Spectral properties of the inhomogeneous Drude-Lorentz model with dissipation

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Abstract

We establish spectral enclosures and spectral approximation results for the inhomogeneous lossy Drude-Lorentz system with purely imaginary poles, in a possibly unbounded Lipschitz domain of $\mathbb{R}^3$. Under the assumption that the coefficients θ_e, θ_m of the material are asymptotically constant at infinity, we prove that spectral pollution due to domain truncation can lie only in the essential numerical range of a $\operatorname{curl}\operatorname{curl}_0 - f(\omega)$ pencil.

As an application, we consider a conducting metamaterial at the interface with the vacuum; we prove that the complex eigenvalues with non-trivial real part lie outside the set of spectral pollution. We believe this is the first result of enclosure of spectral pollution for the Drude-Lorentz model without assumptions of compactness on the resolvent of the underlying Maxwell operator.
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1. Introduction

1.1. The Drude-Lorentz model

This paper concerns the spectra and spectral approximation of a time-harmonic Drude-Lorentz model [1] which commonly occurs in the description of a class of metamaterials. This class includes doubly negative metamaterials, which behave as if the electric permittivity and the magnetic permeability are simultaneously negative. In the early 2000s, it was conjectured that these materials might allow the creation of a *perfect lens* or an *invisibility cloak*, see e.g. [2], [3]. Shortly afterwards, experimental evidence of metamaterial cloaking at microwave frequencies [4] and of optical superlensing [5] was obtained. In the mathematics literature, 'cloaking by anomalous localized resonances' has been intensively studied, see *e.g.,* [6]. Mathematically, some of the counter-intuitive spectral properties of the time-dependent Maxwell system for an interface between a metamaterial and a vacuum are investigated for the non-dissipative case, in a special geometry, in [7,8]. For the dissipative case, in the whole space and in a setting allowing dimension-reduction, we refer to the recent article [9]. Here we consider a more general Drude-Lorentz system

$$\operatorname{curl}\hat{H} = i\omega\left(1 - \frac{(\theta_e)^2}{\omega^2 + i\gamma_e\omega}\right)\hat{E}, \quad -\operatorname{curl}\hat{E} = i\omega\left(1 - \frac{(\theta_m)^2}{\omega^2 + i\gamma_m\omega}\right)\hat{H}, \quad (\nu \times \hat{E})|_{\partial\Omega} = 0,$$

(1.1)

in a bounded or unbounded Lipschitz domain $\Omega \subseteq \mathbb{R}^3$ with outer normal ν. The variable ω is the spectral parameter and θ_e, θ_m are bounded real-valued functions. We describe the essential spectrum and its decomposition into parts connected with the behaviour of the coefficients at infinity and parts due to local dissipative effects. We obtain tight a-priori enclosures for the set in which these different components of the spectrum may lie. Adapting new non-selfadjoint techniques from [10] and [11] to the setting of meromorphic operator-valued functions, we examine how the spectrum behaves under perturbation of the domain Ω. We obtain unexpectedly small enclosures for the sets where spectral pollution [10, Def. 2.2] may appear if an unbounded Ω is approximated by a large, bounded Ω.

We now describe the problem in more detail. Starting from Maxwell's equations

$$\partial_t D = \operatorname{curl} H, \quad \partial_t B = -\operatorname{curl} E, \quad \operatorname{div} D = 0, \quad \operatorname{div} B = 0,$$

relations between (D, B) and (E, H) must be imposed to capture the properties of the medium under consideration, see [12] for an interesting discussion on the diverse constitutive relations and applications to linear bianisotropic media. The Drude-Lorentz model assumes these relations to be given by convolutions

$$D(x, t) = E(x, t) + \int_{t_0}^{t} \chi_e(x, t-s)E(x, s)ds, \quad B(x, t) = H(x, t) + \int_{t_0}^{t} \chi_m(x, t-s)H(x, s)ds.$$

The functions $\chi_e(\cdot, t)$ and $\chi_m(\cdot, t)$ are assumed to be zero for $t < 0$ and are usually described in terms of their Fourier transforms in time; for instance,

$$\hat{\chi}_e(\omega) = -\frac{(\theta_e)^2}{\omega^2 + i\gamma_e\omega} - \sum_{n=1}^{\infty} \frac{(\Omega_n^e)^2}{\omega^2 + i\gamma_n^e\omega - (\lambda_n^e)^2},$$

in which $\lambda_n^e > 0$, $\Omega_n^e \geq 0$, $\gamma_e, \gamma_n^e > 0$, $n \in \mathbb{N} \cup \{0\}$, are constants, and θ_e is some non-negative function. From the equation $\mathrm{curl}\,\hat{H} = i\omega\hat{D} = i\omega(1 + \hat{\chi}_e)\hat{E}$ one then obtains

$$\mathrm{curl}\,\hat{H} = i\omega\left(1 - \frac{(\theta_e)^2}{\omega^2 + i\gamma_e\omega} - \sum_{n=1}^{\infty} \frac{(\Omega_n^e)^2}{\omega^2 + i\gamma_n^e\omega - (\lambda_n^e)^2}\right)\hat{E},$$

together with a corresponding equation for $\mathrm{curl}\,\hat{E}$. In this paper, as in [7], we treat the simplest case, namely the lossy Drude system [13, §6] defined in (1.1).

1.2. Notation

- Let $\Omega \subset \mathbb{R}^3$ be an open set. $L^2(\Omega)^3 = L^2(\Omega, \mathbb{C}^3)$ is the standard Hilbert space of complex-valued vector fields having finite L^2-norm. The L^2-norm will be denoted by $\|\cdot\|$.
- The homogeneous Sobolev or Beppo Levi space $\dot{H}^1(\Omega)$ is defined as the completion of $C_c^\infty(\overline{\Omega})^3$ with respect to the seminorm $\|u\|_{\dot{H}^1(\Omega)} = \|\nabla u\|$.
- $\nabla\dot{H}^1(\Omega) = \{\nabla\varphi \in L^2(\Omega)^3 : \varphi \in \dot{H}^1(\Omega)\}$ will be regarded as a subspace of $L^2(\Omega)^3$.
- $H(\mathrm{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \mathrm{curl}\,u \in L^2(\Omega)^3\}$ is endowed with the norm given by $\|u\|_{H(\mathrm{curl},\Omega)}^2$ $= \|u\|^2 + \|\mathrm{curl}\,u\|^2$.
- $H_0(\mathrm{curl}, \Omega)$ is the closure of $C_c^\infty(\Omega)^3$ with respect to $\|\cdot\|_{H(\mathrm{curl},\Omega)}$. If $\partial\Omega$ is sufficiently regular, it can also be described as

$$H_0(\mathrm{curl}, \Omega) = \{u \in L^2(\Omega)^3 : \mathrm{curl}\,u \in L^2(\Omega)^3, \ \nu \times u = 0 \text{ on } \partial\Omega\}$$

- The differential expression curl is associated with two self-adjoint realisations in $L^2(\Omega)^3$. curl is the maximal one, with domain $\mathrm{dom}(\mathrm{curl}) = H(\mathrm{curl}, \Omega)$; curl_0 the minimal one with domain $\mathrm{dom}(\mathrm{curl}_0) = H_0(\mathrm{curl}, \Omega)$. Note that $(\mathrm{curl}_0)^* = \mathrm{curl}$.
- $H(\mathrm{div}, \Omega) = \{u \in L^2(\Omega)^3 : \mathrm{div}\,u \in L^2(\Omega)\}$, $\|u\|_{H(\mathrm{div},\Omega)}^2 = \|u\|^2 + \|\mathrm{div}\,u\|^2$.
- $H(\mathrm{div}\,0, \Omega)$ is the subspace of $L^2(\Omega)^3$ of vector fields with null (distributional) divergence.
- Given a linear operator $T : \mathcal{H} \supset \mathrm{dom}(T) \to \mathcal{H}$,

$$\sigma(T) = \{\omega \in \mathbb{C} : T - \omega \text{ is not boundedly invertible}\},$$

$$\sigma_{\mathrm{app}}(T) = \{\omega \in \mathbb{C} : \exists(u_n)_n \subset \mathrm{dom}(T), \|u_n\| = 1, \|(T - \omega)u_n\| \to 0\}$$

$$\sigma_e(T) := \{\omega \in \mathbb{C} : \exists(u_n)_n \subset \mathrm{dom}(T), \|u_n\| = 1, u_n \rightharpoonup 0, \|(T - \omega)u_n\| \to 0\}.$$

For non-selfadjoint operators in complex Banach spaces, there are 5 non-equivalent definitions of essential spectrum, see [14, Chp. 9, p.414], denoted by $\sigma_{ek}(T)$, $k = 1, \ldots, 5$. Note that $\sigma_e(T) := \sigma_{e2}(T)$.
- Let $D \subset \mathbb{C}$ be a domain. Given $\omega \mapsto \mathcal{L}(\omega)$, $\omega \in D$, a holomorphic family of closed linear operators with the same ω-independent domain $\mathrm{dom}(\mathcal{L}) = \mathrm{dom}(\mathcal{L}(\omega))$, $\omega \in D$, we define $\sigma(\mathcal{L}) = \{\omega \in D : 0 \in \sigma(\mathcal{L}(\omega))\}$, and similarly we define point, continuous, residual, essential spectrum by replacing σ with σ_x, $x = p, c, r, e$ in the previous formula.

1.3. Operator formulations and main results

The system (1.1) has several operator formulations, which we now outline. The equations hold in a (bounded or unbounded) Lipschitz domain $\Omega \subseteq \mathbb{R}^3$, in which the functions θ_e and θ_m are assumed to be bounded and non-negative. The Fourier transform $\hat{E}$ of the electric field E is supposed to lie in the space $H_0(\mathrm{curl}, \Omega)$, which encodes the boundary condition $\nu \times \hat{E} = 0$ on $\partial\Omega$, while $\hat{H}$ is assumed to lie in $H(\mathrm{curl}, \Omega)$. The first operator formulation of (1.1) is then

$$\mathcal{L}(\omega)\begin{pmatrix} \hat{E} \\ \hat{H} \end{pmatrix} = \mathbf{0},$$

in which $\omega \mapsto \mathcal{L}(\omega)$ is the 2×2 rational block-matrix pencil given by

$$\mathcal{L}(\omega) = \begin{pmatrix} -\omega + \frac{\theta_e^2}{\omega + i\gamma_e} & i\,\mathrm{curl} \\ -i\,\mathrm{curl}_0 & -\omega + \frac{\theta_m^2}{\omega + i\gamma_m} \end{pmatrix}, \quad \mathrm{dom}(\mathcal{L}) = H_0(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega); \quad (1.2)$$

see subsection 1.2 for definitions of the Sobolev spaces, curl, curl_0, etc. It is not difficult to show (see [15]) that the Drude-Lorentz pencil $\mathcal{L}(\omega)$ is the first Schur complement of the 'companion' block operator matrix

$$\mathcal{A} = \begin{pmatrix} A & B \\ B|_{\mathrm{dom}(A)} & -iD \end{pmatrix} \quad (1.3)$$

in $L^2(\Omega)^6$, with domain $\mathrm{dom}(\mathcal{A}) = H_0(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega) \oplus L^2(\Omega)^3 \oplus L^2(\Omega)^3$ and

$$A = \begin{pmatrix} 0 & i\,\mathrm{curl} \\ -i\,\mathrm{curl}_0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \theta_e & 0 \\ 0 & \theta_m \end{pmatrix}, \quad D = \begin{pmatrix} \gamma_e & 0 \\ 0 & \gamma_m \end{pmatrix}; \quad (1.4)$$

in other words,

$$\mathcal{L}(\omega) = A - \omega - B(-iD - \omega)^{-1}B. \quad (1.5)$$

In particular the spectrum of $\mathcal{A}$ coincides with the spectrum of $\mathcal{L}$ outside the two poles $-i\gamma_e, -i\gamma_m$. We will exploit this connection and the results in [11] to further decompose the spectrum of $\mathcal{L}$ into the spectra of two operator pencils. This method allows us to generalise the known spectral analysis of the Drude-Lorentz model in the following ways:

(1) In our assumptions, $0 \in \sigma_e(A)$, where A is defined as in (1.4), since $\nabla \dot{H}_0^1(\Omega) \oplus \nabla \dot{H}^1(\Omega)$ is an infinite-dimensional kernel of A. This is not allowed by many results in the literature, e.g. [15, Proposition 2.2], where it is required that A have compact resolvent.
(2) We allow the domain Ω to be unbounded. Consequently, contributions to $\sigma_e(\mathcal{A})$ are expected from infinity.
(3) We allow the coefficients θ_e, θ_m to be *both* non-constant, even though we assume that they are asymptotically constant.

On the other hand, to avoid very singular situations we restrict ourselves to the case where γ_e and γ_m (namely, the position of the poles) are fixed.

A large part of the spectral analysis has been achieved not by inspecting directly the operator pencil $\mathcal{L}$, but its first Schur complement $\mathcal{S}_1$, defined on $\mathrm{dom}(\mathcal{S}_1) := \{u \in H_0(\mathrm{curl}, \Omega) : \Theta_m(\omega)^{-1} \mathrm{curl}_0 u \in H(\mathrm{curl}, \Omega)\}$ by

$$\mathcal{S}_1(\omega) = \mathrm{curl}\, \Theta_m(\omega)^{-1} \mathrm{curl}_0 - \frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)}, \tag{1.6}$$

for $\omega \in \mathbb{C} \setminus (\{-i\gamma_e, -i\gamma_m\} \cup \overline{W(\Theta_m)})$; here the notation $W(\cdot)$ denotes the numerical range of an operator or a pencil, see Definition 2.3, and

$$\Theta_e(\omega) := \omega^2 + \omega i \gamma_e - \theta_e^2, \quad \Theta_m(\omega) := \omega^2 + \omega i \gamma_m - \theta_m^2, \tag{1.7}$$

are ω-quadratic multiplication pencils. An important technical point is realising that $\mathcal{S}_1(\omega)$, in general, cannot be defined either as an m-accretive operator or as a self-adjoint operator *independently of $\omega \in \mathbb{C} \setminus \{-i\gamma_e, -i\gamma_m\}$*. We overcome this obstacle by introducing a set

$$\Sigma = \{\omega \in \mathbb{C} \mid \mathrm{Re}(\omega)\, \mathrm{Im}(\omega + i\gamma_m/2) \neq 0\}, \tag{1.8}$$

decomposing Σ as a disjoint union $\Sigma = \Sigma_1 \dot\cup \Sigma_2$, and defining $\mathcal{S}_1(\omega)$ in two different ways, depending on whether $\omega \in \Sigma_1$ or in Σ_2. In fact, $i\mathcal{S}_1$ is m-accretive for $\omega \in \Sigma_1 \subset \Sigma$, while it is m-dissipative for $\omega \in \Sigma_2 = \Sigma \setminus \overline{\Sigma_1}$.

Note also that the relation between the spectrum of the operator pencil $\mathcal{L}$ and that of $\mathcal{S}_1$ is completely non-trivial. This is a frequently faced problem in the study of spectra of metamaterials where the dependence on the spectral parameter is non-linear, see e.g. [16] where similar hurdles were encountered in the study of the essential spectrum of a negative metamaterial in a bounded domain. From our perspective, these difficulties are natural consequences of the lack of a *diagonal dominance* pattern (in the sense of [17, Def. 2.2.1]) for the block operator matrices involved. In [16, p.1187] the operator matrix is upper-dominant. In our case, $\mathcal{L}(\omega)$ is off-diagonally dominant, since the off-diagonal entries are differential operators of order 1 while the diagonal entries are of order 0. Unfortunately the off-diagonal entries are not boundedly invertible, so standard theorems relating the spectrum of an operator matrix and that of its Schur complements, such as [17, Thm. 2.3.3], do not apply. We overcome these difficulties by defining the Schur complement $\mathcal{S}_1$ locally and by improving the abstract result [17, Prop. 2.10.1(c)], which would allow only bounded and self-adjoint entries on the main diagonal. We note in passing that the question raised in [16, p.1187] can be partially solved by applying [17, Prop. 2.10.1(b)].

Our first main result is a decomposition of the essential spectrum, see Proposition 2.12 and Theorem 3.7, which is summarised in the following theorem.

Theorem 1.1. *Suppose that Ω is an unbounded Lipschitz open set of $\mathbb{R}^3$ and θ_e, θ_m are asymptotically constant. Let P_∇ be the orthogonal projection from $L^2(\Omega)^3 = \nabla \dot{H}_0^1(\Omega) \oplus H(\mathrm{div}\, 0, \Omega)$ onto $\nabla \dot{H}_0^1(\Omega)$. Let*

$$G(\omega) = -P_\nabla \left(\frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \right) P_\nabla \tag{1.9}$$

Let $\mathcal{S}_\infty$ be the pencil $\mathcal{S}_1$ restricted to divergence-free vector fields and with coefficients θ_e, θ_m constantly equal to their value at infinity. Finally, let Σ be as in (1.8). Then, with σ_{ek} denoting the essential spectrum as in [14, Chp. IX, p.414],

$$\sigma_{ek}(\mathcal{L}) \cap \Sigma = \sigma_{ek}(\mathcal{S}_1) \cap \Sigma = (\sigma_{ek}(\mathcal{S}_\infty) \cup \sigma_{ek}(G)) \cap \Sigma, \quad k = 1, 2, 3, 4, \tag{1.10}$$

where $\sigma_{ek}(\mathcal{S}_\infty)$ is described in Proposition 4.1 and

$$\sigma_{ek}(G) \subset \begin{cases} -i[0, \gamma_e), & \text{if } -\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty \leq 0, \\ -i[0, \gamma_e) \cup \left(-d_e - i\frac{\gamma_e}{2}, d_e - i\frac{\gamma_e}{2}\right) & \text{if } -\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty > 0, \end{cases}$$

with $d_e \in \left(-\sqrt{-\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty}, \sqrt{-\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty}\right)$.

Remark 1.2. In the case where Ω is bounded, decomposition (1.10) continues to hold. In such a case $\mathcal{S}_\infty$ can be defined by freezing the coefficients θ_e, θ_m at arbitrary positive values θ_e^0, θ_m^0; in fact, $\sigma_e(\mathcal{S}_\infty) = \emptyset$ independently of the chosen values θ_e^0, θ_m^0. This can be proved along the lines of [18, Theorem 6]. Therefore, in bounded domains the only contribution to the essential spectrum comes from $\sigma_e(G)$. See also Remark 3.8.

The explicit computation of $\sigma_{ek}(G)$ generally depends upon the regularity of the function θ_e. If it is continuous, we obtain the equality

$$\sigma_{ek}(G) = \text{ran}\left(-\frac{i\gamma_e}{2} - \sqrt{-\frac{\gamma_e^2}{4} + \theta_e^2}\right) \cup \text{ran}\left(-\frac{i\gamma_e}{2} + \sqrt{-\frac{\gamma_e^2}{4} + \theta_e^2}\right).$$

Otherwise, the essential spectrum depends on the geometry of the set of discontinuities of θ_e. If θ_e is a step function, some computations can be found in Section 6 below, which are based on the analytic results for transmission problems of Ola [19] and Pankrashkin [20], initially investigated in the seminal paper [21]. In the example of Section 6, $\sigma_{ek}(G)$ consists of at most six distinct points. For more complicated examples where the discontinuity interfaces are allowed to have non-convex corners, bands of essential spectrum can be generated, see [22]. These problems have been tackled recently using the T-coercivity method, see [23].

Our second fundamental result concerns spectral approximation of $\mathcal{L}$ by the truncation method. This involves replacing Ω with a bounded domain $\Omega_n \subset \Omega$, and $\mathcal{L}$ with $\mathcal{L}_n$, which will be associated with Problem (1.2) with the same *electric boundary conditions*; when $n \to \infty$, Ω_n monotonically increases and covers the whole of Ω. The question is whether $\sigma(\mathcal{L}_n)$ will be 'close to' $\sigma(\mathcal{L})$ as $n \to \infty$. This is already an interesting problem for self-adjoint operators having band-gap spectrum [24]: indeed, the gaps in the essential spectrum may contain eigenvalues of the approximating operators as $n \to \infty$, or equivalently, the spectral gaps may contain points of the *spectral pollution* set, given by

$$\sigma_{\text{poll}}((\mathcal{L}_n)_n) = \{\omega \in \varrho(\mathcal{L}) : \exists I \subset \mathbb{N}, \ I \text{ infinite}, \ \omega_n \in \sigma(\mathcal{L}_n), \ \omega_n \to \omega, \ n \in I, \ n \to \infty\}.$$

For the Maxwell pencil $\mathcal{L}$ defined in (1.5), the presence of spectral pollution for the approximating sequence $(\mathcal{L}_n)_n$ is almost inevitable, since $\mathcal{L}_n$ is a non-self-adjoint, rational pencil of

operators for every n, having non-trivial essential spectrum even in bounded domains. It is then of pivotal importance to determine where spectral pollution may appear; and, on the other hand, which spectral points $\omega \in \sigma(\mathcal{L})$ can be approximated exactly via domain truncation. Theorem 5.13 shows that spectral pollution for the sequence $\mathcal{L}_n$, $n \in \mathbb{N}$ can only occur in the *essential numerical range* $W_e(\mathcal{S}_\infty)$ of the constant coefficient pencil $\mathcal{S}_\infty$ given on a suitable domain by

$$\mathcal{S}_\infty(\omega) = \Theta_{m,\infty}(\omega)^{-1} \operatorname{curl} \operatorname{curl}_0 - \frac{\Theta_{e,\infty}(\omega)}{(i\gamma_e + \omega)(i\gamma_m + \omega)}, \tag{1.11}$$

in which $\Theta_{e,\infty}(\Omega) = \omega^2 + i\gamma_e\omega - (\theta_e^0)^2$, $\Theta_{m,\infty}(\Omega) = \omega^2 + i\gamma_m\omega - (\theta_m^0)^2$, and θ_e^0, θ_m^0 are the values of θ_e and θ_m at infinity. We recall that

$$W_e(\mathcal{S}_\infty) = \{\omega \in \mathbb{C} : \exists u_n \in \operatorname{dom}(\mathcal{S}_\infty(\omega)), \, n \in \mathbb{N},$$

$$\|u_n\| = 1, \, u_n \rightharpoonup 0, \, (\mathcal{S}_\infty(\omega)u_n, u_n) \to 0, \, n \to \infty\}.$$

In particular, the set of spectral pollution is always contained in the union of one-dimensional curves in $\mathbb{C}$, improving in a substantial way abstract enclosures for the spectral pollution set in term of the essential numerical range of $\mathcal{L}$, which in this case would establish only that spectral pollution is contained in the infinite horizontal strip $\mathbb{R} \times [-\gamma_e, 0]$.

The structure of this article is as follows. In Section 2 we first establish a basic numerical range enclosure for the whole of $\sigma(\mathcal{L})$, see Proposition 2.5; we then define the operator pencil $\mathcal{S}_1$ in a rigorous way in Theorem 2.6 and we prove that the spectral properties of $\mathcal{L}$ are retained by $\mathcal{S}_1$ inside Σ, see Proposition 2.12. This result is then exploited to prove a refined numerical range enclosure, see Theorem 2.13. In Section 3 we prove Theorem 3.7, which establishes that $\sigma_e(\mathcal{S}_1) \cap \Sigma$ can be decomposed in the union of the essential spectra of two operator pencils, $\mathcal{S}_\infty$ capturing the behaviour at infinity due to divergence-free vector fields; G capturing the contribution of gradients. Section 4 contain qualitative results regarding the essential spectra of the reduced operators $\mathcal{S}_\infty$ and G. In Section 5 we then prove Theorem 5.13, establishing that spectral pollution for the domain truncation method is contained in $W_e(\mathcal{S}_\infty)$, and an approximation property for the isolated eigenvalues of $\mathcal{L}$. Finally, Section 6 contains explicit computations for the case of locally constant functions $\theta_e(x) = \alpha_e \chi_K(x)$, $\theta_m(x) = \alpha_m \chi_K(x)$, which are identically zero at infinity.

2. Numerical range, Schur complements and spectral enclosures

Let $\mathcal{H} = L^2(\Omega)^3 \oplus L^2(\Omega)^3$. The operators A, B and D appearing in the definition (1.3) of $\mathcal{A}$ have domains $\operatorname{dom}(A) = H_0(\operatorname{curl}, \Omega) \oplus H(\operatorname{curl}, \Omega)$, $\operatorname{dom}(B) = \operatorname{dom}(D) = \mathcal{H}$; the fact that $\operatorname{dom}(B) = \operatorname{dom}(D) = \mathcal{H}$ relies on our assumption that the functions θ_e and θ_m lie in $L^\infty(\Omega, \mathbb{R})$. Since the off-diagonal operators B and D are bounded, $\mathcal{A}$ is a diagonally dominant, closed $\mathcal{J}$-self-adjoint operator matrix, where $\mathcal{J} = \operatorname{diag}(i, -i, i, -i)J$, and J is the standard componentwise complex conjugation. In particular, $\sigma(\mathcal{A}) = \sigma_{app}(\mathcal{A})$. Due to [17, Thm. 2.3.3 (ii)], $\sigma(\mathcal{A}) \setminus \sigma(-iD) = \sigma(\mathcal{L})$, with equality for the point, continuous, and essential spectrum as well.

Proposition 2.1. *Let $M := \max\{\gamma_e, \gamma_m\}$. Then the numerical range $W(\mathcal{A})$ of the block operator matrix $\mathcal{A}$ is contained in the strip $\mathbb{R} \times [-iM, 0]$.*

Proof. If $\omega \in W(\mathcal{A})$, by definition there exists $(u, v) \in \mathcal{H}$, $\|u\|^2 + \|v\|^2 = 1$ such that

$$(Au, u) + 2\operatorname{Re}(Bv, u) - i(Dv, v) = \omega.$$

Hence, $0 \geq \operatorname{Im}\omega = -\operatorname{Re}(Dv, v) = -\gamma_e \|v_1\|^2 - \gamma_m \|v_2\|^2 \geq -\max\{\gamma_e, \gamma_m\}$. $\quad\square$

Remark 2.2. The previous enclosure holds independently on the domain Ω and it holds for non-constant, positive, and bounded γ_e and γ_m by replacing them with $\|\gamma_e\|_\infty$ and $\|\gamma_m\|_\infty$.

On the other hand, $\mathbb{C} \setminus \{-i\gamma_e, -i\gamma_m\} \ni \omega \mapsto \mathcal{L}(\omega)$ defines a pencil of block operator matrices in $\mathcal{H}$, given explicitly by

$$\mathcal{L}(\omega) = \begin{pmatrix} -\omega + \frac{\theta_e^2}{(\omega + i\gamma_e)} & i\operatorname{curl} \\ -i\operatorname{curl}_0 & -\omega + \frac{\theta_m^2}{(\omega + i\gamma_m)} \end{pmatrix} \tag{2.1}$$

where $\operatorname{dom}(\mathcal{L}(\omega)) = H_0(\operatorname{curl}, \Omega) \oplus H(\operatorname{curl}, \Omega)$.

Definition 2.3. Given a linear operator T with domain $\operatorname{dom}(T) \subset H$ on a Hilbert space H, the numerical range of T is

$$W(T) = \{\omega \in \mathbb{C} : \exists u \in \operatorname{dom}(T), \ \|u\| = 1, \ (Tu, u) = \omega\}.$$

Let $\mathcal{T}$ be a pencil of linear operators $\mathcal{T}(\omega)$ with $\operatorname{dom}(\mathcal{T}(\omega)) \subset H$. We define

$$W(\mathcal{T}) = \{\omega \in \mathbb{C} : 0 \in \overline{W(\mathcal{T}(\omega))}\}.$$

Remark 2.4. Note that $W(\mathcal{T})$ is denoted by $W_\Psi(\mathcal{T})$ in the recent article [25, Equation (1.2)], *cf.* [26]. The main reason to use this set in place of the classical one $\widetilde{W}(\mathcal{T}) = \{\omega \in \mathbb{C} : 0 \in W(\mathcal{T}(\omega))\}$ is that in general $\sigma_{\operatorname{app}}(\mathcal{T})$ is not a subset of the closure of $\widetilde{W}(\mathcal{T})$. Instead, it is immediate to check that $\sigma_{\operatorname{app}}(\mathcal{T}) \subset W(\mathcal{T})$.

Proposition 2.5. *The numerical range $W(\mathcal{L})$ is contained in the non-convex subset of $\mathbb{C}$ described by the inequality*

$$0 \leq -\operatorname{Im}\omega \leq \min\left(M, \frac{\gamma_e \|\theta_e\|_\infty^2 + \gamma_m \|\theta_m\|_\infty^2}{(\operatorname{Re}\omega)^2}\right), \tag{2.2}$$

in which $M = \max\{\gamma_e, \gamma_m\}$.

Proof. Let $\omega \in W(\mathcal{L})$; then by definition $0 \in \overline{W(\mathcal{L}(\omega))}$. Let $(u_n, v_n) \in H_0(\operatorname{curl}, \Omega) \oplus H(\operatorname{curl}, \Omega)$, $\|u_n\|^2 + \|v_n\|^2 = 1$, $n \in \mathbb{N}$ and consider the equation

$$-\omega + \left(\frac{\theta_e^2}{(\omega + i\gamma_e)} u_n, u_n\right) + 2\operatorname{Re}(i\operatorname{curl} v_n, u_n) + \left(\frac{\theta_m^2}{(\omega + i\gamma_m)} v_n, v_n\right) = \varepsilon_n$$

with $\varepsilon_n \in \mathbb{C}$, $\varepsilon_n \to 0$ as $n \to \infty$. Upon taking the imaginary part we see that

$$-\operatorname{Im}\omega - \left(\frac{(\operatorname{Im}\omega + \gamma_e)}{|\omega + i\gamma_e|^2}\theta_e^2 u_n, u_n\right) - \left(\frac{(\operatorname{Im}\omega + \gamma_m)}{|\omega + i\gamma_m|^2}\theta_m^2 v_n, v_n\right) = \operatorname{Im}\varepsilon_n. \qquad (2.3)$$

If $\operatorname{Im}\omega > 0$, then $\operatorname{Im}(\omega + \varepsilon_n) \le 0$; this gives a contradiction for $n \to \infty$, hence $\operatorname{Im}\omega \le 0$. Similarly, it cannot happen that both $\operatorname{Im}\omega + \gamma_e < 0$ and $\operatorname{Im}\omega + \gamma_m < 0$; hence $-\max\{\gamma_e, \gamma_m\} \le \operatorname{Im}\omega \le 0$, i.e. $-M \le \operatorname{Im}\omega \le 0$. To obtain (2.2), we observe that (2.3) may be rewritten as

$$(-\operatorname{Im}\omega)\left\{1 + \frac{(\theta_e^2 u_n, u_n)}{|\omega + i\gamma_e|^2} + \frac{(\theta_m^2 v_n, v_n)}{|\omega + i\gamma_m|^2}\right\}$$

$$= \frac{\gamma_e(\theta_e^2 u_n, u_n)}{(\operatorname{Re}\omega)^2 + (\gamma_e + \operatorname{Im}\omega)^2} + \frac{\gamma_m(\theta_m^2 v_n, v_n)}{(\operatorname{Re}\omega)^2 + (\gamma_m + \operatorname{Im}\omega)^2} + \operatorname{Im}\varepsilon_n$$

$$\le \frac{\gamma_e\|\theta_e\|_\infty^2 + \gamma_m\|\theta_m\|_\infty^2}{(\operatorname{Re}\omega)^2} + \operatorname{Im}\varepsilon_n$$

The factor in parentheses $\{\cdot\}$ on the left hand side exceeds 1, so the result follows by taking the limit as $n \to \infty$. $\square$

In order to make further progress we use an additional Schur complement argument on the pencil $\mathcal{L}(\omega)$, which can be considered as a block operator matrix in $\mathcal{H} = L^2(\Omega)^3 \oplus L^2(\Omega)^3$. In terms of the bounded quadratic pencils Θ_e and Θ_m, see (1.7), $\mathcal{L}(\omega)$ has the form

$$\mathcal{L}(\omega) = \begin{pmatrix} -\frac{\Theta_e(\omega)}{(i\gamma_e + \omega)} & i\operatorname{curl} \\ -i\operatorname{curl}_0 & -\frac{\Theta_m(\omega)}{(i\gamma_m + \omega)} \end{pmatrix}.$$

If ω is an eigenvalue of $\mathcal{L}$ with eigenfunction $\binom{E}{H}$, then

$$\begin{cases} -\frac{\Theta_e(\omega)}{(i\gamma_e + \omega)}E + i\operatorname{curl} H = 0, \\ -i\operatorname{curl}_0 E - \frac{\Theta_m(\omega)}{(i\gamma_m + \omega)}H = 0. \end{cases}$$

Assume that $\Theta_m(\omega)$ is boundedly invertible. Formally, we could apply $\Theta_m(\omega)^{-1}$ from the left in the second equation and apply curl; by using the first equation we obtain

$$\operatorname{curl}\Theta_m(\omega)^{-1}\operatorname{curl}_0 E - \frac{\Theta_e(\omega)}{(i\gamma_e + \omega)(i\gamma_m + \omega)}E = 0$$

for all $\omega \in \varrho(\Theta_m)$. However, without further restrictions on ω the operator $\operatorname{curl}\Theta_m(\omega)^{-1}\operatorname{curl}_0$ may not even be accretive.

Theorem 2.6. *Let Σ_1 be the set*

$$\Sigma_1 := \{\omega \in \mathbb{C} \mid \operatorname{Re}(\omega)\operatorname{Im}(\omega + i\gamma_m/2) > 0\}.$$

Then for each $\omega \in \Sigma_1$, the sesquilinear form

$$\mathfrak{s}_1(\omega)[x, y] = (\Theta_m(\omega)^{-1}\operatorname{curl}_0 x, \operatorname{curl}_0 y) - \left(\frac{\Theta_e(\omega)}{(\omega + i\gamma_m)(\omega + i\gamma_e)}x, y\right)$$

is such that $i\mathfrak{s}_1(\omega)$ is quasi-accretive. Similarly, defining

$$\Sigma_2 := \{\omega \in \mathbb{C} \mid \mathrm{Re}(\omega)\,\mathrm{Im}(\omega + i\gamma_m/2) < 0\},$$

if $\omega \in \Sigma_2$ then $-i\mathfrak{s}_1(\omega)$ is quasi-accretive.

Proof. Let $\mathfrak{a}(\omega)$ be the sesquilinear form

$$\mathfrak{a}(\omega)(x, y) = \left(\frac{\Theta_e(\omega)}{(\omega + i\gamma_m)(\omega + i\gamma_e)}x, y \right).$$

Let $\mathfrak{t}(\omega)(x, y) = (\Theta_m(\omega)^{-1}\,\mathrm{curl}_0\, x, \mathrm{curl}_0\, y)$, $x, y \in H_0(\mathrm{curl}, \Omega)$. Since $\mathfrak{a}(\omega)$ is bounded for fixed $\omega \in \Sigma_1$, we see that $e^{i\phi}\mathfrak{a}$ is closed and quasi-accretive with domain $\mathcal{H}$ for all $\phi \in [0, 2\pi)$. Assume that we have already proved that $i\mathfrak{t}(\omega)$ is quasi-accretive for ω lying in the set Σ_1. By [27, Thm. VI.1.27, VI.1.31] the sum $i\mathfrak{s}_1(\omega) = i\mathfrak{a}(\omega) + i\mathfrak{t}(\omega)$ is closed and quasi-accretive on $\mathrm{dom}(\mathfrak{t}(\omega))$.

Hence, it is sufficient to show that if $\omega \in \Sigma_1$ then $i\mathfrak{t}(\omega)$ is quasi-accretive. Equivalently, we must show that $W(\mathfrak{t}(\omega)) \subset \mathbb{H}_- = \{z \in \mathbb{C} : \mathrm{Im}\, z \le 0\}$ for all $\omega \in \Sigma_1$. We note that

$$(\Theta_m(\omega)^{-1}u, u) = \frac{\bar{\omega}^2 - i\gamma_m\bar{\omega} - (\theta_m^2 u, u)}{|\omega^2 + i\gamma_m\omega - (\theta_m^2 u, u)|^2}$$

for every $u \in \mathcal{H}$, $\|u\| = 1$. Now

$$\bar{\omega}^2 - i\gamma_m\bar{\omega} - (\theta_m^2 u, u) = \left(\bar{\omega} - \frac{i\gamma_m}{2} \right)^2 + \frac{\gamma_m^2}{4} - (\theta_m^2 u, u).$$

Suppose first that $\omega \in \Sigma_1$ and $\mathrm{Re}(\omega) > 0$. Then $\omega = -i\gamma_m/2 + re^{i\phi}$, for some $r > 0$, $\phi \in (0, \pi/2)$. (The case where $\omega \in \Sigma_1$ with $\mathrm{Re}(\omega) < 0$ follows from the previous case and a reflection argument with respect to the point $-i\gamma_m/2$.) It follows that

$$\left(\bar{\omega} - \frac{i\gamma_m}{2} \right)^2 + \frac{\gamma_m^2}{4} - (\theta_m^2 u, u) = r^2 e^{-2i\phi} + \frac{\gamma_m^2}{4} - (\theta_m^2 u, u)$$

and upon taking the imaginary part, $\mathrm{Im}(r^2 e^{-2i\phi} + \frac{\gamma_m^2}{4} - (\theta_m^2 u, u)) = \sin(-2\phi) < 0$. Thus,

$$\begin{aligned}
\mathrm{Im}\,\mathfrak{t}(\omega)[x] &= \mathrm{Im}(\Theta_m(\omega)^{-1}\,\mathrm{curl}_0\, x, \mathrm{curl}_0\, x) \\
&\le r^2 \sin(-2\phi)\left(\frac{1}{|\omega^2 + i\gamma_m\omega - \theta_m^2|^2}\,\mathrm{curl}_0\, x, \mathrm{curl}_0\, x \right) \le 0.
\end{aligned} \tag{2.4}$$

So $W(\mathfrak{t}(\omega)) \subset \mathbb{H}_-$ for all $\omega \in \Sigma_1$ and $i\mathfrak{t}(\omega)$ is accretive for all $\omega \in \Sigma_1$. The final claim follows by noting that when $\omega \in \Sigma_2$, $\mathrm{Im}(\mathfrak{t}(\omega)[x]) \ge 0$, by reversing all the inequalities in (2.4). $\quad\square$

Remark 2.7. Theorem 2.6 shows that $\mathfrak{s}_1(\omega)$ has a well-defined m-accretive representation via the first representation theorem for all $\omega \in \mathbb{C} \setminus (i\mathbb{R} \cup (-i\gamma_m/2 + \mathbb{R}))$. We note *en passant* that the singular sets $\{-i\gamma_e, -i\gamma_m\}$ and $W(\Theta_m)$ are both contained in $\mathbb{C} \setminus \Sigma = (i\mathbb{R} \cup (-i\gamma_m/2 + \mathbb{R}))$.

Corollary 2.8. *Let Σ_1 and Σ_2 be as in Theorem 2.6. For every $\omega \in \Sigma_1$, there exists an operator $\mathcal{S}_1(\omega)$ such that $i\mathcal{S}_1(\omega)$ is m-accretive and*

$$i(\mathcal{S}_1(\omega)x, y) = i\mathfrak{s}_1(\omega)[x, y]$$

for all $x \in \mathrm{dom}(S_1)$, $y \in \mathrm{dom}(\mathfrak{s}_1)$, and $\mathrm{dom}(\mathcal{S}_1)$ is a core of $\mathrm{dom}(\mathfrak{s}_1)$. Similarly for $\omega \in \Sigma_2$ there exists an operator $\mathcal{S}_1(\omega)$ such that $-i\mathcal{S}_1(\omega)$ is m-accretive and

$$-i(\mathcal{S}_1(\omega)x, y) = -i\mathfrak{s}_1(\omega)[x, y].$$

Remark 2.9. The operator $\mathcal{S}_1$ defined in Corollary 2.8 is given, for $\omega \in \Sigma := \Sigma_1 \cup \Sigma_2$ defined in (1.8), by

$$\mathcal{S}_1(\omega) = \mathrm{curl}(\Theta_m(\omega)^{-1}\,\mathrm{curl}_0\,\cdot) - \frac{\Theta_e(\omega)}{(i\gamma_m + \omega)(i\gamma_e + \omega)}. \tag{2.5}$$

From (1.7), we see that $\Theta_m(it) < 0$ for all sufficiently large $t \in \mathbb{R}$, and for all $t \geq 0$. Since $\Theta_e(it) < 0$ for all $t \in i\mathbb{R}_+$, $S_1(\omega)$ is self-adjoint and negative for $\omega \in i\mathbb{R}_+$.

Proposition 2.10. *The following properties hold.*

(i) $\sigma(\mathcal{A}) = -\overline{\sigma(\mathcal{A})}$.
(ii) $\sigma(\mathcal{L}) \setminus \{-i\gamma_e, -i\gamma_m\} = -\overline{\sigma(\mathcal{L})} \setminus \{-i\gamma_e, -i\gamma_m\}$.
(iii) $\sigma(\mathcal{S}_1) \cap \Sigma = -\overline{\sigma(\mathcal{S}_1) \cap \Sigma}$

Proof. (i) follows from the equality $\mathcal{A} = Q\mathcal{A}_c Q^{-1}$ for $Q = \mathrm{diag}(-i, -i, 1, 1)$, and $\mathcal{A}_c = -\overline{\mathcal{A}_c}$. (ii) then follows from (i) due to the equality $\sigma(\mathcal{L}) \setminus \sigma(-iD) = \sigma(\mathcal{A}) \setminus \sigma(-iD)$ and the fact that $\sigma(-iD)$ is invariant to the symmetry $\zeta \mapsto -\overline{\zeta}$. (iii) now follows from (ii) in a similar fashion. $\quad\square$

Notation. Define multiplication operators in $L^2(\Omega)^3$ by

$$\begin{aligned}
V_m(\omega) &= \frac{\Theta_m(\omega)}{(\omega + i\gamma_m)}, \quad \omega \neq -i\gamma_m; \\
V_e(\omega) &= -\frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)}, \quad \omega \notin \{-i\gamma_m, -i\gamma_e\}.
\end{aligned} \tag{2.6}$$

Lemma 2.11. *Let $\omega \in \Sigma$. Then $\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}$ is closed and bounded in $L^2(\Omega)^3$; $\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}$ and $\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}$ are closable with bounded closure as operators in $L^2(\Omega)^3$.*

Proof. Since $\mathrm{dom}(\mathcal{S}_1(\omega)) \subset H_0(\mathrm{curl}, \Omega) = \mathrm{dom}(\mathrm{curl}_0)$ we have immediately that $\overline{\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}}$ is closed and bounded, $\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}$ is closable with bounded closure given by $\overline{\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}} = (\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-*})^*$. It therefore remains to prove that $\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}$ is closable and bounded. We will prove that $\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}$ can be extended as a bounded operator in $L^2(\Omega)^3$, therefore implying that it is closable.

Let $B < 0$ be bounded and self-adjoint with the property that $\mathrm{Im}(V_e(\omega) + iB) < 0$, where $V_e(\omega)$ is the multiplication operator defined in (2.6). Then $\mathrm{Im}(\mathcal{S}_1(\omega) + iB) < 0$ for $\omega \in \Sigma_1$ as a consequence of Theorem 2.6. The Lax-Milgram theorem implies immediately that $\mathcal{S}_1(\omega) + iB$

is boundedly invertible in $L^2(\Omega)^3$. However, a further inspection shows that if u is the weak solution of $(\mathcal{S}_1(\omega) + iB)u = \operatorname{curl} g$, $g \in L^2(\Omega)^3$, i.e.,

$$\langle(\Theta_m(\omega))^{-1} \operatorname{curl}_0 u, \operatorname{curl}_0 v\rangle + \langle(V_e(\omega) + iB)u, v\rangle = \langle g, \operatorname{curl}_0 v\rangle, \quad v \in (C_c^\infty(\Omega))^3,$$

then for every $\delta < 1$ we have

$$c_1(\omega)\|\operatorname{curl}_0 u\|^2 + c_2(\omega)\|u\|^2 \le \frac{\|g\|^2}{4\delta} + \delta\|\operatorname{curl}_0 u\|^2,$$

in which

$$c_1(\omega) = \operatorname{ess\,inf}_{x\in\Omega} |\operatorname{Im}\Theta_m(\omega, x)^{-1}|$$

$$= |\operatorname{Re}\omega||2\operatorname{Im}\omega + \gamma_m| \operatorname{ess\,inf}_{x\in\Omega}\left(\frac{1}{|\omega^2 + i\gamma_m\omega - \theta_m^2(x)|^2}\right),$$

$$c_2(\omega) = \inf_{u\in L^2(\Omega)^3} \frac{|\operatorname{Im}(\langle(V_e(\omega) + iB)u, u\rangle)|}{\|u\|^2} > 0.$$

From this we deduce that $u \in H_0(\operatorname{curl}, \Omega)$, hence $(\mathcal{S}_1(\omega) + iB)^{-1}$ maps $\operatorname{curl} L^2(\Omega)^3$ to $H_0(\operatorname{curl}, \Omega)$, or equivalently $\operatorname{curl}_0(\mathcal{S}_1(\omega) + iB)^{-1} \operatorname{curl}$ has bounded closure in $L^2(\Omega)^3$.

Now let $\omega \in \varrho(\mathcal{S}_1) \cap \Sigma_1$. Then $\mathcal{S}_1(\omega)^{-1}$ is a bounded operator in $L^2(\Omega)^3$ and we have the resolvent identity

$$\mathcal{S}_1(\omega)^{-1} = (\mathcal{S}_1(\omega) + iB)^{-1} + i\mathcal{S}_1(\omega)^{-1} B(\mathcal{S}_1(\omega) + iB)^{-1}$$

and then

$$\mathcal{S}_1(\omega)^{-1}\operatorname{curl} = (\mathcal{S}_1(\omega) + iB)^{-1}\operatorname{curl} + i\mathcal{S}_1(\omega)^{-1} B(\mathcal{S}_1(\omega) + iB)^{-1}\operatorname{curl}$$

is bounded since so is the right-hand side, due to the previous discussion. Now, if $u \in L^2(\Omega)^3$ is the weak solution of

$$\mathcal{S}_1(\omega)u = \operatorname{curl}(\Theta_m(\omega))^{-1}\operatorname{curl}_0 u + V_e(\omega)u = \operatorname{curl} g$$

for some $g \in L^2(\Omega)^3$, $V_e(\omega)u \in L^2(\Omega)^3$ implies $\operatorname{curl}\Theta_m(\omega)^{-1}\operatorname{curl}_0 u \in L^2(\Omega)^3$; then, we may immediately conclude that $u \in \operatorname{dom}(\operatorname{curl}\Theta_m(\omega)^{-1}\operatorname{curl}_0) \subset H_0(\operatorname{curl}, \Omega)$. In particular, $\mathcal{S}_1(\omega)^{-1}\operatorname{curl}$ is bounded as an operator from $L^2(\Omega)^3$ to $H_0(\operatorname{curl}, \Omega)$, concluding the proof.   □

Proposition 2.12. $\sigma(\mathcal{L}) \cap \Sigma = \sigma(\mathcal{S}_1) \cap \Sigma$ and $\sigma_x(\mathcal{L}) \cap \Sigma = \sigma_x(\mathcal{S}_1) \cap \Sigma$, where $x \in \{p, c, r, e\}$, *denoting point, continuous, residual, and essential spectrum, respectively.*

Proof. The proof is along the lines of [17, Prop. 2.10.1(c)]. Without loss of generality, we consider only $\omega \in \Sigma_1$. Note that $\mathcal{L}(\omega)$ is the sum of a self-adjoint operator and a $\mathcal{J}$-self-adjoint bounded operator, where $\mathcal{J} = \operatorname{diag}(i, -i)J$, J being the standard complex conjugation; hence, it is easy to check that $\mathcal{L}(\omega)$ is $\mathcal{J}$-self-adjoint for all $\omega \in \Sigma_1$ and $\operatorname{dom}(\mathcal{L}(\omega))$ does not depend

on $\omega \in \Sigma_1$ and it is given by $H_0(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega)$. Due to Theorem 2.6, $i\mathcal{S}_1(\omega)$, $\omega \in \Sigma_1$ is a well-defined m-accretive operator associated with the sesquilinear form $i\mathfrak{s}_1(\omega)$, via the first representation theorem. In particular, $\mathrm{dom}(\mathcal{S}_1(\omega)) \subset H_0(\mathrm{curl}, \Omega)$ does not depend on $\omega \in \Sigma_1$ and it is a core for $H_0(\mathrm{curl}, \Omega)$.

We will first prove that $\sigma(\mathcal{S}_1) \cap \Sigma_1 \subset \sigma(\mathcal{L}) \cap \Sigma_1$ and that $\sigma_p(\mathcal{S}_1) \cap \Sigma_1 = \sigma_p(\mathcal{L}) \cap \Sigma_1$. If $f \in L^2(\Omega)^3$ and $\omega \in \varrho(\mathcal{L}) \cap \Sigma_1$, then the solution u to the equation $\mathcal{L}(\omega)(u, v)^t = (f, 0)^t$, $v := -i(\omega + i\gamma_m)\Theta_m(\omega)^{-1} \mathrm{curl}_0 u$, is in one-to-one correspondence with the solution of $\mathcal{S}_1(\omega)u = f$, therefore implying that $\omega \in \varrho(\mathcal{S}_1)$. This also proves that $\sigma_p(\mathcal{S}_1) \cap \Sigma_1 = \sigma_p(\mathcal{L}) \cap \Sigma_1$ by arguing in a similar way for $f = 0$.

We now prove that $\sigma(\mathcal{S}_1) \cap \Sigma_1 \supset \sigma(\mathcal{L}) \cap \Sigma_1$. Assume that $\omega \in \varrho(\mathcal{S}_1) \cap \Sigma_1$. Then, at least on $L^2(\Omega)^3 \oplus V_m(\omega)H(\mathrm{curl}, \Omega)$, after recalling (2.6), we can write the equality

$$\mathcal{L}(\omega)^{-1} = \begin{pmatrix} \mathcal{S}_1(\omega)^{-1} & -\mathcal{S}_1(\omega)^{-1}i\,\mathrm{curl}\,V_m(\omega)^{-1} \\ V_m(\omega)^{-1}i\,\mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1} & V_m(\omega)^{-1}(I + \mathrm{curl}_0\,\mathcal{S}_1(\omega)^{-1}\,\mathrm{curl}(V_m(\omega))^{-1}) \end{pmatrix} \quad (2.7)$$

so that $\mathcal{S}_1(\omega) = T(\omega) + V_e(\omega)$, $T(\omega) = \mathrm{curl}\,\Theta_m(\omega)^{-1}\,\mathrm{curl}_0$. First note that $L^2(\Omega)^3 \oplus V_m(\omega)H(\mathrm{curl}, \Omega)$ is dense in $L^2(\Omega)^3 \oplus L^2(\Omega)^3$, whenever $\omega \in \Sigma_1$. Hence, it suffices to prove that the right-hand side in (2.7) has bounded closure as an operator in $L^2(\Omega)^3 \oplus L^2(\Omega)^3$. This is an immediate consequence of Lemma 2.11, concluding therefore the proof of the inclusion.

It remains to prove that $\sigma_e(\mathcal{L}) \cap \Sigma = \sigma_e(\mathcal{S}_1) \cap \Sigma$. Due to the previous part of the proof, it is enough to show that $\sigma_p(\mathcal{L}) \cap \sigma_e(\mathcal{L}) \cap \Sigma = \sigma_p(\mathcal{S}_1) \cap \sigma_e(\mathcal{S}_1) \cap \Sigma$. First, we note that $\mathcal{L}(\omega)$ is $\mathcal{J}$-selfadjoint with respect to $\mathcal{J} = \mathrm{diag}(i, -i)J$ and $\mathcal{S}_1(\omega)$ is J-self-adjoint, J being the standard complex conjugation; therefore, [14, Theorem IX.1.6] implies $\sigma_{e1}(\mathcal{L}(\omega)) = \cdots = \sigma_{e4}(\mathcal{L}(\omega))$ (and similarly for $\mathcal{S}_1(\omega)$). We will first show that $\sigma_{e2}(\mathcal{S}_1)) \cap \Sigma \subset \sigma_{e2}(\mathcal{L})) \cap \Sigma$. Let $\omega \in \sigma_{e2}(\mathcal{S}_1)) \cap \Sigma$ and let u_n be a Weyl singular sequence in $\mathrm{dom}(\mathcal{S}_1)$ such that $\mathcal{S}_1(\omega)u_n \to 0$. Then, by setting $v_n = -(\omega + i\gamma_m)(\Theta_m(\omega))^{-1}\,\mathrm{curl}\,u_n$ and $h_n = (u_n, v_n)/(\|u_n\|^2 + \|v_n\|^2)^{1/2}$ we have $\mathcal{L}(\omega)h_n = \|h_n\|^{-1}(\mathcal{S}_1(\omega)u_n, 0)^t \to 0$. Let $t \in \mathbb{R}$ be such that $\mathcal{S}_1(\omega) + it$ is boundedly invertible (this is possible since for $|t|$ sufficiently big $\mathrm{Im}(\mathfrak{s}_1(\omega) + it)[u] < 0$, hence by the Lax-Milgram theorem we conclude that $\mathcal{S}_1(\omega) + it$ is boundedly invertible). Now from $\mathcal{S}_1(\omega)u_n \to 0$ we deduce

$$v_n = -(\omega + i\gamma_m)(\Theta_m(\omega))^{-1}\,\mathrm{curl}_0(\mathcal{S}_1(\omega) + it)^{-1}(\mathcal{S}_1(\omega) + it)u_n \rightharpoonup 0$$

since $\mathrm{curl}_0(\mathcal{S}_1(\omega) + it)^{-1}$ is a bounded operator, $\mathcal{S}_1(\omega)u_n \to 0$ and $u_n \rightharpoonup 0$. Hence h_n is a Weyl sequence for $\mathcal{L}(\omega)$, and $\omega \in \sigma_{e2}(\mathcal{L})$.

We will now show that $\sigma_{e4}(\mathcal{S}_1) \cap \Sigma \supset \sigma_{e4}(\mathcal{L}) \cap \Sigma$. We can assume without loss of generality that we are inside Σ_1. Now assume that $\omega \in \Sigma_1$ but $\omega \notin \sigma_{e4}(\mathcal{S}_1)$. Then there exists a compact operator such that $0 \in \varrho(\mathcal{S}_1(\omega) + K)$. According to (2.7), if $\mathcal{K} = \mathrm{diag}(K, 0)$, then

$$(\mathcal{L}(\omega) + \mathcal{K})^{-1} =$$
$$\begin{pmatrix} (\mathcal{S}_1(\omega) + K)^{-1} & -(\mathcal{S}_1(\omega) + K)^{-1}i\,\mathrm{curl}\,V_e(\omega)^{-1} \\ V_e(\omega)^{-1}i\,\mathrm{curl}_0(\mathcal{S}_1(\omega) + K)^{-1} & V_e(\omega)^{-1}(I + \mathrm{curl}_0(\mathcal{S}_1(\omega) + K)^{-1}\,\mathrm{curl}(V_e(\omega))^{-1}) \end{pmatrix}.$$

In order to conclude, we then just need to show that $\mathrm{curl}_0(\mathcal{S}_1(\omega) + K)^{-1}\,\mathrm{curl}$ is bounded. But this can be proved as in the proof of Lemma 2.11 by first showing that $\mathrm{curl}_0(\mathcal{S}_1(\omega) + K + iB)^{-1}\,\mathrm{curl}$ is bounded for some bounded operator $B < 0$, and then by using the resolvent identity

$$(\mathcal{S}_1(\omega) + K)^{-1} \operatorname{curl} = (\mathcal{S}_1(\omega) + K + iB)^{-1} \operatorname{curl} + i(\mathcal{S}_1(\omega) + K)^{-1} B(\mathcal{S}_1(\omega) + K + iB)^{-1} \operatorname{curl}.$$

Altogether, $(\mathcal{L}(\omega) + \mathcal{K})^{-1}$ is bounded as an operator in $L^2(\Omega)^3 \oplus L^2(\Omega)^3$, hence $\omega \notin \sigma_{e4}(\mathcal{L}) \cap \Sigma_1$. $\quad \square$

According to Proposition 2.12, $\sigma(\mathcal{L}) \cap \Sigma$ can be enclosed in $W(\mathcal{S}_1) \cap \Sigma$.

Theorem 2.13. $\sigma(\mathcal{L}) \cap \Sigma \subset W(\mathcal{S}_1) \cap \Sigma$. *Moreover, the following explicit enclosure holds:*

$$\sigma(\mathcal{L}) \setminus \overline{W(\Theta_m)} \subset \Gamma \setminus \overline{W(\Theta_m)}$$

$$\Gamma := \{\omega \in \mathbb{C} : \operatorname{Re}\omega = 0, \operatorname{Im}\omega \in (-\gamma_e, 0) \setminus \{\gamma_m\}\} \cup \{\omega \in \mathbb{C} : \operatorname{Re}\omega \neq 0,$$

$$\operatorname{Im}\omega \geq -(\gamma_e + \gamma_m)/2, (2.2) \text{ holds}\}$$

Proof. Note that due to Remark 2.9, the operator $\mathcal{S}_1(it)$, $t \in \mathbb{R} \setminus (-\gamma_m, 0)$ is symmetric and since for fixed t it is the sum of a semibounded self-adjoint operator and a bounded self-adjoint operator, $\mathcal{S}_1(it)$ is self-adjoint and semibounded. The proof of Proposition 2.12 therefore extends to $\omega \in i\mathbb{R} \setminus (-i\gamma_m, 0)$, giving $\sigma(\mathcal{L}) \setminus \overline{W(\Theta_m)} = \sigma(\mathcal{S}_1) \setminus \overline{W(\Theta_m)}$ (note that $W(\Theta_m) \subset (-i\gamma_m, 0) \cup (-i\gamma_m/2 - s, -i\gamma_m/2 + s)$ for some $s > 0$ depending on θ_m).

Let then $\omega \in \sigma_{\text{app}}(\mathcal{S}_1) \setminus \overline{W(\Theta_m)}$. There exists $u_n \in \operatorname{dom}(\mathcal{S}_1)$, $\|u_n\| = 1$, $n \in \mathbb{N}$, such that $\mathcal{S}_1(\omega)u_n \to 0$. In particular,

$$
\begin{cases}
\operatorname{Im}\langle \Theta_m(\omega)^{-1} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \rangle = \operatorname{Im}\left\langle \frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} u_n, u_n \right\rangle + \varepsilon_n \\
\operatorname{Re}\langle \Theta_m(\omega)^{-1} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \rangle = \operatorname{Re}\left\langle \frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} u_n, u_n \right\rangle + \varepsilon_n
\end{cases}
\tag{2.8}
$$

with $\varepsilon_n \to 0$ as $n \to \infty$. Let $\omega = x + iy$. Then (2.8) can be written explicitly as

$$
\begin{cases}
\left\langle \frac{x(2y + \gamma_m)}{|\Theta_m(\omega)|^2} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \right\rangle = \frac{x\gamma_m}{x^2 + (y + \gamma_m)^2} \\
\qquad\qquad - \frac{x(2y + \gamma_e + \gamma_m)}{(x^2 + (y + \gamma_m)^2)(x^2 + (y + \gamma_e)^2)} \langle \theta_e^2 u_n, u_n \rangle + \varepsilon_n \\
\left\langle \frac{x^2 - y^2 - \gamma_m y - \theta_m^2}{|\Theta_m(\omega)|^2} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \right\rangle = \frac{x^2 + y^2}{x^2 + (y + \gamma_m)^2} \\
\qquad\qquad - \frac{(x^2 - y^2 - y(\gamma_e + \gamma_m) - \gamma_e\gamma_m)}{(x^2 + (y + \gamma_m)^2)(x^2 + (y + \gamma_e)^2)} \langle \theta_e^2 u_n, u_n \rangle + \varepsilon_n
\end{cases}
\tag{2.9}
$$

If $x = 0$, $y < -\gamma_e$ then the second equation reads

$$\left\langle \frac{-y^2 - \gamma_m y - \theta_m^2}{|\Theta_m(\omega)|^2} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \right\rangle = \frac{y^2}{(y + \gamma_m)^2} - \frac{(-(y + \gamma_e)(y + \gamma_m))}{((y + \gamma_m)^2)(y + \gamma_e)^2)} \langle \theta_e^2 u_n, u_n \rangle + \varepsilon_n$$

and since $y < -\gamma_e < -\gamma_m$, the left-hand side is negative while the right-hand side is strictly positive for sufficiently big n. Similarly, if $y > 0$, the left-hand side is negative while the right-hand side is strictly positive for sufficiently big n, a contradiction. Therefore, if $x = 0$, $y \in (-\gamma_e, 0)$.

Now, assume $x \neq 0$. We can then divide by x in the first equation of (2.9). Then we see immediately that if $y \leq -(\gamma_e + \gamma_m)/2 < -\gamma_m/2$ then the left-hand side is negative while the right-hand side is strictly positive, a contradiction. Therefore, for $x \neq 0$, $y > -(\gamma_e + \gamma_m)/2$. $\quad\square$

3. Decomposition of the essential spectrum

In this section we will adapt the strategy of proof recently used for the analogous decomposition of the essential spectrum for the time-harmonic Maxwell system with non-trivial conductivity in the recent article [11]. For the convenience of the reader we will state and prove all the required results.

Without loss of generality we assume that Ω is unbounded, the bounded case being substantially simpler, see Remark 3.8 below. Let $\Omega_R = \Omega \cap B(0, R)$, for $R > 0$. For any $\delta > 0$ we assume that the functions θ_e and θ_m admit a decomposition

$$\theta_e(x) = \theta_e^c(x) + \theta_e^\delta(x) + \theta_e^0, \quad \theta_m(x) = \theta_m^c(x) + \theta_m^\delta(x) + \theta_m^0 \tag{3.1}$$

for all $x \in \Omega$, where θ_e^c, θ_m^c, have compact support in Ω_R (for some sufficiently large R depending on δ), $\theta_e^\delta, \theta_m^\delta$, are bounded multiplication operators with norm less than δ, and θ_e^0, θ_m^0 are real constants, representing the asymptotic values of θ_e and θ_m. In particular,

$$\lim_{R \to \infty} \sup_{|x| > R} |\theta_*(x) - \theta_*^0| = 0, \quad * = e, m.$$

Corresponding to this decomposition of θ_e and θ_m we also introduce 'limits at infinity' of the functions Θ_e and Θ_m in (1.7), namely

$$\Theta_{e,\infty}(\omega) = \omega^2 + \omega i \gamma_e - (\theta_e^0)^2, \quad \Theta_{m,\infty}(\omega) = \omega^2 + \omega i \gamma_m - (\theta_m^0)^2, \tag{3.2}$$

and of the functions V_e and V_m appearing in (2.6), namely

$$V_{m,\infty}(\omega) = \frac{\Theta_{m,\infty}(\omega)}{(\omega + i\gamma_m)}, \quad V_{e,\infty}(\omega) = \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)}. \tag{3.3}$$

We use the classical Helmholtz decomposition $L^2(\Omega)^3 = \nabla \dot{H}_0^1(\Omega) \oplus H(\operatorname{div} 0, \Omega)$, see e.g. [18, Lemma 11], and we denote by $P_{\ker(\operatorname{div})}$ the associated orthogonal projection onto $H(\operatorname{div} 0, \Omega)$. The following result is stated in [11, Proposition 5.1] and in a less general setting in [18, Lemma 23].

Proposition 3.1. *Let* $m : \Omega \to \mathbb{C}^{3\times3}$ *be a locally bounded function such that*

$$\lim_{R \to \infty} \sup_{\|x\| > R} \|m(x)\| = 0. \tag{3.4}$$

Then $m P_{\ker(\operatorname{div})}$ *is compact from* $(H(\operatorname{curl}, \Omega), \|\cdot\|_{H(\operatorname{curl},\Omega)})$ *to* $(L^2(\Omega)^3, \|\cdot\|_{L^2(\Omega)^3})$.

Proof. Given $\delta > 0$, there exist a bounded operator m_δ (which we identify with the corresponding multiplication operator in $L^2(\Omega)^3$) with $\|m_\delta\| < \delta$, and a function m_c^δ which is compactly

supported in $\Omega_R := \Omega \cap B(0, R)$ for large $R > 0$, such that $m = m_c^\delta + m_\delta$. We claim that $m_c^\delta P_{\text{ker(div)}}$ is compact for every $\delta > 0$. Note that $\|m P_{\text{ker(div)}} - m_c^\delta P_{\text{ker(div)}}\|_{\mathcal{B}(H(\text{curl},\Omega), L^2(\Omega)^3)} \leq \delta$ vanishes as $\delta \to 0$; therefore $m P_{\text{ker(div)}}$ is compact as limit of the compact operators $m_c^\delta P_{\text{ker(div)}}$.

Define χ_R to be a C^∞ cut-off function, $\chi_R = 1$ on $\text{supp}(m_c) \subset \Omega_R$ and $\chi = 0$ in $\mathbb{R}^3 \setminus \overline{\Omega_R}$. There exists $C_R > 0$ such that, for $u \in H(\text{curl}, \Omega)$,

$$\|(\chi_R P_{\text{ker(div)}} u)|_{\Omega_R}\|_{H(\text{curl},\Omega_R) \cap H(\text{div},\Omega_R)} \leq C_R \|u\|_{H(\text{curl},\Omega)},$$

where we have used that $\text{div}(\chi_R P_{\text{ker(div)}} u) = \nabla \chi_R \cdot P_{\text{ker(div)}} u$ and the identity $\text{curl}(\chi_R P_{\text{ker(div)}} u) = \nabla \chi_R \times P_{\text{ker(div)}} u + \chi_R \text{curl}\, u$, which holds since $\text{curl}\, P_{\text{ker(div)}} u = \text{curl}\, u$. Finally, $m_c P_{\text{ker(div)}}$ is seen to be compact by rewriting it as follows

$$m_c P_{\text{ker(div)}} u = m_c \iota (\chi_R P_{\text{ker(div)}} u)|_{\Omega_R};$$

ι is the compact embedding of $H_0(\text{curl}, \Omega_R) \cap H(\text{div}, \Omega_R)$ in $L^2(\Omega_R)^3$, see [28]. $\quad\square$

Remark 3.2. If Ω is bounded, the claim of Proposition 3.1 holds true without assuming (3.4). This is a direct consequence of the compact embedding of $H_0(\text{curl}, \Omega) \cap H(\text{div}\, 0, \Omega)$ into $L^2(\Omega)^3$, whenever Ω is bounded.

Definition 3.3. For $\omega \in \Sigma = \Sigma_1 \cup \Sigma_2$, we define rational pencils of closed operators acting in the Hilbert space $H(\text{div}\, 0, \Omega)$ equipped with the $L^2(\Omega)^3$-norm by

$$\mathcal{C}_m(\omega) := \text{curl}(\Theta_m(\omega))^{-1} \text{curl}_0, \qquad \mathcal{S}_m(\omega) := \mathcal{C}_m(\omega) - V_{e,\infty}(\omega),$$

$$\text{dom}(\mathcal{C}_m(\omega)) = \text{dom}(\mathcal{S}_m(\omega))$$
$$:= \{u \in H_0(\text{curl}, \Omega) \cap H(\text{div}\, 0, \Omega) : (\Theta_m(\omega))^{-1} \text{curl}\, u \in H(\text{curl}, \Omega)\},$$

and

$$\mathcal{C}_\infty(\omega) := (\Theta_{m,\infty}(\omega))^{-1} \text{curl}\, \text{curl}_0, \qquad \mathcal{S}_\infty(\omega) := \mathcal{C}_\infty(\omega) - V_{e,\infty}(\omega),$$
$$\text{dom}(\mathcal{C}_\infty(\omega)) = \text{dom}(\mathcal{S}_\infty(\omega)) := \{u \in H_0(\text{curl}, \Omega) \cap H(\text{div}\, 0, \Omega) : \text{curl}\, u \in H(\text{curl}, \Omega)\}$$

where $V_{e,\infty}(\omega)$ is defined in (3.3).

Notation. For $\omega \in \mathbb{C} \setminus \{-i\gamma_e, -i\gamma_m\}$ define the function

$$f(\omega) = \frac{\Theta_{m,\infty}(\omega)\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \tag{3.5}$$

If θ_m is not differentiable, it may happen that $\text{dom}(\mathcal{S}_m)$ is ω-dependent and even that $\text{dom}(\mathcal{S}_m(\omega)) \cap \text{dom}(\mathcal{S}_\infty(\omega)) = \{0\}$ for suitably chosen $\omega \in \Sigma$. In spite of this, the following result holds (the version for the non-self-adjoint time-harmonic Maxwell system was proved in [11, Proposition 5.4]).

Proposition 3.4. *If θ_e, θ_m satisfy (3.1) and $\mathcal{S}_m$, $\mathcal{S}_\infty$ are as in Definition 3.3, then $\sigma_{ek}(\mathcal{S}_m) = \sigma_{ek}(\mathcal{S}_\infty)$ for $k = 1, 2, 3, 4$, and hence*

$$\sigma_{ek}(\mathcal{S}_m) \cap \Sigma = \left\{ \omega \in \Sigma : \ f(\omega) = t, \ t \in \sigma_{ek}(\mathrm{curl}\,\mathrm{curl}_0\,|_{H(\mathrm{div}\,0,\Omega)}) \right\},$$

where f is the function defined in (3.5).

Proof. We follow the proof of [11, Proposition 5.4]. Let $\omega \in \Sigma$ and to shorten the notation set $z_\omega := V_{e,\infty}(\omega)$. Then $\omega \in \sigma_{ek}(\mathcal{S}_m)$ if and only if $0 \in \sigma_{ek}(\mathcal{C}_m(\omega) - z_\omega)$ and $\omega \in \sigma_{ek}(\mathcal{S}_\infty)$ if and only if $0 \in \sigma_{ek}(\mathcal{C}_\infty(\omega) - z_\omega)$ where, $\mathcal{C}_m, \mathcal{C}_\infty$ are the operator functions defined in Definition 3.3, for $\omega \in \Sigma$.

Since the quadratic form $\mathfrak{c}_m(\cdot)$ associated with $\mathcal{C}_m(\cdot)$ and the form $\mathfrak{c}_\infty$ associated with $\mathcal{C}_\infty(\cdot)$ have the same domain $\mathrm{dom}\,\mathfrak{c}_m(\cdot) = \mathrm{dom}\,\mathfrak{c}_\infty = H_0(\mathrm{curl}, \Omega)$, the second resolvent identity takes the form

$$
\begin{aligned}
(\mathcal{C}_m(\omega) &- z_\omega)^{-1} - (\mathcal{C}_\infty(\omega) - z_\omega)^{-1} \\
&= \left(\mathrm{curl}_0(\mathcal{C}_m(\omega)^* - \overline{z_\omega})^{-1} \right)^* (\Theta_{m,\infty}(\omega)^{-1} - \Theta_m(\omega)^{-1})\,\mathrm{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1}
\end{aligned}
\tag{3.6}
$$

for $\omega \in \Sigma \cap (\varrho(\mathcal{S}_m) \cap \varrho(\mathcal{S}_\infty))$. In fact, for arbitrary $u, v \in L^2(\Omega)^3$ and $\omega \in \Sigma \cap (\varrho(\mathcal{S}_m) \cap \varrho(\mathcal{S}_\infty))$, we can write

$$
\begin{aligned}
\Big\langle \big((\mathcal{C}_m(\omega) &- z_\omega)^{-1} - (\mathcal{C}_\infty(\omega) - z_\omega)^{-1} \big) u, v \Big\rangle \\
&= \Big\langle (\mathcal{C}_m(\omega) - z_\omega)^{-1} u, v \Big\rangle - \Big\langle u, (\mathcal{C}_\infty(\omega) - z_\omega)^{-*} v \Big\rangle \\
&= \Big\langle (\mathcal{C}_m(\omega) - z_\omega)^{-1} u, (\mathcal{C}_\infty(\omega)^* - \overline{z_\omega})(\mathcal{C}_\infty(\omega)^* - \overline{z_\omega})^{-1} v \Big\rangle \\
&\quad - \Big\langle (\mathcal{C}_m(\omega) - z_\omega)(\mathcal{C}_m(\omega) - z_\omega)^{-1} u, (\mathcal{C}_\infty(\omega)^* - \overline{z_\omega})^{-1} v \Big\rangle \\
&= (\mathfrak{c}_\infty(\omega) - \mathfrak{c}_m(\omega)) \big[(\mathcal{C}_m(\omega) - z_\omega)^{-1} u, (\mathcal{C}_\infty(\omega)^* - \overline{z_\omega})^{-1} v \big];
\end{aligned}
$$

together with $\mathfrak{c}_m(\omega) = \langle \Theta_m(\omega)^{-1}\,\mathrm{curl}_0\,\cdot, \mathrm{curl}_0\,\cdot \rangle$ and analogously for $\mathfrak{c}_\infty(\omega)$, the identity (3.6) follows. The first factor on the right-hand side of (3.6) is bounded since $\mathrm{dom}\,\mathcal{C}_m(\cdot) \subset \mathrm{dom}\,\mathrm{curl}_0$. By assumption (3.1), for fixed $\omega \in \Sigma$, condition (3.4) of Proposition 3.1 is satisfied by $(\Theta_m(\omega)^{-1} - \Theta_\infty(\omega)^{-1})$ and thus the operator $(\Theta_m(\omega)^{-1} - \Theta_\infty(\omega)^{-1}) P_{\ker\mathrm{div}}$ is compact from $H(\mathrm{curl}, \Omega)$ to $H(\mathrm{div}\,0, \Omega) \subset L^2(\Omega)^3$. The boundedness of $\mathrm{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1}$ from $H(\mathrm{div}\,0, \Omega)$ to $H(\mathrm{curl}, \Omega)$ follows from

$$
\begin{aligned}
\mathrm{curl}\,\mathrm{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1} &= \mathrm{curl}\,\mathrm{curl}_0(\mathrm{curl}\,\mathrm{curl}_0 - f(\omega))^{-1}\Theta_m(\omega) \\
&= I + f(\omega)(\mathrm{curl}\,\mathrm{curl}_0 - f(\omega))^{-1}
\end{aligned}
$$

where f is defined in (3.5). Now, $f(it) < 0$, $t \in \mathbb{R}$, hence

$$0 \leq I + f(it)(\mathrm{curl}\,\mathrm{curl}_0 - f(it))^{-1} \leq I,$$

and the boundedness for $\omega = it$ follows. For a general $\omega \in (\varrho(\mathcal{S}_\infty) \cap \Sigma)$, we have

$$(\mathcal{C}_\infty(\omega) - z_\omega)^{-1} = (\mathcal{C}_\infty(it) - z_{it})^{-1} + (\mathcal{C}_\infty(it) - z_{it})^{-1}(z_{it} - z_\omega)(\mathcal{C}_\infty(\omega) - z_\omega)^{-1}$$

hence, upon applying $\operatorname{curl}\operatorname{curl}_0$, $\operatorname{curl}\operatorname{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1}$ is seen to be bounded.

Altogether, the operator

$$(\Theta_{m,\infty}(\omega)^{-1} - \Theta_m(\omega)^{-1})\operatorname{curl}_0(\mathcal{C}_\infty - z_\omega)^{-1}$$
$$= (\Theta_{m,\infty}(\omega)^{-1} - \Theta_m(\omega)^{-1})P_{\ker(\operatorname{div})}\operatorname{curl}_0(\mathcal{C}_\infty - z_\omega)^{-1}$$

is compact. Hence, by (3.6), the resolvent difference of $\mathcal{S}_m(\omega)$ and $\mathcal{S}_\infty(\omega)$ is compact and, by [14, Thm. IX.2.4], $\sigma_{ek}(\mathcal{S}_m(\omega)) = \sigma_{ek}(\mathcal{S}_\infty(\omega))$ follows for all $k = 1, 2, 3, 4$, $\omega \in \Sigma$, hence $0 \in \sigma_{ek}(\mathcal{S}_m(\omega))$ if and only if $0 \in \sigma_{ek}(\mathcal{S}_\infty(\omega))$, for $\omega \in \Sigma$. This means that $\sigma_{ek}(\mathcal{S}_m) \cap \Sigma = \sigma_{ek}(\mathcal{S}_\infty) \cap \Sigma$. □

Remark 3.5. In the proof of Proposition 3.4, it is shown that for $\omega = it$, $t > 0$,

$$\|\operatorname{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1}\|_{\mathcal{B}(H(\operatorname{div} 0, \Omega), H(\operatorname{curl}, \Omega))} \le C,$$

where the constant $C > 0$ does not depend on Ω. This will be important in Section 5, where families of domains are considered.

We further state the following abstract result regarding the spectrum of triangular block operator matrices, a proof of which can be found in [11, Theorem 8.1]. Following [14, Chp.IX, p.414], given a linear operator T densely defined in $\mathcal{H}$, we set $\sigma_{e2}^*(T) = \{\omega \in \mathbb{C} : \operatorname{def}(T - \omega) = \infty\}$, with the convention that $\operatorname{def}(T - \omega) = \infty$ if $\operatorname{ran}(T - \omega)$ is not closed.

Theorem 3.6. *Let $\mathcal{A}$ be defined by*

$$\mathcal{A} = \begin{pmatrix} A & 0 \\ C & D \end{pmatrix}$$

with A, D are densely defined, C, D are closable, $\operatorname{dom}(A) \subset \operatorname{dom}(C)$ and $\varrho(A) \ne \emptyset$. Then

$$\big(\sigma_{e2}(A) \setminus \sigma_{e2}^*(\overline{D})\big) \cup \sigma_{e2}(\overline{D}) \subset \sigma_{e2}(\overline{\mathcal{A}}) \subset \sigma_{e2}(A) \cup \sigma_{e2}(\overline{D}), \tag{3.7}$$

and hence

$$\sigma_{e2}(\overline{\mathcal{A}}) \cup \big(\sigma_{e2}(A) \cap \sigma_{e2}^*(\overline{D})\big) = \sigma_{e2}(A) \cup \sigma_{e2}(\overline{D});$$

in particular, if $\sigma_{e2}^*(\overline{D}) = \sigma_{e2}(\overline{D})$ or if $\sigma_{e2}(A) \cap \sigma_{e2}^*(\overline{D}) = \emptyset$, then

$$\sigma_{e2}(\overline{\mathcal{A}}) = \sigma_{e2}(A) \cup \sigma_{e2}(\overline{D}).$$

We are now in position to prove the following theorem, which yields a decomposition of the $\sigma_e(\mathcal{L})$ as the union of the essential spectrum of the constant-coefficient pencil $\mathcal{S}_\infty$ and the essential spectrum of the pencil of bounded multiplication operators $V_e(\cdot)$, compressed to gradient fields. For the convenience of the reader, the relations between the several different operators and their essential spectra are represented in Fig. 1.
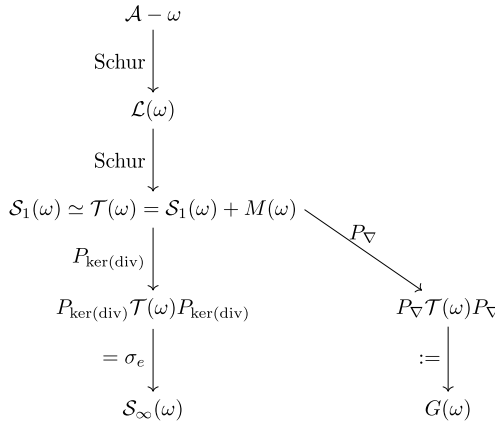
$$\mathcal{A} - \omega$$

$$\downarrow \text{Schur}$$

$$\mathcal{L}(\omega)$$

$$\downarrow \text{Schur}$$

$$\mathcal{S}_1(\omega) \simeq \mathcal{T}(\omega) = \mathcal{S}_1(\omega) + M(\omega) \searrow^{P_\nabla}$$

$$P_{\text{ker(div)}} \downarrow$$

$$P_{\text{ker(div)}} \mathcal{T}(\omega) P_{\text{ker(div)}} \qquad\qquad P_\nabla \mathcal{T}(\omega) P_\nabla$$

$$= \sigma_e \downarrow \qquad\qquad\qquad\qquad := \downarrow$$

$$\mathcal{S}_\infty(\omega) \qquad\qquad\qquad\qquad G(\omega)$$

Fig. 1. Relations between $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{S}_1$. $\sigma_e(\mathcal{S}_1)$ then decomposes in $\sigma_e(\mathcal{S}_\infty) \cup \sigma_e(G)$, according to Theorem 3.7.

Theorem 3.7. *Suppose that* θ_e, θ_m *satisfy the limiting assumption* (3.1). *Let* $P_\nabla := \text{id} - P_{\text{ker(div)}}$ *be the orthogonal projection from* $L^2(\Omega)^3 = \nabla \dot{H}_0^1(\Omega) \oplus H(\text{div}\,0, \Omega)$ *onto* $\nabla \dot{H}_0^1(\Omega)$. *Let* $G(\omega)$ *denote the operator*

$$G(\omega) = -P_\nabla V_e(\omega) P_\nabla, \text{ with } \text{dom}(G(\omega)) = \nabla \dot{H}_0^1(\Omega), \tag{3.8}$$

viewed as an operator from the space $\nabla \dot{H}_0^1(\Omega)$ *to* $\nabla \dot{H}_0^1(\Omega)$. *Then*

$$\sigma_{ek}(\mathcal{S}_1) \cap \Sigma = (\sigma_{ek}(\mathcal{S}_\infty) \cup \sigma_{ek}(G)) \cap \Sigma, \quad k = 1, 2, 3, 4,$$

where $\sigma_{ek}(\mathcal{S}_\infty)$ *is described in Proposition 4.1, and*

$$\sigma_{ek}(G) \subset \begin{cases} -i[0, \gamma_e), & \text{if } -\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty \le 0, \\ -i[0, \gamma_e) \cup \left(-d_e - i\frac{\gamma_e}{2}, d_e - i\frac{\gamma_e}{2}\right) & \text{if } -\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty > 0, \end{cases}$$

with $d_e \in \left(-\sqrt{-\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty}, \sqrt{-\frac{\gamma_e^2}{4} + \|\theta_e^2\|_\infty}\right)$.

Proof. Let $\omega \in \Sigma$. The operator $M(\omega) := (V_e(\omega) - V_{e,\infty}(\omega)) P_{\text{ker(div)}}$ in $L^2(\Omega)^3$ is curl_0-compact by Proposition 3.1, and hence $\mathcal{C}_m(\omega)$-compact, since $\text{dom}(\mathcal{C}_m) \subset \text{dom}(\text{curl}_0)$, where $\mathcal{C}_m(\omega) = \text{curl}\,\Theta_m(\omega)^{-1}\,\text{curl}_0$ is defined in Definition 3.3. Since $\mathcal{S}_1(\omega) = \mathcal{C}_m(\omega) + V_e(\omega)$, with $V_e(\omega)$ bounded multiplication operator, bounded sequences in the $\mathcal{S}_1(\omega)$-graph norm have bounded $\mathcal{C}_m(\omega)$-graph norms. Hence $M(\omega)$ is $\mathcal{S}_1(\omega)$-compact which yields $\sigma_e(\mathcal{S}_1(\omega)) = \sigma_e(\mathcal{S}_1(\omega) + M(\omega))$.

Since $\nabla \dot{H}_0^1(\Omega) \subset \text{ker}(\text{curl}_0)$ and hence $\text{curl}_0 P_\nabla = P_\nabla \text{curl} = 0$, $\nabla \dot{H}_0^1(\Omega)$ is a reducing subspace for $\text{curl}\,\Theta_m(\omega)^{-1}\,\text{curl}_0$. Therefore the operator

$$\mathcal{T}(\omega) := \mathcal{S}_1(\omega) + M(\omega)$$

$$= \text{curl}\,\Theta_m(\omega)^{-1}\,\text{curl}_0 - V_e(\omega) P_\nabla - V_e(\omega) P_{\text{ker(div)}} + (V_e(\omega) - V_{e,\infty}(\omega)) P_{\text{ker(div)}} \tag{3.9}$$

$$= \mathcal{C}_m(\omega) - V_e(\omega) P_\nabla - V_{e,\infty}(\omega) P_{\text{ker(div)}}$$

which is a bounded perturbation of $\mathcal{C}_m(\omega)$ admits an operator matrix representation with respect to the decomposition $L^2(\Omega)^3 = \nabla \dot{H}_0^1(\Omega) \oplus H(\mathrm{div}\,0, \Omega)$ given by

$$
\begin{aligned}
\mathcal{T}(\omega) &= \begin{pmatrix} P_\nabla \mathcal{T}(\omega)|_{\nabla \dot{H}_0^1(\Omega)} & P_\nabla \mathcal{T}(\omega)|_{H(\mathrm{div}\,0,\Omega)} \\ P_{\ker\mathrm{div}} \mathcal{T}(\omega)|_{\nabla \dot{H}_0^1(\Omega)} & P_{\ker\mathrm{div}} \mathcal{T}(\omega)|_{H(\mathrm{div}\,0,\Omega)} \end{pmatrix} \\
&= \begin{pmatrix} -P_\nabla V_e(\omega)|_{\nabla \dot{H}_0^1(\Omega)} & 0 \\ -P_{\ker\mathrm{div}} V_e(\omega)|_{\nabla \dot{H}_0^1(\Omega)} & P_{\ker\mathrm{div}}(\mathcal{C}_m(\omega) - V_{e,\infty}(\omega))|_{H(\mathrm{div}\,0,\Omega)} \end{pmatrix} \\
&= \begin{pmatrix} G(\omega) & 0 \\ P_{\ker\mathrm{div}} V_e(\omega)|_{\nabla \dot{H}_0^1(\Omega)} & \mathcal{S}_m(\omega) \end{pmatrix},
\end{aligned}
\tag{3.10}
$$

with domain $\mathrm{dom}(\mathcal{T}(\omega)) = \nabla \dot{H}_0^1(\Omega) \oplus \mathrm{dom}(\mathcal{S}_m(\omega))$. Apart from $\mathcal{S}_m(\omega)$, the other two matrix entries in $\mathcal{T}(\omega)$ are bounded and everywhere defined, and $\sigma_{e2}(\mathcal{S}_m(\omega)) = \sigma_{e2}^*(\mathcal{S}_m(\omega))$, due to J-self-adjointness. Thus Theorem 3.6 and Proposition 3.4 yield that

$$
\sigma_{e2}(\mathcal{T}(\omega)) = \sigma_{e2}(\mathcal{S}_m(\omega)) \cup \sigma_{e2}(G(\omega)) = \sigma_{e2}(\mathcal{S}_\infty(\omega)) \cup \sigma_{e2}(G(\omega))
$$

and hence, since $\omega \in \Sigma$ was arbitrary,

$$
\sigma_{e2}(\mathcal{S}_1) \cap \Sigma = \sigma_{e2}(\mathcal{S}_1 + M) \cap \Sigma = \sigma_{e2}(\mathcal{T}) \cap \Sigma = (\sigma_{e2}(\mathcal{S}_\infty) \cup \sigma_{e2}(G)) \cap \Sigma. \quad \square
$$

Remark 3.8. If Ω is bounded, we claim that $\sigma_{ek}(\mathcal{S}_1) = \sigma_{ek}(G)$ for $k = 1, 2, 3, 4$. To see this, one applies the Helmholtz decomposition in $\nabla H_0^1(\Omega) \oplus H(\mathrm{div}\,0, \Omega)$ to obtain the block operator matrix representation

$$
\mathcal{T}(\omega) = \begin{pmatrix} P_\nabla \mathcal{S}_1(\omega) P_\nabla & P_\nabla \mathcal{S}_1(\omega) P_{\ker\mathrm{div}} \\ P_{\ker\mathrm{div}} \mathcal{S}_1(\omega) P_\nabla & P_{\ker\mathrm{div}} \mathcal{S}_1(\omega) P_{\ker\mathrm{div}} \end{pmatrix}
$$

of $\mathcal{S}_1(\omega)$. In particular, $\sigma_{ek}(\mathcal{S}_1) = \sigma_{ek}(\mathcal{T})$. Since Ω is bounded, $P_{\ker\mathrm{div}} \mathcal{S}_1(\omega) P_{\ker\mathrm{div}}$ has compact resolvent. The term $P_\nabla \mathcal{S}_1(\omega) P_{\ker\mathrm{div}}$ is compact as a consequence of Remark 3.2, and therefore can be discarded to leave a lower triangular operator matrix. The claim then follows arguing as in the proof of Theorem 3.7, by recalling that $P_\nabla \mathcal{S}_1(\omega) P_\nabla = G(\omega)$.

4. Spectrum of the reduced operators

Recall that $\mathcal{S}_{1,\infty}(\omega)|_{\ker(\mathrm{div})} = \mathcal{S}_\infty(\omega)$ for all $\omega \in \Sigma$, where $\mathcal{S}_\infty$ is defined in Definition 3.3. Since $\mathcal{S}_\infty$ is a constant-coefficients operator, by classical symbol analysis we deduce that

Proposition 4.1. *For f as in* (3.5) *we have*

$$
\sigma_e(\mathcal{S}_\infty) \cap \Sigma = \left\{ \omega \in \mathbb{C} : f(\omega) = t, t \in \sigma_e(\mathrm{curl\,curl}_0\,|_{\ker(\mathrm{div})}) \right\} \cap \Sigma
$$

Proof. Without loss of generality, we may assume $\omega \in \Sigma_1$, the case $\omega \in \Sigma_2$ being similar. We first note that if $\omega \in \Sigma_1$, $\Theta_{m,\infty}(\omega)$ is (boundedly) invertible. Studying the spectrum of $\mathcal{S}_\infty$ is then equivalent to considering directly the pencil $(\mathrm{curl\,curl}_0 - f(\omega))|_{\ker(\mathrm{div})}$, which we will call

again $\mathcal{S}_\infty(\omega)$ with an abuse of notation. Now, $\omega \in \sigma_e(\mathcal{S}_\infty) \cap \Sigma_1$ iff there exists a Weyl singular sequence $u_n \in H(\operatorname{div} 0, \Omega)$ such that $\operatorname{curl} \operatorname{curl}_0 u_n - f(\omega) u_n \to 0$, which holds iff $f(\omega) \in \mathbb{R}$ and $f(\omega) \in \sigma_e(\operatorname{curl} \operatorname{curl}_0 |_{\ker(\operatorname{div})})$. The claim is proved. $\quad\square$

Proposition 4.2.

$$W_e(\mathcal{S}_\infty) \cap \Sigma = \left\{ \omega \in \mathbb{C} : f(\omega) = t, t \in W_e(\operatorname{curl} \operatorname{curl}_0 |_{\ker(\operatorname{div})}) \right\} \cap \Sigma$$

Proof. If (u_n) is a Weyl sequence in $H(\operatorname{div} 0, \Omega)$ with $\|u_n\| = 1$ for all n, then $(\operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n) - t \to 0$ if and only if $(\operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n) - f(\omega) \to 0$ for all $\omega \in \mathbb{C}$ such that $f(\omega) = t$. $\quad\square$

As a consequence of Proposition 4.1 we can study the asymptotics of the spectrum of $\mathcal{S}_\infty$.

Proposition 4.3. *Assume that*

$$f(\omega_n) = \frac{\Theta_{e,\infty}(\omega_n) \Theta_{m,\infty}(\omega_n)}{(i\gamma_e + \omega_n)(i\gamma_m + \omega_n)} = t_n \to +\infty \tag{4.1}$$

as $n \to \infty$, $t_n \in \sigma_e(\operatorname{curl} \operatorname{curl}_0 |_{\ker(\operatorname{div})})$. Then the following are true:

(i) if $\operatorname{Re} \omega_n$ is bounded as $n \to \infty$ then $\operatorname{dist}(\operatorname{Im} \omega_n, \{-i\gamma_e, -i\gamma_m\}) \to 0$. If $\omega_n \to -i\gamma_x$, $x = e, m$ then $\omega_n = -i\gamma_x - \frac{ic}{t_n} + o(1/t_n)$ as $n \to \infty$, where $c \in \mathbb{R} \setminus \{0\}$ is explicitly given by $c = (\theta_x^0)^2\left(-\gamma_x + \frac{(\theta_x^0)^2(\theta_y^0)^2}{\gamma_y - \gamma_x}\right)$, with $y \neq x$, $x, y \in \{e, m\}$.

(ii) If $|\operatorname{Re} \omega_n| \to +\infty$ then $t_n = |\operatorname{Re} \omega_n|^2 + o(|\operatorname{Re} \omega_n|^2)$ as $n \to \infty$ and $\operatorname{Im} \omega_n \to 0$ with the asymptotics $\operatorname{Im} \omega_n = -\frac{1}{\operatorname{Re} \omega_n^2}((\theta_e^0)^2 \gamma_e + (\theta_m^0)^2 \gamma_m) + o(1/\operatorname{Re} \omega_n^2)$.

Proof. (i) $(\operatorname{Im} \omega_n)_n$ is a bounded sequence since the numerical range of $S_{1,\infty}$ is contained in an horizontal strip. Since also $(\operatorname{Re} \omega_n)_n$ is a bounded sequence by assumption, we may assume that up to a subsequence $\omega_n \to \omega_\infty \in \mathbb{C}$. From (4.1) we have that

$$(\omega_n^2 + i\gamma_m \omega_n - (\theta_m^0)^2)(\omega_n^2 + i\gamma_e \omega_n - (\theta_e^0)^2) = t_n(\omega_n + i\gamma_e)(\omega_n + i\gamma_m)$$

so

$$\limsup_{n \to \infty} |t_n(\omega_n + i\gamma_e)(\omega_n + i\gamma_m)| \leq |(\omega_\infty^2 + i\gamma_m \omega_\infty - (\theta_m^0)^2)(\omega_\infty^2 + i\gamma_e \omega_\infty - (\theta_e^0)^2)|$$

which implies that either $\omega_\infty = -i\gamma_e$ or $\omega_\infty = -i\gamma_m$. Set then $\omega_n = -i\gamma_e + \varepsilon_n$ as $n \to \infty$, where $\varepsilon_n \to 0$, $\varepsilon_n \in \mathbb{C}$. Substituting this ansatz in (4.1) and keeping only the zeroth order terms we get

$$-(\theta_e^0)^2(-\gamma_e^2 + \gamma_e \gamma_m - (\theta_m^0)^2) = z(i(\gamma_m - \gamma_e)), \quad z = \lim_{n \to \infty} t_n \varepsilon_n$$

We note that the existence of the limit for z is up to a subsequence, and can be easily deduced from the fact that equation (4.1) is not satisfied if $(t_n \varepsilon_n)_n$ is not a bounded sequence. Hence we have that

$$\varepsilon_n = \frac{z}{t_n} + o(1/t_n) = -\frac{i\,(\theta_e^0)^2}{t_n}\left(-\gamma_e + \frac{(\theta_e^0)^2(\theta_m^0)^2}{\gamma_m - \gamma_e}\right) + o(1/t_n)$$

as $n \to \infty$, concluding the proof of (i).

(ii) To shorten the notation, let us set $x_n = \operatorname{Re}\omega_n$, $y_n = \operatorname{Im}\omega_n$. From equation (4.1), recalling that $(y_n)_n$ is bounded, after taking the real part we see that

$$x_n^4 - t_n x_n^2 = o(x_n^4) \quad \Rightarrow \quad t_n = x_n^2 + o(x_n^2), \quad n \to \infty.$$

A further inspection of equation (4.1) gives that the $o(x_n^2)$-term must be in the form $c_n + o(1)$, where c_n possibly depends on y_n. Using the ansatz $t_n = x_n^2 + c_n$ in the real part of (4.1) and neglecting the lower order terms we get

$$x_n^4 - 6x_n^2 y_n^2 - 3x_n^2 y_n(\gamma_e + \gamma_m) + x_n^2(-(\theta_e^0)^2 - (\theta_m^0)^2 - \gamma_e\gamma_m)$$

$$= x_n^4 + x_n^2(c_n - y_n^2 - y_n(\gamma_e + \gamma_m) - \gamma_e\gamma_m)$$

from which we deduce $-5y_n^2 - 2y_n(\gamma_e + \gamma_m) - (\theta_e^0)^2 - (\theta_m^0)^2 = c_n + o(1)$ as $n \to \infty$. In order to continue the analysis we now turn to the imaginary part of (4.1). More explicitly, we have

$$4x_n^3 y_n - 4x_n y_n^3 + (x_n^3 - 3x_n y_n^2)(\gamma_e + \gamma_m) + 2x_n y_n(-(\theta_e^0)^2 - (\theta_m^0)^2 - \gamma_e\gamma_m)$$

$$- x_n\big(\gamma_m(\theta_e^0)^2 + \gamma_e(\theta_m^0)^2\big) = (x_n^2 + c_n)x_n(2y_n + (\gamma_e + \gamma_m)) \quad (4.2)$$

The term in x_n^3 simplifies. Now, if y_n does not tend to zero as $n \to \infty$, the highest order term in the previous equation is $x_n^3 y_n$, so we get the equation $2x_n^3 y_n = o(x_n^3)$, which is a contradiction. Hence $y_n \to 0$. The candidate highest order terms are $x_n^3 y_n$ and x_n. By direct inspection one checks that if $x_n^3 y_n = o(x_n)$ or $x_n = o(x_n^3 y_n)$ as $n \to \infty$ the previous equation gives a contradiction. So it must be $y_n = \frac{z_n}{x_n^2} + o(1/x_n^2)$ as $n \to \infty$. By using this ansatz in (4.2) and keeping only the highest order term (namely the ones in x_n), it may be proved that

$$-x_n(\gamma_m(\theta_e^0)^2 + \gamma_e(\theta_m^0)^2) + 4z_n x_n = 2z_n x_n + c_n(\gamma_e + \gamma_m)x_n,$$

for $c_n = -(\theta_e^0)^2 - (\theta_m^0)^2 + o(1)$. Thus, $z_n = -\frac{1}{2}((\theta_e^0)^2\gamma_e + (\theta_m^0)^2\gamma_m) + o(1)$ and $y_n = -\frac{1}{2x_n^2}((\theta_e^0)^2\gamma_e + (\theta_m^0)^2\gamma_m) + o(1/x_n^2)$. $\quad\square$

We now turn to the bounded pencil $G(\cdot)$ defined in (3.8). Recall that $V_e(\omega)(x) = \frac{\Theta_e(\omega,x)}{(\omega+i\gamma_e)(\omega+i\gamma_m)}$, $\omega \in \Sigma$, $x \in \Omega$.

Proposition 4.4. *Assume that θ_e is a continuous function in $\overline{\Omega}$. Then*

$$\sigma_e(G) = \{\omega \in \mathbb{C} : \exists x_0 \in \Omega,\ \Theta_e(\omega, x_0) = 0\}$$

$$= \{\omega \in \mathbb{C} : \exists x_0 \in \Omega,\ \omega = -i\gamma_e/2 \pm (\sqrt{-\gamma_e^2 + 4\theta_e^2(x_0)})/2\}$$

Proof. Since $(\omega + i\gamma_e)(\omega + i\gamma_m)$ is constant in $x \in \Omega$, we can replace V_e by Θ_e in the definition of G without changing the spectrum. We first note that the set

$$\{\omega \in \mathbb{C} : \mathrm{Re}(\Theta_e(\omega, x_0)) = 0, \mathrm{Im}(\Theta(\omega, x_1)) = 0,\ x_0 \neq x_1\}$$

actually coincides with $\{\omega \in \mathbb{C} : \exists x_0 \in \Omega : \Theta_e(\omega, x) = 0\}$. In fact, it is easy to check that $\mathrm{Im}(\Theta(\omega, x)) = 2\,\mathrm{Re}\,\omega\,\mathrm{Im}\,\omega + \mathrm{Re}\,\omega\gamma_e$ does not depend on $x \in \Omega$, and therefore $\mathrm{Im}(\Theta(\omega, x_1)) = 0$ for $x_1 \in \Omega$ if and only if $\mathrm{Im}(\Theta(\omega, x)) = 0$ for all $x \in \Omega$.

It is clear that if either $\mathrm{Re}(\Theta_e(\omega))$ or $\mathrm{Im}(\Theta_e(\omega))$ is strictly positive (or strictly negative) in the whole of Ω then by the Lax-Milgram theorem the problem

$$(\Theta_e(\omega)\nabla u, \nabla v) = \langle F, v\rangle,\ \ F \in H^{-1}(\Omega),\ u, v \in \dot{H}_0^1(\Omega)$$

has a unique solution $u_F \in \dot{H}_0^1(\Omega)$. This proves the inclusion

$$\sigma_e(G(\omega)) \subseteq \{\omega \in \mathbb{C} : \exists x_0 \in \Omega : \Theta_e(\omega, x) = 0\}$$

The reverse inclusion (which uses the continuity of θ_e) follows by constructing quasi-modes as in the proof of [18, Proposition 27]. $\square$

5. Limiting essential spectrum and spectral pollution

The aim of this section is to enclose the set of spectral pollution for the domain truncation method applied to the Drude-Lorentz pencil $\mathcal{L}$. We start by recalling three definitions.

Definition 5.1. For a family of operator-valued functions $(F_n(\cdot))$ defined on some set $K \subset \mathbb{C}$, the *limiting approximate point spectrum*, denoted $\sigma_{\mathrm{app}}((F_n)_n)$, is the set of $\omega \in K$ such that there exists a sequence $(u_n)_n$ with $u_n \in \mathrm{dom}(F_n(\omega))$ for each n, $\|u_n\| = 1$, as $n \to \infty$ and $\|F_n(\omega)u_n\| \to 0$ as $n \to \infty$. For a family of operators $(F_n)_n$, one takes $K = \mathbb{C}$ and the requirement is that $\|(F_n - \omega I)u_n\| \to 0$ as $n \to \infty$.

Definition 5.2. For a family of operator-valued functions $(F_n(\cdot))$ defined on some set $K \subset \mathbb{C}$, the *limiting essential spectrum*, denoted $\sigma_e((F_n)_n)$, is the subset of $\sigma_{\mathrm{app}}((F_n)_n)$ consisting of $\omega \in K$ for which the sequence $(u_n)_n$ as in the Definition 5.1 has the additional property $u_n \rightharpoonup 0$, $n \to \infty$.

Definition 5.3. For an family of operator-valued functions $(F_n(\cdot))$ defined on some set $K \subset \mathbb{C}$, the *region of boundedness*, denoted $\Delta_b((F_n)_n)$, is the set of $\omega \in K$ such that $(F_n(\omega))^{-1}$ exists for all sufficiently large n and $\limsup_{n\to\infty} \|(F_n(\omega))^{-1}\| < +\infty$. For a family of operators $(F_n)_n$, then $\Delta_b((F_n)_n)$, is the set of $\omega \in \mathbb{C}$ such that $(F_n - \omega I)^{-1}$ exists for all sufficiently large n and $\limsup_{n\to\infty} \|(F_n - \omega I)^{-1}\| < +\infty$.

Given an unbounded Lipschitz domain Ω, let $(\Omega_n)_n$ be a monotonically increasing sequence of Lipschitz bounded domains exhausting Ω. Note that we do not make any assumption on the topology of Ω; in particular, $\mathbb{R}^3 \setminus \overline{\Omega}$ may have infinitely many connected components. We will denote by $\mathcal{A}_n$, $\mathcal{L}_n$, $\mathcal{S}_{1,n}$, *etc.*, the operators or pencils obtained from $\mathcal{A}$, $\mathcal{L}$, $\mathcal{S}_1$ by replacing Ω

with Ω_n in their domain definitions. In order to avoid cumbersome notation we will not use the subscript n to denote the restriction of multiplication operators to Ω_n.

The domain truncation method consists of finding $\sigma(\mathcal{L}_n)$, $n \in \mathbb{N}$, in the hope that for sufficiently large n these will be good approximations to $\sigma(\mathcal{L})$. Ideally, one would like to prove that $(\mathcal{L}_n)_n$ is a spectrally exact approximation of $\mathcal{L}$, that is for every $\omega \in \sigma(\mathcal{L})$ there exists $\omega_n \in \sigma(\mathcal{L}_n)$, $n \in \mathbb{N}$ such that $\omega_n \to \omega$; and conversely, every limit point of sequences $(\omega_n)_n$ with $\omega_n \in \sigma(\mathcal{L}_n)$, $n \in \mathbb{N}$, lie in $\sigma(\mathcal{L})$. However, this is false in general. Indeed, spectral pollution may appear due to the non-self-adjointness of the operators involved. Therefore, our strategy will be to enclose the set of spectral pollution in a (possibly) small subset of $\mathbb{C}$ and to show that we can approximate exactly the discrete points of $\sigma(\mathcal{L})$ outside the set of spectral pollution.

We begin with a result about the generalised resolvent convergence of the operators involved.

Theorem 5.4. *The following statements hold.*

(i) $\mathcal{A}_n \xrightarrow{gsr} \mathcal{A}$, $n \to \infty$.

(ii) $\mathcal{L}_n(\cdot) \xrightarrow{gsr} \mathcal{L}(\cdot)$ *for all* $\omega \in (\Delta_b((\mathcal{L}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\}) \cap \varrho(\mathcal{L})$, $n \to \infty$

(iii) $\Delta_b((\mathcal{L}_n)_n) \cap \Sigma = \Delta_b((\mathcal{S}_{1,n})_n) \cap \Sigma$

(iv) $\mathcal{S}_{1,n}(\cdot) \xrightarrow{gsr} \mathcal{S}_1(\cdot)$, *for all* $\omega \in (\Delta_b((\mathcal{L}_n)_n) \cap \Sigma) \cap \varrho(\mathcal{L})$, $n \to \infty$.

Proof. (i) According to Proposition 2.1, $W(\mathcal{A}_n) \subset \mathbb{R} \times [-i\gamma_e, 0]$ for all $n \in \mathbb{N}$, and the same enclosure holds for the numerical range of $\mathcal{A}$ as well. We then deduce that $\|(\mathcal{A}_n - \lambda)^{-1}\| \leq \mathrm{dist}(\lambda, W(\mathcal{A}_n))^{-1} \leq (\min\{|\operatorname{Im}\lambda|, |\operatorname{Im}\lambda + \gamma_e|\}^{-1})$ for all $\lambda \in (\mathbb{R} \times [-i\gamma_e, 0])^c$. In particular, $\Delta_b((\mathcal{A}_n)_n) \cap \varrho(\mathcal{A}) \neq \emptyset$. The matrices $\mathcal{A}_n$, $n \in \mathbb{N}$ and $\mathcal{A}$ are diagonally dominant of order 0 for all n since the operators B_n, $n \in \mathbb{N}$ in the matrix representation (1.3) are bounded for all n. Let P_n denote projection from $L^2(\Omega)^3$ to $L^2(\Omega_n)^3$ by restriction, so that $P_n \xrightarrow{s} I_{L^2(\Omega)^3}$ as $n \to \infty$. It is clear that $B_n = B P_n$ converges strongly to B, and similarly $D_n = D P_n \to D$ strongly. Due to [10, Thm. 3.1], to conclude that $\mathcal{A}_n \xrightarrow{gsr} \mathcal{A}$ it is enough to show that there exists a core Φ of $H_0(\mathrm{curl}, \Omega) \oplus H(\mathrm{curl}, \Omega)$ such that $\|\mathcal{A}_n P_n u - \mathcal{A}u\| \to 0$ for all $u \in \Phi$. This last property is satisfied by $C_c^\infty(\Omega)^3 \oplus C_c^\infty(\overline{\Omega})^3$ because $\mathrm{curl}_0 P_n \varphi = \mathrm{curl}_0 \varphi$ for all $\varphi \in C_c^\infty(\Omega)^3$ and sufficiently large n, and $\mathrm{curl} P_n \psi = P_n \mathrm{curl}\,\psi$ in Ω_n for all $\psi \in C_c^\infty(\overline{\Omega})^3$.

(ii) To prove $\mathcal{L}_n(\omega) \xrightarrow{gsr} \mathcal{L}(\omega)$ for $\omega \in (\Delta_b((\mathcal{L}_n)_n) \cap \varrho(\mathcal{L})) \setminus \{-i\gamma_m, -i\gamma_e\}$, $n \in \mathbb{N}$ it is sufficient to write

$$(\mathcal{A}_n - \omega)^{-1} = \begin{pmatrix} \mathcal{L}_n(\omega)^{-1} & -\mathcal{L}_n(\omega)^{-1} B(-iD - \omega)^{-1} \\ (-iD - \omega)^{-1} B \mathcal{L}_n(\omega)^{-1} & (-iD - \omega)^{-1}(I + B\mathcal{L}_n(\omega)^{-1} B(-iD - \omega)^{-1}) \end{pmatrix}$$

hence, since $(\mathcal{A}_n - \omega)^{-1}(P_n F, 0)^t \to (\mathcal{A} - \omega)^{-1}(F, 0)^t$ for all $F \in L^2(\Omega)^6 \oplus L^2(\Omega)^6$, we deduce that $\mathcal{L}_n(\omega)^{-1} P_n F \to \mathcal{L}(\omega)^{-1} F$, $n \to \infty$.

(iii) Let $\omega \in (\Delta_b((\mathcal{L}_n)_n) \cap \Sigma)$, so that $\sup_{n \geq n_0} \|(\mathcal{L}_n(\omega))^{-1}\| < \infty$. The identity

$$(\mathcal{L}_n(\omega))^{-1} = \begin{pmatrix} \mathcal{S}_{1,n}(\omega)^{-1} & -\mathcal{S}_{1,n}(\omega)^{-1} i\,\mathrm{curl}(V_m(\omega))^{-1} \\ (V_m(\omega))^{-1} i\,\mathrm{curl}_0\, \mathcal{S}_{1,n}(\omega)^{-1} & (V_m(\omega))^{-1}(I + \mathrm{curl}_0\, \mathcal{S}_{1,n}(\omega)^{-1} \mathrm{curl}(V_m(\omega))^{-1}) \end{pmatrix}$$

$$(5.1)$$

implies that $\|\mathcal{L}_n(\omega)^{-1}\| \geq \|\mathcal{S}_{1,n}(\omega)^{-1}\|$, $n \geq n_0$, hence $\sup_{n \geq n_0} \|(\mathcal{S}_{1,n}(\omega))^{-1}\| < \infty$; equivalently, $\omega \in (\Delta_b((\mathcal{S}_{1,n})_n) \cap \Sigma)$.

Conversely, if $\omega \in (\Delta_b((\mathcal{S}_{1,n})_n) \cap \Sigma)$ then, by definition of region of boundedness, there exists $n_0 \in \mathbb{N}$ and $C > 0$ such that $\omega \in \varrho(\mathcal{S}_{1,n})$ for $n \geq n_0$ and $\sup_{n \geq n_0} \|\mathcal{S}_{1,n}(\omega)^{-1}\| \leq C$. Hence, if $f \in L^2(\Omega)^3$, the equation $\mathcal{S}_{1,n}(\omega)u_n = P_n f$ has a unique solution $u_n \in L^2(\Omega_n)^3$ with the uniform *a priori* bound $\|u_n\|_{L^2(\Omega_n)^3} \leq C\|f\|_{L^2(\Omega)^3}$. This implies that

$$|\langle \Theta_m(\omega)^{-1} \operatorname{curl}_0 u_n, \operatorname{curl}_0 u_n \rangle| \leq |\langle P_n f, u_n \rangle| + |\langle V_e(\omega)u_n, u_n \rangle|$$

hence

$$c_1(\omega)\|\operatorname{curl}_0 u_n\|^2 \leq \frac{\|f\|^2}{4\delta} + (\delta + \|V_e(\omega)\|)\|u_n\|^2 \leq [C^2(\delta + \|V_e(\omega)\|) + 1/4\delta]\|f\|^2$$

and all the constants appearing in the previous estimate are independent of $n \geq n_0$, so in particular

$$\sup_{n \geq n_0} \|\operatorname{curl}_0 \mathcal{S}_{1,n}(\omega)^{-1}\| \leq C'(\omega),$$

where we can set for example $C' = c_1(\omega)^{-1}[C^2(1 + \|V_e(\omega)\|) + 1/4]$. Now we can repeat the previous estimate starting from elements $f = \operatorname{curl} g \in \operatorname{curl} L^2$, where we note that $\langle P_n \operatorname{curl} g, u_n \rangle = \langle g, \operatorname{curl}_0 u_n \rangle$ can be estimated in term of $\operatorname{curl}_0 u_n$ which is uniformly bounded in terms of the datum by the previous discussion. Altogether we obtain that there exists a constant C'' depending on C' and ω such that

$$\sup_{n \geq n_0} \|\operatorname{curl} \mathcal{S}_{1,n}(\omega)^{-1} \operatorname{curl}_0\| \leq C''$$

Now the claim of the theorem follows from (5.1) since the right-hand side therein is uniformly bounded in n, $n \geq n_0$.

(iv) Finally, $\mathcal{S}_{1,n}(\omega)^{-1} P_n \xrightarrow{s} \mathcal{S}_1(\omega)^{-1}$ for all $\omega \in (\Delta_b((\mathcal{L}_n)_n) \cap \Sigma)$ follows by observing that $\omega \in \Delta_b((\mathcal{L}_n)_n) \cap \Sigma$, hence (ii) implies that $\mathcal{L}_n(\omega)^{-1} P_n \xrightarrow{s} \mathcal{L}(\omega)^{-1}$; therefore (5.1) implies that $\mathcal{S}_{1,n}(\omega)^{-1} P_n \xrightarrow{s} \mathcal{S}_1(\omega)$ (by direct calculation on vectors $(f,0)^t$). $\quad\square$

Corollary 5.5. $\sigma_{\mathrm{poll}}((\mathcal{A}_n)_n) \subset \sigma_e((\mathcal{A}_n)_n)$.

Proof. From Definition (1.3), $\mathcal{A}_n$ is seen to be $\mathcal{J}$-self-adjoint with respect to $\mathcal{J} = \operatorname{diag}(i, -i, i, -i)J$, with $Ju = \bar{u}$ the componentwise complex conjugation. The result is then a consequence of Thm. 2.3 in [29]. $\quad\square$

Proposition 5.6. $\sigma_{\mathrm{poll}}((\mathcal{L}_n)_n) = \sigma_{\mathrm{poll}}((\mathcal{A}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\}$

Proof. [17, Thm. 2.3.3(ii)] implies that, for fixed n, $\sigma(\mathcal{L}_n) = \sigma(\mathcal{A}_n) \setminus \{-i\gamma_e, -i\gamma_m\}$ and similarly $\sigma(\mathcal{L}) = \sigma(\mathcal{A}) \setminus \{-i\gamma_e, -i\gamma_m\}$. Hence, $\sigma_{\mathrm{poll}}((\mathcal{L}_n)_n) \subset \sigma_{\mathrm{poll}}((\mathcal{A}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\}$. Conversely, one may observe that if $\lambda_n \in \sigma(\mathcal{A}_n)$, $\lambda_n \to \lambda \in (\varrho(\mathcal{A}) \setminus \{-i\gamma_e, -i\gamma_m\})$, then for big enough n $\lambda_n \notin \{-i\gamma_e, -i\gamma_m\}$, so $\lambda_n \in \sigma(\mathcal{L}_n)$ and $\lambda_n \to \lambda \in \varrho(\mathcal{L})$. Thus, $\lambda \in \sigma_{\mathrm{poll}}((\mathcal{L}_n)_n)$. $\quad\square$

Remark 5.7. The poles $\{-i\gamma_e, -i\gamma_m\}$ may or may not be in the essential spectrum of $\mathcal{A}$. If B is compactly supported and A restricted to divergence-free vector field has compact resolvent, then

the poles belong to $\sigma_e(\mathcal{A})$, see [15, Prop. 2.2] for a proof in a similar setting. However, for the purposes of this paper we are not interested in the poles that are out of the domain of definition of $\mathcal{L}$.

Proposition 5.8. $\sigma_e((\mathcal{A}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\} = \sigma_e((\mathcal{L}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\}$

Proof. $\mathcal{A}_n$ is diagonally dominant for every n and the norms of the off-diagonal entries do not depend on n; furthermore, $\Delta_b((D_n)_n = \mathbb{C} \setminus \{-i\gamma_e, -i\gamma_m\}$. The result therefore follows from [30, Proposition 2.3.4(i),(iii)]. $\square$

Proposition 5.9. *The following identities hold.*

(i) $\sigma_e((\mathcal{L}_n)_n) \cup \sigma_p(\mathcal{L}) = \sigma_{\mathrm{app}}((\mathcal{L}_n)_n)$;
(ii) $(\sigma_e((\mathcal{S}_{1,n})_n) \cup \sigma_p(\mathcal{S}_1)) \cap \Sigma = \sigma_{\mathrm{app}}((\mathcal{S}_{1,n})_n) \cap \Sigma$.

Proof. The proof is a generalisation of [29, Prop. 2.15(ii)] to families of operators.

Observe that the inclusion $\sigma_p(\mathcal{L}) \subset \sigma_{\mathrm{app}}((\mathcal{L}_n)_n)$ (and the analogous inclusion for $(\mathcal{S}_{1,n})_n$)) are consequence of the gsr convergence of $\mathcal{L}_n$ to $\mathcal{L}$ established in Theorem 5.4. Indeed, we claim that if $\mathcal{L}_n(\omega_0) \xrightarrow{gsr} \mathcal{L}(\omega_0)$ for some $\omega_0 \in \Delta_b((\mathcal{L}_n)_n) \cap \varrho(\mathcal{L})$, then for all $u \in \mathrm{dom}(\mathcal{L})$ there exists a sequence $u_n \in \mathrm{dom}(\mathcal{L}_n)$, $\|u_n\| = 1$ such that $\|u_n - u\| \to 0$, $\|\mathcal{L}_n(\omega)u_n - \mathcal{L}(\omega)u\| \to 0$, $n \to \infty$, $\omega \in \mathbb{C} \setminus \{-i\gamma_e, -i\gamma_m\}$. Assuming the claim is satisfied, if $\omega \in \sigma_p(\mathcal{L})$ with eigenfunction u, then there exists an approximating sequence $(u_n)_n$ as above, and therefore $\sigma_p(\mathcal{L}) \subset \sigma_{\mathrm{app}}((\mathcal{L}_n)_n)$.

To prove the claim, one first realises that if $\omega \in \Delta_b((\mathcal{L}_n)_n) \cap \varrho(\mathcal{L})$ then the sequence $u_n := \mathcal{L}_n(\omega)^{-1} P_n \mathcal{L}(\omega)u$ has the required properties. In the general case, one first note that for $t < 0$ sufficiently big $\omega \in \Delta_b((\mathcal{L}_n(\cdot) + it)) \cap \varrho(\mathcal{L}(\cdot) + it)$, see the proof of Lemma 2.11. Therefore, given $u \in \mathrm{dom}(\mathcal{L} + it) = \mathrm{dom}(\mathcal{L})$, there exists a sequence $u_n \in \mathrm{dom}(\mathcal{L}_n(\cdot) + it) = \mathrm{dom}(\mathcal{L}_n)$ such that $\|u_n - u\| \to 0$ and $\|(\mathcal{L}_n(\omega) + it)u_n - (\mathcal{L}(\omega) + it)u\| \to 0$, and therefore $\|\mathcal{L}_n(\omega)u_n - \mathcal{L}(\omega)u\| \to 0$, as claimed.

The inclusion "$\subset$" in (i) and (ii) is immediate from Definitions 5.1, 5.2, and the previous observation.

We now prove that $\sigma_e((\mathcal{L}_n)_n) \cup \sigma_p(\mathcal{L}) \supset \sigma_{\mathrm{app}}((\mathcal{L}_n)_n)$. Let $\omega \in \sigma_{\mathrm{app}}((\mathcal{L}_n)_n)$, that is, there exists a sequence of elements $u_n \in \mathrm{dom}(\mathcal{L}_n)$, $\|u_n\| = 1$, $n \in \mathbb{N}$, such that $\|\mathcal{L}_n(\omega)u_n\| \to 0$. Since the unit ball in a Hilbert space is weakly compact, we may assume that, up to a subsequence, $u_n \rightharpoonup u$ in $L^2(\Omega)^3 \oplus L^2(\Omega)^3$. If $u = 0$, then $\omega \in \sigma_e((\mathcal{L}_n)_n)$, concluding the proof. Assume then $u \neq 0$. Theorem 5.4(iii) implies that there exists $\omega_0 \in \Delta_b((\mathcal{L}_n)_n) \cap \varrho(\mathcal{L})$ such that $\mathcal{L}_n(\omega_0) \xrightarrow{gsr} \mathcal{L}(\omega_0)$. Now,

$$\mathcal{L}_n(\omega_0)u_n = (\mathcal{L}_n(\omega_0) - \mathcal{L}_n(\omega))u_n + \mathcal{L}_n(\omega)u_n \tag{5.2}$$

and we notice that $\mathcal{L}_n(\omega_0) - \mathcal{L}_n(\omega) := \mathcal{B}(\omega, \omega_0)\mathrm{diag}(P_n, P_n)$, where $\mathcal{B}$ is a bounded 2×2 block operator matrix not depending on n. Taking the inverse of $\mathcal{L}_n(\omega_0)$ in (5.2) gives

$$u_n = \mathcal{L}_n(\omega_0)^{-1}\mathcal{B}\,\mathrm{diag}(P_n, P_n)u_n + \varepsilon_n, \quad \varepsilon_n := \mathcal{L}_n(\omega_0)^{-1}\mathcal{L}_n(\omega)u_n.$$

Now, $\varepsilon_n \to 0$, $n \to \infty$ because $\omega_0 \in \Delta_b((\mathcal{L}_n)_n)$ and $\mathcal{L}_n(\omega)u_n \to 0$, $n \to \infty$. Moreover, the weak convergence $u_n \rightharpoonup u$, the strong convergence $\mathcal{L}_n(\omega_0)^{-1} \xrightarrow{s} \mathcal{L}(\omega_0)^{-1}$ and the uniqueness of the weak limit imply that

$$u = \mathcal{L}(\omega_0)^{-1} \mathcal{B} u,$$

or equivalently, since $\mathcal{B} = \mathcal{L}(\omega_0) - \mathcal{L}(\omega)$, that $\mathcal{L}(\omega)u = 0$. Thus, $\omega \in \sigma_p(\mathcal{L})$.

The proof of $(\sigma_e((\mathcal{S}_{1,n})_n) \cup \sigma_p(\mathcal{S}_1)) \cap \Sigma \supset \sigma_{\mathrm{app}}((\mathcal{S}_{1,n})_n) \cap \Sigma$ in (ii) is analogous to the proof of the same inclusion in (i) with $\mathcal{S}_{1,n}(\omega)$ replacing $\mathcal{L}_n(\omega)$; in place of $\mathcal{S}_{1,n}(\omega_0)$ for some $\omega_0 \in \Delta_b((\mathcal{S}_{1,n})_n)$ we choose $\mathcal{S}_{1,n}(\omega) + it$ for $t < 0$ big enough. Notice that this is possible since $\omega \in \Delta_b((\mathcal{S}_{1,n}(\cdot) + it)_n)$ for $t < 0$ large enough, see proof of Lemma 2.11. The proof is concluded. $\square$

Theorem 5.10. $\sigma_e((\mathcal{L}_n)_n) \cap \Sigma = \sigma_e((\mathcal{S}_{1,n})_n) \cap \Sigma$

Proof. Proposition 5.9 implies

$$\sigma_e((\mathcal{L}_n)_n) \cup \sigma_p(\mathcal{L}) = \sigma_{\mathrm{app}}((\mathcal{L}_n)_n), \quad \sigma_e((\mathcal{S}_{1,n})_n) \cup \sigma_p(\mathcal{S}_1) = \sigma_{\mathrm{app}}((\mathcal{S}_{1,n})_n)$$

Now, the definitions of region of boundedness and of limiting approximate point spectrum, together with the equality $\sigma_{\mathrm{app}}((\mathcal{L}_n)_n) = \sigma_{\mathrm{app}}((\mathcal{L}_n^*)_n)^*$, imply that $\sigma_{\mathrm{app}}((\mathcal{L}_n)_n) = \mathbb{C} \setminus \Delta_b((\mathcal{L}_n)_n)$ and $\sigma_{\mathrm{app}}((\mathcal{S}_{1,n})_n) = \mathbb{C} \setminus \Delta_b((\mathcal{S}_{1,n})_n)$. By Theorem 2.12, $\sigma_p(\mathcal{L}) \cap \Sigma = \sigma_p(\mathcal{S}_1) \cap \Sigma$; and as a consequence of Theorem 5.4(iii), $\Delta_b((\mathcal{S}_{1,n})_n) \cap \Sigma = \Delta_b((\mathcal{L}_n)_n) \cap \Sigma$. Thus, up to intersection with Σ we have

$$\sigma_e((\mathcal{L}_n)_n) \cup \sigma_p(\mathcal{L}) = \sigma_{\mathrm{app}}((\mathcal{L}_n)_n) = \mathbb{C} \setminus \Delta_b((\mathcal{L}_n)_n) = \mathbb{C} \setminus \Delta_b((\mathcal{S}_{1,n})_n)$$

$$= \sigma_{\mathrm{app}}((\mathcal{S}_{1,n})_n) = \sigma_e((\mathcal{S}_{1,n})_n) \cup \sigma_p(\mathcal{S}_1). \quad \square$$

In the following proposition we use the notion of *discrete compactness* for sequences of operators in varying Hilbert spaces. We refer to [10, Def. 2.5] and the references therein for the relevant definitions and properties.

Proposition 5.11. *Given $n \in \mathbb{N}$, the following equality holds.*

$$\sigma_e((\mathcal{S}_{1,n})_n) \cap \Sigma = \big(\sigma_e((\mathcal{S}_{\infty,n})_n) \cup (\sigma_e((G_n)_n))\big) \cap \Sigma),$$

where G_n is defined as in (3.8).

Proof. This can be proved along the lines of [11, Section 7, Section 8]. For the sake of completeness we recall here the main steps of the proof.

Observe that Proposition 3.1 implies that $M_{e,n}(\omega) = (\Theta_e(\omega) - \Theta_{e,\infty}(\omega)) P^n_{\ker(\mathrm{div})}$ is a compact operator from $H(\mathrm{curl}, \Omega_n)$ to $L^2(\Omega_n)^3$ for every n. According to the decomposition of the coefficients (3.1), up to an operator which is vanishing uniformly in n, the sequence $M_{e,n}(\omega)$ is compactly supported in Ω_n for each n and equals $(\Theta_e(\omega) - \Theta_{e,\infty}(\omega))\chi_{\Omega_R \cap \Omega_n} P^n_{\ker(\mathrm{div})}$, where $\Omega_R = \Omega \cap B(0, R)$ contains the compact support of $\Theta_e(\omega) - \Theta_{e,\infty}(\omega)$. This last sequence of operators is clearly discretely compact from $H(\mathrm{curl}, \Omega_n)$ to $L^2(\Omega_n)^3$ because of the compact embedding of $H(\mathrm{curl}, \Omega_R) \cap H(\mathrm{div}\, 0, \Omega_R)$ into $L^2(\Omega_R)$. Since discretely compact perturbations do not modify the limiting essential spectrum, $\sigma_e(((\mathcal{S}_{1,n})_n) \cap \Sigma = \sigma_e((\mathcal{S}_{1,n} + M_{e,n}))_n) \cap \Sigma$. Now we note that

$$\mathcal{S}_{1,n}(\omega) + M_{e,n}(\omega) = \operatorname{curl} \Theta_m(\omega)^{-1} \operatorname{curl}_0 - \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} P_{\ker(\operatorname{div})}$$
$$- \frac{\Theta_e(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} P_\nabla$$

has a triangular block operator matrix representation with respect to the Helmholtz decomposition $\nabla H_0^1(\Omega_n) \oplus H(\operatorname{div} 0, \Omega_n)$. More specifically, if we define $\mathcal{S}_{m,n}(\omega)$, as $\mathcal{S}_m$ with Ω_n replacing Ω in the domain definition, we have

$$\mathcal{S}_{1,n}(\omega) + M_{e,n}(\omega) \simeq \mathcal{T}_n(\omega) = \begin{pmatrix} -P_\nabla^n V_e(\omega) P_\nabla^n & 0 \\ -P_{\ker(\operatorname{div})}^n V_e(\omega) P_\nabla^n & \mathcal{S}_{m,n} \end{pmatrix}.$$

Now since $\mathcal{T}_n$ is a sequence of triangular block operator matrices with bounded off-diagonal entries and with J-selfadjoint diagonal entries, Theorem 3.6 implies that

$$\sigma_e((\mathcal{T}_n)_n) \cap \Sigma = (\sigma_e((-P_\nabla^n V_e(\omega) P_\nabla^n)_n \cup (\sigma_e((\mathcal{S}_{m,n})_n) \cap \Sigma)$$

Now [11, Proposition 7.3] implies that $\sigma_e((\mathcal{S}_{m,n})_n) = \sigma_e((\mathcal{S}_{\infty,n})_n)$. We give here a sketch of the proof. The proof is modelled upon the proof of Proposition 3.4. The idea is to establish that for a suitably chosen $\omega \in \Sigma$ the difference $\mathcal{K}_n(\omega) = \mathcal{S}_{m,n}(\omega)^{-1} - \mathcal{S}_{\infty,n}(\omega)^{-1}$ is discretely compact and that $\mathcal{K}_n(\omega)^* P_n$ is strongly convergent. Then the equality of the limiting essential spectra follows from [29, Thm. 2.12(ii)]. The strong convergence

$$\mathcal{K}_n(\omega)^* P_n = \mathcal{S}_{m,n}(\omega)^{-*} P_n - \mathcal{S}_{\infty,n}(\omega)^{-*} P_n \xrightarrow{s} \mathcal{S}_m(\omega)^{-*} - \mathcal{S}_\infty(\omega)^{-*}$$

for $\omega \in \Delta_b((\mathcal{K}_n)_n) \cap \varrho(\mathcal{S}_m) \cap \varrho(\mathcal{S}_\infty)$ can be proved along the lines of Theorem 5.4(iv).

It remains to prove that $(\mathcal{K}_n(\omega))_n$ is discretely compact. Arguing as in the proof of Proposition 3.4 it may be shown that

$$\mathcal{K}_n(\omega) = (\operatorname{curl}_0(\mathcal{C}_{m,n}(\omega) - \overline{z_\omega})^{-1})^* (\Theta_{m,\infty}(\omega)^{-1} - \Theta_m(\omega)^{-1}) \operatorname{curl}_0(\mathcal{C}_{\infty,n}(\omega) - z_\omega)^{-1} \quad (5.3)$$

As a consequence of proof of Theorem 5.4(iv),

$$(\operatorname{curl}_0(\mathcal{C}_{m,n}(\omega) - \overline{z_\omega})^{-1})^* P_n \xrightarrow{s} (\operatorname{curl}_0(\mathcal{C}_m(\omega) - \overline{z_\omega})^{-1})^*$$
$$\operatorname{curl}_0(\mathcal{C}_{\infty,n}(\omega) - z_\omega)^{-1} P_n \xrightarrow{s} \operatorname{curl}_0(\mathcal{C}_\infty(\omega) - z_\omega)^{-1},$$

for suitably chosen ω. Moreover, due to Remark 3.5 there exists $C > 0$ such that, for $\omega \notin \overline{W(\mathcal{C}_\infty)}$,

$$\sup_{n \in \mathbb{N}} \|\operatorname{curl}_0(\mathcal{C}_{\infty,n}(\omega) - z_\omega)^{-1}\|_{\mathcal{B}(H(\operatorname{div} 0, \Omega_n); H(\operatorname{curl}, \Omega_n))} \leq C,$$

hence $\operatorname{curl}_0(\mathcal{C}_{\infty,n}(\omega) - z_\omega)^{-1} u_n$ is uniformly bounded in $H(\operatorname{curl}, \Omega_n)$ for every sequence $u_n \in L^2(\Omega_n)^3$, $\|u_n\| \leq 1$. Therefore, by (5.3), the discrete compactness of $\mathcal{K}_n(\omega)$ boils down to the discrete compactness of $(\Theta_{m,\infty}(\omega)^{-1} - \Theta_m(\omega)^{-1}) P_{\ker(\operatorname{div})}^n$ from $H(\operatorname{curl}, \Omega_n)$ to $L^2(\Omega_n)^3$. The proof of this last property is identical to the proof of the discrete compactness of the sequence $(\Theta_{e,\infty}(\omega) - \Theta_e(\omega)) P_{\ker(\operatorname{div})}^n$, which was established above. Altogether we have

$$\sigma_e((\mathcal{T}_n)_n) \cap \Sigma = (\sigma_e((-P_\nabla^n V_e(\omega) P_\nabla^n)_n) \cup \sigma_e((\mathcal{S}_{\infty,n})_n)) \cap \Sigma,$$

and the result follows by recalling that $G_n(\omega) = -P_\nabla^n V_e(\omega) P_\nabla^n$. $\quad\square$

We recall the following standard result, a proof of which can be found in [11, Lemma 7.4].

Lemma 5.12. *Let $n \in \mathbb{N}$. The closure of $C_c^\infty(\Omega_n)^3 \cap H(\mathrm{div}\, 0, \Omega_n)$ with respect to the $H(\mathrm{curl}, \Omega_n)$-norm is $H_0(\mathrm{curl}, \Omega_n) \cap H(\mathrm{div}\, 0, \Omega_n)$.*

Theorem 5.13. *The following enclosures hold:*

$$\sigma_{\mathrm{poll}}((\mathcal{L}_n)_n) \cap \Sigma \subset \sigma_e((\mathcal{L}_n)_n) \cap \Sigma \subset \left(W_e(\mathcal{S}_\infty) \cup \sigma_e(G) \right) \cap \Sigma, \tag{5.4}$$

and therefore $(\sigma_{\mathrm{poll}}((\mathcal{L}_n)_n) \cap \Sigma) \subset (W_e(\mathcal{S}_\infty) \cap \Sigma)$. For every isolated $\omega \in (\sigma_p(\mathcal{L}) \cap \Sigma)$ outside $W_e(\mathcal{S}_\infty) \cup \sigma_e(G)$ there exists a sequence $\omega_n \in \sigma(\mathcal{L}_n)$, $n \in \mathbb{N}$, such that $\omega_n \to \omega$ as $n \to \infty$.

Proof. The enclosure of spectral pollution in the limiting essential spectrum follows from Proposition 5.6 and Corollary 5.5. For the enclosure (5.4) we argue as in [11, Theorem 7.5]. If $\omega \in \sigma_e((\mathcal{S}_{\infty,n})_{n\in\mathbb{N}})$, by definition there exist $w_n \in \mathrm{dom}\,\mathcal{S}_{\infty,n}(\omega) \subset H_0(\mathrm{curl}, \Omega_n) \cap H(\mathrm{div}\, 0, \Omega_n)$, $\|w_n\| = 1$, $n \in \mathbb{N}$, such that $w_n \rightharpoonup 0$ and $\mathcal{S}_{\infty,n}(\omega)w_n \to 0$ as $n \to \infty$. Taking the scalar product with w_n, we find that

$$\langle \mathcal{S}_{\infty,n}(\omega)w_n, w_n \rangle = \Theta_{m,\infty}(\omega)^{-1} \|\mathrm{curl}_0\, w_n\|^2 - \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \to 0$$

as $n \to \infty$. By Lemma 5.12, for each $n \in \mathbb{N}$ there exists $v_n \in C_c^\infty(\Omega_n)^3 \cap H(\mathrm{div}\, 0, \Omega_n)$ with $\|v_n - w_n\|^2 \leq 1/n$, $\|\mathrm{curl}(v_n - w_n)\|^2 \leq 1/n$. Let $v_n^0 \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}\, 0, \Omega)$ be the extension of v_n to Ω by zero for $n \in \mathbb{N}$. Then

$$\left| \Theta_{m,\infty}(\omega)^{-1} \|\mathrm{curl}\, v_n^0\|^2 - \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \|v_n^0\|^2 \right|$$

$$\leq \left| \Theta_{m,\infty}(\omega)^{-1} \|\mathrm{curl}\, w_n\|^2 - \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \|w_n\|^2 \right|$$

$$+ \frac{1}{n} \left(|\Theta_{m,\infty}(\omega)^{-1}| + \left| \frac{\Theta_{e,\infty}(\omega)}{(\omega + i\gamma_e)(\omega + i\gamma_m)} \right| \right) \to 0$$

as $n \to \infty$. Since $\|v_n^0\| \to 1$ as $n \to \infty$, upon renormalisation of the elements v_n^0, we obtain $\omega \in W_e(\mathcal{S}_\infty)$.

Next, we prove the inclusion $\sigma_e((P_\nabla^n G_n(\cdot)|_{\nabla \dot{H}_0^1(\Omega_n)})_{n\in\mathbb{N}}) \subset \sigma_e(G)$. If ω lies in $\sigma_e((P_\nabla G_n(\cdot)|_{\nabla \dot{H}_0^1(\Omega)})_{n\in\mathbb{N}})$, there exist $u_n \in \dot{H}_0^1(\Omega_n)$, $\|\nabla u_n\| = 1$, $n \in \mathbb{N}$, such that $\nabla u_n \rightharpoonup 0$ and

$$\| P_{\nabla \dot{H}_0^1(\Omega_n)} \Theta_e(\omega)^{-1} \nabla u_n \| \to 0, \quad n \to \infty.$$

Let $u_n^0 \in \dot{H}_0^1(\Omega)$ be the extension of $u_n \in \dot{H}_0^1(\Omega_n)$ to Ω by zero for $n \in \mathbb{N}$. By standard properties of Sobolev spaces, $\nabla u_n^0 = (\nabla u_n)^0$. Hence the sequence $(u_n^0)_{n \in \mathbb{N}} \subset \dot{H}_0^1(\Omega)$ is such that $\|\nabla u_n^0\| = 1$, $n \in \mathbb{N}$, $\nabla u_n^0 \rightharpoonup 0$ and

$$\| P_{\nabla \dot{H}_0^1(\Omega_n)} \Theta_e(\omega)^{-1} \nabla u_n^0 \| \to 0, \quad n \to \infty.$$

Now the claim follows if we observe that $P_\nabla := P_{\nabla \dot{H}_0^1(\Omega)} f = P_{\nabla \dot{H}_0^1(\Omega_n)} f$ for all $f \in L^2(\Omega)^3$ with $\operatorname{supp} f \subset \Omega_n$.

Finally, we consider the approximation of isolated eigenvalues which lie outside $W_e(\mathcal{S}_\infty) \cup \sigma_e(G)$ but inside Σ. Note first that $\sigma(\mathcal{A}_n) \setminus \{-i\gamma_e, -i\gamma_m\} = \sigma(\mathcal{L}_n) \setminus \{-i\gamma_e, -i\gamma_m\}$, $n \in \mathbb{N}$. Therefore [29, Theorem 2.3], applied to the sequence $(\mathcal{A}_n)_n$ approximating $\mathcal{A}$, yields that for every isolated $\omega \in \sigma(\mathcal{A})$ outside $\sigma_e((\mathcal{A}_n)_n) \cup \sigma_e((\mathcal{A}_n^*)_n)^* = \sigma_e((\mathcal{A}_n)_n)$ there exists $\omega_n \in \sigma(\mathcal{A}_n)$, $n \in \mathbb{N}$, and $\omega_n \to \omega$. Since we have already proved that $\sigma_e((\mathcal{A}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\} = \sigma_e((\mathcal{L}_n)_n) \setminus \{-i\gamma_e, -i\gamma_m\}$ and that $\sigma_e((\mathcal{L}_n)_n) \cap \Sigma \subset (W_e(\mathcal{S}_\infty) \cup \sigma_e(G)) \cap \Sigma$, we deduce that every isolated point in $\omega \in \sigma(\mathcal{A}) \cap \Sigma = \sigma(\mathcal{L}) \cap \Sigma$, outside $W_e(\mathcal{S}_\infty) \cup \sigma_e(G)$ can be approximated by spectral points $\omega_n \in \sigma(\mathcal{A}_n) \setminus \{-i\gamma_e, -i\gamma_m\} = \sigma(\mathcal{L}_n)$, concluding the proof. $\quad\square$

6. Example

We consider the Drude-Lorentz model of a dispersive metamaterial in a cuboid $K = (0, 1) \times (0, L_2) \times (0, L_3)$, embedded in an infinite waveguide $\Omega = (0, +\infty) \times (0, L_2) \times (0, L_3)$, for some $L_2, L_3 > 0$; the region $x_1 > 1$ in Ω is assumed to be a vacuum. We model the discontinuity between the vacuum and the metamaterial as a discontinuity in the Drude-Lorentz parameters θ_e and θ_m; note that when $\theta_e = \theta_m = 0$ we have the standard time-harmonic Maxwell system in the vacuum with permeability and permittivity constant and equal to 1. Specifically, we set

$$\theta_e^2(x) = \alpha_e \chi_K(x), \quad \theta_m^2(x) = \alpha_m \chi_K(x), \quad \gamma_e = t > 0, \ \gamma_m = 1,$$

where α_e, α_m are positive constants. This leads to the coupled pair of operators

$$\mathcal{L}_1(\omega) = \begin{pmatrix} -\omega & i\operatorname{curl} \\ -i\operatorname{curl}_0 & -\omega \end{pmatrix} \qquad \mathcal{L}_2(\omega) = \begin{pmatrix} -\omega + \frac{\alpha_e}{\omega + it} & i\operatorname{curl} \\ -i\operatorname{curl}_0 & -\omega + \frac{\alpha_m}{\omega + i} \end{pmatrix}$$

where $\omega \in \mathbb{C} \setminus \{-i, -it\}$, $\mathcal{L}_i(\omega)$ acts in $L^2(\Omega_i)^3 \oplus L^2(\Omega_i)^3$, $i = 1, 2$, and

$$\Omega_1 = K, \quad \Omega_2 = (1, +\infty) \times (0, L_2) \times (0, L_3).$$

According to our results it is convenient to consider the associated first Schur complements, from which we deduce that if (E, H) is an eigenfunction with eigenvalue ω then

$$\begin{cases} \operatorname{curl}\operatorname{curl}_0 E - \omega^2 E = 0, & \text{in } \Omega_1, \\ \nu \times E = 0, & \text{on } \partial\Omega_1 \cap \partial\Omega, \end{cases}$$

$$\begin{cases} \operatorname{curl}\operatorname{curl}_0 E - f(\omega) E = 0, & \text{in } \Omega_2, \\ \nu \times E = 0, & \text{on } \partial\Omega_2 \cap \partial\Omega, \end{cases}$$

in which $f(\omega)$ is defined by

$$f(\omega) = \frac{(\omega^2 + i\omega t - \alpha_e)(\omega^2 + i\omega - \alpha_m)}{(\omega + i)(\omega + it)}$$

for all $\omega \notin \{-i, -it\}$. Note that the points $\omega = 0$ and the roots of $(\omega^2 + i\omega t - \alpha_e)(\omega^2 + i\omega - \alpha_m)$ will be in the essential spectrum, since any gradient field compactly supported in Ω_i, $i = 1, 2$ will solve both systems. Define, for $n = (n_2, n_3) \in \mathbb{N}^2$,

$$\lambda_n^1(\omega) = \sqrt{\frac{\pi^2 n_2}{L_2^2} + \frac{\pi^2 n_3^2}{L_3^2} - f(\omega)}, \qquad \lambda_n^2(\omega) = \sqrt{\frac{\pi^2 n_2}{L_2^2} + \frac{\pi^2 n_3^2}{L_3^2} - \omega^2}.$$

The compatibility condition $\nu \times \operatorname{curl} E|_{x_1=0^-} = -\nu \times \operatorname{curl} E|_{x_1=0^+}$ implies that every eigenvalue ω must satisfy, for some $n \in \mathbb{N}^2$, the equation

$$\lambda_n^1(\omega) \coth(\lambda_n^1(\omega)) + \lambda_n^2(\omega) = 0. \tag{6.1}$$

Consider now the truncated domains $\Omega_X = (0, X) \times (0, L_2) \times (0, L_3)$. The compatibility condition (6.1) now becomes

$$\lambda_n^1(\omega) \coth(\lambda_n^1(\omega)) + \lambda_n^2(\omega) \coth(\lambda_n^2(\omega)(X - 1)) = 0. \tag{6.2}$$

Equations (6.1), (6.2) can be solved with a standard computational engine. For the computations, we set $t = 4$, $\alpha_e = 400$, $\alpha_m = 10$. (See Fig. 3.)

For this example, Theorem 5.13 implies that spectral pollution can only happen in $W_e(\mathcal{S}_\infty)$. Since $(\omega + i\gamma_m(\omega))\mathcal{S}_\infty(\omega) = \operatorname{curl} \operatorname{curl}_0 - \omega^2$ acting on divergence-free vector fields,

$$W_e(\mathcal{S}_\infty) = -(\operatorname{conv}(\sigma_e(\operatorname{curl} \operatorname{curl}_0)))^{1/2} \cup (\operatorname{conv}(\sigma_e(\operatorname{curl} \operatorname{curl}_0)))^{1/2}.$$

Due to the divergence-free condition, $\operatorname{curl} \operatorname{curl}_0 = -\Delta$ as differential expressions. We can now perform a standard principal symbol analysis to obtain $\sigma_e(\operatorname{curl} \operatorname{curl}_0) = [(\pi / \max\{L_2, L_3\})^2, +\infty)$, and hence

$$\sigma_e(\mathcal{S}_\infty) = W_e(\mathcal{S}_\infty) = \left(-\infty, \frac{-\pi}{\max\{L_2, L_3\}}\right] \cup \left[\frac{\pi}{\max\{L_2, L_3\}}, +\infty\right).$$

Regarding $\sigma_e(G)$, let us set $(\omega^2 + \omega i \gamma_e)/\alpha_e =: z$ and let $\omega \in \sigma_e(G)$ with associated Weyl sequence $\nabla \varphi_n \rightharpoonup 0$ in $L^2(\Omega)^3$, $\varphi_n \in \dot{H}_0^1(\Omega)$, $n \in \mathbb{N}$. We then have $-P_\nabla(z - \chi_K)\nabla \varphi_n \to 0$ in $L^2(\Omega)^3$ if and only if $-\nabla(-\Delta_{\dot{H}_0^1})^{-1} \operatorname{div}(z - \chi_K)\nabla \varphi_n \to 0$ in $L^2(\Omega)^3$, where $-\Delta_{\dot{H}_0^1}$ is the Dirichlet laplacian mapping $\dot{H}_0^1(\Omega)$ to its dual $\dot{H}^{-1}(\Omega)$. Now, note that $-\nabla(-\Delta_{\dot{H}_0^1})^{-1} \operatorname{div}(z - \chi_K)\nabla \varphi_n \to 0$ in $L^2(\Omega)^3$ if and only if $-\operatorname{div}(z - \chi_K)\nabla \varphi_n \to 0$ in $\dot{H}^{-1}(\Omega)$; the 'only if' part follows immediately by applying $\nabla(-\Delta_{\dot{H}_0^1})^{-1}$, while the 'if' part follows from definition of $\dot{H}^{-1}(\Omega)$. Therefore $\sigma_e(G)$ is completely determined by $\sigma_e(-\operatorname{div}(\cdot - \chi_K)\nabla)$ where for every $z \in \mathbb{C}$, $-\operatorname{div}(z - \chi_K)\nabla$ is understood as an operator from $\dot{H}_0^1(\Omega)$ to $\dot{H}^{-1}(\Omega)$. For smooth boundaries, the problem of finding the essential spectrum of such $\operatorname{div}(p(\cdot) - \chi_K)\nabla$ pencils
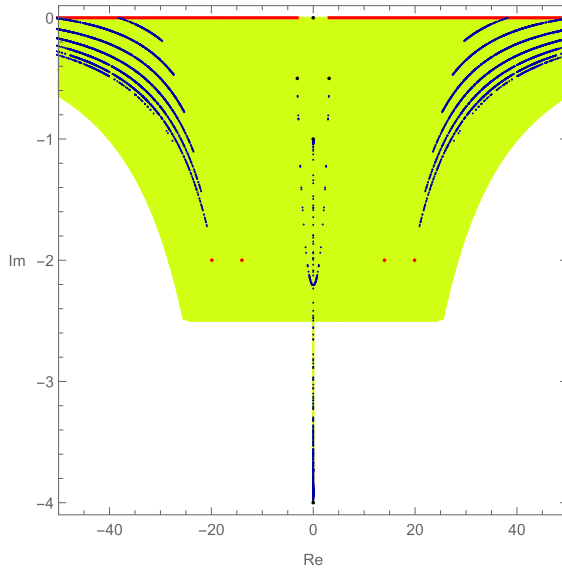
Fig. 2. Spectrum of the Drude-Lorentz model in the waveguide $\Omega = (0, +\infty) \times (0, 1) \times (0, \pi)$. The eigenvalues are in blue, the essential spectrum in red, the poles in black, and the spectral enclosure Γ of Theorem 2.13 in green. (For interpretation of the colours in the figure(s), the reader is referred to the web version of this article.)
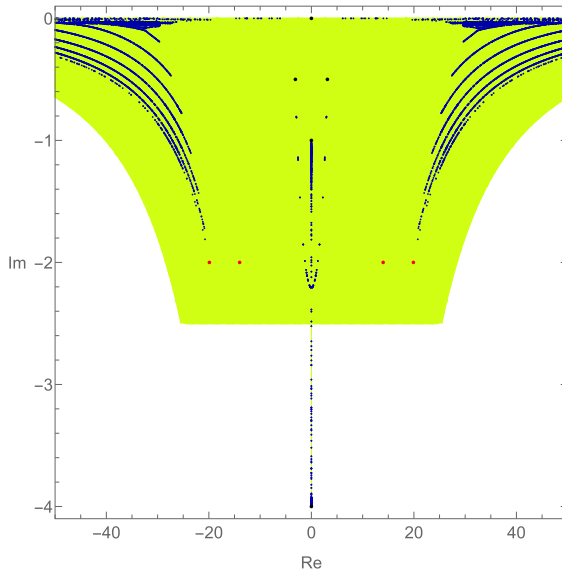

Fig. 3. Spectrum of the Drude-Lorentz model in the truncated waveguide $\Omega_X = (0, X) \times (0, 1) \times (0, \pi)$, with $X = 25$. Accumulation of eigenvalues to the real axis is clearly visible.

has been recently investigated in [20]. It is not too difficult to prove that $z = 0$ and $z = 1$ are in the essential spectrum of G. They correspond to the solutions of the two quadratic equations $\omega^2 + \omega i \gamma_e - \alpha_e = 0$ and $\omega^2 + \omega i \gamma_e = 0$. However, there are further points in the

essential spectrum corresponding to $z = 1/2$. In total, therefore, $\sigma_e(G)$ consists of the six points

$$\sigma_e(G) = \left\{ 0, -i\gamma_e, -i\frac{\gamma_e}{2} \pm \sqrt{\alpha_e - \frac{\gamma_e^2}{4}}, -i\frac{\gamma_e}{2} \pm \sqrt{\frac{\alpha_e}{2} - \frac{\gamma_e^2}{4}} \right\};$$

for the values used in the numerical experiments, namely $\alpha_e = 400$ and $\gamma_e = 4$, only 0 and $-i\gamma_e$ are purely imaginary. The four points of $\sigma_e(G)$ lying off the imaginary axis are marked in red in Fig. 2. Moreover, we claim that the eigenvalues of $\mathcal{L}$ (in blue in Fig. 2) are isolated (and of finite geometric multiplicity), and therefore Theorem 5.13 implies that they are approximated without spectral pollution via domain truncation. For the claim, note that $\sigma_{e1}(\mathcal{L}) = \sigma_{e2}(\mathcal{L})$ due to $\mathcal{J}$-self-adjointness of $\mathcal{L}$. Also, it was proved above that $\sigma_{e1}(\mathcal{L})$ is contained in the union of two real half-lines and six isolated points. Therefore, $\Delta_{e1}(\mathcal{L}) := \mathbb{C} \setminus \sigma_{e1}(\mathcal{L})$ has only one connected component, which has non-trivial intersection with $\varrho(\mathcal{L})$. According to the notation of [14, Chp. IX], $\Delta_{e1}(\mathcal{L}) = \Delta_{e5}(\mathcal{L})$, where $\Delta_{e5}(\mathcal{L}) = \mathbb{C} \setminus \sigma_{e5}(\mathcal{L})$; [14, Theorem 1.5] now implies that any $\omega \notin \sigma_e(\mathcal{L}) = \sigma_{e5}(\mathcal{L})$ is an isolated eigenvalue (of finite geometric multiplicity). The claim is proved.

Data availability

Data access statement: all data is provided in full in the results of this paper.

Acknowledgments

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