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Stress-rate-type strain-limiting models for solids resulting from implicit constitutive theory

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Abstract

The main objective of this work is two-fold. First, we investigate the stress-rate-type implicit constitutive relations for solids within the context of strain-limiting theory of material response. The relations we study are models for generalisations of elastic bodies whose strain depends on the stress and the stress rate. Secondly, we obtain travelling-wave solutions for some special cases that are nonlinear in the stress. These are the first notion of solutions available in the literature for this type of models describing stress-rate-type materials.

Keywords: Stress-rate-type models; Strain-limiting theory; Implicit constitutive relation

1 Introduction

The usage of complex materials in technology forces mathematicians to model material response in such a way that it is not only general enough to explain observed phenomena in experiments, but also as simple as possible from the point of view of mathematical analysis. Due to the complexity in their material response as well as the presence of the vast spectrum of possibilities for modelling them, viscoelastic materials have attracted a serious amount of attention in recent years. In accordance with the requirements of applications, rate-type viscoelastic models that include information about the current values of the kinematical quantities are much more favourable over the integral-type models where one has to keep track of the history of these quantities. The same understanding is valid both for fluids and solids. From the modelling point of view, on the other hand, the independent variables differ depending on whether viscoelastic fluids or solids are modelled. In this manuscript, we aim to model stress-rate-type solids obeying the requirements of strain-limiting theory.

It all started when Rajagopal [1] introduced a new perspective for modelling elastic-material response starting with implicit constitutive relations. The approach attracted a lot of attention immediately due to the fact that it was possible to explain some experimentally observed phenomena where, after linearisation of the strain under the assumption of smallness of the deformation gradient, it was possible to obtain a non-linear relationship between the linearised strain and the stress, which is not possible to obtain in classical elasticity. In those models, no restrictions are put on the state of the stress and hence they are called *strain-limiting models*. From a classical mechanical point of view, they are more appropriate for using the stress as a primitive variable supporting the reasoning that the stress is the cause of the deformation but not the effect.

Many studies exist on implicit constitutive modelling and strain-limiting response of materials including elastic and viscoelastic solids and fluids (we refer the reader to the review article by Şengül [2] and references therein both for mathematical treatments and experimental applications). In this context, mathematical analysis for the viscoelastic response of solids with strain-rate-type constitutive relations have been investigated in articles such as [3–5], and [6]. More recently, the three-dimensional problem with periodic and Dirichlet boundary data have been studied by Bulíček, et al. in [7] and [8]. From an application point of view for such strain-rate-type models one can refer to experimental studies such as [9] where the mechanical behaviour of various types of silk and spider silk are investigated. Recently, in [10], Erbay and Şengül aimed to investigate models resulting from implicit constitutive relations depending on the stress and the stress rate. Following the approach introduced for fluids by Rajagopal and Srinivasa [11], they introduced a stress-rate-type model for solids, and showed that it is thermodynamically consistent. Such rate-type models are suitable for giving explanations to some experimental observations on various materials such as dragline silk (see e.g. [12]) where, following the approach of Holzapfel and Simo [13], a stress-type internal variable that obeys an evolution rule is introduced. As also noted in [6], due to the inherent causal relation modelling of such phenomena by defining the stress as a primitive variable instead of the strain is therefore more appropriate, which is one of the motivations to study such mathematical models.

The aim of this note is to develop a general stress-rate-type model for the response of solids within the context of strain-limiting theory using a thermodynamical point of view. We start by analysing the general non-linear model and search for heteroclinic travelling-wave solutions, which reduces the equation of motion to an ordinary differential equation. Due to having analytical techniques available in the theory of first-order ordinary differential equations only in the cases where the derivative term is explicitly expressed as a function of other variables, we pass onto a special case that is linear in the stress rate. We look at travelling-wave profiles for some special cases of non-linearities chosen to be consistent with the restrictions we obtain during analysis.

2 Derivation of the model

We provide a brief introduction of the mathematical background covering fundamental equations that form the basis of continuum mechanics, and the formulation of implicit constitutive theory. Similar reviews can also be found in [2, 10, 14, 15].

Let $\mathbf{u}$ be the displacement of the body at the position $\mathbf{x} \in \mathbb{R}^3$ of a particle at time t with respect to the undeformed reference configuration $\mathbf{X}$, that is $\mathbf{u} = \mathbf{x} - \mathbf{X}$. The motion is expressed by the deformation map $\chi(\mathbf{X}) = \mathbf{x}$. The deformation gradient tensor $\mathbf{F}$ and the velocity field $\mathbf{v}$ are defined as $\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}}$ and $\mathbf{v} = \dot{\mathbf{x}}$; from these the definitions for linearised strain tensor, ϵ, and the symmetric part of the velocity gradient, $\mathbf{D}$, are obtained as

$$\boldsymbol{\epsilon} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T\right), \qquad \mathbf{D} = \frac{1}{2}\left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)^T\right). \tag{1}$$

The Almasi–Hamel strain tensor $\mathbf{A}$ and Cauchy–Green stretch tensor $\mathbf{B}$, which is defined as $\mathbf{B} = \mathbf{F}\mathbf{F}^T$, are related to ϵ as follows;

$$\mathbf{A} = \boldsymbol{\epsilon} - \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)\left(\frac{\partial \mathbf{u}}{\partial \mathbf{x}}\right)^T, \qquad \mathbf{A} = \frac{1}{2}\left(\mathbf{I} - \mathbf{B}^{-1}\right). \tag{2}$$

The strain-limiting theory of elastic solids is based on the assumption that

$$\left\| \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right\| = \mathcal{O}(\delta), \tag{3}$$

in some appropriate norm with $\delta \ll 1$. This immediately implies that ignoring the higher-order terms in the definition of $\mathbf{A}$ we can approximate it by $\boldsymbol{\epsilon}$. As a result, by (2), we can replace the Cauchy–Green tensor $\mathbf{B}$ with $\mathbf{I} + 2\boldsymbol{\epsilon}$ in the constitutive relation for the response of the material.

This approach has been adopted in several studies investigating the response of elastic solids (see e.g. [16]) starting from the general implicit relations of the form $\mathcal{F}(\mathbf{T}, \mathbf{B}) = 0$. Recently, models for viscoelastic materials have been investigated by Rajagopal and Saccomandi [3] and Erbay and Şengül [4]. In these studies, the response of material is strain-rate type and hence the general constitutive relation to start with is given by $\mathcal{F}(\mathbf{T}, \mathbf{B}, \mathbf{D}) = 0$. In this work, on the other hand, we are interested in the stress-rate-type viscoelastic response and hence the implicit constitutive relation we would like to consider is given by

$$\mathcal{F}(\mathbf{T}, \dot{\mathbf{T}}, \mathbf{B}) = 0. \tag{4}$$

Even though there is a vast literature on rate-type models for material response in the classical setting, there is no systematic way of generating new models that satisfy the requirements of thermodynamics. In [17], Prusa and Rajagopal considered general constitutive relations for fluids between the histories of the Cauchy stress and the relative deformation gradient, and showed that fluids defined through such models are in fact generalisations of rate-type fluids. This was a big step towards understanding new material phenomena by posing new models that are mechanically meaningful and mathematically tractable at the same time. Later, in [18], Rajagopal investigated anisotropy for simple materials where the class of response relations under considerations was implicit relations between the history of the stress, the history of the density, and the history of the deformation gradient. More recently, Erbay and Şengül [10] showed that by considering (4), it is possible to obtain a thermodynamically consistent model for stress-rate-type viscoelastic solids within the context of strain-limiting theory. This is a big step towards understanding new material phenomena by posing new models that are mechanically meaningful and mathematically tractable at the same time. In this work, we would like to generalise the model introduced in [10] as well as show the existence of travelling-wave solutions under certain conditions on the non-linearities, which is the first notion of solutions investigated in this context. More precisely, we would like to consider the constitutive relation

$$\mathbf{B} = \mathcal{H}(\mathbf{T}, \dot{\mathbf{T}}). \tag{5}$$

As noted by Şengül in [2] (see also [10]) under the assumption of isotropy, from (5) one obtains

$$\mathbf{B} = \alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\dot{\mathbf{T}} + \alpha_3\mathbf{T}^2 + \alpha_4\dot{\mathbf{T}}^2 + \alpha_5(\mathbf{T}\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}) + \alpha_6(\mathbf{T}^2\dot{\mathbf{T}} + \dot{\mathbf{T}}\mathbf{T}^2) + \alpha_7(\dot{\mathbf{T}}^2\mathbf{T} + \mathbf{T}\dot{\mathbf{T}}^2) + \alpha_8(\mathbf{T}^2\dot{\mathbf{T}}^2 + \dot{\mathbf{T}}^2\mathbf{T}^2), \tag{6}$$

with the scalar functions α_i $(i = 0, \dots, 8)$ depending on the invariants

$$\operatorname{tr}\mathbf{T}, \quad \operatorname{tr}\dot{\mathbf{T}}, \quad \operatorname{tr}\mathbf{T}^2, \quad \operatorname{tr}\dot{\mathbf{T}}^2, \quad \operatorname{tr}\mathbf{T}^3, \quad \operatorname{tr}\dot{\mathbf{T}}^3,$$
$$\operatorname{tr}(\mathbf{T}\dot{\mathbf{T}}), \quad \operatorname{tr}(\mathbf{T}^2\dot{\mathbf{T}}), \quad \operatorname{tr}(\dot{\mathbf{T}}^2\mathbf{T}), \quad \operatorname{tr}(\mathbf{T}^2\dot{\mathbf{T}}^2).$$

We would like to follow the approach of smallness of the displacement gradient as in (3). However, due to having the stress rate rather than the Cauchy–Green tensor in (4), in this case we need to make the following assumptions;

$$\left\| \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \right\| = \mathcal{O}(\delta), \qquad \left\| \frac{\partial \mathbf{v}}{\partial \mathbf{x}} \right\| = \mathcal{O}(\delta), \quad \delta \ll 1. \tag{7}$$

Using (7), we are able to replace $\mathbf{B}$ with $\mathbf{I} + 2\boldsymbol{\epsilon}$ in the constitutive relation (5).

3 Thermodynamical analysis

In the work of Erbay and Şengül [10], thermodynamical analysis of a model obtained from (5) under the assumptions of (7) was carried out using a Gibbs potential formulation. It was shown that the model

$$\epsilon = h(T) - \kappa T_t, \tag{8}$$

where h is the derivative of the complementary free energy satisfying $h(0) = 0$ and κ is a non-negative constant, is thermodynamically consistent. This meant that a model with (8) as the constitutive relation satisfies the first and second laws of thermodynamics in the form of the Clausius–Duhem inequality. The key idea was that a formulation based on the complementary free energy (or equivalently Gibbs free energy) was used instead of the classical formulation based on the Helmholtz energy function so that it was not necessary to introduce strain-like quantities. As a result, the thermodynamic potentials depend on the stress and the stress rate.

Without loss of generality, assuming that the density changes due to the propagating waves are neglected, similar arguments to those in [10] lead to

$$(\partial_T\phi_c - A)\dot{T} \geq 0, \tag{9}$$

where $A = A(T, \dot{T})$ is the Almasi–Hamel strain tensor and ϕ_c is the complementary energy, which can be shown to be independent of the stress rate, that is, $\phi_c = \phi_c(T)$. This inequality suggests that A consists of two parts, one that is equal to $\partial_T\phi_C$, and the other, say $l(T, \dot{T})$, which satisfies

$$l(T, \dot{T})\dot{T} \geq 0. \tag{10}$$

Since l, in general, depends non-linearly on T and $\dot{T}$, it is difficult to draw a conclusion from (10). However, this is not the case when l depends linearly on the stress rate. Therefore, we will consider the case when

$$A(T,\dot{T}) = h(T) - \gamma(T)\dot{T}, \tag{11}$$

where $h(T) = \partial_T\phi_c$ for some scalar function γ. In this case, one can conclude from the thermodynamic inequality (9) that $\gamma(T) \geq 0$.

4 Travelling-wave solutions

In one-space dimension, the linearised strain can be written as $\epsilon = \partial_x u(x,t)$, where u is the displacement function and ∂_x stands for partial differentiation with respect to x. As expressed in [10], owing to (2), the Almansi–Hamel strain $A(x,t)$ can be rewritten as

$$A = \epsilon - \frac{1}{2}(\partial_x u)^2.$$

With the small-strain assumption, we are able to replace A by ϵ in the equations derived above. Furthermore, in this case, the difference between the quantities measured in the reference and current configurations disappears and the material time derivatives become partial derivatives with respect to time. As a result of separating the Almansi–Hamel strain tensor into elastic and dissipative parts as in the previous section, we consider

$$\epsilon = h(T) - l(T,T_t). \tag{12}$$

Using the equation of motion given as $u_{tt} = \partial_x T$ as well as the fact that in one-space dimension the linearised strain is $\epsilon = \partial_x u$, we obtain

$$\big(h(T) - l(T,T_t)\big)_{tt} = \partial_{xx}T. \tag{13}$$

Passing to the travelling-wave variable $\xi = x - ct$ so that $T = T(\xi)$ and $T_t = -c\,T'$, from (13) we obtain the ordinary differential equation given as

$$T'' + c^2 l''\big(T,-c\,T'\big) = c^2 h''(T), \tag{14}$$

where $'$ stands for differentiation with respect to ξ. We are considering heteroclinic travelling waves taking two constant values at minus and plus infinity, say T_∞^- and T_∞^+, respectively. Therefore, the boundary conditions we require are the following:

$$T'(\xi), T''(\xi) \to 0 \quad \text{as } \xi \to \pm\infty. \tag{15}$$

Integrating (14) introduces an integration constant. Using boundary conditions (15), we find that the integration constant is zero due to

$$l'\big(T,-cT'\big) = l_T\big(T,-cT'\big)T' - c l_{T'}\big(T,-cT'\big)T''.$$

Integrating once more with respect to ξ, and using T_∞^- and T_∞^+ as the two constant states at infinities in order to find the integration constant explicitly, we obtain

$$l\big(T,-cT'\big)=\frac{1}{c^2}\left(\frac{T_\infty^+ + T_\infty^-}{2}-T\right)+\left(h(T)-\frac{h(T_\infty^+)+h(T_\infty^-)}{2}\right)+\frac{l(T_\infty^+,0)+l(T_\infty^-,0)}{2}. \tag{16}$$

Clearly, (16) is a first-order ordinary differential equation that is non-linear in its highest-order derivative term. Therefore, in order to be able to proceed with the solution, it is more convenient to look at the case when the non-linearity is linear in the highest-order derivative term (see also (11)). This means we should have started with the constitutive relation given by

$$\epsilon = h(T)-\gamma(T)T_t. \tag{17}$$

Here, the stress dependence is non-linear, whereas the stress-rate dependence is linear. Once again, using the fact that in the one-dimensional setting, $\epsilon = u_x$ together with the equation of motion $u_{tt}=T_x$, we obtain the partial differential equation

$$h(T)_{tt}-\big(\gamma(T)T_t\big)_{tt}=T_{xx}. \tag{18}$$

Looking at the second term on the left-hand side, we can define a function $\psi(T)$ such that $\psi(T)_t=\gamma(T)T_t$ so that we can rewrite (18) as

$$h(T)_{tt}-\psi(T)_{ttt}=T_{xx}. \tag{19}$$

Defining ψ in this manner requires $\psi_T(T)=\gamma(T)$, which we will use later. Looking at the travelling-wave solutions of the form $T=T(\xi)$ with $\xi=x-ct$ again, we obtain

$$c^2h''(T)+c^3\psi'''(T)=T''. \tag{20}$$

Integrating (20) once and using (15) as before to find the corresponding integration constant, say A_1, we find that $A_1=0$ so that

$$c^2h'(T)+c^3\psi''(T)=T'$$

holds. Integrating one more time we obtain

$$c^2h(T)+c^3\psi'(T)=T+A_2, \tag{21}$$

where A_2 is the integration constant of this step. Now, using (15) again, we obtain that $c^2h(T_\infty^\pm)=T_\infty^\pm+A_2$ and adding these equalities we find

$$A_2=\frac{1}{2}\big(c^2\big(h\big(T_\infty^+\big)+h\big(T_\infty^-\big)\big)-\big(T_\infty^+ + T_\infty^-\big)\big),$$

so that the *wave speed* c is found to satisfy

$$c^2 = \frac{T_\infty^- - T_\infty^+}{h(T_\infty^-) - h(T_\infty^+)}. \tag{22}$$

Using the above value of A_2 in (21) and writing $\psi'(T) = \gamma(T)T'$ we obtain

$$T' = \frac{1}{c^3\gamma(T)}\left[T - \frac{(T_\infty^+ + T_\infty^-)}{2} - c^2\left(h(T) - \frac{h(T_\infty^+) + h(T_\infty^-)}{2}\right)\right]. \tag{23}$$

Equation (23) is a first-order ordinary differential equation and it possesses solutions whose implicit form can be found by a single integration. In order to be able to find solutions explicitly, we need to consider the restrictions we might need on the non-linear functions γ and h. First, it is easy to check that the equilibrium points are $T = T_\infty^\pm$. Moreover, considering the constitutive relation (17), when $\gamma(T) = 0$ there is no stress-rate part, which means we must be in the elastic case. Therefore, in this case, we cannot obtain heteroclinic travelling-wave solutions. Similarly, when $h(T)$ is linear in T, that is, $h(T) = h'(0)T$ with $h'(0) \neq 0$, we can only obtain a constant solution.

We can summarise our findings as follows:

Theorem 1 *Assume that $h \in C^2$, $\psi \in C^3$, and the following conditions hold;*

(a) $\psi : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing function of its variable;*

(b) $h : \mathbb{R} \to \mathbb{R}$ *is such that $h(z) \neq h'(0)T$ with $h'(0) \neq 0$;*

(c) $T(z) \to T_\infty^+$ *as $z \to \infty$, and $T(z) \to T_\infty^-$ as $z \to -\infty$.*

Then, equation (18) possesses heteroclinic, travelling-wave solutions $T(\xi) \in \mathcal{S}$ with the travelling-wave variable $\xi = x - ct$. The two constant states of the waves are given by T_∞^+ and T_∞^-, which are also the equilibrium states of the equation, and c is the wave speed given by (22).

5 Travelling-wave profiles for some special cases

Since heteroclinic travelling waves propagate from one constant state to another when $c^2 > 0$, using (22) we must have either $T_\infty^- > T_\infty^+$ and $h(T_\infty^-) > h(T_\infty^+)$, or $T_\infty^- < T_\infty^+$ and $h(T_\infty^-) < h(T_\infty^+)$. Without loss of generality, we can take the first case and choose

$$T_\infty^- = 1, \qquad T_\infty^+ = 0. \tag{24}$$

In this case, we find that $c^2 = 1/h(1)$ so that (23) reduces to

$$T' = \frac{1}{c^3\gamma(T)}\left[T - \frac{1}{2} - c^2\left(h(T) - \frac{h(0) + h(1)}{2}\right)\right]. \tag{25}$$

Assuming that $h(0) = 0$, this equation reduces to

$$T' = \frac{1}{c^3\gamma(T)h(1)}\big(h(1)T - h(T)\big). \tag{26}$$

Looking at this equation and using (24) as equilibrium points, we can deduce that $\gamma(0) \neq 0$ and $\gamma(1) \neq 0$. Finally, one can show that the fact that $T(\xi)$ is a solution implies that $T(\xi + p)$ is also a solution, for any p. Hence, without loss of generality, we can choose $T(0) = 1/2$.

To find the solution $T(\xi)$ one can rewrite (26) to obtain

$$\frac{c^3\gamma(T)h(1)T'}{h(1)T - h(T)} = 1 \tag{27}$$

and integrating with respect to the travelling-wave variable ξ we obtain

$$c^3 h(1)\int \frac{\gamma(T)T'}{h(1)T - h(T)}\,d\xi = \xi + A_3, \tag{28}$$

where the integration constant A_3 depends on the integrated form of the expression on the left. Letting

$$\Gamma\bigl(T(\xi)\bigr) = c^3 h(1)\int \frac{\gamma(T)}{h(1)T - h(T)}\,dT,$$

one sees from (28) that

$$A_3 = \Gamma\bigl(T(0)\bigr) = \Gamma(1/2). \tag{29}$$

Taking all the restrictions we obtained for γ and h into account, in order to illustrate the travelling-wave profiles, we will first consider when

$$h(T) = T^2 \quad \text{and} \quad \gamma(T) = T^2 + 1. \tag{30}$$

The readers are referred to Appendix A for the calculations leading to the solution given in the implicit form as

$$-T - 2\log(T-1) + \log(T) = \xi + A_3,$$

which is plotted in Fig. 1.

These choices of non-linearities clearly satisfy the restrictions required in Theorem 1 and hence they are important.

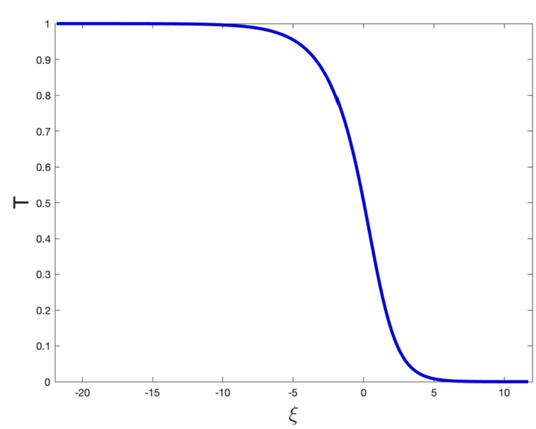

Figure 1 Travelling wave $T(\xi)$ for $h(T) = T^2$, $\gamma(T) = T^2 + 1$

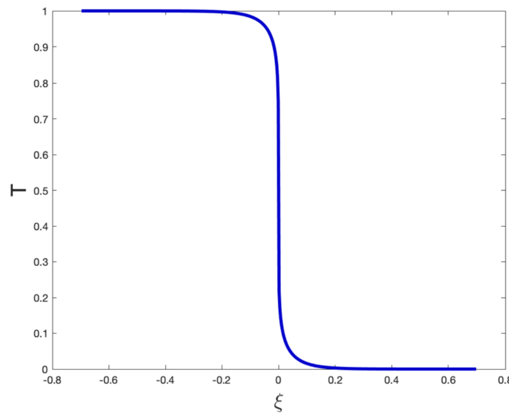

Figure 2 Travelling wave $T(\xi)$ for $h(T) = T^2$, $\gamma(T) = (T - \frac{1}{2})^4$

Next, we consider the case when

$$h(T) = T^2 \quad \text{and} \quad \gamma(T) = \left(T - \frac{1}{2}\right)^4. \tag{31}$$

By keeping h the same and changing only the function γ, we aim to see how dramatically the travelling-wave profile is effected. Moreover, in this case, $\gamma > 0$ is satisfied only for a certain $\mathcal{S}$ rather than the whole real line as in the previous case. For this case, we obtain the solution given implicitly as

$$-\frac{T}{2} + \frac{T^2}{2} - \frac{T^3}{3} - \frac{\log(1-T)}{16} + \frac{\log(T)}{16} = \xi + A_3,$$

for which the calculations are available in Appendix B and the travelling-wave profile is given in Fig. 2.

Next, we consider the case when

$$h(T) = T^2 \quad \text{and} \quad \gamma(T) = \log(T + 2). \tag{32}$$

This choice of functions is due to checking whether it is possible for the non-linearity γ to be a transcendental function. It turns out that it is possible to have the existence of a travelling-wave solution in this case. Here, we obtain

$$-\mathrm{Li}_2\left(-\frac{T}{2}\right) + \mathrm{Li}_2\left(\frac{1-T}{3}\right) - \log(3)\log(T-1) + \log(2)\log(T) = \xi + A_3,$$

for which the travelling-wave profile is shown in Fig. 3. The calculations can be found in Appendix C.

Our final choice for functions h and γ are

$$h(T) = T^2 \quad \text{and} \quad \gamma(T) = \arctan(T) + 1. \tag{33}$$

This choice of γ makes the linearised strain dependent on the arctangent function, which is also the case in the strain-rate response of materials analysed in [6]. It is clear that a

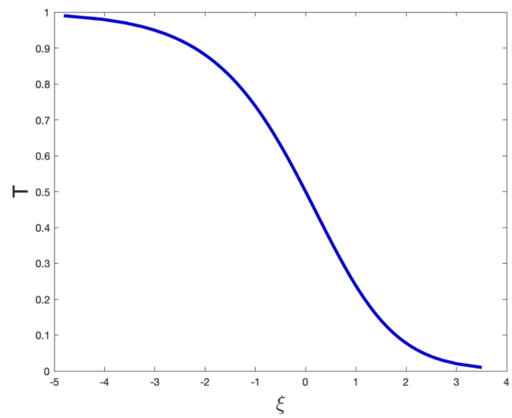

Figure 3 Travelling wave $T(\xi)$ for $h(T) = T^2$, $\gamma(T) = \log(T+2)$

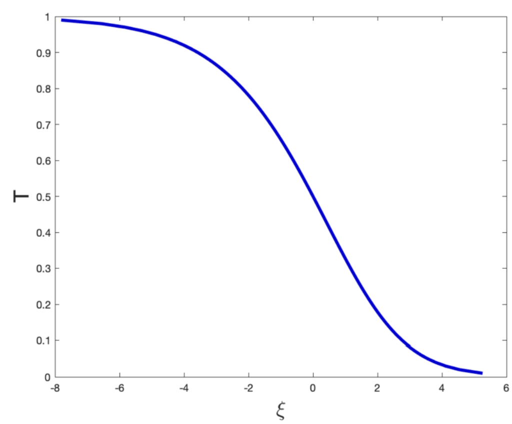

Figure 4 Travelling wave $T(\xi)$ for $h(T) = T^2$, $\gamma(T) = \arctan(T) + 1$

similar travelling-wave profile is obtained here, in the stress-rate model as well. We refer to Appendix D for the derivation of the solution that is implicitly expressed as

$$
\begin{aligned}
&-\log(1-T) + \log(T) + \frac{i}{2}\mathrm{Li}_2(-iT) - \frac{i}{2}\mathrm{Li}_2(iT) \\
&\quad + \frac{i}{2}\left(\mathrm{Li}_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)(-i+T)\right) + \log\left(\left(-\frac{1}{2}-\frac{i}{2}\right)(T-1)\right)\log(1+iT)\right) \\
&\quad - \frac{i}{2}\left(\mathrm{Li}_2\left(\left(\frac{1}{2}-\frac{i}{2}\right)(i+T)\right) + \log\left(\left(-\frac{1}{2}+\frac{i}{2}\right)(T-1)\right)\log(1-iT)\right),
\end{aligned}
$$

and is plotted in Fig. 4.

6 Conclusions

In this work, we introduce a stress-rate-type viscoelastic constitutive relation and analyse it from the point of view of thermodynamical consistency as well as the mathematical analysis of the corresponding heteroclinic travelling-wave solutions. Our findings are important owing to the fact that this is the first time a stress-rate-type model for material response is shown to admit some notion of solutions. There are still a number of related open problems concerning the corresponding partial differential equation (18). This equation is different from classical equations resulting from mechanical theories in two respects, the first one being that the inertia term is non-linear, and the second one that the unknown is not the displacement or the deformation, but the stress. As a result, there is no available way to approach the problem using standard techniques. In the case of the strain-rate modelling, Erbay, Erkip and Şengül [19] obtained the local-in-time existence of solutions and Şengül [20] obtained global solutions under some restrictions of the non-linearities for a slightly related partial differential equation by converting h and hence eliminating the non-linearity on the inertia term as a result of a series change of variables. It would be favourable to attack these equations directly rather than trying to put them in classical form. However, such a mathematical tool does not currently exist. It is also worth mentioning that the partial differential equation obtained in the strain-rate case and equation (18) have completely different behaviours in terms of their stabilities, as shown in [10] in the very simple case of linear h. Therefore, it is expected that a brand new approach should be adopted for (18). Hopefully, such a mathematical tool will be introduced as a result of careful investigations of experimental results on material behaviour.

Appendix A: The case of quadratics

Here, we consider the case when $h(T) = T^2$, $\gamma(T) = T^2 + 1$. From equation (22) we can conclude that $c = 1$. Inserting these into (28) we obtain

$$\int \frac{T^2 + 1}{T - T^2}\, dT = \xi + A_3,$$

which implies

$$\int \left(-1 + \frac{1}{T} - \frac{2}{T - 1}\right) dT = \xi + A_3,$$

and hence $-T - 2\log(T - 1) + \log(T) = \xi + A_3$. Inserting $T(\xi = 0) = 1/2$ we are able to find the integration constant as $A_3 = \log(2) - \frac{1}{2}$.

Appendix B: The case of quadratic and quartic

Here, we consider the case when $h(T) = T^2$, $\gamma(T) = (T - \frac{1}{2})^4$. From equation (22), we have $c = 1$ again. Inserting these into (28) we obtain

$$\int \frac{(T - \frac{1}{2})^4}{T - T^2}\, dT = \xi + A_3,$$

implying

$$\int \frac{(T^2 - T + \frac{1}{4})^2}{T - T^2}\, dT = \xi + A_3.$$

Now, for brevity of calculation we define $k := T^2 - T$. It is important to note that this is not a change of variables and that the integration variable is still dT. We obtain

$$\int \frac{(k+\frac{1}{4})^2}{-k}\,dT = \xi + A_3 \quad \Leftrightarrow \quad \int \frac{k^2+\frac{1}{2}k+\frac{1}{16}}{-k}\,dT = \xi + A_3$$
$$\Leftrightarrow \quad -\int k + \frac{1}{2} + \frac{1}{16}k^{-1}\,dT = \xi + A_3.$$

This gives

$$-\int T^2 - T + \frac{1}{2} + \frac{1}{16}\frac{1}{T-1} - \frac{1}{16}\frac{1}{T}\,dT = \xi + A_3.$$

As a result we obtain

$$-\frac{T}{2} + \frac{T^2}{2} - \frac{T^3}{3} - \frac{\log(1-T)}{16} + \frac{\log(T)}{16} = \xi + A_3.$$

Inserting $T(\xi = 0) = 1/2$ we are able to find the integration constant as $A_3 = -\frac{1}{6}$.

Appendix C: The case of quadratic and logarithm

In this case we consider $h(T) = T^2$, $\gamma(T) = \log(2+T)$. From equation (22), we have $c = 1$ again. Inserting these into (28) we obtain

$$\int \frac{\log(2+T)}{T-T^2}\,dT = \xi + A_3,$$

which is equivalent to

$$\int \frac{-\log(2+T)}{(T-\frac{1}{2})^2 - \frac{1}{4}}\,dT = \xi + A_3.$$

This gives

$$-\int \frac{-\log(2+T)}{T-1}\,dT + \int \frac{-\log(2+T)}{T}\,dT = \xi + A_3.$$

Defining $k := 2 + T$, the left-hand side becomes $\int \frac{\log k}{k-3}\,dk + \int \frac{-\log k}{k-2}\,dk$. Since both integrals are of the form $\int \frac{\log x}{x-a}\,dx$, we evaluate this expression to insert the value for a later. We can also write the same expression by defining $y := x - a$ as follows:

$$\int \frac{\log(y+a)}{y}\,dy = \int \frac{\log(a(\frac{y}{a}+1))}{y}\,dy = \int \frac{\log a + \log(\frac{y}{a}+1)}{y}\,dy$$
$$= \log a \log y + \int \frac{\log(\frac{y}{a}+1)}{y}\,dy.$$

Now, we can define $z = -\frac{y}{a}$ to obtain $\log a \log y + \int \frac{\log(-z+1)}{z}\,dz$. Using the polylogarithm identity

$$\log a \log y + \int \frac{-\mathrm{Li}_1(z)}{z}\,dz = \log a \log y - \mathrm{Li}_2(z) = \log a \log(x-a) - \mathrm{Li}_2\left(1 - \frac{x}{a}\right).$$

Inserting the values in place we obtain

$$-\mathrm{Li}_2\left(-\frac{T}{2}\right)+\mathrm{Li}_2\left(\frac{1-T}{3}\right)-\log(3)\log(T-1)+\log(2)\log(T)=\xi+A_3.$$

Inserting $T(\xi=0)=1/2$ we are able to find the integration constant as $A_3=-\mathrm{Li}_2(-\frac{1}{4})+\mathrm{Li}_2(\frac{1}{6})-\log^2(2)-\log(3)(-\log(2)+i\pi)$.

Appendix D: The case of quadratic and arctangent

We consider $h(T)=T^2$, $\gamma(T)=\arctan(T)+1$. From equation (22) we obtain $c=1$. Inserting these into (28) it follows that

$$\int\frac{\arctan T+1}{T-T^2}\,dT=\xi+A_3.$$

We use the complex logarithm form of the inverse tangent function to obtain

$$\int\frac{\frac{i}{2}\log(\frac{i+T}{i-T})+1}{T-T^2}\,dT=\xi+A_3,$$

which can be rewritten as

$$\begin{aligned}&\frac{i}{2}\int\frac{\log(i+T)}{T}+\frac{\log(i+T)}{1-T}\,dT-\frac{i}{2}\int\frac{\log(i-T)}{T}\\&\quad+\frac{\log(i-T)}{1-T}\,dT+\int\frac{1}{T}+\frac{1}{1-T}\,dT=\xi+A_3.\end{aligned}$$

Calculating the integrals, we obtain

$$\begin{aligned}&-\frac{1}{2}i\left(\mathrm{Li}_2\left(\left(\frac{1}{2}-\frac{i}{2}\right)(i+T)\right)-\mathrm{Li}_2(1-iT)\right.\\&\quad\left.+\left(\log\left(\left(-\frac{1}{2}+\frac{i}{2}\right)(T-1)\right)-\log(iT)\right)\log(T+i)\right).\end{aligned}$$

As a result, we obtain

$$\begin{aligned}&\frac{1}{2}i\left(\mathrm{Li}_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)(-i+T)\right)-\mathrm{Li}_2(iT+1)\right.\\&\quad\left.+\log(-T+i)\left(\log\left(\left(-\frac{1}{2}-\frac{i}{2}\right)(T-1)\right)-\log(-iT)\right)\right)\\&\quad+\log(T)-\log(1-T)=\xi+A_3.\end{aligned}$$

With some simplifications we have

$$\begin{aligned}&\frac{1}{2}i\left(\mathrm{Li}_2\left(\left(\frac{1}{2}+\frac{i}{2}\right)(-i+T)\right)-\mathrm{Li}_2(iT+1)\right.\\&\quad\left.+\log(-T+i)\left(\log\left(\left(-\frac{1}{2}-\frac{i}{2}\right)(T-1)\right)-\log(-iT)\right)\right)\end{aligned}$$

$$- \frac{1}{2}i\left(\mathrm{Li}_2\left(\left(\frac{1}{2}-\frac{i}{2}\right)(i+T)\right) - \mathrm{Li}_2(1-iT)\right.$$
$$\left. + \left(\log\left(\left(-\frac{1}{2}+\frac{i}{2}\right)(T-1)\right) - \log(iT)\right)\log(T+i)\right) - \log(1-T) + \log(T) = \xi + A_3.$$

Inserting $T(0) = 1/2$ we are able to find the integration constant as

$$A_3 = \frac{1}{8}\left(4i\left(-\mathrm{Li}_2\left(\frac{i}{2}\right) + \mathrm{Li}_2\left(-\frac{i}{2}\right) - \mathrm{Li}_2\left(\frac{3}{4}+\frac{i}{4}\right) + \mathrm{Li}_2\left(\frac{3}{4}-\frac{i}{4}\right)\right)\right.$$
$$\left. - \pi\log\left(\frac{5}{4}\right) + 12\log(2)\cot^{-1}(2)\right).$$

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