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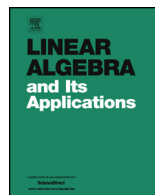
On the sum of left and right circulant matrices

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A B S T R A C T

We consider square matrices arising as the sum of left and right circulant matrices and derive asymptotics of the sequence of their powers. Particular emphasis is laid on the case where the matrix has consecutive integer entries; we find explicit formulae for the eigenvalues and eigenvectors of the matrix in this case and find its Moore-Penrose pseudoinverse. The calculation involves the discrete Fourier transform of integer vectors arising from sum systems and exhibits a resonance phenomenon.

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1. Introduction

A sum system [3,5,9] is a collection of finite sets of integers such that the sums formed by taking one element from each set generate a prescribed arithmetic progression. Such

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systems with two component sets, each of cardinality n, arise naturally in the study of $n \times n$ matrices with symmetry properties and consecutive integer entries.

For a simple example, consider the sum of a right-circulant (Toeplitz) matrix [2,6,11] and a left-circulant (Hankel) matrix

$$\begin{pmatrix} a & b & c \\ c & a & b \\ b & c & a \end{pmatrix} + \begin{pmatrix} d & e & f \\ e & f & d \\ f & d & e \end{pmatrix} = \begin{pmatrix} a+d & b+e & c+f \\ c+e & a+f & b+d \\ b+f & c+d & a+e \end{pmatrix}.$$

The sum will have consecutive integer entries $\{n_0, n_0 + 1, \ldots, n_0 + 8\}$ if and only if

$$\{a, b, c\} + \{d, e, f\} = n_0 + \langle 9 \rangle.$$

We here use the Minkowski set sum $A + B = \{x + y : x \in A, y \in B\}$ and the convention $aA + b = \{ax + b : x \in A\}$ for sets $A, B \subset \mathbb{R}$ and $a, b \in \mathbb{R}$ as well as the notation $\langle n \rangle := \{0, 1, \ldots, n - 1\}$ for any $n \in \mathbb{N}$.

The offset n_0 can easily be added or subtracted, so for standardisation we call a pair of sets $A_1 = \{a_1, \ldots, a_n\}, A_2 = \{b_1, \ldots, b_n\} \subset \mathbb{N}_0$ an $n + n$ *sum system* if

$$A_1 + A_2 = \langle n^2 \rangle,$$

i.e. if

$$\{a_j + b_k : j \in \{1, \ldots, n\}, k \in \{1, \ldots, n\}\} = \{0, 1, \ldots, n^2 - 1\}.$$

Clearly each component set of a sum system contains the number 0 as its smallest element. Moreover, it was shown in Lemma 3.2 of [5] that the component sets of a sum system have the palindromic property

$$a \in A_j \Rightarrow \max A_j - a \in A_j \qquad (j \in \{1, 2\}).$$

Therefore, if n is odd, $n = 2\nu + 1$, then $\frac{1}{2} \max A_j \in A_j$, and by subtracting this middle number from each element of the sum system components, we obtain a system of number sets symmetric around 0.

Example 1. For $\nu = 4$ there exist the three $9 + 9$ sum systems

$$\{\{0, 1, 2, 3, 4, 5, 6, 7, 8\}, \ \{0, 9, 18, 27, 36, 45, 54, 63, 72\}\},$$
$$\{\{0, 1, 2, 9, 10, 11, 18, 19, 20\}, \ \{0, 3, 6, 27, 30, 33, 54, 57, 60\}\}$$
$$\text{and} \quad \{\{0, 1, 2, 27, 28, 29, 54, 55, 56\}, \ \{0, 3, 6, 9, 12, 15, 18, 21, 24\}\},$$

which each generate the set of consecutive integers $\{0, 1, 2, \ldots, 79, 80\}$. The first of these sum systems, where $A_1 = \langle n \rangle$ and $A_2 = n\langle n \rangle$, is called *canonical sum system* and exists

analogously for all $n \in \mathbb{N} + 1$. Subtracting half the largest number from each component set, we obtain the centred systems

$$\{\{-4, -3, -2, -1, 0, 1, 2, 3, 4\}, \ \{-36, -27, -18, -9, 0, 9, 18, 27, 36\}\},$$
$$\{\{-10, -9, -8, -1, 0, 1, 8, 9, 10\}, \ \{-30, -27, -24, -3, 0, 3, 24, 27, 30\}\}$$
$$\text{and} \quad \{\{-28, -27, -26, -1, 0, 1, 26, 27, 28\}, \ \{-12, -9, -6, -3, 0, 3, 6, 9, 12\}\},$$

respectively, the Minkowski set sum of each pair of sets giving the set of consecutive integers $\{-40, -39, \ldots, 0, \ldots 39, 40\}$.

We call a matrix arising as the sum of a left circulant and a right circulant matrix a *sum circulant matrix*. It is clear that a sum circulant matrix whose left and right circulant parts take their entries from the two component sets of a sum system has consecutive integer entries, and it is not hard to see that conversely a sum circulant matrix with consecutive integer entries must have left and right circular parts whose entries arise from a sum system.

In the present paper, we study $n \times n$ sum circulant matrices with odd $n = 2\nu + 1$, the sequence of their powers and their Moore-Penrose pseudoinverses in greater generality. We take particular interest in the case where the entries of their circulant summands are taken from the two centred component sets of an $n + n$ sum system.

Following on from this introduction and motivational results, we organise the paper as follows. In Section 2 we give the definition of circulant Toeplitz and Hankel matrix generators and show how the algebra of circulant Toeplitz matrices reflects the convolution algebra of their generating vectors (the central matrix columns). We also show how the discrete Fourier transform can be used to diagonalise such matrices. These tools are used throughout the remainder of the paper.

In Section 3 we consider the sequence of powers of general sum circulant matrices, showing that, up to suitable rescaling, the subsequence of fourth powers always converges to one of a small number of limit matrices that can be described as circulant Toeplitz matrices with a generating vector of very simple Fourier transform. For the study of the limits and convergents of sum circulant matrices whose generating vectors arise from sum systems, it is therefore of great interest to know the Fourier transforms of these generating vectors.

In Section 4, we use the characterisation of all sum systems in terms of joint ordered factorisations of the pair of integers (n, n), established in [5], to find the Fourier transforms of vectors created from the entries of centred sum system components in increasing order. The result carries over to a certain class of permutations of the vector entries that have covariant Fourier transforms. Finally, in Section 5, these results are used to characterise the eigenvalues (and hence the characteristic polynomials) of sum circulant matrices arising from sum systems and find their Moore-Penrose pseudoinverses.

Our investigations are motivated by the following previous results. In [8] properties of odd-sided $(2\nu + 1) \times (2\nu + 1)$ right circulant (Toeplitz) matrices M_ν with top row

$$(0, \nu, -1, \nu - 1, -2, \nu - 2, \ldots, 2, -\nu - 1, 1, -\nu)$$

were studied, establishing in particular that the Moore-Penrose inverse $M_\nu^{\div}$ is given by the right circulant matrix with top row $(0, \frac{-1}{n}, 0, \ldots, 0, \frac{1}{n})$.

The sequence of powers of an $n \times n$ matrix M, $(M, M^2, M^3, M^4, \ldots)$ satisfies, by the Cayley-Hamilton theorem, the n-term recurrence relation

$$\alpha_n M^{j+n} + \alpha_{n-1} M^{j+n-1} + \alpha_{n-2} M^{j+n-2} + \cdots + \alpha_1 M^{j+1} + \alpha_0 M^j = 0,$$

where $\chi(\lambda) = \sum_{k=0}^{n} \alpha_k \lambda^k$ is the characteristic polynomial of M. If M is a sum circulant matrix, then so is its Moore-Penrose pseudoinverse $M^{\div}$, and the powers of $M^{\div}$ satisfy a reciprocal recurrence to that of M (see Theorem 5.2 and Remark 4 in Section 5 of the present paper). The sequence of powers of $M^{\div}$ can thus be viewed as a continuation of the sequence of powers of M to negative indices.

For the matrices M_ν as defined in the example above, it was shown in [8] that the characteristic polynomial applied to the matrix M_ν (by the Cayley-Hamilton theorem) yields separate equations for the odd and even powers,

$$\sum_{k=0}^{\nu} n^{2(\nu-k)} f_{\nu-k} M_\nu^{2k+1} = 0_n = \sum_{k=0}^{\nu} n^{2(\nu-k)+1} f_{\nu-k} M_\nu^{2k},$$

where $f_s = \frac{1}{2s+1} \binom{\nu+s}{2s}$, $s \in \{0, \ldots, \nu\}$. Thus the entries of the odd matrix powers and the entries of the even matrix powers obey separate recurrences, reflecting a split over the superalgebra of centro-symmetric and centro-antisymmetric $n \times n$ matrices (cf. [4]).

When $\nu = 2$, the sequence generated from powers of M_2^{2k+1} contains interlacing sequences of Fibonacci and Lucas numbers, up to powers of 5. On this basis, a natural definition of higher-dimensional Fibonacci sequences was given in [1].

The recurrence relation obtained from the characteristic polynomial of M_ν was also used [8] to obtain the recurrence for even integer values of the Riemann ζ function,

$$\zeta(2j) = (-1)^{j+1} \left(\frac{j\pi^{2j}}{(2j+1)!} + \sum_{k=1}^{j-1} \frac{(-1)^k \pi^{2j-2k}}{(2j-2k+1)!} \zeta(2k) \right).$$

Further relations for functions related to the ζ function were obtained by expressing these recurrences as Toeplitz determinants [8,7].

Similarly, the reverse recurrence corresponding to the powers of the Moore-Penrose pseudoinverse $M_\nu^{\div}$ contains (up to powers of $n = 2\nu + 1$) the Fleck numbers defined modulo q by

$$F(r, t, q) = \sum_{k \equiv t \bmod q} (-1)^k \binom{r}{k}.$$

For prime powers $q = p^e$, the Fleck numbers obey the congruence relation

$$F(r, t, p^e) = \sum_{k \equiv t \bmod p^e} (-1)^k \binom{r}{k} \equiv 0 \pmod{p^\beta},$$

where $\beta = \left\lfloor \frac{r - p^{e-1}}{\phi(p^e)} \right\rfloor$ and ϕ is Euler's totient function. As indicated above, one can think of these Fleck numbers with their intriguing divisibility properties as being generated by powers of the Moore-Penrose pseudoinverse matrix $M_\nu^{\div}$.

2. Circulant matrix constructors and the discrete Fourier transform

Let $n = 2\nu + 1$ be an odd positive integer and consider the n-dimensional vector space $\mathbb{C}^n$. We treat its elements as column vectors and, as reflection symmetry will play an essential role in the following, we use the indices $-\nu, -\nu + 1, \ldots, \nu - 1, \nu$. For ease of notation, all index calculations are done in the cyclic ring $\mathbb{Z}/(n\mathbb{Z}) = \{-\nu, \ldots, \nu\}$, so $\nu + 1 = -\nu$ etc.

The vector space $\mathbb{C}^n$ has two natural products; the *componentwise product* $\cdot : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$,

$$(u \cdot v)_k = u_k v_k \qquad (k \in \{-\nu, \ldots, \nu\}; u, v \in \mathbb{C}^n)$$

and the *cyclic convolution* $* : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n$,

$$(u * v)_k = \sum_{j=-\nu}^{\nu} u_j v_{k-j} \qquad (k \in \{-\nu, \ldots, \nu\}; u, v \in \mathbb{C}^n),$$

that together with standard vector addition, give it two distinct commutative algebra structures. (Some authors use the dot for the inner product $\sum_{j=-\nu}^{\nu} u_j \overline{v_j}$, but as this can easily be expressed as matrix multiplication in the form $u^T \overline{v}$, we prefer to use the dot for the componentwise product as defined above.)

Let J be the reflected unit matrix, $J_{jk} = \delta_{j,-k}$ ($j, k \in \{-\nu, \ldots, \nu\}$), where δ is the Kronecker delta symbol. We then call $u \in \mathbb{C}^n$ *even* or *odd* if $Ju = u$ or $Ju = -u$, respectively, and split the vector space into its even and odd subspaces, $\mathbb{C}^n = \mathbb{C}_+^n \oplus \mathbb{C}_-^n$, where $\mathbb{C}_\pm^n := \{u \in \mathbb{C}^n : Ju = \pm u\}$. Then $\mathbb{C}_+^n$ is a subalgebra with either of the two products and, with respect to this splitting, both $(\mathbb{C}^n, \cdot)$ and $(\mathbb{C}^n, *)$ are superalgebras.

For the definition of the discrete Fourier transform, it is convenient to introduce the function

$$\overline{e}(x) := e^{-2\pi i x} \qquad (x \in \mathbb{R});$$

then $\overline{e}$ satisfies the functional equations $\overline{e}(x+y) = \overline{e}(x)\overline{e}(y)$ and $\overline{e}(xy) = \overline{e}(x)^y$ ($x, y \in \mathbb{R}$). This function has the property that $\overline{e}(x) = 1 \Leftrightarrow x \in \mathbb{Z}$, and it is also useful to note that

$\overline{e}(n/2) = (-1)^n$ $(n \in \mathbb{Z})$. The *discrete Fourier transform* is the linear bijection on $\mathbb{C}^n$ generated by the matrix F, where

$$F_{jk} = \overline{e}(jk/n) \qquad (j, k \in \{-\nu, \ldots, \nu\}),$$

with inverse

$$F_{jk}^{-1} = \frac{1}{n} \overline{e}(-jk/n) \qquad (j, k \in \{-\nu, \ldots, \nu\}).$$

We remark that the entries of the Fourier transform Fu of a vector $u \in \mathbb{C}^n$ are the values the Laurent polynomial $p_u(z) = \sum\limits_{k=-\nu}^{\nu} u_k z^k$ takes at the n-th roots of unity.

We also define the *Fourier conjugation* $\mathcal{F} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$, $\mathcal{F}M = FMF^{-1}$ for any matrix $M \in \mathbb{C}^{n \times n}$.

Furthermore, we introduce the following methods of generating $n \times n$ matrices from elements of $\mathbb{C}^n$, viz. the *diagonal matrix constructor*

$$D : \mathbb{C}^n \to \mathbb{C}^{n \times n}, \qquad D(u)_{jk} = u_j \, \delta_{jk} \qquad (j, k \in \{-\nu, \ldots, \nu\}; u \in \mathbb{C}^n),$$

the *circulant Toeplitz constructor*

$$T : \mathbb{C}^n \to \mathbb{C}^{n \times n}, \qquad T(u)_{jk} = u_{j-k} \qquad (j, k \in \{-\nu, \ldots, \nu\}; u \in \mathbb{C}^n),$$

and the *circulant Hankel constructor*

$$H : \mathbb{C}^n \to \mathbb{C}^{n \times n}, \qquad H(u)_{jk} = u_{j+k} \qquad (j, k \in \{-\nu, \ldots, \nu\}; u \in \mathbb{C}^n).$$

Then the following statements are easy to verify. Here $\mathbb{C}^{n \times n}$ is given the standard algebra structure with matrix multiplication as product.

Lemma 2.1.

(a) $F : (\mathbb{C}^n, *) \to (\mathbb{C}^n, \cdot)$ *is an algebra isomorphism.*
(b) *The diagonal matrix constructor* $D : (\mathbb{C}^n, \cdot) \to \mathbb{C}^{n \times n}$ *is an algebra homomorphism.*
(c) *The circulant Toeplitz constructor* $T : (\mathbb{C}^n, *) \to \mathbb{C}^{n \times n}$ *is an algebra homomorphism.*
(d) $H = TJ$.
(e) *Fourier conjugation* $\mathcal{F} : \mathbb{C}^{n \times n} \to \mathbb{C}^{n \times n}$ *is an algebra isomorphism; moreover,* $\mathcal{F} \circ T = D \circ F$.
(f) $J : (\mathbb{C}^n, \cdot) \to (\mathbb{C}^n, \cdot)$ *and* $J : (\mathbb{C}^n, *) \to (\mathbb{C}^n, *)$ *are algebra isomorphisms; also,* $FJ = JF$.
(g) $JT = (T \circ J)J$.
(h) *If* $u, v \in \mathbb{C}_+^n$, *then* $H(u)H(v) = T(u)T(v)$ *(and these commute); furthermore,* $T(u)$ *and* $H(v)$ *commute.*

(i) *If $u, v \in \mathbb{C}^n_-$, then $H(u)H(v) = -T(u)T(v)$ (and these commute); furthermore, $T(u)$ and $H(v)$ anticommute.*

We remark that parts (c) and (g) of the above Lemma imply the identities $T(u)T(v) = T(u * v)$, $T(u)H(v) = H(u * v)$, $H(u)T(v) = H(u * (Jv))$ and $H(u)H(v) = T(u * (Jv))$ for $u, v \in \mathbb{C}^n$.

Part (f) of the above Lemma shows that the Fourier transform of an even or odd vector is even or odd, respectively. The Fourier transforms of such vectors can be expressed more conveniently for practical calculation as follows.

Lemma 2.2.

(a) *If $u \in \mathbb{C}^n_-$, then*

$$(Fu)_k = -2i \sum_{j=1}^{\nu} u_j \sin\left(2\pi \frac{jk}{n}\right) \qquad (k \in \{-\nu, \dots, \nu\}),$$

and we have the inverse

$$u_j = \frac{2i}{n} \sum_{k=1}^{\nu} (Fu)_k \sin\left(2\pi \frac{jk}{n}\right) \qquad (j \in \{-\nu, \dots, \nu\}).$$

(b) *If $u \in \mathbb{C}^n_+$, then*

$$(Fu)_k = u_0 + 2 \sum_{j=1}^{\nu} u_j \cos\left(2\pi \frac{jk}{n}\right) \qquad (k \in \{-\nu, \dots, \nu\}),$$

and we have the inverse

$$u_j = \frac{1}{n} \left((Fu)_0 + 2 \sum_{k=1}^{\nu} (Fu)_k \cos\left(2\pi \frac{jk}{n}\right) \right) \qquad (j \in \{-\nu, \dots, \nu\}).$$

3. The asymptotics of powers of sum circulant matrices

In this section we consider sum circulant matrices, i.e. $n \times n$ matrices of the form $M = T(u) + H(v)$, where $u, v \in \mathbb{C}^n_-$.

Lemma 3.1. *Let $u, v \in \mathbb{C}^n_-$ and $M = T(u) + H(v)$. Then, for any $m \in \mathbb{N}_0$,*

$$\mathcal{F}M^2 = D(Fu)^2 - D(Fv)^2,$$

which is a diagonal matrix.

Proof. Observing that

$$\mathcal{F}M = FMF^{-1} = D(Fu) + D(Fv)J,$$

we calculate

$$\mathcal{F}M^2 = FM^2F^{-1} = (D(Fu) + D(Fv)J)^2$$
$$= D(Fu)^2 + D(Fu)D(Fv)J - D(Fv)D(Fu)J - D(Fv)^2$$
$$= D(Fu)^2 - D(Fv)^2. \quad \square$$

Remark 1. By the binomial theorem, the expression in Lemma 3.1 gives the even powers of M,

$$M^{2m} = \mathcal{F}^{-1}\left(\sum_{j=0}^{m}\binom{m}{j}(-1)^j\, D(Fu)^{2(m-j)}\, D(Fv)^{2j}\right),$$

and hence also the odd powers,

$$M^{2m+1} = \mathcal{F}^{-1}\left(\sum_{j=0}^{m}\binom{m}{j}(-1)^j\, D(Fu)^{2(m-j)+1}\, D(Fv)^{2j}\right.$$

$$\left. + \sum_{j=0}^{m}\binom{m}{j}(-1)^j\, D(Fu)^{2(m-j)}\, D(Fv)^{2j+1}J\right).$$

In the case of linearly dependent generating vectors $v = Ku$, these formulae simplify to

$$M^{2m} = (1 - K^2)^m\, \mathcal{F}^{-1}\left(D(Fu)^{2m}\right),$$
$$M^{2m+1} = (1 - K^2)^m\, \mathcal{F}^{-1}\left(D(Fu)^{2m+1}\,(I + KJ)\right).$$

The following theorem, one of our main results, states that the subsequence of fourth powers of a sum circulant matrix with odd generators converges to a simple limit matrix after suitable rescaling.

Theorem 3.1. *Let $u, v \in \mathbb{C}_{-}^n$ and let $M = T(u) + H(v)$ be the corresponding sum circulant matrix. Let $C_{u,v} := \max\limits_{j \in \{-\nu,\dots,\nu\}} |(Fu)_j^2 - (Fv)_j^2|$. Furthermore, let $w_\infty := F^{-1}\hat{w}_\infty$, where $\hat{w}_\infty \in \mathbb{C}_+^n$ is defined by*

$$\hat{w}_{\infty,j} = \begin{cases} 1 & \text{if } |(Fu)_j^2 - (Fv)_j^2| = C_{u,v} \\ 0 & \text{otherwise} \end{cases} \qquad (j \in \{-\nu,\dots,\nu\}).$$

Then

$$\lim_{m \to \infty} \frac{M^{4m}}{C_{u,v}^{2m}} = T(w_\infty).$$

Proof. Due to the odd symmetry of u and v and Lemma 2.2 (a), the entries of Fu and of Fv are purely imaginary and therefore their squares are (non-positive) real. Hence

$$\frac{((Fu)_j^2 - (Fv)_j^2)^{2m}}{C_{u,v}^{2m}} = \left(\frac{|(Fu)_j^2 - (Fv)_j^2|}{C_{u,v}} \right)^{2m} \to \hat{w}_\infty \qquad (m \to \infty).$$

By Lemma 3.1 and Lemma 2.1 (e), we hence find

$$\frac{\mathcal{F}M^{4m}}{C_{u,v}^{2m}} = \frac{(\mathcal{F}M^2)^{2m}}{C_{u,v}^{2m}} = \left(\frac{D(Fu)^2 - D(Fv)^2}{C_{u,v}} \right)^{2m}$$
$$\to D(\hat{w}_\infty) = D(Fw_\infty) = \mathcal{F}T(w_\infty) \qquad (m \to \infty),$$

and the statement of the Theorem follows. $\square$

In the case of linearly dependent generating vectors $v = Ku$, we find

$$(Fu)_j^2 - (Fv)_j^2 = (1 - K^2)(Fu)_j^2 = (K^2 - 1)|(Fu)_j|^2 \qquad (j \in \{-\nu, \dots, \nu\}),$$

bearing in mind that $(Fu)_j$ is purely imaginary in the last step. Proceeding in analogy to the proof of Theorem 3.1, we obtain the following statement.

Theorem 3.2. *Let $u \in \mathbb{C}_-^n \setminus \{0\}$ and $K \in \mathbb{C}$, and let $M = T(u) + KH(u)$ be the corresponding sum circulant matrix. Then*

$$\lim_{m \to \infty} \frac{M^{2m}}{(K^2 - 1)^m C_u^m} = T(w_\infty),$$

where $C_u = \max_{j \in \{-\nu, \dots, \nu\}} |(Fu)_j|^2$ and $w_\infty := F^{-1}\hat{w}_\infty$ with

$$\hat{w}_{\infty,j} = \begin{cases} 1 & \text{if } |(Fu)_j|^2 = C_u \\ 0 & \text{otherwise} \end{cases} \qquad (j \in \{-\nu, \dots, \nu\}.$$

Remark 2. The preceding theorems show that after suitable rescaling, the sequence of fourth powers of the sum circulant matrix M (or, in the case of linearly dependent generating vectors, the sequence of even powers) approaches one of a fairly small number of simply structured circulant Toeplitz matrices. Indeed, as the vector $\hat{w}_\infty$ only has entries 0 and 1 and is even with central entry 0 and at least one pair of non-zero entries, there are only $2^\nu - 1$ possible different vectors $\hat{w}_\infty$, and generically, i.e. excluding cases where $|(Fu)_j^2 - (Fv)_j^2|$ (or, if v is a multiple of u, $|(Fu)_j^2|$) is maximal for two or more index pairs $(-j, j)$, there are only ν different vectors. For example, for $\nu = 2$ (so $n = 5$)

there are only the three possibilities $\hat{w}_\infty \in \{(1,0,0,0,1)^T, (0,1,0,1,0)^T, (1,1,0,1,1)^T)\}$. Thus the fourth powers of all matrices of the form considered in Theorem 3.1 with $n = 5$ have one of 3 (generically, of 2) different asymptotic shapes.

Note that the scaling factors appearing in the limit formula cancel out when ratios of matrix entries are considered. Therefore Theorem 3.1 shows that the ratios of any two matrix entries, keeping their places fixed as we proceed to higher powers, are sequences of convergents to the corresponding ratios in the Toeplitz matrix with generating vector w_∞ obtained as the inverse Fourier transform of $\hat{w}_\infty$. For example, taking $n = 5$ and $u = (-2, -1, 0, 1, 2)^T$ (and v a multiple of u), we find

$$Fu \approx -2i(1.314328, -2.126627, 0, 2.126627, -1.314328)^T.$$

We can immediately read off $\hat{w}_\infty = (0, 1, 0, 1, 0)^T$ and therefore

$$w_\infty = \frac{2}{5} \left(\cos \frac{4\pi}{5}, \cos \frac{2\pi}{5}, 1, \cos \frac{2\pi}{5}, \cos \frac{4\pi}{5} \right)^T.$$

In terms of the Golden Ratio $\phi = \frac{1}{2} \left(1 + \sqrt{5} \right) = -2 \cos \left(\frac{4\pi}{5} \right) = 1.61803 \ldots$ and its inverse $\phi^{-1} = -\frac{1}{2} \left(1 - \sqrt{5} \right) = 2 \cos \left(\frac{2\pi}{5} \right) = 0.61803 \ldots$, we can write this as

$$w_\infty = \frac{-2}{5} \left(\phi, -\phi^{-1}, 1, -\phi^{-1}, \phi \right)^T.$$

4. The Fourier transform of sum system components

If the generating vectors u, v of a sum circulant matrix $M = T(u) + H(v)$ arise from the two components

$$A_1 = \{0 = a_0, \ldots, a_{n-1}\}, A_2 = \{0 = b_0, \ldots, b_{n-1}\}$$

of an $n + n$ sum system as

$$u_j = a_{\nu+j} - \frac{a_{n-1}}{2}, \qquad v_j = b_{\nu+j} - \frac{b_{n-1}}{2} \qquad (j \in \{-\nu, \ldots, \nu\}), \tag{4.1}$$

then the entries of the matrix M are the integers $-2\nu(\nu+1), \ldots, 2\nu(\nu+1)$, each appearing exactly once. Due to the palindromic property of sum systems ([5] Theorem 3.3), $a_k = a_{n-1} - a_{n-1-k}$ and $b_k = b_{n-1} - b_{n-1-k}$ ($k \in \{0, \ldots, n-1\}$), so u and v are odd vectors. In particular, the results of Section 3 apply. It therefore seems an interesting question to find the Fourier transforms Fu and Fv of these vectors. In this section, we express these Fourier transforms in terms of the joint ordered factorisation of (n, n) that generated the sum system. Specifically, let

$$((1, f_1), (2, g_1), (1, f_2), (2, g_2), \ldots, (1, f_L), (2, g_L))$$

be a joint ordered factorisation of (n, n), so

$$\prod_{k=1}^{L} f_k = n, \qquad \prod_{k=1}^{L} g_k = n,$$

with positive integers $f_1, \ldots, f_L, g_1, \ldots, g_{L-1} \geq 2$ and $g_L \geq 1$, where L is some natural number. With the cumulative products

$$\tilde{f}_j = \prod_{k=1}^{j} f_k, \qquad \tilde{g}_j = \prod_{k=1}^{j} g_k \qquad (j \in \{0, \ldots, L\}) \tag{4.2}$$

(so $\tilde{f}_0 = \tilde{g}_0 = 1$ and $\tilde{f}_L = \tilde{g}_L = n$) we can then write the two sum system components as Minkowski set sums

$$A_1 = \sum_{k=1}^{L} \tilde{f}_{k-1} \tilde{g}_{k-1} \langle f_k \rangle, \qquad A_2 = \sum_{k=1}^{L} \tilde{f}_k \tilde{g}_{k-1} \langle g_k \rangle, \tag{4.3}$$

(see [5] Theorem 6.7) and also split the index set as follows,

$$\langle n \rangle = \left\{ \sum_{k=1}^{L} \tilde{f}_{k-1} m_k : 0_L \leq m \leq f - 1_L \right\} \tag{4.4}$$

$$= \left\{ \sum_{k=1}^{L} \tilde{g}_{k-1} m_k : 0_L \leq m \leq g - 1_L \right\}, \tag{4.5}$$

writing $f := (f_1, \ldots, f_L)^T$, $g := (g_1, \ldots, g_L)^T$. We are here using multi-index notation, in particular the partial ordering on $\mathbb{N}_0^L$ defined by

$$x \leq y \iff \forall k \in \{1, \ldots, L\} : x_k \leq y_k;$$

also, $0_L = (0, \ldots, 0)^T \in \mathbb{N}_0^L$ and $1_L = (1, \ldots, 1)^T \in \mathbb{N}_0^L$. In the following, note that we use the symbol $a \mid b$ as an abbreviation of $b \in a\mathbb{Z}$; in particular, $a \mid 0$ $(a \in \mathbb{N})$.

Theorem 4.1. *Let $n = 2\nu + 1$ be an odd natural number, and let $\tilde{f}_k, \tilde{g}_k$ $(k \in \{0, \ldots, L\})$ be the cumulative products, as in (4.2), of a joint ordered factorisation of (n, n). For each $l \in \{-\nu, \ldots, \nu\}$, let k_l^f denote the value of $k \in \{0, \ldots, L\}$ such that $\frac{n}{\tilde{f}_{k-1}} \nmid l$ and $\frac{n}{\tilde{f}_k} \mid l$, and let k_l^g denote the value of $k \in \{0, \ldots, L\}$ such that $\frac{n}{\tilde{g}_{k-1}} \nmid l$ and $\frac{n}{\tilde{g}_k} \mid l$.*

Then the Fourier transforms of the vectors u and v defined as in (4.1) are

$$(Fu)_l = \frac{(-1)^l n i}{2 \sin(\pi l / n)} \, \tilde{g}_{k_l^f - 1}, \quad (Fv)_l = \frac{(-1)^l n i}{2 \sin(\pi l / n)} \, \tilde{f}_{k_l^g} \quad (l \in \{-\nu, \ldots, \nu\} \setminus \{0\}),$$

$(Fu)_0 = (Fv)_0 = 0.$

Corollary 4.1. *In the situation of Theorem 4.1,*

$$(Fu)_l^2 - (Fv)_l^2 = \frac{n^2}{4\sin^2(\pi l/n)} \left(\tilde{f}_{k_l^g}^2 - \tilde{g}_{k_l^f - 1}^2 \right) \qquad (l \in \{-\nu, \ldots, \nu\} \setminus \{0\}).$$

In the homogeneous case $f_k = g_k =: f$ $(k \in \{1, \ldots, L\})$, which corresponds to linearly dependent u, v,

$$(Fu)_l^2 - (Fv)_l^2 = \frac{n^2}{4\sin^2(\pi l/n)} \left(f^2 - 1 \right) \tilde{f}_{k_l^f - 1}^2 \qquad (l \in \{-\nu, \ldots, \nu\} \setminus \{0\}).$$

Remark 3. In the formulae of Theorem 4.1 and Corollary 4.1, the factors $f_{k_l^g}$ and $g_{k_l^f-}$ are determined by the divisibility properties of the index l with respect to the last factors in the joint ordered factorisation. In most cases $k_l^g = k_l^f = L$, but a resonance type phenomenon occurs when g_L or f_L divide l respectively.

In the homogeneous case, we can see that $k_{\pm 1}^f = L$ and therefore $\tilde{f}_{k_{\pm 1}^f - 1} = n/f \geq \tilde{f}_{k_l^f - 1}$ $(l \in \{-\nu, \ldots, \nu\})$. Also, $\left| \frac{n^2}{2\sin^2(\pi l/n)} \right|$ is strictly monotone decreasing in $|l| \in \{1, \ldots, \nu\}$. Hence $|(Fu)_l^2 - (Fv)_l^2|$ is maximal for $l = \pm 1$. By Theorem 3.1, the corresponding vector $\hat{w}_\infty$ will have entries $\hat{w}_{\infty,j} = \delta_{|j|,1}$ $(j \in \{-\nu, \ldots, \nu\})$. Hence Lemma 2.2 gives $w_{\infty,j} = \frac{2}{n} \cos \frac{2\pi j}{n}$ $(j \in \{-\nu, \ldots, \nu\})$ such that

$$\lim_{m \to \infty} \left(\frac{4f^2 \sin^2 \frac{\pi}{n}}{n^4 (f^2 - 1)} \right)^{2m} (T(u) + H(v))^{4m} = T(w_\infty).$$

We prepare the proof of Theorem 4.1 with some preliminary observations.

Lemma 4.1. *Let $n = 2\nu + 1$ and consider the entries of a sum system component $0 = a_0 < \cdots < a_{n-1}$ and the corresponding odd vector $u = (u_{-\nu}, \ldots, u_\nu)$, where*

$$u_j = a_{j+\nu} - \frac{a_{n-1}}{2} \qquad (j \in \{-\nu, \ldots, \nu\}).$$

Then

$$(Fu)_l = \begin{cases} \overline{e}(-\nu l/n) \sum_{k=0}^{n-1} a_k \, \overline{e}(kl/n) & \text{if } l \in \{-\nu, \ldots, \nu\} \setminus \{0\} \\ \sum_{k=0}^{n-1} a_k \, \overline{e}(kl/n) - \frac{na_{n-1}}{2} = 0 & \text{if } l = 0. \end{cases}$$

Proof. We note that due to the palindromic property of the sum system component,

$$\sum_{k=0}^{n-1} a_k = \frac{1}{2} \sum_{k=0}^{n-1} (a_k + a_{n-1-k}) = \frac{n}{2} a_{n-1}.$$

Further,

$$(Fu)_l = \sum_{j=-\nu}^{\nu} \left(a_{j+\nu} - \frac{a_{n-1}}{2} \right) \overline{e}(jl/n)$$

$$= \overline{e}(-\nu l/n) \sum_{k=0}^{n-1} a_k \, \overline{e}(kl/n) - \overline{e}(-\nu l/n) \frac{a_{n-1}}{2} \sum_{k=0}^{n-1} \overline{e}(l/n)^k,$$

and the result follows by observing that $\overline{e}(l/n) = 1 \Leftrightarrow l = 0$ ($l \in \{-\nu, \dots, \nu\}$) and that

$$\sum_{k=0}^{n-1} \overline{e}(l/n)^k = \frac{1 - \overline{e}(ln/n)}{1 - \overline{e}(l/n)} = 0 \qquad (l \in \{-\nu, \dots, \nu\} \setminus \{0\}). \quad \square$$

Lemma 4.2. *Let $n, f, g \in \mathbb{N}$ be such that $f \mid n$ and $g \mid n$, and let $l \in \mathbb{Z}$. Then*
(a)

$$\sum_{j=0}^{g-1} \overline{e}(lfj/n) = \begin{cases} \frac{1 - \overline{e}(lfg/n)}{1 - \overline{e}(lf/n)} & \text{if } \frac{n}{f} \nmid l, \\ g & \text{if } \frac{n}{f} \mid l; \end{cases}$$

in particular,

$$\sum_{j=0}^{g-1} \overline{e}(lfj/n) = 0$$

if $\frac{n}{f} \nmid l$ and $\frac{n}{fg} \mid l$;
(b)

$$\sum_{j=0}^{g-1} j \, \overline{e}(lfj/n) = \begin{cases} \frac{1}{1 - \overline{e}(lf/n)} \left(\overline{e}(lf/n) \frac{1 - \overline{e}(lfg/n)}{1 - \overline{e}(lf/n)} - g \, \overline{e}(lfg/n) \right) & \text{if } \frac{n}{f} \nmid l, \\ \frac{g(g-1)}{2} & \text{if } \frac{n}{f} \mid l; \end{cases}$$

in particular,

$$\sum_{j=0}^{g-1} j \, \overline{e}(lfj/n) = \frac{g}{\overline{e}(lf/n) - 1}$$

if $\frac{n}{f} \nmid l$ and $\frac{n}{fg} \mid l$.

Proof. Part (a) is a direct application of the formula for geometric sums. For part (b), we observe that for $m \in \mathbb{N}$, $q \in \mathbb{C} \setminus \{1\}$

$$\sum_{j=0}^{m-1} jq^j = q \frac{d}{dq} \left(\frac{1 - q^m}{1 - q} \right) = q \left(\frac{1 - q^m}{(1-q)^2} \right) - m \left(\frac{q^m}{1 - q} \right). \quad \square$$

Lemma 4.3. *Let $n \in \mathbb{N}$ be odd, and let $l \in \mathbb{Z}$ be such that $0 < |l| < n$; furthermore, let $f_1 f_2 \cdots f_L = n$ be a factorisation of n with $f_j \in \mathbb{N}$ ($j \in \{1, \ldots, L\}$) and $\tilde{f}_k = \prod_{j=1}^{k} f_j$ ($j \in \{0, \ldots, L\}$). Then, for any $k \in \{1, \ldots, L\}$,*

$$\prod_{j=1}^{L} \left(\sum_{m_j=0}^{f_j-1} (1 + \delta_{jk}(m_j - 1))\, \overline{e}(l\tilde{f}_{j-1}m_j/n) \right)$$
$$= \begin{cases} \dfrac{n}{\tilde{f}_{k-1}\left(\overline{e}(l/n) - 1\right)} \text{if } \dfrac{n}{\tilde{f}_{k-1}} \nmid l \text{ and } \dfrac{n}{\tilde{f}_k} \mid l, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We first observe that at least one of the factors in the product (and hence the whole product) vanishes unless the conditions $(n/\tilde{f}_{k-1}) \nmid l$ and $(n/\tilde{f}_k) \mid l$ are satisfied. Indeed, the factors with $j \neq k$ have the form

$$\sum_{m_j=0}^{f_j-1} \overline{e}(l\tilde{f}_{j-1}m_j/n).$$

By Lemma 4.2 (a), this is equal to 0 if $(n/\tilde{f}_{j-1}) \nmid l$ and $(n/\tilde{f}_j) \mid l$.

We note that $n/\tilde{f}_0 = n$ does not divide l. If $(n/\tilde{f}_1) \mid l$, then the factor with $j = 1$ vanishes. Otherwise, if $(n/\tilde{f}_2) \mid l$, then the factor with $j = 2$ vanishes. Continuing in this way, we reach the question whether $(n/\tilde{f}_k) \mid l$. If it does, we are in the situation we shall consider in more detail later, otherwise, we ask whether $(n/\tilde{f}_{k+1}) \mid l$. If it does, then the factor with $j = k + 1$ vanishes. Otherwise, if $(n/\tilde{f}_{k+2}) \mid l$, then the factor with $j = k + 2$ vanishes. Continuing in this way, we reach the question whether $(n/\tilde{f}_{L-1}) \mid l$. If it does, then the factor with $j = L - 1$ vanishes. Otherwise, the factor with $j = L$ vanishes, since $n/\tilde{f}_L = 1$ does divide l.

This leaves us with the case where $(n/\tilde{f}_{k-1}) \nmid l$ and $(n/\tilde{f}_k) \mid l$. By Lemma 4.2 (b), the factor with $j = k$ in this case takes the form

$$\sum_{m_k=0}^{f_k-1} m_k\, \overline{e}(l\tilde{f}_{k-1}m_k/n) = \frac{f_k}{\overline{e}(l\tilde{f}_{k-1}/n) - 1}.$$

For the preceding factors with $j \in \{1, \ldots, k-1\}$, we have $(n/\tilde{f}_{j-1}) \nmid l$, so by Lemma 4.2 (a) they have the form

$$\sum_{m_j=0}^{f_j-1} \overline{e}(l\tilde{f}_{j-1}m_j/n) = \frac{\overline{e}(l\tilde{f}_j/n) - 1}{\overline{e}(l\tilde{f}_{j-1}/n) - 1}.$$

Thus the first k factors form a telescoping product

$$\prod_{j=1}^{k} \left(\sum_{m_j=0}^{f_j-1} (1 + \delta_{jk}(m_j - 1)) \, \overline{e}(l\tilde{f}_{j-1}m_j/n) \right)$$

$$= \frac{\overline{e}(l\tilde{f}_1/n) - 1}{\overline{e}(l/n) - 1} \frac{\overline{e}(l\tilde{f}_2/n) - 1}{\overline{e}(l\tilde{f}_1/n) - 1} \cdots \frac{\overline{e}(l\tilde{f}_{k-1}/n) - 1}{\overline{e}(l\tilde{f}_{k-2}/n) - 1} \frac{f_k}{\overline{e}(l\tilde{f}_{k-1}/n) - 1}$$

$$= \frac{f_k}{\overline{e}(l/n) - 1}.$$

For the remaining factors with $j \in \{k+1, \ldots, L\}$, we have $(n/\tilde{f}_j) \mid l$, so by Lemma 4.2 (a),

$$\sum_{m_j=0}^{f_j-1} \overline{e}(l\tilde{f}_{j-1}m_j/n) = f_j.$$

Therefore the complete product takes the form

$$\prod_{j=1}^{L} \left(\sum_{m_j=0}^{f_j-1} (1 + \delta_{jk}(m_j - 1)) \, \overline{e}(l\tilde{f}_{j-1}m_j/n) \right) = \frac{f_k \, f_{k+1} \cdots f_L}{\overline{e}(l/n) - 1}$$

$$= \frac{n}{\tilde{f}_{k-1}(\overline{e}(l/n) - 1)}. \quad \square$$

Proof of Theorem 4.1. Using the representation of the first sum system component as a Minkowski sum (4.3), the corresponding splitting of the index set (4.4) and Lemma 4.1, we find for $l \in \{-\nu, \ldots, \nu\} \setminus \{0\}$

$$\overline{e}(\nu l/n) \, (Fu)_l = \sum_{0_L \leq m \leq f - 1_L} \left(\sum_{k=1}^{L} \tilde{f}_{k-1}\tilde{g}_{k-1}m_k \right) \overline{e}\left(\frac{l}{n} \sum_{j=1}^{L} \tilde{f}_{j-1}m_j \right)$$

$$= \sum_{0_L \leq m \leq f - 1_L} \left(\sum_{k=1}^{L} \tilde{f}_{k-1}\tilde{g}_{k-1}m_k \prod_{j=1}^{L} \overline{e}\left(\frac{l}{n} \, \tilde{f}_{j-1}m_j \right) \right)$$

$$= \sum_{0_L \leq m \leq f - 1_L} \left(\sum_{k=1}^{L} \tilde{f}_{k-1}\tilde{g}_{k-1} \prod_{j=1}^{L} (1 + \delta_{jk}(m_j - 1)) \overline{e}\left(\frac{l}{n} \, \tilde{f}_{j-1}m_j \right) \right)$$

$$= \sum_{k=1}^{L} \tilde{f}_{k-1}\tilde{g}_{k-1} \sum_{0_L \leq m \leq f - 1_L} \prod_{j=1}^{L} (1 + \delta_{jk}(m_j - 1)) \overline{e}\left(\frac{l}{n} \, \tilde{f}_{j-1}m_j \right)$$

$$= \sum_{k=1}^{L} \tilde{f}_{k-1}\tilde{g}_{k-1} \prod_{j=1}^{L} \left(\sum_{m_j=0}^{f_j-1} (1 + \delta_{jk}(m_j - 1)) \overline{e}\left(\frac{l}{n} \, \tilde{f}_{j-1}m_j \right) \right).$$

By Lemma 4.3, we thus find

$$(Fu)_l = \overline{e}(-\nu l/n)\, \tilde{f}_{k_l^f-1} \tilde{g}_{k_l^f-1} \frac{n}{\tilde{f}_{k_l^f-1}\,(\overline{e}(l/n)-1)} = \frac{\overline{e}(-l/2)}{\overline{e}(\frac{l}{2n})-\overline{e}(-\frac{l}{2n})}\, n\tilde{g}_{k_l^f-1},$$

and the expression for $(Fu)_l$ claimed in the theorem follows when we observe that $\overline{e}(-l/2) = (-1)^l$ and $\overline{e}(\frac{l}{2n}) - \overline{e}(-\frac{l}{2n}) = -2i\sin l/n$.

Analogously, using the representation of the second component of the sum system in (4.3), the splitting (4.5) and Lemma 4.3, we find

$$\overline{e}(\nu l/n)\,(Fv)_l = \sum_{k=1}^{L} \tilde{f}_k \tilde{g}_{k-1} \prod_{j=1}^{L} \left(\sum_{m_j=0}^{g_j-1} (1+\delta_{jk}(m_j-1))\,\overline{e}\left(\frac{l}{n}\,\tilde{g}_{j-1}m_j\right) \right),$$

and using Lemma 4.3 as above we obtain the expression for $(Fv)_l$ claimed in the theorem.   □

Theorem 4.1 gives the Fourier transforms of vectors u, v arising from the two components of the $n+n$ sum system described by the given joint ordered factorisation, under the hypothesis that the entries of these vectors are arranged in strictly monotone increasing order, i.e. that

$$j_1 < j_2 \quad \Rightarrow \quad u_{j_1} < u_{j_2}, \quad v_{j_1} < v_{j_2} \qquad (j_1, j_2 \in \{-\nu, \dots, \nu\}).$$

However, the formulae of Theorem 4.1 are still useful in cases where the vector entries are permuted in such a way that the Fourier transforms of the vectors are a permutation of the Fourier transforms of the monotone increasing vectors. The following statement gives a sufficient condition for this to be the case.

Lemma 4.4. *Let $m \in \{1, \dots, n-1\}$ be an integer coprime with n, so that $(\,m,\,n\,) = 1$.*

(a) *The mapping $\pi_m : \{-\nu, \dots, \nu\} \to \{-\nu, \dots, \nu\}$, $\pi_m(j) := mj \pmod{n}$ ($j \in \{-\nu,\ \dots,\ \nu\}$) is a permutation.*

(b) *For any $u \in \mathbb{C}^n$ and $\tilde{u} \in \mathbb{C}^n$ defined by $\tilde{u}_j = u_{\pi_m^{-1}(j)}$ ($j \in \{-\nu, \dots, \nu\}$), we have $(F\tilde{u})_k = (Fu)_{\pi_m(k)}$ ($k \in \{-\nu, \dots, \nu\}$).*

Proof. For (a), we only need to show that π_m is injective. Suppose $j_1, j_2 \in \{-\nu, \dots, \nu\}$ are such that $\pi_m(j_1) = \pi_m(j_2)$, i.e. that $m(j_1-j_2) = mj_1 - mj_2 = 0 \pmod{n}$. This implies $j_1 = j_2$, since m is a unit in the ring $\mathbb{Z}/n\mathbb{Z}$.

For (b), note that

$$(F\tilde{u})_k = \sum_{j=-\nu}^{\nu} u_{-\pi_m^{-1}(j)}\,\overline{e}(jk/n) = \sum_{j=-\nu}^{\nu} u_j\,\overline{e}(-\pi_m(j)k/n) = \sum_{j=-\nu}^{\nu} u_j\,\overline{e}(-mjk/n)$$

$$= \sum_{j=-\nu}^{\nu} u_j\,\overline{e}(j\pi_m(k)/n) = (Fu)_{\pi_m(k)} \quad (k \in \{-\nu, \dots, \nu\}). \quad \square$$

The number of different pairs u, v arising from the same sum system by permutations of the type described in Lemma 4.4 is equal to the number of coprime numbers $m \in \{1, \ldots, n-1\}$, i.e. equal to $\phi(n)$, where ϕ is Euler's totient function. Here we assume that the same index permutation is applied to both u and v. However, due to the oddness of the vectors, half of these are just the negatives of the other half.

5. Eigenvalues and pseudoinverses

We begin this section with some observations on the spectra of sum circulant matrices with odd generators.

Theorem 5.1. *Let $u, v \in \mathbb{C}^n_-$. Then the matrix $M = T(u) + H(v)$ has the (structural) eigenvalue 0 with eigenvector $w \in \mathbb{C}^n_+$, with $w_k = 1$ ($k \in \{-\nu, \ldots, \nu\}$). The other eigenvalues of M are*

$$\mu_j^{\pm} = \pm\sqrt{(Fu)_j^2 - (Fv)_j^2} \qquad (j \in \{1, \ldots, \nu\}).$$

If $\mu_j^{\pm} \neq 0$, then $w^{\pm} \in \mathbb{C}_n$ with

$$w_k^{\pm} = (Fu - Fv)_j \cos(2\pi jk/n) \pm i\sqrt{(Fu)_j^2 - (Fv)_j^2}\, \sin(2\pi jk/n)$$

($k \in \{-\nu, \ldots, \nu\}$) are corresponding eigenvectors. In the case of coalescence of the pair to a zero eigenvalue of algebraic multiplicity 2, $\mu_j^+ = \mu_j^- = 0$, then

$$w_k = \sin(2\pi jk/n) \qquad (k \in \{-\nu, \ldots, \nu\}); \tag{5.1}$$

gives an eigenvector $w \in \mathbb{C}^n_-$ if $(Fu)_j = (Fv)_j$,

$$w_k = \cos(2\pi jk/n) \qquad (k \in \{-\nu, \ldots, \nu\}), \tag{5.2}$$

gives an eigenvector $w \in \mathbb{C}^+_n$ if $(Fu)_j = -(Fv)_j$; in particular, if $(Fu)_j = 0 = (Fv)_j$, then both (5.1) and (5.2) are eigenvectors, otherwise the geometric multiplicity of the double eigenvalue $\mu_j^{\pm} = 0$ is equal to 1.

Proof. We begin by noting that $\mathcal{F}M = D(Fu) + D(Fv)J$ has non-zero entries only on the two diagonals. Therefore the subspaces

$$X_j = \{x \in \mathbb{C}^n : x_k = 0 \quad (k \in \{-\nu, \ldots, \nu\} \setminus \{-j, j\})\} \qquad (j \in \{0, \ldots, \nu\})$$

are invariant under the linear mapping generated by the matrix $\mathcal{F}M$, and we can therefore restrict our attention to the eigenvalues and eigenvectors of the matrix on each of these subspaces. As $(Fu)_0 = (Fv)_0 = 0$, the matrix acts as the null mapping on

X_0 and therefore has eigenvalue 0 and the corresponding eigenvector $\hat{w} \in X_0 \subset \mathbb{C}^n$ with $\hat{w}_k = \delta_{k,0}$ ($k \in \{-\nu, \ldots, \nu\}$), using the Kronecker delta symbol. Then $w = F^{-1}\hat{w}$ is an eigenvector of M for this eigenvalue, and by Lemma 2.2 (b) $w_k = 1/n$ for all $k \in \{-\nu, \ldots, \nu\}$. Clearly any non-null multiple of this vector also is an eigenvector for the same eigenvalue.

For $j \in \{1, \ldots, n\}$, the restriction of $\mathcal{F}M$ to the invariant subspace X_j is equivalent to the 2×2 matrix

$$\begin{pmatrix} (Fu)_{-j} & (Fv)_{-j} \\ (Fv)_j & (Fu)_j \end{pmatrix} = \begin{pmatrix} -(Fu)_j & -(Fv)_j \\ (Fv)_j & (Fu)_j \end{pmatrix}.$$

If $(Fu)_j^2 \neq (Fv)_j^2$, then the eigenvalues of this matrix are $\mu_j^{\pm} = \pm\sqrt{(Fu)_j^2 - (Fv)_j^2}$ with eigenvectors

$$\begin{pmatrix} (Fu)_j - (Fv)_j \mp \sqrt{(Fu)_j^2 - (Fv)_j^2} \\ (Fu)_j - (Fv)_j \pm \sqrt{(Fu)_j^2 - (Fv)_j^2} \end{pmatrix},$$

and the eigenvectors of M stated in the Theorem are obtained by applying the inverse Fourier transform in Lemma 2.2 and multiplication by the constant $n/2$.

If $(Fu)_j = (Fv)_j$, then the 2×2 matrix has eigenvalue 0 of algebraic multiplicity 2 and geometric multiplicity 1, with the eigenvector $(-1, 1)^T$. If $(Fu)_j = -(Fv)_j$, then the matrix has the same eigenvalue with eigenvector $(1, 1)^T$. The inverse Fourier transform of Lemma 2.2 (a) and (b), respectively, gives corresponding eigenvectors of M in analogy to the above. $\square$

Example 2. Applying Lemma 4.4 with $m = 2$ to the centred $4 + 4$ sum system given in Example 1 with generators $u = \{-10, -9, -8, -1, 0, 1, 8, 9, 10\}$ and $v = 3u$, where $n = 9$, we see that the index permutation π_2 maps u to the generator $\tilde{u} = \{-8, 9, -1, 10, 0, -10, 1, -9, 8\}$. From $2^{-1} \equiv 5 \pmod 9$, it follows that $\tilde{u}_{\pi_5(j)} = u_j$ ($j \in \{-\nu, \ldots, \nu\}$).

For all $\phi(n)$ possible such mappings π_m, it follows that the eigenvalues of $T(u)$ and $T(u_{\pi_m})$ are the same, so that the respective characteristic polynomials coincide, $\chi(T(u)) = \chi(T(u_{\pi_m}))$. If the same index permutation is applied to the generator v, then Theorem 5.1 shows that the matrices $M = T(u) + H(Ku)$ and $M_{\pi_m} = T(u_{\pi_m}) + H(Ku_{\pi_m})$ have the same eigenvalues, and consequently the same characteristic polynomial, $\chi(M) = \chi(M_{\pi_m})$. However, for two distinct permutations $\pi_m \neq \pi_\ell$, the matrix $M_{\pi_m, \pi_\ell} = T(u_{\pi_m}) + H(v_{\pi_\ell})$ will in general have different eigenvalues and different characteristic polynomial from $M = T(u) + H(v)$.

Finally, we construct the Moore-Penrose pseudoinverses of sum circulant matrices. For a matrix S, the Moore-Penrose pseudoinverse is the unique matrix $S^{\div}$ such that

$SS^{\div}S = S$, $S^{\div}SS^{\div} = S^{\div}$, and $SS^{\div}$ and $S^{\div}S$ are hermitian [10]. The pseudoinverse of a circulant matrix is again a circulant matrix (see [11]).

As before, we consider the case of odd generating vectors $u, v \in \mathbb{C}^n_-$. By Theorem 5.1 the matrix $T(u) + H(v)$ has eigenvalue 0 and is therefore not invertible. As observed in the proof of Lemma 3.1, the Fourier conjugate matrix $\mathcal{F}M = D(Fu) + D(Fv)J$ is very simply structured and in particular has entries 0 off the two diagonals. The following lemma shows that its pseudoinverse is a matrix of the same structure.

Lemma 5.1. *Let* $\hat{u}, \hat{v} \in C^n_-$. *Then, with the vectors* $\hat{u}^{\div}, \hat{v}^{\div} \in C^n_-$ *defined as*

$$
\hat{u}_k^{\div} = \begin{cases} 0 \\ \frac{1}{4\hat{u}_k} \\ \frac{\hat{u}_k}{\hat{u}_k^2 - \hat{v}_k^2} \end{cases} \qquad \hat{v}_k^{\div} = \begin{cases} 0 & \text{if } \hat{u}_k = \hat{v}_k = 0, \\ -\frac{1}{4\hat{v}_k} & \text{if } \hat{u}_k^2 = \hat{v}_k^2 \neq 0, \\ \frac{\hat{v}_k}{\hat{u}_k^2 - \hat{v}_k^2} & \text{if } \hat{u}_k^2 \neq \hat{v}_k^2 \end{cases}
$$

for $k \in \{-\nu, \dots, \nu\}$,

$$
D(\hat{u}^{\div}) + D(\hat{v}^{\div})J = (D(\hat{u}) + D(\hat{v})J)^{\div}.
$$

Proof. Using the facts that $JD(w) = -D(w)J$ and $D(w)D(y) = D(w \cdot y)$ for any $w, y \in \mathbb{C}^n_-$, we find

$$
(D(\hat{u}) + D(\hat{v})J)(D(\hat{u}^{\div}) + D(\hat{v}^{\div})J)(D(\hat{u}) + D(\hat{v})J)
$$

$$
= D(\hat{u} \cdot \hat{u}^{\div} \cdot \hat{u} - \hat{u} \cdot \hat{v}^{\div} \cdot \hat{v} + \hat{v} \cdot \hat{u}^{\div} \cdot \hat{v} - \hat{v} \cdot \hat{v}^{\div} \cdot \hat{u})
$$

$$
+ D(\hat{u} \cdot \hat{u}^{\div} \cdot \hat{v} - \hat{u} \cdot \hat{v}^{\div} \cdot \hat{u} + \hat{v} \cdot \hat{u}^{\div} \cdot \hat{u} - \hat{v} \cdot \hat{v}^{\div} \cdot \hat{v})J,
$$

and it is a straightforward componentwise calculation to check that

$$
\hat{u} \cdot \hat{u}^{\div} \cdot \hat{u} - \hat{u} \cdot \hat{v}^{\div} \cdot \hat{v} + \hat{v} \cdot \hat{u}^{\div} \cdot \hat{v} - \hat{v} \cdot \hat{v}^{\div} \cdot \hat{u} = \hat{u},
$$

$$
\hat{u} \cdot \hat{u}^{\div} \cdot \hat{v} - \hat{u} \cdot \hat{v}^{\div} \cdot \hat{u} + \hat{v} \cdot \hat{u}^{\div} \cdot \hat{u} - \hat{v} \cdot \hat{v}^{\div} \cdot \hat{v} = \hat{v}.
$$

The remaining three properties of the Moore-Penrose pseudoinverse can be verified in analogous manner. □

As a direct consequence, we obtain the following statement.

Theorem 5.2. *Let* $u, v \in \mathbb{C}^n$ *and let* $u^{\div} = F^{-1}\hat{u}^{\div}$, $v^{\div} = F^{-1}\hat{v}^{\div}$, *where* $\hat{u}^{\div}, \hat{v}^{\div}$ *are defined as in Lemma 5.1 with* $\hat{u} = Fu$, $\hat{v} = Fv$. *Then the pseudoinverse of the sum circulant matrix* $M = T(u) + H(v)$ *is*

$$
M^{\div} = T(u^{\div}) + H(v^{\div}).
$$

Proof. By Lemma 2.1 (d), (e), (f) and Lemma 5.1,

$$M^{\div} = F^{-1}(\mathcal{F}M^{\div})F = F^{-1}(D(\hat{u}^{\div}) + D(\hat{v}^{\div})J)F$$
$$= F^{-1}D(Fu^{\div})F + F^{-1}D(Fv^{\div})FJ = T(u^{\div}) + H(v^{\div}). \quad \square$$

Remark 4. Consider a sum circulant matrix M and its Moore-Penrose pseudoinverse $M^{\div}$. In view of Theorems 5.1, 5.2 and Lemma 5.1, a straightforward calculation shows that the non-zero eigenvalues of $M^{\div}$ are the inverses of the non-zero eigenvalues of M, with the same eigenvectors. Also, the eigenvectors for the eigenvalue 0 are the same for both M and $M^{\div}$. These statements are not trivial, as sum circulant matrices are not in general normal.

In the case where M (and hence $M^{\div}$) only has the simple structural eigenvalue 0, it follows that the characteristic polynomials χ of M and $\chi^{\div}$ of $M^{\div}$ are related as

$$\chi^{\div}(\lambda) = \frac{(-\lambda)^{n+1}}{P}\chi\left(\frac{1}{\lambda}\right),$$

where P is the product of the non-zero eigenvalues of M, so $\chi^{\div}$ is essentially the reciprocal of the polynomial χ. Via the Cayley-Hamilton theorem, this shows that the powers of $M^{\div}$ satisfy a recurrence relation that is the reverse of the recurrence for the powers of M and can therefore be considered a continuation of the latter sequence to negative powers. Note that negative powers of M do not exist directly as M is not invertible.

Specifically for sum circulant matrices generated by vectors u, v arising from a joint ordered factorisation as in Eq. (4.1), we find the following pseudoinverses. Note that in this case, by Theorem 4.1, $(Fu)_l \neq 0$ and $(Fv)_l \neq 0$ unless $l = 0$.

Corollary 5.1. Let $n = 2\nu + 1$ be an odd natural number, and let $\tilde{f}_k, \tilde{g}_k$ $(k \in \{0, \ldots, L\})$ be the cumulative products, as in (4.2), of a joint ordered factorisation of (n, n). For each $l \in \{-\nu, \ldots, \nu\}$, let k_l^f denote the value of $k \in \{0, \ldots, L\}$ such that $\frac{n}{\tilde{f}_{k-1}} \nmid l$ and $\frac{n}{\tilde{f}_k} | l$, and let k_l^g denote the value of $k \in \{0, \ldots, L\}$ such that $\frac{n}{\tilde{g}_{k-1}} \nmid l$ and $\frac{n}{\tilde{g}_k} | l$. Let

$$\hat{u}_l^{\div} = \begin{cases} (-1)^l \frac{2i}{n} \sin(\pi l/n) \frac{\tilde{g}_{k_l^f - 1}}{\tilde{f}_{k_l^g}^2 - \tilde{g}_{k_l^f - 1}^2} & \text{if } \tilde{f}_{k_l^g} \neq \tilde{g}_{k_l^f - 1} \\ -\frac{(-1)^l i}{2n\tilde{g}_{k_l^f - 1}} \sin(\pi l/n) & \text{if } \tilde{f}_{k_l^g} = \tilde{g}_{k_l^f - 1} \end{cases}$$

$$\hat{v}_l^{\div} = \begin{cases} (-1)^l \frac{2i}{n} \sin(\pi l/n) \frac{\tilde{f}_{k_l^g}}{\tilde{f}_{k_l^g}^2 - \tilde{g}_{k_l^f - 1}^2} & \text{if } \tilde{f}_{k_l^g} \neq \tilde{g}_{k_l^f - 1}, \\ \frac{(-1)^l i}{2n\tilde{f}_{k_l^g}} \sin(\pi l/n) & \text{if } \tilde{f}_{k_l^g} = \tilde{g}_{k_l^f - 1} \end{cases}$$

for $l \in \{-\nu, \ldots, \nu\} \setminus \{0\}$ and $\hat{u}_0^{\div} = \hat{v}_0^{\div} = 0$.

Then $T(F^{-1}\hat{u}^{\div}) + H(F^{-1}\hat{v}^{\div})$ is the pseudoinverse of $T(u) + H(v)$, where u and v are as in Eq. (4.1).

Specifically in the homogeneous case where all factors in the joint ordered factors are the same and the resulting generating vectors u, v are linearly dependent, we can give an explicit formula for the components of the generating vectors $u^{\div}$, $v^{\div}$ of the Moore-Penrose pseudoinverse.

Theorem 5.3. *Let $n = 2\nu + 1$ be an odd natural number and suppose $n = f^L$ with natural numbers $f > 1, L$. Let $u, v \in \mathbb{C}^n_-$ be the vectors generated as in Eq.* (4.1) *from the sum system with joint ordered factorisation $f_k = g_k = f$ ($k \in \{1, \ldots, L\}$), and let $\hat{u}^{\div}$, $\hat{v}^{\div}$ be defined as in Corollary* 5.1.
Then $u^{\div} := F^{-1}\hat{u}^{\div}$ satisfies

$$u_j^{\div} = \begin{cases} -\frac{1}{n^2(f+1)} \sum\limits_{m=1}^{L-1} \eta_{j,m} & (j \in \{1, \ldots, \nu - 1\}) \\ \frac{L(f-1)+1}{n^2(f^2-1)} & (j = \nu), \end{cases}$$

where

$$\eta_{j,m} := \begin{cases} 1 & \text{if } \frac{(2j-1)f^m}{n} \text{ is an odd integer,} \\ -1 & \text{if } \frac{(2j+1)f^m}{n} \text{ is an odd integer,} \\ 0 & \text{otherwise;} \end{cases}$$

moreover, $u_{-j}^{\div} = -u_j^{\div}$ ($j \in \{1, \ldots, \nu\}$), $u_0^{\div} = 0$ and $v^{\div} := F^{-1}\hat{v}^{\div} = f\, u^{\div}$.

Proof. The main difficulty in using the formulae of Corollary 5.1 is to identify k_l^f and k_l^g for each $l \in \{-\nu, \ldots, \nu\}$. In the homogeneous case, we have $k_l^f = k_l^g := k_l$ and $L - k_l$ is the greatest power of f that divides l. Moreover, $\tilde{f}_k = \tilde{g}^k = f^k$, so $\tilde{f}_k \neq \tilde{g}_{k-1}$ for all k. Consequently, Corollary 5.1 gives

$$\hat{u}_l^{\div} = (-1)^l \frac{2if}{n(f^2-1)} \frac{1}{f^{k_l}} \sin \frac{\pi l}{n} \qquad (l \in \{-\nu, \ldots, \nu\}).$$

As this is an odd vector, the formula of Lemma 2.2 (a) gives

$$u_j^{\div} = -\frac{4f}{n^3(f^2-1)} \sum_{l=1}^{\nu} (-1)^l \frac{n}{f^{k_l}} \sin \frac{2\pi jl}{n} \sin \frac{\pi l}{n} \qquad (j \in \{1, \ldots, \nu\}).$$

In this sum, the factor $\frac{n}{f^{k_l}}$ is the greatest power of f that divides l; so it is equal to 1 if l is not a multiple of f, equal to f if l is a multiple of f, but not of f^2, equal to f^2 if l is a multiple of f^2, but not of f^3 etc. This consideration allows us to expand the sum in a telescopic manner as follows,

$$u_j^{\ddot{\div}} = -\frac{4f}{n^3(f^2-1)} \left(\sum_{l=1}^{(f^L-1)/2} (-1)^l \sin\frac{2\pi jl}{n} \sin\frac{\pi l}{n} \right.$$

$$\left. + \sum_{m=1}^{L-1} f^{m-1}(f-1) \sum_{l=1}^{(f^{L-m}-1)/2} (-1)^l \sin\frac{2\pi j f^m l}{n} \sin\frac{\pi f^m l}{n} \right), \qquad (5.3)$$

$(j \in \{1,\dots,\nu\})$. Now for $m \in \{0,\dots,L-1\}$

$$\sin\frac{2\pi j f^m l}{n} \sin\frac{\pi f^m l}{n} = \frac{1}{2}\cos(2j-1)\frac{\pi f^m l}{n} - \frac{1}{2}\cos(2j+1)\frac{\pi f^m l}{n} \qquad (5.4)$$

and, if $-e^{i(2j\pm1)\pi f^m/n} \neq 1$,

$$\sum_{l=1}^{(f^{L-m}-1)/2} (-1)^l \cos(2j\pm1)\frac{\pi f^m l}{n} = \text{Re} \sum_{l=1}^{(f^{L-m}-1)/2} (-e^{i(2j\pm1)\pi f^m/n})^l$$

$$= \text{Re}\left(-e^{i(2j\pm1)\pi f^m/n}\frac{1-(-e^{i(2j\pm1)\pi f^m/n})^{(f^{L-m}-1)/2}}{1+e^{i(2j\pm1)\pi f^m/n}} \right)$$

$$= -\frac{\text{Re}\, e^{i(2j\pm1)\frac{\pi f^m}{2n}}}{2\cos(2j\pm1)\frac{\pi f^m}{2n}} + \frac{(-1)^{(f^{L-m}-1)/2}}{2\cos(2j\pm1)\frac{\pi f^m}{2n}}\cos((2j\pm1)\frac{\pi f^L}{2n}) = -\frac{1}{2}.$$

However, if $-e^{i(2j\pm1)\pi f^m/n} = 1$, i.e. if $(2j\pm1)f^m/n$ is an (odd) integer, then

$$\sum_{l=1}^{(f^{L-m}-1)/2} (-1)^l \cos(2j\pm1)\frac{\pi f^m l}{n} = \frac{f^{L-m}-1}{2}.$$

Note that this case does not happen for $(2j+1)f^m/n$ and for $(2j-1)f^m/n$ simultaneously, because then their difference $2f^m/n$ would be an (even) integer, which is never the case if $m \in \{1,\dots,L-1\}$. Combining this with Eq. (5.4), we obtain

$$\sum_{l=1}^{(f^{L-m}-1)/2} (-1)^l \sin\frac{2\pi j f^m l}{n} \sin\frac{\pi f^m l}{n} = \frac{f^{L-m}}{4}\eta_{j,m},$$

where $\eta_{j,m}$ is as defined in the Theorem. Clearly $\eta_{j,0} = -\delta_{j,\nu}$ for $j \in \{1,\dots,\nu\}$, so Eq. (5.3) yields

$$u_j^{\ddot{\div}} = -\frac{4f}{n^3(f^2-1)}\left(-\frac{f^L}{4}\delta_{j,\nu} + \sum_{m=1}^{L-1} f^{m-1}(f-1)\frac{f^{L-m}}{4}\eta_{j,m} \right)$$

$$= \frac{f}{n^2(f^2-1)}\delta_{j,\nu} - \frac{1}{n^2(f+1)}\sum_{m=1}^{L-1}\eta_{j,m}.$$

In the case $j=\nu$, the sum evaluates to $-(L-1)$. $\square$

We conclude with an example.

Example 3. We take generating vectors from the $9 + 9$ sum system which arises from the homogeneous joint ordered factorisation $((1,3)(2,3)(1,3)(2,3))$, giving

$$u = \{-10, -9, -8, -1, 0, 1, 8, 9, 10\}^T, \quad \text{and} \quad v = 3u,$$

(cf. Example 1, 2nd centred system in the Introduction). Here $L = 2$ and $f_1 = f_2 = g_1 = g_2 = 3$, so $\tilde{f}_0 = \tilde{g}_0 = 1, \tilde{f}_1 = \tilde{g}_1 = 3, \tilde{f}_2 = \tilde{g}_2 = 9$. This yields $k_1^f = k_2^f = k_4^f = k_1^g = k_2^g = k_4^g = 2$, and $k_3^f = k_3^g = 1$, so that

$$\tilde{f}_{k_1^g} = \tilde{f}_{k_2^g} = \tilde{f}_{k_4^g} = \tilde{f}_2 = 9, \quad \tilde{f}_{k_3^g} = \tilde{f}_1 = 3,$$

and

$$\tilde{g}_{k_1^f - 1} = \tilde{g}_{k_2^f - 1} = \tilde{g}_{k_4^f - 1} = \tilde{g}_1 = 3, \quad \tilde{g}_{k_3^f - 1} = \tilde{g}_0 = 1.$$

Hence we have that $\tilde{f}_{k_\ell^g} \neq \tilde{g}_{k_\ell^f - 1}$ for all $\ell \in \{1, \ldots, 4\}$, and applying Corollary 5.1 gives us

$$\hat{u}^{\div} = \frac{i}{108} \left(\sin\left(\frac{-4\pi}{9}\right), -3\sin\left(\frac{-3\pi}{9}\right), \sin\left(\frac{-2\pi}{9}\right), -\sin\left(\frac{-\pi}{9}\right), 0, -\sin\left(\frac{\pi}{9}\right), \right.$$
$$\left. \sin\left(\frac{2\pi}{9}\right), -3\sin\left(\frac{3\pi}{9}\right), \sin\left(\frac{4\pi}{9}\right) \right).$$

Applying the inverse Fourier transform we find that

$$u^{\div} = \frac{1}{648} \{-5, 0, 2, -2, 0, 2, -2, 0, 5\}, \quad \text{and} \quad v^{\div} = 3u^{\div},$$

where as promised by Theorem 5.2, we do indeed have that

$$T(u^{\div}) + H(v^{\div}) = T(F^{-1}\hat{u}^{\div}) + H(F^{-1}\hat{v}^{\div}) = (T(u) + H(v))^{\div}.$$

Comparing this with Theorem 5.3, we find that $\eta_{1,1} = -1, \eta_{2,1} = 1,$ and $\eta_{3,1} = 0$, which agrees with the above entries for $u^{\div}$.

6. Conclusions

In this work we have established asymptotics for sequences of powers of $n \times n$ sum circulant matrices of the form $M = T(u) + H(v)$, where $u, v \in \mathbb{C}_-^n$, are odd generators. When the generators are taken from a two-dimensional sum system then the matrix has consecutive integer entries; we find explicit formulae for the eigenvalues and eigenvectors of the matrix in this case and find its Moore-Penrose pseudoinverse. The calculation

involves the discrete Fourier transform of integer vectors arising from sum systems and exhibits a resonance phenomenon.

The formulae Moore-Penrose inverse demonstrates an independence with regarding the sum of the left and right circulant matrices and can be constructed from the two individual component matrices.

Further topics for consideration could include taking the generating vectors from alternative combinatorial sets.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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