory developed by Bresse [3] and Rayleigh [4] which included the effect of rotary inertia was next significantly advanced by Timoshenko [6, 7] in the earlier part of the twentieth century when the additional effect of shear deformation was incorporated. This established a landmark in the development of beam theories and in many ways, it overshadowed the earlier beam theories. The expression "Timoshenko beam" has been used in literally thousands of papers in scientific and engineering journals. Interested readers are referred to a selected sample of the literature [8-23], which covers the main aspects of a Timoshenko-Ehrenfest beam.

At this juncture, it should be acknowledged that the so-called Timoshenko beam theory was in fact jointly developed by Timoshenko and Ehrenfest, but because of the untimely and tragic death of Ehrenfest, the story was somehow lost and unfortunately Ehrenfest was not duly accredited. Recently Elishakoff [24, 25] carried out a detailed and open-minded research to trace back the history of the theory, demonstrating with irrefutable evidence that the incorporation of the effects of shear deformation and rotary inertia into the Bernoulli-Euler beam theory was indeed a joint initiative by both Timoshenko and Ehrenfest. In this context, it is worth noting that Timoshenko in his classic paper (see his footnote in [7]) mentioned that the frequency equation of a simply supported beam with the inclusion of the effects of shear deformation and rotary inertia was derived jointly by himself and Ehrenfest. He had made the same acknowledgement much earlier in his book on the theory of elasticity [26]. It should be noted that Timoshenko's acknowledgement of Ehrenfest's contribution is clearly evident in the footnote in the original version of his paper published in the Philosophical Magazine [7], but later versions of the paper appearing in books and periodicals have omitted the footnote. This omission has been discussed by Elishakoff, see pages 43-46 of [25]. Following Elishakoff's research [24, 25], the misunderstanding about the origin of the theory has now been cleared up and the so-called Timoshenko beam is called a Timoshenko-Ehrenfest beam by recent researchers [27-30]. The Timoshenko-Ehrenfest beam theory gives remarkably accurate results, particularly for beams with solid cross-section such as circles and rectangles. This has been confirmed in the literature through the application of higher order beam theories including the mathematical theory of elasticity [31-36] as well as by using finite element analysis [37-40]. Despite the limitation of the Timoshenko-Ehrenfest beam theory in assuming uniform shear stress distribution through the thickness of the beam, which clearly violates the zero-shear stress condition at the outer surface of the beam, the achievable accuracy from the theory is surprisingly high, and intriguingly it appears too good to be true. However, it must be recognised that to compensate for the inconsistency of not satisfying the zero-shear stress condition on the outer surface of the beam, Timoshenko and Ehrenfest introduced a shear correction factor (also called the shape factor) in their theory which improved the accuracy of results. The Timoshenko-Ehrenfest beam theory is well covered in the literature [8-23] and no detailed elaboration is needed here.

The purpose of this paper is to re-examine the Timoshenko-Ehrenfest beam theory and provide further insights into the theory and its applicability. The authors' approach in this paper sheds new light and gives a different perspective to the problem of free vibration behaviour of Timoshenko-Ehrenfest beams. However, in order to prepare the necessary background, to underpin the underlying motivation behind this work and to lead the readers smoothly into the subsequent text, the following comments are made.

The literature [8-23] shows that investigators have characterised a Timoshenko-Ehrenfest beam primarily by two non-dimensional parameters, namely r^2 and s^2 (or r and s) related (in the usual notation) to the length (L), radius of gyration of the cross-section ($\sqrt{I/A}$), bending rigidity (EI) and shear rigidity (kAG) of the beam. These parameters will be explained in detail later. It has been emphasised that the effects of rotary inertia and shear deformation on the free vibration behaviour can be described uniquely by the parameters r^2 and s^2 respectively, so that the effect of each of the two parameters on natural frequencies and mode shapes can be studied either independently or together. Of course, in the case of a simply supported beam for which the mode shapes are sine waves, an explicit expression for the natural frequencies is available [41]. From the expression for natural frequencies, it is evident that for simply supported beams the effect of shear deformation (s^2) is much more pronounced than that of the rotary inertia (r^2) as was shown by Timoshenko and Ehrenfest [6, 7]. However, for other boundary conditions explicit expressions for natural frequencies are not possible even though the frequency equations are available [42]. It is thus not so obvious that in terms of results, the shear deformation term will always dominate the rotary inertia term for all boundary conditions of the beam. Numerical analysis of the frequency equation or the use of finite element analysis in solving the eigenvalue problem suggests that the shear deformation term is always dominant. From a theoretical standpoint, the authors have been intrigued over the years by the question: "Why is the effect of shear deformation always more pronounced than that of the rotary inertia for any boundary condition of a beam?" The question does not appear to have been answered or addressed clearly in the literature. One of the main objectives of this investigation is to provide a clear-cut answer to this question. A secondary objective of this paper is to provide a range of applicability of the Timoshenko theory for realistic problems.

2. THEORY

The two governing partial differential equations of motion of a Timoshenko-Ehrenfest beam undergoing free natural vibration are of second order and they are coupled in bending (flexural) displacement and bending rotation [11, 16, 19, 20, 41]. These equations can be derived using standard procedures, for example Newton's second law, Lagrange's equation, or Hamilton's principle. The two equations are generally combined into one fourth-order partial differential equation which is identically satisfied by both the bending displacement (y) and the bending rotation (θ). Using the co-ordinate system shown in Fig. 1, this differential equation, in the usual notation, is given by [10, 18, 41, 42]

$$EI\frac{\partial^4 w}{\partial x^4}+\rho A\frac{\partial^2 w}{\partial t^2}-\rho I\left(1+\frac{E}{kG}\right)\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4}=0 \tag{1}$$

where

$$w = y \text{ or } \theta \tag{2}$$

and E, G and ρ are respectively the Young's modulus, modulus of rigidity and density of the beam material, A and I are respectively the area and second moment of area of the beam cross-section, k is the shear correction factor [43], x is the distance from the origin (which in Fig. 1 is at the left hand end of the beam) and t is time.

Harmonic oscillation is assumed, so that

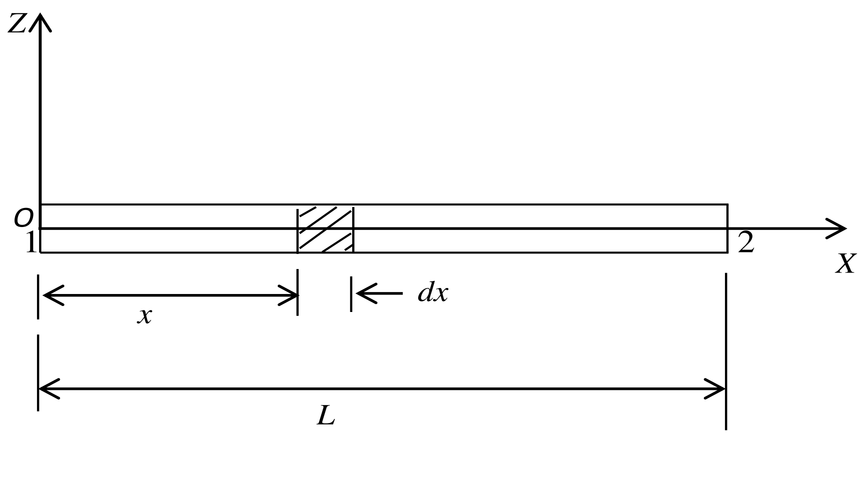

FIGURE 1: Co-ordinate system and notation for a Timoshenko-Ehrenfest beam.

$$w(x,t) = W(x)e^{i\omega t} \tag{3}$$

where $W(x)$ represents the amplitude of the bending displacement $Y(x)$ or the bending rotation $\Theta(x)$ and ω is the angular frequency in rad/s.

Substituting Eq. (3) into Eq. (1) yields the following ordinary differential equation

$$EI\frac{d^4W}{dx^4} - \rho A\omega^2 W + \rho I\omega^2\left(1 + \frac{E}{kG}\right)\frac{d^2W}{dx^2} + \frac{\rho^2 I\omega^4}{kG}W = 0 \tag{4}$$

Introducing the non-dimensional length

$$\xi = x/L \tag{5}$$

where L is the length of the beam, Eq. (4) can be expressed as

$$\left[D^4 + \frac{\rho I\omega^2 L^2}{EI}\left(1 + \frac{E}{kG}\right)D^2 - \frac{\rho A\omega^2 L^4}{EI}\left(1 - \frac{\rho I\omega^2}{kAG}\right)\right]W = 0 \tag{6}$$

where the differential operator D is given by

$$D = \frac{d}{d\xi} \tag{7}$$

Now introducing the following non-dimensional parameters in customary notation

$$b^2 = \frac{\rho A\omega^2 L^4}{EI}, \quad r^2 = \frac{I}{AL^2}, \quad s^2 = \frac{EI}{kAGL^2} \tag{8}$$

Equation (6) can now be written in the familiar form [19, 20]

$$[D^4 + b^2(r^2 + s^2)D^2 - b^2(1 - b^2r^2s^2)]W = 0 \tag{9}$$

Investigators have solved the above differential equation in terms of four arbitrary constants and for a given set of boundary conditions of the beam they eliminated these constants to derive the frequency equation, which is generally an implicit function of the parameters r^2 (rotary inertia) and s^2 (shear deformation). However, an explicit frequency expression is available for the simply supported or sliding boundary condition [41]. Several authors [12, 14, 16] have studied the individual and combined effects of r^2 (rotary inertia) and s^2 (shear deformation) on the free vibration behaviour of Timoshenko-Ehrenfest beams. They have generally concluded that the effects of these parameters on the free vibration behaviour are more pronounced for short and stocky beams for which the cross-sectional dimensions are relatively large and comparable with their lengths. It is also clear that even for slender beams the two effects can be significant for higher frequencies. As expected, both r^2 and s^2 have softening effects on the beam in the sense that they reduce the natural frequencies when compared with the ones obtained by using the Bernoulli-Euler theory. However, no comment with enough clarity seems to have been made in the literature as to why the effect of shear deformation (s^2) is always much more pronounced than that of the rotary inertia (r^2) for all boundary conditions of the beam. Nevertheless, it should be emphasized that Timoshenko in his classic papers [6-7] showed the effect of shear deformation to be four times that of rotary inertia in diminishing the natural frequency of a beam.

The existing literature is largely focused on studying the individual and combined effects of r^2 and s^2 on the free vibration behaviour of a Timoshenko-Ehrenfest beam. By contrast, the present investigation departs from this traditional approach in that it shows that the rotary inertia and shear deformation in the Timoshenko-Ehrenfest beam theory are in fact related, and that they can be essentially combined into one parameter (either r^2 or s^2). The procedure, which is described below, enables one to indicate that the effect of shear deformation (s^2) will always dominate that of the rotary inertia (r^2). It is very simple to understand this, and it may seem rather trivial as will be shown later, but apparently, this simple fact has been overlooked by earlier investigators for a century.

In 1985, Elishakoff and Lubliner [44] used a simplified form of the Timoshenko-Ehrenfest beam theory by neglecting the last term of Eq. (1). Using this simplified form of the Timoshenko-Ehrenfest beam equation, it was possible to obtain closed-form solutions for the random vibrational response of simply supported beams [45]. Later, it was shown [46] that this assumption of ignoring the last term of Eq. (1) is somehow justified because the last term basically amounts to a second order effect, namely, the interaction between the rotary inertia and shear deformation which contributes very little and thus has a relatively minimal influence on the natural frequencies. Interestingly, Goldenveiser et al [47], Kaplunov et al. [48], and Elishakoff et al. [49] demonstrated that, within a formulation based on the first order asymptotic analysis of elasticity theory, the last term of the Timoshenko-Ehrenfest beam equation vanishes. Thus, it appears asymptotically inconsistent to retain this last term in Eq. (1), and its absence has been referred to as the truncated Timoshenko-Ehrenfest beam theory [25]. In this theory Eq. (9) becomes

$$[D^4 + b^2(r^2 + s^2)D^2 - b^2]W = 0 \tag{10}$$

When $W(x) = Y(x)$ and the effects of both rotary inertia and shear deformation are ignored (i.e. $r^2 = s^2 = 0$), Eq. (10) reduces to the Euler-Bernoulli model

$$[D^4 - b^2]W = 0 \tag{11}$$

It is therefore apparent from Eq. (10) that the rotary inertia and shear deformation terms act analogously and hence their effects are asymptotically additive. This is an idea which was already envisaged by Timoshenko and Ehrenfest [6, 7, 26]. Because Eqs. (1) and (9) relate only to the internal points of the beam, the above arguments hold for any boundary conditions at the ends.

From the non-dimensional parameters given by Eqs. (8), one can write

$$\frac{r^2}{s^2} = k\frac{G}{E} \tag{12}$$

For isotropic and homogeneous materials, as in the present case, E and G are related via Poisson's ratio ν as follows [50]

$$G = \frac{E}{2(1+\nu)} \tag{13}$$

Substituting Eq. (13) into Eq. (12) gives

$$\frac{r^2}{s^2} = \frac{k}{2(1+\nu)} = \eta^2 \tag{14}$$

or alternatively,

$$\frac{s^2}{r^2} = \frac{2(1+\nu)}{k} = \frac{1}{\eta^2} \tag{15}$$

For conventional materials $0 \leq \nu \leq 0.5$, while for a beam with a given cross-section $0.5 \leq k \leq 1$. Hence, from Eq. (14), for any given material properties and cross-section, η^2 is a constant in the range $1/6 \leq \eta^2 \leq 1/2$. It is thus clear that the value of r^2 will be always less than half the value of s^2. For isotropic and homogeneous materials such as aluminium and steel, Poisson's ratio ν is close to 0.3 whereas the shear correction factor k is generally within the range $0.4 < k < 0.9$. From these values the range for η^2 can be established as $0.15 < \eta^2 < 0.35$. For a solid rectangular cross-section, k is approximately 2/3. If the Poisson's ratio ν is assumed to have a value of 1/3, then from Eq. (14) η^2 becomes 1/4 so that $s^2 = 4r^2$ or $s = 2r$. This clearly demonstrates the relative importance of the shear deformation term over the rotary inertia one.

Now turning attention to the differential Eq. (9) and substituting $r^2 = \eta^2 s^2$ from Eq. (14), one obtains

$$[D^4 + b^2 s^2 (1+\eta^2) D^2 - b^2 (1 - b^2 s^4 \eta^2)] W = 0 \tag{16}$$

The differential Eq. (16) has three possible solutions depending upon the conditions

$$b^2 s^4 \eta^2 \begin{matrix} < \\ > \\ = \end{matrix} 1 \tag{17}$$

or equivalently

$$\frac{\omega}{\omega_0} \begin{matrix} < \\ > \\ = \end{matrix} \frac{\eta}{\pi^2}\left(\frac{AL^2}{I}\right) \tag{18}$$

where

$$\omega_0 = \sqrt{\frac{\pi^4 EI}{\rho A L^4}} \tag{19}$$

is the lowest natural frequency for a simply supported Euler-Bernoulli beam.

Case (i) : $b^2 s^4 \eta^2 < 1$ covers most of the slenderness ratios and frequency ranges usually encountered. The solution is given by

$$W(\xi) = A_1 \cosh \alpha\, \xi + A_2 \sinh \alpha\, \xi + A_3 \cos \beta\, \xi + A_4 \sin \beta\, \xi \tag{20}$$

where A_1-A_4 are constants and α and β are given by

$$\begin{matrix}\alpha\\ \beta\end{matrix} = \frac{bs}{\sqrt{2}}\left[\mp(1+\eta^2) + \left\{(1+\eta^2)^2 + \frac{4}{b^2s^4}(1-b^2s^4\eta^2)\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{21}$$

Case (ii) : $b^2s^4\eta^2 > 1$ can occur for squat beams or at high frequencies. This case yields an imaginary value for α and the solution is given by

$$W(\xi) = B_1 \cos\alpha^* \xi + B_2 \sin\alpha^* \xi + B_3 \cos\beta\xi + B_4 \sin\beta\xi \tag{22}$$

where B_1-B_4 are constants and α^* and β are given by

$$\begin{matrix}\alpha^*\\ \beta\end{matrix} = \frac{bs}{\sqrt{2}}\left[(1+\eta^2) \mp \left\{(1+\eta^2)^2 + \frac{4}{b^2s^4}(1-b^2s^4\eta^2)\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}} \tag{23}$$

and clearly,

$$\alpha^* = i\ \alpha \tag{24}$$

Case (iii): $b^2s^4\eta^2 = 1$ is a special case which has been termed a cross-over frequency [36]. The differential Eq. (16) becomes

$$D^2\{D^2 + b^2s^2(1+\eta^2)\}W = 0 \tag{25}$$

The solution is given by

$$W(\xi) = C_1 + C_2\xi + C_3 \cos\gamma\,\xi + C_4 \sin\gamma\,\xi \tag{26}$$

where C_1–C_4 are constants and γ is given by

$$\gamma = bs\sqrt{(1+\eta^2)} \tag{27}$$

Cases (i) and (ii) above have been investigated by many researchers [9, 10, 12, 19, 20, 21] whereas there seem to be only two reported attempts [21, 51] for the solution to Case (iii). Interestingly in Ref. [21], the constant C_2 in Eq. (26) vanishes in the solution for the bending displacement but remains in the solution for the bending rotation.

For Case (iii) using Eq. (8), $b^2s^4\eta^2 = 1$ (or $b^2r^2s^2 = 1$), can be written in its extended form as

$$\left(\frac{\rho A\omega^2L^4}{EI}\right)\left(\frac{I}{AL^2}\right)\left(\frac{EI}{kAGL^2}\right) = 1 \tag{28}$$

or

$$\omega\sqrt{\frac{I}{A}}=\sqrt{\frac{kG}{\rho}}=\sqrt{\frac{kE}{2(1+\nu)\rho}} \tag{29}$$

The above equation will be used later to establish the boundaries of the frequencies between the cases when $b^2s^4\eta^2 < 1$ and $b^2s^4\eta^2 > 1$, respectively.

3. DISCUSSION OF RESULTS

The existing literature is inundated with results for the natural frequencies and mode shapes of Timoshenko-Ehrenfest beams [8-23]. Clearly, a massive amount of information is already available. Therefore, the natural frequencies and mode shapes are not presented in this paper, but attention is focused on the intrinsic merit, scope, and applicability of the Timoshenko-Ehrenfest beam theory, instead. The first set of results was obtained from Eq. (15) to show the inter-dependency of the terms s^2 and r^2 for realistic problems. Clearly the ratio s^2/r^2 will be maximum when k is minimum and ν is maximum, and conversely the ratio will be minimum when k is maximum and ν is minimum. Noting that $0 < k < 1$ and $0 < \nu < 0.5$, some practical values for k and ν can be used to establish the limits of the ratio s^2/r^2. Figure 2 shows the variation of s^2 against r^2 for two extreme cases for which the maximum and minimum values of k are 0.8 and 0.4 and those of ν are 0.3 and 0.25. Clearly the values of r^2 and s^2 cannot be chosen totally arbitrarily and most realistic cases are expected to fall within the boundaries between the solid and broken lines shown in Fig. 2, which give ratios s^2/r^2 of 6.5 and 3.125, respectively.

The second set of results was obtained to show the variation of the ratio s^2/r^2 (or $1/\eta^2$) against Poisson's ratio ν for two widely separated values of the shape factor k. This is illustrated in Fig. 3, which clearly indicates that the range of applicability of the ratio is confined within the space between the two lines. To show a related but different perspective Fig. 4 illustrates the variation of the ratio s^2/r^2 against the shape factor k for two extreme values of Poisson's ratio ν. Again, for realistic problems the ratio is confined within the narrow band between the two curves as shown.

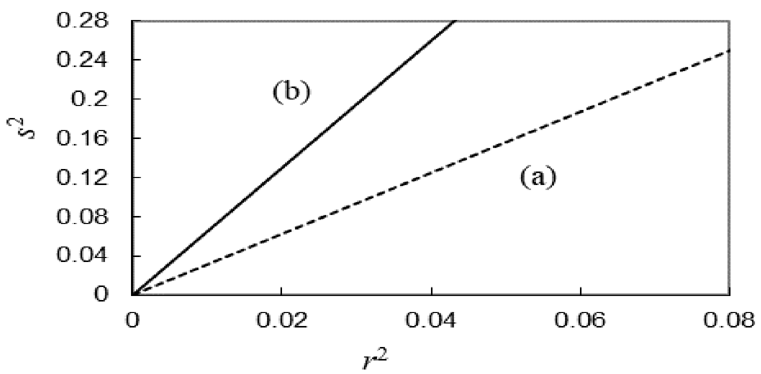

FIGURE 2: Variation of s^2 against r^2 for a Timoshenko-Ehrenfest beam.

(a) ———— $k = 0.8,\ \nu = 0.25$; (b) -------- $k = 0.4,\ \nu = 0.3$

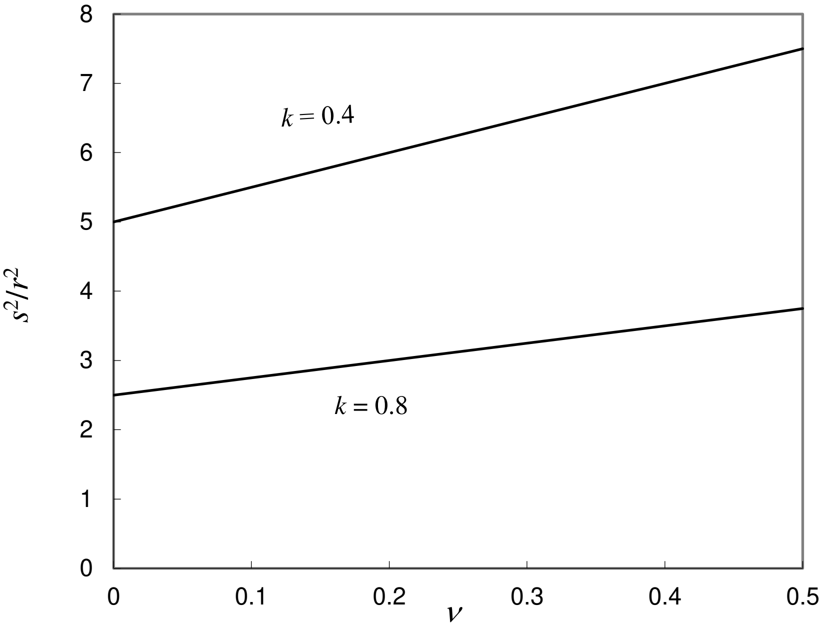

FIGURE 3: Variation of s^2/r^2 against Poisson's ratio ν for $k = 0.4$ and $k = 0.8$.

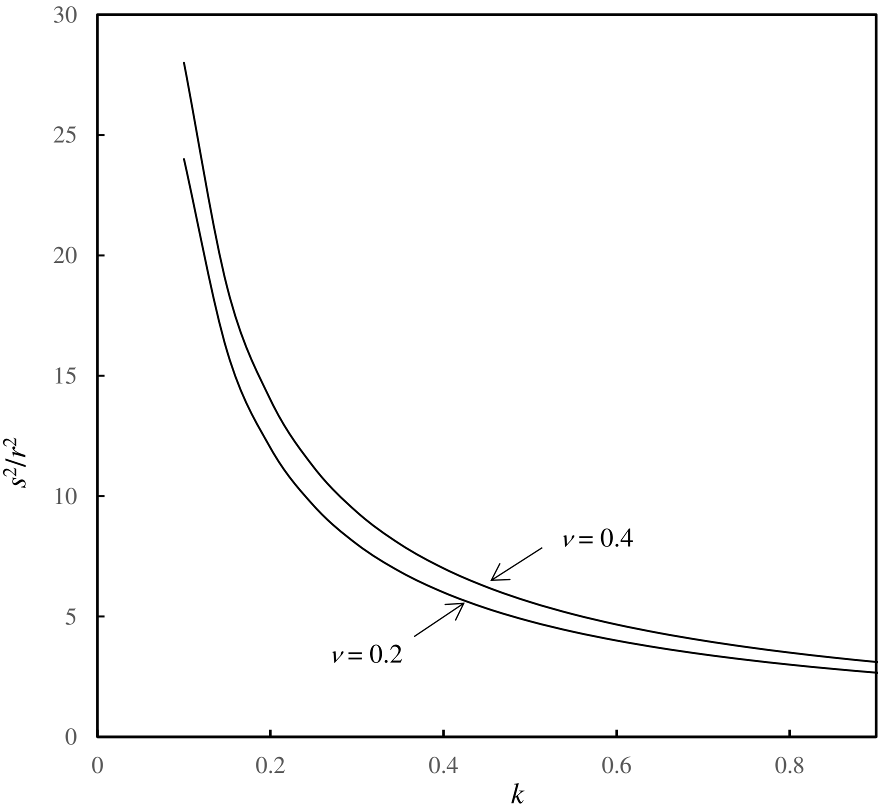

FIGURE 4: Variation of s^2/r^2 against shape factor k for ν =0.2 and $\nu = 0.4$.

The final set of results was obtained to show the boundaries of the three cases, which limit the solutions of the Timoshenko-Ehrenfest beam equation, see Eq. (16). These are $b^2s^4\eta^2 < 1$, $b^2s^4\eta^2 = 1$ and $b^2s^4\eta^2 > 1$ and the corresponding solutions are given by Eqs. (20), (22) and (26). The expression given by Eq. (29), which defines the boundary between $b^2s^4\eta^2 < 1$ and $b^2s^4\eta^2 > 1$, is used to plot the variation of the frequency ω against the radius of gyration $\sqrt{I/A}$. It is interesting to note that for both steel and aluminium, which are widely used as construction materials, the ratios E/ρ (or G/ρ) are very close to each other. This is because the Young's modulus (E) and density (ρ) of steel are both nearly three times more than that of aluminium. Poisson's ratio ν is around 0.3 for both materials and the shape factor k, which is independent of material properties, generally falls within $0.4 < k < 0.8$ [43].

Figure 5 illustrates how ω changes with $\sqrt{I/A}$ for steel (or aluminium) for a range of radii of gyration between 0.005 and 0.2. It is within this range that the effects of shear deformation and rotary inertia are expected to be most important. The data used are E = 210 GPa (or 70 GPa), ρ =8100 kg/m^3 (or 2700 kg/m^3), ν = 0.3 and k = 0.6. The curve is a rectangular hyperbola, as expected (see Eq. (29)), and to the left and right of the curve the conditions $b^2s^4\eta^2 < 1$ and $b^2s^4\eta^2 > 1$ apply, respectively, whereas on the curve $b^2s^4\eta^2 = 1$. The figure provides an additional insight into the frequency range of operation when dealing with Timoshenko-Ehrenfest beam vibration problems. The conditions $b^2s^4\eta^2 \geq 1$ require low slenderness ratios and/or high frequencies, and thus for the majority of problems the natural frequencies of interest are expected to lie in a region which is to the left of the curve, i.e., below the curve. The high frequency range is chosen in plotting the graph of Fig. 5 for a justifiable reason because the curve flattens in the low frequency range.

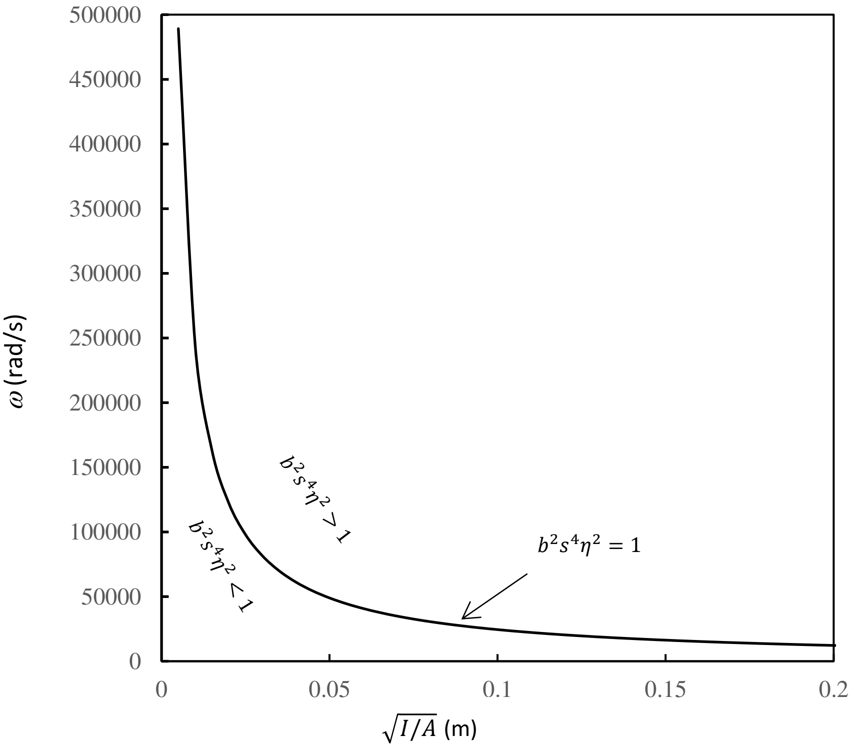

FIGURE 5: Variation of the frequency (ω) against the radius of gyration ($\sqrt{I/A}$).

4. CONCLUSIONS

The Timoshenko-Ehrenfest theory for free vibration of beams has been re-examined to provide further insights. The Timoshenko-Ehrenfest beam is customarily characterised by two independent and apparently unrelated parameters, namely the rotary inertia and the shear deformation, but the authors have shown that the analysis can effectively be carried out by using only one parameter because the rotary inertia and the shear deformation parameters are related. However, the independent effects of shear deformation and rotary inertia can be separately investigated and examined. It has been shown that the effect of the shear deformation will always be more important than that of the rotary inertia for all boundary conditions of a Timoshenko-Ehrenfest beam. The curves presented follow from elementary theories of solid mechanics and provide considerable insights and a new depth of understanding of the Timoshenko-Ehrenfest beam theory. To the best of the authors' knowledge these results have not been presented before. Considering the elementary nature of the background theory used in this paper, this is somewhat surprising.

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NOMENCLATURE

A	area of cross-section, m^2
A_1-A_4	constants of integration, see Eq. (20)
B_1-B_4	constants of integration, see Eq. (22)
b^2	non-dimensional parameter, see Eq. (8)
C_1-C_4	constants of integration, see Eq. (26)
D	$d/d\xi$, see Eq. (7)
E	Young's modulus, N/m^2
G	shear modulus, N/m^2
I	second moment of area, m^4
k	shear correction or shape factor
L	length, m
r^2	non-dimensional parameter, see Eq. (8)
s^2	non-dimensional parameter, see Eq. (8)
t	time, s
W	amplitude of bending displacement, m
w	bending displacement, m
X, Y	coordinate system
Y	amplitude of bending displacement, m
y	flexural displacement, m
α, α^*	see Eqs. (21) and (23)
β	see Eqs. (21) and (23)
γ	see Eq. (27)
Θ	amplitude of bending rotation, rad
θ	bending rotation, rad
ν	Poisson's ratio
ξ	non-dimensional length parameter (*x/L*)
ρ	density of material, kg/m^3

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