# Bidding Dilemma In Keyword Search Auctions

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**Abstract**

We model an incomplete information generalized second price auction for keyword search to analyze the optimal bidding strategies of the participating advertisers. The results also apply to a more general setting where goods are being auctioned off at multiple positions. We characterize all possible pure strategy Bayes–Nash equilibrium in a static GSP auction and show that the consideration of the click through rates ratio plays a key role in determining the equilibrium bidding strategies for the advertisers. Specifically, we find that when the click through rates ratio exceeds a critical value, there will be no pure strategy Bayes–Nash equilibrium. The results from dynamic GSP auction confirm that the existence of both separating strategy and pooling strategy equilibrium also depend on critical values of click through rates ratio such that the dominant bidding strategies in standard dynamic auction become irrelevant in dynamic GSP auctions. Lastly, we find that when search engines do not publish the bidding history (i.e. there is 'minimum disclosure of information'), the advertisers will never try to mimic each other or in other words, there will be no pooling strategy equilibrium.

**Keywords:** Game theory, Keyword auction, Generalized second price auction, Bayes–Nash equilibria, Sponsored search, Click through rate

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# 1 Introduction

Keyword advertising (also known as sponsored search advertising) is a multi–billion industry. In 2021 the aggregate search ad revenue in USA was 66.2 billion dollars[1]. In this industry, companies pay to have their advertisement appear when a consumer searches for a particular key word. Currently, Google is the largest platform for sponsored search advertising with almost 92.49 percent market share [2]. Google sells its keywords in a generalized second price auction. In a generalized second price auction, advertisers bid on keywords and get allocated positions according to their bid. Positions vary in quality in terms of click through rates. A study conducted by 'First Rate', a Google adwords certified partner shows that users clicked on the top link 17 percent of the time while the second spot received 13 percent of the clicks (Sandberg 2012). But another study conducted by Compete.com in shows that for a different set of keywords, the click through rate (CTR) at the top position was 59 percent while the second position has a CTR of only 15 percent (Andrews 2012). Thus getting a higher position, in most cases, means attracting more viewers and clicks but we can not make a definitive statement regarding the difference in click through rates as it varies by industry or keyword. Additionally, the value of getting the top position itself is unclear. On the one hand, it has a higher click through rate meaning more customers. But as a consequence, it comes at a higher price.

Practitioners vary in their beliefs about the importance of outbidding other advertiser for the top position. Gord Hotchkiss (President, Enquiro search solutions) advocates for getting the top spot because "Top sponsored ads received 2 to 3 times the click through rates compared to others"[3]; whereas Brandt Dainow (CEO, ThinkMetrics) says "You can start saving money by not bidding on the top spot." (Dainow 2007) This research aims to provide guidance as to the conditions under which a company should bid to earn the top position and when a company should bid below their valuation to get a lower position.

Major search engines like Google, Bing etc. do not provide bidding history to advertisers in its generalized second price auction[4]. In contrast, product auction sites such as eBay[5] and Sam's club (Goes et al.2012) often provided bidding history to current bidders. Advertisers have expressed an interest in having bidding history available in keyword auctions. Pardoe and Stone (2010) argue, *"knowing the bids of other advertisers for a specific keyword would have helped an advertiser to predict the ad position and cost per click associated with any click."*. This research shows for which types of advertisers the bidding history will be valuable and

[1] https://www.statista.com/statistics/190275/us-online-display-and-search-advertising-forecast-2010-to-2015/
[2] https://gs.statcounter.com/search-engine-market-sharehttps://gs.statcounter.com/search-engine-market-share
[3] http://www.webpronews.com/bidding-on-top-ad-spot-a-waste-of-money-2007-01
[4] http://community.bingads.microsoft.com/ads/en/bingads/f/8/t/74897.aspx
[5] http://pages.ebay.com/help/buy/bidding-overview.html

for which types of advertisers the bidding history will be irrelevant in decision making.

Prior literature gives guidance to as to how companies should bid in auctions. In a simple second price auction, all bidders should bid truthfully. In other words all bidders should bid their valuations for the item up for bid (Vickrey 1961). However, in a static complete information generalized second price auction, truthful bidding is not the dominant strategy (Edelman et al. 2007) as the advertisers can maximize their payoff even without getting the top position. This research aims to show when an advertiser should bid truthfully and when an advertiser should shade the bid in both static and dynamic generalized second price auction with incomplete information.

In practice, most of the online advertisers participate over multiple periods of time and bid on the same keyword or keyphrase in every period. Even when they develop a strategy to mitigate the single period trade–off between getting top position and paying high cost per click, it does not help them in a dynamic setting. There is certainly a dynamic trade–off as well and most of the advertisers overlook it. Conventional wisdom suggests that as the advertisers rarely change their bids over the time, their bidding behavior remain inefficient. The industry insight reveals "*Keyword bidding is also the most time-consuming part of AdWords, which is unfortunate because most advertisers like to set a bid when they start, then never change it.*"[6] While the process of learning is certainly important in a dynamic game, our analysis shows that there has to be enough incentives for the advertisers to be engaged in this learning process. In other words, addressing questions like *when to learn* or *whether to learn at all* is as important as developing a dynamic bidding strategy solely based on learning. If the competing advertisers reveal their true identities early in the game, there would be a significant amount of learning at a higher cost as the advertisers would start bidding aggressively. On the other hand, if the advertisers do not reveal their true identities, each advertiser does not get information to update his belief about competitors' valuations but have a low expected cost per click. In this paper we have developed a dynamic model of incomplete information GSP auction. The model helps us to understand how the forward–looking advertisers deal with the dynamic trade–off points in sponsored search advertising auctions.

The theoretical objective of this paper is three–fold – *(a)* to derive the optimal bidding strategies in a static as well as in a dynamic game of incomplete information and *(b)* to examine the feasibility of bid mimicking, *(c)* to understand the impact of *minimum disclosure of information*(i.e. the search engines do not disclose the bid amounts placed in the previous periods) on advertisers' bidding strategies. The existing

[6]http://www.wordtracker.com/academy/pay-per-click/getting-started/setting-up-adwords

literature recognize that in the keyword search GSP auctions multiple equilibria might exist. Our model provides a detailed analytical framework which identifies the conditions under which equilibrium bids would exist. The specific research questions that we address are - *(a)* under what conditions would the advertisers bid truthfully in static as well as dynamic GSP auction?, *(b)* can advertisers with high valuation (for a keyword) mimic the bidding behavior of advertisers with low valuation and vice-versa? and *(c)* How does the availability of bidding history affect the equilibrium bidding strategies?

For the advertisers, the answer to question *(a)* demonstrates the incentives for adopting a forward–looking bidding behavior. Similarly, the findings from the research question *(b)* show that the advertisers with different valuations can maximize their expected payoffs by adopting same bidding paths. For search engines, the answer to question *(c)* demonstrates that the decision to withhold bidding information will not affect the bidding behavior of the forward–looking advertisers.

With regard to the question *(a)* the advertisers would understand the incentives for adopting a forward–looking bidding behavior. Answer to the question *(b)* explains why the bid shading amount of two advertisers (of different type) might not differ at all. The result establishes the fact that under certain conditions the expected bid from any advertiser tends to be very low. Lastly, the question *(c)* answers whether an advertiser needs to have access to the bidding history or not.

## 2 Literature Review

In our paper we show that in a GSP auction, not all advertisers would bid truthfully – some of the advertisers would shade their bid substantially. The existing literature on auction tells us that in a second price static auction (with single unit demand), the advertisers would always bid truthfully (Vickrey 1961). On the other hand in a dynamic second price auction (with single unit demand in every period), the equilibrium strategy would be to bid the true value in the last period and shade in earlier periods (Krishna 2002). The possibility of truthful bidding in a dynamic game has also been supported by Bergemann and Valim (2003). In a more generalized setting, Albright (1974) explains how dynamic allocations in terms of highest valuations lead to the most efficient arrangeemnt. In the context of static complete information GSP auction both (Edelman, Ostrovsky, and Schwarz 2007) and (Varian 2007) show that the truthful bidding is not a dominant strategy. The authors also prove that even in a GSP auction with complete information,, multiple equilibrium would exist. Existence of multiple equilibria has also been supported by Athey and Nekipelov

(2010). We however find that in a static GSP auction with incomplete information, truthful bidding would be a dominant strategy for the bidders with lowest valuation; however, the higher valuation bidders would not bid truthfully in a static GSP auction. When we extend our analysis to a dynamic setting we find that depending upon parametric conditions, truthful bidding can be a weakly dominating strategy for even the bidders with high valuations.

More recently Gomes and Sweeney (2014) provided a formal analysis of optimal bidding strategies in a static incomplete GSP mechanism. The paper observed that the existing literature is yet to give the readers a complete analysis of the Bayes–Nash equilibrium in a static incomplete information GSP auction. The paper claims that in a specific case of GSP auction with two positions, the existence condition of an efficient equilibrium requires the click through rate at the second position to be sufficiently small. Additionally, the authors also mention that the strictly increasing bid function ensures the existence of a Bayes–Nash equilibrium. Contrary to their claims, we find that if the click through rate at the second position is significantly smaller than the click through rate at the first position then there exists no equilibrium in pure strategies. Furthermore we show that a Bayes–Nash equilibrium may not exist even when the bidding function is monotonically increasing in valuation. The differences in results arise because the bidding function in Gomes and Sweeney (2014) is conditionally monotonic whereas my bidding function is consistently monotonic. The conditional monotonicity arises because in Gomes and Sweeney (2014) the optimization problem is a welfare-maximization problem (i.e. joint payoff maximization problem). As a result when the click through rates are close enough, the welfare-maximization problem does not yield any solution. Mathematically, the bidding function in Gomes and Sweeney (2014) ceases to be monotonic and the equilibrium breaks down. In our model the optimal bidding function is always monotonically increasing in valuation and the equilibrium breaks down only when the difference between the click through rates is significantly large. This large difference between the click through rates forces the advertisers to place an excessively high bid amount which would eventually give them suboptimal expected payoffs. In this case the advertisers would always be better off by deviating to a value below their previously optimal bids.

In terms of direction of bids over time our results are in accordance with the results from standard first price and second price auctions. Milgrom and Weber (2000) and Krishna (2002) show that in first price as well as in second price standard auctions, the bid amounts increase over the time. In fact, in a standard second price sequential auction, the last period dominant strategy is to bid truthfully. Our paper also proves that the forward–looking advertisers place higher bids over the time. However, bidding truthfully in the last period is not a dominant strategy in our case – this result goes hand in hand with the fact that truthful bidding is not a dominant strategy for all except the bidders with the lowest valuation in a static game of GSP auction.

Recently the researchers working on dynamic auctions have started analyzing the role of market transparency. An important question in this regard is – how the bidding strategies under complete disclosure of information be different from the bidding strategies under minimum disclosure of information? In the GSP context, complete disclosure of information means access to position, payment and bid information whereas minimum disclosure of information means access to only position and payment information. Dufwenberg and Gneezy (2002) have shown that in a sequence of standard first price auctions. the bids tend to be higher under minimum disclosure of information (i.e. when bids from the last periods are not revealed to the bidders). Gershkov (2009) shows that sellers would maximize their revenue by revelation of information to all bidders. We find that the bidding strategies of the forward–looking bidders would remain same under both complete disclosure of information and minimum disclosure of information as long as the GSP auctions take place for more than two periods. However, under minimum disclosure of information myopic bidders would place lower bids (compared to the bids they would place under complete disclosure of information) with higher probability.

Two important equilibrium concepts that we have used in the dynamic setting are separating equilibrium and pooling equilibrium. Separating equilibrium takes place when bidders with different valuations place different bids and pooling equilibrium takes place when bidders with different valuations place the same bid. Bergemann and Horner (2010) shows that under first price auction with minimum disclosure of information there would always be an efficient separating equilibrium. Unlike their paper, our paper shows that under second price mechanism with minimum disclosure of information (when bidding information is not available), inefficient pooling equilibria exist. This is precisely because of the fact that in (Bergemann and Horner 2010) bidding information is the only signal for the bidders. In a dynamic GSP auction in spite of the absence of bidding information, the position and payment information would reveal important information to the participating advertisers. So, the signaling mechanism does not necessarily go away if the search engines do not disclose the bidding information; the signal only gets weaker. Our results are in accordance with (Athey and Bagwell 2010) which shows that generally in the context of dynamic auction a separating equilibrium exists when patience is low (i.e. the discount factor is low) and a pooling equilibrium exists when patience is high. Horner and Jamison (2008) show that higher patience often leads to pooling equilibrium even in a different setting (first price, common value auction). Apart from confirming this result in the context of a dynamic GSP auction we also prove that when the patience is really high, a critical value of the click through rates ratio would determine whether a pooling or a separating equilibrium would eventually exist.

This paper complements some of the existing papers on sponsored search literature as it provides further insights on Bayes–Nash equilibrium in static GSP auction. The results demonstrate that the GSP auction is distinct from a standard second price auction. At the same time, the paper shows that the truthful bidding might be a weakly dominant strategy for some of the advertisers – this result is quite in accordance with real life experiences of most of the practitioners. Moreover, our paper further contributes to the existing literature of sponsored search auction by developing and analyzing a model of dynamic GSP auction.

In the next section we describe the model, the basic assumptions and rules of the game; we also setup the framework for the static and dynamic incomplete information games. In section *4* we develop models for static and dynamic GSP auctions. Here we provide new results which go above and beyond the current literature and even overturn some of the existing results. Lastly in section *5* we discuss the theoretical contributions of our paper and the managerial implications of these results.

# 3 The Model

In our model we have three advertisers (X, Y and Z) and two positions (1 and 2) on the search engine landing page. This three bidders–two objects framework has already been used in the traditional auction literature (Bulow and Klemperer 1998). The positions get assigned to the advertisers solely through a generalized second price auction mechanism. The generalized second price mechanism implies that through a simultaneous bidding process the highest bidder will get the first position and pay an amount which is equal to the second highest bidder's bid amount; similarly the second highest bidder will get the second position and pay an amount equal to the third bidder's bid amount. In case of a tie, a random draw decides the final position outcome. While a full information model gives a number of important insights, developing a model of incomplete information keyword auction would be more realistic. Typically the participating advertisers in any keyword auction are not sure about their competitors' valuations for the keyword. We model this situation by developing a parsimonious framework where the advertisers can be of high or low type. The modeling assumption would allow us to precisely identify the optimal strategies for the bidders. To keep the analysis tractable, we use a model with three players and two positions. However, this specification still captures the economics essence of a GSP auction in a more extended set–up.

As explained above, in our model the valuations for the keyword are typically private knowledge (all advertisers only know their own valuations) though every advertiser knows that his competitors can either have high valuation ($V^H$) with probability $p$ or low valuation ($V^L$) with probability $(1-p)$. $V^H$ and $V^L$ can in fact summarize both the valuations of the advertisers and the quality of the advertisement. Probability $p$ is a subjective probability but in our model it is the same for every advertiser. We use $b^H$ and $b^L$ to denote the optimal bids of the high type and the low type advertisers. In the general auction context, the specifications as described above were successfully used by previous research as (Zeithammer 2007), (Budish and Takeyama 2001) and (Jeitschko 1998). We further assume that the click through rates at the two positions $(C_1, C_2)$ are common knowledge as is also assumed in Varian's (2007) model of position auction with complete information. Lastly, we assume that if a bidder is indifferent between participating and not participating, she would still participate in the auction (as Tan and Yilankaya (2006) showed that in such cases, there would not be any consequence of participation). The following table summarizes all the notations in our model.

**Table 1: Parameter and decision variables**

| Symbol | Definition |
|---|---|
| $V^i$ | Advertiser i's valuation for the keyword (it can be either $V^H$ or $V^L$) |
| $b^i$ | Advertiser i's bid amount |
| $C_j$ | Click through rates for position j |
| $p$ | probability of a given bidder being high type |

Table 1: Variable definitions

In a static game all the advertisers bid simultaneously. In a dynamic game the sequence is as follows. In the first stage, all the advertisers simultaneously bid for the keyword. Then the search engine publishes the ranks and the payment information for the first round of auction. Then the advertisers update their beliefs and bid for the second round of auction. The search engine again publishes the ranks and the payment information and the process goes on. The following figure depicts the sequence of a two period dynamic GSP auction.

**Sequence of events**

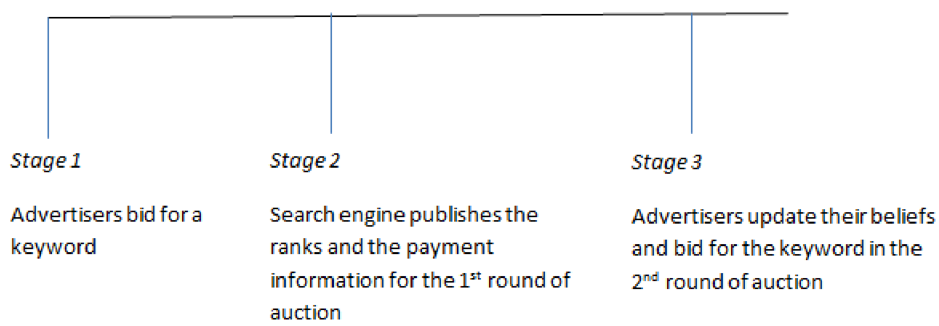


Figure 1: Timing of the game

# 4 Analysis

## 4.1 Static Game

As a starting point we analyze a static game of incomplete information. In this game we denote the equilibrium bid of the high type as $b^H$ and that of the low type as $b^L$. We examine whether there is any profitable deviation and identify the values of $b^H$ and $b^L$ that can be sustained in equilibrium. Let $p_{jk}$ represent the probability that the remaining two bidders are of type $j$ and $k$ where $j, k \in \{h, l\}$. A low type bidder's expected utility from adopting $b^L$ given that the high types bid $b^H$ and the low types bid $b^L$ is presented below.

The advertiser's expected payoff from bidding $b^L$ when $V = V^L$ is

$$E(\pi^L |\, b(L) = b^L) = p_{hh}.0 + \frac{p_{hl}}{2}(V^L - b^L)C_2 + \frac{p_{ll}}{3}(V^L - b^L)(C_1 + C_2) \tag{1}$$

As shown in the appendix, if $b^L < V^L$ then a low–type bidder can profitably deviate by increasing the bid and consequently the probability of winning the auction. If $b^L > V^L$ then a low–type bidder can profitably deviate by decreasing the bid thereby avoiding the possibility of paying more for a keyword than it is worth to the bidder. Thus, the only potential equilibrium bid by low–types is $b^L = V^L$. Note, this results holds true for any $b^H > V^L$. In order to find out the equilibrium bid for the high type advertiser we once again compute the expected payoffs. We proceed in the same fashion as before.

The advertiser's expected payoff from bidding $b^H$ when $V = V^H$ is,

$$E(\pi^H | b(H) = b^H) = \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_{ll}(V^H - b^L)C_1 \quad (2)$$

For any $b^H > V^H$, a high type advertiser will profitably deviate with a lower bid. However, as shown in the Appendix, there exists a range of $b^H$ such that no advertiser can unilaterally deviate to earn higher profit. Thus, there are multiple equilibria. This analysis further shows that the Bayes–Nash equilibrium of GSP auction is not a straightforward generalization of the Bayes–Nash equilibrium of a simple second price auction. We know that in a simple second price auction mechanism advertisers bid truthfully regardless of their type but in case of GSP auction only low type advertisers bid truthfully. Existing literature also says that the GSP auction is analogous to the genralized English auction. Our results further show that unlike in generalized English auction, the bid functions in GSP auction depend on bidders' beliefs. The results are quite important even from a managerial perspective. The search engines would always like to see 'truthful bidding' from every advertiser because that would maximize search engines' revenues. However, after observing the real life practice it has often been speculated that the advertisers generally do not bid truthfully. The question remains whether non–truthful bidding is the optimal strategy for the advertisers irrespective of their valuations for the keyword. Our first result shows some light in that direction.

**Proposition 1.** *In a static incomplete information GSP auction, truthful bidding by a high type advertiser will not occur in equilibrium. A low type advertiser has a unique equilibrium bid, the valuation itself.*

*Proof.* See appendix – Proof of Proposition 1

□

The low type advertiser would not deviate upward as that would lead to negative payoff. For a low type advertiser, deviating downward is equivalent to not participating in the auction. As we have already imposed the participation assumption, a low type advertiser would bid truthfully. However, the high type advertisers have incentive for bid–shading. If the click–through rates at the two positions are close enough, it might be beneficial for a high type advertiser to shade the bid so that his nearest competitor is a low type advertiser. As a result, his gain from a lower cost would outweigh the loss from a lower click through rate. This effect is more prominent when there is a greater likelihood of the competing bidders being low type.

Although Proposition 1 states that truthful bidding by the high type will not occur in pure strategy equilibrium, it is possible that there is no pure strategy equilibrium. The range of bids by high types that can be supported in equilibrium is given by $b^{H*} < b^H < b^{H**}$. However, it is possible that this range does not exist depending on the differences in click–through rates. We summarize this result in the following Proposition.

**Proposition 2.** *In a GSP auction with incomplete information the existence of a pure strategy Bayes–Nash equilibrium depends on the click through rates ratio. If the click–through rate of the top spot is less than double the click–through rate of the second spot, then there are multiple pure strategy Bayes–Nash equilibria. If the click–through rate of the second spot is zero, then there is a unique pure strategy Bayes–Nash equilibrium. Otherwise pure strategy Bayes–Nash equilibrium does not exist.*

*Proof.* See appendix – Proof of Proposition 2

□

This analysis can be extended to a more generalized setting. We can assume that there are $N+1$ advertisers (which implies that each advertiser faces $N$ competitors) and $M$ positions. The relationship between $N$ and $M$ has been defined as $j = N - M$. The results are qualitatively similar and we can exactly identify the range of equilibrium $b^H$. In our generalized model the critical click through rates ratio is,

$$C^* = \left( \frac{\sum_{k=j+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k}(1-p)^k C_{(N+1)-k}}{N-k}}{\sum_{k=j+1}^{N} \frac{\binom{N}{N-k} p^{N-k}(1-p)^k C_{(N+1)-k}}{N+1-k}} \right) \left( \frac{\sum_{i=1}^{M} C_i \sum_{k=0}^{N-M} \frac{\binom{N}{N-k} p^{N-k}(1-p)^k}{(N+1)-k} + \sum_{k=(N-M)+1}^{N-1} \frac{\binom{N}{N-k} p^{N-k}(1-p)^k \sum_{i=1}^{N-k} C_i}{(N+1)-k}}{\sum_{k=0}^{N-1} \binom{N}{N-k} p^{N-k}(1-p)^k} \right)$$

Whenever the click through rate at the topmost position is lower than this critical value, equilibrium bidding strategies for the high type advertisers exist. When $N = 2, M = 2$ we get $C^* = \frac{2}{3}\sum_{i=1}^{2} C_i$ as established earlier. Though we do not provide the proof in this paper, the derivations are straightforward and can be available on request.

The results of Proposition 1 and 2 stem from the interaction of two model features: there is variation in click–through rates across positions and consumers are heterogeneous in their appreciation of this difference. To see this, consider the following special cases. If there is no variation in click through rates (i.e., $C_1 = C_2$ or $C_2 = 0$), or no heterogeneity in valuation (i.e., $V^H = V^L$), then the high type advertisers bid truthfully (this is the result from standard second price auction). Thus, the novel predictions of the GSP with incomplete information rely on the presence of both of these realities. To understand the result, consider the literature on new and used durable goods (e.g., (Desai and Purohit 1999)) in which there is variation in the appreciation of quality. In both contexts, all players prefer the higher quality offering if at equal prices. However, just as consumers with a lower appreciation of quality buy the used good at a lower price, lower valuation bidders win the lower quality bidding position at a lower price. The lower valuation advertisers recognize it will be too costly to attempt to outbid the high valuation advertisers to get the top position. The interesting trade–offs occur for the high valuation advertisers.

When the positions offer similar click–through rates, the high–type advertisers view the positions as reasonable substitutes. Bids are thus shaded because if an advertiser loses the bidding for the top position, there is a possibility that the second position will be won. However, if the click–through rates are sufficiently different, there is no pure strategy equilibrium. The reasoning is as follows. If a high–type advertiser undercuts the remaining high types, the equilibrium bid of a high type advertiser increases as benefit of the top position increases. An increase in the equilibrium bid reduces the expected payoffs of a high type advertiser from playing either $b^H$ or $(b^H + \epsilon)$. However, the expected payoff from playing $(b^H - \epsilon)$ is not a function of $b^H$; a higher $b^H$ does not adversely affect this expected payoff. When $C_1 > 2C_2$, we observe that within the equilibrium bid range playing $(b^H - \epsilon)$ as well as $(b^H + \epsilon)$ would always dominate playing $b^H$. So, within that equilibrium bid range a high type advertiser can always improve his expected payoff by bidding a little less (infinitesimally small) or a little more - as a result, an oscillatory bidding behavior prevails. Consequently, we do not have any equilibrium bid in this case. The following diagrams explain how the high type advertisers would always perform better by deviating from a potential equilibrium.

## Equilibrium in static GSP auction

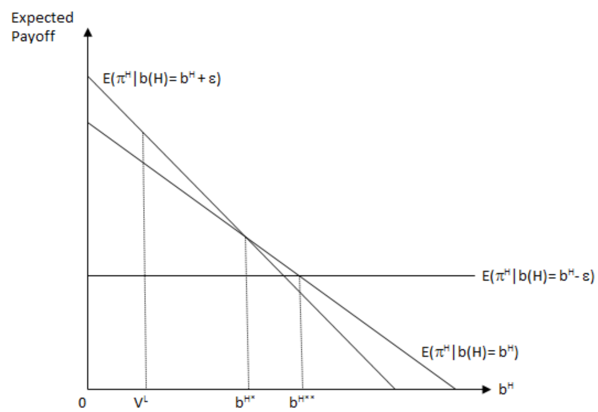


Figure 2: Multiple equilibria

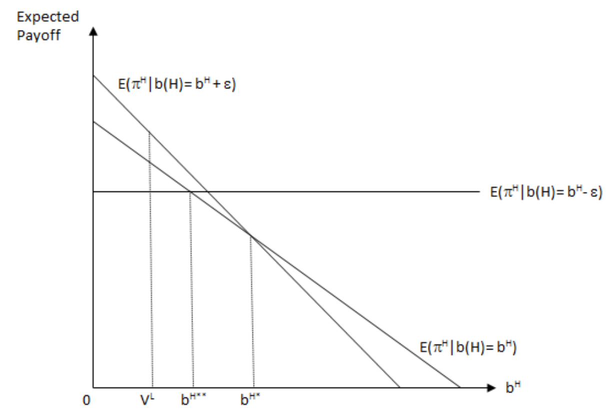


Figure 3: No equilibria

In these diagrams we show the expected payoff lines of a high type advertiser from playing three different strategies – *(i)* bidding $b^H$, *(ii)* bidding $(b^H + \epsilon)$ and *(iii)* bidding $(b^H - \epsilon)$. On the vertical axis we plot the expected payoffs of a high type advertiser and on the horizontal axis we plot a high type advertiser's bids. When $C_1 > 2C_2$, the expected payoff from playing $(b^H - \epsilon)$ is bigger than the expected payoffs from playing either $b^H$ or $(b^H + \epsilon)$ in the equilibrium bid range. In that range a high type advertiser would always gain by bidding $(b^H - \epsilon)$. The stability analysis helps us to clearly identify the cases when GSP auction would not have any pure strategy equilibrium bid for the high type advertisers. This is an important finding which leads to the second proposition.

We also observe that as $p$ approaches 1 high type advertisers bid $V^H$.Varian's (2007) result on GSP auction with complete information proves to be a special case of the current model. In general, as the probability of being a high type advertiser increases, the equilibrium bid of the high type advertisers also increases. We further observe that a positive or negative change in the click through rates do not affect the equilibrium bid of a low type advertiser. However, as the click through rates at both the positions increase so does the equilibrium bid of a high type advertiser. We have already observed that the ratio of the two click through rates has a significant influence on the stability of the equilibrium bids. The comparative static findings show that the ratio $\frac{C_1}{C_2}$ has a monotonic positive relationship with $b^H$. Lastly, we observe that as a low type advertiser's valuation for a keyword increases both types of advertisers bid more aggressively. More specifically, when a low type advertiser's valuation for the keyword tend to be the same for a high type advertiser, then all the advertisers bid truthfully. On the other hand, when the low type advertiser's valuation for the keyword is almost zero, the high type advertisers opt for more bid shading. In this case once again we observe the multiplicity of pure strategy equilibrium strategies. The results are in accordance with Varian (2007). Though Varian did not explicitly mention this result but it can be easily verified that in his model the pure strategy equilibrium bid bounds for any player would become a single equilibrium point (the true valuation) when all the players have same valuation. We also observe that when a high type advertiser's valuation for a keyword increases it does not affect the equilibrium bid of a low type advertiser. However, a high type advertiser in that case places a higher equilibrium bid.

## 4.2 Dynamic Game

### 4.2.1 Separating and Pooling Equilibrium Strategies

In this section, we analyze the optimal bidding strategies in a $T$ period dynamic game. The number of periods ($T$) must be greater than or equal to 3 as under minimum disclosure of information, it requires three periods for an incomplete information dynamic game to be transformed into a complete information dynamic game. Initially we assume that the advertisers have *complete disclosure of information* i.e. the advertisers have accessibility to all bidding, payment and position information. we also assume that the advertisers are forward looking and accordingly optimize their aggregate expected payoff. In the next sub–section, we discuss how the results change when we introduce *minimum disclosure of information* in the model.

From the static game we know that a low type advertiser would not bid more than $V^L$ as his expected payoff would be negative; bidding less than $V^L$ would also not work as his chance of winning any position would be zero. Therefore a low type advertiser has a unique Nash equilibrium bidding strategy in the single stage game. In any period in the dynamic game, a low type advertiser does not bid less than $V^L$ because there is no other type which bids less than $V^L$ in a static incomplete information game – thus, there is no opportunity of mimicking another type by bidding less than $V^L$. The low type advertiser does not bid higher than $V^L$ to mimic the high type advertisers because bidding more than $V^L$ will bring negative expected payoff in a given period. Deviation in any given period will never help the low type advertiser to get a positive payoff in future periods. Thus, the dynamic game becomes a sequence of the same single stage game for a low type advertiser. However, the high type advertiser does not bid truthfully in one period incomplete information GSP auction. Additionally, we know that in a finite horizon game, the players would always play the static

Nash equilibrium strategy in the last period as there will not be any opportunity of punishing the others players for defection (Nicholson and Snyder 2011). This leads to the three possible scenarios in a dynamic GSP auction - *(i)* a high type advertiser mimics the low type advertisers in all $(T-1)$ periods by bidding $V^L$ and then in the $T$th period plays the equilibrium bidding strategy from the one shot game, *(ii)* a high type advertiser reveals his type in the very first period and updates his prior beliefs at the beginning of each of the following periods to bid accordingly, *(iii)* a high type advertiser reveals his type in the $J$th period and updates the prior beliefs on other players' valuations at the beginning of the following period.

As we know that no low type advertiser would bid more than $V^L$ (his expected payoff would be negative), any advertiser bidding higher than $V^L$ would be a high type advertiser. Once the high type advertisers' identities are revealed, all the advertisers play a complete information game in the next period. The strategies in the subsequent periods would be independent of the past and future strategies because once the identities have been revealed, the high type advertisers will not bid $V^L$ and bidding anything other than the complete information game bid would bring lower payoff. In other words, there will not be any trade-off point for the high type advertisers in deciding whether to optimally bid or not. Thus, in every period the advertisers would play a complete information game.

At any given period $t$, high type advertisers may or may not reveal their private information. If no such information is available, then a high type advertiser has two options – he will not reveal his type (bid $V^L$) or he will reveal his type (bid $b^H$ i.e. the equilibrium bid from the static incomplete information game). However, if such information is available then the high type advertiser will place the optimal bid for a complete information game (bid $b^{H_F}$). The static incomplete information game analysis shows that $b^H$ is equal to $\frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2 p}$. The optimal bid for a complete information game, $b^{H_F}$ however depends on the number of other high type advertisers. If there are two other high type advertisers, $b^{H_F}$ is $V^H$ and if there is just one other high type advertiser then $b^{H_F}$ is $\frac{C_1 V^H - C_2(V^H - V^L)}{C_1}$. If there is no other high type advertiser (i.e. all the competitors are low type), $b^{H_F}$ takes any value higher than $V^L$.

We find that adopting pooling strategy for the first $j$ periods (and consequently playing separating strategy for the rest of $(T-j)$ periods) is strictly dominated by either pure separating strategy or pure pooling strategy (see appendix $C$). This implies that a high type advertiser does not reveal his identity in the middle of the dynamic game – either he would reveal his identity at the very beginning or at the very end of the dynamic game. So, we need to compare only two sets of expected payoffs. The following table explains all possible notations used in the expected payoff comparisons.

## Various payoffs under complete disclosure of information

| Symbol | Definition |
|---|---|
| $\pi_F^j$ | Payoff from competing against $j$ number of low type advertisers in a full information game |
| $\pi_F^i$ | Payoff from the game when only high type advertiser $i$ has complete information |
| $\pi_{max}$ | $(V^H - V^L)C_1$ |
| $\pi_k^j$ | Payoff from competing against $j$ number of low type advertisers in the $k$th state; k$\in \{a, b, c, d\}$ |
| $p^j$ | Probability of competing against $j$ number of low type advertisers |

Table 2: Payoff notations under complete disclosure of information

The index $k$ can take four different values – it is $a$ when the high type advertiser $i$ as well as the other high type advertisers bid $b^H$; when $k$ is $b$, the advertiser $i$ bids $V^L$ but the other high type advertisers bid $b^H$; $k$ takes a value of $c$ when all the high type advertisers bid $V^L$ and a value of $d$ represents that the advertiser $i$ bids $b^H$ but the other high type advertisers bid $V^L$.

First we will compare the following two expected payoffs related to the separating equilibrium strategy (without deviation and with deviation). The belief structure here is as follows – whenever one advertiser bids more than $V^L$, other advertisers believe him to be a high type advertiser; if an advertiser bids $V^L$ then he can be either low type or high type (one who mimics a low type). Of course if someone bids more than $V^L$ in a previous period and then bids $V^L$ in a subsequent period, other players would still believe that this advertiser is of high type.

A high type advertiser $i$'s aggregate expected payoff from bidding $b^H$ in the very first period and bidding $b^{H_F}$ in the subsequent periods is (while the other high type advertisers are doing the same in all periods),

$$E(\pi^i_{S,ND}) = \sum_{j=0}^{2} p^j \cdot \pi_a^j + (T-1)\sum_{j=0}^{2} p^j \cdot \pi_F^j \tag{3}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $V^L$ in the very first period is (while the other high type advertisers are bidding $b^H$ in the very first period),

$$E(\pi^i_{S,D}) = \sum_{j=0}^{2} p^j \cdot \pi^j_b + \pi^i_F + (T-2)\sum_{j=0}^{2} p^j \cdot \pi^j_F \tag{4}$$

Since starting from the third period the game becomes a complete information game, the advertisers will play a complete information game in $(T-2)$ periods.

The expected value comparisons (see appendix $C$) show that $E(\pi^i_{S,ND}) > E(\pi^i_{S,D})$ when $\frac{C_1}{C_2} < C^a(p)$ (see appendix $C$).

Now we will compare the following two expected payoffs related to the pooling equilibrium strategy (without deviation and with deviation).

A high type advertiser $i$'s aggregate expected payoff from bidding $V^L$ in the first $(T-1)$ periods and bidding $b^H$ in the last period (while the other high type advertisers are doing the same) is,

$$E(\pi^i_{P,ND}) = (T-1)\sum_{j=0}^{2} p^j \cdot \pi^j_c + \sum_{j=0}^{2} p^j \cdot \pi^j_a \tag{5}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $b^H$ in the very first period (while the other high type advertisers are bidding $V^L$ in the very first period) is,

$$E(\pi^i_{P,D}) = \pi_{max} + \sum_{j=0}^{2} p^j \cdot \pi^j_d + (T-2)\sum_{j=0}^{2} p^j \cdot \pi^j_F \tag{6}$$

The expected value comparisons show that $E(\pi^i_{P,ND}) > E(\pi^i_{P,D})$ when $\frac{C_1}{C_2} > C^b(p,T)$ (see appendix $C$).

However, when $p$ takes a low value, separating equilibrium strategy gives higher payoff for any value of the click through rates ratio. As the probability of competing with low type advertisers increases, a high type advertiser maximizes his payoff by revealing his identity at the very beginning. This ensures the maximum possible payoff in every period. The following proposition summarizes our findings.

**Proposition 3.** *In a $T$ period dynamic GSP auction with complete disclosure of information, pure strategy separating equilibrium takes places if $p \in (0, \underline{p})$. When $p \in (\underline{p}, 1)$, critical click through rates ratio would determine the equilibrium type. $\exists\{C^a, C^b\}$ such that when $p \in (\underline{p}, 1)$, pure strategy separating equilibrium takes place if $\frac{C_1}{C_2} < \min\{C^a, C^b\}$ and pure strategy pooling equilibrium occurs if $\frac{C_1}{C_2} > \max\{C^a, C^b\}$.*

For detailed proof, please see appendix $C$ – Proof of Proposition 3.

When the probability of a competitor being high type is low (i.e. $p$ takes a relatively low value), a high type advertiser would have higher chances of winning in every period. Thus, a high type advertiser has less incentive to mimic a low type advertiser in this case. As a result, for low values of $p$, high type advertisers always reveal their true identity at the very beginning of the game. Here we also make a 'low-cost' equilibrium refinement in order to reduce the multiple static equilibrium strategies and assume that when a high type advertiser reveals himself, he does that by bidding slightly higher than $V^L$. However, for high values of $p$, the strategic decisions would also depend on the critical click through rates ratio. If the ratio is very high, the value of getting the top position is also very high. As a result the expected payoff of a high type advertiser tends to be very low in any given period. So, the high type advertisers would try to bring down the price of the top position by mimicking the low type.

Proposition 3 confirms the standard result from the second price auction literature – the bid amounts increase over the time. However, the results also show that unlike in the dynamic second price auctions, bidding truthfully in the last period is not a dominant strategy. Also, unlike in the static GSP auction, existence of equilibrium in dynamic GSP auction does not depend on a single critical click through rates ratio.

### 4.2.2 Complete vs. Minimum Disclosure of Information

In this subsection we will analyze the dynamic game with minimum disclosure of information. Under minimum disclosure of information the advertisers do not have accessibility to the bidding history. The search engines do not publish the bidding history and bids placed in previous period auctions typically remain private information. The advertisers can only observe their positions and payments from the last period auction.

Minimum disclosure of information affects the dynamic game in two significant ways. Firstly, the high type advertisers would always find it advantageous to deviate from the pooling strategy equilibrium. In the first period if a high type advertiser $i$ bids anything higher than $V^L$ while all other high type advertisers bid $V^L$, then advertiser $i$ would get the top position. More importantly, other high type advertisers will not realize that advertiser $i$ has deviated because in the next period they will not be able to observe $i$'s bid from the first period. So, advertiser $i$ can keep bidding higher than $V^L$ in every period and obtain the top spot (which would also give him the maximum possible payoff in each period). This clearly shows that the pooling equilibrium will not sustain even for a single period.

Secondly, in case of pure strategy separating equilibrium, deviating from the equilibrium leads to two possibilities – bidding $V^L$ or bidding $(b^H - \epsilon)$. In a game with complete disclosure of information, bidding $(b^H - \epsilon)$ is not an option for a high type advertiser $i$ because it will give lower payoff in that period and the other high type advertisers will instantaneously recognize his true identity. However, in a game of minimum disclosure of information, if the advertiser $i$ bids $(b^H - \epsilon)$ and gets the third position then the high type advertiser at the top position would not know advertiser $i$'s true identity (the high type advertiser at the second position would however know $i$'s identity as that advertiser's payment is advertiser $i$'s bid).

Once again we compare the expected payoffs from separating strategy equilibrium and deviation from separating strategy equilibrium. The following table explains all possible notations used in the expected payoff comparisons.

## Various payoffs under minimum disclosure of information

| Symbol | Definition |
|---|---|
| $\Pi_F^j$ | Payoff from competing against $j$ number of low type advertisers in a full information game |
| $\Pi_F^i$ | Payoff from the game when only high type advertiser $i$ has complete information |
| $\Pi_{max}$ | $(V^H - V^L)C_1$ |
| $\Pi_{(K,t)}^j$ | $t$ th period payoff at the $k$th state facing $j$ low type advertisers; k$\in \{a, b, c\}$ |
| $p^j$ | Probability of competing against $j$ number of low type advertisers |

Table 3: Payoff notations under minimum disclosure of information

The index $K$ can take three different values – it is $a$ when the high type advertiser $i$ as well as the other high type advertisers bid $b^H$; when $K$ is $b$, the advertiser $i$ bids $V^L$ but the other high type advertisers bid $b^H$; $K$ takes a value of $c$ when the advertiser $i$ bids $(b^H - \epsilon)$ and all other high type advertisers bid $b^H$.

First we will compare the following three expected payoffs related to the separating equilibrium strategy (without deviation and with deviation), A high type advertiser $i$'s aggregate expected payoff from bidding $b^H$ in the very first period (while the other high type advertisers are doing the same) is,

$$E(\Pi^i_{S,ND}) = \sum_{j=0}^{2} p^j \cdot \Pi^{j\,(a,1)} + \sum_{j=0}^{2} p^j \cdot \Pi^j_{(a,2)} + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_F \tag{7}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $V^L$ in the very first period (while the other high type advertisers are bidding $b^H$ in the very first period ) is,

$$E(\Pi^i_{S,D1}) = \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} + \Pi^i_F + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_F \tag{8}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $(b^H - \epsilon)$ in the very first period (while the other high type advertisers are bidding $b^H$ in the very first period ) is,

$$E(\Pi^i_{S,D2}) = \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} + \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,2)} + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_F \tag{9}$$

The expected value comparisons (see appendix $D$) show that $E(\Pi^i_{S,ND}) > E(\Pi^i_{S,D1})$ when $\frac{C_1}{C_2} < C^c(p)$ and $E(\Pi^i_{S,ND}) > E(\Pi^i_{S,D2})$ when $\frac{C_1}{C_2} < C^d(p)$. Region $IV$ in the following graph shows the feasible range of click through rates ratio ($\frac{C_1}{C_2}$) and probability ($p$) which will lead to a pure strategy separating equilibrium.

## Separating equilibrium under minimum disclosure of information

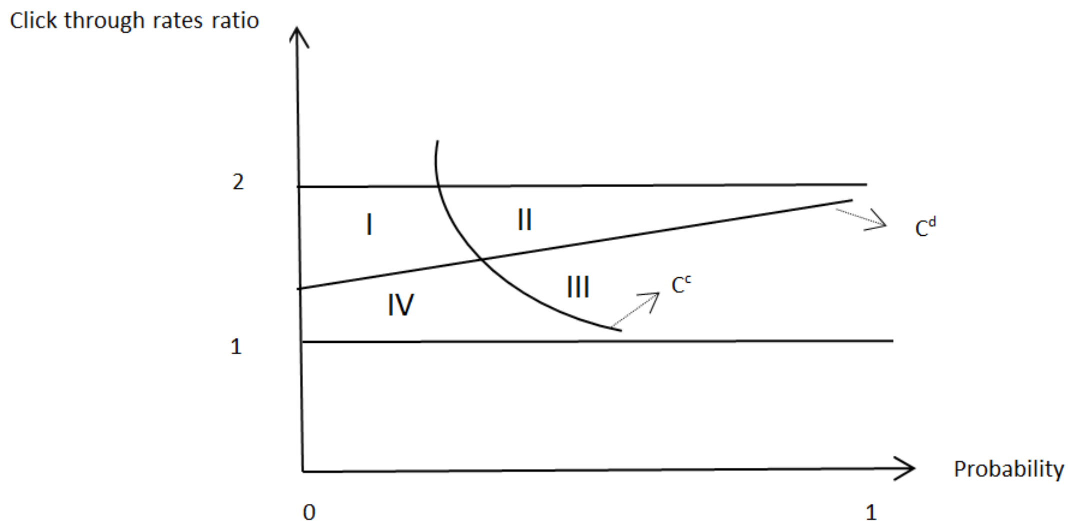


Figure 4: Equilibrium under minimum disclosure of information

The above results give us the the next proposition.

**Proposition 4.** *In a T period dynamic GSP auction with minimum disclosure of information, no pure strategy pooling equilibrium takes place. A pure strategy separating equilibrium takes place if* $\frac{C_1}{C_2} < \min\{C^c, C^d\}$ *As* $C^c$ *is strictly greater than* $C^a$*, the overall likelihood of pure strategy separating equilibrium is higher under minimum disclosure of information.*

For detailed proof, see appendix $D$ – Proof of Proposition 4.

Overall likelihood is being measured by the area in this two dimensional graph. However, as $C^c$ is strictly greater than $C^a$ (i.e. they do not intersect each other within the relevant click through rates ratio zone), the higher area can just be translated as a higher range in one dimension (i.e. the range in $p$). We find that in a multi–period dynamic GSP auction with 'minimum disclosure of information', all the high type advertisers never mimic the other type at the same time i.e. pooling equilibrium does not exist. As a result, the equilibrium total payment under minimum disclosure of information is higher than the equilibrium total payment under complete disclosure of information.

## 5 Discussion and Conclusion

In this paper we develop a model of incomplete information keyword auction and compute all feasible Bayes–Nash equilibria of the game. Our analysis and the results show that the incomplete information GSP auction can not be treated as a straightforward generalization of a simple second price auction. Unlike some of the existing models in the literature, we show that for a certain type of advertisers (advertisers with low valuations for the keywords) truthful bidding would be the only equilibrium bidding strategy. While we agree with the existing literature on the possibility of having multiple Bayes–Nash equilibria, we also explain why some of those equilibrium bidding strategies would not sustain. From a theoretical perspective, this paper has the following contributions – *(i)* in an incomplete information GSP auction, a high type advertiser would bid truthfully only when the difference between the click through rates or the difference between the valuations of a keyword for both types of advertiser is negligible, *(ii)* if the ratio of the click through rates exceeds a critical value, high type advertisers would not have any equilibrium bidding strategy, *(iii)* the existence of equilibrium in a dynamic GSP auction depends on multiple critical values of click through rates ratio; a low $p$ does not necessarily guarantee a separating equilibrium or a high $p$ does not necessarily guarantee a pooling equilibrium and *(iv)* under minimum disclosure of information, high type advertisers would never mimic the low type at the equilibrium.

While we contribute to the existing academic literature on GSP auction by providing a complete analysis of the static and the dynamic incomplete information game, we provide additional insights that may help the advertisers and the search engines. In this regard, our results also have some important substantive contributions. The analysis of the incomplete information game in this paper shows that a generalized second price auction mechanism does not necessarily motivate the advertisers to bid truthfully. Though the search engines like Google or Yahoo advise the advertisers to bid their true valuations, the existing mechanism does not provide the advertisers a strong incentive to do so. To avoid this problem, the search engines can either tweak the existing mechanism or provide additional incentives in the existing system. For example, they can group the slots on a single page and run separate auctions for different slot–groups. Secondly, the Bayes–Nash equilibrium of our model shows that a low cost equilibrium (which is one of the multiple equilibria) would give the advertisers a relatively higher payoff. Thirdly, if the click through rates ratio exceeds a critical value, we observe that there does not exist any pure strategy equilibrium outcome. This has revenue implications for the search engines; perhaps the search engines can do some optimization in terms of number of positions offered per page. Fourthly, it explains why search engines prefer to have minimum disclosure of information in GSP auctions. It has been noted in the literature (Pardoe and Stone (2010)) that in GSP auctions, the advertisers receive minimal information on the activity of the other advertisers which makes the individual bidding process too complicated. However, our results show that under such circumstances, the high type advertisers never mimic the low type advertisers which leads to higher expected aggregate payment made by the advertisers. Lastly, some existing literature like Jansen (2011) argue that the ads positioned at the bottom of a search engine landing page might get almost equal number of clicks that the top positions get - this is known as 'recency effect'. our results show that in presence of such an effect, the high type advertisers would shade their bids by relatively large amounts because the top positions would give same click through rates at higher costs. That means in presence of a 'recency effect', high type advertisers are more likely to mimic the low type advertisers. However, as the differences between click through rates widen, the high type advertisers would bid high. In that case, the search engines should try to accommodate as many positions as possible (without making it visibly congested) on the top of the search landing pages ensuring that the ratio of the click through rates does not exceed the critical value.

# Appendices

## A Proof of proposition 1

To develop a player's belief structure we assume that the valuation of a keyword is a random variable which can take a value of $V^H$ (high valuation) with probability $p$ and a value of $V^L$ (low valuation) with a probability $(1-p)$. Probability $p$ is a subjective probability but in our model it is same for every advertiser. The two events of an advertiser being either a high type or a low type are both mutually exclusive and mutually exhaustive. Using this individual probability distribution we formulate a joint probability distribution as described below,

Expected payoff from bidding $b^L$ when $V = V^L$ is,

$$E(\pi^L|\, b(L) = b^L) = p_{hh}.0 + \frac{p_{hl}}{2}(V^L - b^L)C_2 + \frac{p_{ll}}{3}(V^L - b^L)(C_1 + C_2) \tag{A.1}$$

## Advertiser's belief structure

| Joint probability distribution | In terms of $p$ | Event |
|---|---|---|
| $p_{hh}$ | $p.p$ | Two high type competitors |
| $p_{hl}$ | $2.p.(1-p)$ | One high type competitor, one low type competitor |
| $p_{ll}$ | $(1-p).(1-p)$ | Two low type competitors |

Table 4: Joint probability distribution

The first term in the expression on the right side implies that when a low type advertiser is facing two high type advertisers, he would certainly not get any position; so, his payoff will be zero. However, if he faces one high type competitor and one low type competitor then with probability $\frac{1}{2}$ he would get the second position. If the low type advertiser gets the second position then his net payoff (per click) would be the difference between his valuation for the keyword and the bid. As the second position gives $C_2$ number of clicks, his total payoff would be $(V^L - b^L)C_2$. Lastly, if both of his competitors are of low type then with probability $\frac{1}{3}$ each he would get either the first position or the second position or nothing. The third term captures a low type advertiser's expected payoff in this regard.

Using similar logic we can compute the expected payoffs of the advertisers in various situations. Expected payoff from bidding $(b^L + \epsilon)$ when $V = V^L$ (and no other player is deviating) is,

$$E(\pi^L \,|\, b(L) = b^L + \epsilon) = p_{hh}.0 + p_{hl}(V^L - b^L)C_2 + p_{ll}(V^L - b^L)C_1 \tag{A.2}$$

Expected payoff from bidding $(b^L - \epsilon)$ when $V = V^L$ (and no other player is deviating) is 0. Therefore, a low type player will never bid $(b^L - \epsilon)$. Similarly, expected payoff from bidding $b^H$ when $V = V^H$ is,

$$\begin{aligned} E(\pi^H \,|\, b(H) = b^H) = \frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 \\ + p_{ll}(V^H - b^L)C_1 \end{aligned} \tag{A.3}$$

Expected payoff from bidding $(b^H + \epsilon)$ when $V = V^H$ (and no other player is deviating) is,

$$\begin{aligned} E(\pi^H \,|\, b(H) = b^H + \epsilon) = p_{hh}(V^H - b^H)C_1 + p_{hl}(V^H - b^H)C_1 \\ + p_{ll}(V^H - b^L)C_1 \end{aligned} \tag{A.4}$$

Expected payoff from bidding $(b^H - \epsilon)$ when $V = V^H$ (and no other player is deviating) is,

$$E(\pi^H | \, b(H) = b^H - \epsilon) = p_{hh}.0 + p_{hl}(V^H - b^L)C_2 + p_{ll}(V^H - b^L)C_1 \tag{A.5}$$

In order to be indifferent between bidding $b^L$ and bidding $(b^L + \epsilon)$ the two expected payoffs must be same. Solving the equation $E[\pi^L | b(L) = b^L] = E[\pi^L | b(L) = b^L + \epsilon]$ we get $b^L = V^L$.

On the other hand, a high type player will prefer to bid $b^H$ over $(b^H + \epsilon)$ if

$$\begin{aligned}
\frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_{ll}(V^H - b^L)C_1 \quad &> \\
p_{hh}(V^H - b^H)C_1 + p_{hl}(V^H - b^H)C_1 + p_{ll}(V^H - b^L)C_1 & \\
\Rightarrow b^H > \frac{C_1(3-p)V^H - C_2((3-2p)V^H - 3(1-p)V^L)}{C_1(3-p) - C_2 p} = b^{H^*} &
\end{aligned}$$

Similarly, a high type player will prefer to bid $b^H$ over $(b^H - \epsilon)$ if

$$\begin{aligned}
\frac{p_{hh}}{3}(V^H - b^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b^H)C_1 + \frac{p_{hl}}{2}(V^H - b^L)C_2 + p_{ll}(V^H - b^L)C_1 \quad &> \\
p_{hh}.0 + p_{hl}(V^H - b^L)C_2 + p_{ll}(V^H - b^L)C_1 & \\
\Rightarrow \frac{C_1(3-2p)V^H - C_2((3-4p)V^H - 3(1-p)V^L)}{C_1(3-2p) + C_2 p} = b^{H^{**}} > b^H &
\end{aligned}$$

Therefore, any bid $b^H$ which lies in between $b^{H^*}$ and $b^{H^{**}} > b^H$ could possibly be an equilibrium bid for the high type advertiser. We also observe that $b^{H^*} - V^L = \frac{(C_1(3-p) - C_2(3-2p))(V^H - V^L)}{C_1(3-p) - C_2 p} > 0$ and $b^{H^{**}} - V^L = \frac{(C_1(3-2p) - C_2(3-4p))(V^H - V^L)}{C_1(3-2p) + C_2 p} > 0$.

# B Proof of Proposition 2

We see that $b^{H^*} - b^{H^{**}} = \frac{3(C_1 - 2C2)C_2(1-p)p(V^H - V^L)}{(C_1(3-p) - C_2 p)(C_2 p + C_1(3-2p))}$. The result implies that as long as $C_1 < 2C_2$, $b^{H^*}$ is less than $b^{H^{**}}$. However, if $C_1 > 2C_2$ then $b^{H*}$ is greater than $b^{H^{**}}$.

Now, consider any $b^H$ which satisfies the condition $b^{H^*} \leq b^H \leq b^{H^{**}}$.

$b^{H^*} \leq b^H \Rightarrow E(\pi^H | \, b(H) = b^H + \epsilon) \leq E(\pi^H | \, b(H) = b^H)$

$b^H \leq b^{H^{**}} \Rightarrow E(\pi^H | \, b(H) = b^H - \epsilon) \leq E(\pi^H | \, b(H) = b^H)$

This implies that when $C_1 < 2C_2$, expected payoff from playing $b^H$ will be the highest. So, a high type advertiser would not deviate from $b^H$ when $C_1 < 2C_2$.

Further, consider any $b^H$ which satisfies the condition $b^{H^{**}} \leq b^H \leq b^{H^*}$.

$b^{H^{**}} \leq b^H \Rightarrow E(\pi^H | \, b(H) = b^H) \leq E(\pi^H | \, b(H) = b^H - \epsilon)$

$b^H \leq b^{H^*} \Rightarrow E(\pi^H | \, b(H) = b^H) \leq E(\pi^H | \, b(H) = b^H + \epsilon)$

This implies that when $C_1 > 2C_2$, expected payoff from playing $b^H$ will be the lowest. So, a high type advertiser would always deviate from $b^H$ when $C_1 > 2C_2$.

## C Proof of Proposition 3

When a high type advertiser $i$ does not mimic a low type advertiser in the first period, his expected payoff from the first period is,

$$\begin{aligned}\sum_{j=0}^{2} p^j \cdot \pi_a^j = \frac{p_{hh}}{3}(V^H - b_1^H)(C_1 + C_2) + \frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 \\ + p_{ll}(V^H - V^L)C_1\end{aligned} \tag{C.1}$$

The next period game becomes a complete information game. The payoff from the complete information game is,

$$\begin{aligned}\sum_{j=0}^{2} p^j \cdot \pi_F^j = \frac{p_{hh}}{3} \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1})\frac{C_1}{2} + \frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} \\ + p_{ll}(V^H - V^L)C_1\end{aligned} \tag{C.2}$$

Therefore, the high type advertiser's $T$ periods aggregate expected payoff from playing the separating strategy is,

$$E(\pi^i_{S,ND}) = \sum_{j=0}^{2} p^j \cdot \pi^j_a + (T-1) \sum_{j=0}^{2} p^j \cdot \pi^j_F \tag{C.3}$$

When a high type advertiser $i$ deviates from separating equilibrium strategy in the first period and bids $V^L$, his expected payoff would be,

$$\sum_{j=0}^{2} p^j . \pi^j_b = p_{hh} \cdot 0 + \frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} + +p_{ll}(V^H - V^L)\frac{(C_1 + C_2)}{3} \tag{C.4}$$

In the second period the high type advertiser has complete information but his other high type competitors do not have complete information. So, the high type advertiser's payoff would be,

$$\pi^i_F = p_{hh}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1})C_1 + p_{hl}(V^H - V^L)C_1 + +p_{ll}(V^H - V^L)C_1 \tag{C.5}$$

Here we make the 'low cost equilibrium' refinement and assume that when a high type advertiser competes against two low type advertisers, he bids incrementally higher than $V^L$.

Starting from the third period, the game becomes a complete information game. Thus, the high type advertiser's $T$ periods aggregate expected payoff from deviating from the separating strategy equilibrium is,

$$E(\pi^i_{S,D}) = \sum_{j=0}^{2} p^j \cdot \pi^j_b + \pi^i_F + (T-2) \sum_{j=0}^{2} p^j \cdot \pi_F \tag{C.6}$$

Expected payoff comparison shows that,

$$E(\pi^i_{S,ND}) > E(\pi^i_{S,D})$$
$$\Rightarrow \frac{C_1}{C_2} <$$
$$\frac{-3+32p-43p^2+3\sqrt{1-24p+170p^2-416p^3+441p^4-200p^5+32p^6}}{2(-6+32p-34p^2+8p^3)} = C^a \tag{C.7}$$

Comparative static result shows that as $p$ increases, $C^a$ decreases. When $p$ is equal to the critical value $\underline{p}$, $C^a$ is equal to 2. Solving this equation we get $\underline{p}$ is equal to 0.38.

Now, we will consider the high type advertiser's aggregate expected payoff from the pooling strategy. When the high type advertiser mimics the low type advertiser in the first $(T-1)$ periods, the following would be his expected payoff in the first period,

$$(T-1)\sum_{j=0}^{2} p^j \cdot \pi^j_c = (T-1)\frac{(V^H - V^L)(C_1 + C_2)}{3} \tag{C.8}$$

The last period expected payoff would be same as the expected payoff from one period incomplete information GSP auction i.e. $\sum_{j=0}^{2} p^j \cdot \pi^j_a$

Thus, the high type advertiser's $T$ period aggregate expected payoff from the pooling strategy is,

$$E(\pi^i_{P,ND}) = (T-1)\sum_{j=0}^{2} p^j \cdot \pi^j_c + \sum_{j=0}^{2} p^j \cdot \pi^j_a \tag{C.9}$$

When a high type advertiser $i$ deviates from pooling equilibrium strategy in the first period and bids $b^H$, his expected payoff would be,

$$\pi_{max} = (V^H - V^L)C_1 \tag{C.10}$$

As a result in the second period all other advertisers would have complete information regarding advertiser $i$. Consequently, the other high type advertisers would always bid higher than advertiser $i$. In this case, advertiser $i$'s expected payoff is,

$$\sum_{j=0}^{2} p^j \cdot \pi_d^j = p_{hh} \cdot 0 + p_{hl}(V^H - V^L)C_2 + p_{ll}(V^H - V^L)C_1 \tag{C.11}$$

Starting from the third period, all the advertisers will be playing a complete information game and their payoff would be $\pi^F$. Thus, the high type advertiser's $T$ period aggregate expected payoff from deviating from the pooling strategy equilibrium is,

$$E(\pi^i_{P,D}) = \pi_{max} + \sum_{j=0}^{2} p^j \cdot \pi_d^j + (T-2)\pi_F^j \tag{C.12}$$

Expected payoff comparison shows that,

$$E(\pi^i_{P,ND}) > E(\pi^i_{P,D}) \Rightarrow \frac{C_1}{C_2} > \frac{-B - \sqrt{B^2 - 4AC}}{2A} = C^b \tag{C.13}$$

where,

$$A = -6 + 38p - 30p^2 + 6p^3 + 6T - 20pT + 15p^2T - 3p^3T \tag{C.14}$$

$$B = 3 - 35p + 39p^2 - 9p^3 - 3T + 17pT - 18p^2T - 3p^3T \tag{C.15}$$

$$C = -p + 6p^2 - 6p^3 + pT - 6p^2T + 6p^3T \tag{C.16}$$

Comparative statics shows that as $T$ increases, $C^b$ decreases and as $p$ increases, $C^b$ increases.

Only pure strategy separating equilibrium exists when $E(\pi^i_{S,ND}) > E(\pi^i_{S,D})$ and $E(\pi^i_{P,ND}) < E(\pi^i_{P,D})$; the resulting conditions are $\frac{C_1}{C_2} < C^a$ and $\frac{C_1}{C_2} < C^b$. This implies that as long as $\frac{C_1}{C_2} < \min\{C^a, C^b\}$, there will be only pure strategy separating equilibrium. Following similar logic, we can conclude that as long as $\frac{C_1}{C_2} > \max\{C^a, C^b\}$, there will be only pure strategy pooling equilibrium.

The above analysis shows that regardless of what other players are doing, all high type players would either pursue separating or pooling strategy at the same time since the parametric conditions would remain same for all of them. This result also implies that no high type advertiser would deviate from separating (or pooling) strategy when every other high type advertiser is pursuing separating (or pooling strategy). However, can all the high type advertiser deviate at the same time?

We find that if all the high type advertisers reveal their identities in the $J$th period (where $J < T$), then each high type advertiser's expected aggregate payoff will be,

$$
\begin{aligned}
&E(\pi^i_{SP}) = (J-1)\{\frac{(V^H - V^L)(C_1 + C_2)}{3}\} + \{\frac{p_{hh}}{3}(V^H - b_1^H)(C_1 + C_2) + \\
&\frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 + p_{ll}(V^H - V^L)C_1\} + \\
&(T-J)\{(\frac{p_{hh}}{3} \cdot 1 \cdot 0) + (\frac{p_{hh}}{3} \cdot 1 \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1V^H - C_2(V^H - V^L)}{C_1})\frac{C_1}{2} + \\
&\frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2}) + (\frac{p_{hh}}{3} \cdot 1 \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1V^H - C_2(V^H - V^L)}{C_1})\frac{C_1}{2} \\
&+ \frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} + p_{ll}(V^H - V^L)C_1)\}
\end{aligned}
\tag{C.17}
$$

Comparisons among $E(\pi^i_{SP})$, $E(\pi^i_{P,ND})$ and $E(\pi^i_{S,ND})$ show that,

$$
E(\pi^i_{P,ND}) > E(\pi^i_{SP}) > E(\pi^i_{S,ND}) \Rightarrow 2 > \frac{C_1}{C_2} > \frac{6p(1-p)-1}{1-3(1-p)^2} = C^* > 1 \tag{C.18}
$$

and,

$$
E(\pi^i_{P,ND}) < E(\pi^i_{SP}) < E(\pi^i_{S,ND}) \Rightarrow 1 < \frac{C_1}{C_2} < \frac{6p(1-p)-1}{1-3(1-p)^2} = C^* < 2 \tag{C.19}
$$

There is no $\frac{C_1}{C_2}$ in between 1 and 2 which validates the following inequalities : $E(\pi^i_{SP}) > E(\pi^i_{P,ND}) >$

$E(\pi^i_{S,ND})$ and $E(\pi^i_{SP}) > E(\pi^i_{S,ND}) > E(\pi^i_{P,ND})$.

# D Proof of Proposition 4

If the high type advertiser gets the top position in the first period auction and the second position holder is also a high type advertiser then the type of advertiser at the third position would not be known to the advertiser at the top position. In that case the high type advertiser would face two high type advertisers with probability $p$ and one high type, one low type with probability $(1-p)$. This is a new game of incomplete information when only one rival advertiser's type is unknown. The parameter $b_2^H$ is the equilibrium bid of a high type advertiser in such a static incomplete information game. An analysis of a one period incomplete information game with the above mentioned belief structure reveals that $b_2^H = \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p)-2C_2p}$. The low cost equilibrium bid as derived in static incomplete information game would lead to this expression when $p_{hh} = p$, $p_{hl} = (1-p)$ and $p_{ll} = 0$.

Now, a high type advertiser $i$'s aggregate expected payoff from bidding $b^H$ in the very first period (while the other high type advertisers are doing the same) is,

$$E(\Pi^i_{S,ND}) = \sum_{j=0}^{2} p^j \cdot \Pi^j_{(a,1)} + \sum_{j=0}^{2} p^j \cdot \Pi^j_{(a,2)} + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_{F} \tag{D.1}$$

Where,

$$\begin{aligned}\sum_{j=0}^{2} p^j \cdot \Pi^j_{(a,1)} = &\frac{p_{hh}}{3}(V^H - b_1^H)(C_1+C_2) + \frac{p_{hl}}{2}(V^H - b_1^H)C_1 + \frac{p_{hl}}{2}(V^H - V^L)C_2 \\ &+ p_{ll}(V^H - V^L)C_1\end{aligned} \tag{D.2}$$

$$\sum_{j=0}^{2} p^j \cdot \Pi^j_{(a,2)} = (\frac{p_{hh}}{3} + \frac{p_{hl}}{2})(p(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p}) \frac{C_1 + C_2}{3} + (1-p)((V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p})\frac{C_1}{2} + (V^H - V^L)\frac{C_2}{2})) + p_{ll}(V^H - V^L)C_1 \tag{D.3}$$

$$\sum_{j=0}^{2} p^j \cdot \Pi^j_F = \frac{p_{hh}}{3} \cdot 0 + \frac{p_{hl}}{2}(V^H - \frac{C_1V^H - C_2(V^H - V^L)}{C_1})\frac{C_1}{2} + \frac{p_{hl}}{2}(V^H - V^L)\frac{C_2}{2} + p_{ll}(V^H - V^L)C_1 \tag{D.4}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $V^L$ in the very first period (while the other high type advertisers are bidding $b^H$ in the very first period) is ,

$$E(\Pi^i_{S,D1}) = \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} + \Pi^i_F + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_F \tag{D.5}$$

Where,

$$\sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} = p_{hh} \cdot 0 + p_{hl}(V^H - V^L)\frac{C_2}{2} + p_{ll}(V^H - V^L)\frac{C_1 + C_2}{3} \tag{D.6}$$

$$\Pi^i_F = p_{hh} \cdot (V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2p})C_1 + p_{hl}(V^H - V^L)C_1 + p_{ll}(V^H - V^L)C_1 \tag{D.7}$$

A high type advertiser $i$'s aggregate expected payoff from bidding $(b^H - \epsilon)$ in the very first period (while the other high type advertisers are bidding $b^H$ in the very first period) is,

$$E(\Pi^i_{S,D2}) = \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} + \sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,2)} + (T-2)\sum_{j=0}^{2} p^j \cdot \Pi^j_F \tag{D.8}$$

Where,

$$\sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,1)} = p_{hh} \cdot 0 + p_{hl}(V^H - V^L)C_2 + p_{ll}(V^H - V^L)C_1 \tag{D.9}$$

$$\begin{aligned}
&\sum_{j=0}^{2} p^j \cdot \Pi^j_{(b,2)} = p_{hh}(V^H - V^H)\frac{C_1}{2} \\
&+ p_{hh}(V^H - \frac{C_1(3+p)V^H - C_2((3-p)V^H - 3(1-p)V^L)}{C_1(3+p) - 2C_2 p})\frac{C_2}{2} \\
&+ p_{hl}(V^H - \frac{C_1 V^H - C_2(V^H - V^L)}{C_1})\frac{C_1}{2} + p_{hl}(V^H - V^L)\frac{C_2}{2} + p_{ll}(V^H - V^L)C_1
\end{aligned} \tag{D.10}$$

The expected value comparisons show that,

$$\begin{aligned}
&E(\Pi^i_{S,ND}) > E(\Pi^i_{S,D1}) \\
&\Rightarrow \frac{C_1}{C_2} < \left(\frac{x}{2y} + \frac{(1+i\sqrt{3})w}{(6)X(2^{\frac{2}{3}})X(y(z+\sqrt{4w^3+z^2})^{\frac{1}{3}})}\right. \\
&\left. - \frac{(1-i\sqrt{3})(z+\sqrt{4w^3+z^2})^{\frac{1}{3}}}{(12)X(2^{\frac{1}{3}})X(y)}\right) = C^c
\end{aligned}$$

where,

$$x = 3 - 24p - 13p^2 + 4p^3$$

$$y = 9 - 36p - p^2 + 4p^3$$

$$w = -81 + 1782p - 9666p^2 + 7722p^3 - 1809p^4 - 576p^5 + 144p^6$$

$$z = 1458 - 48114p + 523422p^2 - 2049948p^3 + 1759806p^4$$
$$+ 384426p^5 - 904446p^6 + 158436p^7 + 62208p^8 - 15552p^9$$

and,

$$E(\Pi^i_{S,ND}) > E(\Pi^i_{S,D2}) \Rightarrow \frac{C_1}{C_2} < \frac{9 + 9p + \sqrt{81 - 6p + 25p^2}}{4(3+p)} = C^d.$$

Thus, a pure strategy separating equilibrium occurs if $\frac{C_1}{C_2} < \min\{C^c, C^d\}$.

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