Group incentive compatibility in the housing market problem with weak preferences*

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Abstract

I consider the housing market problem with weak preferences. In this context, I provide a sufficient condition for *weak group strategy proofness*; no group of agents can jointly misreport their preferences such that each agent in the group becomes better-off. Using this sufficient condition, I prove that the *top trading absorbing sets*, *top cycles*, and *highest priority object* rules satisfy *weak group strategy proofness*. Thus, this paper establishes that it is possible to achieve *weak group strategy proofness*, along with other desirable results, for the housing market problem with weak preferences even though *group strategy proofness* is incompatible with *Pareto efficiency* in this setting.

Keywords: housing market, incentive compatibility, top trading cycles, weak preferences.

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1 Introduction

I consider the problem of reallocating objects among a set of agents. Specifically, I consider the problem where each agent has to be assigned exactly one object and is endowed with exactly one object as well. Each agent has preferences over the set of objects. These objects are to be reassigned, among the agents, without any monetary transfers. Such a reallocation problem is referred to as the housing market problem in the literature and was introduced by Shapley & Scarf [19]. This simple economy has several real-life applications such as the allocation of housing [1], offices, seminar slots, and organs for transplant [16].

The *top trading cycles* (TTC) rule, attributed to David Gale, was proposed by Shapley & Scarf [19] for the housing market problem. The TTC rule proceeds by repeating the following until all agents have been removed from the problem: Each agent points at an agent holding her most preferred object. Since each agent is pointing at another agent and there is a finite number of agents, there is at least one cycle. Each agent in the cycle is assigned the object owned by the agent she is pointing at for this housing market problem and removed from the problem.[1] When an agent is removed from the problem, along with an object, her assignment is finalized.

For the case of strict preferences, i.e., the agents cannot express indifferences between objects, several desirable results have been proved for the TTC rule. Roth & Postlewaite [15] show that the outcome of the TTC rule is the unique allocation in the *core* i.e. no group of agents can reallocate their endowments such that at least one agent in the group is made better-off without making any agent in the group worse-off compared to the outcome of the rule. An allocation is called *competitive* if there are prices for the objects such that each agent purchases her most preferred object among the affordable objects and the price of her assignment is the same as the price of her endowment. Roth & Postlewaite [15] also show that the outcome of the TTC rule is the unique *competitive allocation*. With regards to incentive compatibility, the TTC rule is *strategy proof* [14], no agent has an incentive to misreport her preferences, and *group strategy proof* [4], no group of agents has an incentive to jointly misreport their preferences such that at least one agent in the group is made better-off without making any agent in the group worse-off. Moreover, the TTC rule is the only rule that satisfies *Pareto efficiency*, *individual rationality*, and *strategy proofness* simultaneously [9, 20]. Finally, the

[1] The cycles for which trades are conducted in an algorithm are referred to as trading cycles in what follows.

TTC rule is *anonymous*, independent of how the agents are named, and *non-bossy*, no agent can influence the welfare of other agents without affecting her welfare [10].

Considering the more general case of weak preferences, i.e., allowing the agents to be indifferent between objects, for the housing market problem is a natural extension because indifferences can arise in a real-life setting when the agents: (1) do not have enough information to break ties between the objects, or (2) consider different objects to be identically important e.g. organs from multiple donors can be identical for a patient, for transplantation purposes, as biological properties of organs from different donors, blood-type and tissue-type, can be identical. In accounting for indifferences, however, some of the desirable results achievable for the strict preferences, are no longer possible.

In the presence of indifferences, a *core* allocation may not exist [19], *competitive allocation* does not coincide with the *core* [22], and *Pareto efficiency* is incompatible with *group strategy proofness* [6]. Additionally, *Pareto efficiency*, *individual rationality*, and *strategy proofness* are, in general, not compatible [20]. Moreover, following impossibility results hold under the weak preferences: (1) no rule is *Pareto efficient*, *strategy proof*, and *anonymous*, (2) no rule is *Pareto efficient*, *strategy proof*, *individually rational* and *non-bossy*, and (3) no rule is *Pareto efficient*, *strategy proof*, *individually rational*, and *consistent* [5, 7, 8]. *Consistency* states that if there is a group of agents where each agent in the group is assigned the endowment of an agent who is also in the group, then applying the rule on the agents who do not belong to this group, along with their endowments, keeps the assignment of these agents unchanged.

An intuitive method to incorporate indifferences is to arbitrarily break ties between the objects and then applying the *TTC* rule to the induced housing market problem with strict preferences. This straightforward rule is *weakly Pareto efficient*, *individually rational*, *strategy proof*, *non-bossy* and *consistent* [7]. In addition to these results, the rule with arbitrary tie-breaking can be generalized to the setting where the agents are endowed with multiple objects [11, 13]. However, *weak Pareto efficiency* can be particularly weak because any assignment in which at least one agent gets one of her most preferred objects is *weakly Pareto efficient* regardless of how objects are assigned to the other agents. Additionally, examples can be found where, no matter how ties are broken, the outcome of the *TTC* rule with fixed tie-breaking is not *Pareto efficient* [8].

Even though several desirable results are not possible for the housing market problem with weak preferences, some appropriate results are still achievable because the *weak core*, no group of agents can reallocate their endowments such that each agent in the group is made better-off compared to the rule's outcome, is non-empty [19], and the incompatibility of *Pareto efficiency*, *individual rationality*, and *strategy proofness* holds under certain assumptions on the preference domain [21].[2] Utilizing this, much progress has been made for the housing market problem with weak preferences. Alcalde-Unzu & Molis [2] and Jaramillo & Manjunath [8] independently proposed generalizations of the *TTC* rule to account for indifferences; the *top trading absorbing sets* (*TTAS*) and *top cycles* (*TC*) rules, respectively. Both of these rules are *Pareto efficient*, *individually rational*, *strategy proof*, *weak core-selecting*, and *core-selecting* (whenever the *core* is non-empty) [2, 3, 8]. The *TC* rule has a polynomial running time, $O\left(n^6\right)$ implementation where n is the number of agents in the problem, whereas the *TTAS* rule has an exponential running time in the worst case [3, 8]. Plaxton [12] proposed another generalization of the *TTC* rule which is also *Pareto efficient*, *individually rational*, *strategy proof*, *weak core-selecting*, and *core-selecting* (whenever the *core* is non-empty) in addition to having an $O\left(n^3\right)$ implementation.

Aziz & de Keijzer [3] introduced the *generalized absorbing top trading cycle* (*GATTC*) class of rules. Each member of the *GATTC* family satisfies *Pareto efficiency*, *individual rationality*, *weak core-selection*, and *core-selection* (whenever the *core* is non-empty). However, the members of the *GATTC* family may not satisfy *strategy proofness*. Saban & Sethuraman [17] establish sufficient conditions for *strategy proofness* and employ these conditions to provide computationally efficient algorithms for the housing market problem with weak preferences. They provide a class of rules for which each member satisfies *Pareto efficiency*, *individual rationality*, *weak core-selection*, and *strategy proofness*, namely; *common ordering on agents*, *individual ordering on objects* (*CAIO*).[3] Moreover, they propose a member from the *CAIO* family, the *highest priority object* (*HPO*) rule, which is computationally quicker than the *TTAS* and *TC* rules with an $O\left(n^2 \log n + n^2\gamma\right)$ implementation.[4]

In this paper, I focus on the *TTAS*, *TC*, and *HPO* rules owing to their algorithmic similarities. Essentially, the proof of *strategy proofness* for these rules is similar which allows for the possibility of a common sufficient

[2] The impossibility result of [20] is established for the problem where at least one agent is endowed with more than one object. Additionally, the impossibility result relies on the assumption that each agent finds every other object to be strictly better or worse than her endowment.

[3] The *TTAS* and *TC* rules are members of the *CAIO* family [17] while the *CAIO* family is a subset of the *GATTC* family.

[4] Here, γ is the maximum size of an indifference class in the preferences of the agents.

condition for *weak group strategy proofness* for these rules. *Strategy proofness* of Plaxton's mechanism, on the other hand, is proved using a *confluence* property of the algorithm [12], unlike the aforementioned rules. As such, the application of the sufficient condition of *weak group strategy proofness*, provided in this paper, may not be possible for Plaxton's mechanism.

The algorithms for the $TTAS$, TC, and HPO rules share important commonalities. These rules are iterative and each step consists of three phases; *departure*, *pointing*, and *trading*. In the *departure* phase, a group of agents has their assignments finalized and these agents are removed from the algorithm if no more beneficial trades are possible using these agents and their assigned objects. Specifically, it is not possible to increase the welfare of any agent in the group and the only possible way to increase the welfare of some agent outside of the group would decrease the welfare of some agent in the group. Similar to the TTC rule, the $TTAS$, TC, and HPO rules rely on the formation of cycles to determine the trades that should occur at each step. However, in the presence of indifferences, each agent can have multiple most preferred objects (among the remaining ones) at any step of the algorithm. Such agents could belong to multiple cycles and, as such, the algorithm should be capable of determining a unique pointee for these agents so that no agent belongs in more than one trading cycle. So, each cycle may not become a trading cycle in the presence of indifferences i.e. trades may not be conducted for each cycle that is formed. This is in contrast to the strict preferences where each cycle ends up as a trading cycle. The *pointing* phase of these rules is responsible for determining a unique pointee for each agent and the criterion used for this determination is referred to as the unique pointee selection criterion. The unique pointee selection criterion relies on a priority ordering over agents or objects to determine a unique pointee for each agent.[5,6] Finally, in the *trading* phase, the objects are exchanged, among the agents, according to the trading cycles that are formed in the *pointing* phase.

In this paper, I focus on group incentive compatibility in the context of the housing market problem with weak preferences. In this setting, it has been established that *group strategy proofness* is incompatible with *Pareto efficiency* [6]. However, I prove that *weak group strategy proofness* is still achievable. To prove this, I show that if the unique pointee selection criterion of the *pointing* phase of a rule satisfies certain conditions, the rule satisfies *weak group strategy proofness*. In other words, I provide a sufficient condition for *weak group*

[5]There are members of the $CAIO$ which rely on priority orderings over the agents and objects.
[6]The unique pointee selection criterion for the $TTAS$, TC, and HPO rules is independent of the step of the algorithm.

strategy proofness, namely *consistent pointing*, and, using this sufficient condition, I show that the *TTAS*, *TC*, and *HPO* rules are *weakly group strategy proof.*

The remainder of the paper is organized as follows. The model is presented in Section 2 along with some relevant notation. Some properties are formally defined in Section 3. In Section 4, I present three of the existing rules along with the two properties utilized in the sufficient condition for *strategy proofness* presented by Saban & Sethuraman [17]. Section 5 provides the proofs with related discussion and Section 6 concludes the paper. To the best of my knowledge, the possibility of achieving *weak group strategy proofness* has not been studied in the context of the housing market problem with weak preferences.

2 Model and Notation

Let N and O be the sets of agents and objects, respectively. As stated by Jaramillo & Manjunath [8], $|N| = |O|$ can be assumed without loss of generality. Each agent is endowed with an object and the endowment of the agents is represented by the bijection $\omega : N \to O$. For each $i \in N$, agent i's endowment is denoted as ω_i and, for any group of agents $M \subseteq N$, $\omega_M \equiv \{\omega_i : i \in M\}$ i.e. ω_M is the set of endowments of all agents in M.

Let $\mathcal{R}$ be the set of all possible complete and transitive preference relations over O. For a given $R \in \mathcal{R}^N$, the preference relation for agent $i \in N$ is denoted as R_i and for each $a, b \in O$: (1) a being at least as good as b for agent i is represented as aR_ib, (2) a being preferred to b by agent i is represented as aP_ib, and (3) agent i being indifferent between a and b is denoted as aI_ib. Moreover, R_{-i} is used to denote preferences of the agents in N other than agent i. For any $M \subseteq N$, R_M denotes the preferences of all agents in M and $R_{N\backslash M}$ denotes the preferences of all agents in $N\backslash M$. For any $R \in \mathcal{R}^N$, $i \in N$ and $O' \subseteq O$, let $\tau(R_i, O')$ represent the set of agent i's most preferred objects in O' under R_i. Formally, $\tau(R_i, O') \equiv \{a \in O' : aR_ib \ \forall b \in O'\}$.

Let A be the set of all possible allocations i.e. it contains all bijections from N to O. For any allocation $\alpha \in A$, let the object allocated to agent i under α be denoted as α_i. Moreover, for any $M \subseteq N$, let $\alpha_M \equiv \{\alpha_i : i \in M\}$ i.e. α_M is the set of objects assigned to the agents in M under α.

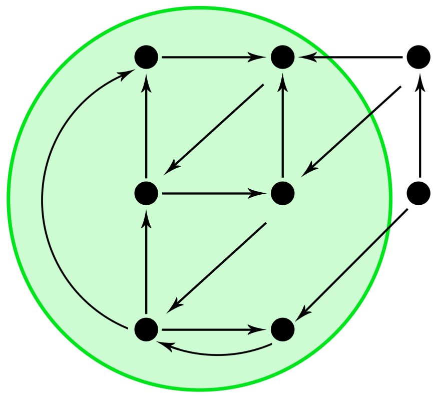
As mentioned in Section 1, the proposed rules for the housing market problem with weak preferences require priority orderings over agents or objects. These priority orderings are complete, transitive, and anti-symmetric. With a slight abuse of notation, I use the same notation for priority orderings over agents and objects: $\prec$. When $\prec$ represents the priority ordering over agents, $i \prec j$ depicts that agent i has higher priority ordering than agent j for $i, j \in N$. When $\prec$ represents priority ordering over objects, $a \prec b$ shows that object a has higher priority ordering than object b for $a, b \in O$.

The quadruple (N, O, R, ω) denotes a housing market problem with the set of agents N, set of objects O, preference profile R, and endowment ω. Without loss of generality, N and O can be considered to be fixed. So, the housing market problem can be simply represented as (R, ω). An allocation rule, $\varphi : \mathcal{R}^N \times A \to A$, gives an assignment for a given housing market problem i.e. for $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega)$ represents the assignment for the housing market problem (R, ω) under the allocation rule φ.

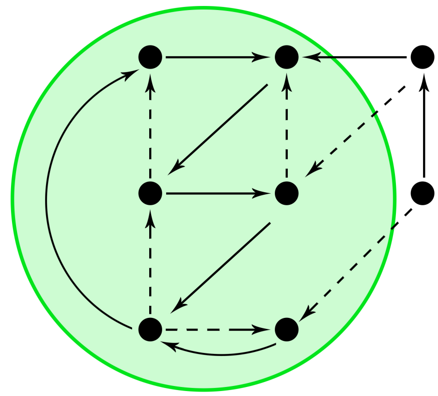
Since the analysis of the housing market problem with weak preferences relies on graph theory, I present some relevant concepts and notation here. Let $G = (V, E)$ be a directed graph where V is a set of vertices and E is a set of directed arcs. For $v, v' \in V$, there is a path from v to v' if there are vertices $v_1, \cdots, v_m$ with a directed arc from v_k to v_{k+1} for $k \in \{1, \cdots, m-1\}$ such that $v_1 = v$ and $v_m = v'$. An ordered set of vertices $(v_1, \cdots, v_m)$ with an arc from v_k to v_{k+1} for $k \in \{1, \cdots, m-1\}$ in the directed graph G is referred to as a chain, denoted as $Ch = (v_1, \cdots, v_m)$. If there is an arc from v_m to v_1, then it is referred to as a cycle which is denoted as $C = (v_1, \cdots, v_m)$.[7] For each $v \in V$, let $E_v \subseteq V$ be the set of vertices with an incoming directed arc from the vertex v. The set of vertices $V' \subseteq V$ is referred to as an absorbing set if: (1) for every $v, v' \in V'$, there is a path from v to v' and a path from v' to v, and (2) there is no path from any $v \in V'$ to $v' \in V \backslash V'$ i.e. $E_v \cap (V \backslash V') = \phi$. Figure 1 (a) provides an example of a directed graph in which the unique absorbing set is depicted in a green shaded circle. Define F as a unique pointee selection criterion where $F(G) = \left(V, E^F\right)$ is a directed graph with the same vertices as the directed graph G and $E^F \subseteq E$ is the set of directed arcs of the directed graph $F(G)$ such that each vertex has exactly one outgoing directed arc i.e. for each $v \in V$, the set of directed arcs from the vertex v in the directed graph $F(G)$, denoted as E_v^F, is a singleton set. Figure 1 (b) shows an application of a unique pointee selection criterion on the directed graph presented in Figure 1 (a).

Consider an arbitrary $(R, \omega) \in \mathcal{R}^N \times A$. Let $\varphi : \mathcal{R}^N \times A \to A$ be a member of the *CAIO* family. Then, the rule φ is iterative and each step has three phases: *departure*, *pointing* and *trading*. In the *departure* phase, the assignment of some of the agents is finalized and they are removed from the problem. Let N_t and O_t

[7] Vertices are not repeated in an ordered set.



(a) A directed graph with 8 vertices and 14 directed arcs is depicted above. The vertices enclosed in the green shaded circle depict the unique absorbing set of this directed graph.



(b) A unique pointee selection criterion, F, has been applied to the directed graph so that each vertex has exactly one outgoing directed arc. The arcs not selected by F are represented as dashed arcs.

Figure 1

be the set of agents and objects, respectively, remaining after the *departure* phase of step t. Let h_t be the bijection which represents the objects held by agents at the beginning of step t. Moreover, $h_{i,t}$ be the object held by agent i at the beginning of step t and, for any $M \subseteq N$, let $h_{M,t} \equiv \{h_{i,t} : i \in M\}$. An agent $i \in N_t$ is referred to as a satisfied agent at step t if $h_{i,t} \in \tau(R_i, O_t)$ i.e. among the objects remaining at step t, agent i holds one of her most preferred objects. Let S_t be the set of all satisfied agents in N_t at step t. An agent who is not satisfied is referred to as an unsatisfied agent and the set of all unsatisfied agents in N_t, at step t, is denoted as U_t. At step t, let $G_t = (N_t, E_t)$ denote the directed graph where the agents in N_t are the vertices and the set E_t consists of directed arcs from each agent $i \in N_t$ to every agent $j \in N_t$ such that $h_{j,t} \in \tau(R_i, O_t)$. That is, in the set E_t, there is a directed arc from each agent to every agent who owns one of her most preferred objects (among the remaining ones). The directed graph G_t is referred to as the *TTC*-graph at step t. Additionally, let AS_t be the set of all absorbing sets of the directed graph G_t. For each $i \in N_t$, let $E_{i,t}$ denote the agents that have an incoming arc from agent i. Then, for each $j \in E_{i,t}$, it must be that $h_{j,t} \in \tau(R_i, O_t)$. Let $Ch(G_t)$ and $C(G_t)$ denote the set of all chains and cycles in the *TTC*-graph G_t, respectively. Additionally, let $Ch_i(G_t)$ be the set of all chains in $Ch(G_t)$ which have agent i as the last agent i.e. if $Ch \in Ch_i(G_t)$ and $Ch = (i_1, \cdots, i_m)$, then $i_m = i$. Moreover, let $C_i(G_t)$ represent the set of all cycles in $C(G_t)$ which include agent i i.e. if $C \in C_i(G_t)$ and $C = (i_1, \cdots, i_m)$, then $i_k = i$ for some $k \in \{1, \cdots, m\}$. In the *pointing* phase, each agent points at a unique agent based on the unique pointee selection criterion associated with the rule φ; denoted as F. Then, $F(G_t) = \left(N_t, E_t^F\right)$ is a directed graph where each agent has exactly one outgoing arc i.e. $E_{i,t}^F$ is a singleton set for each agent i.[8,9] Let $p_{i,t}$ denote the agent pointed at by agent i in the directed graph $F(G_t)$. Let $Ch(F(G_t))$ and $C(F(G_t))$ be the set of all chains and cycles in the directed graph $F(G_t)$, respectively. Additionally, let $Ch_i(F(G_t))$ be the set of chains in $Ch(F(G_t))$ which have agent i as the last agent and $C_i(F(G_t))$ be the set of cycles in $C(F(G_t))$ which include agent i.[10] Since each agent is pointing at exactly one agent, there is at least one cycle in the directed graph $F(G_t)$ i.e. $C(F(G_t)) \neq \phi$. The cycles in $C(F(G_t))$ are referred to as trading cycles because: (1) in the *trading* phase, objects are traded according to these cycles, and (2) each agent can belong to at most one such cycle i.e. $|C_i(F(G_t))| \leq 1$ for any $i \in N_t$. The agents involved in these trades are said to have become part of a trading cycle and their objects are updated in step $t+1$. Formally, for each cycle $C = (i_1, \cdots, i_m)$, in $C(F(G_t))$, the objects of agents in C are updated in step $t+1$ as follows: $h_{i_k,t+1} = h_{i_{k+1},t}$ for $k \in \{1, \cdots, m-1\}$ and $h_{i_m,t+1} = h_{i_1,t}$. The distinction between a cycle and a trading cycle is important because not all cycles in the TTC-graph, G_t, occur as cycles in $F(G_t)$ and trades take place only for the cycles in $C(F(G_t))$ i.e. there may be cycles in $C(G_t)$ which do not occur in $C(F(G_t))$. This can be observed in Figure 1.

3 Some Properties

Consider an arbitrary $(R, \omega) \in \mathcal{R}^N \times A$. An allocation $\alpha \in A$ Pareto dominates $\beta \in A$ if $\alpha_i R_i \beta_i$ for all $i \in N$ and $\alpha_j P_j \beta_j$ for some $j \in N$. An allocation rule, $\varphi : \mathcal{R}^N \times A \to A$, is *Pareto efficient* if for any $(R, \omega) \in \mathcal{R}^N \times A$, $\varphi(R, \omega)$ is not Pareto dominated by any allocation in A.

[8] For the $TTAS$ rule, at any step t, unique pointees are determined only for agents who belong in an absorbing set. Then, $E_{i,t}^{F^{TTAS}}$ can be empty for some agents at step t where F^{TTAS} is the unique pointee selection criterion associated with the $TTAS$ rule.

[9] The unique pointee selection criterion F is provided without a t subscript to highlight that it is independent of the step of the algorithm.

[10] Clearly, it must be true that $Ch(F(G_t)) \subseteq Ch(G_t)$, $C(F(G_t)) \subseteq C(G_t)$, and $C_i(F(G_t))$ is either an empty set or a singleton set.

An allocation rule, φ, is *individually rational* if for all $(R, \omega) \in \mathcal{R}^N \times A$ and $i \in N$, $\varphi_i(R, \omega) R_i \omega_i$ i.e. each agent receives an object which is at least as good as her endowment.

An allocation rule is *strategy proof* if no agent has an incentive to misreport her preferences i.e. for each $i \in N$, $R \in \mathcal{R}^N$ and $R'_i \in \mathcal{R}$, it must be that $\varphi_i(R, \omega) R_i \varphi_i(R', \omega)$ where $R' = (R_{-i}, R'_i)$. In other words, for a *strategy proof* rule, truth-telling is a weakly dominant strategy for each agent.

An allocation rule is *weakly group strategy proof* if no group of agents can jointly misreport preferences such that every agent in the group is made better-off. Formally, for any $M \subseteq N$, there are no $R, R' \in \mathcal{R}^N$ such that $\varphi_i(R', \omega) P_i \varphi_i(R, \omega)$ for each agent $i \in M$ where $R' = \left(R_{N \setminus M}, R'_M\right)$. Additionally, an allocation rule is *group strategy proof* if no group of agents can jointly misreport preferences such that no agent in the group is made worse-off and at least one agent in the group is made better-off. Formally, for any $M \subseteq N$, there are no $R, R' \in \mathcal{R}^N$ such that $\varphi_i(R', \omega) R_i \varphi_i(R, \omega)$ for each agent $i \in M$ and $\varphi_j(R', \omega) P_j \varphi_j(R, \omega)$ for some agent $j \in M$ where $R' = \left(R_{N \setminus M}, R'_M\right)$. It should be obvious that *group strategy proofness* implies *weak group strategy proofness* which implies *strategy proofness* but the converse is not true in general.

For any allocation $\alpha \in A$ and group of agents $M \subseteq N$, α is said to be *blocked* by M if there is $\beta \in A$ such that $\beta_M = \omega_M$ and, for each agent $i \in M$, $\beta_i P_i \alpha_i$ is true. An allocation $\alpha \in A$ is said to be *weakly blocked* by $M \subseteq N$ if there is $\beta \in A$ such that $\beta_M = \omega_M$, $\beta_i R_i \alpha_i$ for all agents $i \in M$, and $\beta_j P_j \alpha_j$ for some agent $j \in M$. An allocation is in the *weak core* if it is not *blocked* by any subset of N whereas an allocation is in the *core* if it is not *weakly blocked* by any subset of N. An allocation rule is said to be *weak core-selecting* if it finds allocations in the *weak core* and *core-selecting* if it finds allocations in the *core*.

4 Existing Rules

In this section, I provide brief descriptions of three of the existing rules proposed for the housing market problem with weak preferences. First, I present the identical *departure* condition for the three rules. Then, a brief description of two important properties for the sufficient condition of *strategy proofness*, given by Saban & Sethuraman [17], is presented. Finally, I present the unique pointee selection criterion for the *TTAS*, *TC*, and *HPO* rules along with some relevant notation.

4.1 ***Departure Condition***

At each step of the algorithm, the assignment of some of the agents is finalized and they are removed from the problem. Unlike the TTC rule, in the presence of indifferences between objects, an agent cannot simply be removed from the problem after she has become part of a trading cycle. Being part of a trading cycle ensures that the agent has one of her most preferred objects (among the remaining ones) i.e. she cannot be made better-off. However, it might still be possible to make other agents better-off without making her worse-off. To illustrate this, consider the following example.

Example 1. Consider the following housing market problem: let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, $\omega = (a, b, c)$ and the preference profile, $R = (R_1, R_2, R_3)$, be given as:

R_1	R_2	R_3
bc	a	b
a	b	c
	c	a

In the above table, each column represents the preferences of an agent. Specifically, agent 1 is indifferent between objects b and c, and prefers each of these objects to object a, agent 2 prefers a to b and b to c, and agent 3 prefers b to c and c to a. The TTC-graph corresponding to the above example is displayed in Figure 2.

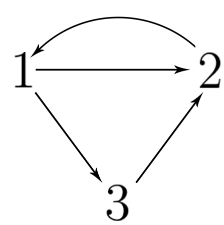


Figure 2: A TTC-graph, corresponding to the housing market problem of Example 1, depicting two cycles $C^1 = (1, 2)$ and $C^2 = (1, 3, 2)$.

For this housing market problem, there is a unique *Pareto efficient* assignment $\alpha = (c, a, b)$. It is clear from Figure 2 that there are two possible cycles; $C^1 = (1, 2)$ and $C^2 = (1, 3, 2)$. Suppose that trades are conducted according to the cycle C^1. Now, if agents 1 and 2 are removed from the problem after becoming part of the trading cycle, the assignment would be $\beta = (b, a, c)$. However, β is not *Pareto efficient* because agent 3 can be made better-off without making any other agent worse-off. This removal criterion is unable to achieve *Pareto efficiency* because, even though agent 1 cannot be made better-off after she has been part of a trading cycle, she could have made agent 3 better-off without affecting her welfare.

As illustrated in Example 1, when indifferences between objects are allowed, becoming part of a trading cycle is not sufficient to ensure *Pareto efficiency.* So, to achieve *Pareto efficiency*, the possibility of improving some other agent's welfare has to be eliminated before an agent, who has been part of a trading cycle, is removed from the problem. In other words, an agent is removed from the problem, her assignment is finalized, if she owns one of her most preferred objects (among the remaining ones)[11] and some other agent cannot be made better-off without making her worse-off.[12] Formally, the *departure* condition, presented at step t for a general rule φ and arbitrary $(R, \omega) \in \mathcal{R}^N \times A$, can be stated as follows: At the start of step t, a set of agents M is removed from the problem if $h_{i,t} \in \tau(R_i, O_{t-1})$ for each agent $i \in M$ and $h_{M,t} = \cup_{i \in M} \tau(R_i, O_{t-1})$. Once agents in M are removed (along with the objects in $h_{M,t}$), another set of agents may be chosen to depart similarly. This process is continued until no more groups can depart.

The assignment of each agent who is removed from the problem is finalized based on the object she is holding at the step of her removal. These agents are referred to as departed or removed from the problem. Then, for the set of agents M who are departing at step t, the assignment of each agent $i \in M$ is $\varphi_i(R, \omega) = h_{i,t}$. The process is repeated until no other set of agents satisfies the *departure* condition. Based on this *departure* condition, each agent in the departing group owns one of her most preferred objects (among the remaining ones), and the only possible way to increase the welfare of some agent, who does not belong in the departing group, by using agents (along with their objects) in the departing group would make some agent in the departing group worse-off. In this manner, the *departure* condition can ensure *Pareto efficiency* [2, 3, 8, 17].[13]

[11] This statement is true for an agent who has been part of a trading cycle because each agent in a trading cycle points at an agent who holds one of her most preferred objects (among the remaining ones).

[12] Under this *departure* condition, some agents can be removed from the problem without ever becoming part of a trading cycle i.e. these agents are assigned their endowment for the housing market problem. This is in contrast to the *TTC* rule for which an agent is removed from the problem only after she has become part of a trading cycle. However, if becoming part of a trading cycle is made a requirement in the *departure* condition, such agents would either form a trading cycle with themselves, or become part of a trading cycle in which welfare of all involved agents remains unchanged.

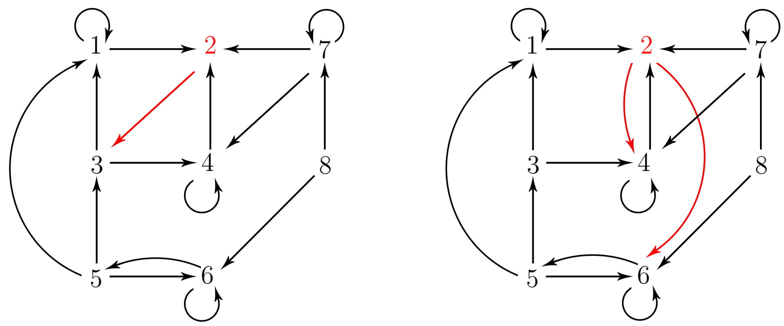
[13] This *departure* condition is identical to the condition used in [8], and equivalent to the conditions of *paired-symmetric absorbing sets* and *terminal sinks* used for the *TTAS* and *HPO* rules in [2] and [17], respectively.

4.2 *Independence of Unsatisfied Agents and Persistence*

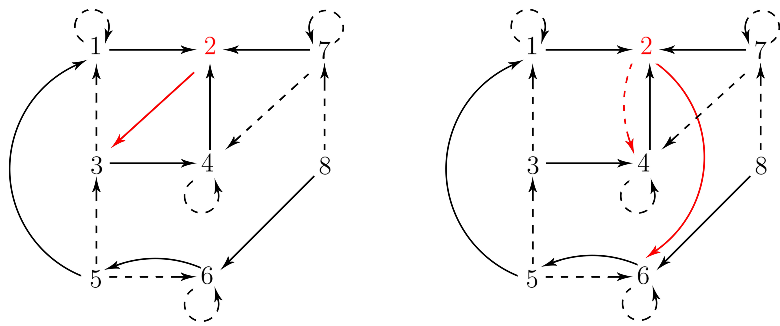
Saban & Sethuraman [17] established a sufficient condition on the unique pointee selection criterion to achieve *strategy proofness* in the context of the housing market problem with weak preferences. They show that the *independence of unsatisfied agents* and *persistence* play an important role in ensuring *strategy proofness* of rules for the housing market problem with weak preferences. In this subsection, these properties are presented for a rule φ that is a member of the $CAIO$ family.

Independence of unsatisfied agents states that the unique pointee selection criterion, F, is independent of the most preferred objects (among the remaining ones) of an unsatisfied agent in the sense that changing the most preferred objects (among the remaining ones) for an unsatisfied agent should not change the unique pointee selected for the other agents. Formally, consider the TTC-graphs $G = (N, E)$ and $G' = \left(N, \acute{E}\right)$ such that $E_j = \acute{E}_j$ for each $j \in N \backslash \{i\}$, and $E_i \neq \acute{E}_i$ where agent i is an unsatisfied agent. In other words, the TTC-graphs, G and G', differ only in the outgoing arcs from the unsatisfied agent i. If the unique pointee selection criterion F satisfies *independence of unsatisfied agents*, then $E_j^F = \acute{E}_j^F$ for each $j \in N \backslash \{i\}$ i.e. the unique pointees selected for the other agents are not affected by changing the most preferred objects (among the remaining ones) of an unsatisfied agent. This condition is important for *strategy proofness* because it eliminates the ability of an unsatisfied agent to manipulate the trading cycles that are formed by misreporting her preferences and, hence, restricts her ability to improve her welfare. Figure 3 illustrates the *independence of unsatisfied agents*. Figure 3 (a) shows two TTC-graphs which differ only in the outgoing arcs of an unsatisfied agent - agent 2. Figure 3 (b) shows the directed graphs after a unique pointee selection criterion, satisfying *independence of unsatisfied agents*, has been applied to the TTC-graphs of Figure 3 (b). It can be observed that the directed graphs of Figure 3 (b) differ only in the unique pointee selected for agent 2.

Persistence requires that if, during the running of the algorithm, an object is made available to an unsatisfied agent, that object stays available to her until she departs or becomes satisfied. Formally, let $G_t = (N_t, E_t)$ be the TTC-graph at step t for the rule φ and let F be the unique pointee selection criterion associated with the rule φ. If at step t of the rule φ, there is a chain $Ch \in Ch\left(F\left(G_t\right)\right)$ such that $Ch = (i_1, \cdots, i_m)$ and $i_m \in U_t$, then the chain Ch "persists" in all of the directed graphs $F\left(G_{\tilde{t}}\right)$ where $\tilde{t} < t_m$ and t_m is the first



(a) Two TTC-graphs which differ only in the outgoing arcs of agent 2 who is an unsatisfied agent.



(b) A unique pointee selection criterion, satisfying *independence of unsatisfied agents*, is applied to the TTC-graphs of Figure 3 (a). The resulting directed graphs differ only in the unique pointee selected for the unsatisfied agent 2. The arcs not selected by the unique pointee selection criterion are represented as dashed arcs.

Figure 3

step in which agent i_m departs or becomes satisfied i.e. $Ch \in Ch\left(F\left(G_{\tilde{t}}\right)\right)$ for all $\tilde{t} < t_m$.[14] Figure 4 provides a visual presentation of *persistence*. According to this condition, if a chain to an unsatisfied agent is formed, under the unique pointee selection criterion F at some step t, then this chain occurs until the unsatisfied agent receives an object at least as good as any of the objects held by the agents in these chains because she can always form a trading cycle by pointing at any of the agents in these chains. If a unique pointee selection

[14] Agent i_m can become satisfied at step t_m if: (1) she becomes part of a trading cycle at step $t_m - 1$, or (2) t_m is the first step in which agent i_m likes her endowment at least as much as each of the remaining objects. Additionally, the distinction between departing and becoming satisfied is necessary because even after an agent becomes satisfied, she is not removed from the problem until she satisfies the *departure* condition.

criterion of a rule does not satisfy *persistence*, it is possible that at some step an object is made available to an unsatisfied agent and she prefers that object to her final assignment. In this case, she would have an incentive to misreport her preferences to become part of a trading cycle in the step this object was made available to her. Hence, *strategy proofness* can be violated if *persistence* does not hold.

Step t

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_{m-1} \longrightarrow i_m$$

$$\Downarrow$$

Step $\tilde{t}$

$$i_1 \longrightarrow i_2 \longrightarrow \cdots \longrightarrow i_{m-1} \longrightarrow i_m$$

Figure 4: For a unique pointee selection criterion that satisfies *persistence*, if a chain is formed to an unsatisfied agent (agent i_m) at step t, that chain "persists" in the following steps $\tilde{t} < t_m$ where t_m is the first step in which agent i_m departs or becomes satisfied.

4.3 *Top Trading Absorbing Sets Rule*

A brief description of the $TTAS$ rule, proposed by Alcalde-Unzu & Molis [2], is given in this subsection. Let $\prec$ be a priority ordering over objects. For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering over objects $\prec$, the outcome of the $TTAS$ rule is denoted as $TTAS^{\prec}(R, \omega)$. It should be noted that the unique pointee selection criterion of the $TTAS$ rule, F^{TTAS}, is defined only for the absorbing sets of the TTC-graph. However, if F^{TTAS} is applied to the whole TTC-graph, rather than just for the absorbing sets, the rule behaves equivalently.[15] So, the equivalent variant of the $TTAS$ rule, where F^{TTAS} is applied to the whole TTC-graph, is presented here.[16] Additionally, in the presentation of the $TTAS$ rule, each agent points at an object and each object points at the agent who holds it [2]. However, the rule is described here so that agents point at other agents instead of objects for notational congruity. Step t of the $TTAS$ rule proceeds as follows:

1. Groups of agents are chosen to depart according to the *departure* condition until no more groups of

[15] This is formally presented in the proof of *strategy proofness* of the $TTAS$ rule [2] and is intuitively explained by Saban & Sethuraman [17].

[16] The equivalent variant of the $TTAS$ rule is used in this paper because the unique pointee selection criterion of the TC and HPO rules is defined for the whole TTC-graph. This allows the application of a common sufficient condition of *weak group strategy proofness* to the three rules.

agents satisfy the *departure* condition. Each departing agent is assigned the object she is holding at step t i.e. if agent i was chosen to depart, then $TTAS_i^{\prec}(R,\omega) = h_{i,t}$.

2. The unique pointee selection criterion for the $TTAS$ rule, F^{TTAS}, is defined as follows: If all of the agent's most preferred objects (among the remaining ones) have been held by her at least m times, she points at whoever owns one of her most preferred objects (among the remaining ones), that she has not held $m+1$ times, with the highest priority under $\prec$. The endowment of an agent is considered as a previously assigned object.

3. Each agent is pointing at exactly one agent under F^{TTAS}. Then, there is at least one trading cycle. At step $t+1$, objects of each agent in a trading cycle at step t are updated i.e. if agent i is in a trading cycle at step t, then $h_{i,t+1} = h_{p_{i,t},t}$.

The $TTAS$ algorithm ends when every agent has departed. The $TTAS$ rule is *Pareto efficient*, *individually rational*, *strategy proof*, *weak core-selecting*, and *core-selecting* (whenever the *core* is non-empty) [2]. However, the $TTAS$ rule can have an exponential running time in the worst case [3].

The unique pointee selection criterion for the $TTAS$ rule, F^{TTAS}, satisfies the *independence of unsatisfied agents* as the unique pointee for each agent is determined independently from the preferences of all the other agents. Moreover, F^{TTAS} enforces *persistence* because an agent i continues to point at an agent j until agent j departs or becomes part of a trading cycle.[17] However, it should be noted that the variant of *persistence* imposed on the $TTAS$ rule differs from *persistence* in the sense that each agent's pointing "persists" until her unique pointee departs or becomes part of a trading cycle whereas *persistence* only requires those chains to "persist" which have an unsatisfied agent as the last agent.[18] The difference between these two properties is highlighted in Figure 5. Figure 5 shows that the unique pointees of agents $i_1, \cdots, i_{m-2}$ can "persist" even after agent i_m, the unsatisfied agent, has departed or become satisfied as long as agent i_{m-1} does not become part of a trading cycle at step $t_m - 1$ where t_m is the first step in which agent i_m departs or becomes satisfied. If agent i_{m-1} does not become part of a trading cycle at step $t_m - 1$, then agents $i_1, \cdots, i_{m-1}$ hold the same

[17] The variant of *persistence* imposed on the $TTAS$ and TC rules implies *persistence* because whenever an agent becomes part of a trading cycle in these rules, she is guaranteed to receive one of her most preferred objects (among the remaining ones) in the next step i.e. unsatisfied agents necessarily become satisfied after becoming part of a trading cycle.

[18] Even though the $TTAS$ and TC rules satisfy a variant of *persistence* which implies *persistence*, the proofs for these two rules rely only on the implied *persistence*.

objects at steps $t_m - 1$ and t_m which means that, under F^{TTAS}, the selected unique pointees for agents $i_1, \cdots, i_{m-2}$ are the same at steps $t_m - 1$ and t_m. However, this is more than what *persistence* requires because agent i_m is not unsatisfied at step t_m.

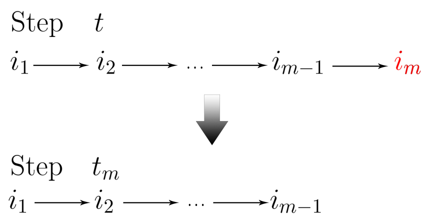


Figure 5: Step t is the first step in which this chain occurs and step t_m is the first step in which agent i_m departs or becomes satisfied such that agent i_{m-1} is not part of a trading cycle at step $t_m - 1$. Then, for the *TTAS* and *TC* rules, agents $i_1, \cdots, i_{m-2}$ have the same unique pointees at step t_m whereas *persistence* does not enforce these pointees at step t_m.

4.4 ***Top Cycles Rule***

In this subsection, the *TC* rule, proposed by Jaramillo & Manjunath [8], is briefly described along with some relevant notation. Let $\prec$ be a priority ordering over agents. For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering $\prec$, the outcome of the *TC* rule is denoted as $TC^{\prec}(R, \omega)$. Step t of the algorithm proceeds as follows:

1. Groups of agents satisfying the *departure* condition are selected to depart until no other group of agents satisfies the *departure* condition. Each departing agent is assigned the object she is holding i.e. if agent i was chosen to depart at step t, then $TC_i^{\prec}(R, \omega) = h_{i,t}$.

2. Unique pointee selection criterion for the *TC* rule, F^{TC}, determines unique pointees for each agent at step t as follows:

   (a) For any agent j who holds the same object as in the previous step, the agents pointing at her in the previous step, point at agent j in the current step under F^{TC}. Formally, if $h_{j,t} = h_{j,t-1}$ for $j \in N_t$, then $p_{i,t} = j$ for all $i \in N_t$ such that $p_{i,t-1} = j$.

(b) If at least one of the most preferred objects (among the remaining ones) of an agent is held by an unsatisfied agent, she points at the unsatisfied agent with the highest priority under $\prec$.

(c) Any agent who is not pointing by this stage must have all of her most preferred objects (among the remaining ones) held by satisfied agents. From 2 (b), each satisfied agent who has one of her most preferred objects (among the remaining ones) held by an unsatisfied agent is pointing at an unsatisfied agent. Denote the set of these agents as LU_1 i.e. $LU_1 = \{i \in S_t : p_{i,t} \in U_t\}$. Sequentially, define $LU_k = \{i \in S_t : p_{i,t} \in LU_{k-1}\}$ for $k > 1$. Induce an ordering $\prec_k$ as follows: for any $i, j \in LU_k$, define $i \sim_k j$ if $i = j$ and $p_{i,t} \sim_{k-1} p_{j,t}$ where $i \sim_1 j$ if $i = j$ and $p_{i,t} = p_{j,t}$ are true, and $i \prec_k j$ if (1) $p_{i,t} \prec_{k-1} p_{j,t}$ holds, or (2) $p_{i,t} \sim_{k-1} p_{j,t}$ and $i \prec j$ are true where $\prec_1 = \prec$. Now, consider an agent who is not pointing yet and has one of her most preferred objects (among the remaining ones) held by some agent in LU_k. Among the agents in LU_k who hold one of her most preferred objects (among the remaining ones), she points at the agent who has a higher priority under $\prec_k$. Continue the process until $LU_k = \phi$.[19]

(d) Any agent who is still not pointing, points at the highest priority agent, other than herself, who holds one of her most preferred objects (among the remaining ones).

3. Since each agent is pointing at exactly one agent, there is at least one trading cycle under the unique pointee selection criterion F^{TC}. In the next step of the TC rule, objects of the agents in a trading cycle are updated i.e. if agent i is part of a trading cycle at step t, $h_{i,t+1} = h_{p_{i,t},t}$.

The second phase of the TC rule ensures that each trading cycle has at least one unsatisfied agent as long as 2 (a) is not in effect. The algorithm is terminated when all of the agents have departed. The TC rule is *Pareto efficient*, *individually rational*, *strategy proof*, *weak core-selecting*, and *core-selecting* (whenever the *core* is non-empty) [3, 8]. Moreover, it has been shown to have a polynomial running time [8].

The TC rule satisfies the *independence of unsatisfied agents* because unique pointees are selected independently from the preferences of unsatisfied agents. Basically, for each agent, a unique pointee is determined to

[19] Only satisfied agents are considered in the definition of LU_k because once the unique pointees are determined for the agents who have one of their most preferred objects (among the remaining ones) owned by an unsatisfied agent, the rule needs to determine unique pointees for the agents who have all of their most preferred objects (among the remaining ones) owned by satisfied agents.

form a path to the closest possible unsatisfied agent under F^{TC}. As this does not depend on the preferences of the unsatisfied agent, because the path ending in an unsatisfied agent consists entirely of satisfied agents (if any), the TC rule satisfies the *independence of unsatisfied agents.* The TC rule explicitly imposes *persistence* in 2 (a) which requires that an agent i continues to point at an agent j as long as agent j holds the same object as the previous step of the rule. This is referred to as *TC-persistence* in this paper. *TC-persistence* implies *persistence* but differs from *persistence* because each agent's pointing "persists" until her pointee departs or becomes part of a trading cycle.[17,18] The difference between *TC-persistence* and *persistence* is highlighted in Figure 5. Figure 5 shows that, under *TC-persistence*, if agent i_{m-1} does not become part of a trading cycle at step $t_m - 1$, then agents $i_1, \cdots, i_{m-1}$ hold the same objects at steps $t_m - 1$ and t_m. Then, under F^{TC}, the unique pointees for agents $i_1, \cdots, i_{m-2}$ are the same at steps t_{m-1} and t_m. However, since agent i_m departs or becomes satisfied at step t_m, she is no longer unsatisfied at step t_m. Therefore, the unique pointees for agents $i_1, \cdots, i_{m-2}$ are not required to be the same at steps t_{m-1} and t_m under *persistence.*

4.5 ***Highest Priority Object Rule***

This subsection briefly describes the *HPO* rule proposed by Saban and Sethuraman [17]. The *HPO* rule uses a priority ordering over objects which is then used to induce an ordering over agents at each step. However, the induced ordering over agents is not used for the *HPO* rule and was used only to show that the *HPO* rule is a member of the *CAIO* family [17]. As such, the induced ordering over agents is ignored in the following description. Let $\prec$ be a priority ordering over objects. For any $(R, \omega) \in \mathcal{R}^N \times A$ and priority ordering over objects $\prec$, the outcome of the *HPO* rule is denoted as $HPO^{\prec}(R, \omega)$. Step t of the *HPO* rule proceeds as follows:

1. Sets of agents are chosen to depart according to the *departure* condition until no more sets of agents satisfy the *departure* condition. Each departing agent is assigned the object she holds at step t i.e. if agent i was chosen to depart at step t, then $HPO_i^{\prec}(R, \omega) = h_{i,t}$.

2. F^{HPO} determines a unique pointee for each agent in N_t in the following manner:

   (a) F^{HPO} enforces *persistence* as follows: For any $j \in U_t$ and $Ch \in Ch_j\left(F^{HPO}(G_{t-1})\right)$, it must be that $p_{i,t} = p_{i,t-1}$ for each $i \in \{i_1, \cdots, i_m\}$ where $Ch = (i_1, \cdots, i_m, j)$.

(b) For each $i \in U_t$ who is not pointing yet, agent i points at the agent who owns the highest priority object in $\tau(R_i, O_t)$ under the priority ordering $\prec$.

(c) Any agent who is not pointing yet must be a satisfied agent. Repeat the following until all satisfied agents are pointing:

i. Let L be the set of all agents who are already pointing. Refer to these agents as labeled agents.

ii. Agent $i \in N_t$ is said to be adjacent to labeled agents if $h_{j,t} \in \tau(R_i, O_t)$ for some $j \in L$ i.e. an agent is adjacent to labeled agents if at least one of her most preferred objects (among the remaining ones) is held by a labeled agent. The set of agents who are adjacent to labeled agents is denoted as AL.

iii. Select the agent in AL who owns the highest priority object, say agent i. Agent i points at the agent in L who owns one of her most preferred objects (among the remaining ones) with the highest priority under $\prec$. Go back to stage i.

3. Since each agent is pointing at exactly one agent, there is at least one trading cycle under F^{HPO}. Moreover, each trading cycle has at least one unsatisfied agent. In step $t+1$ of the *HPO* rule, objects of each agent in a trading cycle of step t are updated i.e. if agent i is part of a trading cycle at step t, $h_{i,t+1} = h_{p_{i,t},t}$.

The *HPO* rule is *Pareto efficient*, *individually rational*, *strategy proof*, and *weak core-selecting* [17]. Moreover, the *HPO* rule can be implemented in $O\left(n^2 \log n + n^2\gamma\right)$ where n is the number of agents in the problem and γ is the maximum number of objects agents are indifferent between for the given preference profile.

The *HPO* rule satisfies *independence of unsatisfied agents* because the unique pointee for each agent is determined independently from the preferences of unsatisfied agents. Moreover, *persistence* is directly imposed on the unique pointee selection criterion of the *HPO* rule.

5 Results and Discussion

In this section, a sufficient condition for *weak group strategy proofness* is presented. Using this sufficient condition, it is established that the *TTAS*, *TC*, and *HPO* rules satisfy *weak group strategy proofness* even though *group strategy proofness* is incompatible with *Pareto efficiency* for the housing market problem with weak preferences [6]. The following example shows that the TTC rule with fixed tie-breaking, which satisfies *weak Pareto efficiency* instead of *Pareto efficiency* [7], does not satisfy *group strategy proofness* regardless of how the ties are broken.

Example 2. Consider the following housing market problem: let $N = \{1, 2, 3\}$, $O = \{a, b, c\}$, $\omega = (a, b, c)$ and the preference profile, $R = (R_1, R_2, R_3)$, be given as:

R_1	R_2	R_3	R_1'	R_1''
bc	a	a	b	c
a	b	c	c	b
	c	b	a	a

Let $P = (P_1, P_2, P_3)$ be the preference profile for which ties in R are broken. Then, $P_2 = R_2$ and $P_3 = R_3$ because R_2 and R_3 do not have any ties. Moreover, $P_1 \in \{R_1', R_1''\}$ i.e. there are exactly two ways of breaking ties in R_1. If the tie-breaking is done so that $P_1 = R_1'$, the outcome of the TTC rule with fixed tie-breaking is (b, a, c). In this case, if agent 1 misreported her preferences as R_1'', the outcome of the TTC rule with fixed tie-breaking would change to (c, b, a) i.e. agent 1's welfare does not change while agent 3 becomes better-off. On the other hand, if ties are broken so that $P_1 = R_1''$, the outcome of the TTC rule with fixed tie-breaking is (c, b, a). In this case, agent 1 can misreport her preferences as R_1' which would change the outcome of the TTC rule with fixed tie-breaking to (b, a, c) i.e. agent 2 becomes better-off and agent 1's welfare remains unchanged.

Example 2 shows that the TTC rule with fixed tie-breaking is unable to achieve *group strategy proofness* even though it satisfies *weak Pareto efficiency* instead of *Pareto efficiency.* This is because the equivalence of *strategy proofness* and *non-bossiness* with *group strategy proofness*, which holds for strict preferences [11, 18], does not hold for weak preferences [7]. *Non-bossiness* is applicable only when an agent's outcome remains unchanged whereas, in the presence of indifferences, the agent can receive a different outcome while maintaining the same welfare level. In this scenario, *non-bossiness* does not apply and, as such, is unable to rule out group manipulation as illustrated in Example 2. However, it is straightforward to show that the TTC rule with fixed tie-breaking does satisfy *weak group strategy proofness.* This result follows directly from *group strategy proofness* of the TTC rule for strict preferences [4] and, as such, a formal proof is not provided.

Proposition 1. The TTC rule with fixed tie-breaking satisfies *weak group strategy proofness.*

The proof of the sufficient condition for *weak group strategy proofness* is not straightforward even though the $TTAS$, TC, and HPO rules reduce to the TTC rule for the restricted case of strict preferences. However, the proof shares many similarities with the proof of the sufficient condition for *strategy proofness* presented in Saban & Sethuraman [17]. As such, I briefly discuss both these proofs to highlight the similarities and the differences.[20]

Similar to the proof of *strategy proofness*, *independence of unsatisfied agents* and *persistence* are important properties for ensuring *weak group strategy proofness.* Note that *persistence* is defined for all unsatisfied agents, appearing at any step of the algorithm, and, as such, it does not need to be modified for dealing with groups of agents to prove *weak group strategy proofness.* Additionally, even though the *independence of unsatisfied agents* is defined for an unsatisfied agent misreporting her preferences, it implies "independence" for any group of unsatisfied agents jointly misreporting their preferences i.e. if the TTC-graphs for two housing market problems differ in the outgoing arcs of only unsatisfied agents, the unique pointees chosen for the other agents are identical as long as the unique pointee selection criterion satisfies *independence of unsatisfied agents.* Saban & Sethuraman [17] show that *strategy proofness* is equivalent to *local invariance* for the members of the $CAIO$ family as these rules satisfy *independence of unsatisfied agents* and *persistence.*

Local invariance is defined as follows: Consider a housing market problem $(R, \omega) \in \mathcal{R}^N \times A$ and a rule φ. Suppose $\varphi_i(R, \omega) = a$ for an arbitrary agent $i \in N$. Let $R' \in \mathcal{R}^N$ be a preference profile such that $R' = (R_{-i}, R'_i)$ where $R'_i|_{O \setminus \{a\}} = R_i|_{O \setminus \{a\}}$ (the preference ordering of agent i is identical for the objects in $O \setminus \{a\}$ under R_i and R'_i) and, for each $b \in O \setminus \{a\}$, $bP_i a \Rightarrow bP'_i a$ and $aR_i b \Rightarrow aP'_i b$. Then, the rule φ satisfies *local invariance* if $\varphi_i(R', \omega) = a$. Following Jaramillo & Manjunath [8], R'_i is referred to as the *local push-up* of R_i at a.

Local invariance reduces the misreported preferences that have to be ruled out as successful misreports for a

[20] These similarities are also shared with the proofs for *strategy proofness* of the $TTAS$ and TC rules.

rule to be *strategy proof*. For a rule that satisfies *local invariance*, if agent i receives a better object (say $a \in O$) by reporting R'_i, instead of her true preference R_i, she can receive the same object by reporting the *local push-up* of R'_i at a (say R''_i). Additionally, when agent i reports R''_i, she is removed from the problem with object a after becoming part of the trading cycle exactly once because she satisfies the *departure* condition after she has been assigned object a under the reported preferences R''_i.[21] As such, *local invariance* allows for a somewhat simpler proof for *strategy proofness* in this context.

After establishing equivalence of *local invariance* and *strategy proofness* for the rules that satisfy the *independence of unsatisfied agents* and *persistence*, Saban & Sethuraman [17] provide conditions for unique pointee selection criteria to ensure *local invariance*. Using these conditions, *strategy proofness* for the rules in the *CAIO* family is established. Unfortunately, *local invariance* cannot be utilized for *weak group strategy proofness* as the latter deals with a group of agents jointly misreporting their preferences while the former is defined only for one agent. Even if *local invariance* is suitably adjusted for groups of agents, it cannot be used in the context of the housing market problem with weak preferences because no rule is *Pareto efficient*, *strategy proof*, *individually rational*, and *non-bossy* [5, 7, 8]. Since members of the *CAIO* family satisfy *Pareto efficiency*, *strategy proofness*, and *individual rationality* [17], these rules cannot be *non-bossy*. So, even when the outcome of an agent remains unchanged, the outcome of the remaining agents can change suggesting that a group variant of *local invariance* may not be suitable for proving *weak group strategy proofness*. Hence, for the sufficient condition, I skip this intermediary step and directly present a condition for the unique pointee selection criterion which is able to ensure *weak group strategy proofness* for the members of the *CAIO* family. Then, I prove *weak group strategy proofness* of the $TTAS$, TC, and *HPO* rules by showing that the unique pointee selection criterion of each of these rules satisfies this condition.

Independence of unsatisfied agents is defined for one unsatisfied agent who is misreporting her preferences. However, this property applies to situations where multiple unsatisfied agents are jointly misreporting their preferences. Specifically, for a rule satisfying *independence of unsatisfied agents*, if two TTC-graphs differ only in the outgoing arcs of unsatisfied agents, then the unique pointees of the other agents are identical if the unique pointee selection criterion satisfies *independence of unsatisfied agents*. The proof is straightforward and

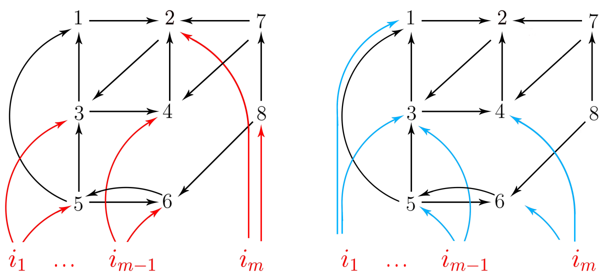
[21] *Local invariance* is an important step in the proofs of *strategy proofness* of the $TTAS$ and TC rules and the sufficient condition of Saban & Sethuraman [2, 8, 17].

requires the construction of a sequence of TTC-graphs so that any two consecutive TTC-graphs differ only in the outgoing arcs of an unsatisfied agent, and the first and final TTC-graphs are the two directed graphs which differ only in the outgoing arcs of unsatisfied agents. This allows the application of the *independence of unsatisfied agents* for the consecutive TTC-graphs and provides the required result which is formally presented in the following lemma.

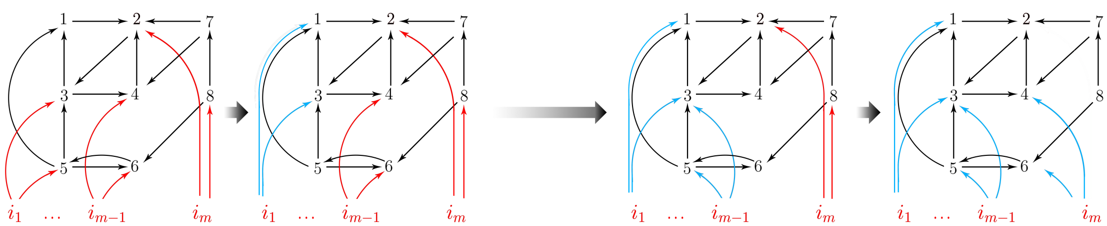
Lemma 1. Let G and G' be two TTC-graphs which differ only in the outgoing arcs of unsatisfied agents. If the unique pointee selection criterion F satisfies the *independence of unsatisfied agents*, then the directed graphs $F(G)$ and $F(G')$ differ only in the outgoing arcs of the unsatisfied agents.

Proof. Let $G = (N, E)$ and $G' = \left(N, \acute{E}\right)$ be the TTC-graphs such that $E_i = \acute{E}_i$ for all $i \in N \backslash M$ and M is a set of unsatisfied agents. It has to be shown that $E_i^F = \acute{E}_i^F$ for each $i \in N \backslash M$ for the directed graphs $F(G)$ and $F(G')$ where the unique pointee selection criterion F satisfies *independence of unsatisfied agents*. Without loss of generality, let $M = \{i_1, \cdots, i_m\}$. An example of the TTC-graphs G and G' is provided in Figure 6 (a). Let $G^0 = G$, and $G^1, \cdots, G^m$ be the TTC-graphs such that, for any $k \in \{1, \cdots, m\}$, G^k and G^0 differ only in the outgoing arcs from the agents $i_1, \cdots, i_k$. Moreover, the outgoing arcs of the agents $i_1, \cdots, i_k$, in the TTC-graph G^k, are identical to the outgoing arcs of these agents in G'. Then, by construction, $G^m = G'$. It should be noted that for $k = 1, \cdots, m$, the TTC-graphs G^{k-1} and G^k differ only in the outgoing arcs of the unsatisfied agent i_k. Figure 6 (b) exhibits this construction for the the two TTC-graphs presented in Figure 6 (a). Then, for $k = 1, \cdots, m$, the directed graphs $F\left(G^{k-1}\right)$ and $F\left(G^k\right)$ differ only in the outgoing arcs of the unsatisfied agent i_k because F satisfies *independence of unsatisfied agents*. The application of a unique pointee selection criterion, satisfying *independence of unsatisfied agents*, on the sequence of TTC-graphs of Figure 6 (b) is depicted in Figure 6 (c). Then, the directed graphs $F\left(G^0\right) = F(G)$ and $F(G^m) = F(G')$ differ only in the outgoing arcs for the unsatisfied agents in M i.e. $E_i^F = \acute{E}_i^F$ for each $i \in N \backslash M$. ■

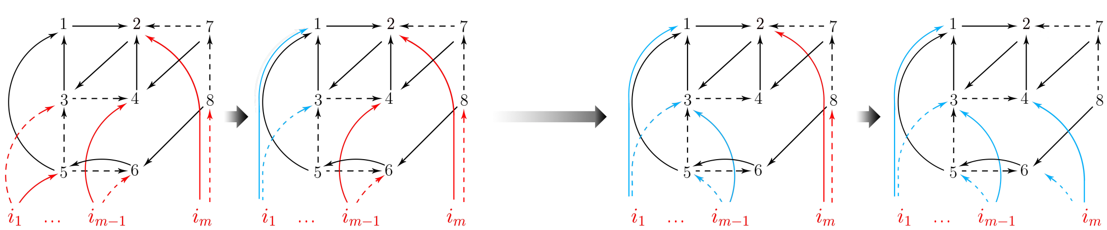
The main proof of this paper proceeds by contradiction for which some additional notation is presented here. For what follows, let $R, R' \in \mathcal{R}^N$ be such that $R' = \left(R_{N \backslash M}, R'_M\right)$ and $M \subseteq N$. Moreover, assume that $\varphi_i(R', \omega) P_i \varphi_i(R, \omega)$ for each $i \in M$ for a rule φ that is a member of the $CAIO$ family. Let N_t and O_t be the set of remaining agents and objects, respectively, after the *departure* phase of step t under R. Similarly, let N'_t and O'_t be the set of remaining agents and objects after the *departure* phase of step t under



(a) The two TTC-graphs, G and G', which differ only in the outgoing arcs of the unsatisfied agents $i_1, \cdots, i_m$.



(b) A sequence of the TTC-graphs is constructed so that the first and last TTC-graphs are the TTC-graphs G and G', respectively. Moreover, consecutive TTC-graphs differ only in the outgoing arcs of an unsatisfied agent. This allows the application of the *independence of unsatisfied agents* for the consecutive TTC-graphs.



(c) A unique pointee selection criterion, satisfying *independence of unsatisfied agents*, is applied to the TTC-graphs of Figure 6 (b). It can be observed that the consecutive directed graphs differ only in the unique pointee of an unsatisfied agent.

Figure 6

R', respectively. Let h_t and h'_t be the bijections stating the objects that are held by agents at step t under R and R', respectively. Let $G_t = (N_t, E_t)$ and $G'_t = \left(N'_t, \acute{E}_t\right)$ represent the TTC-graphs at step t of rule φ under R and R', respectively. Let $Ch\left(G_t\right)$ and $C\left(G_t\right)$ represent the sets of all chains and cycles, respectively, in the TTC-graph G_t while the corresponding sets for G'_t are denoted as $Ch\left(G'_t\right)$ and $C\left(G'_t\right)$. Moreover, let $Ch_i\left(G_t\right)$ be the set of all chains in $Ch\left(G_t\right)$ which have agent i as the last agent and $C_i\left(G_t\right)$ be the set of all cycles in $C\left(G_t\right)$ which include agent i. The corresponding sets for the TTC-graph G'_t are denoted as $Ch_i\left(G'_t\right)$ and $C_i\left(G'_t\right)$.

Let F be the unique pointee selection criterion for the rule φ. The directed graphs with the unique pointee selection criterion applied are denoted as $F\left(G_t\right) = \left(N_t, E_t^F\right)$ and $F\left(G'_t\right) = \left(N'_t, \acute{E}_t^F\right)$ at step t under R and R', respectively. Then, $Ch\left(F\left(G_t\right)\right)$ and $C\left(F\left(G_t\right)\right)$ represent the chains and cycles in the directed graph $F\left(G_t\right)$, respectively. Moreover, $Ch_i\left(F\left(G_t\right)\right)$ is the set of all chains in $Ch\left(F\left(G_t\right)\right)$ which have agent i as the last agent and $C_i\left(F\left(G_t\right)\right)$ is the set of all cycles in $C\left(F\left(G_t\right)\right)$ which include agent i. The corresponding sets for the directed graph $F\left(G'_t\right)$ are denoted as $Ch\left(F\left(G'_t\right)\right)$, $C\left(F\left(G'_t\right)\right)$, $Ch_i\left(F\left(G'_t\right)\right)$, and $C_i\left(F\left(G'_t\right)\right)$. In accordance with the directed graphs $F\left(G_t\right)$ and $F\left(G'_t\right)$, let $p_{i,t}$ and $p'_{i,t}$ denote the agent pointed at by agent i at step t under R and R', respectively. Let $h_{i,t}$ and $h'_{i,t}$ denote the object held by agent i at step t under R and R', respectively. Moreover, define the following:

$$CONN_{i,t} \equiv \{j \in N_t : j \in Ch \text{ for some } Ch \in Ch_i\left(F\left(G_t\right)\right)\}$$

Therefore, $CONN_{i,t}$ is the set of all agents who have a path to agent i in the directed graph $F\left(G_t\right)$ i.e. these agents are either directly or indirectly pointing at agent i at step t under R. Let $CONN_{M,t} = \cup_{i \in M} CONN_{i,t}$ where $M \subseteq N$.

In the following lemma, certain results are established for a member of the $CAIO$ family that violates *weak group strategy proofness*.

Lemma 2. Consider a rule φ belonging to the $CAIO$ family. If there is $M \subseteq N$ such that $\varphi_i\left(R', \omega\right) P_i \varphi_i\left(R, \omega\right)$ for each $i \in M$, where $R' = \left(R_{N \setminus M}, R'_M\right)$, then:

1. $\varphi_i\left(R', \omega\right) P_i \omega_i$ for each $i \in M$ i.e. each agent in M becomes part of a trading cycle at least once under

R',

2. no agent in M departs at step 1 of φ under R or R', and

3. no agent in M is satisfied at step 1 under R.

Proof. Consider a rule φ that is a member of the *CAIO* family. Let $R, R' \in \mathcal{R}^N$ be such that $R' = \left(R_{N \backslash M}, R'_M\right)$ and $\varphi_i\left(R', \omega\right) P_i \varphi_i\left(R, \omega\right)$ for each $i \in M$. Define $\alpha \equiv \varphi\left(R, \omega\right)$ and $\alpha' \equiv \varphi\left(R', \omega\right)$. Since φ is a member of the *CAIO* family it satisfies *individual rationality*, then $\alpha_i R_i \omega_i$ is true for each $i \in M$. By assumption, $\alpha'_i P_i \alpha_i$ for each $i \in M$, then $\alpha'_i P_i \omega_i$ holds for each $i \in M$. So, statement (1) of Lemma 2 is true.

Statement (2) of Lemma 2 for R' follows directly from statement (1) of Lemma 2. To see that statement (2) of Lemma 2 is true for R, consider the following: For contradiction, suppose that agent $i \in M$ is the first agent in M to depart at step 1 under R. Let $\tilde{N}$ and $\tilde{O}$ be the set of agents and objects, respectively, which have departed before agent i. Then, $\alpha_i \in \tau\left(R_i, O \backslash \tilde{O}\right)$. Note that, by the *departure* condition, agents in $\tilde{N}$ are selected to depart with objects in $\tilde{O}$ under R' as well. Since $\alpha_i \in \tau\left(R_i, O \backslash \tilde{O}\right)$, it must be that $\alpha_i R_i \alpha'_i$ which is a contradiction. Therefore, statement (2) of Lemma 2 is true.

Suppose, for contradiction of statement (3) of Lemma 2, that there is an agent $i \in M$ who is satisfied at step 1 under R. Then, $\omega_i \in \tau\left(R_i, O_1\right)$.[22] Since $\alpha'_i P_i \alpha_i$ and $\alpha_i \in \tau\left(R_i, O_1\right)$, then it must be that $\alpha'_i \in O \backslash O_1$. Additionally, agents in $N \backslash N_1$ depart with objects in $O \backslash O_1$ at step 1 under R' and, by the *departure* condition, $\alpha_j = \alpha'_j = \omega_j$ for each $j \in N \backslash N_1$ and $O \backslash O_1 = \{a \in O : a R_j \omega_j \text{ for } j \in N \backslash N_1\}$. Then, if agent i is assigned α'_i under R', some agent in $N \backslash N_1$ would be made worse-off under R' which would violate *individual rationality* because at least one agent in $N \backslash N_1$ would have to be assigned an object in O_1. Therefore, statement (3) of Lemma 2 is true which completes the proof. ■

Next, similar to the proofs of *strategy proofness* by Alcalde-Unzu & Molis [2] and Jaramillo & Manjunath [8], and the proof of the sufficient condition by Saban & Sethuraman [17], for R and R' the steps of the rule φ are divided into two categories: before and after the first agent in the group of jointly misreporting agents departs, becomes satisfied, or becomes part of a trading cycle. Let t be the first step where either of the following is true for some agent $i \in M$ under R :

[22] From statement (2) of Lemma 2, agent i does not depart at step 1.

1. agent i departs at step t,

2. agent i becomes satisfied at step t, or

3. agent i becomes part of a trading cycle at step t.

Let the corresponding step for R' be denoted as t'. Define $\underline{t} = \min\{t, t'\}$. Before step $\underline{t}$, no agent in M departs, becomes satisfied, or becomes part of a trading cycle for the rule φ under R or R'. Additionally, for the steps following step $\underline{t}$, some agent in M has departed, becomes satisfied, or becomes part of a trading cycle for either R or R'. These categories are made so that the proof is more tractable. For all of the steps before step $\underline{t}$, the TTC-graphs for R and R' differ only in the outgoing arcs of unsatisfied agents in M. Thus, the *independence of unsatisfied agents* is applicable for these steps. For the steps after step $\underline{t}$, at least one agent in M has departed, becomes satisfied, or becomes part of a trading cycle under R or R'. As such, the *independence of unsatisfied agents* is not enough and additional restrictions on the unique pointee selection criterion are necessary.

The next result shows that for the members of the *CAIO* family, the state of the algorithm is identical under R and R' before step $\underline{t}$. This is because only agents who are truthfully reporting their preferences are involved in any trading cycles that occur before step $\underline{t}$ for R and R'. In other words, the following result establishes that a group of agents cannot change the state of the algorithm before step $\underline{t}$ by jointly misreporting their preferences. The proof relies on the fact that for each step before $\underline{t}$, the TTC-graphs under R and R' differ only in the outgoing arcs of the agents in M, the group of jointly misreporting agents. Then, by the definition of $\underline{t}$, the agents in M are unsatisfied before step $\underline{t}$ under R and R'. Since every member of the *CAIO* family satisfies the *independence of unsatisfied agents*, this allows the application of Lemma 1 i.e. the directed graphs differ only in the unique pointees selected for the unsatisfied agents in M under R and R'.

Lemma 3. (**$\underline{t}$ equality**) Consider a rule φ belonging to the *CAIO* family. Let $M \subseteq N$ be such that $\varphi_i(R', \omega) P_i \varphi_i(R, \omega)$ for each $i \in M$ where $R' = \left(R_{N \setminus M}, R'_M\right)$ and $\underline{t}$ be the first step an agent in M departs, becomes satisfied, or becomes part of a trading cycle under R or R'. Then, for all $\tilde{t} < \underline{t}$:

1. $N_{\tilde{t}} = N'_{\tilde{t}}$ and $O_{\tilde{t}} = O'_{\tilde{t}}$,

2. $h_{\tilde{t}} = h'_{\tilde{t}}$,

3. $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ holds for each $j \in N_{\tilde{t}} \backslash M$,

4. $h_{\underline{t}} = h'_{\underline{t}}$, and

5. if no agent in M departs at step $\underline{t}$ under R, it must be that $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$.

Proof. Consider a rule φ that is a member of the *CAIO* family. Let $R, R' \in \mathcal{R}^N$ be such that $R' = \left(R_{N\backslash M}, R'_M\right)$ and $\varphi_i\left(R', \omega\right) P_i \varphi_i\left(R, \omega\right)$ for each $i \in M$. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Then, $\underline{t} = \min\{t, t'\}$. Since φ is a member of the *CAIO* family, it satisfies the *independence of unsatisfied agents.*

If $\underline{t} = 1$, Lemma 3 is vacuously true. Now, suppose that $\underline{t} > 1$. Consider $\tilde{t} = 1$. By statement (2) of Lemma 2, no agent in M departs at step 1 under R or R'. Then, the set of departing agents and objects should be the same, under R and R', because $R_j = R'_j$ for each $j \in N\backslash M$. So, $N_1 = N'_1$ and $O_1 = O'_1$. Moreover, since each agent holds her endowment at step 1, $h_1 = h'_1$. Since $\tilde{t} < \underline{t}$, each agent in M is unsatisfied at step 1 under R and R'. Since the rule φ satisfies the *independence of unsatisfied agents*, by Lemma 1, $p_{j,1} = p'_{j,1}$ is true for each $j \in N_1\backslash M$ proving statement (3) of Lemma 3 for $\tilde{t} = 1$.

Now, suppose that statements (1), (2), and (3) of Lemma 3 are true for some step $\tilde{t} < \underline{t} - 1$. I show that statements (1), (2), and (3) of Lemma 3 are true for step $\tilde{t} + 1$. By the induction hypothesis, $N_{\tilde{t}} = N'_{\tilde{t}}$, $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$, and $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ are true for each $j \in N_{\tilde{t}}\backslash M$. Therefore, the directed graphs $F\left(G_{\tilde{t}}\right)$ and $F\left(G'_{\tilde{t}}\right)$ are identical for the agents in $N_{\tilde{t}}\backslash M$. Hence, the same trading cycles occur for the agents in $N_{\tilde{t}}\backslash M$ at step $\tilde{t}$ because agents own the same objects, $h_{\tilde{t}} = h'_{\tilde{t}}$, and agents in $N_{\tilde{t}}\backslash M$ have the same unique pointees under R and R' i.e. $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}}\backslash M$. Additionally, no agent in M is part of a trading cycle at step $\tilde{t} < \underline{t}$ under R and R' i.e. $C_i\left(F\left(G_{\tilde{t}}\right)\right) = C_i\left(F\left(G'_{\tilde{t}}\right)\right) = \phi$ for each $i \in M$. Then, $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$ because $h_{\tilde{t}} = h'_{\tilde{t}}$ and the same trading cycles occur at step $\tilde{t}$ under R and R'.

Since $\tilde{t} + 1 < \underline{t}$, no agent in M departs at step $\tilde{t} + 1$ under R or R'. So, $N_{\tilde{t}+1} = N'_{\tilde{t}+1}$ and $O_{\tilde{t}+1} = O'_{\tilde{t}+1}$ because $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$ and $R_j = R'_j$ for each $j \in N\backslash M$. The *TTC*-graphs $G_{\tilde{t}+1}$ and $G'_{\tilde{t}+1}$ differ only in the outgoing arcs of the agents in M who are unsatisfied at step $\tilde{t}+1 < \underline{t}$ under R and R' because $N_{\tilde{t}+1} = N'_{\tilde{t}+1}$, $O_{\tilde{t}+1} = O'_{\tilde{t}+1}$, $h_{\tilde{t}+1} = h'_{\tilde{t}+1}$, and $R_j = R'_j$ for each $j \in N\backslash M$. Since the rule φ satisfies the *independence* *of unsatisfied agents*, the directed graphs $F\left(G_{\tilde{t}+1}\right)$ and $F\left(G'_{\tilde{t}+1}\right)$ differ only in the outgoing arcs of the unsatisfied agents in M by Lemma 1 i.e. $p_{j,\tilde{t}+1} = p'_{j,\tilde{t}+1}$ holds for each $j \in N_{\tilde{t}+1} \backslash M$.

To prove statement (4) of Lemma 3, note that $h_{\underline{t}-1} = h'_{\underline{t}-1}$, $p_{j,\underline{t}-1} = p'_{j,\underline{t}-1}$ for each $j \in N_{\underline{t}-1} \backslash M$, and no agent in M is part of a trading cycle at step $\underline{t}-1$ under R and R'. So, the same trading cycles occur at step $\underline{t}-1$ under R and R'. Therefore, it can be concluded that $h_{\underline{t}} = h'_{\underline{t}}$.

For statement (5) of Lemma 3, suppose that no agent in M departs at step $\underline{t}$ under R. By statement (1) of Lemma 2, each agent in M becomes part of a trading cycle at least once under R' i.e. no agent in M departs at step $\underline{t}$ under R'. Since $h_{\underline{t}} = h'_{\underline{t}}$ and $R_j = R'_j$ for each $j \in N \backslash M$, it must be that $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$ because only agents in $N \backslash M$ depart at step $\underline{t}$ under R and R'. ■

Note that *persistence* is not required to prove Lemma 3 because it is utilized to keep track of the objects that become available to the unsatisfied agents. Since no agent in M becomes part of a trading cycle before step $\underline{t}$ under R and R', tracking such objects is not necessary.

Lemma 3 establishes that the agents in M cannot manipulate the state of the algorithm before step $\underline{t}$. However, after step $\underline{t}$, the agents in M can depart, become satisfied, or become part of a trading cycle. As such, the two *TTC*-graphs may not share the same vertices or they do not differ only in the outgoing arcs of the unsatisfied agents as some agents in M might be satisfied. Then, the *independence of unsatisfied agents* is no longer applicable and, thus, the algorithm could be susceptible to group manipulation after step $\underline{t}$. Therefore, the next task is to restrict the ability of the group of misreporting agents to manipulate the state of the algorithm after step $\underline{t}$. This is achieved by considering a restriction on the unique pointee selection criterion which is referred to as *consistent pointing*. *Consistent pointing* is defined as follows:

Consistent Pointing: Consider $R, R' \in \mathcal{R}^N$ such that $R' = \left(R_{N\backslash M}, R'_M\right)$ for some $M \subseteq N$. Let t be the first step an agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Let the corresponding step under R' be represented as t'. Suppose $t' < t$. Then, the unique pointee selection criterion, associated with the rule φ - a member of the *CAIO* family - is *consistent* if for all $\tilde{t} \in \{t', \cdots, t-1\}$:

1. $\begin{matrix} N'_{\tilde{t}} \subseteq N_{\tilde{t}} \\ N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{matrix}$ and $\begin{matrix} O'_{\tilde{t}} \subseteq O_{\tilde{t}} \\ O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}} \end{matrix}$,

2. $\begin{array}{c} S_{\tilde{t}} \subseteq S'_{\tilde{t}} \\ S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{array}$ or, equivalently, $\begin{array}{c} U'_{\tilde{t}} \subseteq U_{\tilde{t}} \\ U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{array}$,

3. $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ holds for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, and

4. $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ holds for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

If the unique pointee selection criterion F of the rule φ is *consistent*, the rule φ is said to satisfy *consistent pointing*.[23] The statements of *consistent pointing* might seem complicated which is why an explanation for each of those is provided here: According to statement (1) of *consistent pointing*, $N'_{\tilde{t}} \subseteq N_{\tilde{t}}$ and $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$. This suggests that the sets of remaining agents and objects under R' are a subset of the sets of remaining agents and objects under R, respectively. Therefore, the two TTC-graphs may not have identical vertices and, as such, the *independence of unsatisfied agents* is no longer applicable. In addition, $N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ suggests that any agent who has departed under R', but not under R, must be pointing (directly or indirectly) at some agent in M under R. Similarly $O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ states that any object which has been removed under R', but not under R, must be owned by some agent who is pointing (directly or indirectly) at some agent in M under R. These two conditions suggest that the TTC-graphs under R and R' share the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}-1}$ as vertices. In statement (2) of *consistent pointing*, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$ suggests that there may be some agents who are satisfied under R' but not under R and all such agents are pointing (directly or indirectly) at some agent in M under R because $S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$. This implies that an agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}-1}$ is satisfied under R if and only if she is satisfied under R'. Statement (3) of *consistent pointing* says that the unique pointee decisions of all agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ are identical under R and R'. Finally, statement (4) of *consistent pointing* says that the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ are assigned the same object at step $\tilde{t}+1$ under R and R'. The statements (3) and (4) of consistent pointing imply that identical trading cycles occur for the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ under R and R' for all $\tilde{t} \in \{t', \cdots, t-1\}$ as these agents hold the same objects and have the same unique pointees. This result is formally presented as Lemma 4.

Lemma 4. Consider a rule φ belonging to the *CAIO* family. Let $R, R' \in \mathcal{R}^N$ where $R' = \left(R_{N \backslash M}, R'_M\right)$ and $M \subseteq N$. If, for some step $\tilde{t}$, $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ hold, then $C_j\left(F\left(G_{\tilde{t}}\right)\right) =$

[23] It is noteworthy that *consistent pointing* is the group variant of the *Post-trade inclusion* claim presented by Jaramillo & Manjunath [8] and one of the sufficient conditions for *local invariance* given by Saban & Sethuraman [17].

$C_j\left(F\left(G'_{\tilde{t}}\right)\right)$ and $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ are true where F is the unique pointee selection criterion for the rule φ while $G_{\tilde{t}}$ and $G'_{\tilde{t}}$ are the TTC-graphs at step $\tilde{t}$ under R and R', respectively. Moreover, if $C_j\left(F\left(G_{\tilde{t}}\right)\right) \neq \phi$ for some $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, then the trading cycle $C_j\left(F\left(G_{\tilde{t}}\right)\right)$ is composed entirely of agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

Proof. Consider a rule φ that is a member of the $CAIO$ family. Let $R, R' \in \mathcal{R}^N$ be such that $R' = \left(R_{N\backslash M}, R'_M\right)$ and $M \subseteq N$. Consider any step $\tilde{t}$ such that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ i.e. every agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ has the same unique pointee and holds the same object under R and R'. As the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ are pointing at the same agents under R and R', the same trading cycles are formed for these agents i.e. $C_j\left(F\left(G_{\tilde{t}}\right)\right) = C_j\left(F\left(G'_{\tilde{t}}\right)\right)$ holds for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Since $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ and the same trading cycles occur for these agents, then it must be that $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ holds.

Now, consider an agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ such that $C_j\left(F\left(G_{\tilde{t}}\right)\right) \neq \phi$. By definition of $CONN_{M,\tilde{t}}$, for each $i \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, $p_{i,\tilde{t}} \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ must be true. Therefore, the trading cycle $C_j\left(F\left(G_{\tilde{t}}\right)\right)$ must consist entirely of agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ because each agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ points at another agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. ■

Lemma 4 shows that statements (3) and (4) of *consistent pointing* limit the ability of a group of jointly misreporting agents to manipulate the rule for the steps $\tilde{t} = t', \cdots, t-1$ (given that $t' < t$) in the sense that the same trading cycles occur for the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ under R and R'. In other words, any differences arising from the joint misreporting of agents in M must be for the agents who were pointing (directly or indirectly) at some agent in M under the true preference profile R. Then, the misreporting agents are unable to make additional objects available for themselves under R' because, by misreporting their preferences, they can only influence those agents who were already pointing (directly or indirectly) at some agent in M under R.

Remark. Note that any member of the $CAIO$ family satisfies the statements of *consistent pointing* for all steps $\tilde{t} < \underline{t}$. Statements (1), (3), and (4) of *consistent pointing* at step $\tilde{t}$ follow directly from Lemma 3. Statement (2) of *consistent pointing*, on the other hand, follows from the fact that $O_{\tilde{t}} = O'_{\tilde{t}}$, $h_{\tilde{t}} = h'_{\tilde{t}}$, and $R_j = R'_j$ for each $j \in N \backslash M$ are true by Lemma 3. Moreover, since each agent in M is unsatisfied at step $\tilde{t} < \underline{t}$ under R and R', it must be that $S_{\tilde{t}} = S'_{\tilde{t}}$ and $U_{\tilde{t}} = U'_{\tilde{t}}$ are true.

The next result proves a sufficient condition for *weak group strategy proofness* for the housing market problem with weak preferences. It proves that a member of the *CAIO* family that satisfies *consistent pointing* is *weakly group strategy proof.* Note that Lemma 3 shows that a group of jointly misreporting agents cannot affect the outcome of the algorithm before step $\underline{t} = \min\{t, t'\}$. So, successful manipulation by a group of misreporting agents, if possible, can be done only for steps $\tilde{t} > \underline{t}$. There can be two possibilities: $t \leq t'$ or $t' < t$ i.e. $\underline{t} = t$ or $\underline{t} = t'$, respectively. When $t \leq t'$, it is relatively straightforward to show that if an agent in M departs, becomes satisfied, or becomes part of a trading cycle, under R, before she does under R', she cannot be made better-off by joint misrepresentation of preferences by agents in M. As such, *weak group strategy proofness* cannot be violated when $t \leq t'$. On the other hand, when $t' < t$, *consistent pointing* of the rule suggests that, by jointly misreporting their preferences, the group of agents can only affect the outcome for the agents who were pointing (directly or indirectly) at some agent in M under R suggesting that the agents in M are unable to get additional agents to point (directly or indirectly) at them under the misreported preferences R'. Since the agents who are pointing (directly or indirectly) at agents in M under R continue to do so until step t due to *persistence*,[24] it is not possible to violate *weak group strategy proofness* even when $t' < t$.[25]

Theorem 1. A member of the *CAIO* family satisfying *consistent pointing* satisfies *weak group strategy proofness.*

Proof. Let φ be a member of the *CAIO* family that satisfies *consistent pointing.* Since φ is a member of the *CAIO* family, it satisfies *independence of unsatisfied agents* and *persistence.* Moreover, the results proved in lemmas 1, 2, and 3 are true for the members of the *CAIO* family. For contradiction, suppose that φ does not satisfy *weak group strategy proofness.* Then, there is $M \subseteq N$ and $R, R' \in \mathcal{R}^N$ such that $\alpha'_i P_i \alpha_i$ for each $i \in M$ where $R' = \left(R_{N\setminus M}, R'_M\right)$, $\alpha \equiv \varphi(R, \omega)$, and $\alpha' \equiv \varphi(R', \omega)$. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Define $\underline{t} = \min\{t, t'\}$. The proof proceeds in two steps:

Step 1. It cannot be that $t \leq t'$.

For contradiction, suppose that $t \leq t'$. Then, $\underline{t} = t$. By Lemma 2, no agent in M departs at step $\underline{t}$ under R' because each agent in M has to become part of a trading cycle at least once. Now, consider the following

[24] This is because agents in M are unsatisfied before step t under R.

[25] This is why *consistent pointing* is defined for the case of $t' < t$ because this restriction is not needed when $t \leq t'$.

cases:

Case 1. No agent in M departs at step $\underline{t}$ under R.

Then, by Lemma 3, $N_{\underline{t}} = N'_{\underline{t}}$ and $O_{\underline{t}} = O'_{\underline{t}}$ are true. Let agent $i \in M$ be some agent who becomes satisfied or becomes part of a trading cycle at step $\underline{t}$ under R. Then, it must be that $\alpha_i \in \tau\left(R_i, O_{\underline{t}}\right)$. Since $\alpha'_i \in O_{\underline{t}}$, $\alpha_i R_i \alpha'_i$ which is a contradiction.

Case 2. Some agent in M departs at step $\underline{t}$ under R.

Let agent $i \in M$ be the first agent in M to depart at step $\underline{t}$ under R. By Lemma 3, $N_{\underline{t}-1} = N'_{\underline{t}-1}$, $O_{\underline{t}-1} = O'_{\underline{t}-1}$, and $h_{\underline{t}} = h'_{\underline{t}}$ hold. Then, all agents who depart before agent i at step $\underline{t}$ under R must depart at step $\underline{t}$ under R' as well. Let $\tilde{N}$ and $\tilde{O}$ be the sets of agents and objects, respectively, that are removed from the algorithm before agent i departs at step $\underline{t}$ under R and R'. Then, it must be true that $\alpha_i \in \tau\left(R_i, O_{\underline{t}-1} \backslash \tilde{O}\right)$. Since $\alpha'_i \in O_{\underline{t}-1} \backslash \tilde{O}$, as $\tilde{N}$ and $\tilde{O}$ are removed from the algorithm at step $\underline{t}$ under R' as well, it must be that $\alpha_i R_i \alpha'_i$ which is a contradiction.

Step 2. It cannot be that $t' < t$.

For contradiction, suppose that $t' < t$. If some agent $i \in M$ departs at step $\tilde{t}$ under R' such that $\tilde{t} < t$, then it must be that $\tilde{t} > t'$ by Lemma 2 and $\alpha'_i \in h_{CONN_{M,t-1},t-1}$ by statement (1) of *consistent pointing* because α'_i is removed from the algorithm at step $\tilde{t}$ under R' but cannot be removed before step t under R. To better observe the latter, note that for any step $\ddot{t} \in \{t', \cdots, t-1\}$, as shown in Lemma 4, any trading cycle that occurs at step $\ddot{t}$ under R consists entirely of agents in $N_{\ddot{t}} \backslash CONN_{M,\ddot{t}}$ and occurs at step $\ddot{t}$ under R' as well. Moreover, any trading cycle that occurs at step $\ddot{t}$ under R', but not under R, consists entirely of agents in $CONN_{M,\ddot{t}}$ because $\ddot{t} < t$ and no agent in M becomes part of a trading cycle before step t under R. So, $\alpha'_i \in h_{CONN_{M,\tilde{t}-1},\tilde{t}-1}$ and $h_{CONN_{M,\tilde{t}-1},\tilde{t}-1} \subseteq h_{CONN_{M,t-1},t-1}$ because no agent in M becomes satisfied before step t under R so that once an agent starts pointing (directly or indirectly) at an agent in M, she continues to do so at least until step t. Therefore, it can be concluded that $\alpha'_i \in h_{CONN_{M,t-1},t-1}$.

By definition of step t, at least one agent in M departs, becomes satisfied, or becomes part of a trading cycle at step t under R. Consider the following cases:

Case 1. Some agent in M departs or becomes satisfied at step t under R.

Let agent $i \in M$ be the first such agent in M. Let $\tilde{N}$ and $\tilde{O}$ be the sets of agents and objects, respectively, that are removed before agent i departs or becomes satisfied at step t under R. By *persistence*, it must be the case that $\tilde{N} \subseteq N_{t-1} \backslash CONN_{M,t-1}$ and $\tilde{O} \subseteq O_{t-1} \backslash h_{CONN_{M,t-1},t-1}$ because no agent in $CONN_{M,t-1}$ can depart before some agent in M becomes satisfied under R. Then, by *consistent pointing*, $N'_{t-1} \subseteq N_{t-1}$, $O'_{t-1} \subseteq O_{t-1}$, $h_{j,t} = h'_{j,t}$, and $R_j = R'_j$ for each $j \in N_{t-1} \backslash CONN_{M,t-1}$. Then, $\tilde{N}$ and $\tilde{O}$ are removed from the algorithm at step t under R' as well. So, $O'_{t-1} \backslash \tilde{O} \subseteq O_{t-1} \backslash \tilde{O}$, $O'_t \subseteq O'_{t-1} \backslash \tilde{O}$, and $\alpha_i \in \tau\left(R_i, O_{t-1} \backslash \tilde{O}\right)$. If agent i departed at some step $\tilde{t} < t$ under R', then $\alpha'_i \in h_{CONN_{M,t-1},t-1}$. However, by *persistence*, $h_{CONN_{M,t-1},t-1} \subseteq O_{t-1} \backslash \tilde{O}$. Since $\alpha_i \in \tau\left(R_i, O_{t-1} \backslash \tilde{O}\right)$, it must be that $\alpha_i R_i \alpha'_i$. On the other hand, if agent i departs at some step $\tilde{t} \geq t$ under R', then it must be the case that $\alpha'_i \in O'_{t-1} \backslash \tilde{O}$. Hence, $\alpha_i R_i \alpha'_i$ which is a contradiction.

Case 2. No agent in M departs or becomes satisfied at step t under R.

Then, it must be that some agent in M becomes part of a trading cycle at step t under R. Let agent $i \in M$ be such an agent. Then, $\alpha_i \in \tau(R_i, O_t)$. If agent i departed at some step $\tilde{t} < t$ under R', then $\alpha'_i \in h_{CONN_{M,t-1},t-1}$ and $h_{CONN_{M,t-1},t-1} \subseteq O_t$ by *persistence* because no agent in $CONN_{M,t-1}$ can depart before some agent in M becomes satisfied under R. So, $\alpha_i R_i \alpha'_i$ must be true. If agent i departs at step $\tilde{t} \geq t$ under R', then $\alpha'_i \in O_t$ because $O'_t \subseteq O_t$ since no agent in M departs at step t under R. Then, it can be concluded that $\alpha_i R_i \alpha'_i$.

Steps 1 and 2 provide a contradiction. Therefore, the rule φ satisfies *weak group strategy proofness.* ■

It is noteworthy that the proof of Theorem 1 is similar in structure to the proof of the sufficient condition for *strategy proofness* presented by Saban & Sethuraman [17]. One major difference is that the intermediate step of proving equivalence with *local invariance* is skipped for the sufficient condition of *weak group strategy proofness.* As discussed earlier, such an equivalence may not hold for *weak group strategy proofness* even if *local invariance* is appropriately defined for a group of agents because the rules considered in this paper violate *non-bossiness.*

Next, *weak group strategy proofness* of the *TTAS*, *TC*, and *HPO* rules is established. Since these rules are members of the *CAIO* family [17], by Theorem 1, it only needs to be proved that these rules satisfy *consistent pointing.* Before proving *consistent pointing* of these rules, an intermediate step is proved in Lemma 5 which assists in proving statements (1) and (2) of *consistent pointing.*

Lemma 5. Consider a rule φ belonging to the *CAIO* family. Let $R, R' \in \mathcal{R}^N$ where $R' = \left(R_{N \backslash M}, R'_M\right)$

and $M \subseteq N$. Take t as the first step an agent in M departs, becomes satisfied, or becomes part of a trading cycle under R and let t' be the corresponding step under R'. Suppose that $t' < t$ and, for some step $\tilde{t} \in \{t', \cdots, t-1\}$, the following are satisfied:

1. $\begin{array}{c} N'_{\tilde{t}} \subseteq N_{\tilde{t}} \\ N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{array}$ and $\begin{array}{c} O'_{\tilde{t}} \subseteq O_{\tilde{t}} \\ O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}} \end{array}$,

2. $\begin{array}{c} S_{\tilde{t}} \subseteq S'_{\tilde{t}} \\ S'_{\tilde{t}} \backslash S_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{array}$ or, equivalently, $\begin{array}{c} U'_{\tilde{t}} \subseteq U_{\tilde{t}} \\ U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1} \end{array}$,

3. $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ holds for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, and

4. $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ holds for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

Then, whenever $\tilde{t}+1 < t$, the following are true:

1. $\begin{array}{c} N'_{\tilde{t}+1} \subseteq N_{\tilde{t}+1} \\ N_{\tilde{t}+1} \backslash N'_{\tilde{t}+1} \subseteq CONN_{M,\tilde{t}} \end{array}$ and $\begin{array}{c} O'_{\tilde{t}+1} \subseteq O_{\tilde{t}+1} \\ O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1} \subseteq h_{CONN_{M,\tilde{t}},\tilde{t}+1} \end{array}$, and

2. $\begin{array}{c} S_{\tilde{t}+1} \subseteq S'_{\tilde{t}+1} \\ S'_{\tilde{t}+1} \backslash S_{\tilde{t}+1} \subseteq CONN_{M,\tilde{t}} \end{array}$ or, equivalently, $\begin{array}{c} U'_{\tilde{t}+1} \subseteq U_{\tilde{t}+1} \\ U_{\tilde{t}+1} \backslash U'_{\tilde{t}+1} \subseteq CONN_{M,\tilde{t}} \end{array}$.

Proof. Consider a rule φ which is a member of the *CAIO* family. Let $R, R' \in \mathcal{R}^N$ be such that $R' = (R_{N\backslash M}, R'_M)$ and $M \subseteq N$. Since the rule is a member of the *CAIO* family, it satisfies *persistence*. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Let $t' < t$ and consider some step $\tilde{t} \in \{t', \cdots, t-1\}$.

Suppose that $N'_{\tilde{t}} \subseteq N_{\tilde{t}}$, $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$, $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$, and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ are true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Since $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, then the same trading cycles occur at step $\tilde{t}$ for agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ under R and R' by Lemma 4. Therefore, $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ is true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Since $R_j = R'_j$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, if an agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ departs at step $\tilde{t}+1$ under R, then she departs at step $\tilde{t}+1$ under R' because $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$ and $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ are true. Therefore, $N'_{\tilde{t}+1} \subseteq N_{\tilde{t}+1}$ and $O'_{\tilde{t}+1} \subseteq O_{\tilde{t}+1}$ must hold. Since $\tilde{t}+1 < t$ has been assumed, each agent in M is unsatisfied at step $\tilde{t}+1$ under R. Then, by *persistence*, no agent in $CONN_{M,\tilde{t}}$ departs at step $\tilde{t}+1$ under

R as these agents continue to "persistently" point until step t under R. Therefore, any agent who departs at step $\tilde{t}+1$ under R' but not under R must belong in $CONN_{M,\tilde{t}}$ i.e. $N_{\tilde{t}+1} \backslash N'_{\tilde{t}+1} \subseteq CONN_{M,\tilde{t}}$ must be true. Similarly, $O_{\tilde{t}+1} \backslash O'_{\tilde{t}+1} \subseteq h_{CONN_{M,\tilde{t}},\tilde{t}+1}$ because any object that departs at step $\tilde{t}+1$ under R' but not under R must be held by an agent in $CONN_{M,\tilde{t}}$ at step $\tilde{t}+1$.

Lemma 4 shows that the same trading cycles occur at step $\tilde{t}$ for the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ under R and R'. Moreover, no agent in $CONN_{M,\tilde{t}}$ becomes part of a trading cycle at step $\tilde{t}$ under R because $\tilde{t} < t$. Then, $S_{\tilde{t}+1} \subseteq S'_{\tilde{t}+1}$ because $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ and $R_j = R'_j$ are true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Moreover, $S'_{\tilde{t}+1} \backslash S_{\tilde{t}+1} \subseteq CONN_{M,\tilde{t}}$ because some agents in $CONN_{M,\tilde{t}}$ may become part of a trading cycle at step $\tilde{t}$ under R' but not under R as $\tilde{t} < t$. ■

First, the proof of *consistent pointing* for the $TTAS$ rule is presented as it is simpler compared to the corresponding proofs for the TC and HPO rules.

Proposition 2. The $TTAS$ rule satisfies *weak group strategy proofness.*

Proof. Consider $R, R' \in \mathcal{R}^N$ such that $R' = \left(R_{N\backslash M}, R'_M\right)$ for an arbitrary $M \subseteq N$. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Define $\underline{t} = \min\{t, t'\}$. Suppose $t' < t$. It needs to be proved that the $TTAS$ rule satisfies *consistent pointing.*

Consider any $\tilde{t} \in \{t', \cdots, t-1\}$ and suppose that the statements of *consistent pointing* hold for all steps $\ddot{t} < \tilde{t}$. Then, to complete the proof, the statements of *consistent pointing* need to be proved for step $\tilde{t}$ because the statements of *consistent pointing* are true for all steps before $\underline{t}$ as discussed in the Remark.

Since the statements of *consistent pointing* are true for step $\tilde{t}-1$ and $\tilde{t} < t$, then statements (1) and (2) of *consistent pointing* are true at step $\tilde{t}$ by Lemma 5. Now, consider any agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. It needs to be proved that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$. First, note that, for each $a \in \tau\left(R_j, O_{\tilde{t}}\right)$ and $b \in \tau\left(R_j, O'_{\tilde{t}}\right)$, it must be that aR_jb because $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$ by statement (1) of *consistent pointing* at step $\tilde{t}$. If aP_jb is true, then no object in $\tau\left(R_j, O_{\tilde{t}}\right)$ is available in $O'_{\tilde{t}}$ i.e. $\tau\left(R_j, O_{\tilde{t}}\right) \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ because $O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ is true by statement (1) of *consistent pointing* at step $\tilde{t}$. But this leads to a contradiction because agent j points at an agent who holds an object in $\tau\left(R_j, O_{\tilde{t}}\right)$ at step $\tilde{t}$ under R which implies that agent $j \in CONN_{M,\tilde{t}}$ since $\tau\left(R_j, O_{\tilde{t}}\right) \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$. Therefore, it can be concluded that aI_jb for each $a \in \tau\left(R_j, O_{\tilde{t}}\right)$ and $b \in \tau\left(R_j, O'_{\tilde{t}}\right)$.

Then, it must be that $\tau\left(R_j, O'_{\tilde{t}}\right) \subseteq \tau\left(R_j, O_{\tilde{t}}\right)$ because $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$. Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that agent $j \in N_{\ddot{t}} \backslash CONN_{M,\ddot{t}}$ for each step $\ddot{t} < \tilde{t}$ by *persistence* because once an agent points (directly or indirectly) at an agent in M under R, she continues to do so at least until step t where some agent in M may become satisfied. Then, by statements (3) and (4) of *consistent pointing* for step $\ddot{t} < \tilde{t}$, $p_{j,\ddot{t}} = p'_{j,\ddot{t}}$ and $h_{j,\ddot{t}} = h'_{j,\ddot{t}}$ must be true. Therefore, by Lemma 4, the same trading cycles occur for agent j for all steps $\ddot{t} < \tilde{t}$ under R and R'. Since the same trading cycles occurred for agent j for all steps $\ddot{t} < \tilde{t}$, the objects in $\tau\left(R_j, O'_{\tilde{t}}\right)$ would have been assigned to her the same number of times under R and R'.

Let $a \in \tau\left(R_j, O'_{\tilde{t}}\right)$ be the object that is assigned the least number of times to agent j under R'.[26] Then, agent j points at the agent who owns object a at step $\tilde{t}$ under R'. Under R, it is not possible that agent j points at an agent who holds an object in $O_{\tilde{t}} \backslash O'_{\tilde{t}}$ because $O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ and $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Therefore, agent j points at an agent who holds an object in $\tau\left(R_j, O'_{\tilde{t}}\right)$ at step $\tilde{t}$ under R. As discussed earlier, this agent must hold object a because this object has been assigned the least number of times to agent j under R. Let $p_{j,\tilde{t}} = i$ and $p'_{j,\tilde{t}} = i'$. Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that agent $i \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Then, by *persistence*, agent $i \in N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$ because once an agent points (directly or indirectly) at an agent in M under R, she continues to do so at least until step t. Then, by statement (4) of *consistent pointing* at step $\tilde{t}-1$, it must be that $h_{i,\tilde{t}} = h'_{i,\tilde{t}} = a$. This implies that $i = i'$ i.e. $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ is true.

Since $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ are true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, then $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ holds by Lemma 4. Therefore, the $TTAS$ rule satisfies the statements of *consistent pointing* and, by Theorem 1, it satisfies *weak group strategy proofness.* ■

The proof of Proposition 2 relies on the fact that identical trading cycles occur for the agents who are not pointing (directly or indirectly) at some agent in M under R. Based on this, each object is assigned to this agent the same number of times under R and R'. Since F^{TTAS}, the unique pointee selection criterion associated with the $TTAS$ rule, is based on the number of times each object has been assigned to an agent in the previous steps, it can be established that the $TTAS$ rule satisfies *consistent pointing* and, thus, *weak group strategy proofness.*

The next result shows that the TC rule is *weakly group strategy proof* by establishing that it satisfies *consistent pointing.* The proof relies heavily on the progression of the pointing phase for the TC rule.

[26] If there are multiple such objects, the ties are broken in accordance with the priority ordering over objects.

Proposition 3. The TC rule satisfies *weak group strategy proofness.*

Proof. Consider $R, R' \in \mathcal{R}^N$ such that $R' = \left(R_{N\setminus M}, R'_M\right)$ for an arbitrary $M \subseteq N$. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Define $\underline{t} = \min\{t, t'\}$. Suppose $t' < t$. It needs to be proved that the TC rule satisfies *consistent pointing.*

Consider any $\tilde{t} \in \{t', \cdots, t-1\}$ and suppose that the statements of *consistent pointing* hold for all steps $\ddot{t} < \tilde{t}$. Then, to complete the proof, the statements of *consistent pointing* need to be proved for step $\tilde{t}$ because the statements of *consistent pointing* are true for all steps before $\underline{t}$ as discussed in the Remark.

By the induction hypothesis, the statements of *consistent pointing* are satisfied at step $\tilde{t}-1$. Therefore, statements (1) and (2) of *consistent pointing* are true at step $\tilde{t}$ by Lemma 5. So, it needs to be proved that for each agent $j \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$, $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ and $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ are true. To prove that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ holds for each $j \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$, the pointing phase of the TC rule is considered below:

Case 1. The agents in $N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$ who are pointing based on *TC-persistence* at step $\tilde{t}$ under R.

For any agent $j \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$, let $p_{j,\tilde{t}} = i$. Since agent $j \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$, it must be true that agent $i \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$ and, therefore, agent $i \in N_{\tilde{t}-1} \setminus CONN_{M,\tilde{t}-1}$ otherwise $i \in CONN_{M,\tilde{t}-1}$ and, by *TC-persistence*, it must be that agent $i \in CONN_{M,\tilde{t}}$ because once an agent points (directly or indirectly) at an agent in M, she continues to do so at least until step t under R. Then, it must be that $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ are true by statement (4) of *consistent pointing* at step $\tilde{t}-1$. Similarly, by statement (4) of *consistent pointing* at step $\tilde{t}-2$, $h_{i,\tilde{t}-1} = h'_{i,\tilde{t}-1}$ is true. Then, $h_{i,\tilde{t}} = h_{i,\tilde{t}-1}$ is true if and only if $h'_{i,\tilde{t}} = h'_{i,\tilde{t}-1}$ is true. So, agent j points based on *TC-persistence* at step $\tilde{t}$ under R if and only if she points based on *TC-persistence* at step $\tilde{t}$ under R'. Therefore, $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ is true because $p_{j,\tilde{t}-1} = p'_{j,\tilde{t}-1}$ as $j \in N_{\tilde{t}-1} \setminus CONN_{M,\tilde{t}-1}$, and *$TC$-persistence* implies $p_{j,\tilde{t}} = p_{j,\tilde{t}-1}$ and $p'_{j,\tilde{t}} = p'_{j,\tilde{t}-1}$.

After dealing with the above case, none of the remaining agents in $N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$ are pointing based on *TC-persistence.*

Case 2. The agents in $N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$ who have a unique most preferred object (among the remaining ones) at step $\tilde{t}$ under R.

Consider any agent $j \in N_{\tilde{t}} \setminus CONN_{M,\tilde{t}}$ who has a unique most preferred object (among the remaining ones) at step $\tilde{t}$ under R. By statement (1) of *consistent pointing* at step $\tilde{t}$, $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$ is true. Moreover, since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that $p_{j,\tilde{t}} \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Then, $p_{j,\tilde{t}} \in N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$ otherwise $p_{j,\tilde{t}} \in CONN_{M,\tilde{t}-1}$ and, by *TC-persistence*, $p_{j,\tilde{t}} \in CONN_{M,\tilde{t}}$ which contradicts $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Therefore, $p_{j,\tilde{t}} = p'_{j,\tilde{t}} = i$ because $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ holds by statement (4) of *consistent pointing* at step $\tilde{t}-1$.

Case 3. The agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ who have multiple most preferred objects (among the remaining ones) at step $\tilde{t}$ under R. For this case, I use $j \xrightarrow[\tilde{t}]{R} i$ to represent that agent j points at agent i, after the application of the unique pointee selection criterion F^{TC}, at step $\tilde{t}$ under R. This case relies heavily on the progression of the pointing phase for the TC rule to show that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ is true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

1. Consider an agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ such that $j \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ i.e. agent j points at the unsatisfied agent j_0 at step $\tilde{t}$ under R. This agent is depicted in Figure 7 (a) where the unsatisfied agent is shown in red. Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, agent $j_0 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, and $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ by statement (2) of *consistent pointing* at step $\tilde{t}$, then it must be that agent $j_0 \in U'_{\tilde{t}}$. However, suppose that $j \xrightarrow[\tilde{t}]{R'} j'_0 \neq j_0$. Then, $j'_0 \in U'_{\tilde{t}}$ and $j'_0 \prec j_0$ because $j_0 \in U'_{\tilde{t}}$. This is shown in Figure 7 (b) which shows that even though the unsatisfied agent j_0 is available under R', agent j points at the unsatisfied agent j'_0 suggesting that $j'_0 \prec j_0$. However, by statement (2) of *consistent pointing*, $U'_{\tilde{t}} \subseteq U_{\tilde{t}}$ i.e. agent $j'_0 \in U_{\tilde{t}}$ which contradicts $j \xrightarrow[\tilde{t}]{R} j_0$ because $j'_0 \prec j_0$. As shown in Figure 7 (c), presence of the unsatisfied agent j'_0 under R gives a contradiction because agent j'_0 has a higher priority than agent j_0.

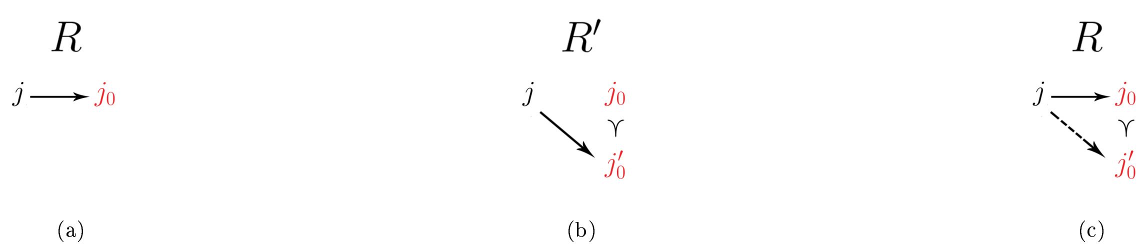


Figure 7: Unique pointee determination for an agent who has at least one of her most preferred objects owned by an unsatisfied agent under R. The unsatisfied agents are depicted in red. The dashed arc shows the contradictory pointing.

2. Now, consider an agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ such that $j \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_1 \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_0 \in U_{\tilde{t}}$ with $j_1 \in S_{\tilde{t}}$. This is depicted in Figure 8 (a). By the preceding argument, it must be that $j_1 \underset{\tilde{t}}{\overset{R'}{\longrightarrow}} j_0$. Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ (so that $j_0, j_1 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$) and, by statement (2) of *consistent pointing* at step $\tilde{t}$, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$ and $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ are true. Therefore, it must be that $j_0 \in U'_{\tilde{t}}$ and $j_1 \in S'_{\tilde{t}}$ are true as shown in Figure 8 (b) where agents j_0 and j_1 are depicted in red and green, respectively.

   Suppose that $j \underset{\tilde{t}}{\overset{R'}{\longrightarrow}} j'_1 \neq j_1$. If agent $j'_1 \in U'_{\tilde{t}}$, then $j'_1 \in U_{\tilde{t}}$ and $h_{j'_1,\tilde{t}} = h'_{j'_1,\tilde{t}}$. This contradicts $j \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_1 \in S_{\tilde{t}}$. So, suppose that $j \underset{\tilde{t}}{\overset{R'}{\longrightarrow}} j'_1 \underset{\tilde{t}}{\overset{R'}{\longrightarrow}} j'_0 \in U'_{\tilde{t}}$ with $j'_1 \in S'_{\tilde{t}}$ where either $j'_1 \prec j_1$ and $j_0 = j'_0$, or $j'_0 \prec j_0$. These possibilities are presented in Figure 8 (b). Since $U'_{\tilde{t}} \subseteq U_{\tilde{t}}$, it must be that agent $j'_0 \in U_{\tilde{t}}$. Now, consider the following cases:

   (a) $h_{j'_1,\tilde{t}} = h'_{j'_1,\tilde{t}}$.

   First, note that it cannot be that $j'_1 \in U_{\tilde{t}}$ as that would contradict $j \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_1 \in S_{\tilde{t}}$. Therefore, $j'_1 \in S_{\tilde{t}}$ which implies that $j'_1 \notin M$ because no agent in M becomes satisfied before step t under R. So, $R_{j'_1} = R'_{j'_1}$ i.e. agent j'_1 is not misreporting her preferences under R'. Since $j'_0 \in U_{\tilde{t}}$, it must be that $j'_1 \underset{\tilde{t}}{\overset{R}{\longrightarrow}} p_{j'_1,\tilde{t}} \preceq j'_0$ and $p_{j'_1,\tilde{t}} \in U_{\tilde{t}}$. This contradicts $j \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_1$ because either $j'_1 \prec j_1$ and $j_0 = p_{j'_1,\tilde{t}}$, or $p_{j'_1,\tilde{t}} \preceq j'_0 \prec j_0$. Both possible contradictions are shown in Figure 8 (c). The first possibility corresponds with $p_{j'_1,\tilde{t}} = j_0$ which is a contradiction because $j'_1 \prec j_1$. The second possibility corresponds with $p_{j'_1,\tilde{t}} \neq j_0$ which leads to a contradiction because $p_{j'_1,\tilde{t}} \preceq j'_0 \prec j_0$.

   (b) $h_{j'_1,\tilde{t}} \neq h'_{j'_1,\tilde{t}}$.

   Let $h'_{j'_1,\tilde{t}} = a$. Since $h_{j'_1,\tilde{t}} \neq a$ and $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$, there is $\hat{j} \in N_{\tilde{t}}$ such that $h_{\hat{j},\tilde{t}} = a \neq h'_{\hat{j},\tilde{t}}$. So, by statement (4) of *consistent pointing* at step $\tilde{t}-1$, it must be that $\hat{j} \in CONN_{M,\tilde{t}-1}$ because $h_{\hat{j},\tilde{t}} \neq h'_{\hat{j},\tilde{t}}$. Note that $\hat{j} \in U_{\tilde{t}}$ contradicts $j \underset{\tilde{t}}{\overset{R}{\longrightarrow}} j_1 \in S_{\tilde{t}}$ because $h_{\hat{j},\tilde{t}} = a$ and $a \in \tau(R_j, O_{\tilde{t}})$. Therefore, agent $\hat{j} \in S_{\tilde{t}}$ i.e. $\hat{j} \notin M$ because $\tilde{t} < t$ and no agent in M is satisfied before step t under R. Therefore, $R_{\hat{j}} = R'_{\hat{j}}$ i.e. agent $\hat{j}$ is not misreporting her preferences under R'.

   Let $\hat{t}$ be the first step such that agent $\hat{j} \in CONN_{M,\hat{t}}$. Then, by statement (4) of *consistent pointing* at step $\hat{t}-1$, it must be that $h_{\hat{j},\hat{t}} = h'_{\hat{j},\hat{t}} = a$ and $\hat{j} \in S_{\hat{t}}$ are true. This is because agent $\hat{j}$ does not become part of a trading cycle at any step $\ddot{t} \in \{\hat{t}, \cdots, t-1\}$ under R as no agent in M becomes part of a trading cycle before step t under R.

Since $h'_{j'_1,\tilde{t}} = a$, there is a step $\ddot{t} \in \left\{\hat{t}, \cdots, \tilde{t}-1\right\}$ such that agent j'_1 is part of a trading cycle under R' and $j'_1 \xrightarrow[\ddot{t}]{R'} \tilde{j}$ such that $h'_{\tilde{j},\ddot{t}} = a$.[27] Note that either agent $\tilde{j} = \hat{j}$ or agent $\tilde{j} \neq \hat{j}$ acquired object a in a trading cycle for some step before step $\ddot{t}$. In either case, agent $\tilde{j} \in S'_{\ddot{t}}$ because $\hat{j} \in S'_{\hat{t}}$ and agents become satisfied after being part of a trading cycle. Therefore, it must be that $j'_1 \xrightarrow[\ddot{t}]{R'} \tilde{j} \in S'_{\ddot{t}}$ which is a contradiction as: (1) $j'_0 \in U'_{\ddot{t}}$ since $j'_0 \in U'_{\tilde{t}}$ and $\ddot{t} < \tilde{t}$, and (2) $h_{j'_0,\ddot{t}} \in \tau\left(R'_{j'_1}, O_{\ddot{t}}\right)$ because $aI'_{j'_1} h_{j'_0,\tilde{t}}$ and $h_{j'_0,\ddot{t}} = h_{j'_0,\tilde{t}}$.

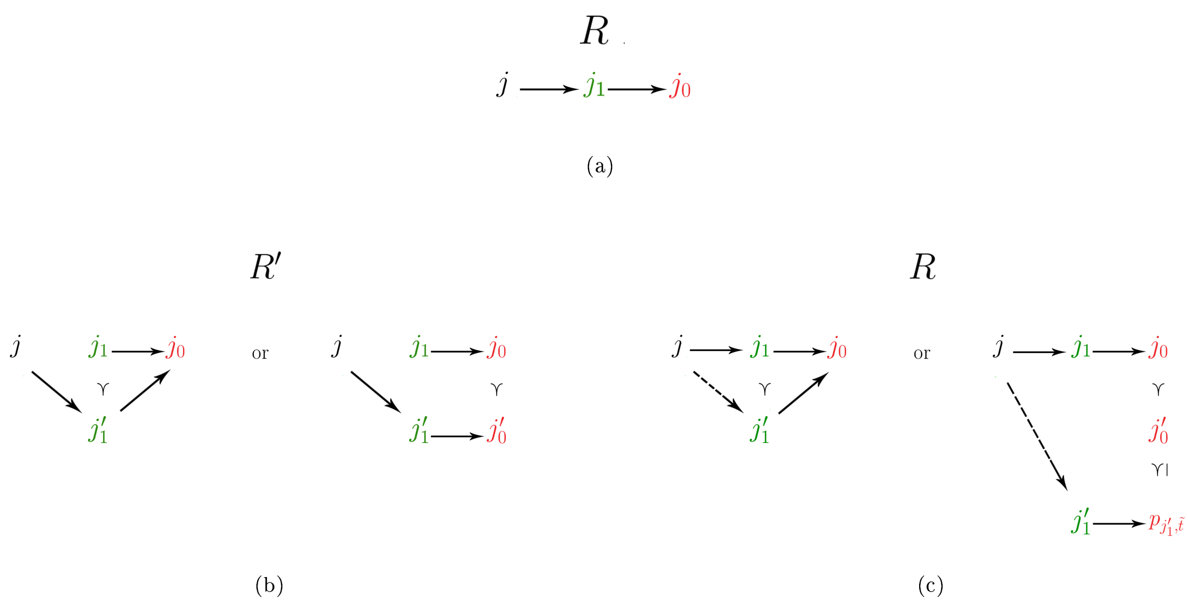


Figure 8: Unique pointee determination for an agent who has all of her most preferred objects owned by satisfied agents and at least one of these satisfied agents has one of her most preferred objects owned by an unsatisfied agent under R. The satisfied and unsatisfied agents are depicted in green and red, respectively. The dashed arcs show the contradictory pointing.

3. **Induction Hypothesis:** For any $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, if $j \xrightarrow[\tilde{t}]{R} j_{k-1} \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_{k-1}\} \subseteq S_{\tilde{t}}$, then $j \xrightarrow[\tilde{t}]{R'} j_{k-1} \xrightarrow[\tilde{t}]{R'} \cdots \xrightarrow[\tilde{t}]{R'} j_1 \xrightarrow[\tilde{t}]{R'} j_0 \in U'_{\tilde{t}}$.

[27] Agent j'_1 becomes satisfied after receiving object a under R'. Since she points at agent j'_0 at step $\tilde{t}$ under R', $aI'_{j'_1} h_{j'_0,\tilde{t}}$ must be true.

To complete the proof, it has to be established that, for any $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, if $j \xrightarrow[\tilde{t}]{R} j_k \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_k\} \subseteq S_{\tilde{t}}$, then $j \xrightarrow[\tilde{t}]{R'} j_k \xrightarrow[\tilde{t}]{R'} \cdots \xrightarrow[\tilde{t}]{R'} j_1 \xrightarrow[\tilde{t}]{R'} j_0 \in U'_{\tilde{t}}$. Consider agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ and suppose that $j \xrightarrow[\tilde{t}]{R} j_k \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_k\} \subseteq S_{\tilde{t}}$ as shown in Figure 9 (a). Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that agent $j_k \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ is true. Then, by the induction hypothesis, $j_k \xrightarrow[\tilde{t}]{R'} \cdots \xrightarrow[\tilde{t}]{R'} j_1 \xrightarrow[\tilde{t}]{R'} j_0 \in U_{\tilde{t}}$ is true. Moreover, by statement (2) of *consistent pointing* at step $\tilde{t}$, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$ i.e. $\{j_1, \cdots, j_k\} \subseteq S'_{\tilde{t}}$. However, suppose that $j \xrightarrow[\tilde{t}]{R'} j'_k \neq j_k$ and let $j \xrightarrow[\tilde{t}]{R'} j'_k \xrightarrow[\tilde{t}]{R'} \cdots \xrightarrow[\tilde{t}]{R'} j'_1 \xrightarrow[\tilde{t}]{R'} j'_0$ which is depicted in Figure 9 (b). Note that agents in $\{j'_0, \cdots, j'_k\}$ are shown in black as their welfare status (satisfied or unsatisfied) is currently undetermined. Now, consider the following cases:

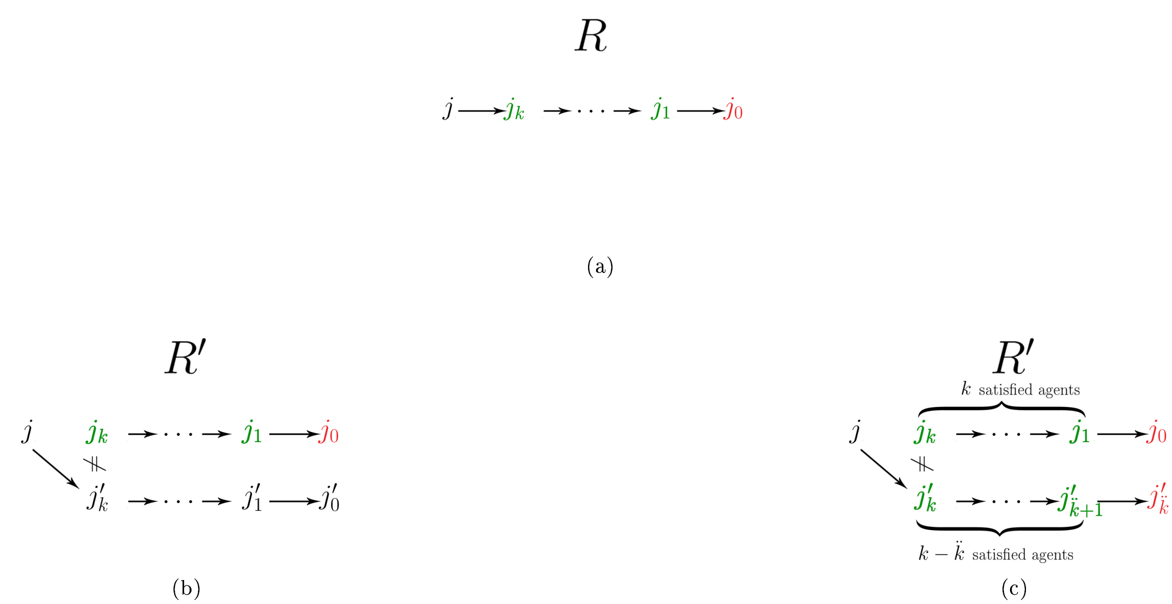


Figure 9: Unique pointee determination for an agent who is k satisfied agents away from the closest unsatisfied agent under R. The satisfied and unsatisfied agents are depicted in green and red, respectively.

(a) $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ for each $i \in \{j'_1, \cdots, j'_k\}$:

Suppose that $j'_k \in U_{\tilde{t}}$. However, this contradicts $j \xrightarrow[\tilde{t}]{R} j_k \in S_{\tilde{t}}$ because $j'_k \in U_{\tilde{t}}$ and $h_{j'_k,\tilde{t}} = h'_{j'_k,\tilde{t}}$ is true. Hence, it must be that $j'_{\ddot{k}} \in S_{\tilde{t}}$.

Now, consider any $j'_{\ddot{k}}$ such that $i \in S_{\tilde{t}}$ is true for each $i \in \left\{j'_{\ddot{k}+1}, \cdots, j'_k\right\}$ and $\ddot{k} \in \{1, \cdots, k-1\}$. Note that $j'_{\ddot{k}} \in U_{\tilde{t}}$ contradicts $j \xrightarrow[\tilde{t}]{R} j_k \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_k\} \subseteq S_{\tilde{t}}$ because $i \in S_{\tilde{t}}$ and $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ are true for each $i \in \left\{j'_{\ddot{k}+1}, \cdots, j'_k\right\}$. The former implies that no agent in $\left\{j'_{\ddot{k}+1}, \cdots, j'_k\right\}$ is misreporting her preferences under R' because no agent in M is satisfied before step t under R and $\tilde{t} < t$. Then, agent $j'_{\ddot{k}+1}$ either points at the unsatisfied agent $j'_{\ddot{k}}$ or a higher priority unsatisfied agent under R. Therefore, an unsatisfied agent is reachable for agent j at step $\tilde{t}$ under R through (at most) $k - \ddot{k}$ satisfied agents if $j \xrightarrow[\tilde{t}]{R} j'_{\ddot{k}}$ compared to the k satisfied agents when $j \xrightarrow[\tilde{t}]{R} j_k$. This is presented in Figure 9 (c) which shows that agent j could reach a closer unsatisfied agent by pointing at agent $j'_{\ddot{k}}$. Therefore, it must be true that $j'_{\ddot{k}} \in S_{\tilde{t}}$. Thus, it can be concluded that $\{j'_1, \cdots, j'_k\} \subseteq S_{\tilde{t}}$ holds. Additionally, by statement (2) of *consistent pointing* at step $\tilde{t}$, $S_{\tilde{t}} \subseteq S'_{\tilde{t}}$ holds i.e. $\{j'_1, \cdots, j'_k\} \subseteq S'_{\tilde{t}}$ which implies that $j'_0 \in U'_{\tilde{t}} \subseteq U_{\tilde{t}}$ otherwise agent j would point at agent j_k at step $\tilde{t}$ under R' instead of agent j'_k.

Since $j \xrightarrow[\tilde{t}]{R'} j'_k \neq j_k$, it must be the case that either $j'_0 \prec j_0$, or $j'_0 = j_0$ with $j'_{\ddot{k}} \prec j_{\ddot{k}}$ for some $\ddot{k} \in \{1, \cdots, k-1\}$ and $j_l = j'_l$ for each $l < \ddot{k}$. These possibilities are shown in Figure 10 (a). The unique pointee path from agent j'_k to the first unsatisfied agent, say agent j''_0, under R should have at least $k-1$ satisfied agents otherwise $j \xrightarrow[\tilde{t}]{R} j_k$ is contradicted, and it should not have more than $k-1$ satisfied agents because $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ for each $i \in \{j'_1, \cdots, j'_k\}$ and $h_{j'_0,\tilde{t}} = h'_{j'_0,\tilde{t}}$ as agent $j'_0 \in U'_{\tilde{t}}$. Therefore, let $j'_k \xrightarrow[\tilde{t}]{R} j''_{k-1} \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j''_1 \xrightarrow[\tilde{t}]{R} j''_0$ where $\left\{j''_1, \cdots, j''_{k-1}\right\} \subseteq S_{\tilde{t}}$ and $j''_0 \in U_{\tilde{t}}$.

By the unique pointee selection criterion of the TC rule, one of the following must be true: (1) $j'_l = j''_l$ for each $l \in \{0, 1, \cdots, k-1\}$, (2) $j''_0 = j'_0$ with $j''_{\ddot{k}} \prec j'_{\ddot{k}}$ for some $\ddot{k} \in \{1, \cdots, k-1\}$ and $j_l = j'_l$ for each $l < \ddot{k}$, or (3) $j''_0 \prec j'_0$. The first possibility is shown in Figure 10 (b). This leads to a contradiction because, based on the unique pointee selection criterion F^{TC}, agent j's unique pointee would be determined as j'_k instead of j_k under R since j'_k was selected as her unique pointee under R' as shown in Figure 10 (a). The second possibility is shown in Figure 10 (c). The chain starting from agent j'_k in Figure 10 (c) differs from the chain depicted in Figure 10 (a). However, the difference is $j''_{\ddot{k}} \prec j'_{\ddot{k}}$ and the subsequent satisfied agents, before the unsatisfied agent j'_0 is reached, are identical. Therefore, because agent j pointed at j'_k under R', as shown in

Figure 10 (a), she should point at j_k' under R which is a contradiction. The third case, shown in Figure 10 (d), allows agent j to reach a higher priority unsatisfied agent while going through the same number of satisfied agents. Then, it must be that agent j points at agent j_k' under R which contradicts $j \overset{R}{\underset{\tilde{t}}{\rightarrow}} j_k$.

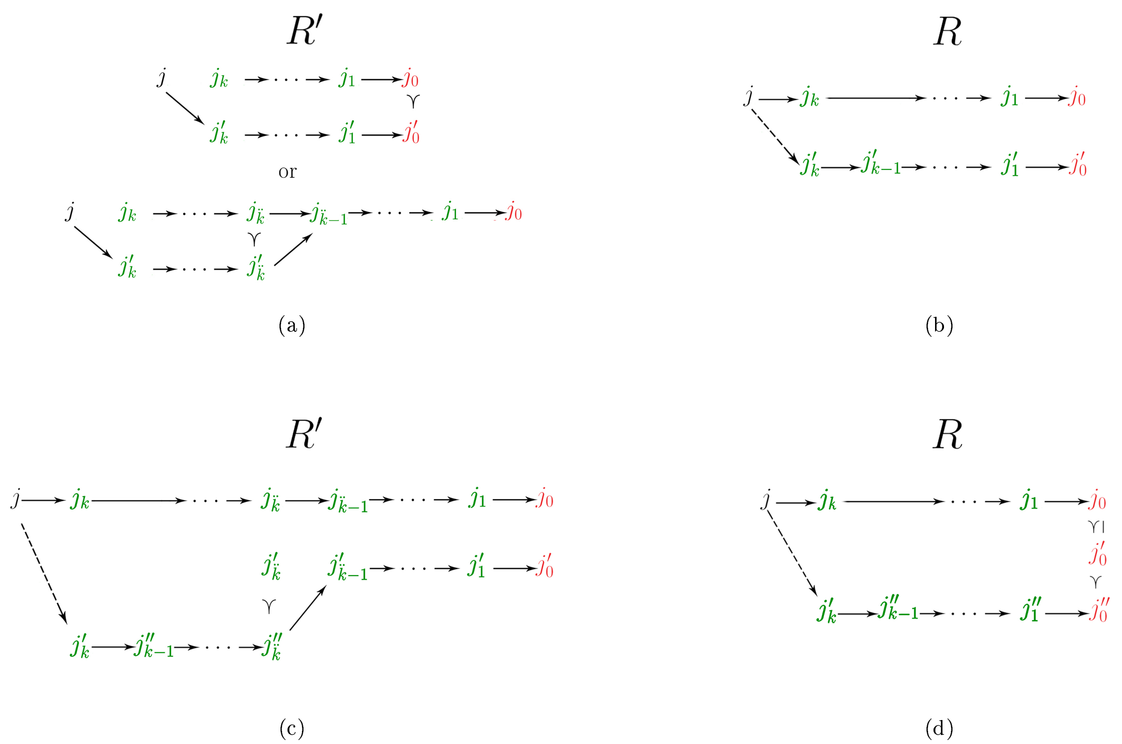


Figure 10: Unique pointee determination for an agent who is k satisfied agents away from the closest unsatisfied agent under R. The satisfied and unsatisfied agents are depicted in green and red, respectively. The dashed arrows show the contradictory pointing.

(b) $h_{i,\tilde{t}} \neq h'_{i,\tilde{t}}$ for some $i \in \{j_1', \cdots, j_k'\}$.

Let $\ddot{k} \in \{1, \cdots, k\}$ be such that $h_{j'_{\ddot{k}},\tilde{t}} \neq h'_{j'_{\ddot{k}},\tilde{t}} = a$ and $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ for each $i \in \left\{\ddot{k}+1, \cdots, k\right\}$. Following the reasoning from 3 (a), it must be that $\left\{j'_{\ddot{k}+1}, \cdots, j_k'\right\} \subseteq S_{\tilde{t}} \subseteq S'_{\tilde{t}}$. Moreover, it cannot be that $j'_{\ddot{k}} \in U'_{\tilde{t}}$ because $U'_{\tilde{t}} \subseteq U_{\tilde{t}}$ and $h_{j'_{\ddot{k}},\tilde{t}} \neq h'_{j'_{\ddot{k}},\tilde{t}}$ has been assumed. So, agent $j'_{\ddot{k}} \in S'_{\tilde{t}}$.

Since $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$, there is an agent $\hat{j} \in N_{\tilde{t}}$ such that $h_{\hat{j},\tilde{t}} = a$ and $h_{\hat{j},\tilde{t}} \neq h'_{\hat{j},\tilde{t}}$ holds because $h'_{j'_{\ddot{k}},\tilde{t}} = a$. It must be that $\hat{j} \xrightarrow[\tilde{t}]{R} \hat{j}_m \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} \hat{j}_1 \xrightarrow[\tilde{t}]{R} \hat{j}_0 \in U_{\tilde{t}}$, $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_m\right\} \subseteq S_{\tilde{t}}$, and $m \geq \ddot{k} - 1$ otherwise $j \xrightarrow[\tilde{t}]{R} j_k \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_k\} \subseteq S_{\tilde{t}}$ is contradicted following the arguments from 3 (a). This can be observed from Figure 11 which shows that if agent j points at the satisfied agent j'_k, she reaches the first unsatisfied agent $\hat{j}_0$ through $k - \ddot{k} + m + 1$ satisfied agents. This would contradict $j \xrightarrow[\tilde{t}]{R} j_k \xrightarrow[\tilde{t}]{R} \cdots \xrightarrow[\tilde{t}]{R} j_1 \xrightarrow[\tilde{t}]{R} j_0 \in U_{\tilde{t}}$ with $\{j_1, \cdots, j_k\} \subseteq S_{\tilde{t}}$ if $k - \ddot{k} + m + 1 < k$ because, by pointing at agent j_k, agent j reaches the first unsatisfied agent j_0 through k satisfied agents. Therefore, it must be that $k - \ddot{k} + m + 1 \geq k$ which implies that $m \geq \ddot{k} - 1$.

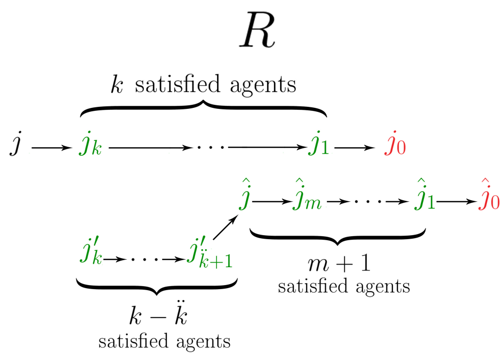


Figure 11: Unique pointee determination for an agent who is k satisfied agents away from the closest unsatisfied agent under R. The satisfied and unsatisfied agents are depicted in green and red, respectively.

By statement (4) of *consistent pointing* at step $\tilde{t} - 1$, it must be that $\hat{j} \in CONN_{M,\tilde{t}-1}$ since $h_{\hat{j},\tilde{t}} \neq h'_{\hat{j},\tilde{t}}$. Let $\hat{t}$ be the first step such that $\hat{j} \in CONN_{M,\hat{t}}$. Then, by statement (4) of *consistent pointing* at step $\hat{t} - 1$, it must that $h_{\hat{j},\hat{t}} = h'_{\hat{j},\hat{t}} = a$ and $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_m \xrightarrow[\hat{t}]{R} \cdots \xrightarrow[\hat{t}]{R} \hat{j}_1 \xrightarrow[\hat{t}]{R} \hat{j}_0 \in U_{\hat{t}}$ with $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_m\right\} \subseteq S_{\hat{t}}$. This is true because no agent in $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_m\right\}$ becomes part of a trading cycle for any step $\ddot{t} \in \left\{\hat{t}, \cdots, t-1\right\}$ as no agent in M becomes part of a trading cycle before step t under R.

It needs to be proved that whenever $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_m \xrightarrow[\hat{t}]{R} \cdots \xrightarrow[\hat{t}]{R} \hat{j}_1$ with $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_m\right\} \subseteq S_{\hat{t}}$, then it must be that $\hat{j} \xrightarrow[\hat{t}]{R'} \check{j}_m \xrightarrow[\hat{t}]{R'} \cdots \xrightarrow[\hat{t}]{R'} \check{j}_1$ with $\left\{\hat{j}, \check{j}_1, \cdots, \check{j}_m\right\} \subseteq S'_{\hat{t}}$ is true whenever $\hat{j} \notin M$. In other words, if an agent, who is not misreporting her preferences under R, is at least m satisfied agents away from the first unsatisfied agent under R, then she is at least m satisfied agents away from the first unsatisfied agent under R'.

Let $m = 1$ i.e. $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_1 \in S_{\hat{t}}$ and, for contradiction, suppose that $\hat{j} \xrightarrow[\hat{t}]{R'} \check{j}_0 \in U'_{\hat{t}}$. However, by statement (2) of *consistent pointing* at step $\hat{t}$, $U'_{\hat{t}} \subseteq U_{\hat{t}}$ is true. Then, $\check{j}_0 \in U_{\hat{t}}$ which contradicts $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_1 \in S_{\hat{t}}$. Now, suppose that the claim is true for $m \leq \hat{m} - 1$ and it needs to be proved for $m = \hat{m}$.

Let $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_{\hat{m}} \xrightarrow[\hat{t}]{R} \cdots \xrightarrow[\hat{t}]{R} \hat{j}_1$ with $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_{\hat{m}}\right\} \subseteq S_{\hat{t}}$. Define $\widehat{N} = \left\{i \in N_{\hat{t}} : h_{i,\hat{t}} \in \tau\left(R_{\hat{j}}, O_{\hat{t}}\right)\right\}$ i.e. $\widehat{N}$ is the set of agents who hold one of agent $\hat{j}$'s most preferred objects at step $\hat{t}$ under R. Note that no agent in $\widehat{N}$ belongs in M because $\widehat{N} \subseteq S_{\hat{t}}$ and each agent in M is unsatisfied before step t under R. Additionally, by assumption, each agent in $\widehat{N}$ is at least $\hat{m} - 1$ satisfied agents away from the first unsatisfied agent under R. Then, by the induction hypothesis, each agent in $\widehat{N}$ is at least $\hat{m} - 1$ satisfied agents away from the first unsatisfied agent under R'. Define $\widehat{N}' = \left\{i \in N'_{\hat{t}} : h'_{i,\hat{t}} \in \tau\left(R_{\hat{j}}, O'_{\hat{t}}\right)\right\}$ and consider the following cases:

Case 1. $\widehat{N}' \subseteq \widehat{N}$.

Since $\widehat{N}' \subseteq \widehat{N}$, the most preferred objects of agent $\hat{j}$ are held by satisfied agents (as $S_{\hat{t}} \subseteq S'_{\hat{t}}$) who are at least $\hat{m} - 1$ satisfied agents away from the first unsatisfied agent at step $\hat{t}$ under R'. Therefore, agent $\hat{j}$ points at a satisfied agent who is at least $\hat{m} - 1$ satisfied agents away from the first unsatisfied agent under R'. Hence, it can be concluded that agent $\hat{j}$ is at least $\hat{m}$ satisfied agents away from the first unsatisfied agent.

Case 2. $\widehat{N}' \nsubseteq \widehat{N}$.

Since $\widehat{N}' \nsubseteq \widehat{N}$, there is an agent $i' \in N'_{\hat{t}}$ such that $i' \in \widehat{N}'$ but $i' \notin \widehat{N}$. In other words, agent i' holds one of agent $\hat{j}$'s most preferred objects at step $\hat{t}$ under R' but not under R. Then, it needs to be proved that agent i' is at least $\hat{m} - 1$ satisfied agents away from the first unsatisfied agent at step $\hat{t}$ under R'. Since $O'_{\hat{t}} \subseteq O_{\hat{t}}$, there is $i \in \widehat{N}$ such that $h_{i,\hat{t}} = h'_{i',\hat{t}}$ and $h_{i,\hat{t}} \neq h'_{i,\hat{t}}$ are true. Moreover, by assumption, it must be that $i \xrightarrow[\hat{t}]{R} i_{\hat{m}-1} \xrightarrow[\hat{t}]{R} \cdots \xrightarrow[\hat{t}]{R} i_1$ with $\{i, i_1, \cdots, i_{\hat{m}-1}\} \subseteq S_{\hat{t}}$

because $i \in \widehat{N}$.

Since $h_{i,\hat{t}} \neq h'_{i,\hat{t}}$, by statement (4) of *consistent pointing* at step $\hat{t}-1$, it must be that $i \in CONN_{M,\hat{t}-1}$. Let t_i be the first step such that $i \in CONN_{M,t_i}$. Then, by statement (4) of *consistent pointing* at step $t_i - 1$, it must be that $h_{i,t_i} = h'_{i,t_i} = h_{i,\hat{t}}$ and $i \xrightarrow[t_i]{R} i_{\hat{m}-1} \xrightarrow[t_i]{R} \cdots \xrightarrow[t_i]{R} i_1$ with $\{i, i_1, \cdots, i_{\hat{m}-1}\} \subseteq S_{\hat{t}}$ are true because no agent in $CONN_{M,t_i}$ becomes part of a trading cycle before step t under R. Then, by the induction hypothesis, agent i is at least $\hat{m}-1$ satisfied agents away from the first unsatisfied agent at step t_i under R'.

Since $h'_{i',\hat{t}} = h_{i,\hat{t}}$, there is a step $\ddot{t}_i \in \left\{t_i, \cdots, \hat{t}-1\right\}$ in which agent i' becomes part of a trading cycle where she receives object $h_{i,\hat{t}}$. However, this requires that agent i' points at a satisfied agent who is at least $\hat{m}-1$ satisfied agents away from the first unsatisfied agent which implies that agent i' is at least $\hat{m}$ satisfied agents away from the first unsatisfied agent under R'. Since agent i' is at least $\hat{m}-1$ satisfied agents away from the first unsatisfied agent, it can be concluded that agent $\hat{j}$ is at least $\hat{m}$ satisfied agents away from the first unsatisfied agent for this case.

Based on the above cases, it can be concluded that $\hat{j} \xrightarrow[\hat{t}]{R'} \check{j}_m \xrightarrow[\hat{t}]{R'} \cdots \xrightarrow[\hat{t}]{R'} \check{j}_1$ with $\left\{\hat{j}, \check{j}_1, \cdots, \check{j}_m\right\} \subseteq S'_{\hat{t}}$ is true whenever $\hat{j} \xrightarrow[\hat{t}]{R} \hat{j}_{\hat{m}} \xrightarrow[\hat{t}]{R} \cdots \xrightarrow[\hat{t}]{R} \hat{j}_1$ with $\left\{\hat{j}, \hat{j}_1, \cdots, \hat{j}_{\hat{m}}\right\} \subseteq S_{\hat{t}}$ holds.

Since $h'_{j'_{\ddot{k}},\hat{t}} = a$ and $h_{\hat{j},\hat{t}} = h'_{\hat{j},\hat{t}} = a$, there is a step $\ddot{t} \in \left\{\hat{t}, \cdots, \tilde{t}-1\right\}$ such that agent $j'_{\ddot{k}}$ becomes part of a trading cycle under R' where $j'_{\ddot{k}} \xrightarrow[\ddot{t}]{R'} \tilde{j}$ such that $h_{\tilde{j},\ddot{t}} = a$. It is possible that $\tilde{j} \neq \hat{j}$ if agent $\tilde{j}$ acquired object a in a trading cycle of some previous step say $\acute{t} \in \left\{\hat{t}, \cdots, \ddot{t}-1\right\}$. Note that agent $\hat{j}$ was at least m satisfied agents away from the closest unsatisfied agent at step $\hat{t}$ under R' because she is at least m satisfied agents from the closest unsatisfied agent at step $\hat{t}$ under R. Hence, for any trading cycle agent $\hat{j}$ becomes part of, after step $\hat{t}$ under R', the agent pointing at her must be at least $m+1$ satisfied agents away from the closest unsatisfied agent since $\hat{j} \in S'_{\hat{t}}$. Therefore, it can be concluded that agent $\tilde{j}$ is at least m satisfied agents away from the first unsatisfied agent at step $\acute{t}$ under R'. Moreover, agent $\tilde{j} \in S_{\ddot{t}}$ because either agent $\tilde{j} = \hat{j} \in S'_{\hat{t}}$ or agent $\tilde{j}$ became part of a trading cycle at step $\acute{t} < \ddot{t}$.

It has already been shown that $\left\{j'_{\ddot{k}}, \cdots, j'_k\right\} \subseteq S'_{\tilde{t}}$. Then, agent $i \in U'_{\tilde{t}}$ for some $i \in \left\{j'_0, \cdots, j'_{\ddot{k}-1}\right\}$ otherwise $j \xrightarrow[\tilde{t}]{R'} j'_k \xrightarrow[\tilde{t}]{R'} \cdots \xrightarrow[\tilde{t}]{R'} j'_1 \xrightarrow[\tilde{t}]{R'} j'_0$ is contradicted. In other words, agent $j'_{\ddot{k}}$ is (at most) $\ddot{k}-1$

satisfied agents away from the closest unsatisfied agent at step $\tilde{t}$ under R' which suggests that agent $j'_{\ddot{k}}$ has a path to an unsatisfied agent through (at most) $\ddot{k}-1$ satisfied agents at step $\ddot{t}$ under R' because $\ddot{t} < \tilde{t}$. This contradicts $j'_{\ddot{k}} \xrightarrow[\ddot{t}]{R'} \tilde{j}$ if $h'_{j'_{\ddot{k}-1},\ddot{t}} = h'_{j'_{\ddot{k}-1},\tilde{t}}$ because, based on this pointing, agent $j'_{\ddot{k}}$ reaches an unsatisfied agent through at least $m+1$ satisfied agents while $m \geq \ddot{k}-1$ suggesting that $m+1 > \ddot{k}-1$ i.e. agent $j'_{\ddot{k}}$ can reach a closer unsatisfied agent by pointing at agent $j'_{\ddot{k}-1}$ at step $\ddot{t}$ under R'. If $h'_{j'_{\ddot{k}-1},\ddot{t}} \neq h'_{j'_{\ddot{k}-1},\tilde{t}}$, then agent $j'_{\ddot{k}-1}$ acquired object $h'_{j'_{\ddot{k}-1},\tilde{t}}$ at some step between $\ddot{t}$ and $\tilde{t}$ under R'. But this would mean that the agent who holds object $h'_{j'_{\ddot{k}-1},\tilde{t}}$ at step $\ddot{t}$ must be (at most) $\ddot{k}-1$ satisfied agents away from the first unsatisfied agent giving the same conclusion as when $h'_{j'_{\ddot{k}-1},\ddot{t}} = h'_{j'_{\ddot{k}-1},\tilde{t}}$.

Therefore, it can be concluded that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ is true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ i.e. statement (3) of *consistent pointing* holds at step $\tilde{t}$. Finally, statement (4) of *consistent pointing* at step $\tilde{t}$ is true by application of Lemma 4 as statement (4) of *consistent pointing* is true at step $\tilde{t}-1$ and statement (3) of *consistent pointing* is true at step $\tilde{t}$. ■

The proof of Proposition 3 relies on the fact that the unique pointee selection criterion of the TC rule, F^{TC}, determines unique pointees so that the closest unsatisfied agent is reached whenever *TC-persistence* is not in effect. If there are multiple such unsatisfied agents or there are multiple paths through which the same unsatisfied agent can be reached, the priority ordering over agents is used to ascertain the unique pointee. Note that, even though the proof of Proposition 3 refers to *TC-persistence*, it does not rely on its differences with *persistence*. The reference is made only to highlight that Jaramillo & Manjunath [8] explicitly impose this restriction on F^{TC}.

The next result establishes *weak group strategy proofness* of the *HPO* rule by showing that it satisfies *consistent pointing*.

Proposition 4. The *HPO* rule satisfies *weak group strategy proofness.*

Proof. Consider $R, R' \in \mathcal{R}^N$ such that $R' = \left(R_{N\backslash M}, R'_M\right)$ for an arbitrary $M \subseteq N$. Let t be the first step for which some agent in M departs, becomes satisfied, or becomes part of a trading cycle under R. Denote the corresponding step under R' as t'. Define $\underline{t} = \min\{t, t'\}$. Suppose $t' < t$. It needs to be proved that the *HPO* rule satisfies *consistent pointing.*

Consider any $\tilde{t} \in \{t', \cdots, t-1\}$ and suppose that the statements of *consistent pointing* hold for all steps $\ddot{t} < \tilde{t}$. Then, to complete the proof, the statements of *consistent pointing* need to be proved for step $\tilde{t}$ because the statements of *consistent pointing* are true for all steps before $\underline{t}$ as discussed in the Remark.

By the induction hypothesis, the statements of *consistent pointing* are satisfied at step $\tilde{t}-1$ and $\tilde{t} < t$, then statements (1) and (2) of *consistent pointing* hold at step $\tilde{t}$ by Lemma 5. By statement (2) of *consistent pointing* at step $\tilde{t}$, $U'_{\tilde{t}} \subseteq U_{\tilde{t}}$ and $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ are true. Then, an agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ points *persistently* at step $\tilde{t}$ under R if and only if she points *persistently* at step $\tilde{t}$ under R'. This is because an agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ is unsatisfied at step $\tilde{t}$ under R if and only if she is unsatisfied at $\tilde{t}$ under R' i.e. $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ implies that any agent who is unsatisfied at step $\tilde{t}$ under R but not under R' must be pointing (directly or indirectly) at an agent in M under R. Moreover, each agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ must belong in $N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$ as otherwise she belongs in $CONN_{M,\tilde{t}-1}$ and, thus, in $CONN_{M,\tilde{t}}$ because no agent in M departs or becomes satisfied before step t under R. Therefore, for each agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that $p_{j,\tilde{t}-1} = p'_{j,\tilde{t}-1}$ by statement (3) of *consistent pointing* at step $\tilde{t}-1$. So, any agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ who is pointing *persistently* at step $\tilde{t}$, it must be that $p_{j,\tilde{t}} = p_{j,\tilde{t}-1}$ and $p'_{j,\tilde{t}} = p'_{j,\tilde{t}-1}$ are true i.e. agent j points *persistently* under R and R'. Therefore, it can be concluded that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ must hold as $p_{j,\tilde{t}-1} = p'_{j,\tilde{t}-1}$ is true.

Now, consider the unique pointees for the unsatisfied agents who are not pointing yet at step $\tilde{t}$ under R and R' in accordance with the unique pointee selection criterion F^{HPO}. Consider any unsatisfied agent j such that $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. By statement (2) of *consistent pointing* at step $\tilde{t}$, agent j is unsatisfied under R and R' because $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ is true. Moreover, every agent in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ also belonged in $N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$ because once an agent points (directly or indirectly) at an agent in M, she continues to do so at least until step t under R. Then, by statement (4) of *consistent pointing* at step $\tilde{t}-1$, for each agent $i \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ it must be the case that $h_{i,\tilde{t}} = h'_{i,\tilde{t}}$ because agent $i \in N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$. Since agent $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it must be that $p_{j,\tilde{t}} \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$. Therefore, it must be that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ because the highest priority object in $\tau\left(R_j, O'_{\tilde{t}}\right)$ must be the highest priority object in $\tau\left(R_j, O_{\tilde{t}}\right)$ as $O'_{\tilde{t}} \subseteq O_{\tilde{t}}$ and $O_{\tilde{t}} \backslash O'_{\tilde{t}} \subseteq h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ by statement (1) of *consistent pointing* at step $\tilde{t}$. Additionally, this object is owned by the same agent at step $\tilde{t}$ under R and R' as $p_{j,\tilde{t}} \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

Let L and L' represent all of the agents who are pointing so far at step $\tilde{t}$ under R and R', respectively. Since

$U'_{\tilde{t}} \subseteq U_{\tilde{t}}$ and $U_{\tilde{t}} \backslash U'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$, it must be that $L' \subseteq L$ and $L \backslash L' \subseteq CONN_{M,\tilde{t}-1}$ are true. Let agent $j_1 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ be the first agent chosen to point from the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ who are not in L under R. Then, agent j_1 must be the first agent chosen to point from the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ who are not in L' under R' because agents are selected for unique pointee determination on the basis of the priority ordering of the object they hold and statement (4) of *consistent pointing* is true at step $\tilde{t}-1$. Since $L' \subseteq L$ and $L \backslash L' \subseteq CONN_{M,\tilde{t}-1}$ are true, for any agent who is selected to point before agent j_1 under R, one of the following must be true:

1. She is selected to point before agent j_1 under R',

2. She has already departed under R', or

3. She holds different objects under R and R'.

Consider an agent $i \in N_{\tilde{t}}$ who has already departed under R'. Then, agent i is already labeled under R because an agent who has departed under R' but not under R must be pointing (directly or indirectly) at an agent in M and she continues to do so at least until step t under R. Therefore, agent $i \in CONN_{M,\tilde{t}}$ and agents in $CONN_{M,\tilde{t}}$ continue to point *persistently* at least until step t under R. Now, consider an agent $i \in N'_{\tilde{t}}$ such that $h_{i,\tilde{t}} \neq h'_{i,\tilde{t}}$. Then, by statement (4) of *consistent pointing* at step $\tilde{t}-1$, it must be that agent $i \in CONN_{M,\tilde{t}-1}$. Therefore, in both cases, agent i became part of $CONN_{M,\ddot{t}}$ at some step $\ddot{t} < \tilde{t}$. Moreover, $h_{i,\tilde{t}}, h'_{i,\tilde{t}} \in h_{CONN_{M,\tilde{t}-1},\tilde{t}}$ because once an agent points (directly or indirectly) at an agent in M, she continues to do so at least until step t under R. Hence, these objects are held by agents who are labeled at step $\tilde{t}$ under R. So, before agent j_1 is chosen to point at step $\tilde{t}$, all of the objects that are held by labeled agents under R' must be held by labeled agents under R as well. Moreover, any agents who are labeled under R but not under R' must belong in $N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$. Agent j_1 points at the labeled agent who holds one of her most preferred objects and, if there are multiple such labeled agents, agent j_1 points at the labeled agent who holds the object with a higher priority under the priority ordering $\prec$. Then, it needs to be established that this object is held by the same agent at step $\tilde{t}$ under R and R'. Since agent $j_1 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, then it must be that $p_{j_1,\tilde{t}} \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ is true. Let $p_{j_1,\tilde{t}} = \widehat{j}_1$. Then, by *persistence*, it must be that $j_1, \widehat{j}_1 \in N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$ otherwise $j_1, \widehat{j}_1 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ is contradicted because once an agent points (directly or indirectly) at an agent in M, she continues to do so at least until step t under R. Then, by statement (4) of *consistent pointing* at step $\tilde{t}-1$, it must be that $h_{\widehat{j}_1,\tilde{t}} = h'_{\widehat{j}_1,\tilde{t}} = a$ is true. Note that agent $\widehat{j}_1$ has not departed at step $\tilde{t}$ under R' because $N_{\tilde{t}} \backslash N'_{\tilde{t}} \subseteq CONN_{M,\tilde{t}-1}$ i.e. any agent who has departed at step $\tilde{t}$ under R' but not under R must belong in $CONN_{M,\tilde{t}-1}$ whereas agent $\widehat{j}_1 \in N_{\tilde{t}-1} \backslash CONN_{M,\tilde{t}-1}$. Furthermore, agent $\widehat{j}_1$ is labeled before agent j_1 is chosen to point at step $\tilde{t}$ under R and R'. Then, it must be that $p_{j_1,\tilde{t}} = p'_{j_1,\tilde{t}}$. Update L and L' to represent all of the agents who are pointing so far at step $\tilde{t}$ under R and R', respectively. Again, $L' \subseteq L$ and $L \backslash L' \subseteq CONN_{M,\tilde{t}-1}$ must be true. Now, let agent $j_2 \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ be the next agent chosen to point from the agents in $N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ who are not in L under R. Following the same reasoning, as for agent j_1, it can be shown that $p_{j_2,\tilde{t}} = p'_{j_2,\tilde{t}}$. Proceeding in this manner, it can be concluded that $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$.

Since $p_{j,\tilde{t}} = p'_{j,\tilde{t}}$ (statement (3) of *consistent pointing* at step $\tilde{t}$) and $h_{j,\tilde{t}} = h'_{j,\tilde{t}}$ (statement (4) of *consistent pointing* at step $\tilde{t}-1$) are true for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$, it can be concluded that $h_{j,\tilde{t}+1} = h'_{j,\tilde{t}+1}$ for each $j \in N_{\tilde{t}} \backslash CONN_{M,\tilde{t}}$ by Lemma 4. This completes the proof of *weak group strategy proofness* for the *HPO* rule. ■

The unique pointee selection criterion for the *HPO* rule, F^{HPO}, determines the unique pointees of the unsatisfied agents after *persistence* has been ensured. Once the unsatisfied agents are dealt with, the unique pointees are determined for the satisfied agents who are not already pointing. In this sense, the proof of *weak group strategy proofness* for the *HPO* rule is simpler than the *TC* rule because it just requires a comparison of the set of labeled agents, under R and R', to show that the *HPO* rule satisfies *consistent pointing* instead of a detailed analysis of the progression of the pointing phase.

Based on the results of this paper and the results that have already been proved for the *TTAS*, *TC*, and *HPO* rules, the following theorem can be stated:

Theorem 2. For a housing market problem with weak preferences, there are computationally efficient rules which are *Pareto efficient*, *weak core-selecting* (hence, *individually rational*), *weakly group strategy proof* (hence, *strategy proof*), and *core-selecting* (whenever the *core* is non-empty).

6 Conclusion

When the housing market problem is considered with weak preferences, several mechanisms have been shown to satisfy desirable properties like *Pareto efficiency*, *individual rationality*, *weak core-selection*, and *strategy proofness*. This paper provides a sufficient condition for *weak group strategy proofness* in the context of the housing market problem with weak preferences. Using this sufficient condition, I show that three of the existing rules - the *top trading absorbing sets*, *top cycles*, and *highest priority object* rules - satisfy *weak group strategy proofness*. Thus, this paper proves that even though *group strategy proofness* is incompatible with *Pareto efficiency* for weak preferences, it is possible to achieve *weak group strategy proofness*. Moreover, since the *top cycles* and *highest priority object* rules are computationally efficient, this paper proves the existence of computationally efficient rules which are *Pareto efficient*, *weak core-selecting* (hence, *individually rational*), *weakly group strategy proof* (hence, *strategy proof*), and *core-selecting* (whenever the *core* is non-empty).

Alternatively, the results of this paper can be interpreted to suggest that *weak group strategy proofness* is not restrictive enough for the housing market problem with weak preferences to identify any of the three existing mechanisms, considered in this paper, to be better than the others. It might be of interest to determine if there are any rules in the class of mechanisms proposed by Saban & Sethuraman [17] which fail to satisfy *weak group strategy proofness*.

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